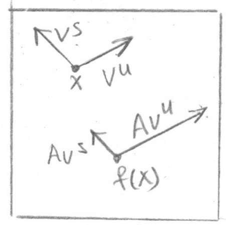
Hyperbolic dynamical systems

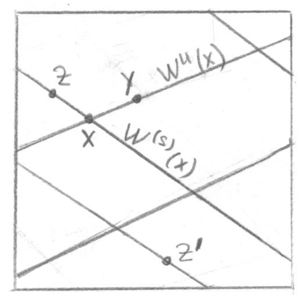
Basic model: a hyperbolic automorphism of $\mathbb{T}^2$.
$L: \mathbb{T}^2 \to \mathbb{T}^2$ given by $L(x) = Ax \mod 1$. $DL \equiv A$.

[The tangent space to $\mathbb{T}^2$ at x is canonically identified with $\mathbb{R}^2$, so we view the differential of a map of $\mathbb{T}^2$ as a matrix.]



Since A is a hyperbolic matrix with e.values μ, λ s.t. $0 < |\mu| < 1 < |\lambda|$, for each $x \in \mathbb{T}^2$ there are subspaces $E^s(x)$ and $E^u(x)$ of the tangent space at x, s.t. for all $n \in \mathbb{N}$,
$\| DL^n(v^s) \| = \| A^n v^s \| = |\mu|^n \|v^s\|$ for all $v^s \in E^s(x)$
$\| DL^n(v^u) \| = \| A^n v^u \| = |\lambda|^n \|v^u\|$ for all $v^u \in E^u(x)$.
The subspaces E^s and E^u are called stable and unstable

Note: we have equalities in dim 2, but in higher dim we have inequalities
$\|A^n v^s\| \leq (c_1) \lambda_s^n \|v^s\|$ and $\|A^n v^u\| \geq (c_2) \lambda_u^n \|v^u\|$,
where c_1, c_2 depend on the norm.



Let $x \in \mathbb{T}^2$. Let us denote the projections of the contracting and expanding lines through x by $W^s(x)$ and $W^u(x)$. For the automorphism L, these are the stable and unstable manifolds of x.

For any $z \in W^s(x)$ close to x along $W^s(x)$,
$d(L^n(x), L^n(z)) = |\mu|^n d(x,y) \to 0$ as $n \to \infty$,
and for any $z \in W^s(x)$, $d(L^n(x), L^n(z)) \to 0$ as $n \to \infty$.
In fact, $W^s(x) = \{ z \in \mathbb{T}^2 : d(L^n(x), L^n(z)) \to 0 \text{ as } n \to \infty \}$ (Explain why)
Similarly, for any $y \in W^u(x)$ close to x along $W^u(x)$
$d(L^{-n}(x), L^{-n}(y)) = |\lambda|^{-n} d(x,y) \to 0$ as $n \to \infty$, and
$W^u(x) = \{ y \in \mathbb{T}^2 : d(L^{-n}(x), L^{-n}(y)) \to 0 \text{ as } n \to \infty \}$.

Now, let us consider a C^1-small perturbation f of L, that is, a diffeomorphism $f: \mathbb{T}^2 \to \mathbb{T}^2$ s.t. for all $x \in \mathbb{T}^2$, $d(f(x), L(x)) < \varepsilon$ and $\| D_x f - A \| < \varepsilon$. (It follows automatically that f^{-1} and L^{-1} are also close)
We know that for any suff. small ε, f is topologically conjugate to L via a homeomorphism h. Thus f has the same topological properties.

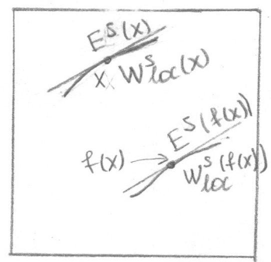
! Even though both L and f are differentiable, h is usually <u>not</u>. Thus we cannot obtain properties of f as a differentiable map just from its top. conjugacy to L.

One can show that for each $x \in X$ there exist subspaces $E^s(x)$ and $E^u(x)$ with $E^s(x) \oplus E^u(x) =$ the tangent space at x such that
E^s and E^u depend continuously on x,
E^s and E^u are Df-invariant, i.e. for each x, $D_x f(E^{s/u}(x)) = E^{s/u}(f(x))$
and there exist constants $0 < \lambda^s < 1 < \lambda^u$ s.t. for all $n \in \mathbb{N}$,

$\| D_x f^n (v^s) \| \le (C_1)\, \lambda_s^n \|v^s\|$ for all $v^s \in E^s(x)$, and

$\| D_x f^n (v^u) \| \ge (C_2)\, \lambda_u^n \|v^u\|$ for all $v^u \in E^u(x)$

<u>Note</u> Often, instead of ↑ one writes $\| Df^{-n}(v^u)\| \le (C_2)\, \tilde{\lambda}_u^n \|v^u\|$, where $0 < \tilde{\lambda}_u < 1$, since this condition better identifies E^u.



There exist smooth curves (in $\mathbb{T}^2$, manifolds in general) $W^s_{loc}(x)$ and $W^u_{loc}(x)$ tangent at x to $E^s(x)$ and $E^u(x)$, resp. s.t. $f(W^s_{loc}(x)) \subset W^s_{loc}(f(x))$ and $f^{-1}(W^u_{loc}(x)) \subset W^u_{loc}(f^{-1}(x))$
They are called local stable and unstable manifolds of x
They can be extended to global stable and unstable manifolds $W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(W^s_{loc}(f^n(x)))$ and $W^u(x) = \bigcup_{n=0}^{\infty} f^n(W^u_{loc}(f^{-n}(x)))$.
We have: $W^s(x) = \{z : d(f^n(x), f^n(z)) \to 0 \text{ as } n \to \infty\}$; $W^u(x) = \{y : d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty\}$

- a diffeomorphism f of $\mathbb{T}^N$ (or of a compact connected Riemannian manifold) is called <u>hyperbolic</u> is for each $x \in \mathbb{T}^N$ there is a splitting of the tangent space into $E^s(x) \oplus E^u(x)$ with properties as above.

- let $f: U \to f(U)$ be a diffeomorphism.
If a splitting $E^s(x) \oplus E^u(x)$ with properties as above exists for all x in a closed f-invariant set Λ, we say that Λ is a hyperbolic set
For example, the invariant set Λ for Smale's horseshoe is a hyperbolic set.