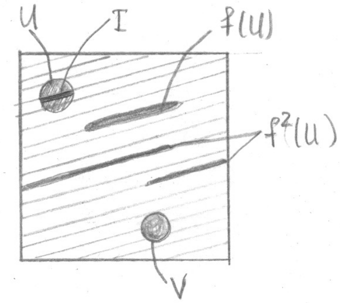
Hyperbolic automorphisms of $\mathbb{T}^2$ (continued)

$f(u) = Au \bmod 1$, where A is a 2×2 hyperbolic integer matrix with $\det \pm 1$.
We showed that the lines L_1 and L_2 in $\mathbb{R}^2$ spanned by the e.vectors of A have irrational slopes, and hence their projections onto $\mathbb{T}^2$ are dense in $\mathbb{T}^2$.

Prop. Let f be a hyperbolic automorphism of $\mathbb{T}^2$.
Then f is top. mixing and hence top. transitive.



Pf. Let U be an open disc in $\mathbb{T}^2$ and let V be an open disc of radius ε. Let L be the expanding line. The projection of L is dense in $\mathbb{T}^2$. Hence the disc U contains a segment I of the projection.
Also, there exists $R>0$ s.t. the projection of the segment of L of length R centered at 0 is ε-dense i.e. intersects every disc of radius ε in $\mathbb{T}^2$ (Why?)
The same holds for the projection of any segment of L of length R.
The length of $f^n(I)$ is $|\lambda|^n \cdot |I| \to \infty$. Hence $|f^n(I)| \geq R$ for all suff. large n.
Therefore, $f^n(I)$, and hence $f^n(U)$, intersects V for all suff. large n.
It follows that f is top. mixing. □

Periodic points of f.

Prop. Let f be a hyperbolic automorphism of $\mathbb{T}^2$. Then $u \in \mathbb{T}^2$ is a periodic point for $f \iff$ both components of u are rational.

Pf. We consider $\binom{x}{y}$ in the unit square $S = [0,1)\times[0,1)$.

($\Leftarrow$) Let $x, y \in \mathbb{Q}$. Then we can write $x = \frac{k}{q}$ and $y = \frac{\ell}{q}$, where $0 \le k, \ell \le q-1$.
Since A is an integer matrix, for any $n \in \mathbb{N}_0$,
$f^n\binom{x}{y}$ = the fractional part of $A^n\binom{x}{y}$ is of the same form.
There are only q^2 points of this form in $S \Rightarrow$ there are $0 \le m < n$ s.t. $f^n\binom{x}{y} = f^m\binom{x}{y}$ and hence $f^{n-m}\binom{x}{y} = \binom{x}{y}$. Thus $\binom{x}{y}$ is periodic with prime period $\le q^2$.

($\Rightarrow$) Suppose that for some $n \in \mathbb{N}$,
$f^n\binom{x}{y} = \binom{x}{y}$, that is, $(A^n - I)\binom{x}{y} = \binom{k}{\ell} \in \mathbb{Z}^2$. Since A has no e. values of absolute value 1, 1 is not an eigenvalue of A^n, and so $A^n - I$ is invertible. Since $A^n - I$ is an integer matrix, its inverse has rational entries, and so $\binom{x}{y} = (A^n - I)^{-1}\binom{k}{\ell} \in \mathbb{Q}^2$. □

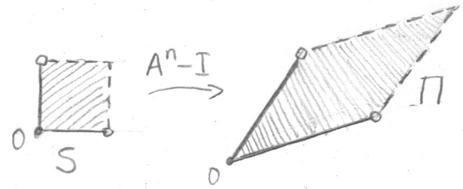
Corollary. Periodic pts of f are dense in $\mathbb{T}^2$.

- f is an example of a chaotic diffeomorphism of a compact smooth manifold.

? How many points of period n does f have?

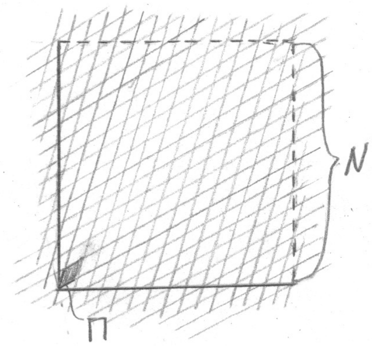
$\binom{x}{y} \in S$ has period $n \iff (A^n - I)\binom{x}{y} = \binom{k}{\ell} \in \mathbb{Z}^2$

Thus the number of points of period n equals the number of integer points in the parallelogram $\Pi = (F^n - I)(S)$. The vertices of Π are in $\mathbb{Z}^2$.



Claim: Let Π be a parallelogram with integer vertices, as shown (only one vertex and two sides are included). Then the number of integer points in Π equals the area of Π.

An argument: Consider the grid obtained by translations of Π. Each paralleogram in it has the same area and the same number of integer points as Π. A large $N \times N$ square as shown contains N^2 integer points. So



$$\# \text{integer pts in } \Pi \approx \frac{N^2}{\# \text{parallelograms inside } \square} = \frac{\text{area of } \square}{\# \text{parallelograms inside } \square} \approx \text{area}(\Pi)$$

In the approximations, the error comes from the parallelograms intersecting the boundary of the square. There $\leq cN$ of them, while $\approx c'N^2$ are inside. It follows that the error $\leq c''/N < 0.5$ for large N. Since # of int. pts and area (Π) are integers, we obtain the equality.

- Thus $P_n(f) = \text{area}(\Pi) = \text{area}((A^n - I)(S)) = |\det(A^n - I)|$

Let λ, μ with $0 < |\mu| < 1 < |\lambda|$ be the e.values of A.

Then λ^n, μ^n are e.values of A^n, and $(\lambda^n - 1), (\mu^n - 1)$ are e. values of $A^n - I$.

Indeed, $(A^n - I) - (\lambda^n - 1)I = A^n - \lambda^n I$ is not invertible

So $P_n(f) = |\det(A^n - I)| = |(\lambda^n - 1)(\mu^n - 1)| = |\lambda^n + \underbrace{\mu^n}_{\to 0} - \underbrace{(\lambda\mu)^n}_{\pm 1} - 1| \approx |\lambda|^n$ for large n.

The exp. growth rate is $p(f) = \lim\limits_{n \to \infty} \frac{\ln P_n(f)}{n} = \ln|\lambda|$.

In the case of positive μ and λ, i.e. $0 < \mu < 1 < \lambda$ we have:

$\mu = 1/\lambda$ and so $P_n(f) = |\lambda^n + \frac{1}{\lambda^n} - 2| = \lambda^n + \frac{1}{\lambda^n} - 2$

since $x + \frac{1}{x} > 2$ for $x > 1$.