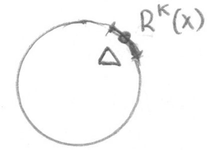
⑥ We continue discussing circle rotations with irrational α.

Uniform distribution (or equidistribution) of a positive semiorbit.

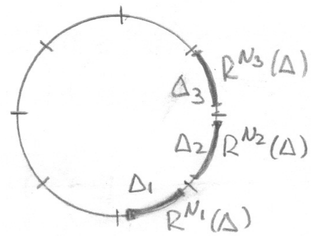
(Asymptotic frequency of visiting an arc (interval) is the same for all arcs of the same length.)



Let $\alpha \notin \mathbb{Q}$. We write R for R_α. Let Δ be an arc in S^1.
We fix $x \in S^1$ and consider its positive semiorbit, $O^+(x)$.
Let $F_\Delta(n)$ be the number of K, $0 \le K < n$, s.t. $R^K(x) \in \Delta$.
The asymptotic frequency of visiting Δ is $\varphi(\Delta) = \lim\limits_{n\to\infty} \frac{F_\Delta(n)}{n}$.

Theorem For any arc Δ in S^1, $\varphi(\Delta) = |\Delta|$.

Pf Let $\overline{\varphi}(\Delta) = \limsup \frac{F_\Delta(n)}{n}$ and $\underline{\varphi}(\Delta) = \liminf \frac{F_\Delta(n)}{n}$.
We show that $\overline{\varphi}(\Delta) \le |\Delta|$ and $\underline{\varphi}(\Delta) \ge |\Delta|$, and hence $\varphi(\Delta) = |\Delta|$.



- Let $|\Delta| = \frac{1}{K}$, where $K \ge 2$. We show that $\overline{\varphi}(\Delta) \le \frac{1}{K-1}$.
  We divide S^1 into $K-1$ arcs Δ_i with $|\Delta_i| = \frac{1}{K-1}$.
  Since $|\Delta| < |\Delta_i|$ and $O^+(x)$ is dense,
  for each $i = 1, \dots, K-1$ there is N_i s.t. $R^{N_i}(\Delta) \subset \Delta_i$.
  Hence for every $n \in \mathbb{N}$, $F_\Delta(n) \le F_{\Delta_i}(n+N_i)$ (Why?)
  Summing these inequalities and letting $N = \max N_i$, we obtain
  $$(K-1)F_\Delta(n) \le \sum_{i=1}^{K-1} F_{\Delta_i}(n+N_i) \le \sum_{i=1}^{K-1} F_{\Delta_i}(n+N) = F_{S^1}(n+N) = n+N.$$
  Hence $\frac{F_\Delta(n)}{n} \le \frac{1}{K-1} \underbrace{\frac{n+N}{n}}_{\to 1}$, and so $\overline{\varphi}(\Delta) = \limsup \frac{F_\Delta(n)}{n} \le \frac{1}{K-1}$
- It follows that if $|\Delta'| = \ell/K$, then $\overline{\varphi}(\Delta') \le \frac{\ell}{K-1} = \frac{\ell}{K} \cdot \frac{K}{K-1}$.
- Let Δ be an arc in S^1. Then for any $\varepsilon > 0$ there is an arc Δ' s.t.
  $\Delta \subsetneq \Delta'$, $|\Delta'| = K/\ell$, and $|\Delta'| \le |\Delta| + \varepsilon$.
  Then $\overline{\varphi}(\Delta) \le \overline{\varphi}(\Delta') \le \frac{\ell}{K} \cdot \frac{K}{K-1} \le (|\Delta| + \varepsilon) \cdot \frac{K}{K-1}$.
  We let $\varepsilon \to 0$. Then $K \to \infty$ and we obtain $\overline{\varphi}(\Delta) \le |\Delta|$.
- $F_\Delta(n) = n - F_{(S^1 \setminus \Delta)}(n)$ for each n. It follows that
  $\underline{\varphi}(\Delta) = 1 - \overline{\varphi}(S^1 \setminus \Delta) \ge 1 - |S^1 \setminus \Delta| = |\Delta|$.
- Thus $\overline{\varphi}(\Delta) \le |\Delta| \le \underline{\varphi}(\Delta)$, and hence $\varphi(\Delta) = |\Delta|$. □

T/F For any Lebesgue measurable set $A \subseteq S^1$ and any $x \in S^1$, the asymptotic frequency of visiting A equals the measure of A.

(F) Let $A = O^+(x)$. Then $\varphi_x(A) = 1$, while $\mu(A) = 0$.

An application: First digits of powers.

Let $m \in \mathbb{N}$, and m is not a power of 10. For example, $m=2$.

Can 9 be the first (leftmost) digit of 2^n? Yes, $2^{53} = 9\ldots$

Can the first digits of 2^n be $\underbrace{33\ldots3}_{100}$?

What are the asymptotic frequencies of first digits of m^n?
Do they even exist? Do they depend on m?

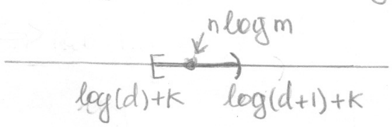
Connection to circle rotations

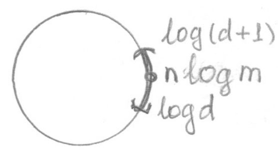
We say that $d \in \mathbb{N}$ gives the first digits of m^n if $m^n = d**\ldots*$

For example, 81 gives the first digits of $2^{13} = 8192$.

Note that $81 \cdot 10^2 \leq 2^{13} < 82 \cdot 10^2$.

$d \in \mathbb{N}$ gives the first digits of $m^n \iff$
$\iff d \cdot 10^k \leq m^n < (d+1) \cdot 10^k$ for some $k \in \mathbb{N} \cup \{0\} \iff$
$\iff \log_{10} d + k \leq n \cdot \log_{10} m < \log_{10}(d+1) + k$. We write $\log$ for $\log_{10}$.





Let $\alpha = \log m$. Since $m \neq 10^\ell$, α is irrational.

We see that d gives the first digits of $m^n \iff$
$\iff R_\alpha^n(0) = 0 + n\alpha \mod 1$ is in the interval $[\log d, \log(d+1))$ in the circle.

Since the positive semiorbit of 0 is dense, there is n s.t. $R_\alpha^n(0)$ is in this interval. Therefore, any $d \in \mathbb{N}$ gives the first digits of m^n for some n.

Moreover, for any $d \in \mathbb{N}$, the asymptotic frequency

$$\varphi(d) = \lim_{n\to\infty} \frac{1}{n}(\#k : 0 \leq k < n \text{ and } m^k \text{ starts with } d) =$$

$$= \lim_{n\to\infty} \frac{1}{n}(\#k : 0 \leq k < n \text{ and } R_\alpha^k(0) \in [\log d, \log(d+1)) \text{ in } S^1)$$

$$= |[\log d, \log(d+1))| = \log(d+1) - \log d = \log\left(\frac{d+1}{d}\right) = \log\left(1 + \frac{1}{d}\right)$$

Thus, $\varphi(1) = \log 2 \approx 0.3$, $\varphi(2) = \log\frac{3}{2} \approx 0.12, \ldots$

Note that $\varphi(d)$ is decreasing.

$\varphi(135) = \log\frac{136}{135} \approx 0.0032$.

Note: these asymptotic frequencies are the same for all $m \neq 10^\ell$.

Next time: translations and linear flows on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, which we can view as the square $\underset{0\quad 1}{\square}$ with opposite sides identified.