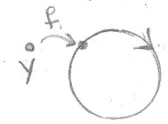
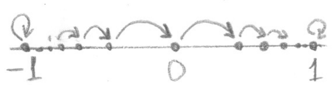
? Does topological transitivity of $f: X \to X$ imply $NW(f) = X$?

No in general. Examples:



$X = S^1 \cup \{y\}$. $f(y) \in S^1$, $f|_{S^1} = R_\alpha$, $\alpha \notin \mathbb{Q}$.
the orbit of y is dense in X, $NW(f) = S^1 \neq X$



$X = \{-1, 1\} \cup \{x_n : n \in \mathbb{Z}\}$
the orbit of 0 is dense in X
$NW(f) = \{-1, 1\}$

OR

$NW(f) = \{1\}$

Note that in each of the examples above X has an isolated point.

Yes, if X has no isolated points.
Then f is top. transitive $\iff$ for any non-empty open sets U, V in X there is $n \in \mathbb{N}$ s.t. $f^n(U) \cap V \neq \emptyset$. Take $V = U$.
another explanation: Let y be a pt. with dense orbit. Then, since X has no isolated points, every $x \in X$ is an ω-limit point for y (Why?) i.e. $\omega(y) = X$. Since $\omega(y) \subseteq NW(f)$, $NW(f) = X$.

? Does top. mixing imply $NW(f) = X$?
Yes Since top. mixing means that for any non-empty open sets U, V in X there is $N \in \mathbb{N}$ s.t. $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$. Again, take $V = U$.

Poincaré Recurrence Theorem.

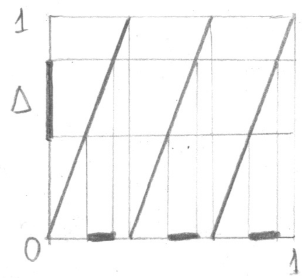
? For $E_m : S^1 \to S^1$ and for a hyperbolic automorphism $L: \mathbb{T}^2 \to \mathbb{T}^2$, recurrent pts are dense (and non-recurrent pts are also dense)
Does the set of recurrent points have full Lebesgue measure?

Def A probability space is a triple $(X, \mathcal{A}, \mu)$, where $\mathcal{A}$ is a σ-algebra and μ is a measure on $\mathcal{A}$ s.t. $\mu(X) = 1$
(If $\mu(X) < \infty$, μ can be rescaled so that $\tilde{\mu}(X) = 1$)
A map $f: X \to X$ is called measure-preserving if for every $A \in \mathcal{A}$, $f^{-1}(A) \in \mathcal{A}$ and $\mu(f^{-1}(A)) = \mu(A)$

Note We require the measure of the preimage of A to equal $\mu(A)$.

Examples. We consider the Lebesgue measure on S^1 and on $\mathbb{T}^2$.

- $R_\alpha: S^1 \to S^1$ and $T_\gamma: \mathbb{T}^2 \to \mathbb{T}^2$ are measure-preserving.
- $E_m: S^1 \to S^1$ is also measure preserving. Indeed for any interval (arc) $\Delta \subsetneq S^1$, the preimage of Δ consists of m disjoint intervals of length $\frac{1}{m}|\Delta|$.



- $L: \mathbb{T}^2 \to \mathbb{T}^2$ given by $L(u) = Au \bmod 1$ is measure-preserving since $|\det A| = 1$.

Poincaré Recurrence Thm.

Let f be a measure-preserving transformation of a probability space $(X, \mathcal{A}, \mu)$, and let A be a measurable set, i.e. $A \in \mathcal{A}$. Then

(a) almost every pt. in A returns to A, i.e. $\mu(\{x \in A : f^n(x) \notin A \text{ for all } n \in \mathbb{N}\}) = 0$,

(b) Moreover, almost every pt. in A returns to A infinitely many times.

Note The result is meaningful for A with $\mu(A) > 0$.

Pf (a) Let $B = \{x \in A : f^n(x) \notin A \text{ for all } n \in \mathbb{N}.\}$

Then B is measurable since $B = A \setminus (\bigcup_{n \in \mathbb{N}} f^{-n}(A))$.

Claim: The sets $f^{-n}(B)$, $n \in \mathbb{N}_0$, are pairwise disjoint.

Suppose $f^{-n}(B) \cap f^{-m}(B) \neq \emptyset$ for some $0 \le m < n$. Then $B \cap f^{n-m}(B) \neq \emptyset$, and so a point from B returns to $B \subseteq A$

Since f is measure-preserving, $\mu(f^{-n}(B)) = \mu(B)$ for each $n \in \mathbb{N}_0$.

Since $\mu(X) = 1$, it follows that $\mu(B) = 0$.

(b) Hw.

Corollary Let X be a compact metric space, let μ be a Borel probability measure on X, and let $f: X \to X$ be a cont. measure-preserving map. Then μ-almost every $x \in X$ is recurrent.

Pf Let $\{U_i : i \in \mathbb{N}\}$ be a countable basis of topology of X. (Take a countable dense set in X and balls of rational radii centered at its points.) Then a point $x \in X$ is recurrent $\iff$ it returns to each U_i that contains it (Explain this!). For each $i \in \mathbb{N}$, the set B_i of points in U_i that do not return to U_i has measure 0 by Poincaré Recurrence Thm. The set of recurrent pts. equals $X \setminus (\bigcup_{i \in \mathbb{N}} B_i)$, and hence it has full measure. $\square$

Corollary For $E_m: S^1 \to S^1$ and $L: \mathbb{T}^2 \to \mathbb{T}^2$ almost every pt (with respect to the Lebesgue measure) is recurrent.