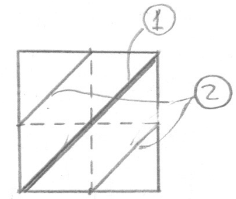
(T/F) If $\gamma_1, \gamma_2 \notin \mathbb{Q}$ and $\frac{\gamma_2}{\gamma_1} \notin \mathbb{Q}$, then the tran the translation $T_\gamma : \mathbb{T}^2 \to \mathbb{T}^2$ is top. transitive, equivalently, minimal.

(F) Let $\gamma_1 = \sqrt{2}$ and $\gamma_2 = \frac{1}{2} + \sqrt{2}$. Then
$T_\gamma^n(0)$ is either $n(\sqrt{2}, \sqrt{2})$ or $(0, \frac{1}{2}) + n(\sqrt{2}, \sqrt{2}) \mod 1$.
So the orbit of 0 lies in the union of two circles.



The assumptions are necessary (explain why),
so we need to impose a stronger condition

<u>Def</u> Numbers $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ are <u>rationally independent</u> if $k_1\alpha_1 + \dots + k_m\alpha_m = 0$, where $k_1, \dots, k_m \in \mathbb{Z}$, implies $k_1 = \dots = k_m = 0$, i.e., none of $\alpha_1, \dots, \alpha_m$ is a linear combination of the other numbers with rational coefficients.

<u>Proposition</u> Let $\gamma = (\gamma_1, \dots, \gamma_n)$. The translation $T_\gamma : \mathbb{T}^n \to \mathbb{T}^n$ is top. transitive, equivalently, minimal $\iff$ $\gamma_1, \dots, \gamma_n, 1$ are rationally independent, that is, $\sum_{i=1}^n k_i\gamma_i \in \mathbb{Z}$, where $k_1, \dots, k_n \in \mathbb{Z}$, implies $k_1 = \dots = k_n = 0$.

<u>Note</u>
- It follows that $\gamma_1, \dots, \gamma_n \notin \mathbb{Q}$ and $\frac{\gamma_i}{\gamma_j} \notin \mathbb{Q}$ for $i \neq j$
- $\sqrt{2}$, $\frac{1}{2} + \sqrt{2}$, 1 are <u>not</u> rationally independent.

We will prove this proposition using

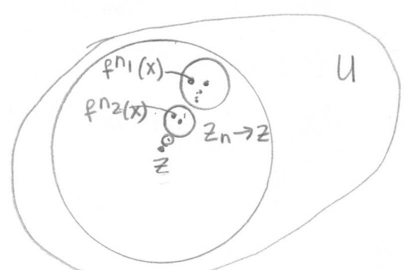
<u>Criteria for top. transitivity</u>

<u>Thm</u> Let X be a compact metric space without isolated points, and let $f : X \to X$ be a homeomorphism. Then the following are equivalent:

(1) f is top. transitive, i.e. there is a point with dense orbit;
(2) there is a point with dense positive semiorbit;
(3) for any non-empty open sets $U, V \subseteq X$ there is $N \in \mathbb{Z}$ s.t. $f^N(U) \cap V \neq \emptyset$;
(4) —— " —— $N \in \mathbb{N}$ —— " ——.

<u>Pf</u> Observations: Since X is a compact metric space, it is separable, i.e. contains a countable dense set (take $\bigcup_{n=1}^{\infty}$ of finite $\frac{1}{n}$-nets)
$\Rightarrow$ there is a countable base of open sets, i.e. open sets $U_1, U_2, \dots$ s.t. for each $x \in X$ and open $U \ni x$, there is $k \in \mathbb{N}$ s.t. $x \in U_k \subseteq U$.
(take balls of rational radii centered at the pts of a countable dense set)

Since X has no isolated points, a dense orbit/semiorbit visits each non-empty open set infinitely many times.



Now we prove the equivalence. Clearly, (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3)

(2) $\Rightarrow$ (4) Let x be a pt with dense $O^+(x)$. Then there is $n \geq 0$ s.t. $f^n(x) \in U$. We can choose $m > n$ s.t. $f^m(x) \in V$. Thus, $f^{m-n}(U) \cap V \neq \emptyset$.

(1) $\Rightarrow$ (4) For some $m, n \in \mathbb{Z}$, $f^n(x) \in U$ and $f^m(x) \in V$. Suppose there is no $k > n$ s.t. $f^k(x) \in V$. Let $W = f^{m-n}(U) \cap V$. It is open $\neq \emptyset$. Then there are inf. many $k < n$ s.t. $f^k(x) \in W$. Take $k < 2m-n$ and $x' = f^{n-m}(f^k(x))$. Then $x' \in U$, $f^{2m-n-k}(x') = f^m(x) \in V$, and $2m-n-k \in \mathbb{N}$.

(4) $\Rightarrow$ (2). ((3) $\Rightarrow$ (1) is similar).

We show that there is $x \in X$ s.t. $O^+(x)$ visits each of the sets $U_1, U_2, \ldots$

There is $N_1 \in \mathbb{N}$ s.t. $f^{N_1}(U_1) \cap U_2 \neq \emptyset$. Since $U_1 \cap f^{-N_1}(U_2)$ is open, there is an open set $V_1 \neq \emptyset$ s.t. $\overline{V_1} \subset (U_1 \cap f^{-N_1}(U_2))$.

Next, there is $N_2 \in \mathbb{N}$ s.t. $f^{N_2}(V_1) \cap U_3 \neq \emptyset$. There is an open set $V_2 \neq \emptyset$ s.t. $\overline{V_2} \subset (V_1 \cap f^{-N_2}(U_3))$. Continuing, we construct a nested sequence of non-empty open sets $V_1 \supset V_2 \supset \ldots$ s.t. $\overline{V_{k+1}} \subset (V_k \cap f^{-N_{k+1}}(U_{k+2}))$ for all k.

Since the sets $\overline{V_k}$ are nested, compact and $\neq \emptyset$, $V = \bigcap_{k=1}^{\infty} \overline{V_k} \neq \emptyset$.

By the construction, for any $x \in V$, $O^+(x)$ visits each U_k. $\square$

Note
- If X is as in the Thm and $f: X \to X$ is a continuous non-invertible map, we have (2) $\Leftrightarrow$ (4), i.e. top. transitivity $\Leftrightarrow$ (4).
- We do not need "no isolated pts" to prove (1) $\Leftrightarrow$ (3)

Corollary Let f be a homeomorphism of a compact metric space. Then f is top. transitive $\Leftrightarrow$ there are no two disjoint non-empty open f-invariant sets in X.

Pf ($\Rightarrow$) Suppose $O(x)$ is dense. Let U, V be open, $\neq \emptyset$, and invariant. Then there are $m, n \in \mathbb{Z}$ s.t. $f^n(x) \in U$ and $f^m(x) \in V$. Hence $f^n(x) = f^{n-m}(f^m(x)) \in f^{n-m}(V) = V$, and so $U \cap V \neq \emptyset$.

($\Leftarrow$) Let U, V be non-empty open sets. Then the sets $\tilde{U} = \bigcup_{n \in \mathbb{Z}} f^n(U)$ and $\tilde{V} = \bigcup_{n \in \mathbb{Z}} f^n(V)$ are open, non-empty, and invariant, and hence not disjoint. Hence there are $m, n \in \mathbb{Z}$ s.t. $f^m(U) \cap f^n(V) \neq \emptyset \Rightarrow f^{m-n}(U) \cap V \neq \emptyset$. Thus (3) holds, and it follows that f is top. transitive. $\square$