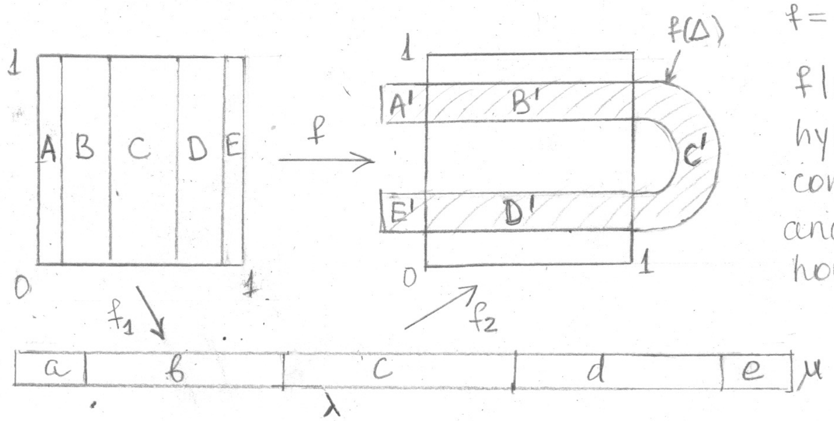
Remark Conjugacy to E_m via coding

Let $f: S^1 \to S^1$ be a continuous expanding map of degree m with $f(0)=0$. Recall that when coding f, at step ℓ we obtained m^ℓ intervals of length $\le 1/\lambda^\ell$. Let us map the endpoints of the corresp. intervals for E_m, $\frac{k}{m^\ell}$ where $0 \le k \le m^\ell$, to these endpoints, preserving the order. Thus we obtain a strictly increasing map h from the set of all endpts for E_m to the set of all endpts for f.
Note that the set of all endpts is dense in $[0,1]$.
Then we can extend h to a cont. strictly increasing map from $[0,1]$ onto $[0,1]$ and thus obtain a homeomorphism of S^1 conjugating E_m and f.
The extension can be done either by showing that h is uniformly cont. on the dense set of endpts, OR by observing that for both E_m and f each pt. that is not an endpt. corresponds to a unique sequence in Ω_m^R, and setting $h(x)$ to be the pt. with the same sequence.
This way of showing top. conjugacy to E_m uses one-dimensionality and does not extend to hyperbolic automorphisms of $\mathbb{T}^2$.

Smale's horseshoe (Smale 60's).

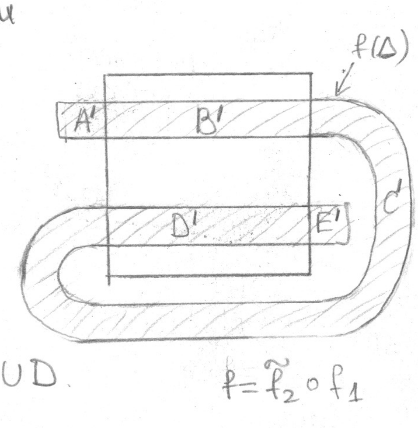
Let Δ be the square $[0,1]\times[0,1]$ in $\mathbb{R}^2$
Consider a diffeomorphism $f: \Delta \to f(\Delta)$ constructed as follows.



$f = f_2 \circ f_1$

$f|_B$ and $f|_D$ are affine hyperbolic maps, contracting vertically by μ and expanding horizontally by λ.

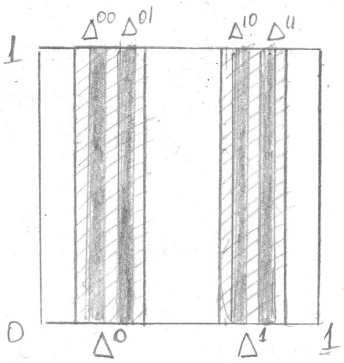
The rectangle D' is "upside down", so it is more convenient to work with the modified construction:

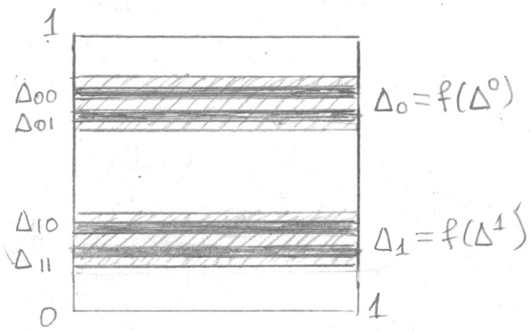


$f = \tilde{f}_2 \circ f_1$

Goal: Describe the maximal f-invariant subset Λ of Δ and code $f|_\Lambda$.
Clearly, $\Lambda = \bigcap_{n=-\infty}^{\infty} f^{-n}(\Delta)$.
We see that $\Delta \cap f(\Delta) = B' \cup D'$ and $f^{-1}(\Delta) = B \cup D$.
We can disregard A', C', and E'.

We denote the vertical rectangles B and D by Δ^0 and Δ^1, and the horizontal rectangles B' and D' by Δ_0 and Δ_1. We have: $f(\Delta^0)=\Delta_0$, $f(\Delta^1)=\Delta_1$





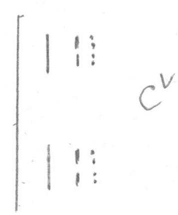
We use the sets Δ^0 and Δ^1 for coding.

$\Delta \cap f(\Delta) = \Delta_0 \cup \Delta_1 = f(\Delta^0) \cup f(\Delta^1)$, where Δ_0 and Δ_1 have height μ.

By drawing Δ_0 and Δ_1 in the square and applying $f = \tilde{f}_2 \circ f_1$, we see that $\Delta \cap f(\Delta) \cap f^2(\Delta)$ consists of 4 rectangles $\Delta_{00}, \Delta_{01}, \Delta_{10}, \Delta_{11}$ of height μ^2, where $\Delta_{ij} = \Delta_i \cap f(\Delta_j) = f(\Delta^i) \cap f^2(\Delta^j)$.

Continuing, we see that $\Delta \cap f(\Delta) \cap \dots \cap f^n(\Delta)$ consists of 2^n rectangles $\Delta_{\omega_0 \dots \omega_n} = \Delta_{\omega_0} \cap \dots \cap f^{n-1}(\Delta_{\omega_n}) = f(\Delta^{\omega_0}) \cap \dots \cap f^n(\Delta^{\omega_n})$ of height μ^n.

Therefore, $\Lambda_1 = \bigcap_{n=0}^{\infty} f^n(\Lambda)$ is the product of a horizontal segment $[0,1]$ with a vertical Cantor set C^V.



Similarly, for each $n \in \mathbb{N}$, $f^{-n}(\Delta)$ consists of 2^n vertical rectangles of width $1/\lambda^n$, and $\Lambda_2 = \bigcap_{n=1}^{\infty} f^{-n}(\Delta)$ is the product of a horizontal cantor set C^H with $[0,1]$.

Thus $\Lambda = \Lambda_1 \cap \Lambda_2 = C^H \times C^V$ is the product of two Cantor sets.
In particular, Λ is totally disconnected.

For $\omega \in \Omega_2$, let $h(\omega) = \bigcap_{n=-\infty}^{\infty} f^{-n}(\Delta^{\omega_n})$. For each $k \in \mathbb{N}$, $\bigcap_{n=-k}^{k} f^{-n}(\Delta^{\omega_n})$ is is a λ^n by μ^n rectangle. Thus $h(\omega)$ is the intersection of a nested sequence of compact sets with diam $\to 0$, and hence consists of a single pt. x.
Each $x \in \Lambda$ corresponds to a unique ω, and $h: \Omega_2 \to \Lambda$ is cont. (Why?)
also, $f(h(\omega)) = f\left(\bigcap_{n\in\mathbb{Z}} f^{-n}(\Delta^{\omega_n})\right) = \bigcap_{n\in\mathbb{Z}} f^{-n+1}(\Delta^{\omega_n}) = \bigcap_{k\in\mathbb{Z}} f^{-k}(\Delta^{\omega_{k+1}}) = h(\sigma(\omega))$.

Thus $h: \Omega_2 \to \Lambda$ is a top. conjugacy between $\sigma: \Omega_2 \to \Omega_2$ and $f|_\Lambda : \Lambda \to \Lambda$.

<u>Corollary</u> Since $f|_\Lambda$ is top. conjugate to $\sigma: \Omega_2 \to \Omega_2$,

- $f|_\Lambda$ is top. mixing $\Rightarrow$ top. transitive
- $P_n(f|_\Lambda) = 2^n$ and the periodic pts of $f|_\Lambda$ are dense in Λ
- $h_{top}(f|_\Lambda) = \ln 2$.