Math Notes

Vince Buffalo

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Thinking about markov chains as accounting of netflow in and netflow out of a state. We should be able to solve it this way too. Reccurence relations as infinite sums (well, over t-¿ N)

popgen stuff: - infinite population limit -¿ modeling frequencies - rescaling time

1 Markov Chains

Beauty of markov chains:

superficially: eigenvalue eigenvector decomp for purely computational reasons, e.g. P^n.

But then, how this relates algebraically to things

probability and LIMIT consequences.

Eigenvalues and Perron-Frobenius Stochastic matrices must have an eigenvalue $\lambda_1 = 1$ and corresponding eigenvector $v_1 = \mathbf{1}$. Why? Each row of a right stochastic matrix must sum to one (in other words, rows of stochastic matrices need to follow the second axiom or probability: unit measure):

$$\sum_j M_{i,j} = 1$$

Alternatively, we can express this in matrix form:

$$\mathbf{1}^T\mathbf{M} = \mathbf{1}^T$$

which makes it clear that 1 is a left eigenvalue of $\mathbf{M}$ since an left eigenvalue is some λ such that $\mathbf{vA} = \lambda\mathbf{v}$.

Furthermore, this eigenvalue λ_1 is the dominant eigenvalue, $\lambda_i < 1, i \neq 1$.

2 Ergodic Theorems and Stationary Distributions

2.1 Absorbing Chains

With a transition matrix P, we can find a permutation matrix M that permutes rows and columns (e.g. MPM^T) such that P is organized into a block matrix:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

where I is an identity matrix of absorbing states, Q contains transitions from non-absorbing to non-absorbing states, and R contains transitions from non-absorbing to absorbing states. Note that the 0 matrix contains the transitions from absorbing states to non-absorbing states, which are necessarily all zero.

$$P^2 = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ R + QR & Q^2 \end{pmatrix}$$

In general:

$$P^t = \begin{pmatrix} I & 0 \\ (I + Q + Q^2 + \ldots + Q^t)R & Q^t \end{pmatrix}$$

Note that since the row of Q all sum to a value less than one (since transitions from non-absorbing states to absorbing are excluded), the dominant eigenvalue of Q, $\lambda_1 < 1$, so the system shrinks. Thus, $t \to \infty$, $Q^t \to 0$. Thus:

$$P^\infty = \begin{pmatrix} I & 0 \\ NR & 0 \end{pmatrix}$$

where $N = (I + Q + Q^2 + Q^3 + \ldots)$; this is called the *fundamental matrix*.

2.1.1 The Fundamental Matrix, and Expected Time until Absorption

One property of interest of an absorbing matrix chain is how may times it visits a state j (given it starts in state i) before being absorbed. This information is available via fundamental matrix described above. We'll cover two ways of deriving this result.

Expectation approach First, if $S_{i,j}$ is a random variable for the number of visits to j starting from i. We can think of $S_{i,j}$ as the sum of indicator variables that the chain enters non-absorbing state j at time k starting from state i, e.g.:

$$S_{i,j} = \mathbb{1}_{\{X(0)=j|X(0)=i\}} + \mathbb{1}_{\{X(1)=j|X(0)=i\}} + \mathbb{1}_{\{X(2)=j|X(0)=i\}} + \cdots$$

$$\mathbb{E}[S_{i,j}] = \sum_{k=0}^{\infty} \mathbb{E}[\mathbb{1}_{\{X(k)=j|X(0)=i\}}]$$

by the fact that $\mathbb{E}[\mathbb{1}_A] = P(A)$ (what Joe Blitzstein calls "the fundamental bridge"):

$$\mathbb{E}[S_{i,j}] = \sum_{k=0}^{\infty} P(X(k) = j|X(0) = i)$$

Note that $P(X(0) = j|X(0) = i) = 1$, and this is if and only if $i = j$; intuitively, this is because the chain starts in state i and we're counting the expected number of visits to $j = i$.

From the general theory of Markov chains, we know that the probability a chain transitions from state i to j in k steps ($P(X(k) = j|X(0) = i)$ is the (i, j) entry of the matrix Q^k. For all states i, j in the sample space, we can express the expected number of visits before absorption in a matrix N:

$$N = \sum_{k=0}^{\infty} Q^k$$

Where N is a matrix containing entries $N_{i,j} = \mathbb{E}[S_{i,j}]$.
Note that N is an infinite sum,

$$N = I + Q + Q^2 + Q^3 + \dots$$

Which we can express:

$$N = I + QN$$

$$N = (I - Q)^{-1}$$

Recursive approach Alternatively, we can find this same formulation through a recursive approach. We note that the expected number of times state i is visited (starting from state i) can be written as:

$$N_{i,i} = 1 + \sum_k N_{i,k} Q_{k,i}$$

where k is all non-absorbing states. First, since the chain is initially in state i, $N_{i,i}$ is incremented. Then, for each of these non-absorbing states k the chain will occupy, the probability it shifts to the state i is $Q_{k,i}$. More generally, we write:

$$N_{i,j} = \delta_{ij} + \sum_k N_{i,k} Q_{k,j}$$

Expected number of periods before absorption With our fundamental matrix N, we can find the expected number of time spent in *any* non-absorbing state by summing over over these non-absorbing states:

$$\mathbb{E}[S_i] = \sum_j N_{i,j}$$