MSAI Probability Home Assignment 5
deadline: 03/12/2023 23:59 AOE

As announced earlier, grading for HWs consists of points and bonus points. Solving bonus (indicated with a star) problems is not required, but recommended. Solving all homeworks' normal problems correctly will give you a score of 7, solving all homeworks' bonus problems correctly will give you additional 2 points to the score.

Hand-written solutions are accepted if the handwriting is clear enough and scanned with sufficient quality, but LaTeX is always preferable.

Problem 1. (2 points) According to the labor laws at city M, the employer must provide **all** employees with a vacation if **at least one** of them has a birthday on that day. Apart from these vacation days, there are none, and the employees work 365 days. The employer asks you to find the optimal number of employees such that the number of working man-days is maximized. What is this number? How many working man-days does it yield?

Problem 2. (1 points) You have n enumerated letters and n enumerated envelopes. You randomly put letters into envelopes. What is the expected value of the number of coinciding numbers of the letter and its envelope?

Problem 3. (2 points) Consider $X \sim Pois(\lambda_1)$ and $Y \sim Pois(\lambda_2)$ and also $X \perp Y$ (independent). Find the distribution of $Z = X + Y$.

Problem 4. (2 points) Consider $X \sim Pois(\lambda_1)$ and $Y \sim Pois(\lambda_2)$ and also $X \perp Y$ (independent). Find the distribution $X|X+Y$ and its PMF $\mathbb{P}(X = k|X + Y = n)$.

Problem 5* . (1 bonus point) Compute expectation of Hypergeometric distribution.

Problem 6* . (4 bonus points) Compute expectation (2 bonus points) and variance (2 bonus points) of Geometric distribution. Hints:

1. Obtain the series for expectation in terms of p and q.
2. Compare the series with geometric series in q, for which we know that it converges, and we know the limit.
3. Differentiate the geometric series. Because the geometric series converges, we can interchange sum and differential operators. Compare the series for expectation with differentiated geometric series.
4. Make the necessary changes and obtain the expectation.
5. Use LOTUS to get series for $\mathbb{E}\left[X^2\right]$.
6. Compare with differentiated geometric series. Make adjustments, differentiate again.
7. Obtain the variance.

Problem 7* . (2 bonus points) Compute expectation (1 bonus points) and variance (1 bonus points) of Negative binomial distribution. Hint: there is a relation between Negative binomial and Geometric distributions.