Flows (continued)

A flow $\{\varphi^t\}$ is top. mixing if for any non-empty open sets $U, V \subset X$ there is $T>0$ such that $\varphi^t(U) \cap V \neq \emptyset$ for all $t > T$.

Prop. A flow $\{\varphi^t\}$ is top. mixing $\Leftrightarrow$ its time-one map φ^1 is top. mixing.

Pf ($\Rightarrow$) Clear.

($\Leftarrow$) Given U, V, we will show that for a large m and all $n \geq N$, $\varphi^{n/m}(U)$ intersects a small ball in V and hence $\varphi^t(U) \cap V \neq \emptyset$ for all $t > N$.

Lemma Let $f: X \to X$ be continuous. If f^m is top. mixing for some $m \in \mathbb{N}$, then f is top. mixing.

Pf Let $U, V \subset X$ be open and non-empty. Since f^m is top. mixing, there exists $N \in \mathbb{N}$ s.t. for every $n \geq N$ we have (Explain!)

for $i = 0, \dots, m-1$, $(f^m)^n(f^{-i}(U)) \cap V = f^{mn-i}(U) \cap V \neq \emptyset$.

Thus $f^k(U) \cap V \neq \emptyset$ for all $k \geq mN$ $\square$.

Now, let $U, V \subset X$ be open and non-empty. Take a ball $B_\varepsilon(x) \subset V$.
We choose $m \in \mathbb{N}$ s.t. $\|\varphi^s - Id\|_{C^0} < \frac{\varepsilon}{2}$ for all $0 \leq s \leq \frac{1}{m}$.
Since φ^1 is top. mixing, so is $\varphi^{1/m}$ by the Lemma.
Hence there is $N \in \mathbb{N}$ s.t. for all $n \geq N$, $\varphi^{n/m}(U) \cap B_{\varepsilon/2}(x) \neq \emptyset$.
Let $t \geq N$. We write $t = \frac{n}{m} + s$, where $0 \leq s \leq \frac{1}{m}$. Since $\varphi^t(U) = \varphi^s(\varphi^{\frac{n}{m}}(U))$, we have $\varphi^t(U) \cap B_\varepsilon(x) \neq \emptyset$, and hence $\varphi^t(U) \cap V \neq \emptyset$. $\square$

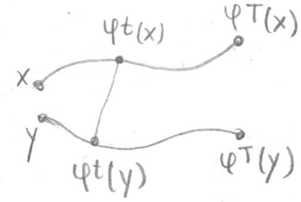
Recurrence properties for $\{\varphi^t\}$

- ω- and α-limit sets. Let $x \in X$. We say that $y \in \omega(x)$ / $\alpha(x)$ if there is a sequence $t_k \to \infty$ / $t_k \to -\infty$ s.t. $\varphi^{t_k}(x) \to y$
- We say that x is positively/negatively recurrent if $x \in \omega(x) / x \in \alpha(x)$.
- Recall: For a map $f: X \to X$, a point x is nonwandering if for any open set $U \ni x$ there is $n \in \mathbb{N}$ s.t. $f^n(U) \cap U \neq \emptyset$, equivalently, there is an arbitrarily large such n.

For a flow $\{\varphi^t\}$, requiring that for each $U \ni x$ there is $t > 0$ s.t. $\varphi^t(U) \cap U \neq \emptyset$ would not yield a meaningful definition since for any sufficiently small $t > 0$, $\varphi^t(U) \cap U \neq \emptyset$.

Def We say that a point $x \in X$ is <u>nonwandering</u> for a flow $\{\varphi^t\}$ if for any open set $U \ni x$ and any $T > 0$ there is $t > T$ such that $\varphi^t(U) \cap U \neq \emptyset$.

Topological entropy for a flow $\Phi = \{\varphi^t\}$



For every $T \geq 0$ we define a distance d_T^{Φ} on X:

$$d_T^{\Phi}(x,y) = \max_{0 \leq t \leq T} d(\varphi^t(x), \varphi^t(y))$$

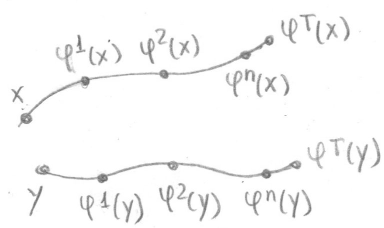
The definition of top. entropy for a flow proceeds as the definition for a map. We define Sep (or Span or Cov) (Φ, ε, T), then take its exponential growth rate as $T \to \infty$, and then take the limit as $\varepsilon \to 0^+$.

So $$h(\Phi) = \lim_{\varepsilon \to 0^+} \limsup_{T \to \infty} \frac{\ln(\text{Sep}/\text{Span}/\text{Cov}(\Phi, \varepsilon, T))}{T}$$

Prop $h(\Phi) = h(\varphi^1)$, so we can define $h(\Phi)$ as $h(\varphi^1)$.

Lemma For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $d(x,y) < \delta$, then $\max\limits_{0 \leq t \leq 1} d(\varphi^t(x), \varphi^t(y)) < \varepsilon$.

Pf Suppose that for some $\varepsilon > 0$ there is no such δ. Then for each $n \in \mathbb{N}$ there are $x_n, y_n \in X$ with $d(x_n, y_n) < \frac{1}{n}$ and $t_n \in [0,1]$ s.t. $d(\varphi^{t_n}(x_n), \varphi^{t_n}(y_n)) \geq \varepsilon$. By compactness of $X \times X \times [0,1]$, there is a convergent subsequence $(x_{n_k}, y_{n_k}, t_{n_k}) \to (x_*, x_*, t_*)$. Using continuity of the map $(x,t) \mapsto \varphi^t(x)$, we obtain a contradiction. $\square$



Pf of the Prop. Let $n = \lfloor T \rfloor$. Then by the Lemma $d_{n+1}^{\varphi^1}(x,y) < \delta(\varepsilon)$ implies $d_T^{\Phi}(x,y) < \varepsilon$.

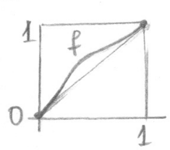
Hence $B_{d_{n+1}^{\varphi^1}}(x, \delta(\varepsilon)) \subseteq B_{d_T^{\Phi}}(x, \varepsilon)$,

and it follows that $h(\varphi^1) \geq h(\Phi)$.

Clearly, $B_{d_n^{\Phi}}(x, \varepsilon) \subseteq B_{d_{n+1}^{\varphi^1}}(x, \varepsilon)$. It follows that $h(\Phi) \geq h(\varphi^1)$. $\square$

Question. For any flow $\{\varphi^t\}$ we can consider its time-one map φ^1. Given a homeomorphism (or a diffeomorphism) f of X, does there exist a flow $\{\varphi^t\}$ on X such that $f = \varphi^1$?

Does such $\{\varphi^t\}$ exist for f



for example, $f(x) = x + \frac{1}{10}\sin^2(\pi x)$?