

Math Notes

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1 Markov Chains

2 Discrete Stochastic Processes

We have an $n \times n$ probability transition matrix P , which projects a vector of state probabilities v to vP . We can gain an intuitive feeling for what this means by looking at a single element of v . Let's say that $v_i(t)$ is the probability a discrete stochastic system is in state i at time t . If we want to know the probability that the system is in state j at time $t + 1$, we can get to this state by using the following conditioning statement:

2.1 Absorbing Chains

With a transition matrix P , we can find a permutation matrix M that permutes rows and columns (e.g. MPM^T) such that P is organized into a block matrix:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

where I is an identity matrix of absorbing states, Q contains transitions from non-absorbing to non-absorbing states, and R contains transitions from non-absorbing to absorbing states. Note that the 0 matrix contains the transitions from absorbing states to non-absorbing states, which are necessarily all zero.

$$P^2 = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ R + QR & Q^2 \end{pmatrix}$$

In general:

$$P^t = \begin{pmatrix} I & 0 \\ (I + Q + Q^2 + \dots + Q^t)R & Q^t \end{pmatrix}$$

Note that since the row of Q all sum to a value less than one (since transitions from non-absorbing states to absorbing are excluded), the dominant eigenvalue of Q , $\lambda_1 < 1$, so the system shrinks. Thus, $t \rightarrow \infty$, $Q^t \rightarrow 0$. Thus:

$$P^\infty = \begin{pmatrix} I & 0 \\ NR & 0 \end{pmatrix}$$

where $N = (I + Q + Q^2 + Q^3 + \dots)$; this is called the *fundamental matrix*.

2.1.1 The Fundamental Matrix, and Expected Time until Absorption

One property of interest of an absorbing matrix chain is how many times it visits a state j (given it starts in state i) before being absorbed. This information is available via fundamental matrix described above. We'll cover two ways of deriving this result.

Expectation approach First, if $S_{i,j}$ is a random variable for the number of visits to j starting from i . We can think of $S_{i,j}$ as the sum of indicator variables that the chain enters non-absorbing state j at time k starting from state i , e.g.:

$$S_{i,j} = \mathbb{1}_{\{X(0)=j|X(0)=i\}} + \mathbb{1}_{\{X(1)=j|X(0)=i\}} + \mathbb{1}_{\{X(2)=j|X(0)=i\}} + \dots$$

$$\mathbb{E}[S_{i,j}] = \sum_{k=0}^{\infty} \mathbb{E}[\mathbb{1}_{\{X(k)=j|X(0)=i\}}]$$

by the fact that $\mathbb{E}[\mathbb{1}_A] = P(A)$ (what Joe Blitzstein calls “the fundamental bridge”):

$$\mathbb{E}[S_{i,j}] = \sum_{k=0}^{\infty} P(X(k) = j | X(0) = i)$$

Note that $P(X(0) = j | X(0) = i) = 1$, and this is if and only if $i = j$; intuitively, this is because the chain starts in state i and we're counting the expected number of visits to $j = i$.

From the general theory of Markov chains, we know that the probability a chain transitions from state i to j in k steps ($P(X(k) = j | X(0) = i)$) is the (i, j) entry of the matrix Q^k . For all states i, j in the sample space, we can express the expected number of visits before absorption in a matrix N :

$$N = \sum_{k=0}^{\infty} Q^k$$

Where N is a matrix containing entries $N_{i,j} = \mathbb{E}[S_{i,j}]$.

Note that N is an infinite sum,

$$N = I + Q + Q^2 + Q^3 + \dots$$

Which we can express:

$$N = I + QN$$

$$N = (I - Q)^{-1}$$

Recursive approach Alternatively, we can find this same formulation through a recursive approach. We note that the expected number of times state i is visited (starting from state i) can be written as:

$$N_{i,i} = 1 + \sum_k N_{i,k} Q_{k,i}$$

where k is all non-absorbing states. First, since the chain is initially in state i , $N_{i,i}$ is incremented. Then, for each of these non-absorbing states k the chain will occupy, the probability it shifts to the state i is $Q_{k,i}$. More generally, we write:

$$N_{i,j} = \delta_{ij} + \sum_k N_{i,k} Q_{k,j}$$

Expected number of periods before absorption With our fundamental matrix N , we can find the expected number of time spent in *any* non-absorbing state by summing over over these non-absorbing states:

$$\mathbb{E}[S_i] = \sum_j N_{i,j}$$