MSAI Probability Home Assignment 6
deadline: 12/12/2023 23:59 AOE

As announced earlier, grading for HWs consists of points and bonus points. Solving bonus (indicated with a star) problems is not required, but recommended. Solving all homeworks' normal problems correctly will give you a score of 7, solving all homeworks' bonus problems correctly will give you additional 2 points to the score.

Hand-written solutions are accepted if the handwriting is clear enough and scanned with sufficient quality, but LaTeX is always preferable.

Problem 1. (3 points) Find the expectation and variance of **exponential distribution** $Exp(\lambda)$. Draw CDF and PDF of $Exp(1)$.

If $X \sim Exp(\lambda)$,

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, x \geqslant 0, \\ 0, \text{else} \end{cases}$$

Problem 2. (1 point) A continuous distribution is said to have the memoryless property if a random variable X from that distribution satisfies

$$\mathbb{P}\left(X \geqslant s + t | X \geqslant s\right) = \mathbb{P}\left(X \geqslant t\right)$$

Prove that exponential distribution is memoryless.

Problem 3. (3 points) Consider 2D real plane $\mathbb{R}^2$. Consider a point with coordinates $(0, d)$. Select an angle φ uniformly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Issue a ray from $(0, d)$ at the angle φ to the y-axis. Denote the point where it intersects x-axis as $(X, 0)$. The distribution of X is called the Cauchy distribution. Find its CDF and PDF. Explain, why it does not have an expected value.

Problem 4^* **.** (2 bonus points) Consider independent random variables $X \sim Exp(\lambda_X)$ and $Y \sim Exp(\lambda_Y)$.

1. Find the CDF of $Z = \min\{X, Y\}$.

2. Let $\lambda_X = \lambda_Y$. Find the CDF of $Z = X + Y$. This will be a (very simple case of) **Gamma distribution**. Hint: use the convolution formula.

Problem 5^* **.** (3 bonus points) Consider $X \sim Exp(1)$ and $Y = -\log X$. Then Y has the distribution called **Gumbel distribution**.

1. Find its CDF.

2. Consider $X_1, X_2, \ldots, X_n \sim Exp(1)$ and define $M_n = \max\{X_1, \ldots, X_n\}$. Show that the CDF of $Y = (M_n - \log n)$ converges to the CDF of Gumbel distribution as $n \to \infty$ (this is what we call convergence of random variables in distribution).