Properties of topological entropy.

Proposition If metrics d and d' on X induce the same topology, then $h_d(f) = h_{d'}(f)$ (top. entropy with respect to d and d' is the same)

Pf Consider the identity map $Id : (X, d') \to (X, d)$. Since d and d' induce the same topology, Id is continuous and hence uniformly cont. So given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ s.t. $d'(x,y) < \delta(\varepsilon)$ implies $d(x,y) < \varepsilon$, and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. It follows that for any $x \in X$ and $n \in \mathbb{N}_0$, $B_{d'_n}(x, \delta(\varepsilon)) \subseteq B_{d_n}(x, \varepsilon)$. Hence $Span_{d'}(f, \delta(\varepsilon), n) \geq Span_d(f, \varepsilon, n)$ for every n, and it follows that $h_{d'}(f) \geq h_d(f)$. Interchanging d and d', we obtain $\leq$. □

Theorem Top. entropy is an invariant of top. conjugacy, i.e. if $f : X \to X$ and $g : Y \to Y$ are conjugate via a homeomorphism $H : X \to Y$, then $h(f) = h(g)$.

Pf Let d^X be a metric on X. Then $d^Y(y_1, y_2) = d^X(H^{-1}(y_1), H^{-1}(y_2))$ is a metric on Y generating the topology of Y. Now, $H : (X, d^X) \to (Y, d^Y)$ is an isometry, and so $d^X_n(x_1, x_2) = d^Y_n(H(x_1), H(x_2))$ for any $x_1, x_2 \in X$ and $n \in \mathbb{N}_0$. It follows that $h(f) = h(g)$. □

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ H \downarrow & & \downarrow H \\ Y & \xrightarrow{g} & Y \end{array}$$

Proposition If $g : Y \to Y$ is a factor of $f : X \to X$, then $h(g) \leq h(f)$.

Pf Let d^X and d^Y be metrics on X and Y, and let $H : X \to Y$ be a semiconjugacy. Since H is cont., it is uniformly cont. Hence given $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ s.t. $d^X(x_1, x_2) < \delta(\varepsilon)$ implies $d^Y(H(x_1), H(x_2)) < \varepsilon$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then for any $x \in X$ and $n \in \mathbb{N}_0$, the image of $B_{d^X_n}(x, \delta(\varepsilon))$ under H is contained in $B_{d^Y_n}(H(x), \varepsilon)$. Since H is surjective, we have $Span(f, \delta(\varepsilon), n) \geq Span(g, \varepsilon, n)$. It follows that $h(f) \geq h(g)$. □

Note The Thm above follows from this proposition.

Ex Since $E_m : S^1 \to S^1$ is a factor of $\sigma_m^R : \Sigma_m^R \to \Sigma_m^R$, $h(E_m) \leq h(\sigma_m^R) = \ln m$.

In fact, • $h(E_m) = \ln m$. Proof 1 (by finding $Sep(E_m, \varepsilon_k, n)$)

Recall that if $d(x,y) \leq \frac{1}{2m}$, then $d(E_m(x), E_m(y)) = m\, d(x,y)$. Hence if $d(E_m^k(x), E_m^k(y)) \leq \frac{1}{2m}$ for $k = 0, \dots, n-1$, then $d_n(x,y) = d(E_m^{n-1}(x), E_m^{n-1}(y)) = m^{n-1} d(x,y)$.

So if $d(x,y) = 1/m^{k+n-1}$, then $d_n(x,y) = 1/m^k$. Let $\varepsilon_k = 1/m^k$.

The set $\{ \ell / m^{n+k-1} : 0 \leq \ell \leq m^{n+k-1} \}$ is ε_k-separated in (S^1, d_n), i.e. it is (n, ε_k)-separated, and there is no (n, ε_k)-sep. set with more than m^{n+k-1} pts (Why?)

So $Sep(E_m, \varepsilon_k, n) = m^{n+k-1}$, and it follows that $h(E_m) = \ln m$. □

Proof 2 The map E_m is expansive, and hence $p(E_m) \leq h(E_m)$. We showed that $p(E_m) = \ln m$, and so $h(E_m) \geq \ln m$. Since $h(E_m) \leq \ln m$, we obtain $h(E_m) = \ln m$. □

Remark The dynamical systems $E_m: S^1 \to S^1$, $\sigma_m: \Omega_m \to \Omega_m$, and $\sigma_A: \Omega_A \to \Omega_A$ with primitive A have "complex" orbit structure. Each has positive entropy: $\ln m$, $\ln m$, and $\ln \underbrace{(\lambda_{max})}_{>1}$, respectively.

Proposition (More properties of top. entropy).

(1) If $\Lambda \subseteq X$ is a closed f-invariant set, then $h(f|_\Lambda) \le h(f)$. (Clear)

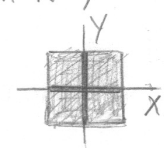
(2) If $X = \bigcup_{i=1}^{m} \Lambda_i$, where Λ_i are closed f-invariant subsets of X (not necessarily disjoint), then $h(f) = \max_{1\le i\le m} h(f|_{\Lambda_i})$.

(3) For any $m \in \mathbb{N}$, $h(f^m) = m \cdot h(f)$.

(4) For $f: X \to X$ and $g: Y \to Y$, consider $f\times g: X\times Y \to X\times Y$ given by $(f\times g)(x,y) = (f(x), g(y))$. Then $h(f\times g) = h(f) + h(g)$.

Pf (Omitting some details)

(4) Let d^X and d^Y be metrics on X and Y. We consider the metric d on $X\times Y$ given by $d((x_1,y_1),(x_2,y_2)) = \max\{d^X(x_1,x_2), d^Y(y_1,y_2)\}$.



Then ε-balls in $X\times Y$ are products of ε-balls in X and in Y. It follows that The same holds for ε-balls in d_n, d_n^X, and d_n^Y. It follows that the product on (n,ε)-separated/spanning sets in X and Y is an (n,ε)-separated/spanning set in $X\times Y$. Therefore,

$\text{Span}(f\times g,\varepsilon,n) \le \text{Span}(f,\varepsilon,n)\cdot \text{Span}(g,\varepsilon,n) \Rightarrow h(f\times g) \le h(f) + h(g)$,

$\text{Sep}(f\times g,\varepsilon,n) \ge \text{Sep}(f,\varepsilon,n)\cdot \text{Sep}(g,\varepsilon,n) \Rightarrow h(f\times g) \ge h(f) + h(g)$.

(2) ($\ge$) by (1). ($\le$) The union of covers of Λ_i's by sets of d_n-diameter $<\varepsilon$ is a cover of X by such sets. Hence $\text{Cov}(f,\varepsilon,n) \le \sum_{i=1}^{m} \text{Cov}(f|_{\Lambda_i},\varepsilon,n)$

For each $n\in\mathbb{N}$, there is $i \in \{1,\dots,m\}$ s.t. $\text{Cov}(f|_{\Lambda_i},\varepsilon,n) \ge \frac{1}{m}\text{Cov}(f,\varepsilon,n)$ (*)

Hence there is i s.t. (*) holds for infinitely many n. Recall that for Cov, $\lim_{n\to\infty} \frac{1}{n}\ln(\text{Cov}(\dots))$ exists and hence equals lim along a subsequence. For i as above,

$\lim_{n\to\infty} \frac{1}{n}\ln(\text{Cov}(f|_{\Lambda_i},\varepsilon,n)) \ge \lim_{n\to\infty}\frac{1}{n}(\ln\frac{1}{m} + \ln \text{Cov}(f,\varepsilon,n)) = \lim_{n\to\infty}\frac{1}{n}\ln(\text{Cov}(f,\varepsilon,n))$

and it follows that $h(f) \le h(f|_{\Lambda_i}) \le \max_i h(f|_{\Lambda_i})$.

(3) ($\le$) An orbit segment of length n for f^m is $x, f^m(x), \dots, f^{m(n-1)}(x)$. Hence $\max_{0\le i<n} d(f^{mi}(x), f^{mi}(y)) \le \max_{0\le j<mn} d(f^j(x), f^j(y))$, and so

$\text{Span}(f^m,\varepsilon,n) \le \text{Span}(f,\varepsilon,mn)$. Therefore,

$\limsup_{n\to\infty}\frac{1}{n}\ln(\text{Span}(f^m,\varepsilon,n)) \le m\cdot \limsup_{n\to\infty}\frac{1}{mn}\ln(\text{Span}(f,\varepsilon,mn)) \le$

$\le m\cdot \limsup_{N\to\infty}\frac{1}{N}\ln(\text{Span}(f,\varepsilon,N))$, and so $h(f^m) \le m\cdot h(f)$.

($\ge$) For each $\varepsilon>0$ there is $\delta(\varepsilon)$ s.t. if $d(x,y)<\delta(\varepsilon)$, then $d(f^i(x), f^i(y)) < \varepsilon$ for $i=0,\dots,m-1$. So if $d(f^{mk}(x), f^{mk}(y)) < \delta(\varepsilon)$, then $d(f^{mk+i}(x), f^{mk+i}(y)) < \varepsilon$ for $i=0,\dots,m-1$. Then $\text{Span}(f^m,\delta(\varepsilon),n) \ge \text{Span}(f,\varepsilon,mn)$ and it follows that $h(f^m) \ge m\cdot h(f)$. $\square$