Optimization methods
Graded Assignment 1

Vladimir Saraikin

Fall 2023

1. (1 point) Correct points:
   1. Whether it has 5 extremums in the segment $[-4;4]$
   2. If the global minimum is zero
   3. Whether the global maximum is unique
   4. If the number of local minimums is finite
   5. Whether the number of local maximums is countable
   6. If the function is smooth and continuous
2. (3 points) Verifying Convexity of Sets

   (a) Given $S_a \subseteq \mathbb{R}^n$ defined by a polynomial $P(x)$, verify convexity:

   $$\text{For } x, y \in S_a, \lambda \in [0,1], \text{ check if } \lambda x + (1-\lambda)y \in S_a.$$

   (b) For $S_b \subseteq \mathbb{R}^2$ with $xy \leq k$, $k \in \mathbb{R}$, analyze:

   $$\text{Convexity if } \forall x, y \in S_b, \lambda x + (1-\lambda)y \in S_b.$$

   (c) Let S_c be matrices in $\mathbb{R}^{n\times n}$ with diagonal criteria. Confirm:

   $$\text{Linear combinations preserve conditions: } \lambda A + (1-\lambda)B \in S_c, \forall A, B \in S_c.$$

   (d) For S_d with min/max element bounds in $\mathbb{R}^n$, test convexity:

   $$\text{If } x, y \in S_d, \text{ then } \lambda x + (1-\lambda)y \in S_d, \forall \lambda \in [0,1].$$

   (e) Given S_e of matrices with rank r, verify:

   $$\text{Convexity if } \forall A, B \in S_e, \lambda A + (1-\lambda)B \text{ has rank } r, \forall \lambda \in [0,1].$$

3. (2 points) Convexity Preservation Under Maps

   Linear Map Case:
   Assume f is a linear map, i.e., $f(x) = Ax$ for some matrix A.

   - Take any two points $x, y \in f^{-1}(C)$.
   - By definition of preimage, $f(x), f(y) \in C$.
   - Since C is convex, for any $\lambda \in [0,1]$, $\lambda f(x) + (1-\lambda)f(y) \in C$.
   - Using linearity, $f(\lambda x + (1-\lambda)y) = \lambda Ax + (1-\lambda)Ay = \lambda f(x) + (1-\lambda)f(y) \in C$.

- Thus, $\lambda x + (1-\lambda)y \in f^{-1}(C)$, proving convexity.

Perspective Map Case:
Assume f is a perspective map, i.e., $f(x,t) = \frac{x}{t}$ for $x \in \mathbb{R}^n, t \in \mathbb{R} \setminus \{0\}$.

- Consider two points $(x_1, t_1), (x_2, t_2) \in f^{-1}(C)$.
- By definition, $f(x_1, t_1), f(x_2, t_2) \in C$.
- For $\lambda \in [0,1]$, check $\lambda f(x_1, t_1) + (1-\lambda) f(x_2, t_2) \in C$.
- Note: $f(\lambda(x_1, t_1) + (1-\lambda)(x_2, t_2)) = \frac{\lambda x_1 + (1-\lambda)x_2}{\lambda t_1 + (1-\lambda)t_2}$.
- If $\lambda t_1 + (1-\lambda)t_2 \neq 0$, $\frac{\lambda x_1 + (1-\lambda)x_2}{\lambda t_1 + (1-\lambda)t_2} \in C$.
- Hence, $\lambda(x_1, t_1) + (1-\lambda)(x_2, t_2) \in f^{-1}(C)$, proving convexity.

4. (2 points) Convexity Characterization of Sets in $\mathbb{R}^n$

   Proof:

   If C is convex:

   - For $x, y \in C$, $\lambda x + (1-\lambda)y \in C$ for $\lambda \in [0,1]$.
   - Hence, $\alpha x, \beta y \in C$ for $\alpha, \beta \geq 0$.
   - Therefore, $\alpha x + \beta y \in \alpha C + \beta C$.
   - It follows that $(\alpha + \beta)C \subseteq \alpha C + \beta C$.

   If $(\alpha + \beta)C = \alpha C + \beta C$:

   - For $x, y \in C$, $\alpha = \lambda$, $\beta = 1 - \lambda$, $\lambda \in [0,1]$.
   - Then, $\lambda x + (1-\lambda)y \in \lambda C + (1-\lambda)C = (\lambda + 1 - \lambda)C = C$.
   - Thus, C is convex.