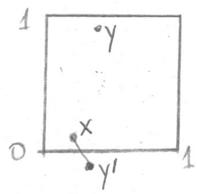
Remarks on distance on $\mathbb{T}^2$ versus the distance on $\mathbb{R}^n$.

For $x, y \in \mathbb{T}^2$, $d_{\mathbb{T}^2}(x,y) = \min\{ d_{\mathbb{R}^2}(x',y') = \|x-y\| : \pi(x') = x \text{ and } \pi(y') = y\}$

We can take $x' = x \in [0,1) \times [0,1)$ and the closest y' that is identified with y. The maximal $d_{\mathbb{T}^2}(x,y)$ is $1/\sqrt{2}$.



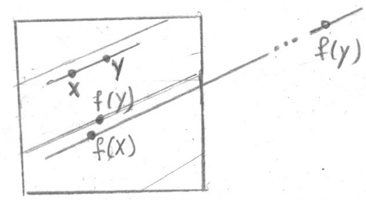
If for $x', y' \in \mathbb{R}^2$ we have $\|x'-y'\| < \frac{1}{2}$, then for $x = \pi(x')$ and $y = \pi(y')$ we have $d_{\mathbb{T}^2}(x,y) = d_{\mathbb{R}^2}(x',y') = \|x-y\|$.

So for any small disc D in $\mathbb{R}^2$, the distances in $\pi(D)$ are the same as in D.

Consider a hyperbolic automorphism $f(x) = Ax \mod 1$. Note that A can have an arbitrarily large eigenvalue λ. Take, for example, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^N$.

For two nearby pts x and y on the projection of the expanding line, we can be sure that $d(f(x), f(y)) = |\lambda|\, d(x,y)$ only if $|\lambda| \cdot d(x,y) < \frac{1}{2}$.

Otherwise we can have the following picture:

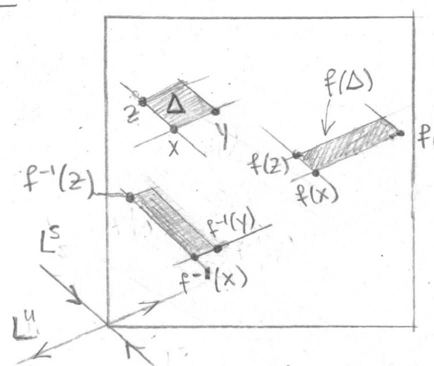


Coding of a hyperbolic automorphism of $\mathbb{T}^2$

We will choose sets Δ_i to be rectangles or paralleograms with sides parallel to the expanding and contracting lines

Their images under f and f^{-1} look like this $\rightarrow$

$f(\Delta)$ and $f^{-1}(\Delta)$ are also parallelograms with sides parallel to L^u and L^s.



The length of the side parallel to L^u $[L^s]$ increases by a factor of $|\lambda|$ [decreases by a factor of $1/|\lambda|$] under f.

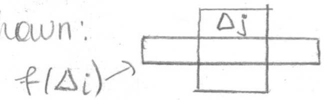
Let us consider Arnold's Cat Map, $f(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x \mod 1$.

Since the matrix is symmetric, the lines L^u and L^s are orthogonal

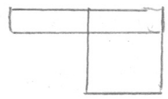
We want to obtain a semiconjugacy h from $\sigma_A : \Omega_A \to \Omega_A$ to $f : \mathbb{T}^2 \to \mathbb{T}^2$ by choosing a suitable partition $\Delta_0, \dots, \Delta_{m-1}$ of $\mathbb{T}^2$ and then setting
$$h((\omega_n)_{n\in\mathbb{Z}}) = \bigcap_{n=-\infty}^{\infty} f^{-n}(\Delta_{\omega_n}).$$

In Ω_A we have: if $a_{ij} = 1$ and $a_{jk} = 1$, i.e. ij and jk are admissible, then ijk is also admissible. So the partition $\{\Delta_i\}$ should have the property that if $f(\Delta_i) \cap \Delta_j \neq \emptyset$ and $f(\Delta_j) \cap \Delta_k \neq \emptyset$, then $f^2(\Delta_i) \cap f(\Delta_j) \cap \Delta_k \neq \emptyset$., and a similar property for f^{-1}.

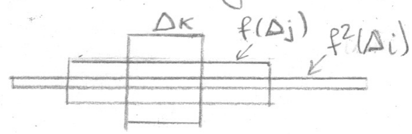
These properties hold if whenever $f(\Delta_i) \cap \Delta_j \neq \emptyset$, $f(\Delta_i)$ goes all the way across Δ_j, as shown:



or

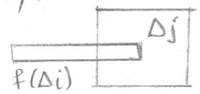


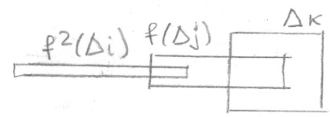
. Then



This is referred to as Markov property.

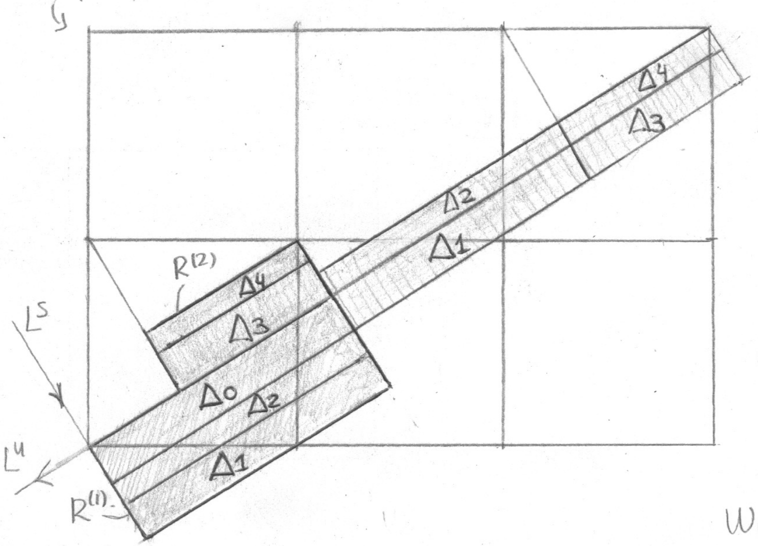
If it does not hold, we can have





To obtain a good partition for Arnold's Cat Map, we first draw L^u and L^s through the vertices on $[0,1]\times[0,1]$ and divide it into two rectangles $R^{(1)}$ and $R^{(2)}$. We draw their images and see that $f(R^{(1)})$ has two pieces in $R^{(1)}$, which would yield more than one x for ω. So we subdivide $R^{(1)}$ and $R^{(2)}$ into rectangles $\Delta_0, \dots, \Delta_4$, for which we have good intersections $f(\Delta_i) \cap \Delta_j$.

(pictures p. 84, 86 of the book)



$f(R^{(1)}) = \Delta_0 \cup \Delta_1 \cup \Delta_3$

$f(R^{(2)}) = \Delta_2 \cup \Delta_4$

Setting $a_{ij} = 1$ if $f(\Delta_i) \cap \Delta_j \neq \emptyset$ (more precisely, if their interiors intersect) and $a_{ij} = 0$ otherwise, we obtain

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

We consider $\sigma_A : \Omega_A \to \Omega_A$.

For each $\omega = (\omega_n)_{n\in\mathbb{Z}}$ in Ω_A, we set $h(\omega) = \bigcap_{n=-\infty}^{\infty} f^{-n}(\Delta\omega_n)$.

The rectangles are contracted by f in the direction of L^s by a factor of $1/\lambda$,

and they are contracted by f^{-1} —— " —— L^u —— " ——

It follows that the diameters of the rectangles $\bigcap_{n=-K}^{K} f^{-n}(\Delta\omega_n)$ tend to 0.

By the Markov property of the partition, these rectangles are non-empty.

It follows that $\bigcap_{n=-\infty}^{\infty} f^{-n}(\Delta\omega_n)$ consists of exactly one point.

The map $h : \Omega_A \to \mathbb{T}^2$ is continuous and surjective, and satisfies $h \circ \sigma_A = f \circ h$. So it is a semiconjugacy. Let $\Gamma = \partial R^{(1)} \cup \partial R^{(2)}$.

If the orbit of x never hits Γ, i.e. $x \notin \bigcup_{n\in\mathbb{Z}} f^{-n}(\Gamma)$, then x corresponds to a unique sequence. The Lebesgue measure of the set of such x equals 1.