Irrational rotation number

Poincaré Classification Theorem (Thm 11.2.7)

Let $f: S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number.

- If f is top. transitive, then f is top. conjugate to the rotation $R_{\tau(f)}$.
- If f is not top. transitive, then $R_{\tau(f)}$ is a factor of f.

Outline of the proof

Let F be a lift of f and let $\tau = \tau(F)$. Let $x \in \mathbb{R}$ and $\bar{x} = \pi(x) \in S^1$.

Let $B = \{F^n(x) + m\}_{m,n\in\mathbb{Z}}$ be the total lift of the orbit of $\bar{x}$.

Define a map $H: B \to \mathbb{R}$ by $H(F^n(x)+m) = n\tau + m$.

One can show that H is strictly increasing (Prop. 11.2.4)

Since $\tau \notin \mathbb{Q}$, the set $\{n\tau \bmod 1\}$ is dense in S^1, and hence the set $H(B) = \{n\tau + m\}_{m,n\in\mathbb{Z}}$ is dense in $\mathbb{R}$ (But B is not necessarily dense in $\mathbb{R}$!)

The map $H: B \to \mathbb{R}$ can be extended to a map $\tilde{H}: \mathbb{R} \to \mathbb{R}$ by setting

$$\tilde{H}(y) = \sup\{H(b) : b \in B \text{ and } b < y\} \underset{\text{as } H(B) \text{ is dense}}{=} \inf\{H(b) : b \in B \text{ and } b > y\}$$

The extension $\tilde{H}$ is continuous and non-decreasing, and $\tilde{H}$ is constant on any interval $I \subset \mathbb{R} \setminus \bar{B}$.

One can check that H, and hence $\tilde{H}$ satisfy

$$\tilde{H}(y+1) = \tilde{H}(y) + 1 \quad \text{and}$$

$$\tilde{H}(F(y)) = \tau + \tilde{H}(y), \quad \text{i.e. } \tilde{H} \circ F = \tilde{R}_\tau \circ \tilde{H}, \quad \text{where } \tilde{R}_\tau(y) = y + \tau.$$

Therefore, $\tilde{H}$ projects to a continuous surjective map $h: S^1 \to S^1$ satisfying $h \circ f = R_{\tau(f)} \circ h$. So h is a conjugacy or a semiconjugacy.

If f is top. transitive, we take x with dense orbit (in fact, then every $x \in S^1$ has dense orbit by Prop. 11.2.5)

Then the set B is dense in $\mathbb{R}$, hence $\tilde{H}$ is injective and so is h.

If f is not top. transitive, then it cannot be top. conjugate to $R_{\tau(f)}$, and so h is a semiconjugacy. □

Note To obtain a transitive f with irrational $\tau(f) = \alpha$, take an orientation-preserving homeomorphism $h: S^1 \to S^1$ and $f = h \circ R_\alpha \circ h^{-1}$.

A non-transitive f with irrational $\tau(f)$ can be obtained as follows:
Take R_τ and choose $x \in S^1$. Then "blow up" the orbit of x. That is,
Take $\{l_n\}_{n\in\mathbb{Z}}$ s.t. $l_n > 0$ and $\sum_{n\in\mathbb{Z}} l_n < 1$. Then insert intervals I_n of length l_n into S^1 so that they are ordered in the same way as the points $x_n = R_\tau^n(x)$. Then define a homeomorphism f s.t. $f(I_n) = I_{n+1}$, $n\in\mathbb{Z}$, and $f|_{S^1 \setminus \cup I_n}$ is semiconjugate to R_τ (endpts of each I_n are mapped to one pt) a semiconjugacy h s.t. $h\circ f = R_\tau \circ h$ is constant on each I_n, i.e. maps it to a point
Every $x \in \bigcup_{n\in\mathbb{Z}} I_n$ is wandering, and so f is not top. transitive.

Circle diffeomorphisms with irrational rotation number.

The total variation of a function $g: S^1 \to \mathbb{R}$ is
$Var(g) = \sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$, where sup is taken over all partitions $0 = x_0 < x_1 < \ldots < x_n = 1$ of $[0,1]$. g has bounded variation if $Var(g) < \infty$.
Every Lipschitz function and hence every C^1 function g on S^1 has bdd variation.

Denjoy Thm (Thm 12.1.2) Let $f: S^1 \to S^1$ be an orientation-preserving C^1 diffeomorphism with $\tau(f) \notin \mathbb{Q}$ and derivative f' of bounded variation. Then f is top. transitive and hence top. conjugate to the rotation $R_{\tau(f)}$.

Note applies to any C^2 diffeomorphism.

Main idea of the proof Suppose f is not top. transitive.
Then $\omega(0)$ is a closed invariant set $\neq S^1$. Take an interval $I \subset S^1 \setminus \omega(0)$
Then $f^n(I)$, $n\in\mathbb{Z}$, are pairwise disjoint (otherwise f has a periodic pt.),
and so $\sum_{n\in\mathbb{Z}} |f^n(I)| \le 1$. However, one can show that there are infinitely many $n \in \mathbb{N}$ s.t. $|f^n(I)| + |f^{-n}(I)| \ge c|I|$, where $c = e^{-\frac{1}{2}Var(\ln f')}$
and so $\sum_{n\in\mathbb{Z}} |f^n(I)| = +\infty$, a contradiction. $\square$

The assumption of the Thm is close to optimal, as demonstrated by
Denjoy Example (Prop. 12.2.1). For every $\tau \in \mathbb{R}\setminus\mathbb{Q}$ and $\alpha \in (0,1)$ there is a non-transitive C^1 diffeomorphism $f: S^1 \to S^1$ with $\tau(f) = \tau$ and α-Hölder continuous derivative.
[means that there is c s.t. $|f'(x) - f'(y)| \le c\,(d(x,y))^\alpha$.
Hölder $\not\Rightarrow$ bdd variation, while Lipschitz $(\alpha=1)$ does]