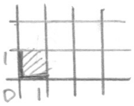
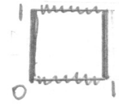
Translations and linear flows on the torus

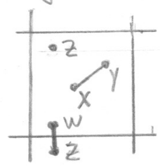
The n-dimensional torus is $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \underbrace{(\mathbb{R}/\mathbb{Z}) \times \ldots \times (\mathbb{R}/\mathbb{Z})}_{n} = \underbrace{S^1 \times \ldots \times S^1}_{n}$

$\mathbb{T}^2$ We identify all squares  in $\mathbb{R}^2$

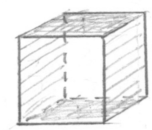
$\mathbb{T}^2$ can be viewed as the unit square  with the opposite sides identified.

Topologically, it is "the surface of a doughnut" 

but the distance comes from the plane



Similarly, $\mathbb{T}^n$ can be viewed as the unit cube in $\mathbb{R}^n$ with the opposite faces identified:



for each $i = 1, \ldots, n$, $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \sim (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$

$\mathbb{T}^n$ with addition mod 1 is a compact abelian topological group.

We will usually use the additive representation of $\mathbb{T}^n$:

$\mathbb{T}^n$ is the set of vectors $(x_1, \ldots, x_n)$, where each coordinate is defined mod 1.

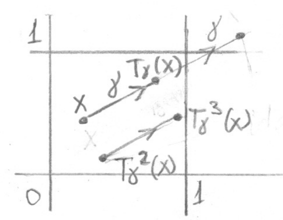
$\mathbb{T}^n$ can also be represented in multiplicative notations:

$$\mathbb{T}^n = \{(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n}) : x_1, \ldots, x_n \in \mathbb{R}\}$$

Translations and linear flows.

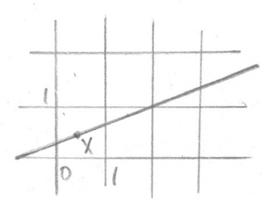
Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$. The map $T_\gamma : \mathbb{T}^n \to \mathbb{T}^n$ given by $T_\gamma(x_1, \ldots, x_n) = (x_1 + \gamma_1, \ldots, x_n + \gamma_n) \bmod 1$ is called a translation on the torus.

(it is a generalization of $R_\alpha : S^1 \to S^1$)

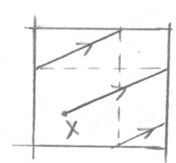


The family of maps $\{T_\gamma^t : t \in \mathbb{R}\}$ given by $T_\gamma^t(x_1, \ldots, x_n) = (x_1 + t\gamma_1, \ldots, x_n + t\gamma_n) \bmod 1$ is called a linear flow on the torus.

The point moves along the line spanned by γ with constant speed.



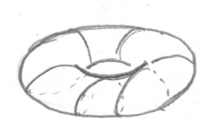
The orbit of x is $\{T_\gamma^t(x) : x \in \mathbb{R}\}$



Note For both T_γ and $\{T_\gamma^t\}$, the orbits of any two points differ by a translation.

So if one is periodic or dense, so is the other.

We can consider the orbit of 0.

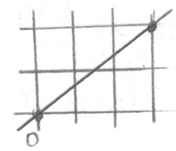


Periodic orbits

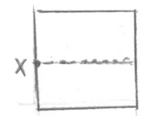
Every point is periodic for the translation $T_\gamma \Leftrightarrow$
there is $m \in \mathbb{N}$ s.t. $(m\gamma_1, \dots, m\gamma_n) \in \mathbb{Z}^n \Leftrightarrow \gamma_1, \dots, \gamma_n$ are rational.

We say that x is periodic for the flow if there is $t>0$ s.t. $T_\gamma^t(x) = x$.
Every point is periodic for the flow if there is $t>0$ s.t. $(t\gamma_1, \dots, t\gamma_n) \in \mathbb{Z}^n$.

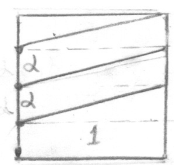
For the flow on $\mathbb{T}^2$ this means the line through 0 spanned by γ hits a point with integer coordinates.
This happens if and only if
either $\gamma_1 = 0$ (i.e. the line is vertical)
or $\frac{\gamma_2}{\gamma_1}$ = the slope of the line is rational.



? Can the translation $T_\gamma : \mathbb{T}^2 \to \mathbb{T}^2$ have orbits that are <u>neither periodic nor dense</u>? <u>Yes</u> let $\gamma = (\sqrt{2}, 0)$.
Then for each $x \in S^1$, the circle $S^1 \times \{x\}$ is invariant, we have an irrational rotation in this circle, and so the orbit of x is dense there.



? Can the flow $\{T_\gamma^t\}$ on $\mathbb{T}^2$ have such orbits?
<u>No</u> If orbits are not periodic, then $\alpha = \frac{\gamma_2}{\gamma_1} \notin \mathbb{Q}$.
The flow induces an irrational rotation on $\{0\} \times S^1$, and it follows that the orbit is dense.



we obtain a dense set in $\{0\} \times S^1$

On $\mathbb{T}^n$, $n \geq 3$, a flow can have such orbits.
For example, let $\gamma = (\gamma_1, \gamma_2, 0)$, where $\frac{\gamma_2}{\gamma_1} \notin \mathbb{Q}$.

? What conditions on γ ensure top. transitivity (equivalently, minimality) of the translation T_γ on $\mathbb{T}^2$?
are these sufficient:

(1) $\gamma_1, \gamma_2 \notin \mathbb{Q}$ <u>No</u>. let $\gamma = (\sqrt{2}, \sqrt{2})$
then the orbit of 0 lies on the diagonal $\{(x,x)\}$

(2) $\frac{\gamma_2}{\gamma_1} \notin \mathbb{Q}$ <u>No</u> let $\gamma = (\frac{1}{2}, \sqrt{2})$
Then $T_\gamma^n(0) = (\frac{n}{2}, n\sqrt{2}) \bmod 1$, and so $x_1 \in \{0, \frac{1}{2}\}$.

?? (3) $\gamma_1, \gamma_2 \notin \mathbb{Q}$ and $\frac{\gamma_2}{\gamma_1} \notin \mathbb{Q}$
Both conditions are necessary (Why?)
are they sufficient? <u>Explain</u>.