

RIGIDITY OF STRONG AND WEAK FOLIATIONS

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ABSTRACT. We consider a perturbation f of a hyperbolic toral automorphism L . We study rigidity related to exceptional properties of the strong and weak stable foliations for f . If the strong foliation is mapped to the linear one by the conjugacy h between f and L , we obtain smoothness of h along the weak foliation and regularity of the joint foliation of the strong and unstable foliations. We also establish a similar global result. If the weak foliation is sufficiently regular, we obtain smoothness of the conjugacy along the strong foliation and regularity of the joint foliation of the weak and unstable foliations. If both conditions hold then we get smoothness of h along the stable foliation. We also deduce a rigidity result for the symplectic case. The main theorems are obtained in a unified way using our new result on relation between holonomes and normal forms.

1. INTRODUCTION

In this paper we consider a perturbation f of a hyperbolic toral automorphism L . We study rigidity related to exceptional properties of the strong and weak stable foliations for f . We recall that an automorphism L of \mathbb{T}^d is hyperbolic if the matrix has no eigenvalues on modulus 1. We denote its stable and unstable subspaces by E^s and E^u .

Let f be a C^∞ diffeomorphism of \mathbb{T}^d which is C^1 close to L . Then f is an Anosov diffeomorphism, i.e., the tangent bundle of \mathbb{T}^d splits into a Df -invariant direct sum of the stable and unstable subbundles \mathcal{E}^s and \mathcal{E}^u , where

$$\|Df|_{\mathcal{E}^s}\| < 1 \quad \text{and} \quad \|Df^{-1}|_{\mathcal{E}^u}\| < 1$$

for some Riemannian metric. Also, f is conjugate to L by a homeomorphism h , i.e.,

$$h \circ f = L \circ h.$$

The conjugacy close to the identity is unique, and it is bi-Hölder but usually not C^1 . Any two conjugacies differ by an affine automorphism of \mathbb{T}^d commuting with L , and hence they have the same regularity.

We denote the stable and unstable foliations for L by W^s and W^u , and for f by \mathcal{W}^s and \mathcal{W}^u . The foliations \mathcal{W}^s and \mathcal{W}^u have uniformly C^∞ leaves, but they are not even

Date: September 17, 2025.

Key words: Anosov diffeomorphism, conjugacy, strong and weak foliations, joint integrability, normal forms.

¹ Supported in part by Simons Foundation grant 855238.

² Supported in part by Simons Foundation grant MP-TSM-00002874.

C^1 foliations in general. A foliation is C^r if it has C^r local foliation charts. We say that the leaves are uniformly C^r if locally they can be C^r embedded with the embeddings varying continuously in the C^r topology. For a foliation \mathcal{W} with uniformly C^r leaves, we say that a function on \mathbb{T}^d is uniformly C^r along \mathcal{W} if locally its restrictions to the leaves are C^r and vary continuously in the C^r topology, see Section 4 for more details and noninteger r .

We recall that the Lyapunov space for an exponent χ of L is the sum of generalized eigenspaces of all eigenvalues of L with modulus e^χ . If L has more than one stable Lyapunov exponent, one can take a dominated splitting $E^s = E^{ss} \oplus E^{ws}$ into strong and weak parts by combining one or more Lyapunov spaces into each part. Since f is C^1 close to L , its stable subbundle also has the corresponding dominated splitting C^0 -close to that of L :

$$(1.1) \quad \mathcal{E}^s = \mathcal{E}^{ss} \oplus \mathcal{E}^{ws}.$$

The bundle \mathcal{E}^{ss} is integrable to the strong foliation \mathcal{W}^{ss} , which has uniformly C^∞ leaves and is a C^∞ subfoliation of the leaves of \mathcal{W}^s . The bundle \mathcal{E}^{ws} is also integrable to the weak foliation \mathcal{W}^{ws} , but in general even the leaves of \mathcal{W}^{ws} are only uniformly $C^{1+\text{H\"older}}$. We always have $h(\mathcal{W}^s) = W^s$ and $h(\mathcal{W}^{ws}) = W^{ws}$ by [G08], but usually not $h(\mathcal{W}^{ss}) = W^{ss}$. Our main goal is to explore what rigidity properties follow if assume that $h(\mathcal{W}^{ss}) = W^{ss}$ or that \mathcal{W}^{ws} is sufficiently regular.

Let h be the conjugacy between L and f close to the identity. Then it can be written as $h(x) = x + \Delta(x)$, where $\Delta : \mathbb{T}^d \rightarrow \mathbb{R}^d$. We will consider the components Δ^* of Δ , where $*$ = u, s, ss, ws , with respect to the splittings

$$\mathbb{R}^d = E^u \oplus E^s = E^u \oplus E^{ss} \oplus E^{ws}.$$

We will also write $h^*(x) = x^* + \Delta^*(x)$, which can be defined globally for the lift of h to the universal cover \mathbb{R}^d , or locally on \mathbb{T}^d . Thus the regularity of h^* is well-defined.

First we consider the rigidity of a strong foliation.

Theorem 1.1 (Rigidity of a strong foliation). *Let L be a hyperbolic automorphism of \mathbb{T}^d with dense leaves of W^{ss} . Let f be a C^∞ diffeomorphism sufficiently C^1 close to L , and let h be a topological conjugacy between f and L . Then for the statements below we have*

$$(1) \iff (2) \iff (3) \implies (5) \implies (4).$$

Moreover, if the leaves of each of the Lyapunov subfoliations of W^{ws} are dense in \mathbb{T}^d , then the five statements are equivalent.

- (1) $h(\mathcal{W}^{ss}) = W^{ss}$,
- (2) \mathcal{W}^u and \mathcal{W}^{ss} are jointly integrable to a foliation \mathcal{W}^{u+ss} ,
- (3) (2) holds and the foliation \mathcal{W}^{u+ss} is conjugate to the linear foliation $W^u \oplus W^{ss}$ by a C^∞ diffeomorphism of \mathbb{T}^d ,

- (4) h is a C^1 diffeomorphism along the leaves of \mathcal{W}^{ws} , and the derivative $D(h|_{\mathcal{W}^{ws}(x)}) : \mathcal{E}_x^{ws} \rightarrow \mathbb{R}^d$ is Hölder continuous on \mathbb{T}^d ,
- (5) h^{ws} is C^∞ on \mathbb{T}^d , and if the leaves of \mathcal{W}^{ws} are uniformly C^q , then h is a uniformly C^q diffeomorphism along \mathcal{W}^{ws} .

In Section 2.1 we give basic examples of perturbations f satisfying the assumptions of Theorem 1.1, illustrating the regularity we obtain and necessity of the extra assumption for the implications (4) \implies (5) and (4) \implies (1). Similar examples for subsequent theorems are also included. Our results essentially recover the main features of the basic models.

Remark 1.2 (Irreducibility conditions). *A matrix $L \in SL(d, \mathbb{Z})$ is called irreducible if it has no nontrivial rational invariant subspaces or, equivalently, if its characteristic polynomial is irreducible over \mathbb{Q} . If L is irreducible, or more generally, weakly irreducible, then all Lyapunov foliations for L have dense leaves, and hence the five statements in Theorem 1.1 are equivalent. We define and discuss weak irreducibility in Section 2.2.*

Joint integrability of \mathcal{W}^u and \mathcal{W}^{ss} plays an important role in the study of ergodic properties of foliation \mathcal{W}^{ss} , and it is also related to the Lyapunov exponents on \mathcal{E}^{ws} . It has been extensively studied, primarily on \mathbb{T}^3 , see e.g. [ALOS, ACEPW, DR24, GaSh20]. In higher dimensions, its relation to rigidity was considered by Gogolev and Shi in [GSh23]. They proved that (1) \iff (2) in general, and that (2) \iff (4) under the assumption that L is irreducible and has at most two-dimensional Lyapunov spaces. Our assumptions on L are considerably weaker and we obtain stronger conclusions (3) and (5). Our approach is completely different, it yields higher smoothness directly, and does not rely on [GSh23] aside from (1) \iff (2). Our techniques are mostly global, but require narrow spectrum of $Df|_{\mathcal{E}^{ws}}$ to use normal forms.

Combining our techniques with the spectral rigidity for $Df|_{\mathcal{E}^{ws}}$ obtained in [GSh23] we obtain the following global version of Theorem 1.1. The bunching assumption (1.2) in the theorem means that nonconformality of Df on \mathcal{E}^{ws} is dominated by the expansion on \mathcal{E}^u , and is trivially satisfied if $\dim \mathcal{E}^{ws} = 1$. A splitting $\mathcal{E}^s = \mathcal{E}^{ss} \oplus \mathcal{E}^{ws}$ is called *absolutely dominated* if there exists $0 < \rho < 1$ such that with respect to some continuous family of Riemannian norms on \mathcal{E}^s we have

$$\|Df(u)\| < \rho < \|Df(v)\| \text{ for all unit vectors } u \in \mathcal{E}^{ss} \text{ and } v \in \mathcal{E}^{ws}.$$

This condition automatically holds in the setting of Theorem 1.1.

Theorem 1.3 (Global rigidity of a strong foliation). *Let L be an irreducible hyperbolic automorphism of \mathbb{T}^d and let f be a C^∞ Anosov diffeomorphism of \mathbb{T}^d conjugate to L by a homeomorphism h . Suppose that f has an absolutely dominated splitting $\mathcal{E}^s = \mathcal{E}^{ss} \oplus \mathcal{E}^{ws}$ and satisfies the bunching condition*

$$(1.2) \quad \|Df|_{\mathcal{E}^{ws}(x)}\| \cdot \|(Df|_{\mathcal{E}^{ws}(x)})^{-1}\| \cdot \|(Df|_{\mathcal{E}^u(x)})^{-1}\| < 1 \text{ for all } x \in \mathbb{T}^d.$$

Then the statements (1), (2), (3'), (4), (5) are equivalent, where (1), (2), (4), (5) are as in Theorem 1.1 and (3') is

- (3') \mathcal{W}^u and \mathcal{W}^{ss} are jointly integrable to a C^∞ foliation \mathcal{W}^{u+ss} whose holonomies preserve a Hölder continuous Riemannian metric on \mathcal{E}^{ws} .

Our approach also allows us to obtain rigidity related to regularity of the weak foliation. We recall that $h(\mathcal{W}^{ws}) = W^{ws}$, and hence \mathcal{W}^u and \mathcal{W}^{ws} are jointly integrable to the foliation $\mathcal{W}^{u+ws} = h^{-1}(W^u \oplus W^{ws})$ with uniformly $C^{1+\text{Hölder}}$ leaves.

Let $r_{ss}(L)$ be the ratio of the top and bottom Lyapunov exponents of L on E^{ss} , i.e.,

$$(1.3) \quad r_{ss}(L) = (\log \rho_{\min}) / (\log \rho_{\max}) \geq 1,$$

where $0 < \rho_{\min} \leq \rho_{\max} < 1$ are the smallest and the largest moduli of the eigenvalues of L on E^{ss} .

Theorem 1.4 (Rigidity of a weak foliation). *Let L be a hyperbolic automorphism of \mathbb{T}^d with dense leaves of W^{ws} . Let f be a C^∞ diffeomorphism sufficiently C^1 close to L , and let h be a topological conjugacy between f and L . If $r > r_{ss}(L)$ and $r \notin \mathbb{N}$, then the following are equivalent.*

- (1) \mathcal{W}^{ws} is a uniformly C^r subfoliation of \mathcal{W}^s ,
- (2) h^{ss} is a uniformly C^∞ diffeomorphism along \mathcal{W}^{ss} , and h^{ss} is C^r on \mathbb{T}^d ,
- (3) The joint foliation \mathcal{W}^{u+ws} is conjugate to the linear foliation $W^u \oplus W^{ws}$ by a C^r diffeomorphism.

If in addition $h(\mathcal{W}^{ss}) = W^{ss}$, then $(1, 2, 3) \implies h$ is uniformly C^∞ along \mathcal{W}^{ss} .

This theorem follows from a more general technical result, Theorem 7.1, where we assume only regularity of the holonomies of \mathcal{W}^{ws} between the leaves of \mathcal{W}^{ss} , rather than regularity of the subfoliation. The distinction is not assuming higher regularity of the leaves of \mathcal{W}^{ws} , which in general are only $C^{1+\text{Hölder}}$.

Combining rigidity of the weak and strong subfoliations, Theorems 1.1 and 1.4, we obtain the following characterizations of smoothness of the conjugacy along \mathcal{W}^s .

Theorem 1.5 (Rigidity of strong and weak foliations). *Let L be a hyperbolic automorphism of \mathbb{T}^d . Suppose that W^{ws} and W^{ss} have dense leaves. Let f be a C^∞ diffeomorphism sufficiently C^1 close to L , and let h be a topological conjugacy between f and L . Then the following are equivalent.*

- (1) $h(\mathcal{W}^{ss}) = W^{ss}$ and \mathcal{W}^{ws} is a uniformly C^∞ subfoliation of \mathcal{W}^s ,
- (2) h is a uniformly C^∞ diffeomorphism along \mathcal{W}^s ,
- (2') h^s is in $C^\infty(\mathbb{T}^d)$ and a diffeomorphism along \mathcal{W}^s ,
- (3) \mathcal{W}^u is conjugate to the linear foliation W^u by a C^∞ diffeomorphism.

Symplectic case. Now we apply the above results to obtain rigidity results for symplectic L and f .

Let L be a symplectic hyperbolic automorphism of \mathbb{T}^d with dominated splittings

$$(1.4) \quad E^s = E^{ss} \oplus E^{ws} \quad \text{and} \quad E^u = E^{uu} \oplus E^{wu}$$

such that $\dim E^{ss} = \dim E^{uu}$ and hence $\dim E^{ws} = \dim E^{wu}$. Here E^{uu} and E^{wu} are strong and weak unstable subbundles. Let f be a C^1 small perturbation of L . Then f has corresponding dominated splittings

$$\mathcal{E}^s = \mathcal{E}^{ss} \oplus \mathcal{E}^{ws} \quad \text{and} \quad \mathcal{E}^u = \mathcal{E}^{uu} \oplus \mathcal{E}^{wu}.$$

The subbundle \mathcal{E}^{ws+wu} is integrable with $C^{1+\text{H\"older}}$ leaves, and $h(\mathcal{W}^{ws+wu}) = W^{ws+wu}$.

Theorem 1.6 (Symplectic rigidity). *Let L be a symplectic hyperbolic automorphism of \mathbb{T}^d . Suppose that the foliations W^{ws} , W^{ss} , W^{wu} , and W^{uu} have dense leaves. Let f be a C^∞ diffeomorphism sufficiently C^1 close to L and preserving a C^∞ symplectic form. Let h be a topological conjugacy between f and L . Then for the statements below we have*

$$(1) \iff (2) \iff (3) \implies (4).$$

Moreover, if the leaves of each of the Lyapunov subfoliations of W^{ws+wu} are dense in \mathbb{T}^d , then the four statements are equivalent.

- (1) h is a C^∞ diffeomorphism,
- (2) $h(\mathcal{W}^{ss}) = W^{ss}$ and $h(\mathcal{W}^{uu}) = W^{uu}$,
- (3) h is α -Hölder with α sufficiently close to 1,
- (4) h is a C^1 diffeomorphism along the leaves of \mathcal{W}^{ws+wu} , and the derivative $D(h|_{\mathcal{W}^{ws+wu}(x)}) : \mathcal{E}_x^{ws+wu} \rightarrow \mathbb{R}^d$ is Hölder continuous on \mathbb{T}^d .

For an irreducible L with one-dimensional weak foliations, Gogolev and Shi showed in the proof of [GSh23, Theorem 6.1] that joint integrability of \mathcal{W}^{uu} and \mathcal{W}^{ss} implies (2). Hence we obtain the following corollary, which extends [GSh23, Theorem 1.4] from $d = 4$ to any $d \geq 4$.

Corollary 1.7. *Let L be an irreducible symplectic hyperbolic automorphism of \mathbb{T}^d with splitting (1.4) such that $\dim E^{ws} = \dim E^{wu} = 1$. Let f be a C^∞ diffeomorphism sufficiently C^1 close to L and preserving a C^∞ symplectic form. If the strong foliations \mathcal{W}^{uu} and \mathcal{W}^{ss} are jointly integrable then f is C^∞ conjugate to L .*

Remark 1.8 (Finite regularity). *Our results hold if f is a C^t diffeomorphism rather than C^∞ , provided that t is sufficiently large, and with the C^∞ regularity of other objects replaced by $C^{t-\delta}$ for any $\delta > 0$. Theorems 1.1 and 1.3 require $t > r_{ws}$ defined similarly to (1.3). In particular, if L has one Lyapunov exponent on E^{ws} then any $t > 1$ suffices. Theorem 1.4 requires $t > r$, Theorem 1.5 requires $t > \max\{r_{ws}, r_{ss}\}$, Theorem 1.6 requires $t > \max\{r_{wu}, r_{uu}, r_{ws}, r_{ss}\}$, and Corollary 1.7 requires $t > \max\{r_{uu}, r_{ss}\}$.*

These are the regularities needed to apply the results on normal forms. The proofs work without any significant modifications.

Normal forms and holonomies. The proofs of the theorems above rely on our new results of on normal forms and holonomies, which are of independent interest. We give preliminaries on normal forms in Section 3 and formulate and prove the new results in Section 4. In the context of two invariant transverse subfoliations \mathcal{W} and \mathcal{V} of \mathcal{W}^s we prove in Theorem 4.4 that if $Df|_{T\mathcal{W}}$ has narrow spectrum and the holonomies of \mathcal{V} between \mathcal{W} are sufficiently smooth then they preserve normal forms, i.e., they are sub-resonance polynomials when written in normal form coordinates. This allows us to use holonomies along \mathcal{V} together with density of its leaves to obtain regularity of the conjugacy along \mathcal{W} . Prior results using normal forms with holonomies were in the context of *neutral* foliations [FKSp11, GKS23] and they do not apply to expanding or contracting foliations. We use our results with both $\mathcal{W} = \mathcal{W}^{ss}$ and $\mathcal{W} = \mathcal{W}^{ws}$. In the latter case, even existence of normal forms is nontrivial since the leaves are of \mathcal{W}^{ws} have low regularity. We overcome this problem in Theorem 4.3 by using smooth holonomies of $\mathcal{V} = \mathcal{W}^{ss}$.

Structure of the paper. In Section 2 we give examples to illustrate the main theorems, and discuss irreducibility. In Section 3 we give preliminaries on normal forms, and in Section 4 formulate and prove the new results. We prove Theorem 1.1 in Section 5, Theorem 1.3 in Section 6, Theorem 1.4 in Section 7, Theorem 1.5 in Section 8, and Theorem 1.6 in Section 9.

2. EXAMPLES AND WEAK IRREDUCIBILITY

2.1. Examples.

(i) We illustrate Theorem 1.1 in a basic setting of a hyperbolic automorphism of \mathbb{T}^3 , which is always irreducible. Let L be a hyperbolic automorphism of \mathbb{T}^3 with eigenvalues

$$0 < \lambda_{ss} < \lambda_{ws} < 1 < \lambda_u,$$

and let e_{ss} , e_{ws} , and e_u be corresponding unit eigenvectors. We consider a C^1 small perturbation f of L of the form

$$f(x) = L(x) + \varphi_u(x)e_u + \varphi_{ss}(x)e_{ss},$$

where φ_u and φ_{ss} are smooth real-valued functions on \mathbb{T}^3 .

Then a conjugacy between f and L can be found in the form

$$h(x) = x + \phi_u(x)e_u + \phi_{ss}(x)e_{ss}.$$

One can take smooth functions φ_u and φ_{ss} for which the corresponding functions ϕ_u and ϕ_{ss} are not smooth, for example trigonometric polynomials as in [L92, Theorem 6.3] and [KtN, Theorem 5.5.5 and Remark 5.5.6].

For such a perturbation, the linear foliation W^{u+ss} is preserved by f and h , and hence $\mathcal{W}^{u+ss} = W^{u+ss} = h(\mathcal{W}^{u+ss})$. While $\mathcal{W}^{ss} \neq W^{ss}$, we have $h(\mathcal{W}^{ss}) = W^{ss}$

since $\mathcal{W}^{ss} = \mathcal{W}^{u+ss} \cap \mathcal{W}^s$ and $h(\mathcal{W}^s) = \mathcal{W}^s$. We see that while h is not smooth, $h(x)^{ws} = x^{ws}$ and hence is smooth on \mathbb{T}^d . One can also see by differentiating term-wise that $\phi_{ss}(x) = -\sum_{k=1}^{\infty} \lambda_{ss}^{k-1} \varphi(f^{-k}x)$ has one derivative in the direction of e_{ws} . This shows that h is C^1 along \mathcal{W}^{ws} , which corresponds to (4) in Theorem 1.1. However, the series for higher derivatives may diverge, and the bundle \mathcal{E}^{ws} is only Hölder in general.

(ii) One can modify this example to show that the implications (4) \implies (5) and (4) \implies (1) in Theorem 1.1 can fail without density of the Lyapunov leaves. We take an automorphism

$$L = L_1 \times L_2 \quad \text{of } \mathbb{T}^5 = \mathbb{T}^3 \times \mathbb{T}^2,$$

where L_1 is as in (i), and L_2 is an automorphism of \mathbb{T}^2 with eigenvalues $0 < \mu_s < 1 < \mu_u = \mu_s^{-1}$ and unit eigenvectors v_s and v_u . If $\mu_s < \lambda_{ws}$, we can take W^{ss} for L as the span of e_{ss} and v_s and W^{ws} as the span of e_{ws} . We consider the perturbation

$$f(x_1, x_2) = (L_1(x_1) + \varphi(x_2)e_{ws}, L_2(x_2)).$$

Then the conjugacy can be written as $h(x_1, x_2) = (x_1 + \phi(x_2)e_{ws}, x_2)$. Thus for a fixed x_2 the conjugacy is a translation on \mathbb{T}^3 in W^{ws} direction, and hence (4) is satisfied. If φ is taken so that ϕ is not C^1 on \mathbb{T}^2 then component h^{ws} is not C^1 on $\mathbb{T}^3 \times \mathbb{T}^2$. Thus (5) fails and hence (1) has to fail too since (1) \implies (5). It can also be directly computed as in [KtN, Remark 5.5.6] that $h(\mathcal{W}^{ss}) \neq W^{ss}$.

(iii) In the setting of (i), we can similarly obtain perturbations satisfying the assumptions of the theorems below. Specifically,

$$f(x) = L(x) + \varphi_u(x)e_u + \varphi_{ws}(x)e_{ss} \quad \text{for Theorem 7.1,}$$

$$f(x) = L(x) + \varphi_{ws}(x)e_{ss} \quad \text{for Theorem 1.4, and}$$

$$f(x) = L(x) + \varphi_u(x)e_u \quad \text{for Theorem 1.5.}$$

These examples illustrate the conclusions we get in these theorems.

2.2. Irreducibility and weak irreducibility.

We recall that $L \in SL(d, \mathbb{Z})$ is *irreducible* if it has no nontrivial rational invariant subspaces or, equivalently, if its characteristic polynomial is irreducible over \mathbb{Q} . The eigenvalues of an irreducible L are simple. Irreducibility of L implies that any L -invariant linear foliation of \mathbb{T}^d is dense in \mathbb{T}^d .

A weaker assumption on L , called *weak irreducibility*, gives denseness of every Lyapunov foliation for L . It was introduced and discussed in [KSW23], see Section 3.3 there. Let ρ_1, \dots, ρ_m be distinct moduli of eigenvalues of L and let $E^{\rho_1}, \dots, E^{\rho_m}$ be the corresponding Lyapunov subspaces. We say that L is *weakly irreducible* if for each i the space $\hat{E}^{\rho_i} = \bigoplus_{j \neq i} E^{\rho_j}$ contains non non-zero elements of \mathbb{Z}^d . Equivalently, there is a set $S \subset \mathbb{R}$ so that for each irreducible over \mathbb{Q} factor of the characteristic polynomial of L the set of moduli of its roots equals S . A weakly irreducible L is not necessarily diagonalizable.

Lemma 2.1 (Weak irreducibility). *A matrix $L \in SL(d, \mathbb{Z})$ is weakly irreducible if and only if for each Lyapunov foliation of L the leaves are dense in \mathbb{T}^d .*

Proof. We will prove the equivalence to the second condition above. Let p_L be the characteristic polynomial of L and $p_L = \prod_{k=1}^K p_k^{d_k}$ be its prime decomposition over \mathbb{Q} .

Let $S = \{\rho_1, \dots, \rho_m\}$ be the set of moduli of roots for each p_k . Suppose that for some i the leaves of the Lyapunov foliation of W^{ρ_i} are not dense in \mathbb{T}^d . Let E be the minimal rational L -invariant subspace containing E^{ρ_i} . Then $E \neq \mathbb{R}^d$. We consider the restriction $B = L|_E$ and the induced operator C on the quotient \mathbb{R}^d/E . Then we have $p_L = p_B \cdot p_C$, and all three are rational polynomials. Hence $p_C = \prod_{k=1}^K p_k^{c_k}$, and so by the assumption p_C has at least one root of modulus ρ_i . Then p_L has more roots of modulus ρ_i , counted with multiplicities, than the corresponding number for p_B . This is impossible since both should be $\dim E^{\rho_i}$, as $E^{\rho_i} \subset E$.

To prove the converse, assume that for each Lyapunov foliation of L the leaves are dense in \mathbb{T}^d . Let $S = \{\rho_1, \dots, \rho_m\}$ be the set of moduli of roots of p_L , and suppose that for some $\rho_i \in S$ and $k_0 \in \{1, \dots, K\}$ no root of p_{k_0} has modulus ρ_i . Let $\mathbb{R}^d = \oplus V_k$ be the splitting into rational L -invariant subspaces $V_k = \ker p_k^{d_k}(L)$. As the eigenvalues of $L|_{V_k}$ are the roots of p_k , we have $E^{\rho_i} \cap V_{k_0} = 0$. This implies that $E^{\rho_i} \subset \oplus_{k \neq k_0} V_k$. Indeed $E^{\rho_i} = \oplus_k (V_k \cap E^{\rho_i})$ since the splittings $\oplus_j E^{\rho_j}$ and $\oplus_k V_k$ have a common refinement. Since $\oplus_{k \neq k_0} V_k$ is a rational L -invariant subspace smaller than \mathbb{R}^d , we conclude that the leaves of corresponding Lyapunov foliation W^{ρ_i} are not dense in \mathbb{T}^d . \square

3. PRELIMINARIES ON NORMAL FORMS

3.1. Smooth extensions and sub-resonance polynomials. Let \mathcal{E} be a continuous vector bundle over a compact metric space \mathcal{M} , let \mathcal{N} be a neighborhood of the zero section in \mathcal{E} , and let f be a homeomorphism of \mathcal{M} . We consider an extension $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{E}$ that projects to f and preserves the zero section. We assume that the corresponding fiber maps $\mathcal{F}_x : \mathcal{N}_x \rightarrow \mathcal{E}_{fx}$ are C^r diffeomorphisms. We will consider $r > 1$, and for $r \notin \mathbb{N}$ we will understand C^r in the usual sense that the derivative of order $N = \lfloor r \rfloor$ is Hölder with exponent $\alpha = r - \lfloor r \rfloor$.

We assume that the fibers \mathcal{E}_x are equipped with a continuous family of Riemannian norms. We denote by $B_{x,\sigma}$ the closed ball of radius $\sigma > 0$ centered at $0 \in \mathcal{E}_x$. For $N \in \mathbb{N}$ and $0 \leq \alpha < 1$ we denote by $C^{N+\alpha}(B_{x,\sigma}) = C^{N+\alpha}(B_{x,\sigma}, \mathcal{E}_{fx})$ the space of functions $R : B_{x,\sigma} \rightarrow \mathcal{E}_{fx}$ with continuous derivatives up to order N on $B_{x,\sigma}$ and, if $\alpha > 0$, with α -Hölder N^{th} derivative at 0.

Definition 3.1. *We say that \mathcal{F} is a $C^{N+\alpha}$ extension of f , where $N \in \mathbb{N}$ and $0 \leq \alpha < 1$, if for some $\sigma > 0$ the fiber maps $\mathcal{F}_x : B_{x,\sigma} \rightarrow \mathcal{E}_{fx}$ are $C^{N+\alpha}$ diffeomorphisms which depend continuously on x in C^N topology with uniformly bounded norms $\|\mathcal{F}_x\|_{C^{N+\alpha}(B_{x,\sigma})}$.*

We say that $\varphi = \{\varphi_x\}_{x \in X}$, where $\varphi_x : B_{x,\sigma} \rightarrow \mathcal{E}_{fx}$, is a $C^{N+\alpha}$ coordinate change if it is a $C^{N+\alpha}$ extension of the identity map on \mathcal{M} preserving the zero section.

For a smooth extension \mathcal{F} we will denote by F its derivative of at the zero section, i.e. $F : \mathcal{E} \rightarrow \mathcal{E}$ is a continuous linear extension of f whose fiber maps are linear isomorphisms $F_x = D_0\mathcal{F}_x : \mathcal{E}_x \rightarrow \mathcal{E}_{fx}$.

Definition 3.2. Let $\varepsilon > 0$ and let

$$(3.1) \quad \chi = (\chi_1, \dots, \chi_\ell), \quad \text{where } \chi_1 < \dots < \chi_\ell < 0.$$

We say that a linear extension F has (χ, ε) -spectrum if there is a continuous F -invariant splitting

$$(3.2) \quad \mathcal{E} = \mathcal{E}^1 \oplus \dots \oplus \mathcal{E}^\ell$$

and a continuous family of Riemannian norms $\|\cdot\|_x$ on \mathcal{E}_x such that

$$(3.3) \quad e^{\chi_i - \varepsilon} \|t\|_x \leq \|F_x(t)\|_{fx} \leq e^{\chi_i + \varepsilon} \|t\|_x \quad \text{for every } t \in \mathcal{E}_x^i$$

and the splitting (3.2) is orthogonal.

We say that a map between vector spaces is *polynomial* if each component is given by a polynomial in some, and hence every, basis. We will consider a polynomial map $P : \mathcal{E}_x \rightarrow \mathcal{E}_y$ with $P(0_x) = 0_y$ and split it into components $(P_1(t), \dots, P_\ell(t))$, where $P_i : \mathcal{E}_x \rightarrow \mathcal{E}_y^i$. Each P_i can be written uniquely as a linear combination of polynomials of specific homogeneous types. A map $Q : \mathcal{E}_x \rightarrow \mathcal{E}_y^i$ has *homogeneous type* $s = (s_1, \dots, s_\ell)$, where s_1, \dots, s_ℓ are non-negative integers, if for any real numbers a_1, \dots, a_ℓ and vectors $t_j \in \mathcal{E}_x^j$, $j = 1, \dots, \ell$, we have

$$(3.4) \quad Q(a_1 t_1 + \dots + a_\ell t_\ell) = a_1^{s_1} \dots a_\ell^{s_\ell} \cdot Q(t_1 + \dots + t_\ell).$$

Definition 3.3. We say that a homogeneous type $s = (s_1, \dots, s_\ell)$ for $Q : \mathcal{E}_x \rightarrow \mathcal{E}_y^i$ is

$$(3.5) \quad \text{sub-resonance if } \chi_i \leq \sum_{j=1}^{\ell} s_j \chi_j.$$

We say that a polynomial map $P : \mathcal{E}_x \rightarrow \mathcal{E}_y$ is sub-resonance if each component P_i has only terms of sub-resonance homogeneous types. We denote by $\mathcal{S}_{x,y}$ the set of all sub-resonance polynomials $P : \mathcal{E}_x \rightarrow \mathcal{E}_y$ with $P(0) = 0$ and invertible derivative at 0.

Clearly, for any sub-resonance relation we have $s_j = 0$ for $j < i$ and $\sum s_j \leq \chi_1 / \chi_\ell$. Hence sub-resonance polynomials have degree at most $d(\chi) = \lfloor \chi_1 / \chi_\ell \rfloor$.

We will denote $\mathcal{S}_{x,x}$ by \mathcal{S}_x , which is a finite-dimensional Lie group with respect to the composition if $\varepsilon > 0$ is sufficiently small [GuKt98]. Any map $P \in \mathcal{S}_{x,y}$ induces an isomorphism between \mathcal{S}_x and \mathcal{S}_y by conjugation. In particular, this holds for any invertible linear map which respects the splitting (3.2).

3.2. Normal forms for contracting extensions. The following theorem was established in [GuKt98, Gu02] for $r \in \mathbb{N} \cup \{\infty\}$, and in [K24] for this setting.

For a given χ as in (3.1), there is $\varepsilon_0 = \varepsilon_0(\chi) > 0$ given by [K24, (3.13)] which ensures that the spectrum is sufficiently narrow and will suffice for all results and proofs below.

Theorem 3.4. [K24, Theorem 4.3] (Normal forms for contracting extensions)

Let $f : X \rightarrow X$ be a homeomorphism of a compact metric space \mathcal{M} , let \mathcal{E} be a continuous vector bundle over \mathcal{M} , let \mathcal{N} be a neighborhood of the zero section in \mathcal{E} , and let $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{E}$ be a C^r extension of f that preserves the zero section. Suppose that the derivative F of \mathcal{F} at the zero section has (χ, ε) -spectrum with $\chi_1/\chi_\ell < r$ and $\varepsilon < \varepsilon_0$.

Then there exists a C^r coordinate change $\varphi = \{\varphi_x\}_{x \in X}$ with diffeomorphisms $\varphi_x : B_{x,\sigma} \rightarrow \mathcal{E}_x$ satisfying $\varphi_x(0) = 0$ and $D_0\varphi_x = \text{Id}$ which conjugates \mathcal{F} to a continuous polynomial extension $\mathcal{P} : \mathcal{E} \rightarrow \mathcal{E}$ of f of sub-resonance type, i.e.,

$$(3.6) \quad \varphi_{fx} \circ \mathcal{F}_x = \mathcal{P}_x \circ \varphi_x, \quad \text{where } \mathcal{P}_x \in \mathcal{S}_{x,fx} \text{ for all } x \in X.$$

If \mathcal{F} is a C^∞ extension then the coordinate change φ is also C^∞ .

Any two such coordinate changes φ_x and φ'_x satisfy $\varphi'_x = \varphi_x \circ g_x$ for some $g_x \in \mathcal{S}_x$.

4. NORMAL FORMS FOR CONTRACTING FOLIATIONS

Let f be a C^r diffeomorphism of a compact manifold \mathcal{M} , where $r = N + \alpha$. We will consider an f -invariant continuous foliation \mathcal{W} of X with *uniformly C^r leaves*. By this we mean that for some $R > 0$ the balls $B^\mathcal{W}(x, R)$ of radius R in the intrinsic Riemannian metric of the leaf can be given by C^r embeddings which depend continuously on x in C^N topology and, if $r \notin \mathbb{N}$, have α -Hölder derivative of order N with uniformly bounded Hölder constant. Similarly, for such a foliation we will say that a function g is *uniformly C^r along \mathcal{W}* if its restrictions to $B^\mathcal{W}(x, R)$ depend continuously on x in C^N topology and have α -Hölder derivative of order N with uniformly bounded Hölder constant. We also allow $r = \infty$, in which case uniformly C^∞ means uniformly C^N for each N .

Definition 4.1 (Normal forms on a contracting foliation).

Let f be a C^1 diffeomorphism of a compact manifold \mathcal{M} , and let \mathcal{E} be a continuous f -invariant subbundle of $T\mathcal{M}$ tangent to an f -invariant topological foliation \mathcal{W} with uniformly C^1 leaves. Suppose that the linear extension $F = Df|_{T\mathcal{W}}$ has (χ, ε) -spectrum.

We say that a family $\{\varphi_x\}_{x \in \mathcal{M}}$ of C^1 diffeomorphisms $\varphi_x : \mathcal{W}_x \rightarrow T_x\mathcal{W}$, which depend continuously on x in C^1 topology, is a normal form for f on \mathcal{W} if for each $x \in \mathcal{M}$ we have $\varphi_x(0) = 0$ and $D_0\varphi_x = \text{Id}$,

$$\mathcal{P}_x = \varphi_{fx} \circ f \circ \varphi_x^{-1} : T_x\mathcal{W} \rightarrow T_{f(x)}\mathcal{W} \text{ is in } \mathcal{S}_{x,fx},$$

and for any $y \in \mathcal{W}_x$ the map $\varphi_y \circ \varphi_x^{-1} : T_x\mathcal{W} \rightarrow T_y\mathcal{W}$ is a composition of a sub-resonance polynomial in $\mathcal{S}_{x,y}$ with a translation.

Identifying $\mathcal{W}(x)$ with $T_x\mathcal{W}$ by φ_x we can view the transition maps $\varphi_y \circ \varphi_x^{-1}$ as maps of $T_x\mathcal{W}$ and see that they are in the finite-dimensional group $\bar{\mathcal{S}}_x$ generated by \mathcal{S}_x and the translations of $T_x\mathcal{W}$.

In Theorems 1.4 and 7.1 we will use normal forms for f on $\mathcal{W} = \mathcal{W}^{ss}$. In this case the leaves of \mathcal{W} are uniformly C^∞ , so we can obtain a natural uniformly C^∞ extension of f by locally identifying $T_x\mathcal{W}$ with $\mathcal{W}(x)$ and apply Theorem 3.4. The following theorem yields a normal form for f on $\mathcal{W} = \mathcal{W}^{ss}$, which is uniformly C^∞ . The proof that $\varphi_y \circ \varphi_x^{-1} \in \bar{\mathcal{S}}_x$ is given in [KS16, KS17].

Theorem 4.2. [K24, Theorem 4.6] (Normal forms for foliations with C^r leaves)

Let f be a C^r , $r \in (1, \infty]$, diffeomorphism of a compact manifold \mathcal{M} , and let \mathcal{W} be an f -invariant topological foliation of \mathcal{M} with uniformly C^r leaves. Suppose that the linear extension $F = Df|_{T\mathcal{W}}$ has (χ, ε) -spectrum with $\chi_1/\chi_\ell < r$ and $\varepsilon < \varepsilon_0$. Then there exists a normal form for f on \mathcal{W} such that $\varphi_x : \mathcal{W}_x \rightarrow T_x\mathcal{W}$ are uniformly C^r diffeomorphisms.

4.1. Normal forms on C^1 leaves. In Theorem 1.1 we will use normal forms for f on $\mathcal{W} = \mathcal{W}^{ws}$. In general, the leaves of $\mathcal{W} = \mathcal{W}^{ws}$ are only $C^{1+\text{H\"older}}$ and so the above result may not apply. In this case we construct the normal form using the next theorem and smoothness of \mathcal{W}^{ss} inside \mathcal{W}^s . The latter yields that holonomies of \mathcal{W}^{ss} are C^∞ , between C^∞ transversals to \mathcal{W}^{ss} inside \mathcal{W}^s .

Theorem 4.3. (Normal forms for foliations with C^1 leaves)

Let f be a C^r diffeomorphism of a compact manifold \mathcal{M} . Let U be an f -invariant topological foliation of \mathcal{M} with uniformly C^r leaves. Let \mathcal{W} and \mathcal{V} be f -invariant topological subfoliations of U with uniformly C^1 leaves transverse in the leaves of U , i.e., $T_xU = T_x\mathcal{W} \oplus T_x\mathcal{V}$ for each $x \in \mathcal{M}$.

Suppose that the holonomies \mathcal{H} of \mathcal{V} inside U are uniformly C^r , and $Df|_{T\mathcal{W}}$ has (χ, ε) spectrum $\chi_1/\chi_\ell < r$ and $\varepsilon < \varepsilon_0$. Then there is a normal form $\{\varphi_x\}$ for f on \mathcal{W} such that for any $x \in \mathcal{M}$ and any $y \in \mathcal{V}(x)$ the lifted holonomy maps

$$(4.1) \quad \bar{H}_{x,y} = \varphi_y \circ \mathcal{H}_{x,y} \circ \varphi_x^{-1} : T_x\mathcal{W} \rightarrow T_y\mathcal{W}$$

are C^r diffeomorphisms, uniformly in x and y .

Proof. To construct the normal form we need to define a C^r extension \mathcal{F} that corresponds to the restriction of f to the leaves of \mathcal{W} . Since the leaves of \mathcal{W} are only C^1 , we use the action of f transversally to \mathcal{V} inside U to define \mathcal{F} . More precisely, for any $x \in \mathcal{M}$ we identify in a uniformly C^r way a small ball $B_\rho(x)$ in $T_x\mathcal{W}$ with a C^r submanifold of $U(x)$ tangent to $T_x\mathcal{W}$ at x . Thus we obtain a uniformly C^r family \mathcal{T}_x of transversals to \mathcal{V} inside U :

$$(4.2) \quad i_x : T_x\mathcal{W} \supset B_\rho(x) \rightarrow U(x), \quad i_x(B_\rho) = \mathcal{T}_x, \quad D_x i_x = \text{Id}.$$

Using these transversals we consider the following holonomy maps of \mathcal{V} :

$$(4.3) \quad \mathcal{H}_x : \mathcal{T}_x \rightarrow \mathcal{W}(x) \text{ and } \tilde{\mathcal{H}}_{fx} : f(\mathcal{T}_x) \rightarrow \mathcal{T}_{fx}.$$

We note that $\tilde{\mathcal{H}}_{fx}$ is uniformly C^r by the assumption on holonomies of \mathcal{V} since $f(\mathcal{T}_x)$ and \mathcal{T}_{fx} are uniformly C^r transversals, but \mathcal{H}_x is only uniformly C^1 as $\mathcal{W}(x)$ is assumed only C^1 . Using these maps and the identifications i_x we define the C^r extension of f

$$\mathcal{F}_x = i_{fx}^{-1} \circ \tilde{\mathcal{H}}_{fx} \circ f|_{\mathcal{T}_x} \circ i_x = (\mathcal{H}_{fx} \circ i_{fx})^{-1} \circ f|_{\mathcal{W}(x)} \circ (\mathcal{H}_x \circ i_x) : T_x \mathcal{W} \rightarrow T_{fx} \mathcal{W}.$$

In fact, the maps $\mathcal{H}_x \circ i_x$ give an atlas of local coordinates on the leaves of \mathcal{W} with C^r transition maps. This gives $\mathcal{W}(x)$ a structure of C^r manifold, which is C^1 consistent with the submanifold structure. We will denote by $\tilde{\mathcal{W}}(x)$ the leaf equipped with this C^r structure. With respect to it, the restriction of f to each leaf, $f_x : \tilde{\mathcal{W}}(x) \rightarrow \tilde{\mathcal{W}}(fx)$, becomes uniformly C^r , as in these local coordinates it coincides with the extension \mathcal{F}_x .

Now this setting becomes essentially the same as in the case of C^r leaves. Theorem 3.4 gives normal form coordinates $\hat{\varphi}_x : T_x \mathcal{W} \rightarrow T_x \mathcal{W}$ for the extension \mathcal{F} and hence

$$(4.4) \quad \varphi_x = \hat{\varphi}_x \circ (\mathcal{H}_x \circ i_x)^{-1} : \mathcal{W}(x) \rightarrow T_x \mathcal{W}$$

is a normal form for f on \mathcal{W} , moreover $\varphi_x \in C^r(\tilde{\mathcal{W}}(x), T_x \mathcal{W})$. The proof that $\varphi_y \circ \varphi_x^{-1} \in \bar{\mathcal{S}}_x$ is the same as in [KS16, KS17].

For any $y \in \mathcal{V}(x)$, the holonomies of \mathcal{V} between the transversals $\tilde{\mathcal{H}}_{x,y} : \mathcal{T}_x \rightarrow \mathcal{T}_y$ are uniformly C^r and are related to the holonomies between the leaves $\mathcal{H}_{x,y} : \mathcal{W}(x) \rightarrow \mathcal{W}(y)$ by $\tilde{\mathcal{H}}_{x,y} = \mathcal{H}_y^{-1} \circ \mathcal{H}_{x,y} \circ \mathcal{H}_x$. By the construction of φ this yields that the lifted holonomies $\bar{H}_{x,y}$ are also uniformly C^r , since

$$(4.5) \quad \bar{H}_{x,y} = \varphi_y \circ \mathcal{H}_{x,y} \circ \varphi_x^{-1} = \hat{\varphi}_y \circ i_y^{-1} \circ \tilde{\mathcal{H}}_{x,y} \circ i_x \circ \varphi_x^{-1} : T_x \mathcal{W} \rightarrow T_y \mathcal{W}.$$

Even though the holonomies $\mathcal{H}_{x,y} : \mathcal{W}(x) \rightarrow \mathcal{W}(y)$ are only C^1 , the holonomies $\mathcal{H}_{x,y} : \tilde{\mathcal{W}}(x) \rightarrow \tilde{\mathcal{W}}(y)$ are C^r . \square

In the setting of the previous theorem, we formulate our main technical result on relationship between holonomies and normal forms. We will apply this theorem with $U = \mathcal{W}^s$ to both $(\mathcal{V}, \mathcal{W}) = (\mathcal{W}^{ss}, \mathcal{W}^{ws})$ and $(\mathcal{V}, \mathcal{W}) = (\mathcal{W}^{ws}, \mathcal{W}^{ss})$.

Theorem 4.4 (Holonomy invariance of normal forms). *In addition to the assumptions of Theorem 4.3, suppose that the foliation U is contracted by f , i.e., $\|Df|_{TV}\| < 1$ for some metric. Then for any $x \in \mathcal{M}$ and $y \in \mathcal{V}(x)$ the lifted holonomy map (4.1) is a sub-resonance polynomial, i.e., $\bar{H}_{x,y} \in \mathcal{S}_{x,y}$.*

4.2. Proof of Theorem 4.4.

We denote $T\mathcal{W}$ by \mathcal{E} and let $\mathcal{E} = \mathcal{E}^1 \oplus \dots \oplus \mathcal{E}^\ell$ be the invariant splitting for $Df|_{\mathcal{E}}$.

First we show that the derivative $D\mathcal{H}_{x,y} = D\bar{H}_{x,y} : \mathcal{E}_x \rightarrow \mathcal{E}_y$ preserves the flag of strong subbundles, i.e., for each $k = 1, \dots, \ell$,

$$D\mathcal{H}_{x,y}(\mathcal{E}_x^{1,k}) \subseteq \mathcal{E}_y^{1,k}, \quad \text{where } \mathcal{E}_y^{1,k} = \mathcal{E}^1 \oplus \dots \oplus \mathcal{E}^k.$$

Since the foliations \mathcal{W} and \mathcal{V} are f -invariant, we have the commutation relation

$$(4.6) \quad f^n \circ \mathcal{H}_{x,y} = \mathcal{H}_{f^n x, f^n y} \circ f^n : \mathcal{W}_x \rightarrow \mathcal{W}_{f^n y}.$$

Differentiating this relation we obtain

$$Df^n|_{\mathcal{E}_y} \circ D\mathcal{H}_{x,y}(x) = D\mathcal{H}_{f^n x, f^n y}(f^n x) \circ Df^n|_{\mathcal{E}_x} : \mathcal{E}_x \rightarrow \mathcal{E}_y.$$

We consider a non-zero vector $u \in \mathcal{E}_x^k$ and let j be largest index so that the j component of $v = D\mathcal{H}_{x,y}(x)u$ is non-zero. We need to show that $j \leq k$. Suppose that $j > k$ and hence $\chi_k + \varepsilon < \chi_j - \varepsilon$ since $\varepsilon < \varepsilon_0$. Then we obtain a contradiction as follows. First we have

$$\|Df^n|_{\mathcal{E}_y} \circ D\mathcal{H}_{x,y}(x) u\| = \|Df^n v\| \geq e^{(\chi_j - \varepsilon)n} \|v_j\|.$$

On the other hand,

$$\|D\mathcal{H}_{f^n x, f^n y}(f^n x) \circ Df^n|_{\mathcal{E}_x} u\| \leq \|D\mathcal{H}_{f^n x, f^n y}(f^n x)\| \cdot \|Df^n|_{\mathcal{E}_x} u\| \leq C e^{(\chi_k + \varepsilon)n} \|u\|,$$

since $\|D\mathcal{H}_{f^n x, f^n y}(f^n x)\|$ is uniformly bounded in n as $\text{dist}_{\mathcal{V}}(f^n x, f^n y)$ decreases. This is impossible for large n since $\chi_k + \varepsilon < \chi_j - \varepsilon$. Therefore $D\mathcal{H}_{x,y}$ preserves the flag.

We take arbitrary $x \in \mathcal{M}$ and $y \in \mathcal{V}(x)$ and consider the lifted holonomy (suppressing the bar in this proof)

$$H_{x,y} = \bar{H}_{x,y} = \varphi_y \circ \mathcal{H}_{x,y} \circ \varphi_x^{-1} : \mathcal{E}_x \rightarrow \mathcal{E}_y.$$

We write its Taylor polynomial of degree $M = \lfloor r \rfloor$ at $0 \in \mathcal{E}_x$ as

$$T_M(H_{x,y})(t) = \sum_{m=1}^M H_{x,y}^{(m)}(t) : \mathcal{E}_x \rightarrow \mathcal{E}_y,$$

where $H_{x,y}^{(m)} : \mathcal{E}_x \rightarrow \mathcal{E}_y$ is a homogeneous polynomial of degree m . Now we show inductively that this Taylor polynomial contains only sub-resonance terms. Writing f^n in normal form coordinates we have

$$\varphi^{f^n x} \circ f^n|_{\mathcal{W}(x)} \circ \varphi_x^{-1} = \mathcal{P}_x^n : \mathcal{E}_x \rightarrow \mathcal{E}_{f^n x},$$

where the polynomial map $\mathcal{P}_x^n(t) = \sum_{m=1}^d P_x^{(m)}(t)$ contains only sub-resonance terms and in particular has degree at most $d = \lfloor \chi_1 / \chi_\ell \rfloor \leq M$. With similar notations for y , the commutation relation (4.6) becomes

$$(4.7) \quad H_{f^n x, f^n y} \circ \mathcal{P}_x^n = \mathcal{P}_y^n \circ H_{x,y} : \mathcal{E}_x \rightarrow \mathcal{E}_{f^n y}.$$

We already proved that the first derivative $H_{x,y}^{(1)} = D_x H_{x,y}$ preserves the strong flag, which means exactly that it is a sub-resonance linear map.

Inductively, we assume that $H_{x,y}^{(m)}$ has only sub-resonance terms for all $x \in \mathcal{M}$, $y \in \mathcal{V}_x$, and $m = 1, \dots, k-1$ and show that the same holds for $H_{x,y}^{(k)}$. We split

$$H_{x,y}^{(k)} = S_{x,y} + N_{x,y}$$

into the sub-resonance part and the rest. It suffices to show that $N_{x,y} = 0$ for all $y \in \mathcal{W}_x$ that are sufficiently close to x . Assuming the contrary, we fix such x and y with $N_{x,y} \neq 0$. We will write N_x for $N_{x,y}$ and $N_{f^n x}$ for $N_{f^n x, f^n y}$. We consider the Taylor

terms of order k in the commutation relation (4.7). They come from the compositions of the respective Taylor polynomials

$$\sum_{j=1}^M H_{f^{n_x}, f^{n_y}}^{(j)} \left(\sum_{m=1}^d P_x^{(m)}(t) \right) \quad \text{and} \quad \sum_{m=1}^d P_y^{(m)} \left(\sum_{j=1}^M H_{x,y}^{(j)}(t) \right).$$

By the inductive assumption, N_x and $N_{f^{n_x}}$ are the only non sub-resonance terms of order k in these polynomials, and all lower order terms are sub-resonance. Since any composition of sub-resonance terms is again sub-resonance, taking non sub-resonance terms of order k on both sides yields the equation

$$(4.8) \quad N_{f^{n_x}}(P_x^{(1)}(t)) = P_y^{(1)}(N_x(t)).$$

We decompose N_x into components $N_x = (N_x^1, \dots, N_x^\ell)$ and let i be the largest index such that $N_x^i \neq 0$, i.e., there exists $t' \in \mathcal{E}_x$ so that $z = N_x^i(t') \in \mathcal{E}_y$ has component $0 \neq z_i \in \mathcal{E}_y^i$. We denote

$$(4.9) \quad w = P_y^{(1)}(z) = D\mathcal{P}_y^n(0)z = Df^n(y)z \in \mathcal{E}_{f^{n_y}} \quad \text{and} \quad w_i = Df^n(y)z_i \in \mathcal{E}_{f^{n_y}}^i$$

its i th component. Then

$$(4.10) \quad \|w_i\| \geq C e^{n(\chi_i - \varepsilon)},$$

where the constant $C = \|z_i\|$ does not depend on n .

Now we estimate from above the i th component of $N_{f^{n_x}}(P_x^{(1)}(t'))$. First,

$$\|P_x^{(1)}(t'_j)\| = \|Df^n(x) t'_j\| \leq \|t'\| e^{n(\chi_j + \varepsilon)} \quad \text{for any } j.$$

Let $N_{f^{n_x}}^s$ be a term of homogeneity type $s = (s_1, \dots, s_\ell)$ in the component $N_{f^{n_x}}^i$. By homogeneity, its norm can be estimated as

$$\|N_{f^{n_x}}^s(P_x^{(1)}(t'))\| \leq \|N_{f^{n_x}}\| \cdot \|t'\|^k \cdot e^{n \sum s_j(\chi_j + \varepsilon)}.$$

Since no term in $N_{f^{n_x}}^i$ is a sub-resonance one, we have $\chi_i > \sum s_j \chi_j$ and hence, since $\varepsilon < \varepsilon_0$, we also have $\chi_i > \sum s_j \chi_j + (n+2)\varepsilon$. Then the left side of (4.8) at t' can be estimated as

$$\|N_{f^{n_x}}^s(P_x^{(1)}(t'))\| \leq C' e^{n(\chi_i - 2\varepsilon)},$$

where the constant C' does not depend on n since the norms of $H_{f^{n_x}, f^{n_y}}^{(k)}$, and hence those of $N_{f^{n_x}}$, are uniformly bounded. This contradicts (4.10) for large n .

Thus we have shown that $T_M(H_{x,y})$, the Taylor polynomial of degree M for $H_{x,y}$ at $0 \in \mathcal{E}_x$, contains only sub-resonance terms for all $x \in \mathcal{M}$ and $y \in \mathcal{V}(x)$. It remains to show that $H_{x,y}$ coincides with $T_M(H_{x,y})$.

In addition to (4.7), the same commutation relation holds for the Taylor polynomials

$$(4.11) \quad T_M(H_{f^{n_x}, f^{n_y}}) \circ \mathcal{P}_x^n = \mathcal{P}_y^n \circ T_M(H_{x,y}).$$

Indeed, the two sides must have the same terms up to order M , but they are sub-resonance polynomials and thus have no terms of order higher than $d \leq M$. Denoting

$$\Delta_n = H_{f^n x, f^n y} - T(H_{f^n x, f^n y})$$

we obtain from (4.7) and (4.11) that

$$(4.12) \quad \mathcal{P}_y^n \circ H_{x,y} - \mathcal{P}_y^n \circ T_M(H_{x,y}) = \Delta_n \circ \mathcal{P}_x^n.$$

We denote

$$\Delta = H_{x,y} - T_M(H_{x,y}) : \mathcal{E}_x \rightarrow \mathcal{E}_y$$

and suppose that $\Delta \neq 0$ for some x, y . Let i be the largest index for which the i th component of Δ is nonzero. Then there exist arbitrarily small $t' \in \mathcal{E}_x$ such that the i th component z_i of $z = \Delta(t')$ is nonzero. Since \mathcal{P}_y^n is a sub-resonance polynomial, the *nonlinear* terms in its i th component can depend only on j components of the input with $j > i$, which are the same for $H_{y,x}$ and $T(H_{x,y})$ by the choice of i . The linear part of \mathcal{P}_y^n is $Df^n(y)$ and it preserves the splitting. Thus the i th component of the left side of (4.12) at t' is $Df^n(y)z_i$. So we can estimate the left side of (4.12) at t'

$$(4.13) \quad \|(\mathcal{P}_y^n \circ H_{x,y} - \mathcal{P}_y^n \circ T(H_{x,y}))(t')\| \geq \|Df^n(y)z_i\| \geq e^{n(\chi_i - \varepsilon)} \|z_i\| \geq e^{n(\chi_1 - \varepsilon)} \|z_i\|.$$

Now we estimate the right side of (4.12). Since $H_{f^n x, f^n y}$ is $C^{M+\alpha}$, there exists a constant C determined by $\|H_{f^n x, f^n y}\|_{C^{M+\alpha}}$ such that

$$(4.14) \quad \|\Delta_n(t)\| \leq C\|t\|^{M+\alpha} \quad \text{for all } t \in \mathcal{E}_{f^n x} \text{ with } \|t\| \leq \delta$$

for a sufficiently small $\delta > 0$. This C can be chosen uniform in n and close x, y .

Also, for a sufficiently small $\delta > 0$ we can estimate \mathcal{P}_x^n as

$$(4.15) \quad \|\mathcal{P}_x^n(t)\| \leq e^{n(\chi_\ell + 2\varepsilon)} \|t\| \quad \text{for all } n \in \mathbb{N} \text{ and } \|t\| < \delta.$$

This follows from the fact that for all $x \in \mathcal{M}$ we have

$$\|D_0 \mathcal{P}_x\| = \|Df|_{\mathcal{E}_x}\| \leq e^{\chi_\ell + \varepsilon} \quad \text{and hence} \quad \|D_t \mathcal{P}_x\| \leq e^{\chi_\ell + 2\varepsilon}$$

for all points $t \in \mathcal{E}_x$ with $\|t\| \leq \delta$ for a sufficiently small $\delta > 0$. Hence \mathcal{P}_x is a $e^{\chi_\ell + 2\varepsilon}$ -contraction on the δ ball in each \mathcal{E}_x , and (4.15) follows.

Combining (4.14) and (4.15) we estimate the right side of (4.12) at t' as

$$\|(\Delta_n \circ \mathcal{P}_x^n)(t')\| \leq C\|\mathcal{P}_x^n(t')\|^{M+\alpha} \leq C\|t'\|^{M+\alpha} e^{n(M+\alpha)(\chi_\ell + 2\varepsilon)}.$$

Now we see that this contradicts (4.13) for large n since $(M+\alpha)\chi_\ell = r\chi_\ell < \chi_1$ and since $\varepsilon < \varepsilon_0$ is sufficiently small. Thus, $\Delta = 0$, i.e., the holonomy map $H_{x,y}$ coincides with its Taylor polynomial of order M . This completes the proof of Theorem 4.4.

5. PROOF OF THEOREM 1.1

We first note that

(3) \implies (2) is clear, and

(1) \implies (2) holds since we always have $h(\mathcal{W}^u) = W^u$

and hence the joint foliation for \mathcal{W}^u and \mathcal{W}^{ss} is $h^{-1}(W^u \oplus W^{ss})$.

The converse (2) \implies (1) is part of [GSh23, Theorem 1.1].

Next we prove that

(1) \implies (5) \implies (4) and

(1) \implies (3), more precisely (5)+(2) \implies (3).

Finally we show (4) \implies (1) under the assumption on density of Lyapunov leaves.

(1) \implies (5). We assume that $h(\mathcal{W}^{ss}) = W^{ss}$. We apply Theorems 4.3 and 4.4 with $U = \mathcal{W}^s$, $\mathcal{W} = \mathcal{W}^{ws}$ and $\mathcal{V} = \mathcal{W}^{ss}$, and denote $\mathcal{E}^{ws} = T\mathcal{W}^{ws}$. Since f is C^1 close to L , the linear extension $Df|_{\mathcal{E}^{ws}}$ has the corresponding (χ, ε) spectrum with small ε . Since \mathcal{W}^{ss} is a C^∞ subfoliation inside the leaves of \mathcal{W}^s , Theorems 4.3 applies and yields the normal form coordinates φ_x on \mathcal{W}^{ws} given by equation (4.4)

$$\varphi_x : \mathcal{W}^{ws}(x) \rightarrow \mathcal{E}_x^{ws}.$$

The maps φ_x are C^1 diffeomorphisms which depend continuously on x in C^1 topology. Further, Theorem 4.4 yields that the holonomies $\mathcal{H} = \mathcal{H}^{ss}$ of \mathcal{W}^{ss} inside \mathcal{W}^s between leaves of \mathcal{W}^{ws} are sub-resonance polynomials in these coordinates (4.1), i.e.,

$$\bar{H}_{x,y} = \varphi_y \circ \mathcal{H}_{x,y} \circ \varphi_x^{-1} : \mathcal{E}_x^{ws} \rightarrow \mathcal{E}_y^{ws} \quad \text{is in } \mathcal{S}_{x,y}.$$

Since $h(\mathcal{W}^{ss}) = W^{ss}$ and $h(\mathcal{W}^{ws}) = W^{ws}$, the two foliations form a global product structure inside the leaves of \mathcal{W}^s conjugate by h to that of the linear foliations. In particular, the holonomies \mathcal{H} are globally defined on the leaves of \mathcal{W}^{ws} . The corresponding linear holonomies H for L are translations along W^{ss} :

$$\text{if } y \in W^{ss}(x) \text{ then } H_{x,y}(z) = z + (y - x),$$

and h conjugates \mathcal{H} and H as follows

$$H_{h(x), h(y)} = h \circ \mathcal{H}_{x,y} \circ h^{-1}.$$

We fix an arbitrary $x \in \mathbb{T}^d$ and $y \in \mathcal{W}^{ws}(x)$. By the assumption, the linear leaf $W^{ss}(x)$ is dense in \mathbb{T}^d . Hence there exists a sequence of vectors $v_n \in E^{ss}$ such that $h(x) + v_n$ converges to $h(y)$. Denoting $y_n = h^{-1}(h(x) + v_n)$ we obtain a sequence of points $y_n \in \mathcal{W}^{ss}(x)$ converging to y .

The corresponding linear holonomies $H_{v_n} = H_{h(x), h(y_n)}$ converge in C^0 to the translation H_v in $W^{ws}(h(x))$ by the vector $v = h(y) - h(x)$. Hence the holonomies \mathcal{H}_{x,y_n} converge in C^0 norm to the homeomorphism

$$\mathcal{H}_v : \mathcal{W}^{ws}(x) \rightarrow \mathcal{W}^{ws}(y) = \mathcal{W}^{ws}(x) \quad \text{such that } H_v = h \circ \mathcal{H}_v \circ h^{-1}.$$

Since the normal form coordinates φ_y depend continuously on y , the corresponding lifted holonomies $\bar{H}_{x,y_n} = \varphi_{y_n} \circ \mathcal{H}_{x,y_n} \circ \varphi_x^{-1} \in \mathcal{S}_{x,y_n}$ converge to the homeomorphism

$$P_{x,y} = \varphi_y \circ \mathcal{H}_v \circ \varphi_x^{-1} = \varphi_y \circ h^{-1} \circ H_v \circ h \circ \varphi_x^{-1} : \mathcal{E}_x^{ws} \rightarrow \mathcal{E}_y^{ws}.$$

The map $P_{x,y}$ is also a sub-resonance polynomial, i.e., $P_{x,y} \in \mathcal{S}_{x,y}$.

Now we lift the restriction of h to $\mathcal{W}^{ws}(x)$ to \mathcal{E}_x using coordinates φ_x

$$\bar{h}_x = h \circ \varphi_x^{-1} : \mathcal{E}_x^{ws} \rightarrow W^{ws}(h(x)),$$

and conjugate by \bar{h}_x the translation H_v of $W^{ws}(h(x))$ to the corresponding map of \mathcal{E}_x

$$\bar{H}_v = (\bar{h}_x)^{-1} \circ H_v \circ \bar{h}_x = \varphi_x \circ h^{-1} \circ H_v \circ h \circ \varphi_x^{-1} = \varphi_x \circ \varphi_y^{-1} \circ P_{x,y} : \mathcal{E}_x^{ws} \rightarrow \mathcal{E}_x^{ws}.$$

By the property of normal form coordinates, the map $\varphi_x \circ \varphi_y^{-1} : \mathcal{E}_y^{ws} \rightarrow \mathcal{E}_x^{ws}$ is a composition of a sub-resonance polynomial in $\mathcal{S}_{y,x}$ with a translation of \mathcal{E}_x^{ws} . Since $P_{x,y}$ is a sub-resonance polynomial in $\mathcal{S}_{x,y}$, we conclude that $\bar{H}_v \in \bar{\mathcal{S}}_x$, the finite dimensional Lie group generated by \mathcal{S}_x and the translations of \mathcal{E}_x . Thus \bar{h}_x conjugates the action of $E^{ws} = \mathbb{R}^k$ by translations of $W^{ws}(h(x))$ with the corresponding continuous action of \mathbb{R}^k by elements of the Lie group $\bar{\mathcal{S}}_x$. This yields the injective continuous homomorphism

$$\eta_x : E^{ws} \rightarrow \bar{\mathcal{S}}_x \quad \text{given by} \quad \eta_x(v) = \bar{H}_v = (\bar{h}_x)^{-1} \circ H_v \circ \bar{h}_x.$$

It is a classical result that a continuous homomorphism between Lie groups is automatically a Lie group homomorphism, and so it is automatically C^∞ , see for example [Ha, Corollary 3.50]. Thus η_x is a C^∞ diffeomorphism onto its image in $\bar{\mathcal{S}}_x$. We conclude that $(\bar{h}_x)^{-1}$ is a C^∞ diffeomorphism between $W^{ws}(h(x))$ and \mathcal{E}_x^{ws} since it is determined by η_x as follows

$$(\bar{h}_x)^{-1}(h(x) + v) = \bar{H}_v(0) = \eta_x(v)(0).$$

Hence $\bar{h}_x : \mathcal{E}_x^{ws} \rightarrow W^{ws}(h(x))$ is also a C^∞ diffeomorphism. Further, since the normal form coordinates φ_x , as well as holonomies and their limits, depend continuously on x , the constructed continuous action on \mathcal{E}_x^{ws} and the corresponding homomorphism η_x also depend continuously on x . This implies that η_x depend continuously on x in C^∞ topology, because it is determined by the corresponding linear homomorphism of the Lie algebras. This yields that $\bar{h}_x : \mathcal{E}_x^{ws} \rightarrow W^{ws}(h(x))$ also depends continuously on x in C^∞ topology. We conclude that the restriction of h to $\mathcal{W}^{ws}(x)$

$$h|_{\mathcal{W}^{ws}(x)} = \bar{h}_x \circ \varphi_x : \mathcal{W}^{ws}(x) \rightarrow W^{ws}(h(x))$$

is as regular as φ_x . If the leaves of \mathcal{W}^{ws} are uniformly C^q , then so are the maps φ_x given by (4.4) since the holonomy \mathcal{H}_x from a smooth transversal to the leaf $\mathcal{W}^{ws}(x)$ is as regular as the leaves. Hence in this case $h|_{\mathcal{W}^{ws}(x)}$ is a uniformly C^q diffeomorphism. In particular, since the leaves of \mathcal{W}^{ws} at least uniformly $C^{1+\text{H\"older}}$, $h|_{\mathcal{W}^{ws}(x)}$ is at least a uniformly $C^{1+\text{H\"older}}$ diffeomorphism.

To complete the proof of (5) we will now show that the component h^{ws} is C^∞ on \mathbb{T}^d . First, h^{ws} is (locally) constant along \mathcal{W}^{u+ss} and hence is uniformly C^∞ along it, since

its leaves are uniformly C^∞ . By Journe lemma [J88],[KtN, Theorem 3.3.1] it suffices to show that h^{ws} is also uniformly C^∞ along smooth transversals to \mathcal{W}^{u+ss} .

As in the construction of normal forms on \mathcal{W}^{ws} we consider the embeddings of small balls $B_\rho^{ws}(x)$ in \mathcal{E}_x^{ws} as a local family of smooth transversals (4.2) and denote

$$i_x : B_\rho^{ws}(x) \rightarrow \mathcal{W}^s, \quad i_x(B_\rho^{ws}(x)) = \mathcal{T}_x, \quad \mathcal{H}_x = \mathcal{H}_x^{\mathcal{W}^{ss}} : \mathcal{T}_x \rightarrow \mathcal{W}^{ws}(x).$$

We recall that the normal form coordinates (4.4) are given by

$$\varphi_x = \hat{\varphi}_x \circ (\mathcal{H}_x \circ i_x)^{-1} : \mathcal{W}^{ws} \rightarrow \mathcal{E}_x^{ws},$$

where $\hat{\varphi}_x : \mathcal{E}_x^{ws} \rightarrow \mathcal{E}_x^{ws}$ is the uniformly C^∞ coordinate change for the extension \mathcal{F}_x . Since we know that both \bar{h}_x and $i_x \circ (\hat{\varphi}_x)^{-1}$ are uniformly C^∞ diffeomorphism and since

$$\bar{h}_x = h \circ (\varphi_x)^{-1} = h \circ \mathcal{H}_x \circ i_x \circ (\hat{\varphi}_x)^{-1} : \mathcal{E}_x^{ws} \rightarrow W^{ws}(hx)$$

we conclude that $h \circ \mathcal{H}_x : \mathcal{T}_x \rightarrow W^{ws}(hx)$ is also uniformly C^∞ , and in particular so is its ws -component. We claim that $(h \circ \mathcal{H}_x)^{ws} = (h|_{\mathcal{T}_x})^{ws}$. Indeed, for any $y \in \mathcal{T}_x$ we have $\mathcal{H}_x(y) \in \mathcal{W}^{ss}(y)$ and, since $h(\mathcal{W}^{ss}) = W^{ss}$, we get $h(\mathcal{H}_x(y)) \in W^{ss}(h(y))$ and thus they have the same ws -component. Therefore, $(h|_{\mathcal{T}_x})^{ws} = h^{ws}|_{\mathcal{T}_x}$ is uniformly C^∞ along this family of transversals to \mathcal{W}^{u+ss} , and we conclude that h^{ws} is C^∞ on \mathbb{T}^d .

This concludes the proof of (1) \implies (5).

(5) \implies (4). Since (5) yields that $h|_{\mathcal{W}^{ws}(x)}$ is a C^1 diffeomorphism, it remains to show that its derivative is Hölder on \mathbb{T}^d . Since h maps $\mathcal{W}^{ws}(x)$ to the linear leaf $W^{ws}(h(x))$, this derivative coincides with the restriction of the derivative of h^{ws} ,

$$D(h|_{\mathcal{W}^{ws}(x)})(x) = D(h^{ws}|_{\mathcal{W}^{ws}(x)})(x) = D(h^{ws})|_{\mathcal{E}_x^{ws}} : \mathcal{E}_x^{ws} \rightarrow E^{ws} \subset \mathbb{R}^d.$$

Since h^{ws} is C^∞ on \mathbb{T}^d by (5), this restriction is as regular as the subbundle \mathcal{E}_x^{ws} , and so at least Hölder continuous on \mathbb{T}^d .

(1) \implies (3). Since (1) implies (2), we have the joint topological foliation $\mathcal{W}^{u+ss} = h^{-1}(W^u \oplus W^{ss})$. Now we use (5), which follows from (1), to show that \mathcal{W}^{u+ss} is conjugate to the linear foliation $W^u \oplus W^{ss}$ by a C^∞ diffeomorphism. We take h to be the conjugacy close to the identity and write

$$h(x) = x + \Delta(x) = x + \Delta^{ws}(x) + \Delta^{u+ss}(x),$$

where $\Delta : \mathbb{T}^d \rightarrow \mathbb{R}^d$ is split into components $\Delta^{ws} : \mathbb{T}^d \rightarrow E^{ws}$ and $\Delta^{u+ss} : \mathbb{T}^d \rightarrow E^u \oplus E^{ss}$. Now we consider the map

$$\tilde{h}(x) = h(x) - \Delta^{u+ss}(x) = x + \Delta^{ws}(x) : \mathbb{T}^d \rightarrow \mathbb{T}^d.$$

We note that both h and \tilde{h} are C^0 close to the identity. The first formula shows that \tilde{h} is an adjustment of h along $W^u \oplus W^{ss}$ and thus, since $h(\mathcal{W}^{u+ss}) = W^u \oplus W^{ss}$, we also have that $\tilde{h}(\mathcal{W}^{u+ss}) = W^u \oplus W^{ss}$. Now we show that \tilde{h} is a C^∞ diffeomorphism of \mathbb{T}^d , and hence it smoothly conjugates \mathcal{W}^{u+ss} to $W^u \oplus W^{ss}$. We can locally write

$$\tilde{h}(x) = x + \Delta^{ws}(x) = x^{u+ss} + x^{ws} + \Delta^{ws}(x) = x^{u+ss} + h^{ws}(x).$$

While the components h^{ws} , x^{u+ss} , and x^{ws} are globally well-defined only for the lifts to \mathbb{R}^d , they make sense locally on \mathbb{T}^d . By (5), the component h^{ws} is C^∞ on \mathbb{T}^d and hence so is \tilde{h} . Thus to prove that \tilde{h} is locally a C^∞ diffeomorphism it suffices to show invertibility of its derivative. We view $D\tilde{h}$ with respect to splittings $\mathcal{E}_x^{ws} \oplus \mathcal{E}_x^{u+ss} \rightarrow E^{ws} \oplus E^{u+ss}$. Since h^{ws} is constant along \mathcal{W}^{u+ss} and a C^1 diffeomorphism along \mathcal{W}^{ws} we see that $D\tilde{h}$ is block triangular with invertible block $\mathcal{E}_x^{ws} \rightarrow E^{ws}$. Now invertibility of $D\tilde{h}$ follows from that of the other diagonal block $D(x^{u+ss})|_{\mathcal{E}_x^{u+ss}}$. The latter is clear since \mathcal{E}_x^{u+ss} is transverse to E^{ws} and thus the derivative of x^{u+ss} , which is just E^{u+ss} -component of the identity, has full rank on \mathcal{E}_x^{u+ss} . The global surjectivity and injectivity of \tilde{h} also follow now since it is C^0 close to the identity. This concludes the proof of (1) \implies (3).

(4) \implies (1) if the leaves of W^1, \dots, W^ℓ are dense in \mathbb{T}^d .

Since f is C^1 close to L , the linear extension $Df|_{\mathcal{E}^{ws}}$ has the corresponding (χ, ε) spectrum and the Hölder continuous splitting

$$(5.1) \quad \mathcal{E}^{ws} = \mathcal{E}^1 \oplus \dots \oplus \mathcal{E}^\ell$$

which is C^0 close to the Lyapunov splitting for L on E^{ws} . We recall that \mathcal{E}^1 is the strongest and \mathcal{E}^ℓ is the weakest. We let

$$\mathcal{E}^{i,ss} = \mathcal{E}^i \oplus \dots \oplus \mathcal{E}^1 \oplus \mathcal{E}^{ss}.$$

Since $\mathcal{E}^{i,ss}$ is a strong subbundle of \mathcal{E}^s , it is tangent to a C^∞ subfoliation $\mathcal{W}^{i,ss}$ inside \mathcal{W}^s . Since we assume that $h|_{\mathcal{W}^{ws}(x)}$ is a C^1 diffeomorphism, each bundle \mathcal{E}^i is tangent to the foliation $\mathcal{W}^i = h^{-1}(W^i)$. The implication (4) \implies (1) follows from the next proposition.

Proposition 5.1. *Assume that h is a C^1 diffeomorphism along \mathcal{W}^{ws} with the derivative*

$$D(h|_{\mathcal{W}^{ws}(x)})(x) : \mathcal{E}_x^{ws} \rightarrow E^{ws} \subset \mathbb{R}^d \text{ Hölder continuous on } \mathbb{T}^d.$$

If $h(\mathcal{W}^{i,ss}) = W^{i,ss}$ then $h(\mathcal{W}^{i-1,ss}) = W^{i-1,ss}$.

We apply the proposition inductively from $i = \ell$ to $i = 1$. For $i = \ell$, the assumption $h(\mathcal{W}^{i,ss}) = W^{i,ss}$ is satisfied since it is $h(\mathcal{W}^s) = W^s$. For $i = 1$ we obtain (1).

The proof of this proposition is similar to that of [GKS11, Proposition 2.5]. However, in [GKS11] the automorphism L was assumed to be irreducible, while we only assume density of the leaves of W^{ss} and the Lyapunov subfoliations of W^{ws} .

Proof of Proposition 5.1. We will use the following notation in this proof:

$$W = W^i, \quad V = W^{i-1,ss}, \quad U = W \oplus V = W^i \oplus W^{i-1,ss} = W^{i,ss},$$

and similarly for the corresponding nonlinear invariant foliations of f ,

$$\mathcal{W} = \mathcal{W}^i, \quad \mathcal{V} = \mathcal{W}^{i-1,ss}, \quad \mathcal{U} = \mathcal{W} \oplus \mathcal{V} = \mathcal{W}^{i,ss}.$$

By the assumption, $h(\mathcal{U}) = U$. We let $\tilde{V} = h(\mathcal{V})$. Then \tilde{V} is a subfoliation of U with continuous leaves. We need to show that $\tilde{V} = V$. Since $W = h(\mathcal{W})$, the foliation \tilde{V} is topologically transverse to W , i.e., any leaf of \tilde{V} and any leaf of W in the same leaf of

U intersect at exactly one point. Thus for any point $a \in \mathbb{T}^d$ and any $b \in \tilde{V}(a)$ we can define the holonomy map $\tilde{H}_{a,b} : W(a) \rightarrow W(b)$ along the foliation \tilde{V} . The key step is to show that $\tilde{H}_{a,b}$ is a parallel translation inside U . This is similar to [GKS11, Lemma 2.6], but in [GKS11] the automorphism L was assumed to be irreducible, which yields conformality of L on W . We modify the argument for the general case when Jordan blocks may create nonconformality.

Lemma 5.2. *For any point $a \in \mathbb{T}^d$ and any $b \in \tilde{V}(a)$ the holonomy map $\tilde{H}_{a,b}$ is a restriction to $W(a)$ of a parallel translation inside U .*

Proof. For any point $c \in \mathbb{T}^d$ and any $d \in \mathcal{V}(c)$ we denote by $\mathcal{H}_{c,d} : \mathcal{W}(c) \rightarrow \mathcal{W}(d)$ the holonomy along the foliation \mathcal{V} . Since \mathcal{V} is a strong subfoliation of \mathcal{U} , it is C^∞ inside the leaves of \mathcal{U} and hence the holonomies $\mathcal{H}_{c,d}$ are C^1 with the derivative

$$D_c \mathcal{H}_{c,d} : T_c \mathcal{W} \rightarrow T_d \mathcal{W}$$

depending Hölder continuously on c and d . Since $\tilde{V} = h(\mathcal{V})$ and $h(\mathcal{W}) = W$ we have

$$\tilde{H}_{a,b} = h \circ \mathcal{H}_{h^{-1}(a), h^{-1}(b)} \circ h^{-1}.$$

It follows from the regularity assumption on $h|_{\mathcal{W}^{ws}}$ that the maps $\tilde{H}_{a,b}$ are also C^1 . To show that $\tilde{H}_{a,b}$ is a parallel translation, we prove that the differential $D\tilde{H}_{a,b} = \text{Id}$. We apply L^n , which contracts U , and denote $a_n = L^n(a)$ and $b_n = L^n(b)$. Since $\tilde{V} = h(\mathcal{V})$ and f preserves the foliation \mathcal{V} , the map L preserves \tilde{V} and we can write

$$\tilde{H}_{a,b} = L^{-n} \circ \tilde{H}_{a_n, b_n} \circ L^n.$$

Differentiating and denoting $D_{a_n} \tilde{H}_{a_n, b_n} = \text{Id} + \Delta_n$ we obtain

$$D_a \tilde{H}_{a,b} = (L^{-n}|_W) \circ (D_{a_n} \tilde{H}_{a_n, b_n}) \circ L^n|_W = \text{Id} + L^{-n}|_W \circ \Delta_n \circ L^n|_W.$$

Since W is a Lyapunov foliation for L , all eigenvalues of L on W have the same modulus and hence the quasiconformal distortion of $L^n|_W$ grows at most polynomially,

$$\|L^{-n}|_W\| \cdot \|L^n|_W\| \leq Cn^{2k} \text{ for all } n,$$

where $k+1$ is the largest size of Jordan blocks of $L^n|_W$. Thus we obtain

$$\|L^{-n}|_W \circ \Delta_n \circ L^n|_W\| \leq Cn^{2k} \|\Delta_n\| \text{ for all } n.$$

It remains to show that $\|\Delta_n\| \rightarrow 0$ exponentially in n . We differentiate the equation $\tilde{H}_{a_n, b_n} = h \circ \mathcal{H}_{h^{-1}(a_n), h^{-1}(b_n)} \circ h^{-1}$ at a_n

$$D_{a_n} \tilde{H}_{a_n, b_n} = (D_{h^{-1}(b_n)} h|_{\mathcal{W}}) \circ (D_{h^{-1}(a_n)} \mathcal{H}_{h^{-1}(a_n), h^{-1}(b_n)}) \circ (D_{a_n} (h|_{\mathcal{W}})^{-1}),$$

and denoting $D_{h^{-1}(a_n)} \mathcal{H}_{h^{-1}(a_n), h^{-1}(b_n)} = \text{Id} + \Delta'_n$ we obtain

$$D_{a_n} \tilde{H}_{a_n, b_n} = (D_{h^{-1}(b_n)} h|_{\mathcal{W}}) \circ D_{a_n} (h|_{\mathcal{W}})^{-1} + (D_{h^{-1}(b_n)} h|_{\mathcal{W}}) \circ \Delta'_n \circ D_{a_n} (h|_{\mathcal{W}})^{-1}.$$

Denoting $D_{h^{-1}(b_n)} h|_{\mathcal{W}} \circ D_{a_n} (h|_{\mathcal{W}})^{-1} = \text{Id} + \Delta''_n$ we conclude that

$$\|\Delta_n\| = \|\text{Id} - D_{a_n} \tilde{H}_{a_n, b_n}\| \leq \|\Delta''_n\| + \|D_{h^{-1}(b_n)} h|_{\mathcal{W}}\| \cdot \|\Delta'_n\| \cdot \|D_{a_n} (h|_{\mathcal{W}})^{-1}\|.$$

By the regularity assumption on $h|_{\mathcal{W}^{ws}}$ and Hölder continuity of $T_x \mathcal{W} = \mathcal{E}_x^i$ we have Hölder dependence of $D_x(h|_{\mathcal{W}}) : T_x \mathcal{W} \rightarrow W$ on x . It follows that that

$$\begin{aligned} & \|D_{h^{-1}(b_n)} h|_{\mathcal{W}}\| \cdot \|D_{a_n}(h|_{\mathcal{W}})^{-1}\| \text{ is uniformly bounded and} \\ & \|\Delta_n''\| \text{ is Hölder in } \text{dist}(a_n, b_n). \end{aligned}$$

Also, $\|\Delta_n'\|$ is Hölder in $\text{dist}(a_n, b_n)$ since $D_c \mathcal{H}_{c,d}$ depends Hölder continuously on c and d and $D_c \mathcal{H}_{c,d} = \text{Id}$. Now since $\text{dist}(a_n, b_n) \rightarrow 0$ exponentially as $n \rightarrow \infty$ we conclude that so does $\|\Delta_n\|$ and hence $D\tilde{H}_{a,b} = \text{Id}$. \square

Now we complete the proof of Proposition 5.1 as in [GKS11, Proposition 2.5]. Let a be a fixed point of L and let B be the unit ball in $V(a)$ centered at a . If $B \subset \tilde{V}(a)$, then $V(a) = \tilde{V}(a)$ by invariance of V and \tilde{V} under L^{-1} . By the assumption, the leaf $V(a)$ is dense in \mathbb{T}^d . It follows that the set of points x such that $V(x) = \tilde{V}(x)$ is dense in \mathbb{T}^d and hence $V = \tilde{V}$. Therefore, it suffices to show that $B \subset \tilde{V}(a)$.

We argue by contradiction. Assume that there is $z_1 \in B$ such that $z_1 \notin \tilde{V}(a)$. Let $x_1 = W(z_1) \cap \tilde{V}(a)$. Since W has dense leaves we can choose a sequence $\{b_n, n \geq 1\}$ in $W(a)$ so that $b_n \rightarrow x_1$ as $n \rightarrow \infty$. Let $y_n = \tilde{H}_{a,x_1}(b_n)$. Continuity of \tilde{V} implies that the sequence $\{y_n\}$ converges to a point $x_2 \in \tilde{V}(a)$. Moreover, Lemma 5.2 implies that $\{x_1, x_2\}$ is a parallel translation of $\{a, x_1\}$.

We continue this procedure inductively to construct the sequence $\{x_n, n \geq 1\}$ in $\tilde{V}(a)$. Let $z_n = W(x_n) \cap V(a)$. Then by the construction

$$d_V(z_n, a) = n \cdot d_V(z_1, a) \quad \text{and} \quad d_W(x_n, z_n) = n \cdot d_W(x_1, z_1),$$

but this is impossible since L contracts V exponentially stronger than W . Indeed, if we take $N(n)$ to be the smallest integer such that $L^{N(n)}(z_n) \in B$, then

$$d_W(L^{N(n)}(x_n), L^{N(n)}(z_n)) \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

which contradicts $\max_{z \in B} d_W(z, W(z) \cap \tilde{V}(a)) < \infty$. Thus $\tilde{V} = V$. \square

6. PROOF OF THEOREM 1.3

The proof of Theorem 1.1 works for the global setting with only minor adjustments. In this case, (2) \implies (1) as well as $h(\mathcal{W}^{ws}) = W^{ws}$ is given by [GSh23, Theorem 1.1]. Further, [GSh23, Theorem 1.2] yields the dominated splitting and spectrum for $Df|_{\mathcal{E}^{ws}}$ matching that of L , which implies that $Df|_{\mathcal{E}^{ws}}$ has (χ, ε) spectrum for any $\varepsilon > 0$. The remaining arguments work without change, except in (1) \implies (3) we only obtain that \tilde{h} is a local C^∞ diffeomorphism, as its injectivity is not clear. Still \tilde{h} gives local foliation charts for \mathcal{W}^{u+ss} . Moreover, the Hölder continuous metric on \mathcal{E}^{ws} invariant under the holonomies of \mathcal{W}^{u+ss} can be pulled by \tilde{h} using the linear foliation as in the proof of Theorem 1.5(3) \implies (2) below, since this is a local property.

7. PROOF OF THEOREM 1.4

Theorem 1.4 follows from Theorem 7.1 below where we assume only the regularity of holonomies of \mathcal{W}^{ws} between the leaves of \mathcal{W}^{ss} .

Theorem 7.1 (Rigidity of weak holonomies). *Let L be a hyperbolic automorphism of \mathbb{T}^d with dense leaves of W^{ws} . Let f be a C^∞ diffeomorphism sufficiently C^1 close to L , and let h be a topological conjugacy between f and L .*

Let $r_{ss}(L)$ be given by (1.3). and let $q > 1$ be a noninteger such that the leaves of \mathcal{W}^{ws} are uniformly C^q . Then for the statements below we have

$$(1) \iff (1') \iff (2) \implies (3) \implies (4).$$

- (1) *Holonomies of \mathcal{W}^{ws} between the leaves of \mathcal{W}^{ss} are uniformly C^r with $r > r_{ss}(L)$,*
- (1') *Holonomies of \mathcal{W}^{ws} between the leaves of \mathcal{W}^{ss} are uniformly C^∞ ,*
- (2) *h^{ss} is a uniformly C^∞ diffeomorphism along \mathcal{W}^{ss} ,
and h^{ss} is C^q on \mathbb{T}^d ,*
- (3) *The joint foliations \mathcal{W}^{u+ws} is conjugate to the linear foliation $W^u \oplus W^{ws}$ by a C^q diffeomorphism,*
- (4) *\mathcal{W}^{ws} is a uniformly C^q subfoliation of \mathcal{W}^s .*

If in addition $h(\mathcal{W}^{ss}) = W^{ss}$, then

$$\begin{aligned} (1, 1', 2) &\iff h \text{ is a uniformly } C^\infty \text{ diffeomorphism along } \mathcal{W}^{ss} \\ &\iff h \text{ is a uniformly } C^r \text{ diffeomorphism along } \mathcal{W}^{ss} \text{ with } r > r_{ss}(L). \end{aligned}$$

7.1. Deducing Theorem 1.4 from Theorem 7.1.

We recall that for a noninteger $r > 1$ a foliation is C^r if and only if its leaves and local holonomy maps are uniformly C^r , see e.g. [PSW97, Theorem 6.1(i)].

Also we always have $h(\mathcal{W}^{ws}) = W^{ws}$, and hence \mathcal{W}^u and \mathcal{W}^{ws} are jointly integrable and the foliation \mathcal{W}^{u+ws} is $h^{-1}(W^u \oplus W^{ws})$.

(1) \implies (2). (1) implies Theorem 7.1(1) and hence Theorem 7.1(2). Since (1) also implies that we can take $q = r$ Theorem 7.1, we obtain that h^{ss} is C^r on \mathbb{T}^d .

(2) \implies (3). (2) implies Theorem 7.1(2) with $q = r$ and hence it yields Theorem 7.1(3) with $q = r$, which is (3).

(3) \implies (1). (3) implies Theorem 7.1(3) with $q = r$ and hence it yields Theorem 7.1(4) with $q = r$, which is (1).

The additional statement for $h(\mathcal{W}^{ss}) = W^{ss}$ also follows from the corresponding part of Theorem 7.1.

7.2. Proof of Theorem 7.1. It follows the same scheme as the proof of Theorem 1.1.

We denote $\mathcal{E}^{ss} = T\mathcal{W}^{ss}$. Since f is C^1 close to L , the linear extension $Df|_{\mathcal{E}^{ss}}$ has (χ, ε) spectrum with small ε . Since \mathcal{W}^{ss} has uniformly C^∞ leaves, we can apply

Theorems 4.2 to $\mathcal{W} = \mathcal{W}^{ss}$ and obtain the normal form coordinates φ_x on \mathcal{W}^{ss}

$$\varphi_x : \mathcal{W}^{ss}(x) \rightarrow \mathcal{E}_x^{ss}.$$

The maps φ_x are C^∞ diffeomorphisms and depend continuously on x in C^∞ topology.

(1) \implies (1'). Suppose that holonomies $\mathcal{H} = \mathcal{H}^{ws}$ of \mathcal{W}^{ws} inside \mathcal{W}^s between the leaves of \mathcal{W}^{ss} are uniformly C^r with $r > r_{ss}(L)$. Hence we can apply Theorem 4.4 with $U = \mathcal{W}^s$, $\mathcal{W} = \mathcal{W}^{ss}$ and $\mathcal{V} = \mathcal{W}^{ws}$. Indeed, the lifted holonomies \bar{H} in (4.1) are also C^r , and so the theorem yields that they are sub-resonance polynomials:

$$(7.1) \quad \bar{H}_{x,y} = \varphi_y \circ \mathcal{H}_{x,y} \circ \varphi_x^{-1} : \mathcal{E}_x^{ss} \rightarrow \mathcal{E}_y^{ss} \quad \text{are in } \mathcal{S}_{x,y},$$

and in particular are C^∞ . Since the coordinates φ_x are also uniformly C^∞ , we conclude that holonomies $\mathcal{H} = \mathcal{H}^{ws}$ are uniformly C^∞ .

(1) \implies (2). Now we use (7.1) to show that h^{ss} is uniformly C^∞ along \mathcal{W}^{ss} . Since we do not assume $h(\mathcal{W}^{ss}) = W^{ss}$, we need to adjust the holonomy argument accordingly. We fix a point $x \in \mathbb{T}^d$ and consider the map

$$(7.2) \quad \begin{aligned} \hat{h}_x : \mathcal{W}^{ss}(x) &\rightarrow W^{ss}(h(x)) \quad \text{given by} \quad \hat{h}_x = H_{h(x)}^{ws} \circ h|_{\mathcal{W}^{ss}(x)}, \quad \text{where} \\ H_{h(x)}^{ws} : h(\mathcal{W}^{ss}(x)) &\rightarrow W^{ss}(h(x)) \text{ is the linear holonomy along } W^{ws}. \end{aligned}$$

We will prove that the maps \hat{h}_x are uniformly C^∞ .

We fix $y \in \mathcal{W}^{ss}(x)$ and take a sequence of points $y_n \in \mathcal{W}^{ws}(x)$ converging to y . This can be done since the leaves of the linear foliation W^{ws} are dense in \mathbb{T}^d and the leaf conjugacy h is a homeomorphism which sends \mathcal{W}^{ws} to W^{ws} .

Since $h(\mathcal{W}^{ws}) = W^{ws}$, the holonomy maps $\mathcal{H}_{x,y_n} : \mathcal{W}^{ss}(x) \rightarrow \mathcal{W}^{ss}(y_n)$ are conjugated to the corresponding linear holonomies $H_{h(x),h(y_n)} : W^{ss}(h(x)) \rightarrow W^{ss}(h(y_n))$,

$$\mathcal{H}_{x,y_n} = (\hat{h}_{y_n})^{-1} \circ H_{h(x),h(y_n)} \circ \hat{h}_x : \mathcal{W}^{ss}(x) \rightarrow \mathcal{W}^{ss}(y_n).$$

We note that $H_{h(x),h(y_n)}$ are translations H_{v_n} by the vectors $v_n = h(y_n) - h(x)$. Since y_n converge to y , and hence $h(y_n)$ converge to $h(y)$, we see that for $v = h(y) - h(x)$,

$$H_{h(x),h(y_n)} = H_{v_n} \quad \text{converge to} \quad H_v : W^{ss}(h(x)) \rightarrow W^{ss}(h(y)).$$

Since \hat{h}_y depend continuously on y , we obtain C^0 convergence

$$\mathcal{H}_{x,y_n} \quad \text{converge to} \quad (\hat{h}_y)^{-1} \circ H_v \circ \hat{h}_x : \mathcal{W}^{ss}(x) \rightarrow \mathcal{W}^{ss}(y) = \mathcal{W}^{ss}(x).$$

For $y \in \mathcal{W}^{ss}(x)$ we have that \hat{h}_y and \hat{h}_x are related by the translation holonomy $H_{\tilde{v}}$

$$\hat{h}_x \circ (\hat{h}_y)^{-1} = H_{\tilde{v}} : W^{ss}(h(y)) \rightarrow W^{ss}(h(x)),$$

where $\tilde{v} = \hat{h}_x(y) - h(y) \in E^{ws}$. We conclude that

$$(7.3) \quad \mathcal{H}_{x,y_n} \quad \text{converges to} \quad \mathcal{H}_{\tilde{v}} := (\hat{h}_x)^{-1} \circ H_{\tilde{v}} \circ \hat{h}_x : \mathcal{W}^{ss}(x) \rightarrow \mathcal{W}^{ss}(x) = \mathcal{W}^{ss}(y),$$

where $H_{\tilde{v}} = H_{\tilde{v}} \circ H_v$ is the translation by $\hat{v} = v + \tilde{v} = \hat{h}_x(y) - h(x)$.

Since φ_y depend continuously on y , using (7.1) and (7.3) we obtain that the corresponding lifted holonomies \bar{H}_{x,y_n} converge to a sub-resonance polynomial $P_{x,y} \in \mathcal{S}_{x,y}$,

$$P_{x,y} = \varphi_y \circ \mathcal{H}_{\hat{v}} \circ \varphi_x^{-1} = \varphi_y \circ (\hat{h}_y)^{-1} \circ H_{\hat{v}} \circ (\hat{h}_x) \circ \varphi_x^{-1} : \mathcal{E}_x^{ss} \rightarrow \mathcal{E}_y^{ss}.$$

Now we lift \hat{h}_x to $\mathcal{W}^{ss}(x)$ to \mathcal{E}_x using coordinates φ_x

$$\bar{h}_x = \hat{h}_x \circ \varphi_x^{-1} : \mathcal{E}_x^{ss} \rightarrow W^{ss}(h(x)),$$

and conjugate the translation $H_{\hat{v}}$ by \bar{h}_x to obtain

$$\bar{H}_{\hat{v}} = (\bar{h}_x)^{-1} \circ H_{\hat{v}} \circ \bar{h}_x = \varphi_x \circ h^{-1} \circ H_{\hat{v}} \circ h \circ \varphi_x^{-1} = \varphi_x \circ \varphi_y^{-1} \circ P_{x,y} : \mathcal{E}_x^{ss} \rightarrow \mathcal{E}_x^{ss}.$$

Since $P_{x,y} \in \mathcal{S}_{x,y}$ and $\varphi_x \circ \varphi_y^{-1} : \mathcal{E}_y^{ss} \rightarrow \mathcal{E}_x^{ss}$ is a composition of a sub-resonance polynomial in $\mathcal{S}_{y,x}$ with a translation of \mathcal{E}_x^{ss} we conclude that $\bar{H}_{\hat{v}} \in \bar{\mathcal{S}}_x$, the finite dimensional Lie group generated by \mathcal{S}_x and the translations of \mathcal{E}_x^{ss} . Thus \bar{h}_x conjugates the action of E^{ss} by translations of $W^{ss}(h(x))$ with the continuous action of E^{ss} by elements of $\bar{\mathcal{S}}_x$. So we get the injective continuous homomorphism

$$\eta_x : E^{ss} \rightarrow \bar{\mathcal{S}}_x \quad \text{given by} \quad \eta_x(\hat{v}) = \bar{H}_{\hat{v}} = (\bar{h}_x)^{-1} \circ H_{\hat{v}} \circ \bar{h}_x.$$

which is C^∞ and depends continuously on x in C^∞ topology. This yields that \hat{h}_x^{-1} and \hat{h}_x are also C^∞ diffeomorphisms that depend continuously on x in C^∞ topology.

This proves that the component h^{ss} is uniformly C^∞ along the leaves of \mathcal{W}^{ss} , as it is easy to see that $\hat{h}_x = h^{ss}|_{\mathcal{W}^{ss}(x)}$ under a local identification of $W^{ss}(h(x))$ with E^{ss} . Since h^{ss} is also locally constant along the transversal leaves \mathcal{W}^{u+ws} , it is as regular along these leaves as they are. Since \mathcal{W}^u has iniformly C^∞ leaves and \mathcal{W}^{ws} has uniformly C^q , the leaves of the joint foliation \mathcal{W}^{u+ws} are uniformly C^q . By Journe lemma we conclude that $h^{ss} \in C^q(\mathbb{T}^d)$.

(2) \implies (1). Since $h^{ss} \in C^\infty(\mathbb{T}^d)$ and $\hat{h}_x = h^{ss}|_{\mathcal{W}^{ss}}$, it follows that the maps \hat{h}_x are uniformly C^∞ along the leaves of \mathcal{W}^{ss} . It is easy to see from the definition (7.2) that \hat{h}_x conjugates the holonomies of \mathcal{W}^{ws} inside \mathcal{W}^s with corresponding linear holonomies: $\mathcal{H}_{x,y} = (\hat{h}_y)^{-1} \circ H_{x,y} \circ \hat{h}_x$ and hence $\mathcal{H}_{x,y}$ are uniformly C^∞ .

The case when $h(\mathcal{W}^{ss}) = W^{ss}$. When $h(\mathcal{W}^{ss}) = W^{ss}$, it is clear from (7.2) that $h|_{\mathcal{W}^{ss}} = \hat{h}_x = h^{ss}|_{\mathcal{W}^{ss}}$ and hence (2) implies that h is a uniformly C^∞ diffeomorphism along \mathcal{W}^{ss} . Conversely, if $h(\mathcal{W}^{ss}) = W^{ss}$ and h is a uniformly C^r diffeomorphism along \mathcal{W}^{ss} with $r > r_{ss}(L)$, then so are the holonomies of \mathcal{W}^{ws} as they are conjugate to the linear holonomies: $\mathcal{H}_{x,y} = (h|_{\mathcal{W}^{ss}(y)})^{-1} \circ H_{x,y} \circ h|_{\mathcal{W}^{ss}(x)}$. This yields (1).

(2) \implies (3). The argument is almost identical to the proof of (3) of Theorem 1.1. We map the joint foliation $\mathcal{W}^{u+ws} = h^{-1}(W^u \oplus W^{ws})$ to $W^u \oplus W^{ss}$ by the map

$$\tilde{h}(x) = h(x) - \Delta^{u+ws}(x) = x + \Delta^{ss}(x) = x^{u+ws} + h^{ss}(x) : \mathbb{T}^d \rightarrow \mathbb{T}^d.$$

This map is C^0 close to the identity and is in $C^q(\mathbb{T}^d)$ since h^{ss} is $C^q(\mathbb{T}^d)$ by (2). Invertibility of its derivative $D\tilde{h}$ follows as in Theorem 1.1 from the fact that $\hat{h}_x = h^{ss}|_{\mathcal{W}^{ss}(x)} : \mathcal{W}^{ss}(x) \rightarrow W^{ss}(h(x))$ is a C^∞ diffeomorphism by the proof of (2).

(3) \implies (4). This follows by intersecting the C^q foliation \mathcal{W}^{u+ws} with uniformly C^∞ leaves of \mathcal{W}^s .

8. PROOF OF THEOREM 1.5

We note that (2) \iff (2') does not require dense leaves.

(2) \implies (2'). Since $h(\mathcal{W}^u) = W^u$, the component h^s is locally constant along \mathcal{W}^u , and hence uniformly C^∞ along \mathcal{W}^u . Together with (2), this yields that h^s is in $C^\infty(\mathbb{T}^d)$.

(2') \implies (2). Since $h(\mathcal{W}^s) = W^s$ we have $h|_{\mathcal{W}^s} = h^s|_{\mathcal{W}^s}$ under a local identification of W^s and E^s , and hence a uniformly C^∞ diffeomorphism.

(1) \implies (2') follows by combining Theorems 1.1 and 1.4. Indeed, $h(\mathcal{W}^{ss}) = W^{ss}$ is Theorems 1.1(1) and hence yields Theorems 1.1(5), so that h^{ws} is in $C^\infty(\mathbb{T}^d)$ and a C^q diffeomorphism along \mathcal{W}^{ws} with some $q > 1$. The second assumption in Theorem 1.5(1) is Theorem 1.4(1) with $r = \infty$ and hence yields Theorem 1.4(2) with $r = \infty$, so that h^{ss} is in $C^\infty(\mathbb{T}^d)$ and a C^∞ diffeomorphism along \mathcal{W}^{ss} . Hence $h^s = h^{ws} + h^{ss}$ is in $C^\infty(\mathbb{T}^d)$ and a diffeomorphism along \mathcal{W}^s .

(2') \implies (3). The argument is almost identical to the proof of (3) of Theorem 1.1. We map the foliation \mathcal{W}^u to W^u by the C^∞ diffeomorphism

$$\tilde{h}(x) = h(x) - \Delta^u(x) = x + \Delta^s(x) = x^u + h^s(x) : \mathbb{T}^d \rightarrow \mathbb{T}^d.$$

This map is C^0 close to the identity and is in $C^\infty(\mathbb{T}^d)$ since h^s is in $C^\infty(\mathbb{T}^d)$ by (2). Invertibility of its derivative $D\tilde{h}$ follows as in Theorem 1.1 from the fact that h^s is a diffeomorphism along \mathcal{W}^s .

(3) \implies (2). This implication requires only that W^u has dense leaves, which is always true for a hyperbolic automorphism. The proof highlights usefulness of conjugacy to a linear foliation. It is an easy version of the holonomy argument where normal form polynomials are replaced by isometries.

Let ϕ be a C^∞ diffeomorphism such that $\phi(\mathcal{W}^u) = W^u$ and let $\mathcal{W}' = \phi(\mathcal{W}^s)$. Since W^u is linear, its holonomies between W^s leaves are isometries with respect to the standard metric on \mathbb{T}^d . For each $x \in \mathbb{T}^d$ both $T_x W^s$ and $T_x \mathcal{W}'$ are transverse to $E^u = TW^u$, and so we can identify them by the projection along E^u . This defines a continuous Riemannian metric g' on $T\mathcal{W}'$ for which holonomies of W^u between \mathcal{W}' leaves are isometries. Moreover, since the leaves of \mathcal{W}^s are uniformly C^∞ , so are the leaves of \mathcal{W}' , and hence g' is uniformly C^∞ on the leaves of \mathcal{W}' . Then $g = \phi_*^{-1}(g')$ defines a continuous Riemannian metric on $T\mathcal{W}^s$ which is uniformly C^∞ along the leaves of \mathcal{W}^s and with respect to which the holonomies of \mathcal{W}^u are isometries.

We note that for each x the isometries of $(\mathcal{W}^s(x), g)$ are C^∞ diffeomorphisms of $\mathcal{W}^s(x)$ and form a finite dimensional Lie group G_x . Since we have that the holonomies

of \mathcal{W}^u are isometries, repeating the holonomy argument as in the proof of Theorem 1.1, we see that $h|_{\mathcal{W}^s(x)}$ conjugates the action of E^s by translations of $W^s(h(x))$ to a continuous action of E^s by isometries in G_x . This yields that $h|_{\mathcal{W}^s(x)}$ are uniformly C^∞ diffeomorphisms.

9. PROOF OF THEOREM 1.6

(1) \implies (2, 3, 4) is clear.

(3) \implies (2) Let $0 < \rho < 1$ be the largest absolute value of eigenvalues of L on E^{ss} and $0 < \rho' < 1$ be the smallest absolute value of eigenvalues of L on E^{ws} . Then if f is C^1 -close to L and $y \in \mathcal{W}^{ss}(x)$ then $\text{dist}(f^n x, f^n y) \leq C(\rho + \varepsilon)^n$ for all $n \in \mathbb{N}$. If h is α -Hölder then for all $n \in \mathbb{N}$

$$\text{dist}(L^n h(x), L^n h(y)) = \text{dist}(h(f^n x), h(f^n y)) \leq C' \text{dist}(f^n x, f^n y)^\alpha \leq C' C^\alpha (\rho + \varepsilon)^{\alpha n}.$$

If α is sufficiently close to 1 so that $(\rho + \varepsilon)^\alpha < \rho'$, this implies that $h(y) \in W^{ss}(h(x))$ and thus $h(\mathcal{W}^{ss}(x)) \subset W^{ss}(h(x))$. Since h is a homeomorphism, $h(\mathcal{W}^{ss}(x))$ contains an open ball in $W^{ss}(h(x))$, and then iterating by f yields $h(\mathcal{W}^{ss}(x)) = W^{ss}(h(x))$.

(4) \implies (2) under the density of leaves assumption for Lyapunov subfoliations of W^{ws+wu} follows from Theorem 1.1 (4) \implies (1) applied to f for \mathcal{W}^{ws} and to f^{-1} for \mathcal{W}^{wu} , yielding $h(\mathcal{W}^{ss}) = W^{ss}$ and $h(\mathcal{W}^{uu}) = W^{uu}$ respectively.

(2) \implies (1). Applying Theorem 1.1 (1) \implies (3) we obtain that the bundle \mathcal{E}^{u+ss} is C^∞ . Similarly using f^{-1} we obtain that \mathcal{E}^{s+uu} is C^∞ and hence so is the intersection $\mathcal{E}^{s+uu} \cap \mathcal{E}^{u+ss} = \mathcal{E}^{ss+uu}$. Since \mathcal{E}^{ss+uu} and \mathcal{E}^{ws+wu} are symplectic orthogonal, we conclude that \mathcal{E}^{ws+wu} and hence the corresponding foliation \mathcal{W}^{ws+wu} are C^∞ . This is the only place where we use that f preserving a C^∞ symplectic form. Since the foliation \mathcal{W}^{ws+wu} is C^∞ , by intersecting it with \mathcal{W}^s and \mathcal{W}^u we obtain that \mathcal{W}^{ws} and \mathcal{W}^{wu} are their respective uniformly C^∞ subfoliations. Thus we obtain that Theorem 1.5(1) is satisfied for both f and f^{-1} and hence yields (2') in each case. Combining them we conclude that $h = h^s + h^u$ is in $C^\infty(\mathbb{T}^d)$ with invertible derivative.

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