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Last time: Corollary Let f be a homeomorphism of a compact metric space X. Then f is top. transitive $\Leftrightarrow$ there are no two disjoint non-empty open f-invariant sets in X.

Prop. Let f be a homeomorphism of a compact metric space X. If f is top. transitive, then there is no invariant non-constant continuous function $\varphi: X \to \mathbb{R}$. Invariant means that $\varphi(f(x)) = \varphi(x)$ for all $x \in X$.

Pf Let $\varphi: X \to \mathbb{R}$ be a continuous invariant function. Since f is invariant, it is constant on the orbit of any point. Let x be a point with dense orbit. Then φ is constant on $O(x)$. Since $O(x)$ is dense in X and f is continuous, it follows that φ is const. on X. □

Note • The converse is false (Hw) • Another proof: use the corollary above.

Minimality of translations and linear flows on $\mathbb{T}^n$

Prop. Let $\gamma = (\gamma_1, \dots, \gamma_n)$. The translation $T_\gamma : \mathbb{T}^n \to \mathbb{T}^n$ is top. transitive, equivalently, minimal $\Leftrightarrow$ $\gamma_1, \dots, \gamma_n, 1$ are rationally independent, that is, $\sum_{i=1}^n k_i \gamma_i \notin \mathbb{Z}$ for any integers $k_1, \dots, k_n$ except for $k_1 = \dots = k_n = 0$.

Pf (By contraposition, ($\Rightarrow$) uses prop. above, ($\Leftarrow$) uses corollary above)

($\Rightarrow$) Suppose that there exist $k_1, \dots, k_n$ not all 0 s.t. $\sum_{i=1}^n k_i \gamma_i = k \in \mathbb{Z}$.
Then there is a non-constant T_γ-invariant function:
Let $\varphi(x) = \sin\left(2\pi \sum_{i=1}^n k_i x_i\right)$. This function is well-defined on $\mathbb{T}^n$ since for each $v \in \mathbb{Z}^n$, $\varphi(x+v) = \varphi(x)$. Clearly, φ is cont. and non-constant.
φ is T_γ-invariant since $\varphi(T_\gamma(x)) = \sin\left(2\pi \sum_{i=1}^n k_i (x_i + \gamma_i)\right) =$
$= \sin\left(2\pi \left(\sum_{i=1}^n k_i x_i\right) + 2\pi k\right) = \varphi(x)$.
Thus by the prop. above, T_γ is not top. transitive.

($\Leftarrow$) Suppose that T_γ is not top. transitive. Then by the Corollary above there exists an open T_γ-invariant set U with Lebesgue measure $0 < \mu(U) < 1$. Consider its characteristic function $\chi = \chi_U : \mathbb{T}^n \to \mathbb{R}$.
Since U is invariant, $\chi(x) = \chi(T_\gamma(x))$ for all $x \in \mathbb{T}^n$.
We consider the Fourier expansion of χ:

$$\sum_{\bar{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n} \hat{\chi}(\bar{k})\, e^{2\pi i (\bar{k} \cdot x)}, \quad \text{where } \hat{\chi}(\bar{k}) = \int_{\mathbb{T}^n} \chi(x)\, e^{-2\pi i (\bar{k} \cdot x)} dx$$

The coefficients of the Fourier expansion of $\chi \circ T_\gamma$ are

$$\widehat{\chi \circ T_\gamma}(\bar{k}) = \int_{\mathbb{T}^n} \chi(T_\gamma(x))\, e^{-2\pi i (\bar{k} \cdot x)} dx = \int_{\mathbb{T}^n} \chi(x+\gamma)\, e^{-2\pi i (\bar{k} \cdot x)} dx \;\; \underline{\underline{y = x+\gamma}}$$

$$= \int_{\mathbb{T}^n} \chi(y)\, e^{-2\pi i \bar{k} \cdot (y-\gamma)} dy = e^{2\pi i (\bar{k} \cdot \gamma)}\, \hat{\chi}(\bar{k}).$$

Since $\chi \circ T_\gamma = \chi$ and the Fourier expansion is unique,
for each $\bar{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n$, $e^{2\pi i \bar{k}\cdot\gamma}\hat{\chi}(\bar{k}) = \hat{\chi}(\bar{k})$, and hence
either $\hat{\chi}(\bar{k})=0$ or $e^{2\pi i \bar{k}\cdot\gamma}=1$, i.e. $\sum_{i=1}^{n} k_i\gamma_i \in \mathbb{Z}$.
Since χ is not constant a.e., $\hat{\chi}(\bar{k})\neq 0$ for some $\bar{k}=0$, and so
$k_1,\dots,k_n,1$ are rationally dependent. □

Now, minimality of the linear flow $\{T_\gamma^t\}$.

Prop. Let $\{f^t\}$ be a flow on a compact metric space X.
If $\{f^t\}$ is top. transitive, then every continuous function $\varphi: X\to\mathbb{R}$
invariant under $\{f^t\}$ is constant. Invariant means: that $\varphi(f^t(x))=\varphi(x)$
for all $x\in X$ and $t\in\mathbb{R}$.

Pf As for $f: X\to X$.

Note If for some $t_0\in\mathbb{R}$ the map $f^{t_0}: X\to X$ is top. transitive, then
the flow is also top. transitive (Why?)

Proposition Let $\gamma=(\gamma_1,\dots,\gamma_n)$. The flow $\{T_\gamma^t\}$ on $\mathbb{T}^n$ is top. transitive,
equivalently, minimal $\iff$ $\gamma_1,\dots,\gamma_n$ are rationally independent, i.e.
$\sum_{i=1}^{n} k_i\gamma_i \neq 0$ for any intergers $k_1,\dots,k_n$ except for $k_1=\dots=k_n=0$.

Pf ($\Rightarrow$) Suppose that $\sum_{i=1}^{n} k_i\gamma_i = 0$ for some integers $k_1,\dots,k_n$, not all 0.
Then $\varphi(x)=\sin\left(2\pi \sum_{i=1}^{n} k_i x_i\right)$ is continuous, non-constant, and invariant
since for any $x\in X$ and $t\in\mathbb{R}$, $\varphi(x+t\gamma)=\sin\left(2\pi\sum k_i x_i + 2\pi t \underbrace{\sum k_i\gamma_i}_{=0}\right)=\varphi(x)$.
Thus the flow is not top. transitive.

($\Leftarrow$) Suppose that $\sum_{i=1}^{n} k_i\gamma_i \neq 0$ for any integers $k_1,\dots,k_n$ that are not all 0.
It suffices to show that for some $t\in\mathbb{R}$, the translation $T_\gamma^t = T_{t\gamma}$ is
top. transitive, equivalently, that there is $t\in\mathbb{R}$ such that
for any integer vector $(k_1,\dots,k_n)\neq 0$, $\sum_{i=1}^{n} k_i(t\gamma_i) = t\sum_{i=1}^{n} k_i\gamma_i \notin \mathbb{Z}$.
By the assumption, $\sum k_i\gamma_i \neq 0$. For any collection of integers $k_1,\dots,k_n,k$,
with $(k_1,\dots,k_n)\neq 0$, there is only one t such that $t\sum k_i\gamma_i = k$, namely $t=\frac{k}{\sum\dots}$
There are countably many such collections of integers, and uncountably
many values of t. Thus there is t s.t. $t\sum_{i=1}^{n} k_i\gamma_i\notin\mathbb{Z}$ for any $(k_1,\dots,k_n)\neq 0$.
□