Theoretical Computer Science 34 (1984) 33-82
North-Holland


# SYNCHRONIZATION TREES

Glynn WINSKEL*

*Department of Computer Science, Carnegie-Mellon University, Pittsburgh, PA 15213, U.S.A.*

**Abstract.** Synchronization trees are a concrete underlying model for much of the work on concurrency. They are trees with labelled arcs; the nodes represent states, the arcs occurrences of events and their labels how the events can synchronize with other events in the environment. The many different ways in which events are allowed to synchronize are captured abstractly by the concept of a synchronization algebra. It says which pairs of labelled events can combine to form an event of synchronization and what label the synchronization event carries. Synchronization trees are trees with arcs labelled by elements of a synchronization algebra. Our approach is based on a natural definition of morphism of trees which essentially expresses how the occurrence of events in one process imply the synchronized occurrence of events in another. Well-known operations on trees arise as categorical constructions. For example, a sum construction is a coproduct on synchronization trees while many familiar parallel compositions of synchronization trees are restrictions of the product in the underlying category of trees. The constructions are continuous with respect to a natural complete partial order structure on trees so one obtains denotational semantics as synchronization trees to a wide range of parallel programming languages, based on the constructions with recursion, in a routine manner by varying the synchronization algebra. Isomorphism of synchronization trees induces a basic congruence on terms of the language. We present a complete proof system for the congruence restricted to nonrecursive terms. The categories of trees are generalized to categories of transition systems. The pleasant categorical set-up which exists between the categories of trees and transition systems makes possible a smooth translation between operational semantics expressed in terms of transition systems and denotational semantics expressed in terms of trees.

## Introduction

We present a collection of categories of labelled trees useful in giving denotational semantics to parallel programming languages such as Milner's "Calculus of Communicating Systems" (CCS) [18], his synchronous CCS, called SCCS [19], and languages derived from Hoare's CSP as presented in [9] and [2]. Enough results are given to provide denotational semantics to any of the languages in [18, 19, 9] though at the rather basic level of labelled trees—called synchronization trees in [18].

Synchronization trees are a basic, very concrete, interleaving model of parallel computation in which processes communicate by mutual synchronization. A synchronization tree is a tree in which the nodes represent states and the arcs represent event occurrences, labelled to show how they synchronize with events in the environment. Tree semantics arise naturally once concurrency is simulated by nondeterministic interleaving and for this reason synchronization-tree semantics underlie

* Present affiliation: Computer Laboratory, University of Cambridge, Cambridge CB2 3QG, United Kingdom.

0304-3975/84/$3.00 © 1984, Elsevier Science Publishers B.V. (North-Holland)

much of the work on the semantics of synchronizing processes. For example, in [18] it is made clear how every equivalence on CCS programs presented there factors through a synchronization-tree semantics while Brookes [2] shows a similar result for the failure-set semantics in [9].

In order to cover a wide range of synchronization disciplines between synchronizing processes we express synchronization disciplines between processes as synchronization algebras. They are algebras on sets of labels which specify how pairs of labelled events combine to form a synchronization event and what labels such combinations carry. They also specify what labelled events can occur asynchronously. The parallel composition is derived from a product in a category of trees; essentially one restricts the product of trees to those synchronized events allowed by the synchronization algebra. By varying the synchronization algebra we obtain many forms of parallel composition in the literature. Other useful operations are defined on synchronization trees. They are all continuous with respect to a natural complete partial order of trees and so can be used to give denotations to processes defined recursively in terms of them by using least-fixed points—the standard tool of Scott–Strachey semantics.

The denotational semantics is related to operational semantics expressed in terms of labelled transition systems used in most of the work on CCS. In this framework, recursion is often handled by introducing loops into the chains of state-to-state transitions. We define a category of transition systems whose product unfolds to the product of trees. Consequently one can define a parallel composition of labelled transition systems which unfolds to parallel composition of trees. Again this is so for a wide variety of synchronization disciplines obtained by varying the synchronization algebra.

There is a natural notion of equivalence on processes; two processes are equivalent if they are represented by isomorphic synchronization trees. A complete set of proof rules are provided for this equivalence on a language of finite processes. Of course these rules will still be valid for any more abstract equivalence based on synchronization trees. Unfortunately we do not consider proof rules for infinite processes or the important phenomenon of divergence (see, e.g., [11, 12]).

Many of the results below follow from [27, 28], which, however, concentrated on showing how to use a broader framework of event structures [21, 26, 27] to give denotational semantics languages of synchronizing processes like CCS. Event structures which include trees are closely related to Petri nets, reflect concurrency naturally and are not committed to interleaving. In [27, 28] it is proved that by interleaving (or serializing) the labelled event structure denotation of a process one obtains its synchronization-tree denotation. The papers [27–29] provide a precise sense in which event structure models and Petri net models of communicating processes specialize down to an interleaving model based on synchronization trees. In the special case of purely synchronous processes (for which the synchronization algebra satisfies the synchronous law below) both the event structure and tree semantics agree.

## 1. A category of trees

Assume in any finite history a process can perform a sequence of events. Because a process need not be deterministic, such a sequence need not be extended in a unique way, but rather form a tree of sequences.

**1.1. Definition.** A *tree* is a subset $T \subseteq A^*$ of finite sequences of some set $A$ which satisfies

(i) $\langle\ \rangle \in T$, and

(ii) $\langle a_0, a_1, \ldots, a_n, \ldots\rangle \in T \Rightarrow \langle a_0, a_1, \ldots, a_n\rangle \in T$.

**Remark.** Condition (i) says: a tree must always contain the null sequence $\langle\ \rangle$, the root node. Condition (ii) says: a tree is closed under the initial subsequence relation. To make the ideas as familiar as possible I have taken a different definition of trees from that given in [27, 28]. However importantly, all the categories here will be equivalent to the categories of the same name introduced in [27, 28]. (Two categories are equivalent if their skeletal categories of isomorphism classes are isomorphic, (see [17]).)

**1.2. Notation.** Let $T$ be a tree with $T \subseteq A^*$. We say *$T$ is over $A$* iff every element of $A$ is in some sequence of $T$. We shall often call elements of $A$ *events*.

The following convention is very useful to avoid treating the null sequence $\langle\ \rangle$ as a special case. Often we shall write a typical sequence as $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ where $n$ is an integer representing the length of the sequence. We shall allow the length $n$ to be 0 when by convention we agree that the above sequence represents $\langle\ \rangle$.

Let $s$ be a sequence $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ and $t$ be a sequence $\langle b_0, b_1, \ldots, b_{m-1}\rangle$. Write their concatenation as

$$st = \langle a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{m-1}\rangle.$$

Let $T$ be a tree. Let $b$ be an element. By $bT$ we mean the tree

$$bT = \{\langle\ \rangle\} \cup \{\langle b\rangle t \mid t \in T\}.$$

Let $T$ be a tree. For $t, t' \in T$ write

$$t \to_T t' \overset{\text{def}}{\Leftrightarrow} \exists a. t' = t\langle a\rangle.$$

When we wish to highlight that an arc is associated with a particular event we draw the event above the arrow so

$$t \overset{a}{\to}_T t' \Leftrightarrow t' = t\langle a\rangle.$$

Clearly the elements $T$ correspond to the nodes of a tree $T$ while arcs correspond to pairs $(t, t')$ where $t \to_T t'$. The nodes are thought of as states of a process and the arcs as occurrences of events. A *morphism* from a tree $S$ to a tree $T$ shows the way in which the occurrence of an event of the process $S$ implies the synchronized occurrence of an event in the process $T$. Formally, it is a map on nodes which preserves the root-node and either preserves or collapses arcs. A special kind of morphism are the *synchronous morphisms* which always preserve arcs.

**1.3. Definition.** A *morphism* of trees from $S$ to $T$ is a map $f: S \to T$ such that

(i) $f(\langle\ \rangle) = \langle\ \rangle$, and

(ii) $s \to_S s' \Rightarrow f(s) = f(s')$ or $f(s) \to_T f(s')$.

A *synchronous morphism* of trees from $S$ to $T$ is a map $f: S \to T$ such that

(i) $f(\langle\ \rangle) = \langle\ \rangle$, and

(ii) $s \to_S s' \Rightarrow f(s) \to_T f(s')$.

Let $f: S \to T$ be a morphism of trees. Assume $s \to_S s'$ in $S$, representing the occurrence of an event $a$ of $S$ so that $s' = s\langle a \rangle$. If $f(s) \to_T f(s')$, there is an event $b$ such that $f(s') = f(s)\langle b \rangle$. Intuitively the occurrence of the event $a$ implies the occurrence of the event $b$, synchronized with that of $a$. If instead $f(s) = f(s')$, then the occurrence of $a$ is not synchronized with an event occurrence in $T$. The latter possibility is disallowed for synchronous morphisms. We shall see that morphisms and synchronous morphisms give rise to a product and synchronous product of trees. Events of the products will essentially be pairs of events of the two trees, representing events of synchronization between two processes. Their occurrence will project via tree morphisms to occurrences of component events in the constituent processes.

**1.4. Proposition.** *Trees with tree morphisms form a category with composition and identities those usual for functions. Similarly, trees with synchronous morphisms form a subcategory.*

**1.5. Definition.** Let **Tr** be the category of trees with tree morphisms. Let $\mathbf{Tr}_{\mathrm{syn}}$ be the subcategory of trees with synchronous morphisms.

**Remark.** The above categories are equivalent but not equal to the categories of the same name in [27, 28].

## 2. Categorical constructions on trees

Some major categorical constructions on **Tr** and $\mathbf{Tr}_{\mathrm{syn}}$ are presented. The basic category theory used can be found in [1] or [17].

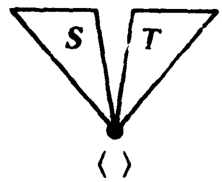
**2.1. Definition** (*coproducts in* **Tr** *and* $\mathbf{Tr}_{\mathrm{syn}}$). Let $S$ and $T$ be trees. Define

$$S + T = \{\langle (0, a_0), \ldots, (0, a_{n-1}) \rangle \mid \langle a_0, \ldots, a_{n-1} \rangle \in S\}$$
$$\cup \{\langle (1, b_0), \ldots, (1, b_{n-1}) \rangle \mid \langle b_0, \ldots, b_{n-1} \rangle \in T\}.$$

Define the obvious injections $i_0: S \to S+T$ and $i_1: T \to S+T$ by

$$i_0: \langle a_0, \ldots, a_{n-1} \rangle \mapsto \langle (0, a_0), \ldots, (0, a_{n-1}) \rangle,$$
$$i_1: \langle b_0, \ldots, b_{n-1} \rangle \mapsto \langle (1, b_0), \ldots, (1, b_{n-1}) \rangle.$$

The coproduct construction just 'glues' trees together at their roots, as shown by the following diagram.




**2.2. Theorem.** *The construction $S+T$, $i_0$, $i_1$ above is a coproduct of $S$ and $T$ in the categories* **Tr** *and* $\mathbf{Tr}_{\mathrm{syn}}$.

**Proof.** Clearly $S+T$ is a tree and $i_0: S \to S+T$ and $i_1: T \to S+T$ are synchronous morphisms. In order for $S+T$, $i_0$, $i_1$ to be a coproduct in **Tr** we require that for arbitrary morphisms $j_0: S \to U$ and $j_1: T \to U$ to a tree $U$ there is a unique morphism $j: S+T \to U$ such that the following diagram commutes:

$$\begin{tikzcd} & S+T \arrow[d, "j"] & \\ S \arrow[ur, "i_0"] \arrow[dr, "j_0"'] & & T \arrow[ul, "i_1"'] \arrow[dl, "j_1"] \\ & U & \end{tikzcd}$$

This is clearly the case for $j$ defined by

$$j(v) = \begin{cases} j_0(\langle a_0, \ldots, a_{n-1} \rangle) & \text{if } v = \langle (0, a_0), \ldots, (0, a_{n-1}) \rangle, \\ j_1(\langle b_0, \ldots, b_{n-1} \rangle) & \text{if } v = \langle (1, b_0), \ldots, (1, b_{n-1}) \rangle. \end{cases}$$

If $j_0$, $j_1$ are synchronous, so is $j$. Consequently $S+T$, $i_0$, $i_1$ is a coproduct in $\mathbf{Tr}_{\mathrm{syn}}$ too. $\square$

**2.3. Definition** (*general coproducts*). Let $\{T_i \mid i \in I\}$ be an indexed set of trees. Define their coproduct by

$$\sum_{i \in I} T_i = \bigcup_{i \in I} \{\langle (i, a_0), \ldots, (i, a_{n-1}) \rangle \mid \langle a_0, \ldots, a_{n-1} \rangle \in T_i\}.$$

Define the obvious injections $in_i: T_i \to \sum_{i \in I} T_i$ by $in_i(\langle a_0, \ldots, a_{n-1} \rangle) = \langle (i, a_0), \ldots, (i, a_{n-1}) \rangle$ for $i \in I$.

When the indexed set $I$ is null, we understand $\sum_{i \in I} T_i = \sum \emptyset$ to be the null tree $\{\langle\ \rangle\}$.

**2.4. Theorem.** *The construction $\sum_{i\in I} T_i$, $in_i$ for $i \in I$, above forms a coproduct of $\{T_i \mid i \in I\}$ in the categories* **Tr** *and* $\mathbf{Tr}_{\text{syn}}$.

**Proof.** The proof is very similar to that of Theorem 2.2. □

It is easier to define the product of trees in the category $\mathbf{Tr}_{\text{syn}}$ than the product in **Tr**. We call the product in $\mathbf{Tr}_{\text{syn}}$ the *synchronous* product. The synchronous product of two trees basically 'zips' their sequences together.

**2.5. Definition** (*synchronous product in the category* $\mathbf{Tr}_{\text{syn}}$). Let $S$ and $T$ be trees. Define their *synchronous product* by

$$S \otimes T = \{\langle (a_0, b_0), (a_1, b_1), \ldots, (a_{n-1}, b_{n-1}) \rangle \mid \langle a_0, a_1, \ldots, a_{n-1} \rangle \in S \\ \& \langle b_0, b_1, \ldots, b_{n-1} \rangle \in T\}.$$

Define projections $\pi_0 : S \otimes T \to S$ and $\pi_1 : S \otimes T \to T$ by

$$\pi_0 : \langle (a_0, b_0), \ldots, (a_{n-1}, b_{n-1}) \rangle \mapsto \langle a_0, \ldots, a_{n-1} \rangle,$$

$$\pi_1 : \langle (a_0, b_0), \ldots, (a_{n-1}, b_{n-1}) \rangle \mapsto \langle b_0, \ldots, b_{n-1} \rangle.$$

**2.6. Theorem.** *The construction $S \otimes T$, $\pi_0$, $\pi_1$ above is a product of $S$ and $T$ in the category* $\mathbf{Tr}_{\text{syn}}$.

**Proof.** Clearly $S \otimes T$ is a tree and $\pi_0 : S \otimes T \to S$ and $\pi_1 : S \otimes T \to T$ are synchronous morphisms. For $S \otimes T$, $\pi_0$, $\pi_1$ to be a product in $\mathbf{Tr}_{\text{syn}}$ we require the property: For arbitrary synchronous morphisms $f_0 : U \to S$ and $f_1 : U \to T$ from a tree $U$ there is a unique synchronous morphism $f : U \to S \otimes T$ making the following diagram commute:

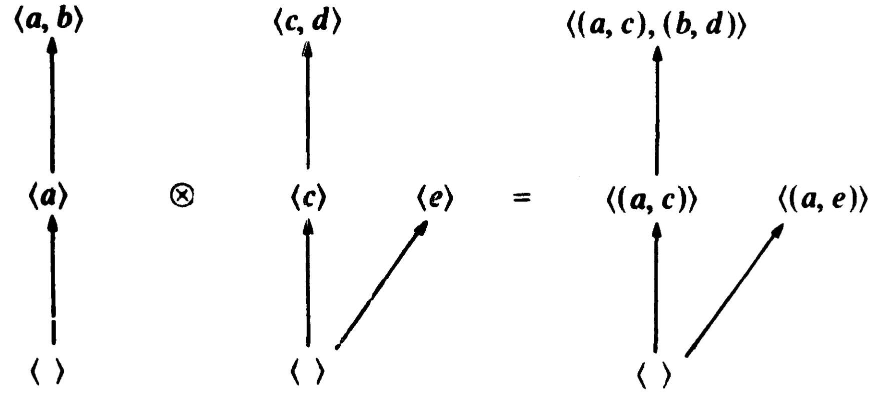
$$\begin{tikzcd}
 & S \otimes T \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
S & & T \\
 & U \arrow[uu, "f"] \arrow[ul, "f_0"] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

Because $f_0$, $f_1$ are synchronous, for $u \in U$ the sequences $f_0(u)$ and $f_1(u)$ have the same length. Thus we can define

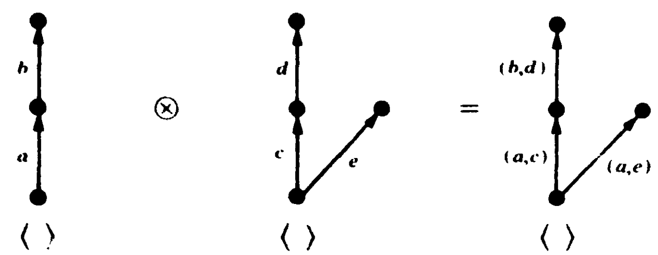
$$f(u) = \langle (a_0, b_0), \ldots, (a_{n-1}, b_{n-1}) \rangle,$$

where $f_0(u) = \langle a_0, \ldots, a_{n-1} \rangle$ and $f_1(u) = \langle b_0, \ldots, b_{n-1} \rangle$. Obviously, $f : U \to S \otimes T$ is a synchronous morphism making the above diagram commute, and clearly it is the unique morphism doing so. □

**2.7. Example**



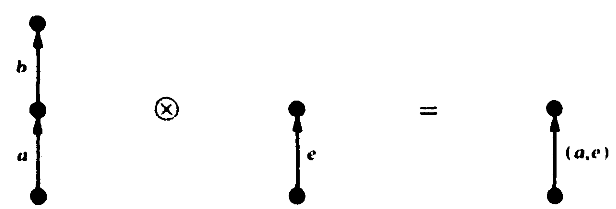

Or, labelling arcs by the events they are associated with we obtain




For example,

$$\pi_0(\langle (a, c), (b, d)\rangle) = \langle a, b\rangle, \qquad \pi_1(\langle (a, c), (b, d)\rangle) = \langle c, d\rangle.$$

Notice how projections 'unzip' sequences of pairs in the synchronous product. Clearly, we have the following synchronous product,




so projections need not be onto—consider the projection $\pi_0 : \langle (a, e)\rangle \mapsto \langle a\rangle$.

**2.8. Notation.** To give an explicit construction of a product in the category **Tr** we use partial functions. Represent undefined by the symbol $*$ and regard a partial function from $A$ to $B$ as a total function from $A$ to $B \cup \{*\}$. Write a partial function, represented by $\theta : A \to B \cup \{*\}$, as $\theta : A \to_* B$—we shall always assume $* \notin B$ for such functions. Compose partial functions as follows: Let $\theta : A \to_* B$ and $\phi : B \to_* C$. Define their composition $\phi\theta : A \to_* C$ to be

$$\phi\theta(a) = \begin{cases} \phi(\theta(a)) & \text{if } \theta(a) \neq *, \\ * & \text{otherwise.} \end{cases}$$

Denote by $\mathbf{Set}_*$ the category of sets (not containing $*$) with partial functions as morphisms. Now $\mathbf{Set}_*$ itself has a useful product. The product in $\mathbf{Set}_*$ of two sets $A$ and $B$ is given by

$$A \times_* B = \{(a, *) \mid a \in A\} \cup \{(a, b) \mid a \in A \;\&\; b \in B\} \cup \{(*, b) \mid b \in B\}$$

with projections $\rho_0 : A \times_* B \to A$ and $\rho : A \times_* B \to B$ given by $\rho_i(x_0, x_1) = x_i$ for $i = 0, 1$.

We wish to extend a partial function $\theta : A \to_* B$ on sets to a function $\bar{\theta} : A^* \to B^*$ on sequences. So by induction on the length of sequences, we define

$$\bar{\theta}(\langle\,\rangle) = \langle\,\rangle \quad \text{and} \quad \bar{\theta}(\langle a \rangle) = \begin{cases} \langle\,\rangle & \text{if } \theta(a) = * \\ \langle \theta(a) \rangle & \text{otherwise} \end{cases} \quad \text{for } a \in A,$$

$$\bar{\theta}(st) = (\bar{\theta}(s))(\bar{\theta}(t)) \quad \text{for } s, t \in A^*.$$

Now we define the product in $\mathbf{Tr}$.

**2.9. Definition** (*product in the category* $\mathbf{Tr}$). Let $S$ and $T$ be trees. Assume $S$ is over $A$ and $T$ is over $B$. Define $S \times T$ to consist of sequences over $A \times_* B$ which project via extensions of $\rho_0 : A \times_* B \to_* A$ and $\rho_1 : A \times_* B \to_* B$ to sequences in $S$ and $T$ as follows:

$$u \in S \times T \Leftrightarrow u \in (A \times_* B)^* \;\&\; \overline{\rho_0}(u) \in S \;\&\; \overline{\rho_1}(u) \in T.$$

Define *projections* $\pi_0 : S \times T \to S$ and $\pi_1 : S \times T \to T$ by taking $\pi_0(u) = \overline{\rho_0}(u)$ and $\pi_1(u) = \overline{\rho_1}(u)$ for $u \in S \times T$.

**2.10. Theorem.** *The construction $S \times T$, $\pi_0$, $\pi_1$ above is a product in the category* $\mathbf{Tr}$.

**Proof.** Clearly $S \times T$ is a tree and $\pi_0 : S \times T \to S$ and $\pi_1 : S \times T \to T$ are morphisms. Assume $f_0 : U \to S$ and $f_1 : U \to T$ are morphisms from a tree $U$. We require that there is a unique morphism $f : U \to S \times T$ making the following diagram commute:

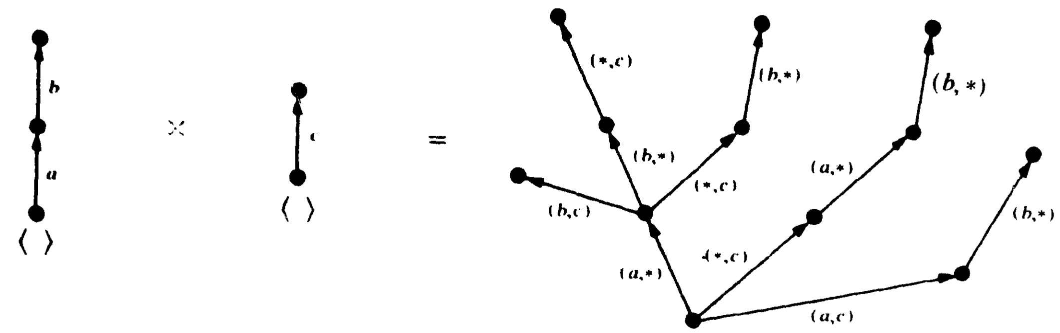
$$\begin{tikzcd} & S \times T \arrow[dl, "\pi_0"'] \arrow[dr, "\pi"] & \\ S & & T \\ & U \arrow[ul, "f_0"] \arrow[ur, "f_1"'] \arrow[uu, "f"] & \end{tikzcd}$$

Define $f(u)$ by induction on $u$,

$$f(u\langle e \rangle) = \begin{cases} f(u) & \text{if } f_0(u\langle e \rangle) = f_0(u) \text{ and } f_1(u\langle e \rangle) = f_1(u), \\ f(u)\langle (a, *) \rangle & \text{if } f_0(u\langle e \rangle) = f_0(u)\langle a \rangle \text{ and } f_1(u\langle e \rangle) = f_1(u), \\ f(u)\langle (*, b) \rangle & \text{if } f_0(u\langle e \rangle) = f_0(u) \text{ and } f_1(u\langle e \rangle) = f_1(u)\langle b \rangle, \\ f(u)\langle (a, b) \rangle & \text{if } f_0(u\langle e \rangle) = f_0(u)\langle a \rangle \text{ and } f_1(u\langle e \rangle) = f_1(u)\langle b \rangle. \end{cases}$$

A simple induction on $u$ shows that $\pi_j f(u) = f_j(u)$ for $j = 0, 1$. Obviously, $f: U \to S \times T$ is a morphism. Assume $h: U \to S \times T$ is another morphism making the diagram commute. Another simple induction on sequences $u$ shows $f(u) = h(u)$, establishing the uniqueness of $f$. Consequently $S \times T$, $\pi_0$, $\pi_1$ is a product in the category **Tr**. □

**2.11. Example.** We show the product of two simple trees. We label arcs by their associated events.




The projections $\pi_0$, $\pi_1$ act, for example, so that

$$\pi_0 : \langle (*, c), (a, *), (b, *) \rangle \mapsto \langle a, b \rangle,$$

$$\pi_1 : \langle (*, c), (a, *), (b, *) \rangle \mapsto \langle c \rangle.$$

Notice how the projections 'unzip' sequences of pairs of events with $*$. By introducing $*$ we allow the possibility of asynchrony; events in the product of two trees are not forced to occur in step if they are to occur at all.

In the categories **Tr** and $\mathbf{Tr}_{\mathrm{syn}}$ there are pleasing relations between product and coproduct. This result indicates the relation between the parallel compositions of synchronization trees (in, e.g., [18, 2]) and the product of trees.

**2.12. Theorem.** *Let S and T be trees. Then*

$$S = \bigcup_{a \in A} aS_a \cong \sum_{a \in A} aS_a \quad \textit{and} \quad T = \bigcup_{b \in B} bT_b \cong \sum_{b \in B} bT_b$$

*for some sets of events A and B and trees $S_a$ and $T_b$ indexed by $a \in A$ and $b \in B$ respectively. We have the following characterization of the product of S and T:*

$$S \times T = \bigcup_{a \in A} (a, *)S_a \times T \cup \bigcup_{a \in A,\, b \in B} (a, b)S_a \times T_b \cup \bigcup_{b \in B} (*, b)S \times T_b$$

$$\cong \sum_{a \in A} (a, *)S_a \times T + \sum_{a \in A, b \in B} (a, b)S_a \times T_b + \sum_{b \in B} (*, b)S \times T_b;$$

*and the following characterization of their synchronous product:*

$$S \otimes T = \bigcup_{a \in A,\, b \in B} (a, b)S_a \otimes T_b \cong \sum_{a \in A,\, b \in B} (a, b)S_a \otimes T_b.$$

**Proof.** Clearly the tree $S=\bigcup_{a\in A} aS_a$, where $S_a=\{t \mid \langle a\rangle t\in S\}$ for some subset $A$ of events. As the sets $aS_a$ are disjoint, $S\cong\sum_{a\in A} aS_a$. Similarly, the tree $T=\bigcup_{b\in B} bT_b\cong\sum_{b\in B} bT_b$ for some subset $B$ of events.

Let $u$ be a sequence of events of the product which project via partial functions $\rho_0$, $\rho_1$ to events of $S$ and $T$—we use the notation of Definition 2.9. We have

$$u\in S\times T \Leftrightarrow \overline{\rho_0}(u)\in S \;\&\; \overline{\rho_1}(u)\in T$$

$$\Leftrightarrow u=\begin{cases} \langle(a,*)\rangle u' & \text{for } a\in A \;\&\; \overline{\rho_0}(u')\in S_a \;\&\; \overline{\rho_1}(u')\in T, \text{ or}\\ \langle(a,b)\rangle u' & \text{for } a\in A \;\&\; b\in B \;\&\; \bar\rho_0(u')\in S_a \;\&\; \bar\rho_1(u')\in T_b, \text{ or}\\ \langle(*,b)\rangle u' & \text{for } b\in B \;\&\; \bar\rho_0(u')\in S \;\&\; \bar\rho_1(u')\in T_b. \end{cases}$$

This gives the above characterization of the product. The characterization of the synchronous product follows similarly. □

We define an operation of restriction in the next section. The synchronous product is a restriction of the product to those events with no undefined component (i.e., a component $*$). Parallel compositions will be defined as a restriction of the product. In fact, the parallel composition of synchronization trees appropriate to Milner's synchronous calculi will be a restriction of the synchronous product $\otimes$.

## 3. Complete partial orders of trees

We consider two natural complete partial orderings on trees. One is based on the idea of restricting a tree to a subset of events—an operation natural in itself—and the other is just inclusion of trees. Our operations on trees will be continuous with respect to both orderings so we shall be able to define trees recursively following now standard lines—see, e.g., [25]—by taking least fixed-points in either of the two cpo's.

**3.1. Definition** (*restriction*). Let $T$ be a tree. Let $B$ be a set. Define the *restriction* of $T$ to $B$, written $T\restriction B$, by

$$t\in T\restriction B \Leftrightarrow t\in T \;\&\; t\in B^*.$$

In other words, the restriction of a tree to a subset of events is just the subtree consisting of sequences in $T$ for which all elements are in $B$. Restriction induces a partial order on trees: one tree is below another if it is a restriction of the other. This ordering makes a complete partial order (cpo) of trees, apart from the fact that trees form a class and not a set. Of course there is another natural cpo of trees induced by simple inclusion. All the above operations on trees are continuous with respect to the two cpo structures.

**3.2. Definition.** Let $S$ and $T$ be trees over $A$ and $B$ respectively. Define

$$S \leq T \Leftrightarrow A \subseteq B \ \&\ S = T \upharpoonright A.$$

**3.3. Lemma.** *Let $S$ and $T$ be trees over the same set of events. If $S \leq T$, then $S = T$.*

**Proof.** Assume $S$ and $T$ are both over the set of events $A$. Then $T = T \cap A^* = S$. □

**3.4. Theorem.** (i) *The relation $\leq$ is a partial order with least element the null tree, $\{\langle\ \rangle\}$. Let $T_0 \leq T_1 \leq \cdots T_n \leq \cdots$ be an $\omega$-chain of trees. Then it has a least upper bound $\bigcup_{n\in\omega} T_n$.*

(ii) *The null tree $\{\langle\ \rangle\}$ is the $\subseteq$-least tree, i.e., for all trees $T$, $\{\langle\ \rangle\} \subseteq T$. Let $T_0 \subseteq T_1 \subseteq \cdots T_n \subseteq \cdots$ be an $\omega$-chain of trees. Then it has a least upper bound $\bigcup_{n\in\omega} T_n$.*

**Proof.** (i) Obviously $S \leq S$ for any tree $S$ so $\leq$ is reflexive. If $S \leq T \leq S$, then $S \subseteq T \subseteq S$ so $\leq$ is antisymmetric. If $S \leq T \leq U$, where $S$, $T$, $U$ are trees over $A$, $B$, $C$ respectively, then $S = T \cap A^* = U \cap B^* \cap A^* = U \cap A^*$ so $S \leq U$, making $\leq$ transitive. Thus $\leq$ is a partial order.

Clearly, $\{\langle\ \rangle\} \leq T$, for all trees $T$.

Let $T_0 \leq T_1 \leq \cdots \leq T_n \leq \cdots$ be an $\omega$-chain of trees $T_n$ with $T_n$ over events $A_n$. Then as $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n \subseteq \cdots$ we obtain that $T = \bigcup_{n\in\omega} T_n$ is a tree over $A = \bigcup_{n\in\omega} A_n$. By the following argument $T$ is an upper bound of each $T_n$.

Suppose $t \in T \cap A_n^*$. Then $t \in T_m$ for some $m \geq n$. As $T_n \leq T_m$ we must have $t \in T_n$. Thus $T_n \leq T$ for every $n$, so $T$ is an upper bound of $\{T_n \mid n \in \omega\}$. Now we show that $T$ is the least upper bound. Suppose $T_n \leq U$ for all $n$ with $U$ a tree. Clearly, $T \subseteq U$. If $u \in U \cap A^*$, then $u \in A_n^*$ for some $n$. Hence, as $T_n < U$ we have $u \in T_n$. So $u \in T$ too. This makes $T \leq U$ and so $T$ is the least upper bound of $\{T_n \mid n \in \omega\}$ with respect to $\leq$.

The remaining part, (ii), is obvious. □

**3.5. Definition.** Say a unary operation operation $op$ on trees is $\leq$- (respectively $\subseteq$-) *monotonic* iff $S \leq T \Rightarrow op(S) \leq op(T)$ (respectively $S \subseteq T \Rightarrow op(S) \subseteq op(T)$).

Say a unary operation operation $op$ on trees is $\leq$- (respectively $\subseteq$-) *continuous* iff it is $\leq$- (respectively $\subseteq$-) monotonic and preserves least upper bounds of $\omega$-chains of trees, i.e., if $T_0 \leq T_1 \leq \cdots \leq T_n \leq \cdots$ (respectively $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n \subseteq \cdots$) is an $\omega$-chain of trees, then $op(\bigcup_{n\in\omega} T_n) = \bigcup_{n\in\omega} op(T_n)$.

If $op$ is an $n$-ary operation on trees, say it is $\leq$- (respectively $\subseteq$-) *monotonic* iff it is monotonic in each argument separately. If $op$ is an $n$-ary operation on trees, say it is $\leq$- (respectively $\subseteq$-) *continuous* iff it is continuous in each argument separately.

The next lemma provides useful necessary and sufficient conditions for an operation to be $\leq$-continuous; the operation should be $\leq$-monotonic and act continuously on the sets of events associated with trees, where the sets of events are ordered by inclusion.

**3.6. Lemma.** *Let op be a unary operation on trees. The operation op is $\leq$-continuous iff*

(i) *the operation op is monotonic, and*

(ii) *if $T_0 \leq T_1 \leq \cdots \leq T_n \leq \cdots$ is an $\omega$-chain of trees, then the events of $op(\bigcup_{n\in\omega} T_n)$ are included in the events of $\bigcup_{n\in\omega} op(T_n)$.*

**Proof.** '$\Rightarrow$': Obvious.

'$\Leftarrow$': Suppose (i) and (ii) above hold. Let $T_0 \leq T_1 \leq \cdots \leq T_n \leq \cdots$ be a chain of trees such that each tree $T_n$ is over events $A_n$. The chain has lub $\bigcup_n T_n$. By monotonicity $\bigcup_{n\in\omega} op(T_n)$ is a tree and $\bigcup_{n\in\omega} op(T_n) \leq op(\bigcup_{n\in\omega} T_n)$. From (ii) we know that the trees $\bigcup_{n\in\omega} op(T_n)$ and $op(\bigcup_{n\in\omega} T_n)$ are over the same set of events. Thus by the above lemma they are equal. $\square$

**3.7. Theorem.** *Each operation $T \mapsto bT$, $T \mapsto T \restriction B$, $+$, $\otimes$, $\times$, for an arbitrary element $b$ and set $B$, is $\leq$-continuous and $\subseteq$-continuous. The operation of restriction is continuous on sets of events ordered by inclusion, i.e., if $T$ is a tree and if $B_0 \subseteq \cdots \subseteq B_n \subseteq \cdots$ is an $\omega$-chain of sets, then $T \restriction (\bigcup_{n\in\omega} B_n) = \bigcup_{n\in\omega} (T \restriction B_n)$.*

**Proof.** The continuity of these operations with respect to $\leq$ is best proved using Lemma 3.6. Continuity with respect to $\subseteq$ is easier to show. We only show the continuity of $\times$ with respect to $\leq$. Assume $S$, $S'$ and $T$ to be trees over $A$, $A'$ and $B$ respectively. Then $S \times T$, $S' \times T$ are over events $A \times_* B$ and $A' \times_* B$. Let $\rho_0 : A' \times B \to A'$ and $\rho_1 : A' \times B \to B$ be the partial functions projecting events in the product to their component events in $S'$ and $T$ respectively.

In showing monotonicity it is sufficient to consider just one argument which by symmetry we can assume to be the left. Suppose $S \leq S'$. We require $S \times T \leq S' \times T$. This follows by

$$
\begin{aligned}
u \in S \times T &\Leftrightarrow u \in (A \times_* B)^* \;\&\; \overline{\rho_0}(u) \in S \;\&\; \overline{\rho_1}(u) \in T \\
&\Leftrightarrow u \in (A \times_* B)^* \;\&\; \overline{\rho_0}(u) \in S' \;\&\; \overline{\rho_1}(u) \in T \\
&\Leftrightarrow u \in (S' \times T) \restriction (A \times_* B).
\end{aligned}
$$

Now assume $S_0 \leq \cdots \leq S_n \leq \cdots$ is a chain of trees so that $S_n$ is over events $A_n$. Let $e$ be an event of $(\bigcup_{n\in\omega} S_n) \times T$. Then $e$ has the form $(a, *)$, $(a, b)$ or $(*, b)$. Thus $e$ is an event of $\bigcup_{n\in\omega} (S_n \times T)$. Thus $\times$ is continuous in its first and, by symmetry, its second argument. Thus $\times$ is $\leq$-continuous.

The remainder of the proof is left to the reader. $\square$

Consequently, each of the above operations can be used in the recursive definition of trees.

## 4. Synchronization algebras

We shall label events of processes to specify how they interact with the environment. We shall obtain trees in which the arcs are labelled just like the synchronization trees of CCS in [18]. However, our approach is more abstract. We shall label trees by elements of a ***synchronization algebra*** which shows how labelled events synchronize with labelled events in the environment. Associated with any particular synchronization algebra is a particular parallel composition of synchronization trees. So, by specializing to particular synchronization algebras we obtain Milner's parallel composition of synchronization trees [18], the parallel composition that underlies his synchronous calculi [19], and the parallel compositions defined in [2] which underlie the parallel compositions on failure sets given in [9].

The intuitions behind synchronization algebras are given in [27, 28]. To recap, a synchronization algebra is a binary, commutative, associative operation $\bullet$ on a set of labels which always includes two distinguished elements $*$ and 0. The binary operation $\bullet$ says how labelled events combine to form synchronization events and what labels such combinations carry. No real events are ever labelled by $*$ or 0. However, their introduction allows us to specify the way labelled events synchronize without recourse to partial operations on labels. (These two forms of undefined should not be confused with another 'undefined' $\perp$ used in the theory of domains.)

The constant 0 is used to specify when synchronizations are disallowed. If two events labelled $\lambda$ and $\lambda'$ are not supposed to synchronize, then their composition $\lambda \bullet \lambda'$ is 0. For this reason 0 does indeed behave like a zero with respect to the 'multiplication' $\bullet$.

We have aready seen the constant $*$ in the definition of product. Recall the partial functions $\rho_0$, $\rho_1$ which projected from the events in the product to events in one of the components. An event $(e_0, *)$ in the product $S \times T$ of trees $S$ and $T$ projected down to the event $e_0$ in $S$ and the undefined 'event' $* = \rho_1((e_0, *))$ in $T$. This meant the event $e_0$ of $S$ occurred asynchronously, unsynchronized with any event of $T$. In a synchronization algebra, the constant $*$ is used to specify when a labelled event can or cannot occur asynchronously. An event labelled $\lambda$ can occur asynchronously iff $\lambda \bullet *$ is not 0. We insist that the only divisor of $*$ is $*$ itself, essentially because we do not want a synchronization event to disappear. (The reader may find it helpful to glance ahead to the definition of parallel composition of synchronization trees given in Definition 6.9.)

**4.1. Definition.** A *synchronization algebra* (SA) is an algebra $(L, \bullet, *, 0)$ where $L$ is a set of *labels* so $L \backslash \{*, 0\} \neq \emptyset$ and $\bullet$ is a binary commutative associative operation on $L$ which satisfies

(i) $\forall \lambda \in L. \lambda \bullet 0 = 0$, and

(ii) $* \bullet * = *$ and $\forall \lambda, \lambda' \in L. \lambda \bullet \lambda' = * \Rightarrow \lambda = *$.

Synchronization algebras have an obvious divisor relation which intuitively says when one labelled event can be a component of a synchronization event.

**4.2. Definition.** Let $(L, \bullet, *, 0)$ be an SA. For $\lambda, \lambda' \in L$ define

$$\lambda \; div \; \lambda' \Leftrightarrow \lambda = \lambda' \text{ or } \exists \mu \in L. \lambda \bullet \mu = \lambda',$$

where by $\lambda \; div \; \lambda'$ we say "$\lambda$ divides $\lambda'$".

**4.3. Lemma.** *Let $(L, \bullet, *, 0)$ be a synchronization algebra. Then the following properties hold:*

(i) *the constants $*$ and $0$ are distinct,*
(ii) *the relation div is reflexive and transitive, i.e., a preorder,*
(iii) $\lambda \; div * \Rightarrow \lambda = *$,
(iv) $0 \; div \; \lambda \Rightarrow \lambda = 0$,
(v) $\alpha_0 \; div \; \beta_0 \;\&\; \alpha_1 \; div \; \beta_1 \Rightarrow (\alpha_0 \bullet \alpha_1) \; div \; (\beta_0 \bullet \beta_1)$.

**Proof.** (i) We can take $\alpha \in L\backslash\{*, 0\}$. Then if $0 = *$, we would have $\alpha \bullet 0 = 0 = *$ which implies $\alpha = *$. This contradicts the choice of $\alpha$ making $0 \neq *$.
(ii) follows by associativity,
(iii) follows by property (ii) in the definition of synchronization algebra,
(iv) follows as 0 is a zero,
(v) follows by commutativity and associativity. □

We might wish to specify that no event can occur asynchronously. An event will be labelled by a non-$*$, non-0 label so this can be specified by ensuring the composition of such labels with $*$ always gives 0. Milner's synchronous calculi [19] fit into this scheme, as we shall see later in Lemma 6.19. In Theorem 6.12 we shall make use of another law on synchronization algebras. It expresses when $\bullet$ behaves like the least upper bound with respect to *div*, or, the same thing, when $\bullet$ is the operation of least common multiple (LCM) for the 'multiplication' $\bullet$.

**4.4. Definition.** Let $(L, \bullet, *, 0)$ be an SA. We say $L$ is *synchronous* when it satisfies the law

$$\forall \lambda \in L\backslash\{*\}. \lambda \bullet * = 0.$$

We say $(L, \bullet, *, 0)$ satisfies the LCM law when

$$\forall \alpha, \beta, \gamma \in L. \alpha \; div \; \gamma \;\&\; \beta \; div \; \gamma \Rightarrow (\alpha \bullet \beta) \; div \; \gamma.$$

As examples and for future reference we now present some synchronization algebras. We present the algebras in the form of multiplication tables. In fact the synchronization algebras correspond to the parallel composition of CCS and the two forms of parallel composition in [9, 2]. A full justification of these facts appears later. For the moment through, the reader can probably see what each synchronization algebra is saying so we shall try to give the intuition. The tie-up with Milner's monoids and groups of actions for his synchronous calculi will be made later.

**4.5. Example**(*the synchronization algebra for CCS* [18]). *Pure CCS—no value passing*: In CCS events are labelled by $\alpha, \beta, \ldots$ or by their complementary labels

$\bar{\alpha}$, $\bar{\beta}$, ... or by the label $\tau$. The idea is that only two events bearing complementary labels may synchronize to form a synchronization event labelled by $\tau$. Events labelled by $\tau$ cannot synchronize further; in this sense they are invisible to processes in the environment though their occurrence may lead to internal changes of state. All labelled events may occur asynchronously. Hence the synchronization algebra for CCS takes the following form. We call the algebra $L_1$.

| $\bullet$ | $*$ | $\alpha$ | $\bar{\alpha}$ | $\beta$ | $\bar{\beta}$ | ... | $\tau$ | 0 |
|---|---|---|---|---|---|---|---|---|
| $*$ | $*$ | $\alpha$ | $\bar{\alpha}$ | $\beta$ | $\bar{\beta}$ | ... | $\tau$ | 0 |
| $\alpha$ | $\alpha$ | 0 | $\tau$ | 0 | 0 | ... | 0 | 0 |
| $\bar{\alpha}$ | $\bar{\alpha}$ | $\tau$ | 0 | 0 | 0 | ... | 0 | 0 |
| $\beta$ | $\beta$ | 0 | 0 | $\tau$ | 0 | ... | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | ... | $\vdots$ | $\vdots$ |

*With value passing*: Suppose values $v \in V$ are passed during synchronization. Take labels of the form $*$, 0, $\alpha v$ (receiving a value $v$ on line $\alpha$) and $\bar{\alpha} v$ (sending a value $v$ on line $\alpha$) with a synchronization algebra like that above but now with $\bar{\alpha} v$ the complement of $\alpha v$. More precisely, take $L_1(V)$ to be the synchronization algebra $(L, \bullet, *, 0)$ where $L = (L_1 \setminus \{\tau, *, 0\} \times V) \cup \{\tau, *, 0\}$ with composition given by

$$\lambda \bullet \lambda' = \begin{cases} \tau & \text{if } \lambda = \alpha v \text{ and } \lambda' = \bar{\alpha} v, \\ \tau & \text{if } \lambda = \bar{\alpha} v \text{ and } \lambda' = \alpha v, \\ \lambda & \text{if } \lambda' = *, \\ \lambda' & \text{if } \lambda = *, \\ 0 & \text{otherwise.} \end{cases}$$

We shall see that $L_1(V)$ can be viewed as a simple quotient algebra of the (direct) product of two synchronization algebras, one being $L_1$ and the other a straightforward extension of the set of values $V$ to a synchronization algebra.

**4.6. Example** (*the synchronization algebra for* $\|$ *in* [9, 2]). In [9] and [2] events are labelled by $\alpha$, $\beta$, ... or $\tau$. For the parallel composition $\|$ in [9, 2] events must 'synchronize on' $\alpha$, $\beta$, .... In other words non-$\tau$-labelled events cannot occur asynchronously. Rather, an $\alpha$-labelled event in one component of a parallel composition must synchronize with an $\alpha$-labelled event from the other component in order to occur; the two events must synchronize to form a synchronization event again labelled by $\alpha$. The S.A. for this parallel composition takes the following form. We call the algebra $L_2$.

| $\bullet$ | $*$ | $\alpha$ | $\beta$ | ... | $\tau$ | 0 |
|---|---|---|---|---|---|---|
| $*$ | $*$ | 0 | 0 | ... | $\tau$ | 0 |
| $\alpha$ | 0 | $\alpha$ | 0 | . | 0 | 0 |
| $\beta$ | 0 | 0 | $\beta$ | . . | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | ... | $\vdots$ | $\vdots$ |

**4.7. Example** (*the synchronization algebra for* ||| *in* [9, 2]). The parallel composition ||| in [9] and [2] is called the 'interleaving' operation in [9, 2]. The reason is that no synchronizations are allowed, but every event can occur asynchronously, so in the framework of [9, 2] where processes are coerced so they perform only one event at a time the parallel composition ||| interleaves the sequences of events of the two component processes. Events are labelled exactly as they are for $L_2$ but the synchronization algebra takes a different form, shown below. We call this algebra $L_3$.

| $\bullet$ | $*$ | $\alpha$ | $\beta$ | $\dots$ | $\tau$ | $0$ |
|---|---|---|---|---|---|---|
| $*$ | $*$ | $\alpha$ | $\beta$ | $\dots$ | $\tau$ | $0$ |
| $\alpha$ | $\alpha$ | $0$ | $0$ | $\dots$ | $0$ | $0$ |
| $\beta$ | $\beta$ | $0$ | $0$ | $\dots$ | $0$ | $0$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\dots$ | $\vdots$ | $\vdots$ |

Of course synchronization algebras can be viewed as standard algebras with an operation $\bullet$ and two constants $*$ and 0. Looked at in this way they come ready equipped with the usual definition of homomorphism (made to preserve the composition and the constants), and the attendant categorical constructions like (direct) product. But does this mathematical definition match the interpretation we put to the operation $\bullet$ and constants $*$ and 0? I think not, and tentatively suggest the following definitions are more suitable. They regard synchronization algebras as *partial algebras* (see [8]) which have partial operations preserved by homomorphisms only when they are defined; think of composition as being undefined when it gives 0. Consequently 0 is preserved in rather a strict way. One class of homomorphisms result if we impose a similar strict law for $*$—we call these *strict*—and another if we require simply that $*$ is preserved.

**4.8. Definition.** Let $A = (L_A, \bullet_A, *_A, 0_A)$ and $B = (L_B, \bullet_B, *_B, 0_B)$ be synchronization algebras. A *homomorphism* of synchronization algebras from $A$ to $B$ is a function $h: L_A \to L_B$ such that the following conditions hold:

(i) $\alpha \bullet \alpha' \neq 0 \Rightarrow h(\alpha \bullet_A \alpha') = h(\alpha) \bullet_B h(\alpha')$,

(ii) $h(\alpha) = 0_B \Leftrightarrow \alpha = 0_A$,

(iii) $h(*_A) = *_B$.

We say a homomorphism $h$ is *strict* when $h(\lambda) = *_B \Leftrightarrow \lambda = *_A$.

**4.9. Proposition.** *Synchronization algebras with homomorphisms form a category with composition the usual composition of functions and identity homomorphisms the identity functions. Synchronization algebras with strict homomorphisms form a subcategory.*

**Proof.** We check the composition of homomorphisms is a homomorphism. Suppose $h: A \to B$ and $g: B \to C$ are homomorphisms. Assume $\alpha \bullet \alpha' \neq 0$ in $A$. Then $h(\alpha \bullet \alpha') = h(\alpha) \bullet h(\alpha') \neq 0$ in $B$. So $gh(\alpha \bullet \alpha') = g(h(\alpha) \bullet h(\alpha')) = gh(\alpha) \bullet gh(\alpha')$ in $C$. Clearly $gh(\alpha) = 0 \Leftrightarrow h(\alpha) = 0 \Leftrightarrow \alpha = 0$ and $gh(*) = g(*) = *$. Thus $gh$ is a homomorphism. The remainder of the proof is left to the reader. $\square$

**4.10. Definition.** Write SA for the category of synchronization algebras with homomorphisms.

We show the form of products in the category SA and its subcategory with strict homomorphisms. Products of synchronization algebras provide one way to construct more complex algebras from more simple ones.

**4.11. Definition.** Let $A = (L_A, \bullet_A, *_A, 0_A)$ and $B = (L_B, \bullet_B, *_B, 0_B)$ be synchronization algebras.

Define the *product of synchronization algebras*, $A \times B$, to be $(L, \bullet, *, 0)$ given by

(i) $L = (L_A \backslash \{0_A\}) \times (L_B \backslash \{0_B\}) \cup \{(0_A, 0_B)\}$,

(ii) $$(\alpha, \beta) \bullet (\alpha', \beta') = \begin{cases} (0_A, 0_B) & \text{if } \alpha \bullet \alpha' = 0 \text{ or } \beta \bullet \beta' = 0, \\ (\alpha \bullet_A \alpha', \beta \bullet_B \beta') & \text{otherwise,} \end{cases}$$

(iii) $* = (*_A, *_B)$ and $0 = (0_A, 0_B)$.

Define projection homomorphisms $h_A : A \times B \to A$, $h_B : A \times B \to B$, by $h_A(\alpha, \beta) = \alpha$ and $h_B(\alpha, \beta) = \beta$, respectively.

**4.12. Definition.** Let $A = (L_A, \bullet_A, *_A, 0_A)$ and $B = (L_B, \bullet_B, *_B, 0_B)$ be synchronization algebras.

Define the *strict product of synchronization algebras*, $A \otimes B$, to be $(L', \bullet, *, 0)$ given by

(i) $L' = (L_A \backslash \{*_A, 0_A\}) \times (L_B \backslash \{*_B, 0_B\}) \cup \{(*_A, *_B), (0_A, 0_B)\}$,

(ii) $$(\alpha, \beta) \bullet (\alpha', \beta') = \begin{cases} (0_A, 0_B) & \text{if } \alpha \bullet \alpha' = 0 \text{ or } \beta \bullet \beta' = 0, \\ (\alpha \bullet_A \alpha', \beta \bullet_B \beta') & \text{otherwise,} \end{cases}$$

(iii) $* = (*_A, *_B)$ and $0 = (0_A, 0_B)$.

Define projection homomorphisms $h'_A : A \times B \to A$, $h'_B : A \times B \to B$, by $h'_A(\alpha, \beta) = \alpha$ and $h'_B(\alpha, \beta) = \beta$, respectively.

Notice that $A \otimes B$ has sort a subset of the sort of $A \times B$ and that it is closed under all operations $\bullet$ of $A \times B$. It is a subalgebra (of partial algebras) in the sense of [8]. It is also the restriction of the larger algebra to a subset, another way of constructing new synchronization algebras from old.

**4.13. Theorem.** *Let $A$ and $B$ be synchronization algebras. The construction $A \times B$, $h_A$, $h_B$ is a categorical product of $A$ and $B$ in* SA. *The construction $A \otimes B$, $h'_A$, $h'_B$ is a categorical product of $A$ and $B$ in the subcategory with strict homomorphisms.*

For the proof of Theorem 4.13, see Appendix A.

Another way to obtain new synchronization algebras is to quotient by a congruence relation. A congruence relation on a synchronization algebra is an equivalence relation $\equiv$ such that

$$\lambda \equiv \lambda' \;\&\; \mu \equiv \mu' \;\&\; \lambda \bullet \mu \neq 0 \;\&\; \lambda' \bullet \mu' \neq 0 \Rightarrow \lambda \bullet \mu \equiv \lambda' \bullet \mu'.$$

Given a synchronization algebra and a congruence relation $\equiv$, the quotient consists of new labels the equivalence clases of $\equiv$ with $\bullet$-composition induced by the representatives. We illustrate how the synchronization algebra for CCS with value passing arises as the quotient of a strict product. Firstly a non-null set of values $V$ extends to a synchronization algebra $V^{\bullet}$ with extra elements $*$ and 0 by taking $v \bullet v = v$ for $v \in V$ and $v \bullet * = * \bullet v = v$ for $v \in V \cup \{*\}$ and $v \bullet 0 = 0 \bullet v = 0$ for $v \in V \cup \{*, 0\}$.

**4.14. Proposition.** *Let $L_1$ be the synchronization algebra for CCS given in Example 4.5. Let $h: L_1 \otimes V^{\bullet} \to L_1$ be the strict projection homomorphism from the strict product. Take the relation $\equiv$ on $L_1 \otimes V^{\bullet}$ to be given by*

$$\lambda \equiv \lambda \Leftrightarrow h(\lambda) = h(\lambda') = \tau.$$

*Then $\equiv$ is a congruence relation and the quotient $(L_1 \otimes V^{\bullet})/\equiv$ is isomorphic to $L_1(V)$, the synchronization algebra for CCS with value passing given in Example 4.5.*

Of course, one can specify that more complicated operations are performed on values than just send and receive.

We stress that the definitions of homomorphisms on synchronization algebras are tentative. Constructions like $\otimes$ on synchronization algebras appear useful but may not be as general as one would like. The axioms on synchronization algebras arose by considering an abstract way to formalize the range of synchronization disciplines between labelled events. Possibly there is a class of algebras for specifying how processes are connected, or linked, together. That the physical linkage can be quite complicated and yet still be highly structured is demonstrated in [7]. Typically processes may be linked by abstract channels or physical wires connected to linkage points or *ports* of the processes. To specify how they are linked by channels or wires the ports are assigned names or labels; perhaps ports to be linked carry the same label, as in [9], or complementary labels as in [18]. An algebra on these labels might specify the geometric layout of the processes, how the processes are physically linked or wired together. But then along the channels or wires values may meet and interact; for example, in hardware the values may be voltage contributions due to processes wired together. The interaction of these values might be specified by a synchronization algebra. (The table giving this interaction in hardware is generally called the logic—it may be Boolean, have undefined values, floating values, strong and weak values, etc.) Such processes interact through the synchronization of events, where an event is a value at a port. Of course only events which are physically linked can interact. When they do, the resultant value communicated will be determined by the component values. This suggests that the synchronization algebra associated with processes should be a product of the 'linkage algebra' and the synchronization algebra of values. At present this is rather speculative but it does suggest we explore a wider class of algebras and, from our experience with synchronization algebras, that the algebras should be partial.

## 5. Synchronization trees

A synchronization tree is a tree with arcs labelled by elements of synchronization algebra. It is convenient to label arcs via the underlying events from which the tree is built.

**5.1. Definition.** Let $L$ be a synchronization algebra. An *L-synchronization tree* is a pair $(T, l)$ where $T$ is a tree over $A$ and $l: A \to L \backslash \{*, 0\}$.

**5.2. Notation.** Let $(T, l)$ be an $L$-sychronization tree. Write $t \xrightarrow{\lambda} t'$ when $t \to t'$ and $l(a) = \lambda$ for the unique $a$ such that $t' = t\langle a \rangle$.

Frequently we shall omit the prefix "$L$-" when discussing synchronization trees. When it is important the appropriate synchronization algebra should be clear from the context.

We produce a category of synchronization trees by restricting the tree-morphisms in accord with the synchronization algebra. We insist the label of the image of an arc should divide the label of the arc because the image of an event is imagined to be a component of the event. Of course, an arc may be collapsed in the image corresponding to the intuition that the event is not synchronized with any event of the image. But then we insist $*$ divides the original label.

**5.3. Definition.** Let $L$ be a synchronization algebra. Define an *L-morphism* of $L$-synchronization trees from $(S, l_S)$ to $(T, l_T)$ to be a map $f: S \to T$ such that

$$f(\langle\ \rangle) = \langle\ \rangle$$

and

$$s \xrightarrow{\lambda} s' \Rightarrow (f(s) = f(s')\ \&\ * \ \mathit{div}\ \lambda) \text{ or } (f(s) \xrightarrow{\lambda'} f(s')\ \&\ \lambda' \ \mathit{div}\ \lambda).$$

**5.4. Proposition.** *Let L be a synchronization algebra. Then L-synchronization trees with L-morphisms form a category under the usual function composition and with the usual identity functions.*

*Let $(S, l_S)$ and $(T, l_T)$ be two L-synchronization trees. Then $(S, l_S)$ and $(T, l_T)$ are isomorphic in this category iff there is a bijection $f: S \to T$ such that*

$$s \to s' \Leftrightarrow f(s) \to f(s')$$

*and such that labels of corresponding arcs divide each other.*

*In particular, if div is an antisymmetric relation on L (i.e., $\lambda$ div $\lambda'$ div $\lambda \Rightarrow \lambda = \lambda'$), then $(S, l_S)$ and $(T, l_T)$ are isomorphic iff there is a bijection $f: S \to T$ such that*

$$s \xrightarrow{\lambda} s' \Leftrightarrow f(s) \xrightarrow{\lambda} f(s').$$

**Proof.** That $L$-synchronization trees with $L$-morphisms, for a synchronization algebra $L$, form a category routinely follows from the facts that **Tr** is a category and *div* is a reflexive transitive relation on labels. The characterizations of isomorphism follow directly from the definition of $L$-morphism. □

**5.5. Definition.** Write $\mathbf{Tr}_L$ for the category of $L$-synchronization trees with $L$-morphisms.

**Remark.** Note this category is equivalent but not equal to the category $\mathbf{Tr}_L$ in [27, 28].

**5.6. Proposition.** *Let L be a synchronization algebra. If $f: (S, l_S) \to (T, l_T)$ is an L-morphism of synchronization trees, then $f: S \to T$ is a morphism of trees. Assume that L is synchronous, so $\lambda \bullet * = 0$ for all $\lambda \in L \setminus \{*\}$. Then for any L-morphism $f: (S, l_S) \to (T, l_T)$ the map $f: S \to T$ is a synchronous morphism of trees.*

**Proof.** Clearly if $L$ is synchronous $*$ $\not{div}$ $\lambda$ for any label $\lambda \in L \setminus \{*, 0\}$. Thus $L$-morphisms cannot collapse arcs. □

Thus we see how assumptions made on the synchronization algebra influence the morphisms we allow. In fact, particular synchronization algebras give us categories isomorphic to **Tr** and $\mathbf{Tr}_{\text{syn}}$.

**5.7. Proposition.** *Let A and S be the synchronization algebras given by*

| $A$ $\bullet_A$ | $*$ | $T$ | $0$ |
|---|---|---|---|
| $*$ | $*$ | $T$ | $0$ |
| $T$ | $T$ | $T$ | $0$ |
| $0$ | $0$ | $0$ | $0$ |

| $S$ $\bullet_S$ | $*$ | $T$ | $0$ |
|---|---|---|---|
| $*$ | $*$ | $0$ | $0$ |
| $T$ | $0$ | $T$ | $0$ |
| $0$ | $0$ | $0$ | $0$ |

*Then $\mathbf{Tr}_A \cong \mathbf{Tr}$ and $\mathbf{Tr}_S \cong \mathbf{Tr}_{\text{syn}}$.*

**Proof.** Because $*$ *div* $T$ in $A$ morphisms may collapse arcs while in $S$, because $*$ $\not{div}$ $T$ they must be preserved. □

## 6. Operations on synchronization trees

Assume $(L, \bullet, *, 0)$ is a synchronization algebra. Define the following operations on ($L$-)synchronization trees.

**6.1. Definition** (*lifting*). *Let* $\lambda \in L\backslash\{*, 0\}$ and $(T, l)$ be a synchronization tree. Define $\lambda(T, l)$ to be the synchronization tree $(T', l')$ where

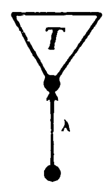
$$t \in T' \Leftrightarrow t = \langle\ \rangle \text{ or } t = \langle(0, \lambda), (1, a_0), \ldots, (1, a_{n-1})\rangle$$

for some $\langle a_0, \ldots, a_{n-1}\rangle \in T$, and the new labelling function acts so $l'((0, \lambda)) = \lambda$ and $l'((1, a)) = l(a)$.

Extend lifting to morphisms as follows: Assume $f: (T, l_T) \to (T', l'_T)$ is a morphism of synchronization trees and $\lambda \in L\backslash\{*, 0\}$. Define $\lambda f$: $\lambda(T, l_T) \to \lambda(T', l'_T)$ by

$$(\lambda f)(t) = \begin{cases} \langle\ \rangle & \text{if } t = \langle\ \rangle, \\ \langle(0, \lambda), (1, b_0), \ldots, (1, b_{m-1})\rangle & \text{if } t = \langle(0, \lambda), (1, a_0), \ldots, (1, a_{n-1})\rangle\ \& \\ & \quad f(\langle a_0, \ldots, a_{n-1}\rangle) = \langle b_0, \ldots, b_{m-1}\rangle. \end{cases}$$

The process represented by $\lambda T$ must first do a $\lambda$-labelled event before becoming the process represented by a copy of $T$. In pictures we can draw lifting as follows:



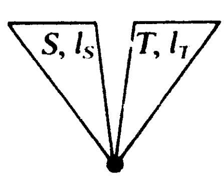

**6.2. Theorem.** *Let* $\lambda \in L\backslash\{*, 0\}$. *The operation of lifting is a functor* $\lambda: \mathbf{Tr}_L \to \mathbf{Tr}_L$.

**Proof.** The proof is obvious.

**6.3. Definition** (*sum*). Let $(S, l_S)$ and $(T, l_T)$ be synchronization trees. Define their *sum* by

$$(S, l_S) + (T, l_T) = (S + T, l), \quad \text{where } l(c) = \begin{cases} l_S(a) & \text{if } c = (0, a), \\ l_T(b) & \text{if } c = (1, b). \end{cases}$$

The sum just sticks trees together at their roots. We can draw the sum $(S, l_S) + (T, l_T)$ as follows:



**6.4. Definition** (*indexed sum*). Let $(T_i, l_i)$ be a set of synchronization trees indexed by $i \in I$. Define their *sum* by

$$\sum_{i \in I} (T_i, l_i) = \left(\sum_{i \in I} T_i, l\right),$$

where $l(c) = l_i(a)$ if $c = (i, a)$ for $i \in I$.

Sum has obvious injection morphisms such that it is a coproduct in the category of synchronization trees. Consequently the construction will extend naturally to a functor.

**6.5. Theorem.** *Let $(S, l_S)$ and $(T, l_T)$ be L-synchronization trees. Let $i_0: S \to S + T$ and $i_1: T \to S + T$ be the injections—as given in the definition of coproduct. Then $i_0$, $i_1$ are L-morphisms and $(S, l_S) + (T, l_T)$, $i_0$, $i_1$ is a coproduct in the category $\mathbf{Tr}_L$ of synchronization trees.*

*Similarly, $\sum_{i \in I}(T_i, l_i)$ with injections $in_i$ for $i \in I$ is a coproduct where $(T_i, l_i)$ is an I-indexed set of synchronization trees with injections $in_i: (T_i, l_i) \to \sum_{i \in I} (T_i, l_i)$—as given in the definition of indexed coproduct.*

**Proof.** These properties follow from the corresponding properties in the underlying category of trees. $\square$

**6.6. Definition** (*restriction*). Let $\Lambda \subseteq L\backslash\{*, 0\}$ satisfy the property

$$\lambda \in \Lambda \;\&\; \lambda \; div \; \lambda' \;\&\; \lambda \;\; div \; \lambda \;\Rightarrow\; \lambda' \in \Lambda.$$

Let $(T, l)$ be a synchronization tree over $A$. Define

$$(T, l) \upharpoonright \Lambda = (T \upharpoonright B, l'),$$

where

$$B = \{b \in A \mid l(b) \in \Lambda\} \quad \text{and} \quad l'(b) = l(b) \qquad \text{for } b \in B.$$

The operation $(T, l) \upharpoonright \Lambda$ restricts events to those which are labelled by elements of $\Lambda$. There are several alternative definitions of restriction in the literature [18, 19, 9, 2]. Ours is chosen to be general and such that it still preserves isomorphism; it is like that in [19]. I do not know how to extend restriction to a functor in a natural way. (At some cost in artificiality restriction can be presented as an equalizer.)

**6.7. Definition** (*relabelling*). Let $\Xi: L \to L$ be a strict homomorphism of the synchronization algebra $L$. Let $(T, l)$ be a synchronization tree. Define $(T, l)[\Xi] = (T, \Xi l)$.

For $\Xi: L \to L$ a strict homomorphism, extend relabelling to morphisms as follows: Assume $f: (S, l_S) \to (S', l'_S)$ is a morphism of synchronization trees. Define $f[\Xi]: (S, l_S)[\Xi] \to (S', l'_S)[\Xi]$ by $(f[\Xi])(s) = f(s)$.

We have chosen this definition of relabelling because it extends to a functor on $\mathbf{Tr}_L$. (Of course there are other possible definitions which are also continuous with respect to $\leq_L$ given below. One example is the make-$\alpha$-labels-into-$\tau$-labels definition of hiding given in [9, 2].)

**6.8. Theorem.** *Let $\Xi: L \to L$ be a strict homomorphism on the synchronization algebra $L$. The operation $[\Xi]: \mathbf{Tr}_L \to \mathbf{Tr}_L$ is a functor on synchronization trees.*

**Proof.** Recall what it means for $\Xi$ to be a strict homomorphism on $L$. $\Xi: L \to L$ and $\Xi$ preserves $\bullet$, $*$, 0 and

$$\forall \lambda \in L.(\Xi(\lambda) = 0 \Rightarrow \lambda = 0) \;\&\; (\Xi(\lambda) = * \Rightarrow \lambda = *).$$

These properties ensure that $T[\Xi]$ is a synchronization tree for a synchronization tree $T$. Because $\lambda' \; div \; \lambda \Rightarrow \Xi(\lambda') \; div \; \Xi(\lambda)$ the map $[\Xi]$ produces morphisms from morphisms. Thus it is clearly a functor. $\square$

**6.9. Definition** (*parallel composition*). Let $(S, l_S)$ and $(T, l_T)$ be synchronization trees. Assume $S$ is over $A$ and $T$ is over $B$. Then $S \times T$ is over $A \times_* B$, the product in $\mathbf{Set}_*$ with projections $\rho_0: A \times_* B \to A$ and $\rho_1: A \times_* B \to B$. Define the parallel composition of $(S, l_S)$ and $(T, l_T)$ by

$$(S, l_S) \; Ⓛ \; (T, l_T) = (S \times T \upharpoonright C, l),$$

where

$$C = \{c \in A \times_* B \mid l_S\rho_0(c) \bullet l_T\rho_1(c) \neq 0\} \quad \text{and} \quad l(c) = l_S\rho_0(c) \bullet l_T\rho_1(c).$$

*Note*: We assume that the function compositions occur in $\mathbf{Set}_*$; so if, for example, $\rho_0(c) = *$, then $l_S\rho_0(c) = *$.

Extend Ⓛ to morphisms as follows. Let $f: (S, l_S) \to (S', l_{T'})$ and $g: (T, l_T) \to (T', l_{T'})$ be two morphisms in $\mathbf{Tr}_L$. Define $f \; Ⓛ \; g = f \times g$, the image of $f$ and $g$ under the product functor $\times$ on $\mathbf{Tr}$.

In fact this definition makes Ⓛ into a functor.

**6.10. Theorem.** *The operation Ⓛ is a functor Ⓛ$: \mathbf{Tr}_L^2 \to \mathbf{Tr}_L$ on synchronization trees.*

**Proof.** Let $f: S \to S'$ and $g: T \to T'$ be $L$-morphisms. We show by induction on the length of $u'$ that if $u \xrightarrow{\lambda} u'$ in $S Ⓛ T$ then $f \times g(u') \in S' Ⓛ T'$ and

$$f \times g(u) = f \times g(u') \;\&\; * \; div \; \lambda \quad \text{or} \quad f \times g(u) \xrightarrow{\lambda'} f \times g(u') \;\&\; \lambda' \; div \; \lambda.$$

It follows that $f Ⓛ g: S Ⓛ T \to S' Ⓛ T'$ is a morphism.

Either (a) $f\times g(u)=f\times g(u')$, or (b) $f\times g(u)\to f\times g(u')$. If (a), then $f(u)=f(u')$ (and $g(u)=g(u')$) so $*\ div\ \lambda$. Otherwise (b), in which case let $c$ be the unique event such that $u\langle c\rangle=u'$. Write its component-events in $S$ and $T$ as $c_0$ and $c_1$ respectively—one of $c_0$ and $c_1$ may be $*$. Let $l_S(c_0)=\lambda_0$ and $l_T(c_1)=\lambda_1$. Similarly, let $c'$ be the unique event of $S'\times T'$ such that $(f\times g(u))\langle c'\rangle=f\times g(u')$. Assume the component events of $c'$ to have labels $\lambda_0'$ and $\lambda_1'$ in $S'$ and $T'$ respectively. As $f$ and $g$ are $L$-morphisms $\lambda_0'\ div\ \lambda_0$ and $\lambda_1'\ div\ \lambda_1$. By Lemma 4.3(v) we obtain $\lambda'=\lambda_0'\bullet\lambda_1'\ div\ \lambda_0\bullet\lambda_1=\lambda$. Thus $\lambda_0'\bullet\lambda_1'\neq 0$ by Lemma 4.3(iv) so $c'$ is an event of $S'Ⓛ T'$. Inductively, this ensures that $f\times g(u')\in S'Ⓛ T'$ and clearly that $f\times g(u)\xrightarrow{\lambda'} f\times g(u')\ \&\ \lambda'\ div\ \lambda$.

Thus Ⓛ takes $L$-morphisms to $L$-morphisms. Its functorial properties follow from those of $\times$ in the underlying category of trees. □

Thus, apart from restriction, all the above operations extend to functors on $\mathbf{Tr}_L$ in an obvious way.

Generally the parallel composition of synchronization trees is defined recursively—see, e.g., [18, 2]. Instead we can give a recursive characterization of our definition of parallel composition, which fortunately agrees with those in the literature when we specialize to particular synchronization algebras. Because here we serialize all event occurrences, parallel composition, like product, can be expressed as an indexed sum.

**6.11. Theorem.** *Let $S$ and $T$ be $L$-synchronization trees. Then*

$$S\cong\sum_{i\in I}\lambda_i S_i \quad \text{and} \quad T\cong\sum_{j\in J}\mu_j T_j$$

*for some indexed sets of labels and synchronization trees. Moreover, the parallel composition of $S$ and $T$ can be characterized as follows:*

$$S Ⓛ T\cong\sum_{\lambda_i\bullet *\neq 0}(\lambda_i\bullet *)(S_i Ⓛ T)+\sum_{\lambda_i\bullet\mu_j\neq 0}(\lambda_i\bullet\mu_j)(S_i Ⓛ T_j)+\sum_{*\bullet\mu_j\neq 0}(*\bullet\mu_j)(S Ⓛ T_j).$$

**Proof.** The theorem follows from Theorem 2.12 and Definition 6.9. □

The above result means that we can show how by specializing to particular synchronization algebras we obtain various parallel compositions of synchronization trees present in the literature. Before this we pause to show how parallel composition relates to product in the categories of synchronization trees. Although there are obvious projection functions, parallel composition does not always coincide with product. It does, however, when the operation $\bullet$ in the algebra behaves like the least common multiple (LCM) operation, given in Definition 4.4.

**6.12. Theorem.** *Let $(S, l_S)$ and $(T, l_T)$ be two $L$-synchronization trees over $A$ and $B$ respectively. Let $\pi_0'=\pi_0\upharpoonright(S Ⓛ T)$ and $\pi_1'=\pi_1\upharpoonright(S Ⓛ T)$ be the obvious restrictions* *of the projections $\pi_0: S \times T \to S$ and $\pi_1: S \times T \to T$ to the parallel composition. Then $S \textcircled{L} T$, $\pi_0'$, $\pi_1'$ is a product in the category $\mathbf{Tr}_L$ if*

$$(\forall \gamma \in L)(\forall \alpha \in l_S A)(\forall \beta \in l_T B). \alpha \; div \; \gamma \; \& \; \beta \; div \; \gamma \Rightarrow (\alpha \bullet \beta) \; div \; \gamma.$$

*It follows that parallel composition is always a categorical product in $\mathbf{Tr}_L$ iff the synchronization algebra satisfies*

$$\forall \alpha, \beta, \gamma \in L. \alpha \; div \; \gamma \; \& \; \beta \; div \; \gamma \Rightarrow (\alpha \bullet \beta) \; div \; \gamma.$$

**Proof.** Let $(S, l_S)$, $(T, l_T)$ be two $L$-synchronization trees and $\pi_0': (S, l_S) \textcircled{L} (T, l_T) \to (S, l_S)$ and $\pi_1': (S, l_S) \textcircled{L} (T, l_T) \to (T, l_T)$ be restrictions of the projections $\pi_0: S \times T \to S$ and $\pi_1: S \times T \to T$ in $\mathbf{Tr}$.

Suppose

$$(\forall \gamma \in L)(\forall \alpha \in l_S A)(\forall \beta \in l_T B). \alpha \; div \; \gamma \; \& \; \beta \; div \; \gamma \Rightarrow (\alpha \bullet \beta) \; div \; \gamma.$$

Assume $f_0: (U, l_U) \to (S, l_S)$ and $f_1: (U, l_U) \to (T, l_T)$ are $L$-morphisms. Let $h: U \to S \times T$ be the unique morphism of trees such that $\pi_0 h = f_0$ and $\pi_1 h = f_1$. We show that $h$ is an $L$-morphism, $h: (U, l_U) \to (S, l_S) \textcircled{L} (T, l_T)$. Then $h$ is certainly the unique $L$-morphism such that $\pi_0' h = f_0$ and $\pi_1' h = f_1$.

Clearly, $h(\langle\,\rangle) = \langle\,\rangle$. We show by induction on the length of $u' \in U$ that if $u \xrightarrow{\gamma} u'$, then

$$h(u') \in S \textcircled{L} T \quad \text{and} \quad h(u) = h(u') \; \& \; * \; div \; \gamma \text{ or } (h(u) \xrightarrow{\delta} h(u') \; \& \; \delta \; div \; \gamma).$$

It follows that $h$ is an $L$-morphism.

Suppose $u \xrightarrow{\gamma} u'$. If $h(u) = h(u')$, then $f_0(u) = f_0(u')$ and $* \; div \; \gamma$. Otherwise $h(u)\langle c \rangle = h(u')$ for some event $c = (a, b)$ of the product. As $f_0$ and $f_1$ are $L$-morphisms, $\alpha = l_S(a) \; div \; \gamma$ and $\beta = l_T(b) \; div \; \gamma$. (We allow $a$, $b$ to be $*$ in which case the labelling is $*$.) By assumption $\alpha \bullet \beta \; div \; \gamma$ so $\alpha \bullet \beta \neq 0$. This makes $c$ an event of the parallel composition. Thus $h(u') \in S \textcircled{L} T$, completing the induction.

Suppose $L$ satisfies the LCM law. Then by the previous argument $S \textcircled{L} T$, $\pi_0'$, $\pi_1'$ is the product of synchronization trees $S$, $T$ in the category $\mathbf{Tr}_L$. Conversely, suppose for arbitrary synchronization trees $S$, $T$ we have $S \textcircled{L} T$, $\pi_0'$, $\pi_1'$ is a product in $\mathbf{Tr}_L$. Suppose $\alpha \; div \; \gamma$ and $\beta \; div \; \gamma$ in $L$. Clearly, if $\gamma = 0$ or $\alpha = \beta = *$, then $\alpha \bullet \beta \; div \; \gamma$ so assume $\gamma \neq 0$ and $\neg(\alpha = \beta = *)$. Suppose $\alpha = *$ (so $\beta \neq *$). Take $S$ to be the null tree and $T$ to be the synchronization tree consisting of a single $\beta$-labelled arc. Let $U$ be the synchronization tree consisting of a single arc labelled by $\gamma$. Take $f_0: U \to S$ to be the unique morphism to the null tree and $f_1: U \to T$ to be the unique arc-preserving morphism. A unique morphism $h: U \to S \textcircled{L} T$ exists such that $\pi_0' h = f_0$ and $\pi_1' = f_1$. Thus $\alpha \bullet \beta \; div \; \gamma$. If $\alpha \neq *$ and $\beta \neq *$, then taking $S$ to consist of a single arc labelled by $\alpha$ and $T$ to consist of a single arc labelled by $\beta$ a similar argument shows $\alpha \bullet \beta \; div \; \gamma$. $\square$

Let us run through, in a series of propositions, some parallel compositions in the literature. We refer to the synchronization algebras $L_1$, $L_2$, $L_3$ of the earlier Examples 4.5, 4.6 and 4.7.

**6.13. Proposition** (Parallel composition in CCS). *Let $L_1$ be the synchronization algebra for CCS presented above. Write the parallel composition* $\textcircled{L_1}$ *as* $|$, *as in* [18]. *Then two $L_1$-synchronization trees*

$$S \cong \sum_{i \in I} \lambda_i S_i \quad and \quad T \cong \sum_{j \in J} \mu_j T_j$$

*have a parallel composition given by*

$$S \mid T \cong \sum_i \lambda_i (S_i \mid T) + \sum_{\lambda_i = \bar{\mu}_j \text{ or } \mu_j = \bar{\lambda}_i} \tau(S_i \mid T_j) + \sum_j \mu_j (S \mid T_j).$$

*Because, for instance, $\alpha$ div $\tau$, yet $0 = \alpha \bullet \alpha$ and $0 \not\mathrm{div}\ \tau$ the parallel composition $|$ for CCS does not coincide with product in the category of synchronization trees.*

A similar proposition holds for the synchronization algebra of CCS with value passing—recall the synchronization algebra in Example 4.4. Two processes synchronize iff one sends and the other receives a common value on the same line.

Now we examine the parallel compositions $\|$ and $|||$ given in [2] to support the failure set semantics in [9]. Here $\|$ only conicides with product in the appropriate category of synchronization trees if no events in the components are labelled by $\tau$.

**6.14. Proposition** (Parallel composition $\|$ in [2]). *Let $L_2$ be the synchronization algebra presented above. Write the parallel composition* $\textcircled{L_2}$ *as* $\|$, *as in* [2]. *Then two $L_2$-synchronization trees*

$$S \cong \sum_i \lambda_i S_i + \sum_k \tau S_k \quad and \quad T \cong \sum_j \lambda_j T_j + \sum_l \tau T_l,$$

*where $\lambda_i$, $\lambda_j$ are non-$\tau$ labels, have a parallel composition given by*

$$S \parallel T \cong \sum_{i,j:\lambda_i = \lambda_j} \lambda_i (S_i \parallel T_j) + \sum_k \tau(S_k \parallel T) + \sum_l \tau(S \parallel T_l).$$

*The synchronization algebra does not satisfy the LCM law above because for instance $\tau$ div $\tau$ and yet $\tau \bullet \tau = 0 \not\mathrm{div}\ \tau$. However, for trees without $\tau$-labels $\|$ coincides with product in the category of $L_2$-synchronization trees.*

**6.15. Proposition** (Parallel composition $|||$ in [2]). *Let $L_3$ be the synchronization algebra presented above. Write the parallel composition* $\textcircled{L_3}$ *as* $|||$, *as in* [2]. *Then two $L_3$-synchronization trees*

$$S \cong \sum_{i \in I} \lambda_i S_i \quad and \quad T \cong \sum_{j \in J} \mu_j T_j$$

*have a parallel composition given by*

$$S \,|||\, T \cong \sum_i \lambda_i (S_i \,|||\, T) + \sum_j \mu_j (S \,|||\, T_j).$$

*For $L_3$ we have $\alpha$ div $\alpha$ and yet $\alpha \bullet \alpha = 0$ so $(\alpha \bullet \alpha \not\mathrm{div}\ \alpha)$. Therefore, $|||$ does not coincide with product in the category of $L_3$-synchronization trees.*

The papers [9] and [2] contain another operation $\square$ called 'conditional composition' which can also be thought of as a parallel composition. The idea is that both components of a conditional composition can proceed independently performing $\tau$-labelled events until one component makes a communication with the environment—performs a non-$\tau$ labelled event—when future communication must henceforth be with that component. There are two choices for the subsequent behaviour of the other component: one is that it may continue to perform $\tau$-events (the idea in [9]) and another that even these invisible events are stopped (the idea in [2]). From the point of view of the failure-set equivalence in [9, 2] these distinctions make no difference but they are detected by a synchronization tree semantics. We present the first alternative and leave the second to the reader—or see [2]. We choose to obtain $\square$ as a restriction of $|||$.

**6.16. Definition.** Let $(S, l_S)$ and $(T, l_T)$ be synchronization trees labelled by elements of $L_2$ (or $L_3$). Define $(S, l_S) \square (T, l_T)$ to be the synchronization tree consisting of sequences $\langle c_0, \ldots, c_{n-1}\rangle$ of $(S, l_S) ||| (T, l_T)$ which satisfy

$$(\forall i. l_S\rho_0(c_i) = * \text{ or } l_S\rho_0(c_i) = \tau) \text{ or } (\forall i. l_T\rho_1(c_i) = * \text{ or } l_T\rho_1(c_i) = \tau)$$

with the labelling $l$ given by $l((a, *)) = l_S(a)$ and $l((*, b)) = l_T(b)$.

**6.17. Proposition.** *Let $S$ and $T$ be $L_2$-synchronization trees so*

$$S \cong \sum_i \lambda_i S_i + \sum_k \tau S_k \quad \text{and} \quad T \cong \sum_j \mu_j T_i + \sum_l \tau T_l.$$

*Then*

$$S \square T \cong \sum_i \lambda_i (S_i \square (T \restriction \tau)) + \sum_j \mu_j ((S \restriction \tau) \square T_j) + \sum_k \tau(S_k \square T) + \sum_l \tau(S \square T_l),$$

*where for instance $T \restriction \tau$ abbreviates $T \restriction \{\tau\}$.*

As a final example we exhibit how Milner's synchronous calculi fit into the picture. In [19], algebras of actions are presented. They are closely related to synchronization algebras, though because the algebras do not contain $*$ they cannot express asynchrony in the direct way synchronization algebras can. The most general algebras of actions described in [19] are Abelian monoids of the form $(M, \bullet, 1)$. The identity element serves to label delay events. These are essential to the way asynchrony is handled in [19]; there the asynchrony of an event is modelled by allowing the event to be preceded by an arbitrary number of delay events. Contrast the direct way asynchrony is modelled using synchronization algebras to restrict the events in the product; the asynchrony of an event with respect to a process is expressed by the event having no component event from that process.

We show how Milner's monoids of actions determine synchronization algebras which satisfy the synchronous law of Definition 4.4.

**6.18. Definition.** Let $(M, \bullet_M, 1)$ be an Abelian monoid (assumed to not contain $*$ or 0).

Define $L[M]$ to be the algebra $(M \cup \{*, 0\}, \bullet, *, 0)$ where $\bullet$ extends the monoid operation $\bullet_M$ so $* \bullet * = *$, $* \bullet \mu = \mu \bullet * = 0$ for $\mu \in M \cup \{0\}$, and $0 \bullet \mu = \mu \bullet 0 = 0$ for $\mu \in M \cup \{*, 0\}$ and $\mu \bullet \mu' = \mu \bullet_M \mu'$ for $\mu, \mu' \in M$.

Define a divisor relation on $(M, \bullet_M, 1)$ by

$$\mu \; div_M \; \mu' \Leftrightarrow \mu = \mu' \text{ or } \exists \nu. \mu \bullet_M \nu = \mu'.$$

**6.19. Lemma.** *The algebra $L[M]$ defined above is a synchronization algebra which satisfies the synchronous law $\forall \lambda \neq *. \lambda \bullet * = 0$. Further, the algebra $L[M]$ satisfies the LCM law $\alpha \; div \; \gamma \; \& \; \beta \; div \; \gamma \Rightarrow \alpha \bullet \beta \; div \; \gamma$ iff $M$ satisfies the LCM law $\alpha \; div_M \; \gamma \; \&$ $\beta \; div_M \; \gamma \Rightarrow \alpha \bullet \beta \; div_M \; \gamma$.*

**Proof.** The lemma follows because the composition $\bullet$ of $L[M]$ is simply the extension of $\bullet_M$ to the extra elements $*$ and 0. □

**6.20. Proposition.** *Let $L$ be a synchronization algebra which satisfies the synchronous law. Then the parallel composition of $L$-synchronization trees*

$$S \cong \sum_i \lambda_i S_i \quad \text{and} \quad T \cong \sum_j \mu_j T_j$$

*has the form*

$$S \text{Ⓛ} T \cong \sum_{\lambda_i \bullet \mu_j \neq 0} (\lambda_i \bullet \mu_j)(S_i \text{Ⓛ} T_j).$$

*So then parallel composition Ⓛ is obtained by restricting $\otimes$, the synchronous product.*

*Let $(M, \bullet_M, 1)$ be an Abelian monoid. Write $\times_M$ for the parallel composition with respect to the synchronization algebra $L[M]$. Then for two $M$-labelled synchronization trees*

$$S \cong \sum_i \lambda_i S_i \quad \text{and} \quad T \cong \sum_j \mu_j T_j$$

*we have*

$$S \times_M T \cong \sum_{i,j} (\lambda_i \bullet_M \mu_j)(S_i \times_M T_j).$$

*The operation $\times_M$ coincides with product in the category of synchronization trees iff the operation $\bullet_M$ in $(M, \bullet_M, 1)$ behaves like an LCM. If $(M, \bullet_M, 1)$ is an Abelian group, $\times_M$ coincides with product.*

**Proof.** When the synchronization algebra satisfies the synchronous law parallel composition takes the above form by Theorem 6.11. By Example 2.11 this is a restriction of the synchronous product. The remaining facts directly follow from the definition of $L[M]$, Theorem 6.12 and Lemma 6.19. (Clearly an Abelian group satisfies the LCM law.) □

## 7. Denotational semantics

We present a denotational semantics to a simple parallel programming language which involves the constructs we have defined earlier. The class of languages is parameterized by the synchronization algebra $L$.

**7.1. Definition.** Let $L$ be a synchronization algebra. The language $\mathbf{Proc}_L$ is given by the following grammar:

$$t ::= NIL \mid x \mid \lambda t \mid t + t \mid t \restriction \Lambda \mid t[\Xi] \mid t \textcircled{L} t \mid \mathbf{rec}\, x.t$$

where $x$ is in some set of variables $X$ over processes, $\lambda \in L \backslash \{*, 0\}$, $\Lambda \subseteq L \backslash \{*, 0\}$ is closed under $div \cap div^{-1}$, and $\Xi$ is a strict homomorphism of $L$.

In order to give a meaning to the recursively defined processes of the form $\mathbf{rec}\, x.t$ we use the fact that the operations are continuous with respect to a cpo of synchronization trees. Fortunately, the two cpo's of trees $\leqslant$ and $\subseteq$ extend naturally to synchronization trees in such a way that the operations of the previous section are continuous.

**7.2. Definition.** Let $L$ be a synchronization algebra. Define the orderings $\leqslant_L$ and $\subseteq_L$ on synchronization trees by

$$(S, l_S) \leqslant_L (T, l_T) \Leftrightarrow S \leqslant T \;\&\; l_S = l_T \restriction A,$$

$$(S, l_S) \subseteq_L (T, l_T) \Leftrightarrow S \subseteq T \;\&\; l_S = l_T \restriction A.$$

**7.3. Theorem.** *The null synchronization tree* $(\{\langle\ \rangle\}, \emptyset)$ *is the least L-synchronization tree with respect to both orderings* $\leqslant_L$ *and* $\subseteq_L$. *Both orderings* $\leqslant_L$ *and* $\subseteq_L$ *possess least upper bounds of* $\omega$*-chains; the lub of a chain* $(T_0, l_0), (T_1, l_1), \ldots, (T_n, l_n), \ldots$ *with respect to either order takes the form* $(\bigcup_n T_n, \bigcup_n l_n)$.

*All the operations lifting* $T \mapsto \lambda T$, *sum* $+$, *restriction* $T \mapsto T \restriction \Lambda$, *relabelling* $T \mapsto T[\Xi]$ *and parallel composition* $\textcircled{L}$, *of Section* 6, *are continuous with respect to* $\leqslant_L$ *and* $\subseteq_L$, *i.e., they preserve lubs of* $\omega$*-chains.*

**Proof.** The cpo properties of $\leqslant_L$ and $\subseteq_L$ follow directly from the cpo properties of $\leqslant$ and $\subseteq$.

The continuity of the operations on synchronization trees follows from the continuity of the operations on trees from which they are derived, e.g., parallel composition is a restriction of the product so its continuity with respect to $\leqslant_L$ is proved as follows.

Let $T_0 \leqslant_L \cdots \leqslant_L T_n \leqslant_L \cdots$ be an $\omega$-chain of synchronization trees such that $T_n$ is over events $A_n$ labelled by $l_n$. We write its lub as $\bigcup_n T_n$ over events $A = \bigcup_n A_n$ with labelling $l = \bigcup_n l_n$. Let $S$ be a synchronization tree with events $B$ labelled by $l_S$. We use $\rho_0 : A \times_* B \to_* A$ and $\rho_1 : A \times_* B \to_* B$ to represent the obvious projection funcions on events. The parallel composition of $\bigcup_n T_n$ and $S$ is the restriction of their product to events $C=\{e\in A\times_* B \mid l\rho_0(e)\bullet l_S\rho_1(e)\neq 0\}$ so we obtain

$$
\begin{aligned}
\left(\bigcup_n T_n\right) Ⓛ S &= \left(\bigcup_n T_n\right)\times S\restriction C\\
&= \left(\bigcup_n (T_n\times S)\right)\restriction C \quad \text{by the continuity of } \times\\
&= \bigcup_n (T_n\times S\restriction C) \quad \text{by the continuity of restriction}\\
&= \bigcup_n (T_n\times S\restriction C_n) = \bigcup_n T_n Ⓛ S.
\end{aligned}
$$

as required. □

Thus we can give a denotational semantics to $\mathbf{Proc}_L$ by representing recursively defined processes as the least fixed points of continuous functionals.

**7.4. Definition** (*denotational semantics for* $\mathbf{Proc}_L$). Let $L$ be a synchronization algebra. Define an *environment* for process variables to be a function $\rho: X\to \mathbf{Tr}_L$. For a term $t$ and an environment $\rho$, define the denotation of $t$ with respect to $\rho$ written $[\![t]\!]\rho$ by the following structural induction. Note that syntactic operators appear on the left and their semantic counterparts on the right:

$$
\begin{aligned}
&[\![NIL]\!]\rho = (\{\langle\ \rangle\},\emptyset) && [\![t\restriction \Lambda]\!]\rho = [\![t]\!]\rho\restriction \Lambda\\
&[\![x]\!]\rho = \rho(x) && [\![t[\Xi]]\!]\rho = [\![t]\!]\rho[\Xi]\\
&[\![\lambda t]\!]\rho = \lambda([\![t]\!]\rho) && [\![t_1 Ⓛ t_2]\!]\rho = [\![t_1]\!]\rho Ⓛ [\![t_2]\!]\rho\\
&[\![t_1+t_2]\!]\rho = [\![t_1]\!]\rho + [\![t_2]\!]\rho && [\![\mathbf{rec}x.t]\!]\rho = \mathit{fix}\,\Gamma
\end{aligned}
$$

where $\Gamma:\mathbf{Tr}_L\to\mathbf{Tr}_L$ is given by $\Gamma(T)=[\![t]\!]\rho[T/x]$ and $\mathit{fix}$ is the least-fixed-point operator so that $\mathit{fix}\,\Gamma=(\bigcup_n T_n, \bigcup_n l_n)$ where $(T_0, l_0)=(\{\langle\ \rangle\},\emptyset)$ and $(T_{n+1}, l_{n+1}) = \Gamma(T_n, l_n)$ inductively.

**Remark.** A straightforward structural induction shows that $\Gamma$ above is indeed continuous with respect to either order $\leq_L$ or $\subseteq_L$ so the denotation of a recursively defined process is really the least fixed point of the associated functional $\Gamma$.

Choosing $L$ to be the appropriate synchronization algebra we immediately obtain denotational semantics for CCS and SCCS.

Of course we cannot expect all languages to fit into the simple scheme $\mathbf{Proc}_L$; for instance the CSP-language of [9, 2] does not quite because it as two parallel compositions corresponding to two synchronization algebras on the same set of labels. However, the semantics for this language and that for CCS with value-passing follow similar lines to that for $\mathbf{Proc}_L$.

We point out how to extend the language $\mathbf{Proc}_L$ to value-passing. We assume the synchronization algebra is that of CCS with value passing, as given in Example 4.5. Include terms of the form $\bar{\alpha}v.t$ with denotation $\bar{\alpha}v[\![t]\!]\rho$ to represent the sending of a value $v$. Include terms of the form $\alpha\vartheta.t$, where $\vartheta$ is a variable over the set of values $V$, with denotation $\sum_{v\in V}\alpha v[\![t]\!]\rho$ to represent the receipt of a value. Terms can be taken to include constants from $V$, value-variables like $\vartheta$, conditional expressions, etc. so the language can be quite rich—see [18] for the full language of CCS and examples.

Some languages like those in [6] have a parallel composition which depends on *sorts* being associated with processes. They need a slightly more intricate definition of parallel composition which uses combinations of our parallel composition, with respect to some synchronization algebra, together with restriction and relabelling.

## 8. Labelled transition systems

We show how categories of synchronization trees fit into the broader categories of labelled transition systems. Transition system have often been used to give operational semantics to programming languages. For example, semantics for Milner's CCS are often based on them and Plotkin [23, 24] shows how widely they can be applied in giving semantics to languages. This section provides a bridge between operational semantics in terms of transition systems and denotational semantics expressed in terms of trees.

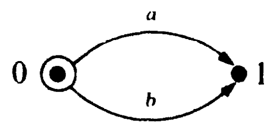
**8.1. Definition.** A *transition system* is a 4-tuple $(S, i, A, Tran)$ where $S$ is a set of *states* with *initial state* $i$, $A$ is a set of *events*, $Tran \subseteq S\times A\times S$ is the *transition relation*, elements of which are called *transitions*, which satisfy

(i) $(\forall a\in A)(\exists s, s'\in S).(s, a, s')\in Tran$,

(ii) $(s, a, s')\in Tran\ \&\ (s, a, s'')\in Tran \Rightarrow s'=s''$.

Intuitively a transition system represents a process which can make transitions between states starting from an initial state. Here we assume, as with trees, that it can only perform one event at a time. The first axiom we impose says that every event is associated with some transition and the second axiom says that from a state the occurrence of an event is associated with a unique transition and so, of course, leads to a unique state. Thus transitions from a state correspond to occurrences of events from that state. (Note, however, that this will not be the case for 'idle' transitions which we shall introduce soon.) Of course, transition systems are more general than trees because the transitive closure of the transition relation may contain loops. In fact this is often the way recursion is handled when using transition systems.

**8.2. Notation.** Let $(S, i, A, Tran)$ be a transition system. We draw the transitions between two states as arrows—there may be more than one. For example, the transition system $(\{0, 1\}, 0, \{a, b\}, \{(0, a, 1), (0, b, 1)\})$ would be drawn as follows:




and we can write the transition $(0, a, 1)$ as $0 \overset{a}{\to} 1$, so events serve to index transitions between pairs of states.

It is convenient to extend the set of transitions in a formal way so that they include the possibility of inaction at any state. We already have a symbol for such inaction, the symbol $*$. Of course, inaction does not take a state to another state so we extend the set *Tran* just by elements of the form $(s, *, s)$. We call such transitions ***idle*** transitions because they are not associated with any event occurrence. For a transition system as above write the idle transitions as

$$Tran_* = Tran \cup \{(s, *, s) \mid s \in S\}.$$

Idle transitions are *not* to be thought of as events of inaction performed by a process; they are not associated with any event of the process at all.

Morphisms on transition systems are defined analogously to those on trees. The intuition is the same. A morphism from a transition system $T$ to a transition system $U$ specifies how the occurrence of an event in $T$ implies the synchronized occurrence of an event in $U$. States of $T$ image to states of $U$. However, there may well be occurrences of events in $T$ which are not represented by any event occurrences in $U$. The transitions associated with such event occurrences image to idle transitions introduced above. Hence we define a morphism as consisting of two parts one a function on states and the other a partial function on events which induces a function on transitions, including the idle ones. Following the definition on trees, we say a morphism is synchronous if it is a total function on events and so never sends a non-idle transition to an idle one.

**8.3. Definition.** A *morphism* from a transition system $(S_0, i_0, A_0, Tran_0)$ to a transition system $(S_1, i_1, A_1, Tran_1)$ is a pair $(f_S, f_E)$ where $f_S: S_0 \to S_1$ is a function on states such that

$$f_S(i_0) = i_1$$

and where $f_E: A_0 \to_* A_1$ is a partial function on events which satisfies

$$(s, a, s') \in Tran_0 \Rightarrow (f_S(s), f_E(a), f_S(s')) \in Tran_{1*}.$$

Say the morphism $(f_S, f_E)$ is *synchronous* if $f_E$ is a total function.

**8.4. Proposition.** *Transition systems with morphisms as defined above form a category*

***under the pairwise composition of functions*** $(f_S, f_E) \circ (g_S, g_E) \overset{\text{def}}{=} (f_S g_S, f_E g_E)$ ***where composition on the state functions is the usual composition on total functions and the composition on the event functions is that for partial functions and identity morphisms are pairs of identity functions. Transition systems with synchronous morphisms form a subcategory.***

**Proof.** The proof follows routinely. □

**8.5. Definition.** Let **Tran** denote the category of transition systems with the above definition of morphism and $\mathbf{Tran}_{\mathrm{syn}}$ the subcategory with synchronous morphisms.

Let us see the form products take in the category **Tran**. The projection functions will provide examples of typical morphisms.

**8.6. Definition** (*the product of transition systems*). Let $(S_0, i_0, A_0, Tran_0)$ and $(S_1, i_1, A_1, Tran_1)$ be transition systems. Define their *product*, $(S_0, i_0, A_0, Tran_0) \times (S_1, i_1, A_1, Tran_1) = (S, i, A, Tran)$ by taking:

(i) States $S = S_0 \times S_1$, the product in **Set** with projections $\pi_j$: $S \to S_j$ for $j = 0, 1$,

(ii) initial state $i = (i_0, i_1)$,

(iii) events $A = A_0 \times_* A_1$, the product in $\mathbf{Set}_*$ with projections $\rho_j: A \to_* A_j$ for $j = 0, 1$.

(iv) transitions $(s, c, s') \in Tran \Leftrightarrow \begin{cases} (\pi_0(s), \rho_0(c), \pi_0(s')) \in Tran_{0_*} \ \& \\ (\pi_1(s), \rho_1(c), \pi_1(s')) \in Tran_{1_*}. \end{cases}$

Define the *projections* $\Pi_j: (S, i, A, Tran) \to (S_j, i_j, A_j, Tran_j)$ by taking $\Pi_j = (\pi_j, \rho_j)$ for $j = 0, 1$.

A similar construction has been used to build the 'product machine' of [5]. It really is a product.

**8.7. Theorem.** *The construction* $(S_0, i_0, A_0, Tran_0) \times (S_1, i_1, A_1, Tran_1)$, $\Pi_0, \Pi_1$ *above is a categorical product in the category* **Tran** *of transition systems.*

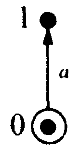
**Proof.** It immediately follows from the definition of product that it is a transition system—the axioms follow from their truth in the components—and that the projections are morphisms of transition systems.

Let $T_j$ abbreviate $(S_j, i_j, A_j, Tran_j)$ for $j = 0, 1$. Suppose that $U$ is a transition system and that $f = (f_S, f_E): U \to T_0$ and $g = (g_S, g_E): U \to T_1$ are morphisms. In order for $T_0 \times T_1$ to be a product we require that there is a unique morphism $h = (h_S, h_E): U \to T_0 \times T_1$ such that $\Pi_0 h = f$ and $\Pi_1 h = g$. This is so when we define $h$ as follows:

$$h_S(u) = (f_S(u), g_S(u)) \quad \text{and} \quad h_E(c) = (f_E(c), g_E(c))$$

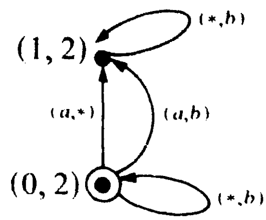
for $u$ a state of $U$ and $c$ an event of $U$. □

**8.8. Example.** The product of the transition systems




and


takes the form




We shall see how the definition of product of transition systems in **Tran** generalizes that of trees in **Tr**. In fact a product of transition systems will unfold to a product of trees. Transition systems also have a coproduct, perhaps not quite what is expected as its unfolding will turn out to not coincide with the coproduct of trees.

**8.9. Definition** (*the coproduct of transition systems*). Let $(S_0, i_0, A_0, Tran_0)$ and $(S_1, i_1, A_1, Tran_1)$ be transition systems. Define their *coproduct* $(S_0, i_0, A_0, Tran_0) + (S_1, i_1, A_1, Tran_1) = (S, i, A, Tran)$ by taking

(i) $S = (S_0 \times \{i_1\}) \cup (\{i_0\} \times S_1)$ with injections $in_j : S_j \to S$, for $j = 0, 1$, given by $in_0(s) = (s, i_1)$ and $in_1(s) = (i_0, s)$,

(ii) $i = (i_0, i_1)$,

(iii) $A = (\{0\} \times A_0) \cup (\{1\} \times A_1)$ the disjoint union of the sets of events with injections $\alpha_j : A_j \to A$ given by $\alpha_j(a) = (j, a)$ for $j = 0, 1$, and

$$\text{(iv)} \quad t \in Tran \Leftrightarrow \begin{cases} \exists (s, a, s') \in Tran_0 . t = (in_0(s), \alpha_0(a), in_0(s')) \text{ or} \\ \exists (s, a, s') \in Tran_1 . t = (in_1(s), \alpha_1(a), in_1(s')). \end{cases}$$

Define the injections $I_j : (S_j, i_j, A_j, Tran_j) \to (S, i, A, Tran)$ by $I_j = (in_j, \alpha_j)$ for $j = 0, 1$.

**8.10. Theorem.** *The construction above is a coproduct in the categories* **Tran** *and* $\mathbf{Tran}_{\mathrm{syn}}$ *of transition systems.*

**Proof.** It is easy to see that the coproduct construction gives a transition system and that the injections are indeed (synchronous) morphisms. Suppose $f: T_0 \to U$ and $g: T_1 \to U$ are morphisms from transition systems $T_j$ abbreviating $(S_j, i_j, A_j, Tran_j)$, for $j = 0, 1$, to a transition system $U$. Define a morphism $h = (h_S, h_E): T_0 + T_1 \to U$ by taking

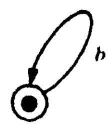
$$h_S(s) = \begin{cases} f_S(s_0) & \text{if } s = in_0(s_0) \\ g_S(s_1) & \text{if } s = in_1(s_1) \end{cases} \quad \text{and} \quad h_E(c) = \begin{cases} f_E(a) & \text{if } c = \alpha_0(a), \\ g_E(a) & \text{if } c = \alpha_1(a). \end{cases}$$

It is easily seen that $h$ is the unique morphism of transition systems so that $hI_0 = f$

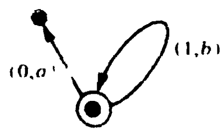
and $hI_1 = g$. Moreover, if $f$ and $g$ are synchronous, then so is $h$. Therefore, the construction is a coproduct in **Tran** and $\mathbf{Tran}_{\mathrm{syn}}$. □

**8.11. Example.** The coproduct of the transition systems


and




takes the form




Clearly a tree can be viewed as a transition system.

**8.12. Definition.** Let $S$ be a tree over the set $A$. Define $\mathcal{TS}\, S$ to be $(S, \langle\ \rangle, E, Tran)$ where

$$E = \{(s, a) \in S \times A \mid s\langle a \rangle \in S\}$$

and

$$Tran = \{(s, (s, a), s') \mid s\langle a \rangle = s' \in S\}.$$

Extend $\mathcal{TS}$ to a functor by defining it to act on morphisms of trees as follows: Let $f: S \to U$ be a tree morphism. Define $(\mathcal{TS} f): \mathcal{TS}\, S \to \mathcal{TS}\, U$ by taking

$$(\mathcal{TS} f)_S(s) = f(s) \quad \text{and} \quad (\mathcal{TS} f)_E(s, a) = \begin{cases} (f(s), b) & \text{if } f(s\langle a \rangle) = f(s)\langle b \rangle, \\ * & \text{otherwise.} \end{cases}$$

**Remark.** Notice that $\mathcal{TS}$ would not have extended to a functor, in the above definition, if we had taken $A$ instead of $E$ as the events. The reason is the following: In the category **Tr** morphisms respect only the node-arc structure, and not the event sets, which are respected by the more discriminating morphisms in **Tran**.

Not only can trees be viewed as transition systems, but also transition systems can be unfolded to trees. This is well known. The unfolding is determined by the categorical set-up. It is characterized to within isomorphism as the right adjoint to the obvious functor $\mathcal{TS}$ taking trees to transition systems (see [1] or [17]). In other words, given a transition system its unfolding is cofree over it with respect to $\mathcal{TS}$ the natural identification of trees with a form of transition system.

**8.13. Definition.** Let $(S, i, A, Tran)$ be a transition system. Define its *unfolding*

$\mathcal{U}(S, i, A, Tran)$ to be the tree

$$\{\langle a_0, a_1, \ldots, a_{n-1}\rangle \mid \exists s_0, s_1, \ldots, s_n \in S. s_0 = i \ \&\ \forall j < n. (s_j, a_j, s_{j+1}) \in Tran\}.$$

Define the *folding* morphism $\phi = (\phi_S, \phi_E) : \mathcal{TS}\,\mathcal{U}((S, i, A, ran)) \to (S, i, A, Tran)$ by taking $\phi_E(u, a) = a$ on events and defining $\phi_S$ by induction as follows:

$$\phi_S(\langle\ \rangle) = i \quad \text{and} \quad \phi_S(u\langle a\rangle) = s,$$

where $s$ is the unique state such that $(\phi_S(u), a, s) \in Tran$.

Thus $\phi$ folds a state $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ in the unfolding to the state $s_n$ where $s_0, s_1, \ldots, s_n \in S$ is the unique sequence of states such that $s_0 = i \ \&\ \forall j < n. (s_j, a_j, s_{j+1}) \in Tran$.

**8.14. Theorem.** *Let $(S, i, A, Tran)$ be a transition system. Then $\mathcal{U}(S, i, A, Tran)$ is a synchronization tree and $\phi$ defined above is a morphism of transition systems. In fact, $\mathcal{U}(S, i, A, Tran)$, $\phi$ is cofree over $(S, i, A, Tran)$ with respect to the functor $\mathcal{TS}$, i.e., for any morphism $f: \mathcal{TS}\, V \to (S, i, A, Tran)$ with $V$ a tree, there is a unique morphism $g: V \to \mathcal{U}(S, i, A, Tran)$ in* **Tr**, *such that $f = \phi(\mathcal{TS}\, g)$:*

$$\begin{tikzcd} \mathcal{U}(S, i, A, Tran) \\ V \arrow[u, "g"] \end{tikzcd} \qquad \begin{tikzcd} (S, i, A, Tran) & \mathcal{TS}\,\mathcal{U}(S, i, A, Tran) \arrow[l, "\phi"'] \\ & \mathcal{TS}\, V \arrow[ul, "f"] \arrow[u, "\mathcal{TS}\, g"'] \end{tikzcd}$$

*Consequently, $\mathcal{U}$ extends to a right adjoint of $\mathcal{TS}$.*

**Proof.** Let $V$ be a tree and $f: \mathcal{TS}\, V \to (S, i, A, Tran)$ be a morphism of transition systems.

Define $g: V \to \mathcal{U}(S, i, A, Tran)$ by induction as follows:

$$g(\langle\ \rangle) = \langle\ \rangle,$$

$$g(v\langle b\rangle) = \begin{cases} g(v)\langle f_E(v, b)\rangle & \text{if } f_E(v, b) \neq *, \\ g(v) & \text{otherwise.} \end{cases}$$

Clearly $g$ is a morphism of trees. We require $\phi \circ (\mathcal{TS}\, g) = f$ and of course this follows if we can show $(\phi \circ (\mathcal{TS}\, g))_S = f_S$ and $(\phi \circ (\mathcal{TS}\, g))_E = f_E$.

We first show $(\phi \circ (\mathcal{TS}\, g))_S = f_S$. We show $\phi_S \circ g(v) = f_S(v)$ by induction on $v \in V$. Obviously, $\phi_S \circ g(\langle\ \rangle) = f_S(\langle\ \rangle)$ establishing the basis of the induction.

Now we show the inductive step, that $\phi_S \circ g(v\langle b\rangle) = f_S(v\langle b\rangle)$ if the induction hypothesis $\phi_S \circ g(v) = v_S(v)$ holds. From the definition of $g$ there are two cases to consider, when $f_E(v, b) \neq *$ and when $f_E(v, b) = *$.

Assume $f_E(v, b) \neq *$. From the definition of $g$ we get $g(v\langle b\rangle) = g(v)\langle f_E(v, b)\rangle$. From the definition of $\phi_S$ we obtain $\phi_S \circ g(v\langle b\rangle) = s$ where $s$ is the *unique* state such that $(\phi_S(g(v)), f_E(v, b), s) \in Tran$. As $f$ is a morphism of transition systems we must have $(f_S(v), f_E(v, b), f_S(v\langle b\rangle)) \in Tran$ too. The induction hypothesis provides $\phi_S(g(v)) = f_S(v)$. Thus $\phi_S(g(v\langle b\rangle)) = s = f_S(v\langle b\rangle)$.

Now assume $f_E(v, b) = *$. The definition of $g$ gives $g(v\langle b\rangle) = g(v)$. So $\phi_S(g(v\langle b\rangle)) = \phi_S(g(v)) = f_S(v)$ by induction. As $f$ is a morphism, $f_S(v) = f_S(v\langle b\rangle)$. Thus $\phi_S \circ g(v\langle b\rangle) = f_S(v\langle b\rangle)$.

This shows that $(\phi \circ (\mathcal{TS} g))_S = f_S$. We now show $(\phi \circ (\mathcal{TS} g))_E = f_E$. This is part of a more general fact which also establishes the uniqueness of $g$; a morphism of trees $h: V \to \mathcal{U}(S, i, A, Tran)$ satisfies $g$'s recursive definition iff $(\phi \circ (\mathcal{TS} h))_E = f_E$. More precisely we show the following:

Let $h: V \to \mathcal{U}(S, i, A, Tran)$ be a morphism of trees. Then $(\phi \circ (\mathcal{TS} h))_E = f_E$ iff

$$h(\langle\ \rangle) = \langle\ \rangle,$$

$$h(v\langle b\rangle) = \begin{cases} g(v)\langle f_E(v, b)\rangle & \text{if } f_E(v, b) \neq *, \\ h(v) & \text{otherwise} \end{cases}$$

for $v \in V$ and $b$ an event of the tree $V$ such that $v\langle b\rangle \in V$.

'*If*': Let $v\langle b\rangle \in V$. From the assumption and the definition of $(\mathcal{TS} h)_E$ we obtain

$$(\mathcal{TS} h)_E(v, b) = \begin{cases} (g(v), f_E(v, b)) & \text{if } f_E(v, b) \neq *, \\ * & \text{otherwise.} \end{cases}$$

From the definition of $\phi_E$ we immediately have $\phi_E \circ (\mathcal{TS} h)_E(v, b) = f_E(v, b)$.

'*Only if*': By the definition of $\phi_E$ and $(\mathcal{TS} h)_E$ we have

$$\phi_E \circ (\mathcal{TS} h)_E(v, b) = \begin{cases} a & \text{if } h(v\langle b\rangle) = h(v)\langle a\rangle, \\ * & \text{otherwise.} \end{cases}$$

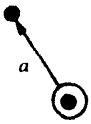
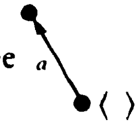
But, by assumption, $\phi_E \circ (\mathcal{TS} h)_E(v, b) = f_E(v, b)$ which implies the result.

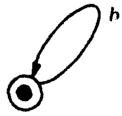
Clearly it now follows that $(\phi \circ (\mathcal{TS} g))_E = f_E$. So $\phi \circ g = f$. The uniqueness of $g$ follows too. Assume $h: V \to \mathcal{U}(S, i, A, Tran)$ is a morphism such that $\phi \circ h = f$. Then $(\phi \circ (\mathcal{TS} h))_E = f_E$. By a simple induction using the above result with $h(\langle\ \rangle) = \langle\ \rangle$ we obtain $h(v) = g(v)$ for all $v \in V$.

Hence, the theorem is proved. □

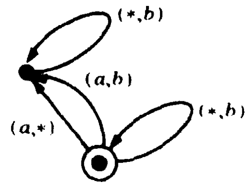
Right adjoints have the pleasant property that they preserve limits, so in particular they preserve products (see [1] or [17]). This means that if we take the product of two transition systems and then unfold them we obtain the same tree, to within isomorphism, as if we unfold them first and then take their product in the category of trees. This is significant for us because we derive parallel compositions from products by restricting the events. It will mean that we can define a parallel composition directly on labelled transition systems and know that it unfolds to the parallel composition of the synchronization trees which are the unfoldings. In view of these facts the following example is not surprising.

**8.15. Example**

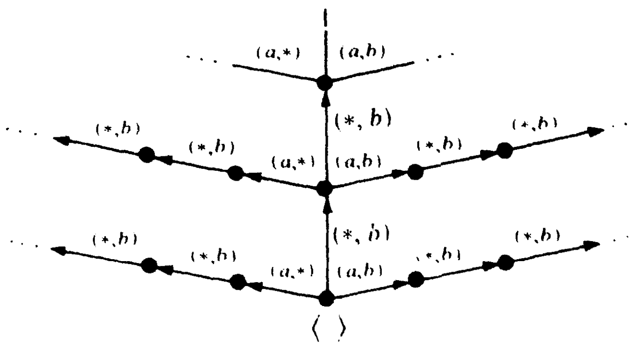
The transition system  unfolds to the tree 

The transition system  unfolds to the tree 

Their product

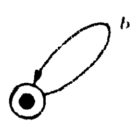
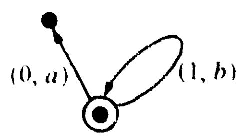



unfolds to the following tree which is isomorphic to the product (in **Tr**) of the two tree unfoldings:

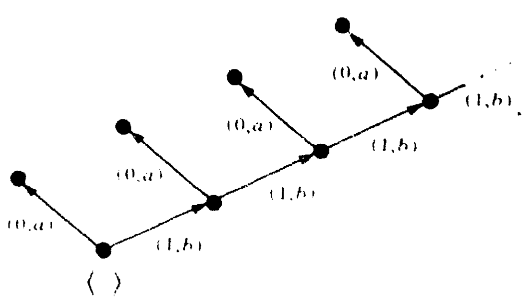


Right adjoints preserve limits but they do not necessarily preserve colimits. And in fact the unfolding functor $\mathcal{U}$ does not preserve coproducts as the following example shows.

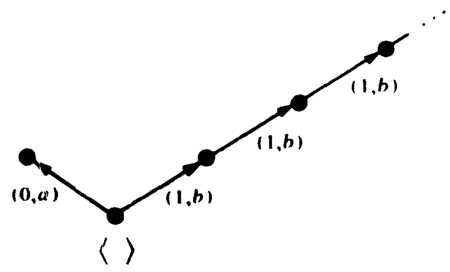
**8.16. Example.** The coproduct of the transition systems

 and  is 

which unfolds to the following tree:



But their two unfoldings have the following coproduct in **Tr**:




Now we label events by elements of a synchronization algebra to specify how they interact with the environment.

**8.17. Definition.** Let $L$ be a synchronization algebra. An *L-labelled transition system* is a 5-tuple $(S, i, A, Tran, l)$ where $(S, i, A, Tran)$ is a transition system and $l$ is a *labelling function* $l: A \to L \backslash \{*, 0\}$.

Just as with trees we can restrict morphisms on transition systems in accord with labellings of the transitions by elements of a synchronization algebra.

**8.18. Definition.** Let $L$ be a synchronization algebra. Let $(S_0, i_0, A_0, Tran_0, l_0)$ and $(S_1, i_1, A_1, Tran_1, l_1)$ be $L$-labelled transition systems. An $L$-morphism from $(S_0, i_0, A_0, Tran_0, l_0)$ to $(S_1, i_1, A_1, Tran_1, l_1)$ is a morphism of transition systems $f: (S_0, i_0, A_0, Tran_0) \to (S_1, i_1, A_1, Tran_1)$ such that $l_1 f_E(a)$ *div* $l_0(a)$ for all $a \in A_0$.

The condition satisfied by $L$-morphisms of transition systems simply expresses that the label of the image of an event must divide the label of the event.

**8.19. Proposition.** *Let $L$ be a synchronization algebra. Then $L$-labelled transition systems with $L$-morphisms form a category with composition the pairwise composition of functions and identities pairs of identity functions.*

**8.20. Definition.** Let $L$ be a synchronization algebra. Let $\mathbf{TRAN}_L$ be the category of labelled transition systems.

Not surprisingly labelled transition systems unfold to labelled trees or synchronization trees simply by extending the unfolding operation to cope with labels. Similarly synchronization trees can be viewed as sorts of labelled transition systems by extending the operation $\mathcal{TS}$.

**8.21. Definition.** Let $L$ be a synchronization algebra. Define the operation $\mathcal{TS}_L: \mathbf{Tr}_L \to \mathbf{TRAN}_L$ by $\mathcal{TS}_L: (T, l) \mapsto (\mathcal{TS}\, T, l)$.

Define the *unfolding* operation on labelled transition systems by taking $\mathcal{U}_L: (S, i, A, Tran, l) \mapsto (T, l')$ where $T = \mathcal{U}(S, i, A, Tran)$, and $l'$ is $l$ restricted to the events of $T$. (Not all events $A$ necessarily appear in branches of $T$.)

**8.22. Proposition.** *In fact, $\mathcal{TS}_L$ extends to a functor with respect to which $\mathcal{U}_L$ gives the cofree object; thus $\mathcal{U}_L$ extends to a right adjoint of $\mathcal{TS}_L$.*

**Proof.** The proof follows from Theorem 8.14. □

Just as with synchronization trees one can define operations on labelled transition systems and use these to give a semantics to a variety of parallel programming languages. The most interesting operation is parallel composition which we obtain by restricting the transitions of the product of transition systems in accord with their labelling.

**8.23. Definition** (*parallel composition of labelled transition systems*). Let $L$ be a synchronization algebra. Let $(S_0, i_0, A_0, Tran_0, l_0)$ and $(S_1, i_1, A_1, Tran_1, l_1)$ be $L$-labelled transition systems. Define their *parallel composition* $(S_0, i_0, A_0, Tran_0, l_0)Ⓛ(S_1, i_1, A_1, Tran_1, l_1)$ to be $(S, i, A', Tran', l)$ formed from the product of transition systems as follows—we use the notation of Definition 8.6:

(i) $S$ is the states of their product with the same initial state $i$,

(ii) $A' = \{c \in A_0 \times_* A_1 \mid l_0\rho_0(c) \bullet l_1\rho_1(c) \neq 0\}$ is a subset of events of the product,

(iii) labelled by $l: A' \to L\backslash\{*, 0\}$; $a \mapsto l_0\rho_0(c) \bullet l_1\rho_1(c)$,

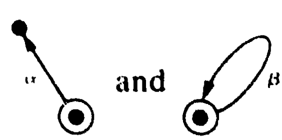
(iv) with transitions $Tran' = S \times A' \times S \cap Tran$ which are a subset of the transitions $Tran$ of the product.

Because the operation of unfolding preserves products and the parallel compositions of synchronization trees and labelled transition systems are restrictions determined in the same way from the labelling we obtain the following reassuring fact.

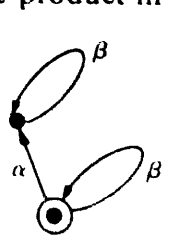
**8.24. Proposition.** *Let $L$ be a synchronization algebra. The parallel composition of labelled transition systems $T_0$ and $T_1$ unfolds to the parallel composition of the unfoldings*:

$$\mathcal{U}_L(T_0 Ⓛ T_1) \cong \mathcal{U}_L(T_0) Ⓛ \mathcal{U}_L(T_1).$$

**8.25. Example.** Let $L_1$ be the synchronization algebra for CCS. The parallel composition of the labelled transition systems




is the appropriate restriction of the product in Example 8.8 and takes the form



By Example 8.16 we know that the unfolding of a coproduct of transition systems is not necessarily the coproduct of their unfoldings; we must look elsewhere for a definition of the sum of labelled transition systems if we wish it to unfold correctly to the sum of the synchronization-tree unfoldings. We can define the *Milner sum* of two transition systems as follows.

**8.26. Definition.** Let $(S_0, i_0, A_0, Tran_0)$ and $(S_1, i_1, A_1; Tran_1)$ be transition systems. Define their *Milner sum* $(S_0, i_0, A_0, Tran_0) +_M (S_1, i_1, A_1, Tran_1) = (S, i, A, Tran)$ by taking:

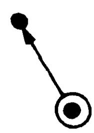
(i) $S = \{0\} \times S_0 \cup \{1\} \times S_1 \cup \{(2, (i_0, i_1))\}$,

(ii) $i = (2, (i_0, i_1))$,

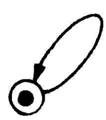
(iii) $A = (\{0\} \times A_0) \cup (\{1\} \times A_1)$,

(iv) $$t \in Tran \Leftrightarrow \begin{cases} \exists (s, a, s') \in Tran_0 . t = ((0, s), (0, a), (0, s')) \text{ or} \\ \exists (s, a, s') \in Tran_1 . t = ((1, s), (1, a), (1, s')) \text{ or} \\ \exists (i_0, a, s) \in Tran_0 . t = (i, (0, a), (0, s)) \text{ or} \\ \exists (i_1, a, s) \in Tran_1 . t = (i, (1, a), (1, s)). \end{cases}$$

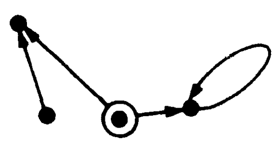
**8.27. Example.** The Milner sum of the transition systems



and



is the transition system:



The Milner sum of two transition systems does unfold to the sum of the two unfoldings. Note too that provided the transition systems have no loops back to the initial state their coproduct does unfold nicely. Of course the same construction works if the transition systems are labelled. It is easy to define operations on labelled transition systems which unfold to the remaining operations on synchronization trees given in Section 6.

As presented, transition systems are still an interleaving model of concurrency because they allow the occurrence of only one event at a time. One can, however, generalize transition systems to reflect concurrency. For example, one can view Petri nets as kinds of transition systems in which transitions are sets of concurrently firing events (see, e.g., [4]). The definition of morphism can be generalized to reflect this extra information about concurrency while maintaining, in essence, the results of this section (see [29]).

## 9. Proof rules

Naturally one wishes to use semantics to prove properties of programs. This can often be reduced to the problem of whether or not two programs have equivalent behaviour with respect to some natural notion of equivalence. Thus much work is involved with inventing natural equivalences and proof rules for them (see, e.g., [18, 2, 11]).

Consider the programming language $\mathbf{Proc}_L$ for some synchronization algebra $L$. There is an obvious equivalence on closed terms of the language: Say two closed terms are equivalent iff they have isomorphic denotations. (The idea extends to open terms; say two terms are equivalent if the closed terms obtained by an arbitrary assignment of closed terms to free variables are always equivalent.)

**9.1. Definition.** Let $L$ be a synchronization algebra. Let $t$ and $t'$ be closed terms of $\mathbf{Proc}_L$. Write

$$t \sim t' \Leftrightarrow [\![t]\!]\rho \cong [\![t']\!]\rho$$

for some arbitrary environment $\rho$.

We immediately know some properties of the equivalence. Firstly it really is an equivalence—it is reflexive, symmetric and transitive—because these properties hold for isomorphism, and then the commutativity and associativity of sum + with respect to $\sim$ directly follow from the properties of coproduct. Less immediate are the commutativity and associativity of parallel composition Ⓛ, but these facts easily follow from the corresponding properties of product $\times$ of trees and $\bullet$ in the synchronization algebra $L$. Because all our operations preserve isomorphism—all but restriction are functors anyhow and functors must preserve isomorphism—we know that the equivalence $\sim$ is also a congruence with respect to the operations of $\mathbf{Proc}_L$.

**9.2. Proposition.** *The equivalence $\sim$ on closed terms of $\mathbf{Proc}_L$ is a congruence with respect to the operations lifting $T \mapsto \lambda T$, sum $+$, restriction $T \mapsto T\lceil\Lambda$, relabelling $T \mapsto T[\Xi]$ and parallel composition of $\mathbf{Proc}_L$.*

Particular laws follow from particular properties of the synchronization algebra $L$. One useful property, when it is valid, is that of the distributivity of parallel composition over sum. This property holds for the equivalence $\sim$ precisely when the synchronization algebra satisfies the synchronous law.

**9.3. Proposition.** *Let $L$ be a synchronization algebra. The following conditions are equivalent:*

(i) *$L$ satisfies the synchronous law, i.e., $\lambda \bullet * = 0$ for $\lambda$ an element of $L\backslash\{*\}$,*
(ii) *$NIL$ is a Ⓛ-zero, i.e., $NIL Ⓛ t \sim NIL$ for $t$ a term of* $\mathbf{Proc}_L$,
(iii) *Parallel composition distributes over sum, i.e., $t Ⓛ (u+v) \sim (t Ⓛ u) + (t Ⓛ v)$, for terms $t$, $u$, $v$ of* $\mathbf{Proc}_L$.

**Proof.** (i)⇔(ii): If $L$ is synchronous, no events of the form $(*, e)$ are allowed in the parallel composition so $NIL Ⓛ t \sim NIL$ for any term $t$. Conversely, if $NIL Ⓛ t \sim NIL$ for any term $t$, then in particular $NIL Ⓛ \lambda NIL \sim NIL$ and this isomorphism ensures $* \bullet \lambda = 0$.

(i)⇒(iii): The distribution of Ⓛ over + directly follows from the expansion rule of Proposition 6.20.

(iii)⇒(i): Suppose (iii) and $\alpha \bullet * = \beta \neq 0$ for some $\alpha \in L\backslash\{*\}$. Then

$$\beta NIL \sim \alpha NIL Ⓛ NIL \sim \alpha NIL Ⓛ (NIL + NIL)$$

$$\sim (\alpha NIL Ⓛ NIL) + (\alpha Ⓛ NIL) \sim \beta NIL + \beta NIL.$$

But this is impossible so $\alpha \bullet * = 0$ for $\alpha \in L\backslash\{*\}$, making $L$ synchronous. □

Of course a semantics for a language of synchronizing processes may well ensure that parallel composition distributes over sum without the synchronization algebra being synchronous. The above result only implies that any abstract semantics which factors through our synchronization tree semantics will satisfy the distributivity. For example, the synchronous calculi SCCS do because the equivalences in [19] could be based on synchronization trees and the synchronization algebras associated with monoids of actions are synchronous (see Definition 6.16).

Now we present a sound and complete proof system for the nonrecursive processes of $\mathbf{Proc}_L$.

**9.4. Definition.** Let $L$ be a synchronization algebra. Let the language $\mathbf{Simp}_L$ consist of the following subset of $\mathbf{Proc}_L$:

$$t ::= NIL \mid \lambda t \mid t + t \mid t \lceil \Lambda \mid t[\Xi] \mid t Ⓛ t$$

where $\lambda \in L\backslash\{*, 0\}$, $\Lambda \subseteq L\backslash\{*, 0\}$ is closed under $div \cap div^{-1}$ and $\Xi: L \to L$ is a strict homomorphism.

**9.5. Notation.** We use the convention that

$$\sum_{i<n} \lambda_i s_i = \lambda_0 s_0 + \cdots + \lambda_{n-1} s_{n-1},$$

where $n > 0$ with the understanding that the sum represents $NIL$ when $n = 0$. Our notation assumes the associativity of +, one of the rules below.

**9.6. Definition** (*proof rules for* **Simp**$_L$). Let $s$, $t$, $u$, $v$ range over terms of **Simp**$_L$.

(1) *Rules of equivalence*:

$$s \sim s, \qquad \frac{s \sim t}{t \sim s}, \qquad \frac{s \sim t,\ t \sim u}{s \sim u}.$$

(2) *Substitutivity*:

$$\frac{s \sim s'}{op(s) \sim op(s')},$$

where $op$ is an operation of lifting, restriction or relabelling.

$$\frac{s \sim s',\ t \sim t'}{op(s, t) \sim op(s', t')},$$

where $op$ is the operation sum or parallel composition.

(3) *Divisor rules*:

$$\lambda t \sim \lambda' t,$$

where $\lambda$, $\lambda' \in L \backslash \{*, 0\}$ and $\lambda\ div\ \lambda'$ and $\lambda'\ div\ \lambda$.

(4) *Rules for restriction*:

$$NIL \restriction A \sim NIL, \qquad (s+t)\restriction A \sim s \restriction A + t \restriction A,$$

$$(\lambda t)\restriction A \sim \begin{cases} \lambda(t \restriction A) & \text{if } \lambda \in A, \\ NIL & \text{if } \lambda \notin A, \end{cases}$$

where $A \subseteq L \backslash \{*, 0\}$ is closed under the relation $div \cap div^{-1}$ and $\lambda \in L \backslash \{*, 0\}$.

(5) *Rules for relabelling*:

$$NIL[\Xi] \sim NIL, \qquad (\lambda t)[\Xi] \sim \Xi(\lambda)t, \qquad (s+t)[\Xi] \sim s[\Xi] + t[\Xi],$$

where $\Xi : L \to L$ is a strict homomorphism of $L$ and $\lambda \in L \backslash \{*, 0\}$.

(6) *Rules for sum*:

$$s + NIL \sim s, \qquad s + t \sim t + s, \qquad s + (t + u) \sim (s + t) + u.$$

(7) *Expansion rules for parallel composition*:

$$\frac{s \sim \sum_{i<n} \lambda_i s_i, \qquad t \sim \sum_{j<m} \mu_j t_j}{s \textcircled{L} t \sim \sum_{\lambda_i \bullet * \neq 0} (\lambda_i \bullet *)(s_i \textcircled{L} t) + \sum_{\lambda_i \bullet \mu_j \neq 0} (\lambda_i \bullet \mu_j)(s_i \textcircled{L} t_j) + \sum_{* \bullet \mu_j \neq 0} (* \bullet \mu_j)(s \textcircled{L} t_j)}.$$

**9.7. Theorem.** *Let L be a synchronization algebra. Let s and t be terms of* **Simp**$_L$. *They have isomorphic denotations as synchronization trees in* **Tr**$_L$ *iff they are provably equivalent according to the proof system above.*

**Proof.** Previous results ensure that the rules are sound. The above rules are sufficient to convert any term of **Simp**$_L$ to one of the normal form $\sum_{i<n} \lambda_i s_i$ in which each $s_i$ is itself of normal form. The normal form corresponds in an obvious way to a

synchronization tree. The isomorphism of two denotations is then provable by inductively using the divisor rule. □

In the case where the synchronization algebra is synchronous the expansion rules above can be replaced by simpler rules expressing the commutativity and associativity of parallel composition and rules as in Proposition 9.3 which say *NIL* is a $\textcircled{L}$-zero and that parallel composition distributes over sum. More precisely the expansion rules can be replaced by the following rules:

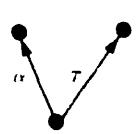
$$NIL \textcircled{L} t \sim NIL, \qquad s \textcircled{L} t \sim t \textcircled{L} s, \qquad (s \textcircled{L} t) \textcircled{L} u \sim s \textcircled{L} (t \textcircled{L} u),$$

$$t \textcircled{L} (u+v) = (t \textcircled{L} u) + (t \textcircled{L} v), \qquad (\lambda s) \textcircled{L} (\mu t) \sim \begin{cases} (\lambda \bullet \mu) s \textcircled{L} t & \text{if } \lambda \bullet \mu \neq 0, \\ NIL & \text{otherwise.} \end{cases}$$

Of course the above proof rules are rather limited; they only work for finite processes and for a somewhat primitive notion of equivalence. Still many more abstract ideas of equivalence are or could be based on synchronization trees. Proof rules for the more abstract equivalences would have to imply the rules above. It is even arguable that synchronization trees give the basic interleaving semantics making identifications of processes which all other interleaving semantics should also make. The argument does not quite push home, however, because of the phenomena of divergence. We explain the problem.

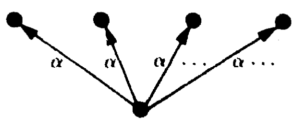
One technique for making a synchronization-tree semantics more abstract is to identify a process with the set of assertions it satisfies. The assertions may be in some fragment of modal logic and express the possible or inevitable behaviour of a process. A recursively defined process is denoted by the least upper bound of a chain of iterates obtained by repeated application of a continuous functional to the $\perp$-process. One would like that the set assertions satisfied by the recursively defined process is the union of those sets of assertions true for the iterates. Unfortunately, this is not the case for synchronization trees when taking modal assertions which express the inevitable behaviour of a process (see, e.g., [10] and [11]). Suppose one iterate was the synchronization tree


We cannot say of the process that it is inevitably prepared to make an $\alpha$-communication because some later iterate could be




A satisfaction relation defined between trees and assertions does not respect any approximation ordering on trees. The problem is that trees alone do not carry enough structure to reflect where their growth is complete and incomplete and without such extra knowledge we cannot be sure of any nontrivial assertions about the inevitable behaviour of the process. Of course one can extend trees or transition systems by extra structure to express those states which are incompletely defined, generally called 'divergent' (see [12, 11], for example). I am not certain how the work above generalizes to trees or transition systems which take account of divergence.

Although our approach ignores divergence there is a defence. Each closed program of $\mathbf{Proc}_L$ is given a denotation as a synchronization tree. This tree faithfully represents the completed program and we can consider those assertions which it satisfies and then take this set of assertions as its more abstract denotation. As an example, the process $P = \mathbf{rec}x.(\alpha NIL + x)$ is denoted by the infinitely branching tree




which according to one reasonable definition would satisfy an assertion saying that the process would inevitably be prepared to make an $\alpha$-communication. Contrast the situation in [11] where, essentially, they denote a process by the set of assertions it satisfies. Because in [11] it is ensured that all the functions in the denotational semantics are continuous they cannot attribute this inevitable behaviour to $P$. This is not to say the equivalence in [11] is wrong, just different.

Finally, I hope that the relation between parallel composition and product will be useful in proving properties of processes with synchronized communication. It is certainly useful in proving relations between semantics in the different categories of Petri nets [29], event structures [27, 28], trees and transitions systems. But also, I hope that the projection functions will be useful in formalizing the practice of proving properties of a parallel composition by projecting-down to the component processes, proving properties there and then combining the properties to yield the required proof.

## 10. Related work

Winskel [27, 28] shows how the above results for trees hold in the more general framework of event structures. Event structures are related to Petri nets in [21, 22]. They exhibit the causal independence and dependence of events and provide a basic model of parallel processes which does not rely on interleaving. In [27, 28] it is shown that they bear a smooth relation with trees; there is a natural interleaving, or serializing, operation on event structures which essentially imposes an extra causal constraint on the occurrence of events by ensuring events occur synchronized, in-step, with the ticks of a clock—it is a synchronous product on event structures. Then one can, for example, easily prove that a noninterleaving semantics for $\mathbf{Proc}_L$ in terms of labelled event structures interleaves to the synchronization tree semantics we provide here. A recent paper [29] on a new category of Petri nets extends the work [27, 28] and the work here. All the different categories are related by adjunctions so we can go quite far in translating between the different modes of expression.

The categories here and those mentioned above might be criticized for being too concrete because they distinguish too many processes. For example, $S+S$ is not generally isomorphic to $S$ even though it is hard to see a programming context in which they could be distinguished. Hopefully, there are categories with objects which reflect a more abstract notion of behaviour with pleasant relations to the ones here. In [11], it is shown how equivalence classes with respect to three natural equivalences on behaviour can be represented by a form of labelled tree. In [15], morphisms very like those here are defined on equivalence classes of trees with respect to Milner's observational equivalence, which essentially treats $\tau$-labelled events as invisible.

And then there are relations with path expressions and trace languages [6, 16]. Obviously, a synchronization tree determines a set of sequences of labels showing the possible communications. Only recently I noticed that ideas very similar to that of the morphisms presented here are found in the literature on languages of traces used to model concurrent processes (see [14]).

In [20], the finite delay property is considered for a synchronous calculus with an Abelian group of actions. The basic idea is to prune away disallowed infinite derivations from the labelled-transition-system semantics. One can generalize synchronization trees to reflect this in the unfolding. Take a generalized tree to consist of finite and infinite sequences. Infinite sequences hang as limit points at the ends of $\omega$-chains of nodes. By not insisting that every $\omega$-chain of nodes have a limit one specifies by their absence those infinite derivations which are not permitted. In a way exactly analogous to the above one obtains a category of generalized trees whose product, when labelled appropriately, is the parallel composition; it coincides with the unfolding of the transition system given in [20] with the correct infinite derivations removed. A transition system semantics similar to Milner's is presented by Plotkin [24] to give an operational semantics to constructs like a fair parallel operation. Interestingly in proving that the operational and denotational semantics are equivalent Plotkin uses projection functions from the parallel composition to the component processes.

As indicated in the previous section one can obtain more abstract semantics by 'filtering-out' those properties of interest for a specfic problem. (See [11] for a good example of this idea. Think of a property as an assertion one might make about the behaviour of a program.) This begs two questions: Is there a class of basic models from which all interesting properties can be extracted? What are the interesting properties of concurrent programs? Petri nets and event structures are more basic models than trees because they express much more about the causal relations between events. It is not yet clear, however, what interesting class of assertions force one to use event structures or nets instead of trees.

Unfortunately, trees, event structures and Petri nets are indifferent, as they stand, to notions of divergence as presented for example in [12] and [11]. This means that a satisfaction relation defined between trees or event structures and assertions about their inevitable behaviour cannot respect an approximation ordering on trees or event structures. In order to capture divergence in event structures it seems one needs somehow to extend their structure to include local places of growth, just as how, with trees sometimes $\perp$ is put at the leaf-nodes to show how they may extend in the approximation ordering. At first glance this idea is very like that of places in concrete data structures (see [13, 3, 26]).

## Appendix A. The proof of Theorem 4.13

Theorem 4.13 is repeated here for convenience.

**Theorem 4.13.** *Let $A$ and $B$ be synchronization algebras. The construction $A\times B$, $h_A$, $h_B$ is a categorical product of $A$ and $B$ in* SA. *The construction $A\otimes B$, $h'_A$, $h'_B$ is a categorical product of $A$ and $B$ in the subcategory with strict homomorphisms.*

**Proof.** Let $A=(L_A, \bullet_A, *_A, 0_A)$ and $B=(L_B, \bullet_B, *_B, 0_B)$ be synchronization algebras.

We first show $A\times B$, $h_A$, $h_B$ is a product in the category SA. We use the notation of Definition 4.11.

We make sure $A\times B$ is an SA: It is obvious that $\bullet$ is commutative. The following steps show $\bullet$ is associative:

$$(\alpha,\beta)\bullet((\alpha',\beta')\bullet(\alpha'',\beta''))=$$

$$=\begin{cases}(\alpha,\beta)\bullet 0 & \text{if } \alpha'\bullet\alpha''=0_A \text{ or } \beta'\bullet\beta''=0_B,\\ (\alpha,\beta)\bullet(\alpha'\bullet\alpha'',\beta'\bullet\beta'') & \text{otherwise}\end{cases}$$

$$=\begin{cases}0 & \text{if } \alpha'\bullet\alpha''=0_A \text{ or } \beta'\bullet\beta''=0_B,\\ 0 & \text{if } \alpha\bullet\alpha'\bullet\alpha''=0_A \text{ or } \beta\bullet\beta'\bullet\beta''=0_B,\\ (\alpha\bullet\alpha'\bullet\alpha'',\beta\bullet\beta'\bullet\beta'') & \text{otherwise}\end{cases}$$

$$=\begin{cases}0 & \text{if } \alpha\bullet\alpha'\bullet\alpha''=0_A \text{ or } \beta\bullet\beta'\bullet\beta''=0_B,\\ (\alpha\bullet\alpha'\bullet\alpha'',\beta\bullet\beta'\bullet\beta'') & \text{otherwise}\end{cases}$$

$$=((\alpha,\beta)\bullet(\alpha',\beta'))\bullet(\alpha'',\beta'').$$

Clearly $L\backslash\{*,0\}\neq 0$; by definition, $(0_A,0_B)\bullet(\alpha,\beta)=(0_A,0_B)$; by definition $*\bullet *=(*_A\bullet *_A, *_B\bullet *_B)=*$ while

$$(\alpha,\beta)\bullet(\alpha',\beta')\Rightarrow \alpha\bullet\alpha'=*_A\ \&\ \beta\bullet\beta'=*_B \Rightarrow \alpha=\alpha'=*_A\ \&\ \beta=\beta'=*_B$$

so $*$ is the unique divisor of $*$.

We check that the projections $h_A$ and $h_B$ are homomorphisms. Suppose $(\alpha,\beta)\bullet(\alpha',\beta')\neq 0$. Then $(\alpha,\beta)\bullet(\alpha',\beta')=(\alpha\bullet\alpha',\beta\bullet\beta')$ where $\alpha\bullet\alpha'\neq 0_A$ and $\beta\bullet\beta'\neq 0_B$. Thus we have $h_A((\alpha,\beta)\bullet(\alpha',\beta'))=h_A(\alpha\bullet\alpha',\beta\bullet\beta')=\alpha\bullet\alpha'=$

$h_A((\alpha, \beta)) \bullet h_A((\alpha', \beta'))$. Also

$$h_A(\alpha, \beta) = 0_A \Leftrightarrow \alpha = 0_A \Leftrightarrow \alpha = 0_A \,\&\, \beta = 0_B \quad \text{for } (\alpha, \beta) \in L.$$

And $h_A(*) = h_A(*_A, *_B)$, which shows that $h_A$ is a homomorphism. Similarly $h_B$ is a homomorphism.

Assume there are homomorphisms $f_A : C \to A$ and $f_B : C \to B$ for a synchronization algebra $C = (L_C, \bullet_C, *_C, 0_C)$. In order to show $A \times B$, $h_A$, $h_B$ is a product we require there exists a unique $f: C \to A \times B$ making the following diagram commute:

$$\begin{tikzcd}
& A \times B \arrow[dl, "h_A"'] \arrow[dr, "h_B"] & \\
A & & B \\
& C \arrow[ul, "f_A"] \arrow[ur, "f_B"'] \arrow[uu, "f"'] &
\end{tikzcd}$$

Define $f(c) = (f_A(c), f_B(c))$. Clearly, provided $f$ is a homomorphism it is the unique one such that the above diagram commutes. If $c \in L_C$, then either

$$c = 0_C \,\&\, f(c) = (f_A(c), f_B(c)) = 0$$

or

$$c \neq 0 \,\&\, f_A(c) \neq 0_A \,\&\, f_B(c) \neq 0_B$$

so $f(c) \in L$, making $f$ a function $L_A \to L_B$. We now argue that $f$ is also a homomorphism. Suppose $c \bullet c' \neq 0$. Then

$$f(c \bullet c') = (f_A(c \bullet c'), (f_B(c \bullet c')) = (f_A(c) \bullet f_A(c'), (f_B(c) \bullet f_B(c'))$$

where $f_A(c) \bullet f_A(c') \neq 0_A$ and $f_B(c) \bullet f_B(c') \neq 0_B$ as $f_A$ and $f_B$ are homomorphisms. Therefore

$$f(c \bullet c') = (f_A(c), f_B(c)) \bullet (f_A(c'), f_B(c')) = f(c) \bullet f(c').$$

We have

$$f(c) = 0 \Leftrightarrow f_A(c) = 0_A \,\&\, f_B(c) = 0_B \Leftrightarrow c = 0_C.$$

Also $f(*_C) = (f_A(*_A), f_B(*_B)) = (*_A, *_B)$. And so $f$ is a homomorphism, as required for $A \times B$, $h_A$, $h_B$ to be a product in SA.

The verification that $A \otimes B$, $h'_A$, $h'_B$ is a product in the subcategory is similar and therefore omitted; one simply checks that the constructions stay inside the subcategory.

## References

[1] M.A. Arbib and E.G. Manes, *Arrows, Structures and Functors: The Categorical Imperative* (Academic Press, New York, 1975).

[2] S.D. Brookes, On the relationship of CCS and CSP, *ICALP 1983, 10th Coll. on Automata, Languages and Programming*, Lecture Notes in Computer Science **154** (Springer, Berlin, 1983).
[3] G. Berry and P.L. Curien, Sequential algorithms on concrete data types, Rept. of Ecole Nationale Supérieure des Mines de Paris, Centre de Mathématiques Appliquées, Sophia Antipolis, 1981.
[4] W. Brauer, ed., *Net Theory and Applications*, Lecture Notes in Computer Science **84** (Springer, Berlin, 1980).
[5] E.M. Clarke, E.A. Emerson and A.P. Sistla, Automatic verification of finite state concurrent systems using temporal logic specifications: A practical approach, *Proc. 10th ACM Conf. on Principles of Programming Languages* (1983).
[6] R.H. Campbell and A.N. Habermann, *The Specification of Process Synchronisation by Path Expressions*, Lecture Notes in Computer Science **16** (Springer, Berlin, 1974).
[7] L. Cardelli and G. Plotkin, An algebraic approach to VLSI design, in: *VLSI 81* (Academic Press, New York, 1981).
[8] G. Gratzer, *Universal Algebra*, University Series in Higher Mathematics (Van Nostrand, Reinhold, New York, 1968).
[9] C.A.R. Hoare, S.D. Brookes and A.W. Roscoe, A theory of communicating processes, Tech. Rept. PRG-16, Programming Research Group, University of Oxford, 1981; *J. ACM*, to appear.
[10] M.C.B. Hennessy and R. Milner, On observing nondeterminism and concurrency, in: *Lecture Notes in Computer Science* **85** (Springer, Berlin, 1979).
[11] M.C.B. Hennessy and R. de Nicola, Testing equivalences for processes, Internal Rept., University of Edinburgh, 1982.
[12] M.C.B Hennessy and G. Plotkin, A term model for CCS, *Proc. 9th Conf. on Mathematical Foundations of Computer Science*, Lecture Notes in Computer Science **88** (Springer, Berlin, 1980).
[13] G. Kahn and G. Plotkin, Domaines concrètes, Rapport IRIA-LABORIA, No. 336, 1978.
[14] E. Knuth, Gy. Györy and L. Ronyai, A study of the projection operation, *Proc. Workshop on Petri Nets*, Informatik-Fachberichte **52** (Springer, Berlin, 1982).
[15] A. Labella and A. Peterossi, Towards a categorical understanding of parallelism, Rept., Istituto di Analisi dei Sistemi ed Informatica del C.N.R., Rome, 1983.
[16] P. Lauer, P. Torrigiani and M. Shields, COSY, a system specification language based on paths and processes, *Acta Informatica* **12** (1979).
[17] S. Maclane, Categories for the working mathematician, in: *Graduate Texts in Mathematics* (Springer, Berlin, 1972).
[18] R. Milner, *A Calculus of Communicating Systems*, Lecture Notes in Computer Science **92** (Springer, Berlin, 1980).
[19] R. Milner, On relating synchrony and asynchrony, Rept. CSR-75-80, Dept. Computer Science, University of Edinburgh, 1981.
[20] R. Milner, A finite delay operator in synchronous CCS, Internal Rept. CSR-116-82, University of Edinburgh, 1982.
[21] M. Nielsen, G. Plotkin and G. Winskel, Petri nets, event structures and domains, *Proc. Conf. on Semantics of Concurrent Computation*, Lecture Notes in Computer Science **70** (Springer, Berlin, 1979).
[22] M. Nielsen, G. Plotkin and G. Winskel, Petri nets, event structures and domains, Part I, *Theoret. Comput. Sci.* **13** (1981) 85–108.
[23] G. Plotkin, A structural approach to operational semantics, Rept. DAIMI FN-19, Computer Science Dept., Aarhus University, 1981.
[24] G. Plotkin, A powerdomain for countable non-determinism, in: *Lecture Notes in Computer Science* **140** (Springer, Berlin, 1982).
[25] D. Scott, Domains for denotational semantics, in: *Lecture Notes in Computer Science* **140** (Springer, Berlin, 1982).
[26] G. Winskel, Events in computation, Ph.D. Thesis, University of Edinburgh, 1980.
[27] G. Winskel, Event structure semantics of CCS and related languages, in: *Lecture Notes in Computer Science* **140** (Springer, Berlin, 1982).
[28] G. Winskel, Event structure semantics of CCS and related languages, Rept., Computer Science Dept., University of Aarhus, 1982.
[29] G. Winskel, A new definition of morphism on Petri nets, in: *Lecture Notes in Computer Science* **166** (Springer, Berlin, 1984).