Contractions

Def. Let X be a metric space. A map $f: X \to X$ is a contraction if there is a number $\lambda < 1$ s.t.

$$d(f(x), f(y)) \le \lambda\, d(x,y) \quad \text{for all } x, y \in X \qquad (*)$$

We call such f a λ-contraction.

Ex
- $f: I \to I$, where I is an interval and f is differentiable on I, is a contraction $\iff$ there is $\lambda < 1$ s.t. $|f'(x)| \le \lambda$ for all $x \in I$
- $f: M \to M$, where M is a complete Riemannian manifold and f is a C^1 map, is a contraction $\iff$ there is $\lambda < 1$ s.t. $\|D_x f\| \le \lambda$ for all $x \in M$.

Note A contraction is Lipschitz and in particular continuous.

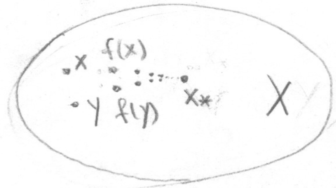
Thm (The Contraction Mapping Principle).
Let X be a complete metric space and let $f: X \to X$ be a λ-contraction. Then f has a unique fixed point $x_* \in X$, and for every $x \in X$ $f^n(x) \to x_*$ exponentially (with exponential speed), specifically, $d(f^n(x), x_*) \le \lambda^n d(x, x_*)$ for all $n \in \mathbb{N}$.

Pf Let $x \in X$. We show that $(f^n(x))$ is a Cauchy sequence.
It follows from $(*)$ that $d(f^n(x), f^n(y)) \le \lambda^n d(x,y)$ for all $x, y \in X$ and $n \in \mathbb{N}$.
Let $m > n \ge 0$. Then

$$d(f^n(x), f^m(x)) \le d(f^n(x), f^{n+1}(x)) + \dots + d(f^{m-1}(x), f^m(x)) \le$$
$$\le (\lambda^n + \dots + \lambda^{m-1})\, d(x, f(x)) \le \Big(\sum_{k=n}^{\infty} \lambda^k\Big) d(x, f(x)) = \frac{\lambda^n}{1-\lambda} d(x, f(x))$$

As $\lambda^n \to 0$ as $n \to \infty$, it follows that $(f^n(x))$ is Cauchy, and hence, since X is complete, it has a limit, $x_* \in X$.
x_* is fixed since by continuity of f, $f(x_*) = f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = x_*$.
Uniqueness of a fixed pt follows from $(*)$.
$d(f^n(x), x_*) = d(f^n(x), f^n(x_*)) \le \lambda^n d(x, x_*)$. $\square$

Remarks
- If we cannot solve $f(x_*) = x_*$, we can approximate x_* by $f^n(x)$ for *any* $x \in X$.
  For any $m > n \ge 0$, $d(f^n(x), f^m(x)) \le \frac{\lambda^n}{1-\lambda} d(x, f(x))$. It follows that $d(f^n(x), x_*) \le \frac{\lambda^n}{1-\lambda} d(x, f(x))$, and so we have an error bound.
- For any $x, y \in X$, $d(f^n(x), f^n(y)) \le \lambda^n d(x,y)$, so the orbits of x and y converge exponentially



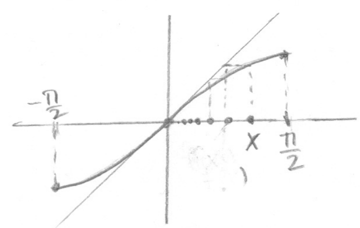
- The thm does not hold without the assumptions.
  - If X is not complete, a contraction on X may have no fixed pt
    Ex $X=(0,1]$, $f(x)=\frac{x}{2}$
  - If X is complete but f satisfies only $d(f(x),f(y))<d(x,y)$ for all $x\neq y$, there may be no fixed pt. Give an example (HW).

Proposition (Stability of contractions)
[Changing a contraction slightly does not move the fixed pt. much]
Let X be a complete metric space and let $f: X\to X$ be a λ-contraction with fixed pt. x_*. Then for any $\varepsilon>0$ there is $\delta\in(0,1-\lambda)$ s.t. for any map $g: X\to X$ with (1) $d(f(x),g(x))<\delta$ for all $x\in X$, and
(2) $d(g(x),g(y))\le(\lambda+\delta)\,d(x,y)$ for all $x,y\in X$,
g is also a contraction and its fixed pt y_* satisfies $d(x_*,y_*)<\varepsilon$.

Pf Since $(\lambda+\delta)<1$, g is a contraction, so $g^n(x)\to y_*$ for any $x\in X$, and in particular $g^n(x_*)\to y_*$.
Also, since $f(x_*)=x_*$, $d(x_*,g(x_*))<\delta$ by (1). So we have
$d(x_*,y_*)=d(x_*,\lim_{n\to\infty} g^n(x_*))\le\sum_{n=0}^{\infty} d(g^n(x_*),g^{n+1}(x_*))\le$
$\le\sum_{n=0}^{\infty}(\lambda+\delta)^n\cdot d(x_*,g(x_*))<\frac{1}{1-(\lambda+\delta)}\cdot\delta$
Given $\varepsilon>0$, $\frac{\delta}{1-\lambda-\delta}\le\varepsilon$ for all suff. small δ (find such δ) $\square$.

Weak contractions
We say that $f: X\to X$ is a weak contraction if
$(**)$ $d(f(x),f(y))<d(x,y)$ for all $x\neq y$ in X.

Ex $f(x)=\sin x$ on $[-\frac{\pi}{2},\frac{\pi}{2}]=X$
$|f'(x)|<1$ for all $x\neq0$, $f'(0)=1$.
For all $x\in X$, $f^n(x)\to 0=x_*$.



Prop. Let X be a compact metric space and let $f: X\to X$ be a weak contraction.
Then f has a unique fixed pt. $x_*\in X$, and for every $x\in X$, $f^n(x)\to x_*$,
(but the convergence is not necessarily exponential)

Proof next time. Some ideas:
- to show existence of a fixed pt, consider the min of $h(x)=d(x,f(x))$
- uniqueness is easy
- then show $f^n(x)\to x_*$.