(21)

Hyperbolic automorphisms of $\mathbb{T}^2$

These maps are projections of hyperbolic linear maps of $\mathbb{R}^2$ onto $\mathbb{T}^2$.
They have complex orbit structure, in particular, they are top. mixing and hence top. transitive, and periodic points are dense.
Also, they are structurally stable, i.e. any map of $\mathbb{T}^2$ that is C^1-close to such f is top. conjugate to f.

We would like to project an invertible linear map $F: \mathbb{R}^2 \to \mathbb{R}^2$ to an invertible map f of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, so that the diagram with $\pi(v) = v \bmod 1$ is commutative:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, shift left, "F"] \arrow[d, "\pi"'] & \mathbb{R}^2 \arrow[l, shift left, "F^{-1}"] \arrow[d, "\pi"] \\ \mathbb{T}^2 \arrow[r, shift left, "f"] & \mathbb{T}^2 \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

$F(u) = Au$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det A = ad - bc \neq 0$.
Then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

F projects $\Leftrightarrow$ if $v_1 - v_2 \in \mathbb{Z}^2$ then $F(v_1) - F(v_2) = F(v_1 - v_2) \in \mathbb{Z}^2$
$\Leftrightarrow$ if $v \in \mathbb{Z}^2$, then $F(v) \in \mathbb{Z}^2 \Leftrightarrow A$ is an integer matrix.

Similarly, F^{-1} projects $\Leftrightarrow A^{-1}$ is an integer matrix.
For an integer matrix A, A^{-1} is also integer $\Leftrightarrow \det A = \pm 1$
Indeed, $(\Leftarrow)$ clear, $(\Rightarrow)$ If $ad - bc = k \neq \pm 1$, then k divides a, b, c, d, but then k^2 divides $ad - bc \Rightarrow k^2$ divides $a, b, c, d, \dots$

- Thus A is an integer matrix with $\det A = \pm 1$.
  Often, we consider A with det 1, i.e. $A \in SL(2, \mathbb{Z})$. In this case, F is orientation-preserving.
- The invertible map $f: \mathbb{T}^2 \to \mathbb{T}^2$ given by $f(u) = Au \bmod 1$ is called an <u>automorphism of $\mathbb{T}^2$</u>. (It is an automorphism of $\mathbb{T}^2$ viewed as an additive group).
- F maps the unit square $[0,1] \times [0,1]$ to a parallelogram with integer vertices and area 1. Since $|\det A| = 1$, F is area-preserving, and so is f.

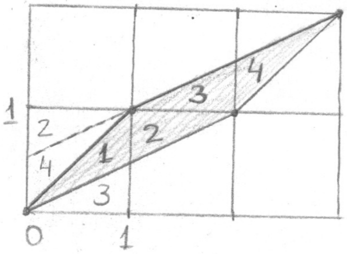
Now, suppose that the integer matrix A with $\det A = \pm 1$ is <u>hyperbolic</u>, i.e. its eigenvalues λ and μ satisfy $0 < |\mu| < 1 < |\lambda|$.
Then $f(u) = Au \bmod 1$ is called a <u>hyperbolic automorphism of $\mathbb{T}^2$</u>.

For a hyperbolic A, there are two invariant lines through 0 spanned by the e.vectors, one expanding and one contracting.

A famous example: Arnold's cat map. (60's)

$A=\begin{pmatrix}2&1\\1&1\end{pmatrix}$, $\det A=1$, $A^{-1}=\begin{pmatrix}1&-1\\-1&2\end{pmatrix}$, $F\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x+y\\x+y\end{pmatrix}$

F maps the unit square to the following parallelogram P of area 1

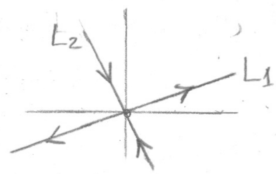


Note that the squares divide P into 4 pieces that fit into $[0,1]\times[0,1]$. As all squares are identified, P wraps around $\mathbb{T}^2$.

Arnold showed the effects of this map by applying it to an image of a cat,

see a picture from "Ergodic Problems of Classical Mechanics" by Arnold and Avez (1968). With repeated applications of f, the picture gets more and more scrambled.

The eigenvalues of A are $\lambda=\frac{3+\sqrt{5}}{2}$ and $\mu=\frac{3-\sqrt{5}}{2}$.



Since $0<\mu<1<\lambda$, A is hyperbolic.

Corresponding eigenvectors: $v_1=\begin{pmatrix}1\\ \frac{\sqrt{5}-1}{2}\end{pmatrix}$, $v_2=\begin{pmatrix}1\\ \frac{-\sqrt{5}-1}{2}\end{pmatrix}$

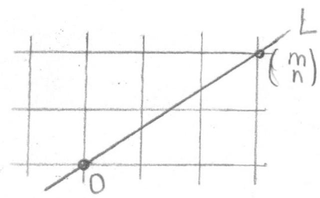
In this example, v_1 and v_2 are orthogonal since A is symmetric.

Note that $\lambda,\mu\notin\mathbb{Q}$ and v_1,v_2 have irrational slopes. The latter implies that the projections of L_1 and L_2 onto $\mathbb{T}^2$ are dense in $\mathbb{T}^2$.

Proposition Let A be a hyperbolic integer 2×2 matrix with $\det A=\pm1$. Then e.values of A are irrational and e.vectors of A have irrational slopes.

Pf Suppose that an e.vector v of A is either vertical or has rational slope.

Consider the invariant line L spanned by v.



Then L intersects $\mathbb{Z}^2$ at a point other than 0.

Let $\begin{pmatrix}m\\n\end{pmatrix}$ be a point in $L\cap\mathbb{Z}^2$ at the smallest positive distance from 0.

If F contracts L, $F\begin{pmatrix}m\\n\end{pmatrix}\neq0$ is an integer point on L closer to 0 than $\begin{pmatrix}m\\n\end{pmatrix}$. If F expands L, then $F^{-1}\begin{pmatrix}m\\n\end{pmatrix}$ is.

In either case, we obtain a contradiction, so v has irrational slope.

Now, let λ be an e.value of A. If λ is rational, then $(A-\lambda I)v=0$ has a solution v with rational components, and hence either rational or undefined slope, which is impossible. So $\lambda\notin\mathbb{Q}$. □

Questions to start thinking about:

? What are the periodic pts of f?

? How many of period n?

? Why is f top. mixing?

? $h(f)=?$