Continuous-time dynamical systems (Flows).

Let X be a compact metric space. A <u>flow</u> $\Phi = \{\varphi^t\}_{t\in\mathbb{R}}$ is a continuous map $(x,t) \mapsto \varphi^t(x)$ from $X\times\mathbb{R}$ to X such that $\varphi^0 = Id$ and $\varphi^s \circ \varphi^t = \varphi^{s+t}$ for all $s,t\in\mathbb{R}$.

It follows that $\{\varphi^t\}$ is a one-parameter group w.r.to composition, for each $t\in\mathbb{R}$, $\varphi^{-t} = (\varphi^t)^{-1}$, and so φ^t is a homeomorphism, and φ^t depends continuously on t in the C^0 topology.

For a fixed t, the map φ^t is called the <u>time-t map</u> of the flow. The pair (X, φ^t) is a discrete-time dynamical system.

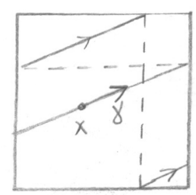
Often, the maps φ^t are diffeomorphisms of a smooth manifold X, for example, $X = S^1$ or $\mathbb{T}^n$, and the map $(x,t)\mapsto\varphi^t(x)$ is smooth.

<u>Ex 1</u> $\{R_\alpha^t\}_{t\in\mathbb{R}}$ on S^1, $R_\alpha^t(x) = x + t\alpha \mod 1$

Points move along S^1 with constant speed $|\alpha|$.

<u>Ex 2</u> Linear flow $\{T_\gamma^t\}_{t\in\mathbb{R}}$ on $\mathbb{T}^2$

$\gamma = (\gamma_1,\gamma_2)$, $T_\gamma^t(x_1,x_2) = (x_1 + t\gamma_1, x_2 + t\gamma_2) \mod 1$



<u>Note</u> γ and $\gamma + (m,n)$, $0 \neq (m,n)\in\mathbb{Z}^2$ define the same translation on $\mathbb{T}^2$, but not the same flow.

For a point $x\in X$, the <u>orbit</u> of x is $O(x) = \{\varphi^t(x) : t\in\mathbb{R}\}$, and the positive and negative semiorbits are $O^+(x) = \{\varphi^t(x) : t\geq 0\}$
$O^-(x) = \{\varphi^t(x) : t\leq 0\}$.

A point $x\in X$ is fixed for the flow $\{\varphi^t\}$ if $\varphi^t(x) = x$ for all $t\in\mathbb{R}$, and x is <u>periodic</u> if there is $t>0$ s.t. $\varphi^t(x) = x$. If x is periodic but not fixed, the smallest $t>0$ s.t. $\varphi^t(x) = x$ is the <u>minimal period</u> of x.

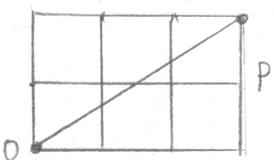
<u>Ex 1</u> $\{R_\alpha^t\}$ on S^1. Let $\alpha\neq 0$. Then every $x\in S^1$ is periodic with minimal period $\frac{1}{|\alpha|}$. Note that for $\alpha\notin\mathbb{Q}$, the time-one map $R_\alpha^1 = R_\alpha$ has no periodic points.

<u>Ex 2</u> $\{T_\gamma^t\}$ on $\mathbb{T}^2$. We recall that

every x is periodic $\Leftrightarrow$ 0 is periodic $\Leftrightarrow$ $\gamma_1 = 0$ or $\frac{\gamma_2}{\gamma_1}\in\mathbb{Q}$.

Let $\gamma \neq (0,0)$. What is the minimal period of 0?
It is the smallest $t>0$ s.t. $t(\gamma_1,\gamma_2)=(p,q)\in\mathbb{Z}^2$.
If $\gamma_1=0$, $t=1/|\gamma_2|$; if $\gamma_2=0$, $t=1/|\gamma_1|$

Find t for $\gamma_1\neq 0$ and $\gamma_2\neq 0$ (Ex)
Observe that p and q have to be relatively prime integers.

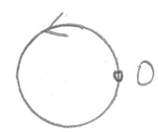


A flow $\{\varphi^t\}$ is top. transitive if there is a point with dense orbit.

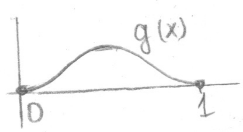
Ex 2 Recall: $\{T_\gamma^t\}$ on $\mathbb{T}^2$ is top. transitive $\Leftrightarrow$
$\Leftrightarrow$ γ_1 and γ_2 are rationally independent $\Leftrightarrow$ $\frac{\gamma_2}{\gamma_1}\in\mathbb{R}\setminus\mathbb{Q}$.

Note If for some $t_0\in\mathbb{R}$ the time-t map is top. transitive, then
The flow $\{\varphi^t\}$ is top. transitive since $\{(\varphi^{t_0})^n(x): n\in\mathbb{Z}\}\subseteq\{\varphi^t(x): t\in\mathbb{R}\}$.
The converse is not true.
The following example shows that the converse is not true

Ex 3 Consider a flow $\{\varphi^t\}$ on S^1 s.t. 0 is fixed, and for every $x\neq 0$, $\varphi^t(x)$ is strictly monotone and $\varphi^t(x)\to 0$ as $t\to\infty$ and as $t\to-\infty$.

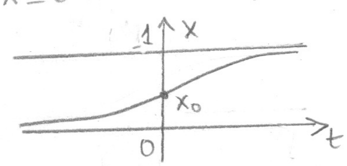


Such a flow can be defined on $[0,1]$ by a differential equation
$\frac{dx}{dt}=g(x)$, where



for example, $g(x)=x^2(1-x)^2$
(squares so that g is smooth on S^1)

This diff equation has two constant solutions, $x\equiv 0$ and $x\equiv 1$, and every solution with $x(0)\in(0,1)$ is strictly increasing, $\to 1$ as $t\to\infty$ and $\to 0$ as $t\to-\infty$



The flow $\{\varphi^t\}$ satisfies $\varphi^t(x_0)=x_0+\int_0^t g(\varphi^s(x_0))\,ds$.

Let $0\neq x_0\in S^1$. Then the orbit of x_0 under the flow is $S^1\setminus\{0\}$, which is dense in S^1. For any $t_0\neq 0$, the orbit of x_0 under the map φ^{t_0} is a monotone sequence $\{(\varphi^{t_0})^n\}$ that converges to 0 as $n\to\pm\infty$, and so the orbit is not dense in S^1.

A flow $\{\varphi^t\}$ is top. mixing if for any non-empty open sets $U,V\subset X$ there is $T>0$ s.t. $\varphi^t(U)\cap V\neq\emptyset$ for all $t\geq T$.

Clearly, $\{\varphi^t\}$ is top. mixing $\Rightarrow$ φ^1 is top. mixing. ? Is the converse true?
? How to define nonwandering pts for $\{\varphi^t\}$. ?