Exponential growth rate of the number of periodic points

For a dynamical system $f: X \to X$, let $P_n(f)$ be the number of periodic points of period n.

Ex (1) For a contraction on a complete metric space, $P_n(f) = 1$ for every $n \in \mathbb{N}$.
(2) For $E_m: S^1 \to S^1$, $P_n(E_m) = m^n - 1$ — " —
(3) For the full shift $\sigma^{(R)}: \Omega_m^{(R)} \to \Omega_m^{(R)}$, $P_n(\sigma^{(R)}) = m^n$ — " —
(4) For a subshift $\sigma_A^{(R)}: \Omega_A^{(R)} \to \Omega_A^{(R)}$, $P_n(\sigma_A^{(R)}) = \mathrm{Tr}(A^n)$ — " —

We consider $f: X \to X$ such that $P_n(f) < \infty$ for all n.
The exponential growth rate for the sequence $P_n(f)$ is

$$p(f) = \limsup \frac{\ln(\max\{P_n(f), 1\})}{n}.$$

We take $\max\{P_n(f), 1\}$ instead of just $P_n(f)$ since $P_n(f)$ can be 0.

Ex of $\sigma_A: \Omega_A \to \Omega_A$ s.t. $P_{2k+1}(\sigma_A) = 0$ for all $k \in \mathbb{N}_0$, while $P_{2k}(\sigma_A)$ is increasing.

$0 \rightleftarrows 1$ $P_{2k+1}(\sigma_A) = 0$, but $P_{2k}(\sigma_A) = 2$ for all k, so let us modify it.

$0 \rightleftarrows 1 \rightleftarrows 2$ $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $P_{2k+1}(\sigma_A) = 0$. $P_{2k}(\sigma_A)$ is increasing — find a formula for it.

Note If $\lim_{n\to\infty} \frac{\ln(P_n(f))}{n} = \lambda$, then for each $\varepsilon > 0$ there is N s.t. for all $n > N$, $e^{n(\lambda-\varepsilon)} < P_n(f) < e^{n(\lambda+\varepsilon)}$

Ex In the examples (1), (2), (3), the limit as above exists, and $p(f)$ equals
(1) 0; (2, 3) $\ln(m)$.

Next, we will find $p(\sigma_A)$ for the case of a primitive matrix A using:

Perron-Frobenius Theorem

Let A be a square matrix with non-negative entries such that for some $N \in \mathbb{N}$ all entries of A^N are positive (i.e. A is primitive). Then

- A has one (up to multiplication by a scalar) eigenvector v with positive coordinates;
- A has no other eigenvector with non-negative coordinates;
- The eigenvalue λ corresponding to v is positive and simple (i.e. a simple root of the characteristic polynomial of A)
- $\lambda > |\mu|$ for every other eigenvalue μ of A.

[Pf The book section 1.9 d or another source]

Corollary Let A be a primitive $m\times m$ matrix with entries 0 or 1. Then

(1) $\lambda > 1$, where λ is as in the theorem

(2) $\lim_{n\to\infty} \frac{\ln(P_n(\sigma_A))}{n}$ exists and equals λ. Thus $h(\sigma_A) = \ln(\lambda)$

Pf (1) Take N such that all entries of A^N are positive.
Since A is a 0-1 matrix, all entries $a_{ij}^{(N)}$ of A^N are ≥ 1.
Let v be an eigenvector with positive coordinates corresp. to λ.
Since $Av = \lambda v$, $A^N v = \lambda^N v$. Hence for each i,
$\lambda^N v_i = (A^N v)_i = \sum_{j=0}^{m-1} a_{ij}^{(N)} v_j \geq \sum_{j=0}^{m-1} v_j > v_i$. So $\lambda^N > 1$, and hence $\lambda > 1$ as λ is positive.

(2) Let $\lambda, \mu_1, \dots, \mu_{m-1}$ be eigenvalues of A, listed with multiplicities.
Since $\lambda > |\mu_i|$ for each i, there is $\varepsilon > 0$ s.t. $|\mu_i| < \lambda - \varepsilon$ for each i.
The e.values of A^n are $\lambda^n, \mu_1^n, \dots, \mu_{m-1}^n$. Therefore
$\mathrm{Trace}(A^n) = \lambda^n + \sum_{i=1}^{m-1} \mu_i^n$, where $\left|\sum_{i=1}^{m-1} \mu_i^n\right| \leq (m-1)(\lambda-\varepsilon)^n < \frac{1}{2}\lambda^n$
for all suff. large n. So $\frac{\lambda^n}{2} \leq \mathrm{Trace}(A^n) \leq \frac{3}{2}\lambda^n$ for all suff. large n, and hence $\lim_{n\to\infty} \frac{\ln(P_n(\sigma_A))}{n} = \lim_{n\to\infty} \frac{\ln(\mathrm{Trace}(A^n))}{n} = \ln(\lambda)$ $\square$

n-step topological Markov chains. alphabet: $\{0, \dots, m-1\}$ as before
There is a list of forbidden words of length $n+1$ (equivalently, $\leq n+1$), i.e., a map $\mathcal{A}: \{0, \dots, m-1\}^{n+1} \to \{0,1\}$, 0 - forbidden, 1 - admissible
$\Omega_{\mathcal{A}}$ = the set of all sequences of symbols $\{0, \dots, m-1\}$ that do not contain any forbidden words. Clearly, $\Omega_{\mathcal{A}}$ is shift-invariant.
$\Omega_{\mathcal{A}}$ is obtained by removing cylinders from Ω_m, and so $\Omega_{\mathcal{A}}$ is closed.

Every n-step top. Markov chain $\sigma: \Omega_{\mathcal{A}} \to \Omega_{\mathcal{A}}$ is topologically conjugate to a 1-step Markov chain $\sigma: \tilde{\Omega}_A \to \tilde{\Omega}_A$ constructed as follows:
Consider the alphabet consisting of words of length n in $\{0, \dots, m-1\}$ and the 0-1 $m^n \times m^n$ matrix A with
$A_{(i_1,\dots,i_n),(j_1,\dots,j_n)} = 1$ if $j_k = i_{k+1}$ for $k = 1, \dots, n-1$ and $\mathcal{A}(i_1, \dots, i_n, j_n) = 1$ and 0 otherwise.

$$\underbrace{i_1, i_2, \dots, i_{n-1}, i_n,}$$
$$\underbrace{j_1, \dots, j_{n-2}, j_{n-1}, j_n,}$$

a conjugacy $h: \Omega_{\mathcal{A}} \to \tilde{\Omega}_A$ is given by

$$h(\dots, \underbrace{\omega_0, \overbrace{\omega_1, \dots, \omega_{n-1}, \omega_n}^{x_1}}_{x_0}, \dots) = (\dots, x_0, x_1, x_2, \dots), \text{ where } x_k = \omega_k \dots \omega_{k+n-1}$$