③

Proposition Let X be a compact metric space and let $f: X \to X$ be a weak contraction, i.e. $d(f(x), f(y)) < d(x,y)$ for all $x \neq y$ in X. $(**)$
Then f has a unique fixed pt $x_* \in X$ and for every $x \in X$, $f^n(x) \to x_*$ as $n \to \infty$.

Pf Existence of a fixed pt:
Let $h(x) = d(x, f(x))$. Then h is continuous on X (Explain why)
Since X is compact, h attains its min value at some $x_* \in X$.
Want: $f(x_*) = x_*$, i.e. $h(x_*) = 0$.
Suppose $f(x_*) \neq x_*$. Then by $(**)$,
$$h(f(x_*)) = d(f(x_*), f(f(x_*))) < d(x_*, f(x_*)) = h(x_*),$$
contradicting the minimality. Thus x_* is fixed.
Uniqueness of a fixed pt. follows from $(**)$ (Explain).
Let $x \neq x_*$. Consider the sequence (a_n), where $a_n = d(f^n(x), x_*)$.
Then $a_n \geq 0$ for all n and (a_n) is decreasing.
Hence (a_n) converges to some $a \geq 0$. If $a = 0$ we are done.
Let $a > 0$. By compactness of X, $(f^n(x))$ has a subsequence $(f^{n_k}(x))$ converging to some $y \in X$, and $d(y, x_*) = a > 0$.
Then by continuity of f, $f^{n_k+1}(x) \to f(y)$, and $d(f(y), x_*) < a$.
Hence $d(f^{n_k+1}(x), x_*) < a$ for all suff. large k, a contradiction.
Thus $a = 0$ and $f^n(x) \to x_*$. $\square$.

Note In general, the convergence is not necessarily exponential, and we do not have an estimate for the speed of convergence.

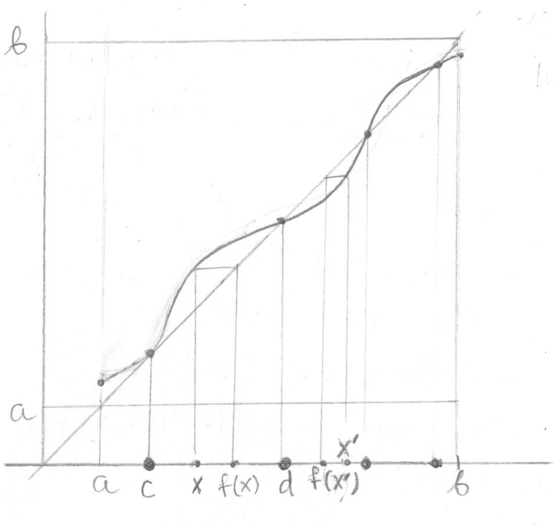
Fixed points

Let X be a metric space, and let $f: X \to X$ be continuous.
Let $\mathrm{Fix}(f) = \{x \in X : f(x) = x\}$. Then $\mathrm{Fix}(f)$ is closed.
Indeed, suppose $x_n \in \mathrm{Fix}(f)$, $n \in \mathbb{N}$, and $x_n \to x \in X$.
Then, by continuity of f, $f(x) = f(\lim x_n) = \lim f(x_n) = \lim x_n = x$
The set $\mathrm{Fix}(f)$ can be empty.

Let $n \in \mathbb{N}$ and let $\mathrm{Per}_n(f) = \{x \in X : f^n(x) = x\}$ be the set of periodic pts of f of period n. Then $\mathrm{Per}_n(f)$ is also closed since $\mathrm{Per}_n(f) = \mathrm{Fix}(f^n)$.

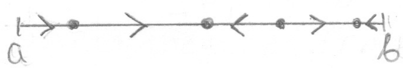
Increasing interval maps.

$f:[a,b]\to[a,b]$ continuous and non-decreasing (also called increasing), i.e. $f(x)\le f(y)$ for all $x,y\in[a,b]$ s.t. $x<y$.



Every continuous $f:[a,b]\to[a,b]$ has a fixed point: Consider $h(x)=f(x)-x$ and apply the Intermediate Value Thm.

The set of fixed pts of f can be finite, countably infinite, or uncountable. In fact, any closed set $E\subseteq[a,b]$ is $\mathrm{Fix}(f)$ for some continuous increasing f.



Prop. Let $f:[a,b]\to[a,b]$ be a cont. non-decreasing map. Then for each $x\in X$, $(f^n(x))$ converges to a fixed pt. of f as $n\to\infty$.

pf (Fill in the details).
If $\mathrm{Fix}(f)=[a,b]$, we are done. Otherwise, since $\mathrm{Fix}(f)$ is closed, its complement is open in $[a,b]$ and hence consists of disjoint open intervals and also $[a,a')$ if $f(a)\ne a$ and $(b',b]$ if $f(b)\ne b$.
Let x be in such an open interval (c,d), where c and d are fixed.
Since f is non-decreasing, $f(c,d)\subseteq[c,d]$
On (c,d), either $f(x)>x$ or $f(x)<x$. Let $f(x)>x$.
Then the sequence $(f^n(x))_{n\ge0}$ is increasing and bdd above by d.
Hence it has a limit, $y\in[x,d]$. Since y must be fixed, $y=d$.
Similarly consider $f(x)<x$, and $[a,a')$, $(b',b]$. $\square$

Now, suppose that $f:[a,b]\to[a,b]$ is cont., strictly increasing, $f(a)=a$ and $f(b)=b$. Then f is invertible.
In this case, for each $x\in[a,b]$ that is not fixed, $(f^n(x))_{n\in\mathbb{Z}}$ is strictly monotone, x lies in an open interval between two adjacent fixed points, $f^n(x)$ converges to one of these fixed points as $n\to\infty$, and to the other as $n\to-\infty$.
We say that x is positively asymptotic to one fixed point, and negatively asymptotic to the other fixed pt.