Structural stability.

? Let f be a cont. map of a compact metric space. Is there $\varepsilon>0$ s.t. for any cont. $g: X\to X$ with $d_{C^0}(f,g)=\sup_{x\in X} d(f(x),g(x))<\varepsilon$, g is top. conjugate to f?

No. Making a C^0-small change one can, for example, get $g=Id$ in a nbhd. of a fixed point for f, change the number of periodic pts, get a non-invertible g for an invertible f, etc. So g would not be top. conjugate to f.
However, there is hope if Df and Dg are also close.

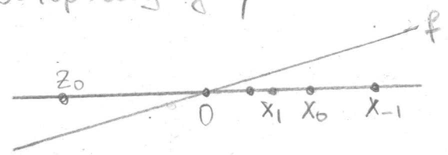
Let M be a compact smooth manifold, for example, S^1 or $\mathbb{T}^N$, and let $f: M\to M$ be a C^1 map (i.e. continuously differentiable, and hence cont.)

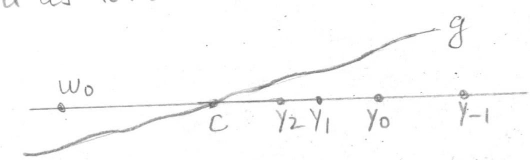
Def. We say that f is (C^1) structurally stable if for any C^1 map $g: M\to M$ sufficiently C^1-close to f, g is topologically conjugate to f.

Note. C^1-close means that g is close to f and Dg is close to Df.
Such g is called a C^1-small perturbation of f.
One can also consider a C^r, $r>1$, map f and call it C^m, $1\le m\le r$, structurally stable if for any g that is C^m close to f, g is top. conjugate to f.

Ex (with non-compact M). Let f be a linear contraction on $\mathbb{R}$, specifically, $f(x)=ax$, where $0<a<1$. Then f is structurally stable.

Pf Idea: any g suff. C^1-close to f is also a contraction.
Let $g:\mathbb{R}\to\mathbb{R}$ satisfy $|g'(x)-f'(x)|<\min\{\frac{a}{2},\frac{1-a}{2}\}$ for all x. (We can also assume that $g(x)$ is close to $f(x)$ for all x, but it plays no role in the argument.)
Then $0<\alpha\le g'(x)\le\beta<1$ for all $x\in\mathbb{R}$. Thus g is a contraction and hence has a unique fixed pt. c (close to 0 if g is close to f)
A top. conjugacy can be constructed as follows:





Set $h(0)=c$, choose $x=x_0\in(0,\infty)$ and $y=y_0\in(c,\infty)$ and set $h(x_n)=y_n$.
Let h be linear from $[x_1,x_0]$ onto $[y_1,y_0]$, and then extend h to (x_{i+1},x_i), $i\in\mathbb{Z}$, as in Lecture 12. Similarly define h on $(-\infty,0)$.

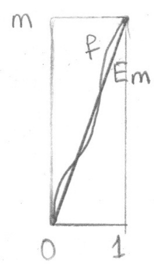
Note. A similar argument works for any $f:\mathbb{R}\to\mathbb{R}$ satisfying $0<a_1\le f'(x)\le a_2<1$ for all x.

Ex The following maps are not structurally stable:

- Id, since the only map top. conjugate to Id is Id;
- $R_\alpha : S^1 \to S^1$, $T_\gamma : \mathbb{T}^N \to \mathbb{T}^N$ (Hw)

Ex $E_m : S^1 \to S^1$, $m \geq 2$ is structurally stable.

Pf Let $f : S^1 \to S^1$ is a C^1-small perturbation of E_m ($f(x) \approx mx$ and $f'(x) \approx m$)
Such f has a fixed pt. c near 0 (Explain why).
Let $\tilde{f}(x) = f(x+c) - c$. Then $\tilde{f}(0) = 0$ and $\tilde{f}$ is top. conj to f (via $\tilde{h} = ?$)
Thus we can assume that $f(0) = 0$.



Consider the lifts E and F of E_m and f to $\mathbb{R}$ with $E(0) = F(0) = 0$ (Extend the maps from $[0, m]$ to $\mathbb{R}$). Then $E(x) = mx$ and $F(x) = mx + \varphi(x)$, where φ is a small 1-periodic function with $\varphi(0) = 0$.

First, we would like to show that there exists a continuous $h : S^1 \to S^1$ close to Id so that

① $E_m \circ h = h \circ f$ and $h(0) = 0$.

We write $h = Id + \tilde{h}$, where $\tilde{h}(0) = 0$, and consider its lift to $\mathbb{R}$ $Id + H$, where H is a 1-periodic function with $H(0) = 0$.
Thus we want to find H such that

② $E \circ (Id + H) = (Id + H) \circ F$, equivalently
$\cancel{mx} + mH(x) = \cancel{mx} + \varphi(x) + H(F(x))$ for all x.

Therefore, $H(x) = \frac{1}{m} H(F(x)) + \frac{1}{m} \varphi(x) =: T(H)(x)$, i.e. $H = T(H)$.
Let Y be the space of continuous 1-periodic functions on $\mathbb{R}$ with $g(0) = 0$.
The map T is a contraction on Y. Indeed, if $H \in Y$ then $T(H) \in Y$ and for any $H_1, H_2 \in Y$, $\|T(H_1) - T(H_2)\|_{C^0} = \sup_{x \in \mathbb{R}} |T(H_1(x)) - T(H_2(x))| =$
$= \frac{1}{m} \sup_{x \in \mathbb{R}} |H_1(F(x)) - H_2(F(x))| \leq \frac{1}{m} \sup_{z \in \mathbb{R}} |H_1(z) - H_2(z)| \leq \frac{1}{m} \|H_1 - H_2\|_{C^0}$
Since Y is a complete metric space, T has a unique fixed pt. $H = T(H)$ in Y.
Thus H is a solution of ② and hence $h = \pi(Id + H)$ is a solution of ①.

Note. Since φ is C^0-small, we can see that H is also C^0 small. It follows from $H = \lim_{n \to \infty} T^n(0)$ and $T(0) = \frac{1}{m} \varphi$.

Thus we obtained a semiconjugacy h. The argument above does not give its invertibility. We will prove that there exists $\hat{h} : S^1 \to S^1$ such that
$\hat{h} \circ E_m = f \circ \hat{h}$ and then show that $\hat{h} = h^{-1}$.