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Topological entropy.

Topological entropy is a number that reflects complexity of the orbit structure of a dynamical system. It is the same for topologically conjugate systems, i.e. it is an invariant of topological conjugacy.

Let (X,d) be a compact metric space and let $f: X \to X$ be a continuous map. For $x, y \in X$, we consider the orbit segments of length n

$x, f(x), f^2(x), \dots, f^{n-1}(x)$
$y, f(y), f^2(y), \dots, f^{n-1}(y)$ and define $d_n(x,y) = \max\limits_{0 \le k \le n-1} d(f^k(x), f^k(y))$

Properties of d_n.

- $d_1 = d$, and $d_{n+1} \ge d_n$ for all $n \ge 1$.
- For each $n \in \mathbb{N}$, d_n is a distance on X (check)
- The ball $B_{d_n}(x, \varepsilon)$ is $\{y \in X : d(f^k(x), f^k(y)) < \varepsilon$ for $k = 0, \dots, n-1\}$
- The metrics d_n, $n \ge 1$, induce the same topology on X, in particular, (X, d_n) is compact for each n.

  Pf Let $n \in \mathbb{N}$. Then for any $x \in X$ and $\varepsilon > 0$, $B_d(x, \varepsilon) \supseteq B_{d_n}(x, \varepsilon)$. Since f is cont., so are f^k, $k = 1, \dots, n-1$. Hence for any $x \in X$ and $\varepsilon > 0$ there is $\delta > 0$ s.t. $d(x,y) < \delta$ implies $d(f^k(x), f^k(y)) < \varepsilon$ for $k = 0, \dots, n-1$, that is, $B_d(x, \delta) \subseteq B_{d_n}(x, \varepsilon)$. Thus d_n and d induce the same topology. □

A definition of top. entropy.

Fix $\varepsilon > 0$, and for $n \in \mathbb{N}$ let

$Sep(f, \varepsilon, n)$ = the max number of points in an ε-separated set in (X, d_n)
= the max number of pts in X s.t. the distance d_n between any two $\ge \varepsilon$.
= the max number of orbit segments of length n that are distinguishable with precision ε.

Since (X, d_n) is compact, $Sep(f, \varepsilon, n) < \infty$. Clearly, $Sep(f, \varepsilon, n) \ge 1$.

Let $h(f, \varepsilon) = \limsup\limits_{n \to \infty} \dfrac{\ln(Sep(f, \varepsilon, n))}{n}$ = the exp. growth rate of $Sep(f, \varepsilon, n)$.

If $\varepsilon_1 < \varepsilon_2$, then $Sep(f, \varepsilon_1, n) \ge Sep(f, \varepsilon_2, n)$ for every n, and hence $h(f, \varepsilon_1) \ge h(f, \varepsilon_2)$. By the monotonicity, $\lim\limits_{\varepsilon \to 0} h(f, \varepsilon)$ exists (finite or $+\infty$)

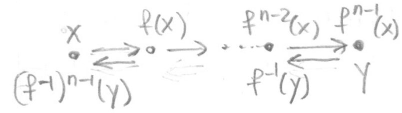
$h(f) = \lim\limits_{\varepsilon \to 0+} h(f, \varepsilon)$ is the <u>topological entropy</u> of f.

<u>Note</u>
- $h(f) \ge 0$ for every f.
- Since $\lim\limits_{\varepsilon \to 0+} h(f, \varepsilon)$ exists, we can find $h(f)$ by taking $\lim\limits_{k \to \infty} h(f, \varepsilon_k)$ for any sequence $\varepsilon_k \to 0^+$.

? Suppose that f is a homeomorphism. $h(f^{-1}) = ?$

- If f is a homeomorphism, then $h(f^{-1}) = h(f)$.

Indeed, there is a one-to-one correspondence between the orbit segments of length n for f and f^{-1}, and so $Sep(f,\varepsilon,n) = Sep(f^{-1},\varepsilon,n)$.



Ex $R_\alpha : S^1 \to S^1$. $h(R_\alpha) = 0$ (Note: R_α has a "simple" orbit structure)

Indeed, for any $x, y \in S^1$, $d_n(x,y) = d(x,y) \Rightarrow Sep(f,\varepsilon,n) = Sep(f,\varepsilon,1)$.

Proposition If $f: X \to X$ is an isometry or a contraction, then $h(f) = 0$.

Pf For such f, $d(f^k(x), f^k(y)) \le d(x,y)$, hence $d_n(x,y) = d(x,y)$, and so $Sep(f,\varepsilon,n) =$ the max number of points in an ε-separated set in $X = N(\varepsilon)$

$h(f,\varepsilon) = \limsup_{n\to\infty} \frac{\ln(N(\varepsilon))}{n} = 0$, and so $h(f) = \lim_{\varepsilon\to 0} h(f,\varepsilon) = 0$. □

Other ways to define top. entropy

Instead of $Sep(f,\varepsilon,n)$, we can use any of the following:

$Span(f,\varepsilon,n) =$ the min number of points in an ε-spanning set in (X, d_n), i.e. a set s.t. for any $x \in X$ there is y in this set with $d_n(x,y) < \varepsilon$
$=$ the min number of orbit segments of length n that approximate any orbit segment of length n up to ε.

$Cov(f,\varepsilon,n) =$ the minimal number of sets of d_n-diameter $< \varepsilon$ needed to cover X.

Proposition $Sep(f,2\varepsilon,n) \underset{1}{\le} Cov(f,2\varepsilon,n) \underset{2}{\le} Span(f,\varepsilon,n) \underset{3}{\le} Sep(f,\varepsilon,n)$

Pf 1 two pts at a distance $\ge 2\varepsilon$ cannot be covered by one set of diam. $< 2\varepsilon$.

3 any max ε-separated set is ε-spanning, otherwise we can add a point.

2 Let A be a min ε-spanning set. Then the balls of radius ε centered at the points of A cover X. By compactness, there exists $\varepsilon' < \varepsilon$ s.t. the balls of radius ε' centered at the pts of A still cover X. (Explain why.) Their diameters (as sets) $\le 2\varepsilon' < 2\varepsilon$, so 1 holds □.

It follows that using $Span(f,\varepsilon,n)$ or $Cov(f,\varepsilon,n)$ in the definition of top. entropy yields the same result.

Also, we can consider diameters $\le \varepsilon$ in the definition of $Cov(f,\varepsilon,n)$ since $Cov(f,2\varepsilon,n) \le Cov_{(\le\varepsilon)}(f,\varepsilon,n) \le Cov(f,\varepsilon,n)$.