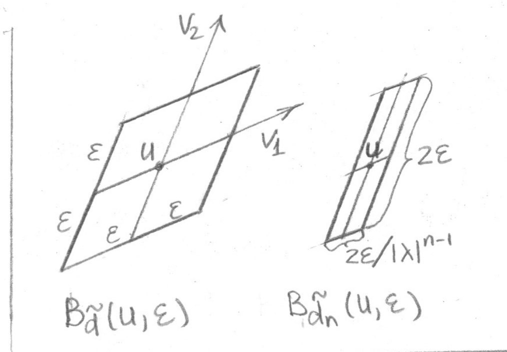
Hyperbolic automorphisms of $\mathbb{T}^2$: topological entropy

Intuitively, contraction does not contribute to the growth of the number of distinguishable orbit segments, only expansion does.

<u>Proposition</u>: Let A be a 2×2 integer matrix with det ± 1 and e.values λ and μ with $0<|\mu|<1<|\lambda|$. Then for the hyperbolic automorphism of $\mathbb{T}^2$ $f(u)=Au \bmod 1$, $h(f)=\ln|\lambda|$.

<u>Pf</u> Let us consider the following convenient metric:
Let v_1, v_2 be unit e.vectors of A corresp. to λ and μ, respectively.
For any $v, v' \in \mathbb{R}^2$, $v-v'$ can be uniquely written as $v-v'=a_1v_1+a_2v_2$.
Let $\tilde{d}(v,v')=\max\{|a_1|,|a_2|\}$.
The ball $B_{\tilde{d}}(v,\varepsilon)$ in $\mathbb{R}^2$ is a parallelogram, and the same holds in $\mathbb{T}^2$ for all small ε.
The $\tilde{d}_n$-ball $B_{\tilde{d}_n}(u,\varepsilon)$ is also a parallelogram.
Its area $=$ base$\times$height $=\dfrac{c4\varepsilon^2}{|\lambda|^{n-1}} \le \dfrac{4\varepsilon^2}{|\lambda|^{n-1}}$.
Hence at least $\dfrac{|\lambda|^{n-1}}{4\varepsilon^2}=\dfrac{1}{4\varepsilon^2|\lambda|}\cdot|\lambda|^n=\alpha|\lambda|^n$ balls of $\tilde{d}_n$-radius ε are needed to cover $\mathbb{T}^2$.



Therefore, $\mathrm{Cov}(f,\varepsilon,n)\ge\alpha|\lambda|^n$, and hence $h(f)\ge\ln|\lambda|$.

On the other hand, for any suff. small ε, any ε-ball in $\tilde{d}_n$ that intersects $[0,1]\times[0,1]$ lies entirely in $[-1,2]\times[-1,2]$. So at most $\dfrac{9}{c4\varepsilon^2/|\lambda|^{n-1}}=\beta|\lambda|^n$ closed ε-balls cover $\mathbb{T}^2$. Hence $\mathrm{Cov}(f,3\varepsilon,n)\le\beta|\lambda|^n \Rightarrow h(f)\le\ln|\lambda|$. $\square$

Hyperbolic automorphisms of $\mathbb{T}^N$, $N\ge 2$

A – an integer $N\times N$ matrix with $\det A\neq 0$.
$A^{-1}=\dfrac{1}{\det A}C^T$, where $C_{ij}=(-1)^{i+j}M_{ij}$, and M_{ij} is the determinant of the matrix obtained by removing the i^{th} row and j^{th} column of A.

<u>Prop.</u> For an integer $N\times N$ matrix A, A^{-1} is also integer $\Leftrightarrow \det A=\pm 1$.
<u>Pf</u> HW.

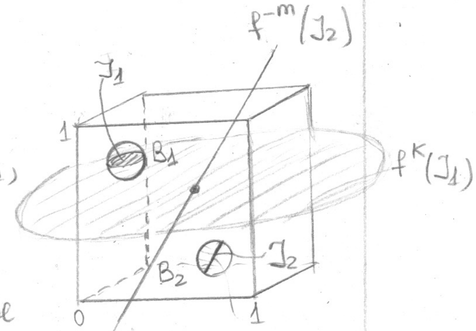
Thus for an integer $N\times N$ matrix A with $\det A=\pm1$, the map $F(v)=Av$ on $\mathbb{R}^N$ projects to an invertible map f on $\mathbb{T}^N$, $f(u)=Au \bmod 1$.
Suppose that A is <u>hyperbolic</u>, i.e. has no eigenvalues of absolute value 1.
Then f is called a hyperbolic automorphism of $\mathbb{T}^N$.

<u>Note</u> For $N=2$, such A has 2 distinct real eigenvalues and is diagonalizable over $\mathbb{R}$. For $N>2$, A may have complex e.values, e.values of multiplicity >1, and is not necessarily diagonalizable.

For simplicity, let us assume that A is diagonalizable over $\mathbb{R}$. Then $\mathbb{R}^N = V^u \oplus V^s$, where the subspace V^u [V^s] is the sum of all eigenspaces corresponding to e.values of $|\ldots|>1$ [<1] (u and s stand for unstable and stable). A expands V^u and contracts V^s.

<u>Remark</u> One can introduce a norm on $\mathbb{R}^N$ so that for any $v_1 \in V^u$ and $v_2 \in E^s$, $\|Av_1\| \geq \min\limits_{|\lambda|>1} |\lambda| \cdot \|v_1\|$ and $\|Av_2\| \leq \max\limits_{|\lambda|<1} |\lambda| \cdot \|v_2\|$. For the Euclidean norm, for any $n \in \mathbb{N}$ we have $\|A^n v_1\| \geq C_1 (\lambda_u)^n \|v_1\|$, where $\lambda_u = \min\limits_{|\lambda|>1} |\lambda|$.
In the general case, V^u is the sum of generalized eigenspaces corresp. to λ with $|\lambda|>1$, and $\|A^n v_1\| \geq C'(\lambda_u - \varepsilon)^n \|v_1\|$.

Properties of hyperbolic automorphisms of $\mathbb{T}^N$



- f is top. mixing and hence top. transitive.
  Idea of the proof: let B_1 and B_2 be open ε-balls. Consider a piece J_1 of the affine space parallel to V^u in B_1, and J_2 parallel to V^s in B_2. Since J_1 is expanded by f^K and J_2 is expanded by f^{-m}, for any suff. large K and m, $f^K(J_1)$ and $f^{-m}(J_2)$ will "go across" the cube and hence intersect. It follows that $f^{K+m}(B_1) \cap B_2 \neq \emptyset$.

- The projections of the subspaces V^u and V^s onto $\mathbb{T}^N$ are dense.
  Idea of the proof: Given $\varepsilon > 0$, explain why $\pi(V^u)$ is ε-dense in $\mathbb{T}^N$ by covering $\mathbb{T}^N$ by finitely many ε-balls.

- $v \in \mathbb{T}^N$ is periodic $\iff$ all components of v are rational.
  <u>Pf</u> as for $N=2$.

- $P_n(f) = \#$ of integer pts in the image of the unit cube $[0,1)^N$ under $(A^n - I)$
  $= \text{Volume}\,((A^n - I)(\text{unit cube})) = |\det(A^n - I)| = \left|\prod_{i=1}^{N} (\lambda_i^n - 1)\right| \approx$
  $= \prod_{|\lambda_i|>1} |\lambda_i|^n$ for large n, and $p(f) = \sum_{|\lambda_i|>1} \ln|\lambda_i|$.
  In the product and sum above, λ_i are counted with multiplicities.

- $h(f) = \sum_{|\lambda_i|>1} \ln|\lambda_i| = p(f)$.
  <u>Pf</u> For A diagonalizable over $\mathbb{R}$, very similar to $N=2$. Some modifications are needed for the general case.