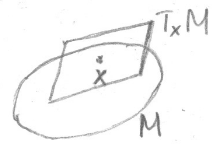
Geodesics and geodesic flow – an overview...

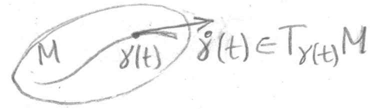
Let M be a (compact) connected differentiable manifold, for example, S^2 or $\mathbb{T}^2$ or .

Let g be a smooth Riemannian metric on M, i.e. a family $\{g_x : x \in M\}$ of inner products $g_x = \langle \cdot, \cdot \rangle_x$ on the tangent space $T_x M$ that depend smoothly on x.



Let $\gamma : [a,b] \to M$ be a piecewise continuously differentiable curve. The length of γ is

$$L(\gamma) = \int_a^b \|\dot\gamma(t)\|_{\gamma(t)}\, dt = \int_a^b \sqrt{\langle \dot\gamma(t), \dot\gamma(t) \rangle_{\gamma(t)}}\, dt.$$

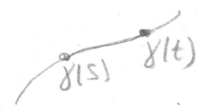


We can define a metric d_g on M by

$$d_g(x,y) = \inf \left\{ L(\gamma) : \begin{array}{l} \gamma \text{ is a piecewise continuously differentiable} \\ \text{curve from } x \text{ to } y \end{array} \right\}$$

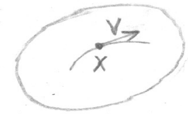
A curve $\gamma : [a,b] \to M$ is a unit speed <u>geodesic</u> if for every $t_0 \in [a,b]$ there is an interval $\mathcal{J} \subseteq [a,b]$ containing t_0 such that for all $s \le t$ in $\mathcal{J}$,

$$d_g(\gamma(s), \gamma(t)) = L(\gamma|_{[s,t]}) = t - s.$$



So $\gamma|_{[s,t]}$ is the shortest curve from $\gamma(s)$ to $\gamma(t)$, and $\|\dot\gamma(t)\| = 1$ for all t.
γ is a local isometry from $[a,b]$ to M.

<u>Note</u> γ is a solution of a certain second-order ODE.
It follows that for (x,v), where $x \in M$ and v is a unit tangent vector at x there exists a unique unit speed geodesic $\gamma : (-\delta, \delta) \to M$ such that $\gamma(0) = x$ and $\dot\gamma(0) = v$.



If (M, d_g) is complete, then γ extends to a unit speed geodesic $\gamma : (-\infty, \infty) \to M$.

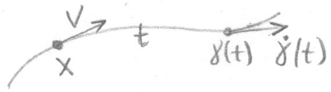
<u>Ex</u> M = the Euclidean plane. Geodesics – straight lines

<u>Ex</u> $M = \mathbb{T}^2$ Geodesics – projections of straight lines

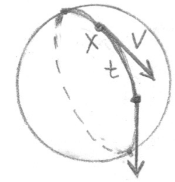
<u>Ex</u> $M = S^2$ (the standard two-sphere)
Geodesics – great circles.

Suppose that every geodesic in M extends to $\gamma : (-\infty, \infty) \to M$.

Geodesic flow is the flow on the unit tangent bundle of M, SM, defined as follows. Given $x \in M$ and $v \in S_x M$, consider the unique unit speed geodesic γ s.t. $\gamma(0) = x$ and $\dot\gamma(0) = v$. Travel distance/time t along γ and take the point $\gamma(t)$ and the unit vector $\dot\gamma(t) \in S_{\gamma(t)} M$. Set $\varphi^t(x,v) = (\gamma(t), \dot\gamma(t))$.



Ex



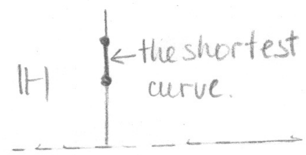
Poincaré half-plane (a model of hyperbolic geometry)

$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ with the inner product in $T_{\underset{z}{(x,y)}} \mathbb{H} \approx \mathbb{R}^2 \approx \mathbb{C}$

$\langle u, v \rangle_{(x,y)} = \frac{1}{y^2} \langle u, v \rangle_{\mathbb{R}^2}$ or $\langle a+bi, \tilde a + \tilde b i \rangle_z = \frac{1}{(\operatorname{Im} z)^2} \operatorname{Re}\big((a+bi)(\tilde a - \tilde b i)\big)$

Vertical half-lines are geodesics.

$t \mapsto ie^t$ gives a unit-speed, or natural, parametrization of $\{iy : y > 0\} = I$.

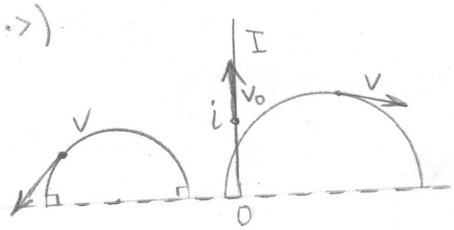


What are the other geodesics?

If γ is a geodesic and F is an isometry of $(\mathbb{H}, \langle\cdot,\cdot\rangle)$, then $F \circ \gamma$ is also a geodesic.

Möbius transformations $F(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$ $(=1)$ are isometries of $(\mathbb{H}, \langle\cdot,\cdot\rangle)$.

Mobius transformations map (circles and lines) to (circles and lines), moreover, for any semicircle C with the center on the real axis and any unit tangent vector v at a point of C, there is a Möbius transf. that maps I to C and the unit vector v_0 at i to v. For any $z \in \mathbb{H}$ and a unit vector v at z, there is a unique vertical half-line or a semicircle with the center on the real axis through z with unit tangent vector v at z. Thus these curves are precisely the geodesics in $\mathbb{H}$.

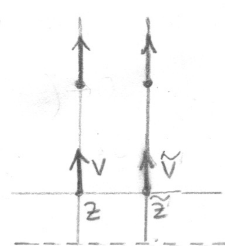


Let $\{\varphi^t\}$ be the geodesic flow on $S\mathbb{H}$

Let v be a vertical unit vector at $z = x+iy$

Then for any vertical unit $\tilde v$ at $\tilde z = \tilde x + iy$

$d_{S\mathbb{H}}(\varphi^t(z,v), \varphi^t(\tilde z, \tilde v)) \to 0$ as $t \to \infty$.



See Section 5.4 for details and further discussion.