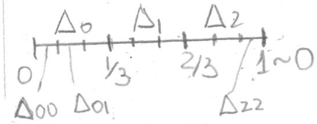
Coding

Example Recall the discussion of $E_3: S^1 \to S^1$, $E_3(x) = 3x \mod 1$



Writing numbers in base 3, we obtained a semiconjugacy h from $\sigma_3: \Omega_3^R \to \Omega_3^R$ to $E_3: S^1 \to S^1$,
$h(\omega_0\omega_1\ldots) = 0.\omega_0\omega_1\omega_2\ldots_3$.

$x = 0.\omega_0\omega_1\ldots_3 \Longleftrightarrow x \in \Delta_{\omega_0} \cap \Delta_{\omega_0\omega_1} \cap \Delta_{\omega_0\omega_1\omega_2} \cap \ldots \Rightarrow x \in \Delta_{\omega_0},\ E_3(x) \in \Delta_{\omega_1},\ E_3^2(x) \in \Delta_{\omega_2}, \ldots$

$\Leftarrow$ for $x \neq k/3^l$. Let $x = \frac{1}{3}$, then $E_3^n(x) \in \Delta_0$ for all n, but $x \neq 0.0000\ldots_3$

So we need to be careful with the endpoints of the intervals.

Using this semiconjugacy, we found a point with dense orbit and an uncountable closed invariant set, the Cantor set on S^1. $h: \Omega^R_{\{0,2\}} \to C$ is almost one-to-one.

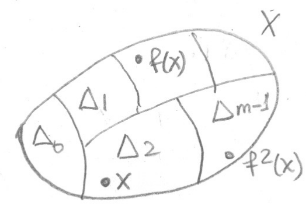
Coding in general

Goal: given a dynamical system $f: X \to X$, obtain a top. conjugacy or semiconjugacy $h: \Omega \to X$ from a symbolic dyn. system $\sigma: \Omega \to \Omega$ to $f: X \to X$. Usually, $\sigma: \Omega \to \Omega$ is either $\sigma_m^{(R)}: \Omega_m^{(R)} \to \Omega_m^{(R)}$ or $\sigma_A^{(R)}: \Omega_A^{(R)} \to \Omega_A^{(R)}$.

Why? We understand σ_m and σ_A, and if there is a semiconjugacy, then

- top. transitivity/mixing of σ implies that for f,
- $h_{top}(f) \le h_{top}(\sigma)$, $=$ if conjugacy,
- h(closed invariant set for h) $=$ —"— for f.

General approach



Partition X into sets $\Delta_0, \ldots, \Delta_{m-1}$ (that may overlap along their boundaries). To each $x \in X$, assign (ω_n) if $f^n(x) \in \Delta_{\omega_n}$ for every n, i.e. $x \in \bigcap_n f^{-n}(\Delta_{\omega_n})$.

Some difficulties arise because of common boundaries.

Want: for each $\omega \in \Omega$ there is exactly one such x, i.e. $\bigcap_n f^{-n}(\Delta_{\omega_n}) = \{x\}$. Then we set $h(\omega) = x$. Also, we want h to be one-to-one on a large set.

Note With this construction, $\Delta_i = h(C_i^0)$, where C_i^0 is a cylinder in Ω. Thus Δ_i have to be closed, and so common boundaries are likely.

If X is connected, e.g. S^1 or $\mathbb{T}^N$, we cannot obtain a conjugacy as $\Omega_{m/A}$ is totally disconnected. But we can have conjugacy for the restriction to a Cantor-like invariant set.

One can obtain a semiconjugacy for several classes of systems.

Coding of expanding maps of S^1

- **Lift and degree.** Let $f: S^1 \to S^1$ be continuous. A *lift* of f to $\mathbb{R}$ is a cont. map $F: \mathbb{R} \to \mathbb{R}$ s.t. $\pi \circ F = f \circ \pi$, where $\pi(x) = x \bmod 1$.

Ex For $f = E_m$, $F(x) = mx + k$, $k \in \mathbb{Z}$.

Any two lifts of f differ by an integer. The number $F(x+1) - F(x)$ is independent of x and the lift. (Why?). It is called the *degree* of f.

Ex The degree of E_m is m.

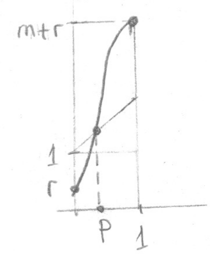
- Now, let $f: S^1 \to S^1$ be a continuous expanding map of degree $m \geq 2$. (Expanding: there is $\lambda > 1$ s.t. for all suff. close x, y, $d(f(x), f(y)) \geq \lambda\, d(x,y)$)

We want to find a suitable partition of S^1 into intervals.

Any lift of F is strictly increasing (Why?)

Let F be the lift with $F(0) \in [0,1)$. Then the graph of F intersects the line $y = x+1$, and hence f has a fixed pt. p.

Conjugating by a rotation, we can assume that $p = 0$.



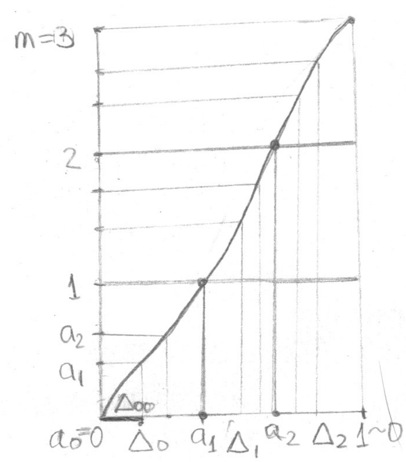
Let $0 = a_0 < a_1 < \ldots < a_{m-1} < 1$ be the points s.t. $F(a_k) = k$, $k = 0, \ldots, m-1$

Let $\Delta_k = [a_k, a_{k+1}]$, $k = 0, \ldots, m-1$.

Each Δ_k maps onto S^1, and $|\Delta_k| \leq 1/\lambda$.

In each Δ_k, there are uniquely determined points $a_k = a_0^{(k)} < a_1^{(k)} < \ldots < a_{m-1}^{(k)} < a_{k+1}$ that are mapped to $a_0, a_1, \ldots, a_{m-1}$ mod 1, respectively.

Thus we obtain intervals Δ_{ki}, $i = 0, \ldots, m-1$ s.t. $f(\Delta_{ki}) = \Delta_i$, and $|\Delta_{ki}| \leq 1/\lambda^2$.



We continue the process. At step ℓ we have m^ℓ intervals of length $\leq 1/\lambda^\ell$.

(!) Careful: $f^{-1}(\Delta_0) = \Delta_{00} \cup \{a_1\}$, so when we define h we discard such pts. Instead of $f^{-1}(\Delta_0)$, we take just $\Delta_{00} = \overline{\operatorname{int}(f^{-1}(\Delta_0))}$, and similarly for other preimages containing an "extra" point.

So given $\omega = (\omega_n)_{n \geq 0} \in \Omega_m^{(R)}$, we set $h(\omega) = \Delta_{\omega_0} \cap \Delta_{\omega_0\omega_1} \cap \Delta_{\omega_0\omega_1\omega_2} \cap \ldots$

$h(\omega)$ is the intersection of a sequence of closed nested intervals with length $\to 0$. Hence $h(\omega)$ consists of exactly one point.

Thus $h(\omega)$ is well-defined. It is onto and continuous, since close sequences are mapped to close pts. Thus $h: \Omega_m^R \to S^1$ is a semiconjugacy.

$\square$.