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Circle rotations

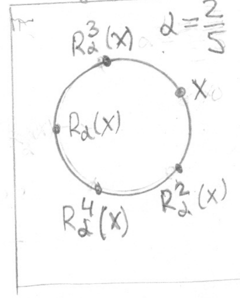
$S^1 = \mathbb{R}/\mathbb{Z}$. For $\alpha \in \mathbb{R}$, $R_\alpha : S^1 \to S^1$ is given by $R_\alpha(x) = x + \alpha \mod 1$

Note: For any $x, y \in S^1$, the orbit of y is the translation of the orbit of x by $y - x$, so all orbits look the same.

rational α

Prop. Let $\alpha = \frac{p}{q}$, where $p \neq 0$, $q \in \mathbb{N}$, and p and q are relatively prime. Then each $x \in S^1$ is periodic with prime (i.e. smallest) period q.

Pf $(R_\alpha)^q = R_{\alpha q} = R_p = R_0 = Id$, so $(R_\alpha)^q(x) = x$ for every x.
Suppose $(R_\alpha)^k(x) = R_{k\alpha}(x) = x$ for some $1 \leq k \leq q-1$.
Then $k\alpha = k\frac{p}{q} = m$ for some $m \in \mathbb{Z}$. Hence $kp = qm$
Since p and q are rel. prime, it follows that q divides k, which is impossible. Thus q is the prime period of x. $\square$



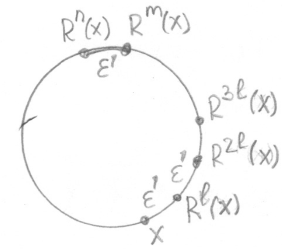
Now, let α be irrational.
Then there are no periodic points, moreover, for any $x \in X$ and $m \neq n$ in $\mathbb{Z}$, $R_\alpha^m(x) \neq R_\alpha^n(x)$, and so all points of the orbit of x are distinct (Explain).

? Does the orbit skip any intervals in S^1? No!

Thm Let α be irrational. Then for any $x \in S^1$ the positive semiorbit of x is dense in S^1.

Note The negative semiorbit is also dense since it is the positive semiorbit under $R_{-\alpha}$, and of course the whole orbit is dense.

Pf We fix $x \in S^1$ and write R for R_α. Let $\varepsilon > 0$.
Want: $m \neq n$ s.t. $d(R^m(x), R^n(x)) < \varepsilon$.
Take $N > \frac{1}{\varepsilon}$ so that $N\varepsilon > 1$.
The N points $x, R(x), \ldots, R^{N-1}(x)$ are distinct
$\Rightarrow$ there are $0 \leq m < n \leq N-1$ s.t. $d(R^m(x), R^n(x)) = \varepsilon' < \varepsilon$.
Let $\ell = n - m$. Since R preserves distances,
$\varepsilon' = d(x, R^\ell(x)) = d(R^\ell(x), R^{2\ell}(x)) = d(R^{2\ell}(x), R^{3\ell}(x)) = \ldots$
It follows that for any $y \in S^1$ there is $z = R^{k\ell}(x)$ s.t. $d(y, z) < \varepsilon$. $\square$



Topological transitivity and minimality.

Let X be a compact metric (or topological) space and let $f : X \to X$ be continuous.
If f is invertible, then f^{-1} is automatically continuous, i.e. a homeomorphism.

Def $f : X \to X$ is *topologically transitive* if there is $x \in X$ such that its orbit is dense in X.

Def $f : X \to X$ is *minimal* if the orbit of *every* $x \in X$ is dense in X.

Note minimality $\Rightarrow$ top. transitivity
$\not\Leftarrow$ (we will see an example soon)
However, ($\Leftarrow$) holds for translation on topological groups.

a topological group G is a topological space that is also a group such that the maps $(g,h)\mapsto gh$ from $G\times G$ to G and
$g\mapsto g^{-1}$ from G to G are continuous.

a left translation on G is the map $L_{g_0}: G\to G$ given by $L_{g_0}(g)=g_0 g$

S^1 with addition mod 1 is a (compact abelian) top. group,
and R_α is a translation on S^1.

Proposition If a translation L_{g_0} on a top. group G is top. transitive, then it is minimal.

Pf Let $g,h\in G$, and let O_g, O_h be their orbits. Since $g_0^n h = g_0^n g(g^{-1}h)$ for all n,
$O_h = O_g(g^{-1}h)$, and hence $\overline{O_h} = \overline{O_g}(g^{-1}h)$. Hence $\overline{O_g}=X \Rightarrow \overline{O_h}=X$. $\square$.

? Why is minimal called minimal?
Because we cannot restrict f to a proper subset of X, as we will see.

Def For an invertible $f: X\to X$, we say that a set $A\subseteq X$ is invariant if $f(A)=A$.
For a non-invertible ———— " ———— if $f(A)\subseteq A$.

Let $A\neq\emptyset, X$ be a closed invariant set. Then we can restrict f to this smaller set and obtain a dynamical system $f: A\to A$.

Ex For R_α with rational α, the orbit of a periodic pt is a closed inv. set.
For R_α with irrational α, there are no closed inv. sets $\neq\emptyset, X$. (Why?)

Proposition $f: X\to X$ is minimal $\Leftrightarrow$ there are no closed invariant sets other than $\emptyset$ and X.

Pf ($\Rightarrow$) Let $A\neq\emptyset$ be a closed invariant set.
Let $x\in X$ and let $O(x)$ be its orbit. Then by invariance $O(x)\subseteq A$.
Since A is closed, $\overline{O(x)}\subseteq A$. But $\overline{O(x)}=X$ by minimality, so $A=X$.

($\Leftarrow$) Let $x\in X$. Then $O(x)$ is invariant and hence $\overline{O(x)}$ is invar. (Why?)
Thus $\overline{O(x)}$ is a closed non-empty invar. set.
Hence $\overline{O(x)}=X$, i.e. $O(x)$ is dense in X. $\square$.