Quantum State Engineering by Shortcuts to Adiabaticity

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Abstract

In this term paper I discuss about the ways to engineer non-classical states of bosonic modes by exploiting a coherent exchange of excitations with a two level system governed by the Jaynes-Cummings model. The protocol aims to speed up the quantum adiabatic process by introducing shortcuts to adiabaticity. The short evolution time makes sure that the protocol is resistant to decoherence. I also discuss about the generation of Fock states, cat like states and photon shifted states and their implementations in Ytterbium ion qubit.

Contents

1 Introduction

The first generation of non classical states was attained in a trapped ion experiment. In this paper I will discuss about the implementaion of generation of non-classical states in a well controlled Yt-terbium qubit. Unlike classical states, non-classical states are prone to decoherence and are fragile against noise sources. Hence, fast and robust protocols are needed for their successful generation. At present, the Stimulated Raman Adiabatic Passage (STIRAP) and dynamical decoupling schemes are employed. But, these are also prone to noise and dissipation owing to their slow evolution time.

To overcome these drawbacks, protocols at coherent level are designed. These are called Shortcuts to Adiabaticity (STA) which aim at speeding up the quantum adiabatic process. Their short evolution time make them resistant to reservoir effects. We add auxiliary driving of the Hamiltonian to speed up the process while approximately maintaining the adiabatic process. This is called Counterdiabatic driving. This suppresses non-adiabatic transitions between instantaneous eigenstates.

In this paper, I first discuss about the basic prerequisites on elementary quantum mechanics and quantum optics. Then, I discuss the STA protocols to generate non-classical states like Fock states, schrödinger cat like states and Photon shifted states. Finally the numerical simulations, robustness and implementations are discussed.

2 Basic Theory

2.1 The Jaynes-Cummings Model

The Jaynes-Cummings(JC) model describes the interaction of a quantised radiation field E with a level system. The hamiltonian of such phenomenon for two level system in the JC model is given by the sum of unperturbed hamiltonian of free field and free atom along with perturbed Hamiltonian which describes the interaction between radiation and two level system.

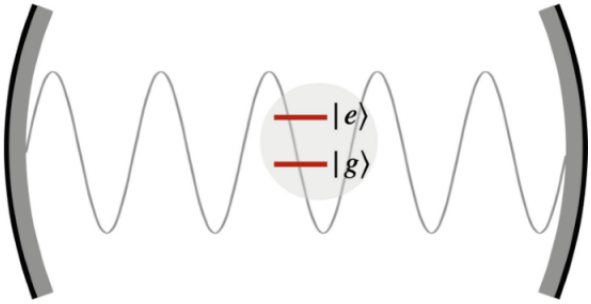

Figure 1: Two level system interacting with radiation field in a cavity under the JC model

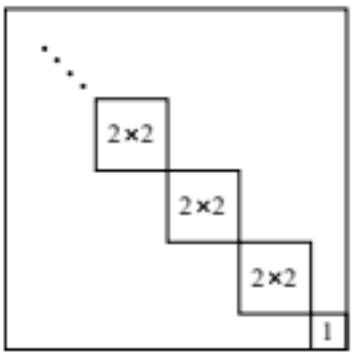

Figure 2: Block diagonal structure of the hamiltonian

$H_{JC} = H_A + H_F + H_I$ where, $H_A = \hbar w_{12}\sigma_z$ is the free atom hamiltonian, $H_F = \sum \hbar w_k a_k^\dagger a_k$ is the all mode radiation field hamiltonian and $H_I = \sum \hbar(\lambda\sigma_+ a_k + \lambda^* a_k^\dagger \sigma_-)$ represents interaction characterised by the coupling parameter λ in the Rotating Wave Approximation.

Consider only one mode of radiation field for simplicity. The basis states are tensor products of the atomic bases i.e $|e\rangle$ and $|g\rangle$ and fock state basis of radiation i.e $|n\rangle$. The bases now becomes $|g, n\rangle$ and $|e, n+1\rangle$ for n = 0, 1,.. . The pauli matrices $\sigma_z = |e\rangle\langle e| + |g\rangle\langle g|$ The ladder operators are $\sigma^+ = |e\rangle\langle g|$ & $\sigma^- = |g\rangle\langle e|$.

The total number of excitations N is the sum of atomic $(\sigma^+\sigma^-)$ and field excitation $(a^\dagger a)$. It can be shown that the total number of excitation N is a constant of motion i.e $[H, N] = 0$. Also, the interaction couples only two states for each excitation of field i.e $|g, n\rangle$ and $|e, n+1\rangle$ for all n except n = 0. This means that the state $|g, 0\rangle$ is not coupled to any other state. These two fact make $H_{JC}(t)$ block diagonal in the vector space spanned by $\{|e, n\rangle, |g, n+1\rangle\}$ where $|n\rangle$ $(n = 0, 1, ...)$ is the n^{th} excitated Fock state of the mode.

$$H_{\mathrm{n}} = \begin{pmatrix} \langle e, n| H_{JC}(t) |e, n\rangle & \langle e, n| H_{JC}(t) |g, n+1\rangle \\ \langle g, n+1| H_{JC}(t) |g, n+1\rangle & \langle g, n+1| H_{JC}(t) |g, n+1\rangle \end{pmatrix}$$

$$H_{\mathrm{n}} = \tfrac{1}{2}\begin{pmatrix} \hbar\Delta + (2n+1)w & 2\lambda^*\sqrt{n+1} \\ 2\lambda\sqrt{n+1} & -\hbar\Delta + (2n+1)w \end{pmatrix}$$

where $\Delta = w_{12} - w$ is the detuning. The eigenstates of H_n are called dressed states $|\psi_n^{\pm}\rangle$ which are linear combinations of bare states i.e the eigenstates of unperturbed hamiltonian.

The eigenvalues of H_n are $E_{\pm} = \hbar\Big(w(n+\frac{1}{2}) \pm \frac{\Omega(n,\Delta)}{2}\Big)$where, $\Omega(n,\Delta)^2 = 4\lambda^2(n+1) + \Delta^2$. On solving for eigenvectors in the TISE, $H\,|\psi_n^{\pm}\rangle = E_{\pm}\,|\psi_n^{\pm}\rangle$ we get the dressed states as,

$$|\psi_n^{+}\rangle = \cos\theta_n\,|g, n+1\rangle + \sin\theta_n\,|e, n\rangle$$
$$|\psi_n^{-}\rangle = -\sin\theta_n\,|g, n+1\rangle + \cos\theta_n\,|e, n\rangle$$
$$\text{where, } \tan\theta_n = \tfrac{2\lambda\sqrt{n+1}}{\Omega(n,\Delta)-\Delta}$$

2.2 Adiabatic Evolution

We say that we have an adiabatic change if the timescale τ for change in the physical parameter associated with the system is much greater than the natural timescale T associated with the system. The intuition of adiabatic process in Quantum system is discussed in the Appendix A by considering analogies in classical mechanics.

Lets consider a hamiltonian H(t). Assume that the spectrum is discrete and there is no crossing of instantaneous eigenstates. Suppose there exists $|\psi(t)\rangle$ which satisfies the TISE at all instants i.e $H(t)|\psi(t)\rangle = E(t)|\psi(t)\rangle$ then the state $|\psi(t)\rangle$ is called an instantaneous eigenstate. These instantaneous eigenstates do not solve the TDSE but we can try build an ansatz out of $|\psi(t)\rangle$. Let the ansatz be,

$$|\Psi(t)\rangle = c(t)\exp\Big(\tfrac{1}{i\hbar}\int_0^t E(t')\,dt'\Big)\,|\psi(t)\rangle$$

On solving the TDSE using this trial wavefunction we get,

$$i\hbar\tfrac{\partial|\Psi(t)\rangle}{\partial t} = H(t)\,|\Psi(t)\rangle$$

$$\Rightarrow \dot{c}\,|\psi(t)\rangle = -c(t)\,|\dot{\Psi}(t)\rangle$$

Assuming c(t = 0) = 1 and taking innerproduct with $\langle(t)|\,c(t)$ becomes,

$$c(t) = \exp\Big(-\int_0^t \langle\psi(t)|\,|\dot{\psi}(t)\rangle\,dt'\Big)$$

This c(t) is just a phase and it can be shown with elementary mathematical analysis. Then the approximate solution of the TDSE is,

$$|\Psi(t)\rangle \approx \exp\Big(i\gamma(t)\Big)\exp\Big(i\nu(t)\Big)\,|\psi(t)\rangle \text{ where,}$$

$\gamma(t) = \exp\Big(\int_0^t \langle\psi(t)|\,|\dot{\psi}(t)\rangle\,dt'\Big)$ is the called the geometrical or Berry's phase and

$\nu(t) = \exp\Big(-\frac{1}{\hbar}\int_0^t E(t')dt'\Big)$ is the usual dynamical phase due to time evolution.

The approximate solution of the TDSE is only valid under certain conditions and those conditions define the protocol adiabatic evolution. When finding the component of c(t) along the basis vectors we calculated the component along only one vector $|\psi(t)\rangle$. This is because if the hamiltonian is varying slowly then the component along other basis vectors can be taken to be very small hence ignored. Refer to Appendix B for a systematic proof of the above statement.

Now, we can state the adiabatic theorem as follows, If at a certain time the system is in state $|\psi(t)\rangle$ and if the hamiltonian is varying slowly for a certain time interval T then,

$$|\psi(t+T)\rangle \approx |\psi(t)\rangle$$

This means that the probability to transition to other states is highly restricted. The system would remain in the same state until the adiabatic conditions are satisfied.

2.3 Landau Zener Transitions

Energy level crossing are of considerable interest from the beginning. It's applications varies from the analysis of bond formation in quantum theory and at present these are used in the generation of ultracold molecules.

Let's consider a Toy model of a 2x2 system given by the Hamiltonian,

$$H(t) = \begin{pmatrix} \frac{\alpha t}{2} & 0 \\ 0 & -\frac{\alpha t}{2} \end{pmatrix} \text{ with } \alpha > 0. \; |\psi_1(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$E_1(t) = \tfrac{\alpha t}{2} \text{ and } E_2(t) = -\tfrac{\alpha t}{2} \qquad |\psi_2(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The instantaneous eigenstates are actually time independent. Eigenstates at a later time is given by,

$$|\psi_2(t)\rangle = \exp\left(\frac{1}{i\hbar}\int_0^t E_2(t'))\,dt'\right) \Rightarrow |\psi_2(t)\rangle = \exp(\frac{i\alpha t^2}{4\hbar})\,|\psi_2(0)\rangle$$

$$|\psi_1(t)\rangle = \exp\left(\frac{1}{i\hbar}\int_0^t E_1(t'))\,dt'\right) \Rightarrow |\psi_1(t)\rangle = \exp(\frac{-i\alpha t^2}{4\hbar})\,|\psi_1(0)\rangle$$

This tells us that the the eigenstate does not change with time. The state at initial will remain the eigenstate forever. Let us perturb the Hamiltonian with a constant perturbation of V_{12}. Now the Hamiltonian becomes,

$$H(t) = \begin{pmatrix} \frac{\alpha t}{2} & V_{12} \\ V_{12}^* & -\frac{\alpha t}{2} \end{pmatrix}$$

We can also express the Hamiltonian as,

$$H(t) = \frac{\alpha t}{2}\sigma_z + \Re(V_{12})\sigma_x - \Im(V_{12})\sigma_y$$

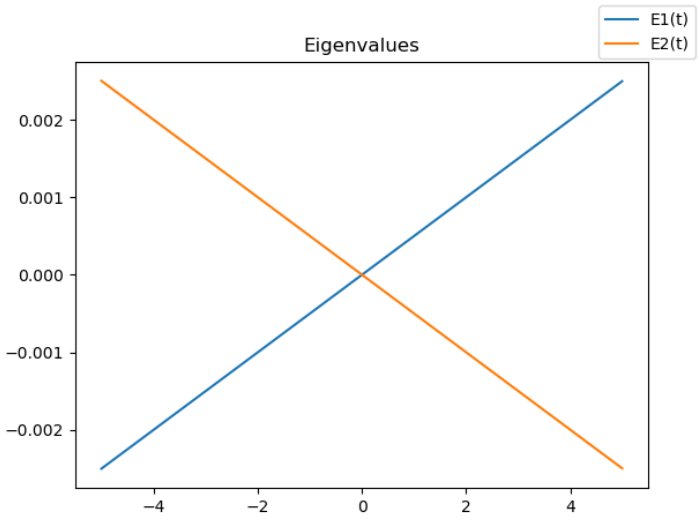

Figure 3: No crossing of energy levels

$$E \pm (t) = \pm\sqrt{\mid V_{12} \mid^2 + \frac{\alpha^2 t^2}{4}}$$

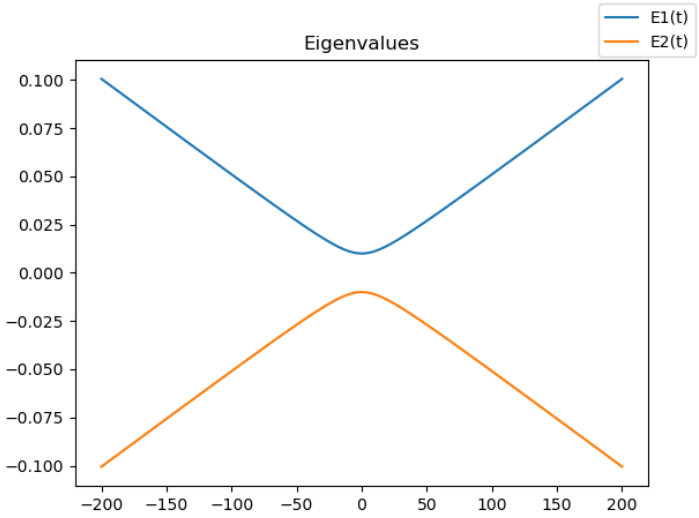

Figure 4: As the perturbation decreases the probability of level crossing increases.

The eigenvectors will become $|\pm n(t)\rangle$ with energies $E\pm(t)$. It can be shown that the instantaneous eigenstate changes from $|\psi_1\rangle$ to $|\psi_2\rangle$ and vice versa i.e $|n(-\infty)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $|n(\infty)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-n(-\infty)\rangle =$

$\begin{pmatrix}1\\0\end{pmatrix}$ and $|-n(\infty)\rangle = \begin{pmatrix}1\\0\end{pmatrix}$. If the process is adiabatic then there won't be level crossing as discussed in the previous section.

Consider $\tau_d = \frac{|V_{12}|}{|\alpha|}$. It can be interpreted as the timescale associated with the change in Hamiltonian. Also, $\frac{|V_{12}|}{\hbar} \equiv w_{12}$ is the Rabi frequency of the two level system and it can be interpreted as the natural timescale of the system. From the discussion in the previous section this process is adiabatic when $w_{12}\tau_d \gg 1$ i.e $\frac{2\pi}{w_{12}} = T_{12} \ll 1$ i.e the adiabatic change is much slower than the natural timescale of the system. The probability for non-adiabatic transition is given by

$$P_{n.ad.} = \exp\Big(-2\pi w_{12}\tau_d\Big) = \exp\Big(\frac{-2\pi \mid V_{12} \mid^2}{\hbar \mid \alpha \mid}\Big)$$

A more detailed analysis of the Landau-Zener transitions is given in the Appendix C.

3 Shortcuts to Adiabaticity

Shortcuts to adiabaticity are fast routes to final results of slow, adiabatic changes of controlling parameters of the system. The shortcuts depends on specific time dependences of the control parameters and/or addition of auxiliary time dependent interesactions with respect to the reference hamiltonians.

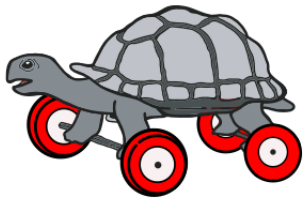

Figure 5: The bare hamiltonian is driven fast by auxiliary hamiltonian just like this tortoise is being driven by the wheels

The basic idea of Counterdiabatic driving is to add some auxiliary interactions to some reference hamiltonian $H_0(t)$ so that the dynamics follow the adiabatic evolution driven by H_0 approximately. It is like banking the road so that vehicles can go faster without sliding off the road.

Consider a reference hamiltonian,

$$H_0(t) = \textstyle\sum_n |n(t)\rangle\, E_n(t)\, \langle n(t)|$$

For adiabatic evolution, there is no transition between states. We can write a state $|n(0)\rangle$ approximatey as,

$$|\psi_n(t)\rangle = e^{i\zeta_n(t)}\, |n(t)\rangle$$

The adiabatic phase is

$$\zeta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')dt' + i\int_0^t dt' \langle n(t')|\partial_{t'} n(t')\rangle$$

Now we find the hamiltonian $H(t)$ for which the above mentioned approximate states are exact eigenstates. This is achieved by solving TDSE,

$$i\hbar\partial_t |\psi_n(t)\rangle = H(t)|\psi_n(t)\rangle$$

To find the hamiltonian lets design the time evolution operator U(t). Since $|\psi_n(t)\rangle$ is a solution of above equation, the time evolution operator operator from t = 0 to t = t could be,

$$U(t) = \sum_j |\psi_j(t)\rangle \langle\psi_j(0)|$$

$$U(t) = \sum_n e^{i\zeta_n(t)} |n(t)\rangle \langle n(0)|$$

It can be easily shown that,

$$H(t) = i\hbar\dot{U}U^\dagger$$

$$H(t) = H_0(t) + H_{CD}(t)$$

$$H_{CD}(t) = i\hbar\sum_n \Big[|\partial_t n(t)\rangle \langle n(t)| - \langle n(t)|\partial_t n(t)\rangle |n(t)\rangle \langle n(t)| \Big]$$

Lets consider an example to understand the process,

$$H_0(t) = \frac{\hbar}{2}\begin{pmatrix} -\Delta(t) & \Omega_R(t) \\ \Omega_R(t) & \Delta(t) \end{pmatrix}$$

where $\Delta(t)$ is the detuning and $\Omega_R(t)$ is the Rabi frequency. The instantaneous dressed eigenstates are,

$$|\lambda_-(t)\rangle = -\sin\left(\frac{\theta(t)}{2}\right)|1\rangle + \cos\left(\frac{\theta(t)}{2}\right)|2\rangle$$
$$|\lambda_+(t)\rangle = \cos\left(\frac{\theta(t)}{2}\right)|1\rangle + \sin\left(\frac{\theta(t)}{2}\right)|2\rangle$$

where the mixing angle satisfies $\tan(\theta(t)) = -\frac{\tilde{\Omega(t)}}{\Delta(t)}$ and eigenvalues are $E_\mp = \mp\hbar\tilde{\Omega}/2$, where the quantity $\tilde{\Omega} = \sqrt{\Delta(t)^2 + \Omega(t)^2}$. The eigenstates obey, $\langle n(t)|\partial_t n(t)\rangle |n(t)\rangle \langle n(t)| = 0$. So,

$$H_{CD}(t) = i\hbar\sum_n |\partial_t n(t)\rangle \langle n(t)|$$

$$H_{CD}(t) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\tilde{\Omega(t)} \\ i\tilde{\Omega(t)} & 0 \end{pmatrix}$$

What the happens here is that we drive the system for some time t_f with $H_{CD}(t)$ starting from t_i = 0. For t < t_i, the states are evolved adiabatically and the eigenstates are approximate. After driving is turned on, the approximate eigenstate becomes exact eigenstate of H(t) and the system does not evolve under adiabatic approximation. During this time period the evolution is sped up. After t > t_f the perturbation is switched off and the eigenstates at time t_f becomes the adiabatically approximated eigenstate of $H_0(t)$. The Berry's phase $\zeta(t)$ can be optimizied to minimize the energy costs.

4 Engineering Non-Classical States

4.1 The Counterdiabatic Driving

The Hamiltonian of the two level system under consideration can be expressed in the Jaynes-Cummings model as

$$H_{JC}(t) = \frac{w_q(t)}{2}\sigma_z + wa^\dagger a + \lambda(t)(a\sigma^+ + a^\dagger\sigma^-).$$

where $w_q(t)$ corresponds to the two level frequency and $\lambda(t)$ is the coupling parameter which represents the interaction strength between the two levels. In this model we assume that the frequency of the two level system is driven with $w_q(t)$ and the coupling is varied as $\lambda(t)$ while the field frequency w is kept constant. Following the discussion of the JC model in section 2.1 the H_{JC} can be expressed as,

$$H_{\mathrm{JC}}(t) = \frac{-w_q(t)}{2}\left|g,0\right\rangle\left\langle g,0\right| \oplus_n H_n(t)$$

where $H_n(t) = \frac{(2n+1)w}{2}\mathbf{1} + \frac{\delta(t)}{2}\sigma_z + \lambda(t)\sqrt{n+1}\sigma_x$. In adiabatic driving the $w_q(t)$ and $\lambda(t)$ vary slowly compared to timescale of the system which is proportional to the minimum energy gap of $H_{JC}(t)$. We can add an additional term to the bare hamiltonian which speeds up the evolution while suppressing non-adiabatic transitions. This technique is called Counterdiabatic driving as discussed in previous section. From Berry's formulation, the $H_{CD}(t)$ is ,

$$H_{CD}(t) = i\sum_{n,\sigma=\pm}(\left|\partial_t\right\rangle(n,\sigma(t))\left\langle n,\sigma(t)\right| - \left\langle n,\sigma(t)|\partial_t(n,\sigma(t)\right\rangle\left|n,\sigma(t)\right\rangle\left\langle n,\sigma(t)\right|)$$

$$\Rightarrow \mathrm{H}_{CD}(t) = i\sum_{n,\sigma=\pm}\left[\partial_t\Phi_{n,\sigma}(t), \Phi_{n,\sigma}(t)\right]$$

where $|n,\sigma(t)\rangle$ denotes the dressed-atom eigenstates of $H_{JC}(t)$ and $\Phi_n(t) = |n,\sigma(t)\rangle\langle n,\sigma(t)|$. The Counterdiabatic driving as a result of the above formulation will be,

$$H_{CD}(t) = \frac{\dot\lambda(t)\delta(t) - \lambda(t)\dot w_q(t)}{\delta^2(t) + \Omega_n^2(t)}\left(ia^\dagger\sigma^- - ia\sigma^+\right)$$

where $\Omega_n(t) = 2\lambda(t)\sqrt{n+1}$ is the Rabi freuqency. This additional driving in the σ_y direction suppresses the non-adiabatic excitations for an arbitrarily fast adiabatic evolution. Similar to $H_{CD}(T)$ we have a localised hamiltonian which has the same structure as $H_{JC}(t)$ which makes it easy to implement. Refer Appendix D for a detailed analysis on the formulation of $H_{LCD}(t)$ and $H_{CD}(t)$. The effective hamiltonian $H_{CD}^{STA} = H_{JC}(t) + H_{CD}(t)$ must be equal to the bare hamiltonian at the start and the end of Counterdiabatic driving. The same applied to $H_{LCD}(t)$. This imposes the following conditions,

$$\begin{aligned}
\dot\lambda(0) = \dot\lambda(\tau) = 0\\
\ddot\lambda(0) = \ddot\lambda(\tau) = 0\\
\dot w_q(0) = \dot w_q(\tau) = 0\\
\ddot w_q(0) = \ddot w_q(\tau) = 0
\end{aligned}$$

The population transfer between $|e, n\rangle$ and $|g, n+1\rangle$ follows Landau-Zener transition. This requires $w_q(t)$ to change sign during evolution while $\lambda(t) \neq 0$ for some t along with $\lambda(0) = \lambda(\tau) = 0$.

4.2 STA Protocol

A suitable protocol has to be chosen for $w_q(t)$ and $\lambda(t)$ based on conditions imposed by Counterdiabatic driving and Landau-Zener(LZ) transitions. The atomic frequency has to be monotonic for LZ transitions. This would mean that $w_q(t)$ has to be a polynomial in t. Let it be, $w_q(t) = \sum_j C_j t^j$. On applying all the boundary conditions and solving the equation, we get $w_q(t) = w_q(0) + 10\Delta w_q t^3 - 15\Delta w t^4 + 6\Delta w t^5$.

Similary, the interaction strength has to vary periodically. The function that satisfies all the boundary conditions of λ is $\lambda(t) = (\lambda_m - \lambda_0)cos^4[\pi(1+2t)/2] + \lambda_0$ where λ_m and λ_0 denote the maximum and initial value of the coupling constant.

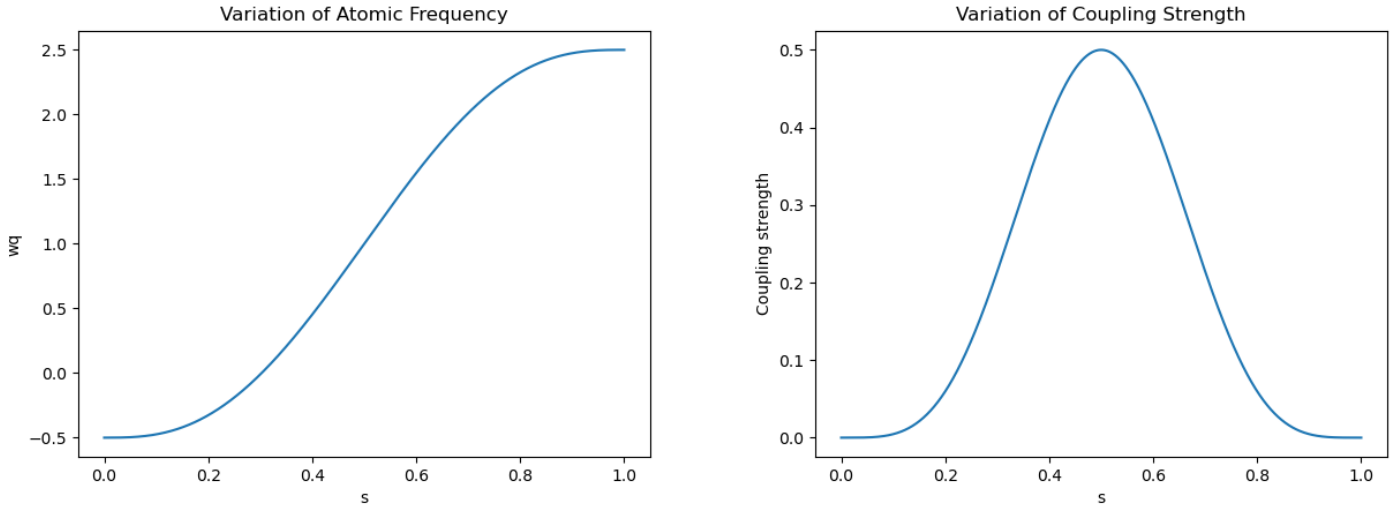

Figure 6: profile of time dependent protcol parameters where $s = t/\tau$

4.3 Fock States

The protocol defined in the previous section allows for a perfect state transfer between $|e, n\rangle$ and $|g, n+1\rangle$, which can be used to generate the Fock state $|N\rangle$. In this specific case, the time independent evolution bare hamiltonian H_{JC} performs in a similar manner as our H_{CD} protocol. To get an arbitrary Fock state $|N\rangle$, we start with an initial state $|e, 0\rangle$. This is driven to $|g, 1\rangle$ by the STA protocol. Upon a π pulse on the state we drive it to $|e, 1\rangle$. Continuing in a similar fashion for N times the state $|e, N\rangle$ is achieved.

The duration of each STA + π cycle is t_c where $t_c = \tau + 2t_\pi$. τ is the time spent in the STA evolution and $2t_\pi$ is the time spent in the π pulse. We assume that $\tau = t_\pi$ The state evolution under

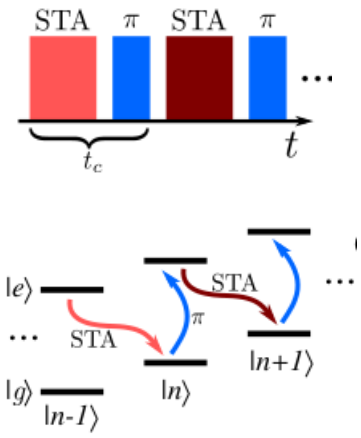

Figure 7: Scheme for generating Fock states using STA and π pulse

various values of $w\tau$ were simulated. The Pi pulse is modelled as a Gaussian wave. The state during the application of this pulse evolves under the hamiltonian,

$$H_\pi(t) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\exp -\frac{(t-t_\pi)^2}{2\sigma_\pi^2}\sigma_x$$

The performance of this protocol is computed using the Mandel Q parameter which reveals the non classicality of the resulting state.

$$Q(t) = \frac{\langle n^2(t)\rangle - \langle n(t)\rangle^2}{\langle n(t)\rangle} - 1$$

$$\text{with } \langle n(t)\rangle = \langle\psi(t)|\, a^\dagger a\, |\psi(t)\rangle$$

The STA protocol results in $Q(t) < 0$ and $Q(nt_c) = -1$ from which we can infer the sub-poissonian behavior of the boson statistics. On evolution under $H_{JC}(t)$, the statistics become super-poissonian unless the evolution is done very slow i.e $w\tau \to \infty$. In this case we get the sub-poissonian statistics as expected.

We also compute the spin purity $p(t) = Tr\left[\rho_s^2(t)\right]$ of the reduced density matrix of the system i.e $\rho_s(t) = Tr_b\left[\rho^2(t)\right]$ where b represents the bosonic mode. It can be seen that $p(nt_c = 1$ as at integral multiples of t_c the state is exactly in $|e, N\rangle$. $p((n-1)t_c + \frac{\tau}{2}) = \frac{1}{2}$, because at this halfway point of STA evolution the state is in a superposition of $|e, n\rangle$ and $|g, n+1\rangle$.

The above method to prepare Fock state relies on an initial vacuum state of field $\rho(0) = |e\rangle\langle e| \otimes \rho_{th}$ where ρ_{th} follows the canonical distribution. You can see the variation of fidelity and infidelity with β_{th} and n. For large β_{th} value the fidelity is high because at low temperature the fluctuations in the vacuum state reduces. The physical parameters used for simulation are $\lambda_0 = 0, \lambda_m = w/4, w_q(0) = 3w_q(\tau) = 3w/2$ and $w\tau = 5$.

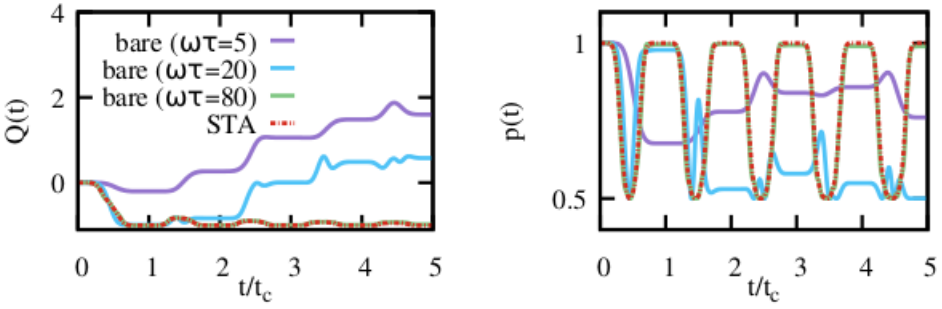

Figure 8: Evolution of mandel parameter and purity of reduced spin state

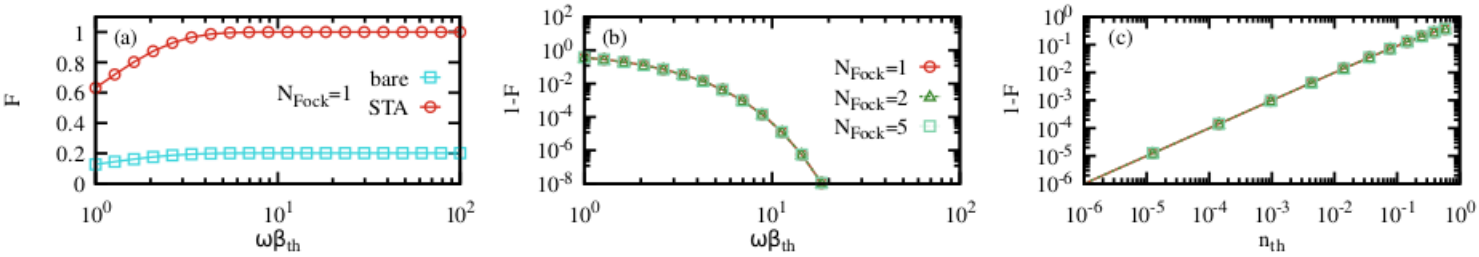

Figure 9: Fidelity and Infidelity variations due to thermal state

4.4 Cat State

Cat states are quantum states which are superpositions of two diametrically opposite states. It is formulated after the Gedanken experiment proposed by Schrödinger where a hypothetical cat is considered both alive and dead as result of its fate being linked to a random quantum event that may or may not occur. These states have application in quantum information processing and have been realised in various physical systems.

$$|cat\rangle_e \propto |\alpha\rangle + |-\alpha\rangle$$

The above state is one such cat state which is a superposition of two opposite phased coherent states. In order to realise cat state using our protocol, we start from a particular fock state and apply a $\frac{\pi}{2}$ pulse. This splits the state into two different n subspace and further STA is performed. For an initial state $|e, N\rangle$, the previous step leads to the superposition of $|e, N-1\rangle$ and $e^{i\phi}|g, N+1\rangle$ where π is the phase gathered during the evolution.

Upon appliction of another $\frac{\pi}{2}$ pulse followed by a projective measurement $M_r = |r\rangle\langle r| \oplus \mathbf{1}_b$ onto the spin state, the resulting state becomes $|\psi_{N-1,N+1}\rangle \propto (|N-1\rangle + e^{i\phi}|N+1\rangle)$. The obstacle in preparing states which are further apart in fock state using STA is that the $w_q(t)$ and $\lambda(t)$ depends on the specific n of the subspace. This results in fidelity slighlty less than 1.

The general state prepared is $|\psi_{nm}\rangle = \frac{1}{\sqrt{2}}(|n\rangle + e^{i\phi}|m\rangle)$. For $|\psi_{0,4}\rangle$ that we generated, the fidelity $F = \langle\psi_{0,4}|\rho_f|\psi_{0,4}\rangle$ is greater than 0.9992. However this fidelity reduces for states that are further

apart. The state $|\psi_{0,6}\rangle$ is prepared with a fidelity of 0.988 and $|\psi_{0,6}\rangle$ with a fidelity of 0.99998. On the other hand, engineering these states with time dependent bare hamiltonian results in fidelities worse than STA. The time independent evolution leads to even worse fidelity as the time independent hamiltonian is unable to correctly realize the cat state because we need population transfer in two different subspaces.

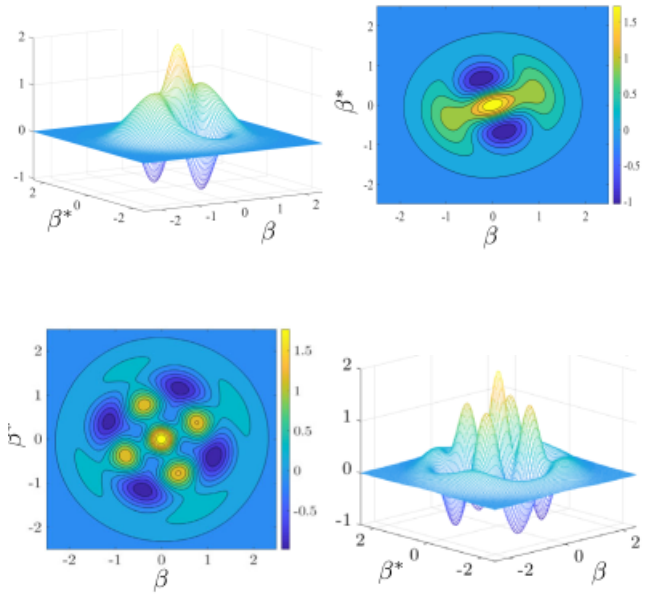

Figure 10: Wigner functions for cat states $|\psi_{0,2}\rangle$ and $|\psi_{0,4}\rangle$

Further, the wigner function $W(\beta,\beta^*) = 2Tr\Big[\rho_f D(\beta)e^{i\pi a^\dagger a}D^\dagger(\beta)\Big]$ where $D(\beta)$ represents the displacement function is calculated for the final state $|\psi_{0,4}\rangle$. It displays its characteristic nature i.e distinguishable local-state components whose strong quantum interference results in the negativity of $W(\beta,\beta^*)$. The wigner function for cat states $|\psi_{0,2}\rangle$ and $|\psi_{0,6}\rangle$ were also calculated.

4.5 Photon Shifted States

Photon shifted states are generated by addition and subtraction of bosonic excitations. At the basic level they can be realised as $|\psi_{ph-add}\rangle \propto a^\dagger|\psi\rangle$ and $|\psi_{sub-add}\rangle \propto a|\psi\rangle$. These can be considered as basic arithmetic operations and are important in quantum based technologies.

The photon shifted states can be generated using a single STA driving. Let the initial coherent state be $|e,\alpha\rangle = D(\alpha)|e,0\rangle$. By applying STA driving in n = 0 subspace, the population of vacuum state shifts to $|1\rangle$. Now each π pulse transfers the population to higher-excitation states.

The state that we generate are more non-classical than $|\psi_{ph-add}\rangle$. This is evident from the negativity function $\mathrm{N} = (1/2\pi)\int d^2\beta\Big[\ |\ W(\beta,\beta^*)\ |\ -W(\beta,\beta^*)]$. The large N value suggest that photon shifted states are more non classical than photon added states.

5 Numerical Simulations

First the Time Dependent Schrödinger Equation has to be solved for both bare and STA hamiltonians. From time dependent perturbation theory,

$$i\hbar\dot{c}_m = \sum_n e^{iw_{mn}t}\delta H_{mn}(t)$$

For the bare hamiltonian,

$$\dot{C}_e = -\frac{iw_q(t)Ce}{2} - ie^{-iwt}\lambda(t)Cg$$
$$\dot{C}_g = \frac{iw_q(t)Cg}{2} + ie^{iwt}\lambda(t)Ce$$

Similarly solve the TDSE for STA evolution, π pulse evolution and $\pi/2$ pulse evolution using RK4 or any other method. The obtained probability coefficients can be used to measure other distributions such as wigner distribution, Fidelity,etc.

6 Robustness

As the scheme is built on STA protocols which evolves faster, the protocol is naturally robust against decoherence. In particular we can achieve a desired non-classical state with high fidelity under a broad range of noises such as decoherence, dephasing, damping, spontaneous emission etc. This is evident from the plots of infidelity.

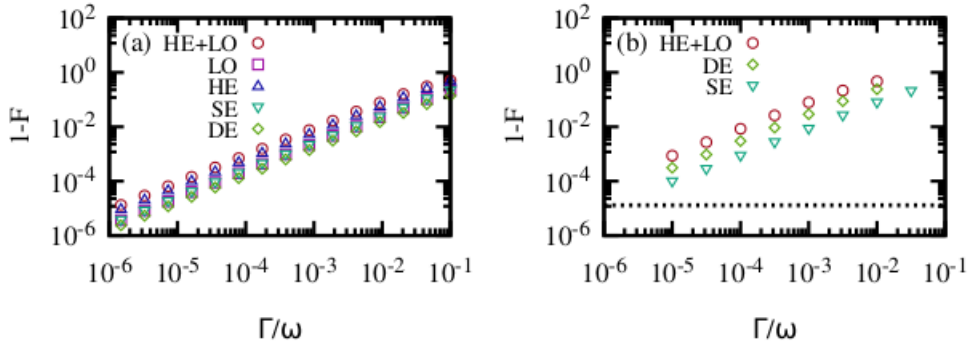

Figure 11: Infidelity due to different decoherence processes for Fock states and Cat states

7 Experimental Feasability

This scheme can be realised in a variety of physical system where JC interaction between a two level system and a bosonic field can be controlled. Consider an ion trap implementation. We utilize a well controllable qubit such as the hyperfine states of the $S_{1/2}$ manifold of a $^{171}Yb^{+}$ ion whose frequency is 12.6428 GHz. The trapped ion is confined in a harmonic potential with frequency 2MHz.

Applying two counter propagating Raman laser beams the internal levels of the ion could be coupled with vibrational mode. In the interaction picture we have the JC model. A possible set of realistic parameters to implement the scheme is given by $w/2\pi \approx 50$, such that $\lambda_m \approx w/4$ which leads to $\tau \approx 0.2ms$ for $w\tau = 10$. For such short interval, decoherence effects are not expected to play a relevant role and you can rely on suitable dynamical coupling schemes to protect your system from decoherence process.

8 Conclusion

The general framework that we have developed is fast, robust and accurate in the preparation of non-classical states in spin-boson systems. Since the framework is based on interactions in spin-boson systems, these can be easily realized and improvised using state of the art setups. The concept of STA is already being exploited in Quantum thermodynamics and phase transitions. This paper illustrates the advantage of STA in engineering quantum states and could lead to new robust protocols to engineer states.

9 References

Quantum state engineering by shortcuts-to-adiabaticity in interacting spin-boson systems
Shortcuts to adiabaticity: concepts, methods, and applications
Quantum Physics III MIT OCW
A simple approach to the Landau-Zener formula
Quantum Optics by Girish.S.Agarwal

Appendices

A Adiabatic Approximation in Classical Mechanics

Consider a classical harmonic oscillator with varying frequency $w(t)$.

$$H(x, p, w) = \frac{p^2}{2m} + \frac{1}{2}mw(t)^2x^2$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t}$$

From Hamilton's equation of motion (or) Ehrenfest's theorem (Quantum mechanically),

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = mw\dot{w}x^2$$

We can say that we have an adiabatic change if the time scale τ for change in the physical parameter w(t) is much greater than period of oscillation i.e $\frac{2\pi}{w(t)}$. Since w(t) changes slowly, H(t) also changes

slowly. The ratio of H(t) to w(t) would change vary slowly over a time period and we can call this the adiabatic invariant which is almost constant in adiabatic changes of w(t).

$$I(t) = \frac{H(t)}{w(t)}$$

$$I(t+T) - I(t) = \int_t^{t+T} \frac{dI}{dt'}\, dt' = \frac{\dot{w}}{w^2}$$

Also, we can consider the geometric interpretation of the adiabatic invariant in the phase space. The harmonic oscillator has a nearlyy elliptical geometry in the phase space if w(t) is adiabatic. The area of the phase space is the area of ellipse i.e $2\pi\frac{E}{w}$. It's kind of intuitive that the area in phase space does not change for small changes in w(t) because phase space represents total accessible states for energy upto E and it will change only if the oscillator's energy changes. If the energy change is adiabatic then the phase space remains almost constant.

$$\oint p dx = 2\pi I$$

Extending the same idea to quantum mechanical systems,

$$I = \frac{E}{w} = \hbar(n + \tfrac{1}{2})$$

Consider the harmonic oscillator potential in the WKB approximation. It has two turning points a,b. Then we have the Bohr-Sommerfield quantization condition i.e

$\int_a^b p(x)\, dx' = \hbar(n + \frac{1}{2})\pi$ This can be rewritten as,

$$\oint p dx = 2\pi\hbar(n + \tfrac{1}{2})$$

This means that the quantum number does not change under adiabatic evolution. This implies that the perturbation must be either constant or slowly varying for adiabatic evolution to take place. Under adiabatic evolution, the principle quantum number does not change hence, the quantum state does not change and remain constant.

B Systematic Analysis of Adiabatic Evolution

Consider the state at time t,

$$|\Psi(t)\rangle = \textstyle\sum_n c_n(t)\, |\psi_n(t)\rangle$$

From TDSE,

$$i\hbar \textstyle\sum_n (\dot{c}_n\, |\psi(t)\rangle + c_n\, |\dot{\psi}(t)\rangle) = \sum_n c_n(t) E_n(t)\, |\psi_n(t)\rangle$$

Taking inner product with $\langle\psi_k(t)|$

$$i\hbar\dot{c}_k = \Big(E_k - i\hbar\, \langle\psi_k|\dot{\psi}_k\rangle\, \Big) c_k - i\hbar \textstyle\sum_{n\neq k} \langle\psi_k|\dot{\psi}_n\rangle\, c_n$$

The matrix element of H(t) can be related to $\langle\psi_k|\dot{\psi}_n\rangle$ in the instantaneous eigenvalue equation by taking time derivative of the $H(t)\,|\psi_n\rangle = E_n(t)\,|\psi_n(t)\rangle$ and taking its inner product with$\langle\psi_k(t)|$ for $k \neq n$. The equation then becomes,

$$i\hbar\dot{c}_k = \Big(E_k - i\hbar\,\langle\psi_k|\dot{\psi}_k\rangle\Big)c_k - i\hbar\textstyle\sum_{n\neq k}\frac{\dot{H}_{kn}}{E_n - E_k}c_n$$

If the term proportional$\frac{\dot{H}_{kn}}{E_n - E_k}$ vanishes then the approximation made in Section 2.2 is valid. This can be achieved by setting slowly varying perturbations (or) by taking states with higher energy differences.

C Probability for Non-Adiabatic Transition

Assume the decay constant to be γ. The probability per unit time i.e the transition rate for excitation out of the state is $\Gamma = \Omega^2\frac{\gamma}{\Delta^2+\frac{\gamma^2}{4}}$. Let P(t) be population in excited state. Then,

$$P(t+\delta t) = [1-\Gamma\delta t]P(t) \approx e^{-\Gamma\delta t}P(t)$$

$$P(t) = \exp\Big(-\textstyle\int_{t_i}^{t_f}\Gamma(t)\delta t\Big)$$

$$P(t) = \exp\Big(-\tfrac{2\Omega^2}{\Delta}\textstyle\int_{\Delta(t_i)}^{\Delta(t_f)}\frac{\gamma/2}{\Delta^2+(\gamma/2)^2}d\Delta\Big)$$

$$P(t) = \exp\tfrac{2\Omega^2}{\Delta}\ [\arctan(2\Delta(t_f)/\gamma) - \arctan(2\Delta(t_i)/\gamma)]$$

For $\Delta(t_i) \ll\gamma/2 \ll\Delta(t_f)$ the probability reduces to,

$$P(t) = \exp\Big(-2\pi\Omega^2/\Delta\Big)$$

same as that discussed in section 2.3. So, the detuning is ramped from far below resonance through the level crossing at $\Delta = 0$ and ending up far above resonance.

D Local Counterdiabatic Hamiltonian

The Local Counterdiabatic Hamiltonian is obtained by performing a time dependant unitary transformation to the total hamiltonian

$$U(t) = e^{-if(t)\sigma_z}$$

$$|\tilde{\psi(t)}\rangle = U^\dagger\,|\psi(t)\rangle$$

The modified hamiltonian is $H_{LCD}(t) = U^\dagger(t)(H(t) - i\hbar\dot{U}^\dagger(t))U(t)$. By choosing f(t) to be $\frac{1}{2}\arctan(\frac{2H_{CD}}{\Delta(t)})$, the $H_{LCD}(t)$ has the same for as the JC hamiltonian but with different parameters. This makes it easy to implement counteradiabatic driving locally.