Full shift on m symbols (continued)

$\sigma : \Omega_m \to \Omega_m$ and $\sigma^R : \Omega_m^R \to \Omega_m^R$

- have exactly m fixed points, these are constant sequences;
- have exactly m^n periodic points of period n, these are periodic sequences with period n, and their number = the number of words of length n.
- periodic points are dense in $\Omega_m^{(R)}$
  given any $\omega = (\omega_n)$ and $\varepsilon > 0$, take n s.t. $2^{-n} < \varepsilon$ and
  in Ω_m^R take $\underbrace{\omega_0 \dots \omega_n}, \underbrace{\omega_0 \dots \omega_n}, \dots$
  in Ω_m take $\dots \underbrace{\omega_{-n} \dots \omega_n}, \underbrace{\omega_{-n} \dots \omega_n}, \dots$
- the dynamical systems are top. transitive
  (a sequence has a dense orbit $\iff$ it contains every finite word);
  moreover, points with dense orbits are dense: take $\dots ** \underbrace{\omega_{-n} \dots \omega_n}, \underbrace{0} \dots \underbrace{m-1}, \underbrace{00} \dots$
- so $\sigma : \Omega_m \to \Omega_n$ and $\sigma^R : \Omega_m^R \to \Omega_m^R$ are chaotic.

Topological mixing.

Def Let $f : X \to X$ be a continuous map of a compact metric space.
We say that $f : X \to X$ is topologically mixing if for any non-empty open sets $U, V \subseteq X$, there is $N \in \mathbb{N}$ s.t. $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

Note: "Mixing means mixing". Let $U \neq \emptyset$ be open, for example $U = B_\varepsilon(x)$.
Let us cover X with finitely many open sets (balls) $V_1, \dots, V_K$.
Then there exists $M \in \mathbb{N}$ s.t. for all $n \geq M$,
$f^n(U) \cap V_i \neq \emptyset$ for each $i = 1, \dots, K$.

Topological mixing $\Rightarrow$ there is $x \in X$ with dense $\{x, f(x), f^2(x), \dots\}$,
$\Rightarrow$ top. transitivity (for both invertible and non-invertible f)
Top. transitivity $\not\Rightarrow$ top. mixing Ex: $R_\alpha : S^1 \to S^1$ with $\alpha \notin \mathbb{Q}$.
Ex $E_m : S^1 \to S^1$ is top. mixing since $E_m^n(U) = S^1$ for all suff. large n.

Proposition $\sigma^R : \Omega_m^R \to \Omega_m^R$ and $\sigma : \Omega_m \to \Omega_m$ are top. mixing.

Proof It suffices to prove the property for any cylinders U and V.
Ω_m^R: for any cylinder $U = C^{n_1, \dots, n_k}_{\alpha_1, \dots, \alpha_k}$, $(\sigma^R)^n(U) = \Omega_m^R$ for all $n > n_k$.
Ω_m: Let $U = C^{n_1, \dots, n_k}_{\alpha_1, \dots, \alpha_k}$ and $V = C^{\ell_1, \dots, \ell_j}_{\beta_1, \dots, \beta_j}$. Then $f^n(U) = C^{n_1 - n, \dots, n_k - n}_{\alpha_1, \dots, \alpha_k}$

We take $N \in \mathbb{N}$ s.t. $n_k - N < \ell_1$. Then for each $n \geq N$,
the sequence $\dots \underset{\substack{\uparrow \\ n_1 - n}}{\alpha_1} \dots \underset{\substack{\uparrow \\ n_2 - n}}{\alpha_2} \dots \underset{\substack{\uparrow \\ n_k - n}}{\alpha_k} \dots \underset{\substack{\uparrow \\ \ell_1}}{\beta_1} \dots \underset{\substack{\uparrow \\ \ell_j}}{\beta_j} \dots$ is in $f^n(U) \cap V$. $\square$

Subshifts of finite type, also called topological Markov chains.

[In general, a subshift is a restriction of $\sigma^{(R)}$ to a closed invariant subset of $\Omega_m^{(R)}$. Subshifts of finite type are an important class.]

We consider a subset of "admissible" sequences, defined as follows.
Let $A=(a_{ij})$ be an $m\times m$ matrix, with entries $a_{ij}=0$ or 1.
We assume that each column and each row contains at least one 1.
A is called an <u>adjacency matrix</u>. We consider the set

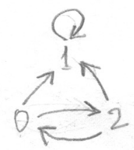
$$\Omega_A^{(R)} = \{\omega\in\Omega_m^{(R)} : a_{\omega_n\omega_{n+1}} = 1 \text{ for all } n\in \mathbb{N}_0 (\mathbb{Z})\} \subseteq \Omega_m^{(R)}$$

That is, in a sequence $\omega\in\Omega_A^{(R)}$, a symbol j can follow a symbol i if and only if $a_{ij}=1$. Note that each i can follow some j, and can be followed by some k.

<u>Ex</u> $m=3$ $\quad A=\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix}$ $\qquad (1,1,1,\dots)\in\Omega_A^R$, $(0,0,0,\dots)\notin\Omega_A^R$, $(0,2,0,2,\dots)\in\Omega_A^R$

The <u>directed graph</u> Γ_A corresponding to A: vertices $0,1,\dots,m-1$, and we draw an arrow from i to j if j can follow i.

<u>Ex</u> Γ_A



a sequence in Ω_A^R corresponds to a walk along the directed edges of Γ_A

Clearly, $\Omega_A^{(R)}$ is invariant under the shift. It is also closed. The dynamical system $\sigma^{(R)}: \Omega_A^{(R)}\to\Omega_A^{(R)}$ is called a <u>subshift of finite type</u> or a <u>topological Markov chain</u>. Note that there is a finite list of "forbidden" words ij, and a sequence is in $\Omega_A^{(R)}$ if and only if it does not contain them.

(?) How many fixed points does $\sigma^{(R)}: \Omega_A^{(R)}\to\Omega_A^{(R)}$ have?
As many as there are 1's on the diagonal of A, i.e. Trace(A)

(?) How many points of period n?