Recurrence properties ("Returning")

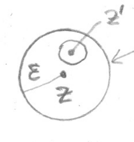
Standing assumption: f is a continuous map of a compact metric space X.

Def Let $x \in X$. A point $y \in X$ is called an *ω-limit point* for x if there exists a sequence $n_k \to \infty$ s.t. $f^{n_k}(x) \to y$.
The set of all ω-limit points for x is called the *ω-limit set* for x and denoted $\omega(x)$.

If f is invertible, we can also define α-limit pts and set for x. $z \in X$ is an *α-limit pt.* for x if there is a sequence $n_k \to -\infty$ s.t. $f^{n_k}(x) \to z$. The set of all such z is denoted $\alpha(x)$.

Ex $R_\alpha : S^1 \to S^1$
If $\alpha \in \mathbb{Q}$, for each $x \in S^1$, $\omega(x) = \alpha(x) =$ the orbit of x (a finite set)
If $\alpha \notin \mathbb{Q}$, —— " ——, $\omega(x) = \alpha(x) = S^1$.

Properties of the set $\omega(x)$ (similar for $\alpha(x)$)

- $\omega(x) \neq \emptyset$ by compactness of X.
- $\omega(x)$ is f-invariant by continuity of f ($f^{n_k}(x) \to y \Rightarrow f^{n_k+1}(x) \to f(y)$)
- $\omega(x)$ is closed since its complement is open $\longrightarrow$ 

$\leftarrow$ not visited by $f^n(x)$, $n \geq N$.
- $\omega(x) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{k \geq n} f^k(x)}$ (Check)

Recurrent points

Def A point $x \in X$ is called *(positively) recurrent* if $x \in \omega(x)$,
i.e. there is a sequence $n_k \to \infty$ s.t. $f^{n_k}(x) \to x$
Equivalently, for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ s.t. $d(f^n(x), x) < \varepsilon$ (Why?).

For an invertible f, we say that x is negatively recurrent if $x \in \alpha(x)$, and recurrent if it is both positively and negatively recurrent.

Ex
- Every periodic point is recurrent
- If X has no isolated points, then every pt. with dense positive semiorbit is positively recurrent.
- For $R_\alpha : S^1 \to S^1$ every x is recurrent
- For $E_m : S^1 \to S^1$, the points $x = \frac{k}{3^n}$, $0 < k < 3^n$, are *not* recurrent.
- For a hyperbolic automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$,
  $x \neq 0$ on $W^s(0)$ are *not* pos. recurrent
  $x \neq 0$ on $W^u(0)$ are *not* neg. recurrent
  $x \neq 0$ on $W^s(0) \cap W^u(0)$ are neither pos. nor neg. recurrent

$W^s(0)$, $W^u(0)$, and $W^s(0) \cap W^u(0)$ are dense in $\mathbb{T}^2$.

<u>Note</u> a point can be positively, but not negatively recurrent. (Example - ?)

(?) Does $f: X \to X$ have at least one (positively) recurrent point? <u>Yes</u>.

We say that a closed non-empty set $Y \subseteq X$ is <u>minimal</u> if $f(Y) \subseteq Y$, i.e. Y is forward-invariant, and there is no closed $\emptyset \neq A \subsetneq Y$ s.t. $f(A) \subseteq A$.
The latter is equivalent to: for every $x \in Y$, $O^+(x)$ is dense in Y.

<u>Note:</u> If Y is minimal, then $f(Y) = Y$. Otherwise we can take $A = f(Y)$

<u>Prop</u> If $Y \subseteq X$ is minimal, then every $x \in Y$ is positively recurrent.

<u>Pf</u> Suppose that for some $x \in Y$, there is no $n_k \to \infty$ s.t. $f^{n_k}(x) \to x$.
Then $A = \overline{O^+(f(x))}$ is closed, $\neq \emptyset$, $f(A) \subseteq A \subseteq Y$, and $A \neq Y$ since $x \notin A$. □

<u>Prop</u> X contains a minimal set.

<u>Pf</u> We will use <u>Zorn's Lemma</u>:
Let C be a partially ordered set. If every totally ordered subset of C has an upper bound in C, then C has a maximal element.
Reversing the order on C, it follows that if ... has a lower bound in C then C has a minimal element ...
(i.e. $A \in C$ s.t. there is no $A' \in C$ with $A' \leq A$ and $A' \neq A$)

Let C be the collection of all non-empty closed forward-invariant subsets of X, partially ordered by inclusion, and let C' be a totally ordered subcollection.
Any finite intersection of elements of C' is non-empty, and it follows by compactness of X that the intersection of all elements of C' is non-empty.
This intersection is in C, and it is a lower bound for C'.
Thus C has a minimal element, Y, which is a minimal subset for X.

<u>Corollary</u> $f: X \to X$ has at least one positively recurrent point.

<u>Note</u> This is not necessarily the case for non-compact X.
Consider, for example, $f(x) = x+1$ on $\mathbb{R}$ or $f(x) = \frac{x}{2}$ on $(0, 1]$.

<u>Note</u> The sets of positively recurrent / negatively recurrent / recurrent points are not necessarily closed.
Consider, for example $E_3: S^1 \to S^1$ or a hyperbolic automorphism $f: \mathbb{T}^2 \to \mathbb{T}^2$. In each case, periodic points are dense $\Rightarrow$ recurrent points are dense, but non-recurrent pts are also dense.