Topological entropy (continued)

Examples of calculation

- Last time: If $f: X \to X$ is a contraction or an isometry, then $h(f) = 0$.
  In particular, circle rotations and translations on $\mathbb{T}^n$ have top. entropy 0.

- Full one- and two-sided shifts on m symbols.
  Consider $\sigma_m : \Omega_m \to \Omega_m$ with $d(\omega, \omega') = 2^{-\min\{|i| : \omega_i \neq \omega'_i\}}$. Let $\varepsilon^k = 2^{-k}$.
  Recall that $B_d(\omega, 2^{-k}) = C^{-k,\dots,k}_{\omega_{-k},\dots,\omega_k}$. For any $n \in \mathbb{N}_0$, $d_n(\omega,\omega') < 2^{-k} \Leftrightarrow$
  $\Leftrightarrow \sigma^i(\omega')$ coincides with $\sigma^i(\omega)$ at the indices $-k,\dots,k$ for $i = 0,\dots,n-1$.
  Therefore, $B_{d_n}(\omega, 2^{-k}) = C^{-k,\dots,k,\dots,k+n-1}_{\omega_{-k},\dots,\omega_{k+n-1}}$.
  There are m^{2k+n} distinct balls like this, they cover Ω_m, and their d_n-diameters $(\sup\{d_n(\omega',\omega'') : \omega',\omega'' \in B_{d_n}(\omega, 2^{-k})\})$ equal $2^{-k-1} < 2^{-k}$.
  Since any two points in distinct balls are $\geq 2^{-k}$ apart in d_n,
  at least m^{2k+n} sets of d_n-diameter $< 2^{-k}$ are needed to cover Ω_m.
  Thus $Cov(\sigma_m, 2^{-k}, n) = m^{2k+n}$. Hence $h_{cov}(\sigma_m, 2^{-k}) = \lim_{n\to\infty} \frac{\ln(Cov(\sigma_m, 2^{-k}, n))}{n} = \ln m$.
  and $h(\sigma_m) = \lim_{k\to\infty} h_{cov}(\sigma^m, 2^{-k}) = \ln m$. $\square$
  Similarly, $h(\sigma_m^{(R)}) = \ln m$.

<u>Note</u> We can also show that $Span(\sigma_m, 2^{-k}, n) = m^{2k+n}$.
For each word $\omega_{-k},\dots,\omega_k,\dots,\omega_{k+n-1}$, choose one sequence ω.
These sequences form an 2^{-k}-spanning set in (Ω_m, d_n),
and there are no 2^{-k}-spanning set with fewer elements (why?)

<u>Note</u> For Cov, one can show that for any $f: X \to X$ (with standard assumptions)
for any $\varepsilon > 0$, $\lim_{n\to\infty} \frac{\ln(Cov(f,\varepsilon,n))}{n}$ exists. (See lemma 3.1.5)
The main idea is that the sequence $a_n = \ln(Cov(f,\varepsilon,n))$ is subadditive, i.e. $a_{m+n} \leq a_m + a_n$, and hence $(\frac{a_n}{n})$ converges.

- Subshift of finite type (top. Markov chain) $\sigma_A^{(R)} : \Omega_A^{(R)} \to \Omega_A^{(R)}$
  As always, we assume that A has no zero columns or rows. (*)
  By the previous discussion,
  $Cov(\sigma_A, 2^{-k}, n)$ = the number of <u>non-empty</u> cylinders $C^{-k,\dots,k+n-1}_{\omega_{-k},\dots,\omega_{k+n-1}}$ in Ω_A
  <u>by (*)</u> the number of admissible words of length $2k+n$
  $= \sum_{i,j=0}^{m-1}$ (the number of admissible words $i \dots j$ of length $2k+n$).

So $\mathrm{Cov}(\sigma_A, 2^{-k}, n) = \sum_{i,j=0}^{m-1} a_{ij}^{(n+2k-1)} = \sum_{i,j=0}^{m-1} (A^{n+2k-1})_{ij}$

Since all $a_{ij} \geq 0$, all $a_{ij}^{(\ldots)} \geq 0$, and so $\sum a_{ij}^{(\ldots)} = \sum |a_{ij}^{(\ldots)}|$.

Both $\sum_{i,j=0}^{m-1} |x_{ij}|$ and the operator norm $\|(x_{ij})\|$ of the matrix (x_{ij}) are norms on $\mathbb{R}^{m^2}$. Since any two norms on $\mathbb{R}^{m^2}$ are equivalent, there exist $c, c' > 0$ s.t. $c\|A^{n+2k-1}\| \leq \sum_{i,j=0}^{m-1} a_{ij}^{(\ldots)} \leq c' \|A^{n+2k-1}\|$.

Fact (see an outline below) For a square matrix A,

$\lim_{N\to\infty} \|A^N\|^{1/N}$ = the spectral radius of $A = \max\{|\lambda| : \lambda$ is an e.value of $A\}$. (*)

Hence $h_{cov}(\sigma_A, 2^{-k}) = \lim_{n\to\infty} \frac{1}{n} \ln(\mathrm{Cov}(\sigma_A, 2^{-k}, n)) = \lim_{n\to\infty} \frac{1}{n} \ln \|A^{n+2k-1}\| =$

$= \lim_{n\to\infty} \frac{n+2k-1}{n} \cdot \frac{1}{n+2k-1} \ln \|A^{n+2k-1}\| = \ln\left(\lim_{n\to\infty} \|A^{n+2k-1}\|^{\frac{1}{n+2k-1}}\right) = \ln(|\lambda|_{max})$

Thus $h(\sigma_A) = \lim_{k\to\infty} h_{cov}(\sigma_A, 2^{-k}) = \ln(|\lambda|_{max})$. Similarly, $h(\sigma_A^{(R)}) = \ln(|\lambda|_{max})$ □.

Note For σ_m and σ_A with primitive A, we see that $p(\sigma_m) = \ln m = h(\sigma_m)$ and $p(\sigma_A) = \ln(\lambda_{max}) = \ln(|\lambda|_{max}) = h(\sigma_A)$.
It is not true in general that $p(f) = h(f)$, and even that $\leq, \geq$.
However we have:

Prop If f is an expansive homeomorphism (or just a continuous map) of a compact metric space X, then $p(f) \leq h(f)$.

Pf let δ_0 be s.t. if $x \neq y$, then $d(f^n(x), f^n(y)) \geq \delta_0$ for some n.
Then the set of periodic pts of period n in δ_0-separated in (X, d_n).
Indeed, if $d(f^k(x), f^k(y)) < \delta_0$ for $k = 0, .., n-1$ for n-periodic x, y, then $d(f^k(x), f^k(y)) < \delta_0$ for all k, and hence $x = y$.
Thus $P_n(f) \leq \mathrm{Sep}(f, \delta_0, n) \leq \mathrm{Sep}(f, \varepsilon, n)$ for all $\varepsilon \leq \delta_0$, and it follows that $p(f) \leq h(f)$. □.

? We defined h using a metric d on X. Does $h(f)$ depend on the metric or only on the topology induced by the metric?

(*) ($\geq$) let λ have the largest $|..|$. Since λ^N is an e.value of A^N, $\|A^N\| \geq |\lambda|^N$.

($\leq$) Suppose that $|\mu| > \lim \|A^N\|^{1/N}$. Then there is $0 < c < 1$ s.t. for all suff. large N, $c|\mu| > \|A^N\|^{1/N}$, and hence $\frac{\|A^N\|}{|\mu|^N} < c^N$. It follows that the series $\sum_{N=0}^{\infty} \frac{A^N}{\mu^N}$ converges and gives the inverse of $(I - \frac{1}{\mu}A)$. Thus $A - \mu I$ is invertible, and μ is not an e.value.