A STEP-BY-STEP GUIDE TO THE BLACK-LITTERMAN MODEL

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The Black-Litterman asset allocation model, created by Fischer Black and Robert Litterman of Goldman, Sachs & Company, is a sophisticated method used to overcome the problem of unintuitive, highly-concentrated, input-sensitive portfolios. Input sensitivity is a well-documented problem with mean-variance optimization and is the most likely reason that more portfolio managers do not use the Markowitz paradigm, in which return is maximized for a given level of risk. The Black-Litterman Model uses a Bayesian approach to combine the subjective views of an investor regarding the expected returns of one or more assets with the market equilibrium vector (the prior distribution) of expected returns to form a new, mixed estimate of expected returns.[1] The resulting new vector of returns (the posterior distribution) is described as a complex, weighted average of the investor's views and the market equilibrium.

Having attempted to decipher several articles about the Black-Litterman Model, I have found that none of the relatively few articles on the Black-Litterman Model provide enough step-by-step instructions for the average practitioner to derive the new vector of expected returns. Adding to the difficulty, the few existing articles lack consistency in their mathematical notations and no one article provides sufficient detail for establishing the values of the model's parameters. In addition to touching on the "intuition" behind the Black-Litterman Model, this paper consolidates critical insights contained in the various works on the Black-Litterman Model and focuses on the details of actually combining market equilibrium expected returns with "investor views" to generate a new vector of expected returns.

THE INTUITION

The goal of the Black-Litterman Model is to create stable, mean-variance efficient portfolios, based on an investor's unique insights, which overcome the problem of input-sensitivity. According to Lee (2000), the Black-Litterman Model also "largely mitigates" the problem of estimation error-maximization (see Michaud (1989)) by spreading the errors throughout the vector of expected returns.

The most important input in mean-variance optimization is the vector of expected returns; however, Best and Grauer (1991) demonstrate that a small increase in the expected return of one of the portfolio's assets can force half of the assets from the portfolio. In a search for a reasonable starting point, Black and Litterman (1992) and He and Litterman (1999) explore several alternative forecasts of expected returns: historical returns, equal "mean" returns for all assets, and risk-adjusted equal mean returns. They demonstrate that these alternative forecasts lead to extreme portfolios – portfolios with large long and short positions concentrated in a relatively small number of assets.

The Black-Litterman Model uses "equilibrium" returns as a neutral starting point. Equilibrium returns are calculated using either the CAPM (an equilibrium pricing model) or a reverse optimization method in which the vector of implied expected equilibrium returns ($\boldsymbol{\Pi}$) is extracted from known information.[*] Using matrix algebra, one solves for $\boldsymbol{\Pi}$ in the formula, $\boldsymbol{\Pi} = \boldsymbol{\delta}\, \boldsymbol{\Sigma}\, \boldsymbol{w}$, where $\boldsymbol{w}$ is the vector of market capitalization weights; $\boldsymbol{\Sigma}$ is a fixed covariance matrix; and, $\boldsymbol{\delta}$ is a risk-aversion coefficient.[2, 3] If the portfolio in question is "well-diversified" relative to the market proxy used to calculate the CAPM returns (or if the market capitalization weighted components of the portfolio in question are considered the market proxy),[4] this method of extracting the implied expected equilibrium returns produces an expected return vector very similar to the one generated by the Sharpe-Littner CAPM. In fact, Best and Grauer (1985)

[*] For the rest of the article, $\boldsymbol{\Pi}$ is referred to as the "Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$)" or $\boldsymbol{\Pi}$.

outline the necessary assumptions to calculate CAPM-based estimates of expected returns that match the Implied Equilibrium returns.

The majority of articles on the Black-Litterman Model have addressed the model from a global asset allocation perspective; therefore, this article presents a domestic example, based on the Dow Jones Industrial Average (DJIA). Table 1 contains three estimates of expected total return for the 30 components of the DJIA: Historical, CAPM, and Implied Equilibrium Returns.[5] Rather than using the Best and Grauer (1985) assumptions to derive a CAPM estimate of expected return that exactly matches the Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$), a relatively standard CAPM estimate of expected returns is used to illustrate the similarity between the two vectors and the differences in the portfolios that they produce. The CAPM estimate of expected returns is based on a 60-month beta relative to the DJIA times-series of returns, a risk-free rate of 5%, and a market risk premium of 7.5%.[6]

Table 1: DJIA Components – Estimates of Expected Total Return

Symbol	Historical Return Vector	CAPM Return Vector	Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$)
aa	17.30	15.43	13.81
ge	16.91	12.15	13.57
jnj	16.98	9.39	9.75
msft	23.95	14.89	20.41
axp	15.00	15.65	14.94
gm	4.59	13.50	12.83
jpm	5.31	15.89	16.46
pg	7.81	8.04	7.56
ba	-4.18	14.16	11.81
hd	29.38	11.59	12.52
ko	-0.57	10.95	10.92
sbc	10.10	7.76	8.79
c	24.55	16.59	16.97
hon	-0.05	16.89	14.50
mcd	2.74	10.70	10.44
t	-1.24	8.88	10.74
cat	7.97	13.08	10.92
hwp	-4.97	14.92	14.45
mmm	9.03	10.43	8.66
utx	13.39	16.51	15.47
dd	0.44	12.21	10.98
ibm	21.99	13.47	14.66
mo	10.47	7.57	6.86
wmt	30.23	10.94	12.77
dis	-2.59	12.89	12.41
intc	13.59	15.83	18.70
mrk	8.65	8.95	9.22
xom	11.10	8.39	7.88
ek	-17.00	11.08	10.61
ip	1.24	14.80	12.92
Average	9.07	12.45	12.42
Std. Dev.	10.72	2.88	3.22
High	30.23	16.89	20.41
Low	-17.00	7.57	6.86

The Historical Return Vector has a much larger standard deviation and range than the other two vectors. The CAPM Return Vector is quite similar to the Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$) (the correlation coefficient ($\boldsymbol{\rho}$) is 85%). Intuitively, one would expect two highly correlated return vectors to lead to similarly correlated portfolios.

In Table 2, the three estimates of expected return from Table 1 are combined with the historical covariance matrix of returns ($\boldsymbol{\Sigma}$) and the risk aversion parameter ($\boldsymbol{\delta}$), to find the optimum portfolio weights.[7]

Table 2: DJIA Components – Portfolio Weights

Symbol	Historical Weight	CAPM Weight	Implied Equilibrium Weight	Market Capitalization Weight
aa	223.86%	2.67%	0.88%	0.88%
ge	-65.44%	9.80%	11.62%	11.62%
jnj	-70.08%	6.11%	5.29%	5.29%
msft	3.54%	3.22%	10.41%	10.41%
axp	-15.38%	5.54%	1.39%	1.39%
gm	5.76%	3.44%	0.79%	0.79%
jpm	-213.39%	1.94%	2.09%	2.09%
pg	92.00%	-1.33%	2.99%	2.99%
ba	-111.35%	4.71%	0.90%	0.90%
hd	280.01%	0.11%	3.49%	3.49%
ko	-151.58%	5.70%	3.42%	3.42%
sbc	17.11%	-4.28%	3.84%	3.84%
c	293.90%	5.11%	7.58%	7.58%
hon	15.65%	2.71%	0.80%	0.80%
mcd	-61.68%	1.32%	0.99%	0.99%
t	-86.44%	4.04%	1.87%	1.87%
cat	-70.67%	5.10%	0.52%	0.52%
hwp	-163.02%	6.60%	1.16%	1.16%
mmm	56.84%	4.73%	1.35%	1.35%
utx	-23.80%	4.38%	0.88%	0.88%
dd	-131.99%	1.03%	1.29%	1.29%
ibm	36.92%	5.57%	6.08%	6.08%
mo	136.78%	1.31%	2.90%	2.90%
wmt	21.03%	0.89%	7.49%	7.49%
dis	5.75%	-2.35%	1.23%	1.23%
intc	97.81%	-1.96%	6.16%	6.16%
mrk	144.34%	4.61%	3.90%	3.90%
xom	218.75%	4.10%	7.85%	7.85%
ek	-148.36%	2.04%	0.25%	0.25%
ip	-113.07%	4.76%	0.57%	0.57%
High	293.90%	9.80%	11.62%	11.62%
Low	-213.39%	-4.28%	0.25%	0.25%

Not surprisingly, the Historical Return Vector produces an extreme portfolio. However, despite the similarity between the CAPM Return Vector and the Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$), the vectors produce two rather distinct portfolios (the correlation coefficient ($\boldsymbol{\rho}$) is 18%). The CAPM-based portfolio contains four short positions and almost all of the weights are significantly different than the benchmark market capitalization weighted portfolio. As one would expect (since the process of extracting the Implied Equilibrium returns given the market capitalization weights was reversed), the Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$) leads back to the market capitalization weighted portfolio. In the absence of views that differ from the Implied Equilibrium return, investors should hold the market portfolio. The Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$) is the market-neutral starting point for the Black-Litterman Model.

THE BLACK-LITTERMAN FORMULA

Prior to advancing, it is important to introduce the Black-Litterman formula and provide a brief description of each of its elements. Throughout this article, *k* is used to represent the number of views and *n* is used to express the number of assets in the formula.

$$E[R] = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q\right] \tag{1}$$

Where:

E[R] =New (posterior) Combined Return Vector (*n x 1* column vector)
τ = Scalar
Σ = Covariance Matrix of Returns (*n x n* matrix)
P = Identifies the assets involved in the views (*k x n* matrix or *1 x n* row vector in the special case of 1 view)
Ω = Diagonal covariance matrix of error terms in expressed views representing the level of confidence in each view (*k x k* matrix)
Π = Implied Equilibrium Return Vector (*n x 1* column vector)
Q = View Vector (*k x 1* column vector)

(***'*** indicates the transpose and $^{-1}$ indicates the inverse.)

INVESTOR VIEWS

More often than not, an investment manager has specific views regarding the expected return of some of the assets in a portfolio, which differ from the Implied Equilibrium return. The Black-Litterman Model allows such views to be expressed in either absolute or relative terms. Below are three sample views expressed using the format of Black and Litterman (1990).

View 1: Merck (mrk) will have an absolute return of 10% (Confidence of View = 50%).
View 2: Johnson & Johnson (jnj) will outperform Procter & Gamble (pg) by 3% (Confidence of View = 65%).
View 3: General Electric (ge) and Home Depot (hd) will outperform General Motors (gm), Wal-Mart (wmt) and Exxon (xom) by 1.5% (Confidence of View = 30%).

View 1 is an example of an absolute view. From Table 1, the Implied Equilibrium return of Merck is 9.22%, which is 88 basis points lower than the view of 10%. Thus, View 1 tells the Black-Litterman Model to set the return of Merck to 10%.

Views 2 and 3 represent relative views. Relative views more closely approximate the way investment managers feel about different assets. View 2 says that the return of Johnson & Johnson will be 3 percentage points greater than the return of Procter & Gamble. In order to gauge whether this will have a positive or negative effect on Johnson & Johnson relative to Procter & Gamble, it is necessary to evaluate their respective Implied Equilibrium returns. From Table 1, the Implied Equilibrium returns for Johnson & Johnson and Procter & Gamble are 9.75% and 7.56%, respectively, for a difference of 2.19%. The view of 3%, from View 2, is greater than the 2.19% by which Johnson & Johnson's return exceeds Procter & Gamble's return; thus, one would expect the model to tilt the portfolio towards Johnson & Johnson relative to Procter & Gamble. In general (and in the absence of constraints and additional views), if the view exceeds the difference between the two Implied Equilibrium returns, the model will tilt the portfolio towards the outperforming asset, as illustrated in View 2.

View 3 demonstrates that the number of outperforming assets need not match the number of assets underperforming and that the terms “outperforming” and “underperforming” are relative. The results of views that involve multiple assets with a range of different Implied Equilibrium returns are less intuitive and generalizations are more difficult. However, in the absence of constraints and other views, the assets of the view form two separate mini-portfolios, a long and a short portfolio. The relative weighting of each nominally outperforming asset is proportional to that asset’s market capitalization divided by the sum of the market capitalization of the other nominally outperforming assets of that particular view. Likewise, the relative weighting of each nominally underperforming asset is proportional to that asset’s market capitalization divided by the sum of the market capitalizations of the other nominally underperforming assets. The net long positions less the net short positions equal 0. The mini-portfolio that actually receives the positive view may not be the nominal outperforming asset(s) from the expressed view. In general, if the view is greater than the weighted average Implied Equilibrium return differential, the model will tend to overweight the “outperforming” assets.

From View 3, the nominally “outperforming” assets are General Electric and Home Depot and the nominally “underperforming” assets are General Motors, Wal-Mart and Exxon. From Table 3a, the weighted average Implied Equilibrium return of the mini-portfolio formed from General Electric and Home Depot is 13.33%. And, from Table 3b, the weighted average Implied Equilibrium return of the mini-portfolio formed from General Motors, Wal-Mart and Exxon is 10.39%. Thus, the weighted average Implied Equilibrium return differential is 2.93%.

Table 3a: View 3 – Nominally “Outperforming” Assets

Symbol	Market Capitalization	Relative Weight	Implied Equilibrium Return Vector (Π)	Weighted Return
ge	398104740	76.91%	13.57	10.43
hd	119533620	23.09%	12.52	2.89
	517638361	100.00%	Total	13.33

Table 3b: View 3 – Nominally “Underperforming” Assets

Symbol	Market Capitalization	Relative Weight	Implied Equilibrium Return Vector (Π)	Weighted Return
gm	26997494	4.89%	12.83	0.63
wmt	256505414	46.44%	12.77	5.93
xom	268832829	48.67%	7.88	3.83
	552335738	100.00%	Total	10.39

Given that View 3 states that General Electric and Home Depot will outperform General Motors, Wal-Mart and Exxon by only 1.5% (a reduction from the current weighted average Implied Equilibrium differential of 2.93%), the view appears to actually represent a reduction in the performance of General Electric and Home Depot relative to General Motors, Wal-Mart and Exxon. Skipping ahead temporarily, this point is illustrated below in the final column of Table 4, where the nominally outperforming assets of View 3 receive short positions and the nominally underperforming assets receive long positions.

BUILDING THE INPUTS

One of the more confusing aspects of the model is moving from the stated views to the actual inputs used in the Black-Litterman formula. First, the model does not require that investors specify views on all assets. However, due to the rules of matrix multiplication, the number of views (k) cannot exceed the number of assets (n). In the DJIA example, the number of views (k) = 3; thus, the View Vector ($\boldsymbol{Q}$) is a *3 x 1* column vector. The model assumes that there is a random, independent, normally-distributed error term ($\boldsymbol{\varepsilon}$) with a mean of 0 associated with each view. Thus, a view has the form $\boldsymbol{Q} + \boldsymbol{\varepsilon}$.

DJIA Example: General Case: (2)

$$Q+\varepsilon=\begin{bmatrix}10\\3\\1.5\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad Q+\varepsilon=\begin{bmatrix}Q_1\\\vdots\\Q_k\end{bmatrix}+\begin{bmatrix}\varepsilon_1\\\vdots\\\varepsilon_k\end{bmatrix} \tag{2}$$

The error term ($\boldsymbol{\varepsilon}$) or terms, when there are 2 or more views, do not directly enter the Black-Litterman Formula. However, the variance of each error term ($\boldsymbol{\omega}$) does enter the formula.† The variance of each error term ($\boldsymbol{\omega}$) is equal to the reciprocal of the *level of confidence* (***LC***) of the view multiplied by a *calibration factor* (***CF***). The variances of the error terms ($\boldsymbol{\omega}$) form $\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is a diagonal covariance matrix with 0's in all of the off-diagonal positions. The larger the variance of the error term ($\boldsymbol{\omega}$), the greater the relative uncertainty of the view. The off-diagonal elements of $\boldsymbol{\Omega}$ are assumed to be 0 because the error terms are residuals, which are assumed to be independent of one another. The process of deriving the calibration factor (***CF***) that enables the use of an intuitive 0% to 100% level of confidence rating is discussed below. Each diagonal element ($\boldsymbol{\omega}$) of $\boldsymbol{\Omega}$ equals ***1/LC*CF***.

DJIA Example: General Case: (3)

$$\Omega=\begin{bmatrix}\left(1/.50 * CF\right) & 0 & 0\\ 0 & \left(1/.65 * CF\right) & 0\\ 0 & 0 & \left(1/.30 * CF\right)\end{bmatrix} \qquad \Omega=\begin{bmatrix}\omega_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \omega_k\end{bmatrix} \tag{3}$$

The expressed views in the column vector $\boldsymbol{Q}$ are matched to specific assets by Matrix $\boldsymbol{P}$. Each expressed view results in a *1 x n* row vector. Thus, *k* views result in a *k x n* matrix. In the three-view DJIA example, in which there are 30 stocks, $\boldsymbol{P}$ is a *3 x 30* matrix.

DJIA Example of Matrix $\boldsymbol{P}$ (Based on Satchell and Scowcroft (2000)): (4)

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	½	0	0	0	-⅓	0	0	0	½	0	0	0	0	0	0	0	0	0	0	0	0	0	-⅓	0	0	0	-⅓	0	0

General Case:

$$P=\begin{bmatrix}P_{1,1} & \cdots & P_{1,n}\\ \vdots & \ddots & \vdots\\ P_{k,1} & \cdots & P_{k,n}\end{bmatrix}$$

† $\boldsymbol{\omega}$, which represents the variance of an *individual* error term, should not be confused with $\boldsymbol{\acute{\omega}}$ or $\boldsymbol{w}$. $\boldsymbol{\acute{\omega}}$ represents the *average* variance of the error terms ($\boldsymbol{\omega}$) and $\boldsymbol{w}$ represents the weight vector.

The first row of Matrix ***P***, in the DJIA example, represents View 1, the absolute view. View 1 only involves one asset, Merck. Sequentially, Merck is the 27th asset in the DJIA, which corresponds with the "1" in the 27th column of row 1. View 2 and View 3 are represented by row 2 and row 3, respectively. In the case of relative views, each row sums to 0. The nominally outperforming assets receive positive weightings, while the nominally underperforming assets receive negative weightings.

For views involving 3 or more assets, such as View 3, two materially different methods are presented in the literature. Satchell and Scowcroft (2000) use an equal weighting scheme, which is presented in row 3 of Matrix ***P***. Under this system, the weightings are proportional to 1 divided by the number of respective assets outperforming or underperforming. For example, View 3 has three nominally underperforming assets, each of which receives a negative ⅓ weighting. View 3 also contains two nominally outperforming assets, each receiving a positive ½ weighting. This weighting scheme ignores the relative size of the assets involved in the view. For example, General Electric is more than three times the size of Home Depot; yet, the Satchel and Scowcroft method reduces their respective weights equally. Thus, in percentage terms, Home Depot experiences a large decrease in its weighting relative to the decrease in General Electric, which may result in undesired and unnecessary active risk (tracking error).

Contrasting with the Satchell and Scowcroft (2000) equal weighting scheme for views involving 3 or more assets, in He and Litterman (1999), the relative weightings of the assets entering Matrix ***P*** are proportional to their market capitalizations.[8] More specifically, the relative weighting of each individual asset is proportional to the asset's market capitalization divided by the total market capitalization of either the outperforming or underperforming assets of that particular view. Returning to the third column of Table 3a, the relative market capitalization weights of the nominally outperforming asset are 0.7691 for General Electric and 0.2309 for Home Depot. And, from Table 3b, the relative market capitalization weights of the nominally underperforming assets are -0.0489 for General Motors, -0.4644 for Wal-Mart, and -0.4867 for Exxon. These figures have been used to create a new Matrix ***P*** (located in Appendix A), which is used in all of the subsequent calculations.

While none of the views in DJIA example involved the same asset more than once, one may include the same asset in more than one view. For example, Johnson & Johnson, the subject of View 2, could have also been included in View 1, View 3, or any subsequent views.

Matrix ***P***, the still unknown Combined Return Vector ***E[R]***, the View Vector (***Q***), and the Error Term Vector (***ε***) form a system of linear constraints. Recall that the Error Term Vector (***ε***) does not directly enter the Black-Litterman formula. It is not necessary to separately build this system of linear constraints; it is implicit in the Black-Litterman formula and is presented here, in its general form only (due to space constraints), to provide additional insight into the workings of the model.

General Case: (5)

$$\begin{bmatrix} P_{1,1} & \cdots & P_{1,n} \\ \vdots & \ddots & \vdots \\ P_{k,1} & \cdots & P_{k,n} \end{bmatrix} \begin{bmatrix} E[R_1] \\ \vdots \\ E[R_n] \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix}$$

The final and most subjective value that enters the Black-Litterman formula is the scalar (***τ***). Unfortunately, there is very little guidance in the literature for setting the scalar's value. Both Black and Litterman (1992) and Lee (2000) address this issue: since the uncertainty in the mean is less than the uncertainty in the return, the scalar (***τ***) is close to zero. Conversely, Satchell and Scowcroft (2000) say the value is often set to 1.[9]

Considering that the Black-Litterman Model is described as a complex, weighted average of the Implied Equilibrium Return Vector (***Π***) and the investor's views (***Q***), in which the relative weightings are a function of the scalar (***τ***) and the average confidence level of the views, the greater the level of confidence in the expressed views, the closer the new return vector will be to the views. If the investor is less confident in the expressed views, the new return vector will be closer to the Implied Equilibrium Return Vector (***Π***). The scalar (***τ***) is more or less inversely proportional to the relative weight given to the Implied Equilibrium returns.

He and Litterman (1999) calibrate the confidence so that the ratio of ***ώ/τ*** is equal to the variance of the view portfolio. When there is only one view, ***ώ*** equals ***ω***, where ***ω*** equals ***1/LC*CF***. The variance of the view portfolio is the sum of the elements of the *k x k* matrix product of ***PΣP′***.[10] When more than one view is expressed, there are two possible interpretations of ***ώ***. ***ώ*** can be interpreted as either the sum *or* the average value of the diagonal elements (***ω***) of the covariance matrix of the error term (***Ω***), where the sum represents the total uncertainty in the views, and the average represents the average uncertainty of the views. Previously, it was established that each diagonal element (***ω***) of the covariance matrix of the error term (***Ω***) is equal to the reciprocal of the level of confidence (***LC***) of that particular view multiplied by a calibration factor (***CF***). The reciprocals of 90%, 50%, and 10% are 1.11, 2.0, and 10.0, respectively; thus, views with significant confidence result in relatively small ***ω***'s. Low levels of confidence can have a significant impact on the model if the sum of the diagonal elements (***ω***) of the covariance matrix of the error term (***Ω***) is used to set the value of the scalar (***τ***). For this reason, greater stability is achieved by setting the value of ***ώ*** equal to the average value of the diagonal elements (***ω***) of the covariance matrix of the error term (***Ω***), as in Formula 6.

General Case: (6)

$$\acute{\omega} = \frac{\sum_{i=1}^{k} \left({}^{1}\!/\!_{LC_i} * CF \right)}{k}$$

Based on my interpretation of He and Litterman (1999), the *initial* value of the scalar (***τ***) should equal the *average* value of the variance of the error terms (***ω***) divided by the variance of the view portfolio.

General Case: (7a and 7b)

$$\tau = \frac{\dfrac{\sum_{i=1}^{k} \left({}^{1}\!/\!_{LC_i} * CF \right)}{k}}{P\Sigma P'} \qquad\qquad P\Sigma P' = \frac{\dfrac{\sum_{i=1}^{k} \left({}^{1}\!/\!_{LC_i} * CF \right)}{k}}{\tau}$$

However, at this point, the variance of the view portfolio is the only observable variable. Formula 7a can be rearranged to form Formula 7b. From Formula 7b, it is evident that for any possible positive value for the average value of the variance of the error terms (***ω***), the value of the scalar (***τ***) must change so that the ratio is equal to the variance of the view portfolio. It is my opinion that this constant ratio provides the ideal reference point for the calibration of the variances of the error terms. The calibration factor (***CF***) should enable the 0% to 100% level of confidence (***LC***) scale to lead to intuitive portfolios. Conceptually, the level of confidence (***LC***) scale can be viewed as a normally distributed value with a mean of 50% and a standard deviation of 16.33%, in which 0% and 100% correspond to values three standard deviations from the mean. When the model is properly calibrated, the deviations from the market capitalization weights should approach 0 as the average level of confidence (***LC***) approaches 0%. Similarly, as the average level of confidence (***LC***) approaches 100%, the magnitude of the deviations from the market capitalization weights that are recommended at the baseline 50% level of confidence (***LC***), should approximately double. This is accomplished by setting the value of the scalar ($\boldsymbol{\tau}$) equal to 1, and inverting the right-hand side of Formula 7a. When the scalar equals 1, $\boldsymbol{P\Sigma P'}$ equals $\boldsymbol{\omega}$.

General Case: (8)

$$\tau = \frac{P\Sigma P'}{\dfrac{\sum_{i=1}^{k}\left(\frac{1}{LC_i} * CF\right)}{k}}$$

Combining this information with the expected average level of confidence (***LC***) of 50%, it is possible to derive the calibration factor (***CF***), which is equal to 0.1403 for the DJIA example.

DJIA Example: General Case: (9)

$$CF = \frac{.2806}{\dfrac{1}{50\%}} \qquad CF = \frac{P\Sigma P'}{\dfrac{1}{50\%}}$$

Once the calibration factor (***CF***) is determined, the variance of the error term (which is set equal to 1 divided by the level of confidence multiplied by the calibration factor) for each view is calculated and the value of the scalar ($\boldsymbol{\tau}$) is derived. As one would expect, setting the level of confidence (***LC***) of each view equal to 50% creates an average value of the diagonal elements ($\boldsymbol{\omega}$) of $\boldsymbol{\Omega}$ of 0.2806, which is equal to the variance of the view portfolio and results in a value of 1 for the scalar ($\boldsymbol{\tau}$). Using Formula 10, the average value of the diagonal elements ($\boldsymbol{\omega}$) of $\boldsymbol{\Omega}$ for the respective levels of confidence (***LC***) of the three views in the DJIA example (50%, 65%, and 30%) is 0.321. Using 0.321 as the divisor in Formula 8, the value of the scalar ($\boldsymbol{\tau}$) is 0.873.

DJIA Example: General Case: (10)

$$\Omega = \begin{bmatrix} \left(\frac{1}{.50} * .14\right) & 0 & 0 \\ 0 & \left(\frac{1}{.65} * .14\right) & 0 \\ 0 & 0 & \left(\frac{1}{.30} * .14\right) \end{bmatrix} \qquad \Omega = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_k \end{bmatrix}$$

Using Formula 8, it is easy to see that as the average value of the diagonal elements ($\boldsymbol{\omega}$) of $\boldsymbol{\Omega}$ increases relative to the sum of the elements of the variance of the view portfolio matrix ($\boldsymbol{P\Sigma P'}$), the scalar ($\boldsymbol{\tau}$) decreases. As the scalar ($\boldsymbol{\tau}$) decreases, more weight is given to the Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$) and less weight is given to the investor's views. A graph of the DJIA portfolios based on levels of confidence (***LC***) of 50%, 10%, and 90% is presented in Appendix B.

Interestingly, Black and Litterman (1990) explain that placing 100% weight on the views (100% confidence in each of the views) does not cause the model to ignore the Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$), unless the "number of views is equal to the dimensionality of the expected return vector and 100% confidence is given to the views."

THE NEW COMBINED VECTOR

Having specified the value of the scalar (***τ***), all of the inputs are then entered into the Black-Litterman Formula and the New Combined Vector of Returns (***E[R]***) is derived. The covariance matrix of historical returns (***Σ***), the only input not shown in the text, is provided in Appendix A. Table 4 contains seven columns: the DJIA Components, the New Combined Return Vector (***E[R]***), the Implied Equilibrium Return Vector (***Π***), the difference between ***E[R]*** and ***Π***, the New Recommended Weights ($\hat{w}$) based on the New Combined Return Vector(***E[R]***), the Original Market Capitalization Weights (***w***) and the difference between $\hat{w}$ and ***w***.

Even though the expressed views only directly involved 8 of the 30 DJIA components [General Electric (ge), Johnson & Johnson (jnj), General Motors (gm), Procter & Gamble (pg), Home Depot (hd), Wal-Mart (wmt), Merck (mrk) and Exxon (xom)], the individual returns of all 30 DJIA components changed from their respective Implied Equilibrium returns.[11] In fact, it is not uncommon for a single view to cause the return of every asset in the portfolio to change from its Implied Equilibrium return, since each individual return is linked to the other returns via the covariance matrix of returns (***Σ***).

Table 4: Return Vectors and Resulting Portfolio Weights

Symbol	New Combined Return Vector (***E[R]***)	Implied Equilibrium Return Vector (***Π***)	Difference (***E[R]*** - ***Π***)	New Weight ($\hat{w}$)	Market Capitalization Weight (***w***)	Difference ($\hat{w}$ - ***w***)
aa	13.78	13.81	-0.03	0.88%	0.88%	--
ge	13.52	13.57	-0.05	10.77%	11.62%	-0.85%
jnj	9.92	9.75	0.17	6.28%	5.29%	0.99%
msft	20.40	20.41	-0.01	10.41%	10.41%	--
axp	14.98	14.94	0.04	1.39%	1.39%	--
gm	12.80	12.83	-0.03	0.84%	0.79%	0.05%
jpm	16.43	16.46	-0.03	2.09%	2.09%	--
pg	7.47	7.56	-0.09	2.00%	2.99%	-0.99%
ba	11.87	11.81	0.05	0.90%	0.90%	--
hd	12.42	12.52	-0.10	3.23%	3.49%	-0.25%
ko	11.02	10.92	0.09	3.42%	3.42%	--
sbc	8.89	8.79	0.10	3.84%	3.84%	--
c	17.01	16.97	0.03	7.58%	7.58%	--
hon	14.48	14.50	-0.02	0.80%	0.80%	--
mcd	10.52	10.44	0.08	0.99%	0.99%	--
t	10.69	10.74	-0.05	1.87%	1.87%	--
cat	11.02	10.92	0.10	0.52%	0.52%	--
hwp	14.31	14.45	-0.14	1.16%	1.16%	--
mmm	8.67	8.66	0.02	1.35%	1.35%	--
utx	15.49	15.47	0.02	0.88%	0.88%	--
dd	10.98	10.98	0.00	1.29%	1.29%	--
ibm	14.65	14.66	-0.01	6.08%	6.08%	--
mo	6.91	6.86	0.05	2.90%	2.90%	--
wmt	12.83	12.77	0.06	8.00%	7.49%	0.51%
dis	12.44	12.41	0.03	1.23%	1.23%	--
intc	18.72	18.70	0.02	6.16%	6.16%	--
mrk	9.44	9.22	0.22	4.68%	3.90%	0.77%
xom	7.91	7.88	0.04	8.38%	7.85%	0.53%
ek	10.57	10.61	-0.04	0.25%	0.25%	--
ip	12.97	12.92	0.05	0.57%	0.57%	--
			Sum	100.77%	100.00%	0.77%

One of the best features of the Black-Litterman Model is illustrated in the final column of Table 4. Only the weights of the 8 DJIA components for which views were expressed changed from their original market capitalization weights. Furthermore, the directions of the changes are intuitive.

From a macro perspective, the new portfolio can be viewed as the sum of two portfolios, where Portfolio 1 is the original market capitalization weighted portfolio (the DJIA), and Portfolio 2 is a series of long and short positions based on the expressed views. As discussed earlier, Portfolio 2 can be sub-divided into mini-portfolios, each associated with a specific view. The relative views result in mini-portfolios with offsetting long and short positions that sum to 0. View 1, the absolute view, increases the weight of Merck without an offsetting stock position, resulting in portfolio weights that no longer sum to unity (weights that sum to 1).

Unfortunately, the intuitiveness of the Black-Litterman Model is difficult to see with added investment constraints, such as constraints on unity, risk, beta, and short selling.[12] Black and Litterman (1999) suggest that, in the presence of constraints, the investor input the New Combined Return Vector into a mean-variance optimizer.

FINE TUNING THE MODEL

Having worked with the Black-Litterman Model for three years, Bevan and Winkelmann (1998) offer additional insights to fine tune the model. Working from a global asset allocation perspective, they recommend calibrating the expected returns to produce a portfolio with a Sharpe Ratio of 1.0, which they say is equivalent to a 1 standard deviation event and roughly in line with the historical average of the G7 countries. Such a generalization from country markets to an individual stock-based example would require considerable refinement.

Of greater interest, Bevan and Winkelmann (1998) offer additional guidance in setting the weight given to the view portfolio. During my discussion of the scalar (τ), its value was set based on the ratio of the average uncertainty of the views relative to the variance of the view portfolio: the ratio of the average value of the calibrated diagonal elements (ω) of the covariance matrix of the error term (Ω) and the sum of the elements of the k x k matrix product of $P\Sigma P'$. After deriving an initial Combined Return Vector ($E[R]$) using the Black-Litterman Model and calculating the optimum portfolio weights, Bevan and Winkelmann (1998) calculate the anticipated Information Ratio of the new portfolio. They recommend a maximum anticipated Information Ratio of 2.0, citing that a return greater than 2 standard deviations above the market capitalization weighted benchmark is unlikely. If the Information Ratio is above 2.0, decrease the weight given to the views (decrease the value of the scalar).

Returning to the DJIA example, Table 5 compares the anticipated risk-return characteristics of the Black-Litterman portfolio (the new weights produced by the New Combined Return Vector) with the market capitalization weighted portfolio.[13] The return and standard deviation of the new portfolio increase slightly. The ex ante Information Ratio is well below the recommended maximum of 2.0.

Table 5: Portfolio Statistics

	Market Capitalization Weighted Portfolio	New Black-Litterman Portfolio
Return	13.23%	13.53%
Std. Dev.	19.12%	19.37%
Excess Return	8.23%	8.32%
Tracking Error	-	0.49%
Beta	1.000	1.003
Residual Risk	-	0.49%
Sharpe Ratio	0.430	0.434
Information Ratio	-	0.108

Next, the results of the expressed views should be evaluated to confirm that there are no unintended results from the views. For example, investors confined to unity may want to remove the absolute view, View 1. Positive absolute views without off-setting negative absolute views, and vice versa, can also lead to undesired portfolios. For example, removing View 1 from the DJIA decreases the variance of the view portfolio by nearly 50%, which, in turn, reduces the calibration factor (***CF***) by a proportional amount. Finally, Bevan and Winkelmann (1998) recommend that investors make "micro-level" changes by adjusting the confidence level of the individual views.

Investors may want to evaluate their ex post Information Ratio for additional guidance when setting the weight on views vs. the weight on the Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$). A quantitative manager who receives "views" from a variety of analysts, or sources, could set the level of confidence of a particular view based on that particular analyst's information coefficient. According to Grinold and Kahn (1999), a manager's information coefficient is the correlation of forecasts with the actual returns. This would give greater relative importance to the more skillful analysts.

In addition to adjusting the confidence level of the individual views, another possible micro-level refinement is the development of a more sophisticated method of weighting the "outperforming" and "underperforming" mini-asset portfolios. While the He and Litterman (1999) market capitalization weighting method is clearly superior to the Satchell and Scowcroft (2000) equal weighting method, both approaches create mini-portfolios that do not maximize returns for their given levels of risk. An advanced weighting system could lead to deviations from the benchmark holdings that maximize the Information Ratio or an alternative objective function.

Most of the examples in the literature, including this one, use a simple covariance matrix of historical returns. However, looking forward, investors should use the best possible estimate of the covariance matrix of returns. Litterman and Winkelmann (1998) outline the method they prefer for estimating the covariance matrix of returns, as well as several alternative methods of estimation. Qian and Gorman (2001) have extended the Black-Litterman Model, enabling investors to express views on volatilities and correlations in order to derive a conditional estimate of the covariance matrix of returns. They claim that the conditional covariance matrix stabilizes the results of mean-variance optimization.

CONCLUSION

The Black-Litterman Model enables investors to combine their unique views with the Implied Equilibrium Return Vector to form a New Combined Return Vector. The investor can control the relative weight placed on the views vs. the equilibrium. At the micro level, the strength placed on each of the views is controlled by the individual level of confidence given to that view. The New Combined Return Vector leads to intuitive, well-balanced portfolios. Investors who do not have views on expected returns should hold the market portfolio. If investment constraints exist, investors should use the Black-Litterman Model to form the New Combined Return Vector, and then input the vector into a mean-variance optimizer. Having overcome the two most often cited weaknesses of mean-variance optimization, input-sensitivity and estimation error-maximization, users of the Black-Litterman Model are able to realize the benefits of the Markowitz paradigm.

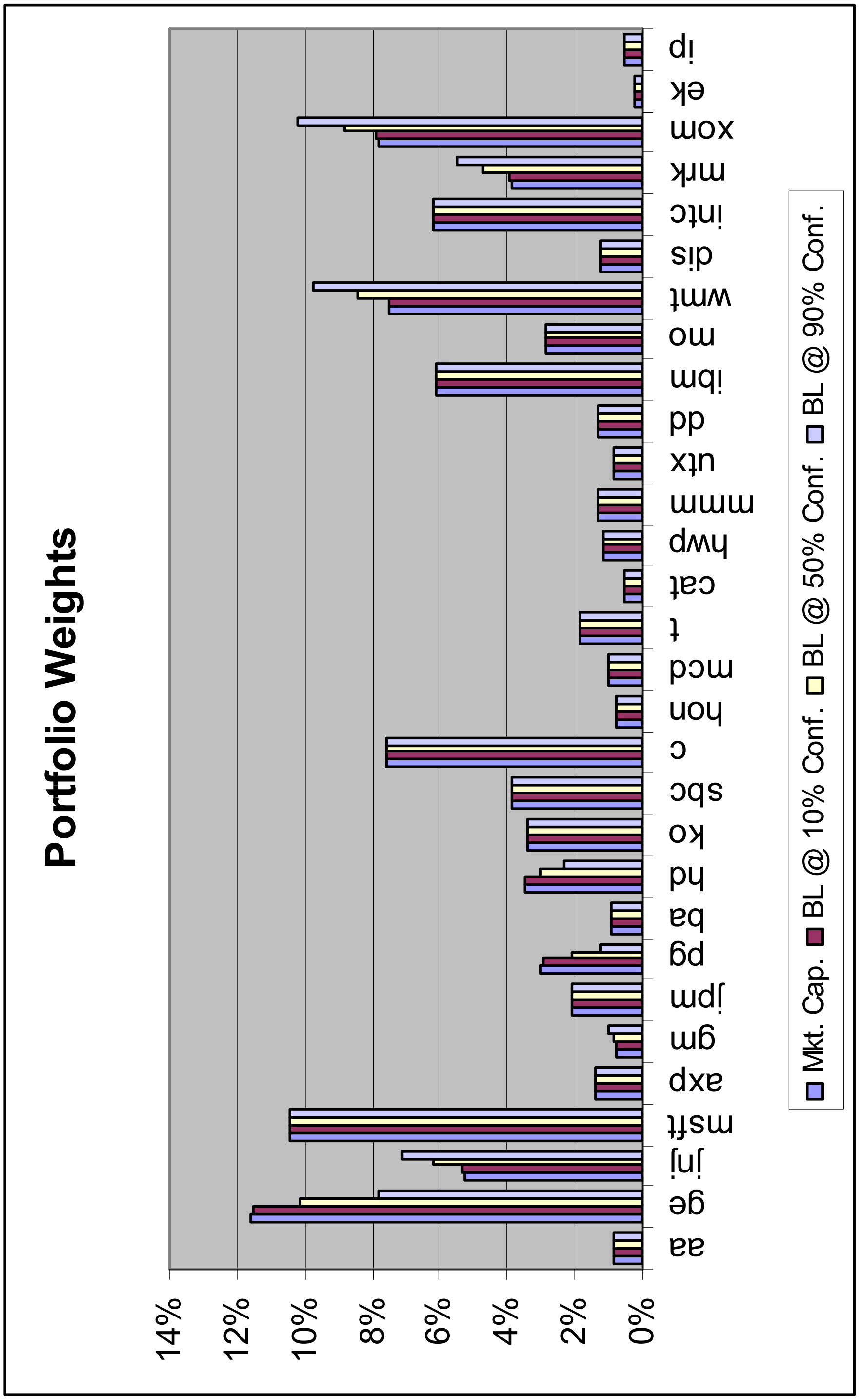
Appendix A

Matrix ***P*** (Based on He and Litterman (1999))

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	.76907	0	0	0	-.04888	0	0	0	.23092	0	0	0	0	0	0	0	0	0	0	0	0	0	-.46440	0	0	0	-0.48672	0	0

Covariance Matrix of Historical Returns ($\boldsymbol{\Sigma}$)

	aa	ge	jnj	msft	axp	gm	jpm	pg	ba	hd	ko	sbc	c	hon	mcd	t	cat	hwp	mmm	utx	dd	ibm	mo	wmt	dis	intc	mrk	xom	ek	ip
aa	0.147	0.030	0.008	0.063	0.047	0.047	0.065	0.001	0.041	0.038	0.005	0.003	0.055	0.083	0.014	0.014	0.075	0.102	0.050	0.073	0.052	0.060	0.014	0.023	0.036	0.103	-0.010	0.017	0.074	0.091
ge	0.030	0.069	0.021	0.070	0.045	0.033	0.047	0.012	0.026	0.049	0.016	0.019	0.051	0.040	0.026	0.020	0.015	0.035	0.014	0.056	0.030	0.034	0.004	0.041	0.032	0.041	0.014	0.009	0.025	0.029
jnj	0.008	0.021	0.070	0.018	0.032	0.009	0.007	0.032	0.024	-0.001	0.041	0.021	0.031	0.010	0.027	-0.012	0.023	-0.018	0.016	0.026	0.013	0.006	0.019	0.026	0.006	0.006	0.048	0.010	0.011	0.018
msft	0.063	0.070	0.018	0.247	0.040	0.039	0.069	-0.016	0.017	0.061	0.012	0.018	0.057	0.031	0.018	0.066	0.010	0.084	0.003	0.058	0.030	0.087	0.001	0.049	0.037	0.150	0.006	0.009	0.045	0.038
axp	0.047	0.045	0.032	0.040	0.103	0.052	0.075	0.032	0.074	0.049	0.051	0.020	0.087	0.073	0.043	0.011	0.053	0.041	0.026	0.076	0.047	0.034	0.006	0.041	0.058	0.049	0.041	0.027	0.028	0.058
gm	0.047	0.033	0.009	0.039	0.052	0.129	0.076	0.012	0.045	0.037	0.007	-0.008	0.073	0.056	0.027	0.061	0.028	0.080	0.017	0.054	0.020	0.046	-0.007	0.015	0.069	0.088	0.009	0.008	0.035	0.021
jpm	0.065	0.047	0.007	0.069	0.075	0.076	0.160	0.019	0.032	0.059	0.036	-0.003	0.110	0.069	0.030	0.042	0.050	0.072	0.022	0.076	0.040	0.056	0.005	0.054	0.065	0.093	0.020	0.007	0.035	0.044
pg	0.001	0.012	0.032	-0.016	0.032	0.012	0.019	0.102	0.024	0.003	0.048	0.003	0.023	0.038	0.018	-0.017	0.006	0.000	0.013	0.012	0.021	-0.015	0.010	0.021	-0.008	-0.023	0.032	0.008	0.005	0.010
ba	0.041	0.026	0.024	0.017	0.074	0.045	0.032	0.024	0.136	0.017	0.039	0.017	0.056	0.067	0.031	-0.003	0.055	0.026	0.048	0.079	0.041	0.026	0.013	0.011	0.057	0.037	0.036	0.027	0.021	0.056
hd	0.038	0.049	-0.001	0.061	0.049	0.037	0.059	0.003	0.017	0.109	0.002	0.013	0.042	0.026	0.028	0.051	0.021	0.055	0.012	0.064	0.022	0.032	-0.018	0.062	0.031	0.022	-0.008	0.002	0.006	0.036
ko	0.005	0.016	0.041	0.012	0.051	0.007	0.036	0.048	0.039	0.002	0.100	0.026	0.060	0.032	0.032	0.002	0.014	-0.001	0.019	0.027	0.029	0.006	0.017	0.029	0.025	0.001	0.048	0.020	0.005	0.028
sbc	0.003	0.019	0.021	0.018	0.020	-0.008	-0.003	0.003	0.017	0.013	0.026	0.093	0.016	0.022	0.023	0.041	0.018	-0.014	0.010	0.016	0.005	0.003	0.017	0.026	0.005	-0.025	0.035	0.012	0.003	0.035
c	0.055	0.051	0.031	0.057	0.087	0.073	0.110	0.023	0.056	0.042	0.060	0.016	0.134	0.063	0.034	0.030	0.050	0.047	0.032	0.073	0.050	0.060	0.000	0.052	0.067	0.072	0.024	0.021	0.031	0.046
hon	0.083	0.040	0.010	0.031	0.073	0.056	0.069	0.038	0.067	0.026	0.032	0.022	0.063	0.210	0.056	-0.003	0.080	0.070	0.038	0.095	0.064	0.048	0.036	0.027	0.053	0.070	0.035	0.029	0.058	0.100
mcd	0.014	0.026	0.027	0.018	0.043	0.027	0.030	0.018	0.031	0.028	0.032	0.023	0.034	0.056	0.075	0.012	0.030	0.006	0.011	0.049	0.022	0.017	0.006	0.034	0.029	0.010	0.035	0.014	0.006	0.041
t	0.014	0.020	-0.012	0.066	0.011	0.061	0.042	-0.017	-0.003	0.051	0.002	0.041	0.030	-0.003	0.012	0.168	-0.018	0.047	-0.010	0.004	-0.021	0.053	-0.012	0.018	0.030	0.050	-0.010	-0.005	0.005	0.016
cat	0.075	0.015	0.023	0.010	0.053	0.028	0.050	0.006	0.055	0.021	0.014	0.018	0.050	0.080	0.030	-0.018	0.111	0.008	0.046	0.060	0.059	0.028	0.016	0.031	0.026	0.035	0.019	0.024	0.041	0.080
hwp	0.102	0.035	-0.018	0.084	0.041	0.080	0.072	0.000	0.026	0.055	-0.001	-0.014	0.047	0.070	0.006	0.047	0.008	0.226	0.017	0.059	0.026	0.086	0.010	0.008	0.065	0.142	-0.023	0.013	0.051	0.032
mmm	0.050	0.014	0.016	0.003	0.026	0.017	0.022	0.013	0.048	0.012	0.019	0.010	0.032	0.038	0.011	-0.010	0.046	0.017	0.063	0.041	0.041	0.016	0.016	0.012	0.022	0.011	0.005	0.015	0.021	0.047
utx	0.073	0.056	0.026	0.058	0.076	0.054	0.076	0.012	0.079	0.064	0.027	0.016	0.073	0.095	0.049	0.004	0.060	0.059	0.041	0.132	0.046	0.043	0.015	0.057	0.066	0.052	0.019	0.018	0.032	0.067
dd	0.052	0.030	0.013	0.030	0.047	0.020	0.040	0.021	0.041	0.022	0.029	0.005	0.050	0.064	0.022	-0.021	0.059	0.026	0.041	0.046	0.084	0.018	0.007	0.024	0.020	0.020	0.016	0.023	0.025	0.060
ibm	0.060	0.034	0.006	0.087	0.034	0.046	0.056	-0.015	0.026	0.032	0.006	0.003	0.060	0.048	0.017	0.053	0.028	0.086	0.016	0.043	0.018	0.118	0.022	0.020	0.044	0.107	-0.016	0.021	0.039	0.038
mo	0.014	0.004	0.019	0.001	0.006	-0.007	0.005	0.010	0.013	-0.018	0.017	0.017	0.000	0.036	0.006	-0.012	0.016	0.010	0.016	0.015	0.007	0.022	0.097	-0.008	-0.004	-0.004	0.017	0.007	0.026	0.016
wmt	0.023	0.041	0.026	0.049	0.041	0.015	0.054	0.021	0.011	0.062	0.029	0.026	0.052	0.027	0.034	0.018	0.031	0.008	0.012	0.057	0.024	0.020	-0.008	0.099	0.011	0.009	0.021	0.004	-0.005	0.031
dis	0.036	0.032	0.006	0.037	0.058	0.069	0.065	-0.008	0.057	0.031	0.025	0.005	0.067	0.053	0.029	0.030	0.026	0.065	0.022	0.066	0.020	0.044	-0.004	0.011	0.107	0.069	0.024	0.014	0.037	0.028
intc	0.103	0.041	0.006	0.150	0.049	0.088	0.093	-0.023	0.037	0.022	0.001	-0.025	0.072	0.070	0.010	0.050	0.035	0.142	0.011	0.052	0.020	0.107	-0.004	0.009	0.069	0.284	0.007	0.007	0.089	0.050
mrk	-0.010	0.014	0.048	0.006	0.041	0.009	0.020	0.032	0.036	-0.008	0.048	0.035	0.024	0.035	0.035	-0.010	0.019	-0.023	0.005	0.019	0.016	-0.016	0.017	0.021	0.024	0.007	0.102	0.006	-0.002	0.024
xom	0.017	0.009	0.010	0.009	0.027	0.008	0.007	0.008	0.027	0.002	0.020	0.012	0.021	0.029	0.014	-0.005	0.024	0.013	0.015	0.018	0.023	0.021	0.007	0.004	0.014	0.007	0.006	0.028	0.001	0.024
ek	0.074	0.025	0.011	0.045	0.028	0.035	0.035	0.005	0.021	0.006	0.005	0.003	0.031	0.058	0.006	0.005	0.041	0.051	0.021	0.032	0.025	0.039	0.026	-0.005	0.037	0.089	-0.002	0.001	0.130	0.041
ip	0.091	0.029	0.018	0.038	0.058	0.021	0.044	0.010	0.056	0.036	0.028	0.035	0.046	0.100	0.041	0.016	0.080	0.032	0.047	0.067	0.060	0.038	0.016	0.031	0.028	0.050	0.024	0.024	0.041	0.136


Portfolio Weights
14%
12%
10%
8%
6%
4%
2%
0%
aa
ge
jnj
msft
axp
gm
jpm
pg
ba
hd
ko
sbc
c
hon
mcd
t
cat
hwp
mmm
utx
dd
ibm
mo
wmt
dis
intc
mrk
xom
ek
ip
Mkt. Cap.
BL @ 10% Conf.
BL @ 50% Conf.
BL @ 90% Conf.

[1] Black and Litterman (1990) state that Theil (1971) explains the "mixed estimation" procedure for combining views with a prior.

[2] The formula, $\boldsymbol{\Pi} = \boldsymbol{\delta}\, \boldsymbol{\Sigma}\, \boldsymbol{w}$, is found in almost all of the articles related to the Black-Litterman Model. Readers unfamiliar with matrix algebra will be surprised at how easy it is to solve for an unknown vector using Excel's matrix functions (MMULT, TRANSPOSE, and MINVERSE). For a primer on Excel matrix procedures, please go to http://www.stanford.edu/~wfsharpe/mia/mat/mia_mat4.htm or http://www.massacademy.org/Courses/CS/notes/excel/excel03/excel03.html

[3] According to Black and Litterman (1992), $\boldsymbol{\delta}$ is the proportionality constant as defined in Black (1989). He and Litterman (1999) state that $\boldsymbol{\delta}$ is the risk aversion parameter representing the world average risk tolerance. Both Satchell and Scowcroft (2000) and Best and Grauer (1985) set $\boldsymbol{\delta} = (r_m - r_f) / \sigma_m^2$. Assuming a constant market risk premium ($r_m - r_f$) of 7.5% coupled with the 5-year average historical standard deviation of the DJIA time-series of returns (18.25%) produces a risk aversion figure of 2.25 (which is used for this paper). Alternatively, the standard deviation of the DJIA could be based on the historical covariance matrix of returns ($\boldsymbol{\Sigma}$) using the market capitalization weights ($\boldsymbol{w}$). Under this alternative, $\sigma_m^2 = w' \Sigma w$, where $\boldsymbol{w}$ is the *n x 1* column vector of the market capitalization weights and $\boldsymbol{w}'$ is the transpose of the vector. Using this method and the same market risk premium of 7.5%, the standard deviation is 19.12% and $\boldsymbol{\delta} = 2.05$. The estimates of standard deviation are different because the components of the DJIA have changed during the relative time frame (5 years), causing a discrepancy between the historical covariance matrix of the current DJIA components and the historical time-series of DJIA.

[4] Using a market proxy with different risk-return characteristics than the market capitalization weighted portfolio (for which one is extracting the Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$)) to set the value of the risk-aversion parameter ($\boldsymbol{\delta}$) can lead to an unintuitive vector of returns. For example, using a risk aversion parameter ($\boldsymbol{\delta}$) based on the risk-return characteristics of the S&P 500 to extract the Implied Equilibrium return vector ($\boldsymbol{\Pi}$) for the NASDAQ 100 leads to a rather extreme vector of returns.

[5] Throughout this article, total returns are used rather than excess returns. The small differences in the procedure are noted along the way. In this case, after deriving the Implied Equilibrium Return Vector ($\boldsymbol{\Pi}$), using the formula $\boldsymbol{\Pi} = \boldsymbol{\delta}\, \boldsymbol{\Sigma}\, \boldsymbol{w}$, one must add the risk-free rate to each of the excess return figures.

[6] Although they are not presented, CAPM estimates of return based on S&P 500 betas were also calculated. The S&P 500-based CAPM estimates of return closely approximated the DJIA-based estimates of CAPM in Table 1. The constant risk-free rate of 5% approximates the yield on the 10-year Treasury Bond at the end of 2001.

[7] From He and Litterman (1999), the unconstrained maximization problem $\max_w w'\mu - \delta w'\Sigma w/2$ has the solution of $w = (\delta\Sigma)^{-1}\mu$, where $\boldsymbol{\delta}$ is the risk aversion parameter, $\boldsymbol{\Sigma}$ is the covariance matrix of returns, and μ is the expected return vector. When using total returns, one must subtract the risk-free interest rate from each of the total returns prior to solving for the weight vector ($\boldsymbol{w}$).

[8] In He and Litterman (1999), the market capitalization method is not explicitly explained; however, from Footnote 4 and Charts 2B and 2C of their article, it is evident that they are using a market capitalization method.

[9] Satchell and Scowcroft (2000) include an advanced mathematical discussion of a method for establishing a conditional value for the scalar ($\boldsymbol{\tau}$) that is similar to the method used by He and Litterman (1999).

[10] Alternatively, one could sum each column of the *k x n* Matrix $\boldsymbol{P}$ matrix to produce $\boldsymbol{P^*}$, a *1 x n* row vector representing the aggregate weights of the views. In which case $\boldsymbol{P^* \Sigma P^{*\prime}}$ is the variance of the view portfolio.

[11] Although the new return figure for Dupont (dd) does not appear to change in Table 4, the return is different beginning with the third decimal place.

[12] In the DJIA example, it was established that the new portfolio can be segmented into two separate portfolios, one of which contained short positions in the four underperforming stocks. However, none of these actually required a net short position since the total position in each of these four stocks remained positive. More dramatic views can result in a net short position.

[13] The data in Table 5 is based on the Implied Betas based on the covariance matrix of historical returns and the mean-variance data of the market capitalization weighted benchmark portfolio. From Grinold and Kahn (1999):

Active Risk $\Psi_P = \sqrt{\omega_P^2 + \beta_{PA}^2 * \sigma_B^2}$

Residual Risk $\omega_P = \sqrt{\sigma_P^2 + \beta_{PA}^2 * \sigma_B^2}$

Active Portfolio Beta $\beta_{PA} = (\beta_P - 1)$

Where:

σ_B is the Variance of the Benchmark Portfolio

σ_P is the Variance of the Portfolio

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