## Subshifts of finite type (top. Markov chains) - continued.

? How many periodic pts. of period $n$ does $\sigma: \Omega_A^{(R)} \to \Omega_A^{(R)}$ have?
Is it the number of admissible words of length $n$? - No

$\omega \in \Omega_A^{R}$ has period $n \iff \omega = \underbrace{\omega_0 \dots \omega_{n-1}}\,\underbrace{\omega_0 \dots \omega_{n-1}}\,\underbrace{\dots\dots}\,\dots$

Thus there is a one-to-one correspondence between such $\omega$ and admissible words of length $n+1$ beginning and ending with the same symbol.

? How many admissible words $\underbrace{i \dots j}$, an in particular $\underbrace{i \dots i}$, of length $n+1$ are there? For $n=1$, $a_{ij}$, which is 0 or 1.

<u>Proposition</u> The number of admissible words of length $n+1$ beginning with $i$ and ending with $j$, which is the number of admissible paths of length $n$ from $i$ to $j$ in the corresponding directed graph $\Gamma_A$, equals the $(i,j)$ entry of the matrix $A^n = (a_{ij}^{(n)})$.

<u>Pf</u> By induction. Holds for $n=1$. Suppose holds for some $n \in \mathbb{N}$. Then
The number of ways to get from $i$ to $j$ in $n+1$ steps in $\Gamma_A$ =
$$= \sum_{k=0}^{m-1} (\#\text{ways to get from } i \text{ to } k \text{ in } n \text{ steps}) \cdot \underbrace{(\#\text{ways to get from } k \text{ to } j \text{ in 1 step})}_{0 \text{ or } 1} =$$
$$= \sum_{k=0}^{m-1} a_{ik}^{(n)} \cdot a_{kj} = a_{ij}^{(n+1)} \qquad \square$$

<u>Corollary</u> The number of periodic pts of period $n$ equals $\mathrm{Trace}(A^n)$.

<u>Conditions for top. transitivity and mixing</u> (will write for $\Omega_A$)
For a dense orbit, we want to list all admissible words and connect them.

<u>Def</u> $\sigma: \Omega_A \to \Omega_A$ is <u>irreducible</u> if for any $i, j \in \{0, \dots, m-1\}$
there is a path from $i$ to $j$ in the directed graph $\Gamma_A$, equivalently,
there exists $n = n(i,j)$ s.t. the $(i,j)$ entry of $A^n$ is positive ($>0$).

<u>Ex</u> For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\Gamma_A$: $0 \rightleftarrows 1$, and $\sigma: \Omega_A \to \Omega_A$ is irreducible.

<u>Proposition</u> If $\sigma: \Omega_A \to \Omega_A$ is irreducible, then
(1) It is top. transitive, moreover, there is a point with dense positive semiorbit.
(2) Periodic points are dense in $\Omega_A$.

<u>Pf</u> (1) We construct a point with dense positive semiorbit as follows:
List all <u>admissible</u> words of length 1, then 2, then 3, ...
By irreducibility, we can connect any two consecutive admissible words. $\underbrace{0}\,*\dots*\,\underbrace{1}\,*\dots*\,\underbrace{m-1}\,*\dots*\,\underbrace{\ }\,\underbrace{\ }\,\underbrace{\ }\dots$
Then we extend the sequence to the left and obtain
$\omega \in \Omega_A$ with dense $O^+(\omega)$.

(2) Given $\omega \in \Omega_A$ and $n \in \mathbb{N}$, we construct a periodic point in $B(\omega, 2^{-n})$ as follows: Recall that $B(\omega, 2^{-n}) = C^{-n,\dots,n}_{\omega_{-n},\dots,\omega_n}$.
By irreducibility, we can extend $\omega_{-n}\dots\omega_n$ to an admissible word ending with $\omega_{-n}$: $\underbrace{\omega_{-n}\dots\omega_n\dots,\omega_{-n}}$. The periodic sequence $\omega'$ obtained by repeating $\underbrace{\quad}$ is in $B(\omega, 2^{-n})$. $\square$.

Def A square matrix A with non-negative entries is called primitive if there is $N \in \mathbb{N}$ s.t. every entry of $A^N$ is positive.

Ex $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ ? Does $A^n$ have positive entries for all $n \geq 2$?

Lemma let A be a square matrix with non-negative entries. If all entries of $A^N$ are positive for some $N \in \mathbb{N}$, then all entries of $A^n$ are positive for all $n \geq N$.
Pf (Ex). Note that primitive implies no zero columns or rows.

Proposition If A is primitive, then $\sigma: \Omega_A \to \Omega_A$ is top. mixing.
Pf It suffices to consider non-emty cylinders $U = C^{n_1,\dots,n_k}_{\alpha_1,\dots,\alpha_k}$ and $V = C^{l_1,\dots,l_j}_{\beta_1,\dots,\beta_j}$ in $\Omega_A$.
Since A is primitive, there is $N \in \mathbb{N}$ s.t. for each $n \geq N$, there is an admissible word of length $n+1$ beginning with $\alpha_k$ and ending with $\beta_1$.
Take M s.t. $l_1 - (n_k - M) > N$, i.e. there is a gap of size $> N$ between $\sigma^M(U)$ and V. Then for every $n \geq M$, there is an admissible sequence in $\sigma^n(U) \cap V$. Thus $\sigma: \Omega_A \to \Omega_A$ is top. mixing $\square$.

? Consider $\sigma^R: \Omega_A^R \to \Omega_A^R$, without assuming that A is primitive.
Can we argue as follows: For U and V as in the proof above, for every $n > n_k$, $(\sigma^R)^n(U) = \Omega_A^R$, and hence $(\sigma^R)^n(U) \cap V \neq \emptyset$. ?
No, $(\sigma^R)^n(U) \neq \Omega_A^R$ in general. Sequences in $(\sigma^R)^n(U)$ contain only symbols reachable from $\alpha_k$. To have $(\sigma^R)^n(U) \cap V \neq \emptyset$ for all suff. large n, we still need $\beta_1$ to be reachable from $\alpha_k$ in n steps for all $n \geq$ some M.
So we still need A to be primitive.