<u>Question</u> Given a homeomorphism (or a diffeomorphism) $h: X \to X$, does there exist a flow $\{\varphi^t\}$ on X such that $f = \varphi^1$?

Typically, <u>No</u>. Some obstacles:

(0) Suppose $X = X_1 \dot\cup X_2$, where X_1, X_2 are open, non-empty, and disjoint. If there is $x \in X_1$ s.t. $f(x) \in X_2$, then there is no $\{\varphi^t\}$ s.t. $f = \varphi^1$ (Explain why). So let us assume that X is connected.

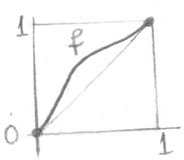
(1) If $f = \varphi^1$ for a flow $\{\varphi^t\}$, then for every t the map φ^t commutes with f. Often, (almost) only the iterates f^n, $n \in \mathbb{Z}$, commute with f.

(2) Suppose f has a periodic point $x \neq f(x)$ of period $k > 1$. If $f = \varphi^1$, then for every $t \in \mathbb{R}$, $f^k(\varphi^t(x)) = \varphi^{k+t}(x) = \varphi^t(f^k(x)) = \varphi^t(x)$. Since $x \neq f(x)$, x is not fixed by the flow $\{\varphi^t\}$, and so f has other periodic pts of period k arbitrarily close to x. Therefore, if f has an isolated periodic pt. $x \neq f(x)$ of period k, then $f \neq \varphi^1$

(3) If $f = \varphi^1$ for a flow $\{\varphi^t\}$, then f is homotopic to the Id. Indeed, $H: X \times [0,1] \to X$ given by $H(x,t) = \varphi^t(x)$ is a homotopy. automorphisms of $\mathbb{T}^n$, $n \geq 2$, that are $\neq$ Id are not homotopic to Id.

<u>Example</u> of $f = \varphi^1$.

Let $f: [0,1] \to [0,1]$ be



for example,
$f(x) = x + \frac{1}{10}\sin(\pi x)$ or
$f(x) = x + \frac{1}{10}\sin^2(\pi x)$

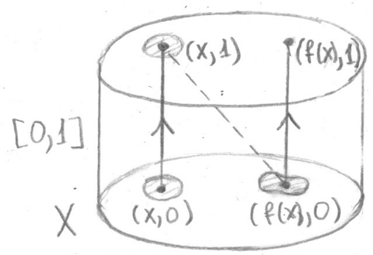
Consider the flow $\{\tilde\varphi^t\}$ given by $\frac{dx}{dt} = g(x)$ as in Lec. 36,

and let $\tilde f = \tilde\varphi^1$. Then $\tilde f$ looks similar to f, and f and $\tilde f$ are top. conjugate (see the last part of Lec. 12).
Let $h: [0,1] \to [0,1]$ be a homeomorphism s.t. $f = h \circ \tilde f \circ h^{-1}$.
Then $\{\varphi^t\}$ with $\varphi^t = h \circ \tilde\varphi^t \circ h^{-1}$ is a flow on $[0,1]$, and $f = \varphi^1$.

? Would it work if we set $\varphi^t(x) = x + \frac{1}{10}\sin^{(2)}(t\pi x)$.
<u>No</u> The condition $\varphi^t \circ \varphi^s = \varphi^{t+s}$ for all s,t would not hold.
also, for a fixed $x \in (0,1)$, $\{t\pi x\} = (-\infty, \infty)$, and so $\varphi^t(x)$, as a function of t, is not monotone. But it should be (Why?)

Suspension flow over $f: X \to X$.

Let f be a homeomorphism of a compact metric space X.
First, let us describe a special case when the roof function is $r \equiv 1$.
Consider the quotient space $X_f = X \times [0,1] \,/\, (x,1) \sim (f(x),0)$



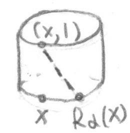
Quotient topology on X_f:
$U \subset X_f$ is open $\iff \pi^{-1}(U)$ is open in $X \times [0,1]$
Note then $B_\varepsilon(x)$ in $X \times \{1\}$ is identified by the homeomorphism f with an open nbhd of $f(x)$ in $X \times \{0\}$

The suspension flow $\{\varphi^t\}$ on X_f is as follows:

Points move along vertical segments with unit speed until they reach $(x,1)$, which is identified with $(f(x),0)$, then continue with unit speed along the vertical segment at $f(x)$, ...
For $x \in X$ and $0 \le s < 1$, $\varphi^t((x,s)) = (f^n(x), s')$, where $n = \lfloor s+t \rfloor$ and $s' = s+t-n$
While f is not φ^1, we have $\varphi^1((x,s)) = (f(x), s)$ for all $x \in X$, $s \in [0,1)$.

Ex. $X = S^1$ and $f = Id$. Then $X_f = \mathbb{T}^2$ and $\varphi^t = T^t_{(0,1)}$

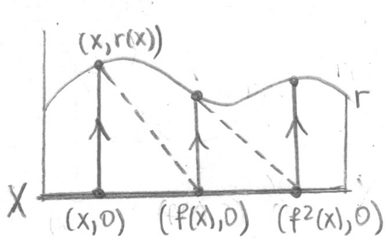
Ex $X = S^1$ and $f = R_\alpha$. Then X_f homeomorphic to $\mathbb{T}^2$
and the map $(x,s) \mapsto (x - \alpha s, s)$ from $\mathbb{T}^2$ to X_f
gives a top. (smooth) conjugacy between $\{\varphi^t\}$ and $T^t_{(\alpha,1)}$.
So $T^t_{(\alpha,1)}$ can be viewed as a suspension flow over R_α



Ex $X = S^1$, $f(x) = 1-x$. Then X_f is the Klein bottle.

In general, instead of $r \equiv 1$, we consider a continuous (or smooth) function $r: X \to (0,\infty)$, called a roof (or ceiling) function. Since X is compact, r is bounded away from 0. We consider the quotient space

$$\tilde{X} = X_{r,f} = \{(x,s) : x \in X,\ 0 \le s \le r(x)\} \,/\, (x, r(x)) \sim (f(x), 0)$$



as before, points move along vertical segments with unit speed until they reach the "roof". The flow $\{\varphi^t\}$ is called a suspension flow or a flow under a function.

? What are the properties of $\{\varphi^t\}$? What can we say about periodic pts, top. transitivity, top. mixing?

? Is the flow with a roof function r top. conjugate to the one with $r \equiv 1$? Are they equivalent in some weaker sense?