Proof of structural stability of E_m (continued)

We are considering $f: S^1 \to S^1$ close to E_m with $f(0)=0$, and the lifts of E_m and f to $\mathbb{R}$, $Ex = mx$ and $F(x) = mx + \varphi(x)$, where φ is a C^1-small 1-periodic function with $\varphi(0)=0$.
We showed that there exists a continuous $h: S^1 \to S^1$ close to Id with $h(0)=0$ s.t. $E_m \circ h = h \circ f$. In the argument, we used only C^0-smallness of φ.

Now we obtain a cont. $\hat{h}: S^1 \to S^1$ close to Id with $h(0)=0$ s.t. $\hat{h} \circ E_m = f \circ \hat{h}$.
Writing $\hat{h} = Id + \tilde{h}$ and lifting it to $\mathbb{R}$ as $Id + H$, where $H(0)=0$, we get

$$(Id+H)\circ E = (E+\varphi)\circ(Id+H)$$

$$\Longleftrightarrow E + H\circ E = E + E\circ H + \varphi(Id+H) \qquad [E \text{ is linear, but } \varphi \text{ is not}]$$

$$\Longleftrightarrow \varphi(Id+H) = H\circ E - E\circ H = T(H), \quad (\star)$$

where $T(H)(x) = H(mx) - mH(x) = -m\left(H(x) - \frac{1}{m}H(mx)\right)$

so $T(H) = -m(Id - \Delta)$, where $\Delta(H)(x) = \frac{1}{m}H(mx)$

T is a linear operator on the space Y of 1-periodic continuous functions on $\mathbb{R}$ with $g(0)=0$, and T is invertible. Indeed, since $\|\Delta\| \le \frac{1}{m}$, $\|\Delta^n\| \le \|\Delta\|^n \le \frac{1}{m^n}$ for all $n \in \mathbb{N}$. Hence the series $-\frac{1}{m}\sum_{n=0}^{\infty} \Delta^n$ converges to a bounded linear operator on Y, which is the inverse of T. We see that $\|T^{-1}\| \le \frac{1}{m}\sum_{n=0}^{\infty}\frac{1}{m^n} \le 1$.

Let $S(H) = \varphi(Id+H)$. We rewrite $(\star)$ as $S(H) = T(H)$, or $H = T^{-1}(S(H))$
$\|S(H_1) - S(H_2)\|_{C^0} = \sup_x |\varphi(x+H_1(x)) - \varphi(x+H_2(x))| \le \|\varphi'\|_{C^0}\cdot\|H_1 - H_2\|_{C^0}$
for all $H_1, H_2 \in Y$. If f is sufficiently C^1-close to E_m, then $\|\varphi'\|_{C^0} \le \lambda < 1$, and hence $\|(T^{-1}\circ S)(H_1) - (T^{-1}\circ S)(H_2)\|_{C^0} \le \|T^{-1}\|\cdot\|\varphi'\|_{C^0}\|H_1 - H_2\|_{C^0} \le \lambda\|H_1 - H_2\|_{C^0}$. Thus $T^{-1}\circ S$ is a contraction on Y, and hence $(\star)$ has a unique solution $H \in Y$. Projecting $Id + H$ onto S^1, we obtain $\hat{h}$.

Therefore, there exist continuous $h, \hat{h}: S^1 \to S^1$ close to Id such that $E_m \circ h = h \circ f$ and $\hat{h} \circ E_m = f \circ \hat{h}$. By the construction, they are onto S^1.
We have: $\quad h\circ\hat{h}\circ E_m = \underline{h\circ f}\circ h = E_m \circ h \circ \hat{h}$.
Let $g = h\circ\hat{h}$. Then $g: S^1 \to S^1$ is cont., close to Id, and commutes with E_m.
We lift g to $\mathbb{R}$ as $Id + G$, where G is a continuous 1-periodic function.
Note that such G is bounded on $\mathbb{R}$.

$g \circ E_m = E_m \circ g$ yields $(Id+G)\circ E = E\circ(Id+G)$, that is
$\cancel{mx} + G(mx) = \cancel{mx} + m\,G(x)$. Thus $G(mx) = m\,G(x)$ for all x.
It follows that $G(m^n x) = m^n G(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.
Since G is bdd, this is impossible unless $G \equiv 0$, i.e. $g = h\circ\hat{h} = Id$.
Hence $\hat{h} = h^{-1}$. Thus h is invertible, and so it is a top. conjugacy between f and E_m. □.

<u>Remark</u>: To obtain a top conjugacy between f and E_m, it suffices that $d(f(x), E_m(x)) < \frac{1}{2}$ for all x, and so "$f(S^1)$ wraps around S^1 exactly m times", and that $|f'(x) - E_m'(x)| < 1$ for all x so that $\|\varphi'\|_{C^0} \le \alpha < 1$. This condition ensures that f is expanding. C^1-closeness of f to E_m yields h close to Id.

<u>Structural stability of hyperbolic automorphisms of $\mathbb{T}^2$.</u>

Let $L(u) = Au \bmod 1$ be a hyperbolic automorphism of $\mathbb{T}^2$, and let f be a diffeomorphism of $\mathbb{T}^2$ suff. C^1-close to L. Then f is top. conjugate to L.

<u>Outline of the proof</u>

We can reduce the problem to the case of $f(0)=0$.
We consider the lifts of L and f to $\mathbb{R}^2$: Av and $F(v) = Av + \varphi(v)$,
where φ is a C^1-small $\mathbb{Z}^2$-periodic function with $\varphi(0)=0$.
We seek $h = Id + \tilde{h}$ satisfying $L\circ h = h\circ F$. Lifting to $\mathbb{R}^2$ we obtain

$$A(Id+H) = (Id+H)\circ F \text{, equivalently, } H = A^{-1}H\circ(A+\varphi) + A^{-1}\varphi \quad (\ast\ast),$$

where H and φ are $\mathbb{Z}^2$-periodic. Note that A^{-1} is <u>not</u> a contraction.
Let v_1 and v_2 be e.vectors of A corresp. to its e.values λ_1 and λ_2
with $0 < |\lambda_2| < 1 < |\lambda_1|$. We decompose φ and H as

$$\varphi = \varphi_1 v_1 + \varphi_2 v_2 \text{ and } H = H_1 v_1 + H_2 v_2, \text{ where } \varphi_{1,2}, H_{1,2} : \mathbb{R}^2 \to \mathbb{R} \text{ are } \mathbb{Z}^2\text{-periodic.}$$

Then $(\ast\ast)$ becomes two equations:

① $H_1 = \lambda_1^{-1} H_1\circ(A+\varphi) + \lambda_1^{-1}\varphi_1$ and ② $H_2 = \lambda_2^{-1} H_2\circ(A+\varphi) + \lambda_2^{-1}\varphi_2$.

① is $H_1 = T_1(H_1)$, where T is a contraction $\Rightarrow$ has a unique solution H_1

② we rewrite as $\lambda_2 H_2 = H_2 \underbrace{(A+\varphi)}_{F} + \varphi_2$ and then $\lambda_2 H_2\circ F^{-1} - \varphi_2\circ F^{-1} = H_2$.
Thus $T_2(H_2) = H_2$, where T_2 is a contraction $\Rightarrow$ has a unique solution H_2.

Then one obtains $\hat{h}$ s.t. $\hat{h}\circ L = f\circ\hat{h}$, and finally shows that $\hat{h} = h^{-1}$. □.

<u>Remark</u> Structural stability is related to "hyperbolicity", which, roughly speaking means exponential expansion in some directions and exponential contraction in the other directions.