(12)

Topological conjugacy and semiconjugacy.

Def Let X and Y be compact metric spaces and let $f: X \to X$ and $g: Y \to Y$ be continuous maps.

We say that $g: Y \to Y$ is a factor of $f: X \to X$ if there exists a continuous surjective map $h: X \to Y$ such that the following diagram is commutative; i.e. $h \circ f = g \circ h$.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

If there is a homeomorphism $h: X \to Y$ s.t. $h \circ f = g \circ h$, equivalently, $f = h^{-1} \circ g \circ h$, we say that f and g are topologically conjugate and call h a top. conjugacy.

T/F If $g: Y \to Y$ is a factor of $f: X \to X$, then for every $n \in \mathbb{N}$ g^n is a factor of f^n. T Pf. by induction:

If $h \circ f^n = g^n \circ h$, then $h \circ f^{n+1} = h \circ f^n \circ f = g^n \circ h \circ f = g^n \circ g \circ h = g^{n+1} \circ h$.

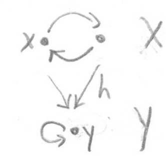
Properties of a semiconjugacy

(1) If $x \in X$ is fixed by f, then $y = h(x)$ is fixed by g.

(2) If $f^n(x) = x$ for some $x \in X$ and $n \in \mathbb{N}$, then for $y = h(x)$, $g^n(y) = y$.

Pf $g^n(y) = g^n(h(x)) = h(f^n(x)) = h(x) = y$. □.

? Is the prime period of y the same as for x? Not necessarily, it can be smaller.

Ex 

(3) If the orbit of $x \in X$ is dense in X, then the orbit of $y = h(x)$ is dense in Y (Hw)

(4) Hence if $f: X \to X$ is top. transitive / minimal, then g is top. transitive/minimal.

(5) If $A \subseteq X$ is f-invariant, then $B = h(A)$ is g-invariant.

Pf Let $y \in B$. Then there is $x \in A$ s.t $y = h(x)$. Since A is f-invariant, $f(x) \in A$, and hence $g(y) = g(h(x)) = h(f(x)) \in B$. □.

Note that if A is closed, then $B = h(A)$ is also closed (since X is compact and h is continuous).

Properties of topological conjugacy

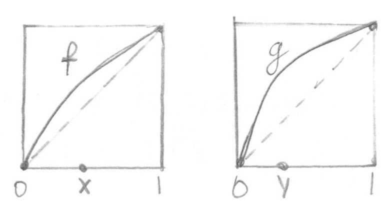
Suppose that $f: X \to X$ and $g: Y \to Y$ are top. conjugate via h.

(0) The spaces X and Y are homeomorphic.
(1) x is fixed $\iff$ $h(x)$ is fixed
(2) x is periodic with prime period n $\iff$ $h(x)$ —— " ——
(3) $\mathcal{O}(x)$ is dense in X $\iff$ $\mathcal{O}(h(x))$ is dense in Y
(4) $f: X \to X$ is top. trans/min $\iff$ $g: Y \to Y$ is —— " ——
(5) a set $A \subset X$ is f-invar. $\iff$ $h(A)$ is g-invar.

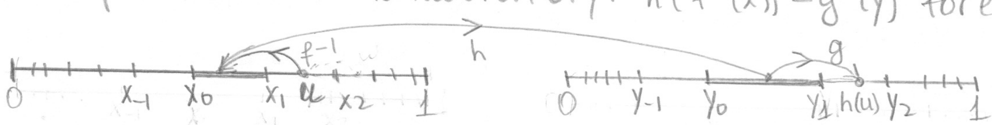
In general, topologically conjugate dynamical systems have identical topological properties, i.e. are "qualitatively the same".
Note that top. conjugacy is an equivalence relation on dynamical systems.
In general, there is no list of properties so that if for $f: X \to X$ and $g: Y \to Y$ they are the same, then f and g are top. conjugate, and it may be difficult to prove top. conjugacy. Sometimes, one can find/construct h.

Examples

- E_m and E_n for $m \neq n$ are not top. conjugate since they have different number of fixed pts, $m-1$ and $n-1$, respectively.
- We considered a semiconjugacy from $\sigma: \Omega_m^R \to \Omega_m^R$ to $E_m: S^1 \to S^1$.
  Does there exist a conjugacy? No
  σ has m fixed pts, $000\ldots$, $11\ldots$, $\ldots$, $(m-1)(m-1)\ldots$, while E_m has $m-1$.
  Another explanation: S^1 is connected, while Ω_m^R is not, so they are not homeomorphic.
- In general, R_α and R_β with $\alpha \neq \beta$ are not top. conj,
  but R_α and $R_{1-\alpha} = R_{-\alpha}$ are. Let $h(x) = -x$, then $h \circ R_\alpha = R_{-\alpha} \circ h$ (check).
- Let $f, g: [0,1] \to [0,1]$ be continuous strictly increasing with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ and $f(x), g(x) > x$ for all $x \in (0,1)$.
  Then f and g are top. conjugate.



A construction of h: let $h(0) = 0$ and $h(1) = 1$. We choose any $x, y \in (0,1)$ and map the orbit of x to the orbit of y: $h(f^n(x)) = g^n(y)$ for each $n \in \mathbb{Z}$.



We denote $x_n = f^n(x)$ and $y_n = f^n(y)$. For any $n \in \mathbb{Z}$, $f(x_{n-1}, x_n) = (x_n, x_{n+1})$
First we define h on (x_0, x_1) by setting it to be a linear map onto (y_0, y_1).
Next, we define h on (x_1, x_2). For $u \in (x_1, x_2)$, we set $h(u) = g(h(f^{-1}(u)))$.
Continuing, we define h on each of the intervals (x_n, x_{n+1}) and obtain a homeomorphism $h: [0,1] \to [0,1]$ s.t. $h \circ f = g \circ h$.