Linear maps in the plane

An application: The differential $D_p f$ at a fixed point p of a differentiable map $f: \mathbb{R}^2 \to \mathbb{R}^2$ (or $f: U \to U$, where $U \subset \mathbb{R}^2$) or $f: \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is a smooth manifold of dim. 2.

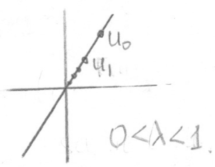
A linear map $F: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $F(x) = Ax$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
0 is a fixed point for F.

Eigenvalues of A: $0 = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (\operatorname{Tr} A)\lambda + \det A$.
$D = (\operatorname{Tr} A)^2 - 4 \det A$.

- $D > 0$ 2 distinct real e.values, λ and μ, A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$
- $D = 0$ 1 real e.values of multiplicity 2, A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$
- $D < 0$ 2 complex conjugate e.values, $\alpha + \beta i$, and $\alpha - \beta i$,
  and A is similar to $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \sqrt{\alpha^2+\beta^2} \begin{pmatrix} \alpha/r & -\beta/r \\ \beta/r & \alpha/r \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

The action of the "model" matrices on $\mathbb{R}^2$.

(1) $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda\, \mathrm{Id}$, $\lambda \neq 0$. $F(u) = \lambda u$.
Every straight line through 0 is mapped to itself. So these lines are invariant under F. All iterates of $u \neq 0$ lie on the same line.



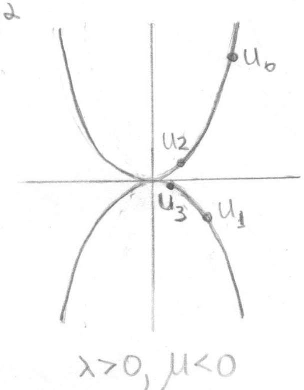
(2) $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \neq \mu$ and $\lambda, \mu \neq 0$.
x-axis and y-axis are invariant, but no other lines hrough 0 are.

Prop. The curves $|y| = C|x|^\alpha$, where α is so that $|\lambda|^\alpha = |\mu|$ are invariant under F.

Pf Check that if $|y| = C|x|^\alpha$, then $|\mu y| = C|\lambda x|^\alpha$. □

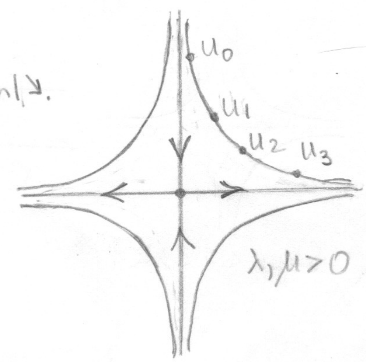
(2a) $0 < |\mu| < |\lambda| < 1$, so $\alpha > 1$.
F is a contraction with $\lambda_{contr.} = |\lambda|$.

The curve $|y| = C|x|^\alpha$ has 4 branches. If $\lambda, \mu > 0$, then $u_n = F^n(u)$ remain on one branch. Otherwise, u_n jumps between branches.



(2b) $0 < |\mu| < 1 < |\lambda|$, so $\alpha < 0$.
Such A is called hyperbolic.

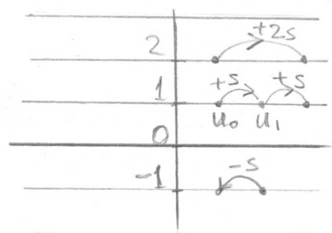
In this case, $|x_n| \nearrow$ and $|y_n| \searrow$.
$|y| = C|x|^\alpha$ with $\alpha < 0$:



(3) $A=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$, $\lambda\neq 0$. $A=\lambda\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}$,

(3a) $A=\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}$, $s\neq 0$. $F\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+sy\\ y\end{pmatrix}$

The x-axis is fixed, horizontal lines are invariant. This map is called a <u>shear</u>.



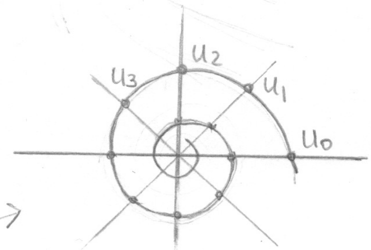
$A=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ is similar to $B=\begin{pmatrix}\lambda & \lambda\\ 0 & \lambda\end{pmatrix}=\lambda\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. $B^n=\lambda^n\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$, so

$B^n\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\lambda^n(x+ny)\\ \lambda^n y\end{pmatrix}=\begin{pmatrix}x_n\\ y_n\end{pmatrix}$ For $|\lambda|<1$, $|x_n|\to 0$ and $|y_n|\to 0$, and $|y_n|$ is decreasing.

There are invariant curves for this case as well.

(4) $A=r\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, $r>0$, $\theta\neq\pi k$.

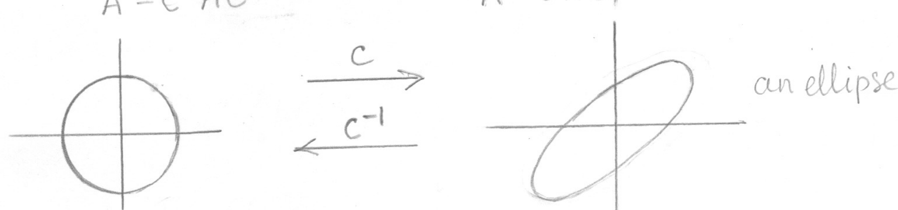
Counterclockwise rotation by θ and scaling by r. Invar. curves are spirals.
For example, let $\theta=\frac{\pi}{4}$ and $r<1$.



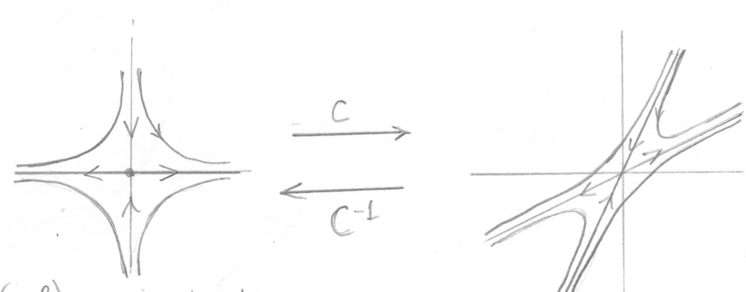
If $r=1$, the invariant curves are circles centered at 0.

Let A be a matrix with non-zero eigenvalues.
Then $\tilde{A}=C^{-1}AC$ for a model matrix $\tilde{A}$, and C gives a <u>linear</u> conjugacy between the linear maps $\begin{pmatrix}x\\y\end{pmatrix}\mapsto\tilde{A}\begin{pmatrix}x\\y\end{pmatrix}$ and $\begin{pmatrix}x\\y\end{pmatrix}\mapsto A\begin{pmatrix}x\\y\end{pmatrix}$.

$\tilde{A}=C^{-1}AC$ $\qquad$ $A=C\tilde{A}C^{-1}$



(4) $r=1$



(2b) $0<\mu<\lambda<1$