(4)

Attracting fixed points

Last time: an example of $f:[a,b]\to[a,b]$ with 4 fixed pts;

In each open interval, $(f^n(x))$ converges monotonically to one of the endpoints.

c_2 and c_4 are attracting fixed points,
c_3 is a repelling fixed point, c_1 is neither attracting nor repelling.
Note that there is an open interval around c_2 (similarly c_4)
that is mapped into itself, and for any x in it, $f^n(x)\to c_2$.

Let I be an interval, and let $f: I\to I$ be a differentiable function with $f(c)=c$.
If $|f'(c)|<1$, then c is attracting
If $|f'(c)|>1$, then c is repelling
If $|f'(c)|=1$, then c can be attr., rep., or neither.

Let $U\subseteq\mathbb{R}^n$ be open, let $f: U\to U$ be differentiable with $f(c)=c$.
If $\|D_cf\|<1$, then c is attracting
If $\|D_cf\|>1$, c is not necessarily repelling Ex 
If D_cf has an eigenvalue λ with $|\lambda|>1$, then c is not attracting.

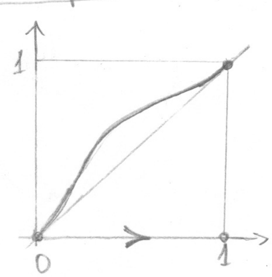
The fixed point of a contraction is attracting.

(?) How to define an attracting fixed pt for $f: X\to X$

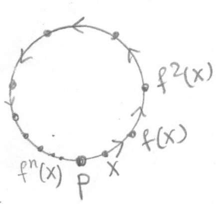
Let $f(p)=p$. We definitely want the following:
There exists $r>0$ s.t. $f^n(x)\to p$ as $n\to\infty$ for all $x\in B_r(p)$.

Requiring just this does not yield a good definition, as the following example shows.

Example



First consider a differentiable $f:[0,1]\to[0,1]$ as shown with $f'(0)=f'(1)=1$. For example,
$f(x)=x+\frac{1}{10}\sin^2(\pi x)$ with $f'(x)=1+\frac{2\pi}{10}\sin(\pi x)\cos(\pi x)$
Then for each $x\in(0,1)$, $(f^n(x))$ is strictly increasing and $f^n(x)\to 1$ as $n\to\infty$



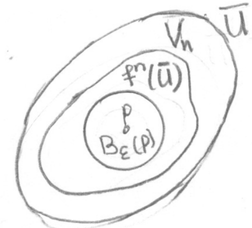
Now, we identify 0 and 1 and obtain a diffeomorphism of the circle S^1 with a single fixed point p. For every $x\in S^1$, $f^n(x)\to p$ as $n\to\infty$
However, we would not call such p an attracting fixed pt. since every x "to the right" of p, (as shown) no matter how close to p, moves away from p and leaves any small interval around p.

a definition

Let X be a compact [or, more generally, locally compact] metric space, and let $f: X \to X$ be a continuous map with $f(p) = p$.

<u>Def</u> We say that p is an <u>attracting</u> fixed pt. if there is an open set $U \ni p$ [with $\bar{U}$ compact] s.t. $f(\bar{U}) \subseteq U$ and $\bigcap_{n=0}^{\infty} f^n(U) = \{p\}$.

<u>Claim</u> It follows that for every $x \in \bar{U}$, $f^n(x) \to p$ as $n \to \infty$.



<u>Pf</u> Since $f(\bar{U}) \subseteq U$, we have $\bar{U} \supseteq f(\bar{U}) \supseteq f^2(\bar{U}) \supseteq \dots$

Let $V_n = \bar{U} \setminus f^n(\bar{U})$. Then $V_1 \subseteq V_2 \subseteq V_3 \subseteq \dots$, $(*)$, $\bigcup_{n=0}^{\infty} V_n = \bar{U} \setminus \{p\}$, and V_n are open in $\bar{U}$ since $f^n(\bar{U})$ is compact and hence closed.

Let $\varepsilon > 0$ be s.t. $B_\varepsilon(p) \subset U$. Then $\{V_n\}$ is an open cover of $\bar{U} \setminus B_\varepsilon(p)$. Since this set is compact, it has a finite subcover, and by $(*)$ there is N s.t. $\bar{U} \setminus B_\varepsilon(p) \subseteq V_N$. Hence $f^n(\bar{U}) \subseteq B_\varepsilon(p)$ for all $n \geq N$. Thus $f^n(x) \to p$ for all $x \in \bar{U}$. $\square$

<u>Note:</u> there is no such U in the Example.

<u>Equivalent definitions</u> (showing equivalence takes some work).

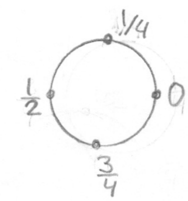
<u>Def 2</u> p is attracting if there is an open set $V \ni p$ s.t. $\bigcap_{n=0}^{\infty} f^n(V) = \{p\}$.

<u>Def 3</u> p is attracting if ① there is $r > 0$ s.t. $f^n(x) \to p$ for all $x \in B_r(p)$ and ② for each $\varepsilon > 0$ there is $\delta > 0$ s.t. $f^n(B_\delta(p)) \subseteq B_\varepsilon(p)$ for all $n \geq 0$, that is, if $d(x,p) < \delta$, then $f^n(x) \in B_\varepsilon(p)$ for all $n \geq 0$.

Circle rotations

The circle: $S^1 = \mathbb{R}/\mathbb{Z}$ with addition mod 1.

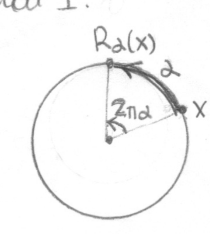
Each equivalence class has a unique representative in $[0,1)$, and $0 \sim 1$, so we can view S^1 as $[0,1]$ with 0 and 1 identified.



<u>Note</u> One can also represent S^1 as $\{z \in \mathbb{C} : |z| = 1\} = \{e^{2\pi i x} : x \in \mathbb{R}\}$

For $\alpha \in \mathbb{R}$, the map $R_\alpha : S^1 \to S^1$ is defined by $R_\alpha(x) = x + \alpha \mod 1$.

We have: $R_0 = Id$, $(R_\alpha)^n = R_{n\alpha}$, R_α is invertible with $(R_\alpha)^{-1} = R_{-\alpha}$.



- For a rational α, all x are periodic with the same prime period. Let $\alpha = \frac{p}{q}$, $p \neq 0$, $q > 1$, p, q relatively prime. Then the prime period is q (Why?)
- For an irrational α, there are no periodic pts. How is an orbit distributed?