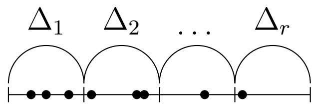
MSAI Statistics Home Assignment 5-6
soft deadline: 25/04/2024 23:59 AOE
hard deadline: 30/05/2024 19:00 GMT+3

As announced earlier, grading for HWs consists of points and bonus points. Solving bonus (indicated with a star) problems is not required, but recommended. Solving all homeworks' normal problems correctly will give you a score of 7, solving all homeworks' bonus problems correctly will give you additional 2 points to the score.

Hand-written solutions are accepted if the handwriting is clear enough and scanned with sufficient quality, but LaTeX is always preferable. This homework includes a python task, which can be solved in Google Colab or in a local Jupyter Notebook. It is thus handy to solve everything (both LaTeX and code) in a single Jupyter Notebook.

Solutions obtained with the use of ChatGPT and similar models can be accepted if the solution is clearly indicated as such, and model version and prompt is provided. If the solution is found to be from ChatGPT and similar models without indication and model/prompt detail, the teachers will evaluate the problem as zero points.

Problem 1. (6 points) Let's prove that Pearson χ^2 test statistic follows $\chi^2(r-1)$ distribution.



Suppose that we have a sample $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}$ of real numbers. Let's split the real line $\mathbb{R}$ into r intervals Δ_i and let's count how many samples reside in each interval:

$$\nu_i = \sum_k \mathbb{I}\mathrm{nd}\{\mathrm{x_k \in \Delta_i}\}$$

Under null hypothesis we are able to estimate the probabilities of a sample appearing in each interval under the null hypothesis:

$$p_i = \int_{\Delta_i} p(x)\mathrm{dx}$$

Then the Pearson χ^2 test statistic is:

$$T(X) = \sum_{i=1}^{r} \frac{(\nu_i - np_i)^2}{np_i} = \sum_{i=1}^{r} \xi_i^2$$

- (1 point) Find ξ_i
- (1 point) Show that $\xi = (\xi_1, \dots, \xi_r)$ is degenerate, i.e. $\exists b : \xi^\top b = 0$
- (1 point) Use CLT to prove that ξ_i has normal distribution and find its parameters
- (3 points) Find $\mathbb{E}[\xi_i \xi_j]$ for $i = j$ and $i \neq j$

This almost concludes the proof. The rest of the proof is as follows:

Now we know the elements of $B = \mathbb{E}[\xi\xi^\top]$. We can use the multivariate CLT to have that $\frac{1}{\sqrt{n}}\xi \to \eta \sim \mathcal{N}(0, B)$.

Next, we can show that $B = E - bb^\top$. With a bit of linear algebra with can also prove that B has $r-1$ eigenvalues equal to 1 and 1 eigenvalue equal to 0. Therefore there exists orthogonal matrix U such that $UBU^\top = \mathrm{diag}(\underbrace{1, \dots, 1}_{\mathrm{r-1}}, 0)$.

Finally, if we consider random variable $\hat{\eta} = U\eta$, we will see that $\hat{\eta} \sim \mathcal{N}(0, UBU^\top)$. The covariance matrix is diagonal, therefore elements of $\hat{\eta}$ are independent and follow the univariate normal distribution: $\hat{\eta}_i \sim \mathcal{N}(0,1)$. Therefore, $T(X) = \sum_{i=1}^{r} \xi_i^2 \to \sum_{i=1}^{r-1} \hat{\eta}_i^2 \sim \chi^2(r-1)$.

Problem 2. (2 points) Suppose you have the following sample:

interval	0	1	2	3	4	5	6	7	8	9
counts	74	92	83	79	80	73	77	75	76	91

Use the Pearson χ^2 test to test hypothesis at $\alpha = 0.05$ level

$$H_0 : X \sim U[0,9]$$
$$H_1 : X \nsim U[0,9]$$

Problem 3. (2 points) Suppose you have the following two independent samples, $X \sim \mathcal{N}(\mu_x, \sigma^2)$ and $Y \sim \mathcal{N}(\mu_y, \sigma^2)$ (see Table 1). Use Student's two-sample t-test to test the following hypothesis at $\alpha = 0.05$ level (you may assume equal variances):

$$H_0 : \mu_x = \mu_y$$
$$H_1 : \mu_x \neq \mu_y$$

	X	Y
1	-1.75	-0.29
2	-0.33	0.09
3	-1.26	1.70
4	0.32	-1.09
5	1.53	-0.44
6	0.35	-0.29
7	-0.96	0.25
8	-0.06	-0.54
9	0.42	-1.38
10	-1.08	0.32

Table 1: Data for Problem 3.

Problem 4. (2 bonus points) In the Table 2 below, X and Y are the reaction times to light and sound signal of the test subjects.

Use Wilcoxon signed rank test to test the following hypothesis at $\alpha = 0.05$ level:

$$H_0 : \mu_x = \mu_y$$
$$H_1 : \mu_x \neq \mu_y$$

For the ease of computation, instead of Wilcoxon statistc W, use standardized test statistic

$$T = \frac{W - \mathbb{E}[W]}{\sqrt{\mathbb{V}\mathrm{ar(W)}}} = \frac{W}{\sqrt{\frac{1}{6}n(n+1)(2n+1)}} \sim \mathcal{N}(0,1)$$

Problem 5. (4 bonus points) Let $X_1, \ldots, X_n \sim \mathcal{N}(\theta, 1)$. Consider the following test

$$H_0 : \theta = \theta_0 = 0$$
$$H_1 : \theta = \theta_1 = 1$$

Let the rejection region be $R = \{X : T(X) > c\}$, where $T(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$.

	X	Y
1	176	168
2	163	215
3	152	172
4	155	200
5	156	191
6	178	197
7	160	183
8	164	174
9	169	176
10	155	155
11	122	115
12	144	163

Table 2: Data for Problem 4.

- (2 bonus points) Find c so that test has size α
- (2 bonus points) Find test power W

Problem 6. (4 bonus points) Let $X_1, \ldots, X_n \sim \mathcal{N}(\theta, \sigma^2)$, where σ^2 is known. Consider the following test

$$H_0 : \theta = \theta_0$$
$$H_1 : \theta \neq \theta_0$$

- (2 bonus points) Compute likelihood ratio $\Lambda(X)$
- (1 bonus point) Compute $\lambda(X) = 2 \log \Lambda(X)$, find its distribution
- (1 bonus point) Specify rejection region $R = \{X : \lambda(X) > c\}$ using test size α. Introduce change of variables such that the test statistic in new variables follows standard normal distribution. Specify the rejection region R in new variables.