Lecture Notes

on

Dynamical Systems,

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ICTS

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Lecture 1

Notes on
Dynamical Systems
AMIT APTE
02 January 2019
ICTS

Class timings: On Mon and Wed from 4:15 to 5:45 pm in Feynmann Lecture Hall
Tutorials: Once in a week.

Notations for the course

1. **Continuous time:**

$$t \in I \subset \mathbb{R}$$

where differential eqns are specified.
'X'= phase space of the system, e.g position and momentum of hamiltonian.

$$(x, p) \in \mathbb{R}^{2d}$$

for a d-dimensional system.

$$x \in X \subset \mathbb{R}^d$$
$$\dot{x} = \frac{dx}{dt}$$

[1]

$$\dot{x} = f(x, t; \lambda)$$

for cont. time

$$f : I \times X \times \mathbb{R}^p$$

where p is no of parameters $\lambda's$ For a fixed λ, f defines a vector field on

$$X \subset \mathbb{R}^d$$

is associated to every point $(t, x) \in I \times X$.
There is a vector $f(x, t)$ with components $f_1, f_2 \cdots f_n$ A solution will lie in a curve/trajectory/orbit given by

$$x : I \to \mathbb{R}^d \quad i.ex(t)$$

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2. **Discrete time:**
$n \in I \subset \mathbb{Z}$ where maps will define the evolution.

$$x_{n+1} = f_n(x_n, \lambda)$$

for discrete time

$$f_n : X \times \mathbb{R}^p \to \mathbb{R}^d$$

The solution will be a series of [2]

$$x_0, x_1, x_2, x_3 \cdots x_n$$

When a soln is said to exist? Depends on the following

- Boundedness
- Stability

[1] x will always denote a d-dimensional vector, without the vector symbol
[2] There is no vector field like $f(x_n, t)$ for discrete maps

- statistical properties of soln (ergodicity, mixing, etc)
- dependence on initial condns
- Long time Assymptotic behaviour $t \to \infty \quad n \to \infty$

Other things to be noted are

- stablility means with respect to initial conditions.
- Dependence on parameters lead to bifurcations.
- Effect of perturbations to the systems.

Critical points (constant in time)
1. For continuous time: p is a critical point if the vector field vanishes i.e

$$f(t, p) = 0$$

is used mostly when if the function is independent of t. It follows that $x(t) = p$ is a solution. Hence p is a fixed point.
2. For discrete time: p is a critical point if it is a fixed point of map f. Consider n independent maps $f_n \equiv f$

$$f(p) = p$$

again $x_n = p$ is a solution

Periodic orbit
1. For continuous time:$x(t)$ is a period orbit if x is a periodic function of t and if T is the smallest such number , then it is called the time period of the system. 2. For discrete time:

$$x_{n+N} = x_n \quad \forall 0 \ll n \ll N$$

Note that period 1 solutions (if exists) are fixed points

Linear vs Non-linear systems
f is a linear function of x. Most commonly written as a matrice

$$f(x, t) = A(t)\, x$$

Solution is linear function of initial conditions
Is there a non-linear system for which the solution is linear fn of I.C ?

Autonomous vs Non-Autonomous system
Autonomous is when function f is independent of time i.e

$$f(t, x) \equiv f(x)$$

$$f_n(x) \equiv f(x)$$

[3] For autonomous system, $\dot{x} = f(x), \quad f : D \subset \mathbb{R}^d \to \mathbb{R}^d$ defines the vector field $f_1(x), f_2(x), \cdots, f_d(x)$ at any point $x \in D$.
In a Non-autonomous system in d-dimensional phase space is equivalent to Autonomous system of d+1-dimensional phase space.
Simplest systems are linear and autonomous...

$$\dot{x} = A\, x \qquad x_{n+1} = A\, x_n$$

then the solution is

$$x(t) = \exp\left[At\right] x(0) \qquad x_n = A^n x_0$$

For non-autonoous system $\dot{x} = f(x)$

$$f : D \in \mathbb{R}^{d+1} \to \mathbb{R}d$$

defines a vector field on D given by

$$(f_1(x, t), \cdots, f_d(x, t), 1)$$

[3] In discrete system, autonomous is very common

which can be thought as

$$\dot{x} = f(x, t)$$
$$\dot{t} = 1$$

Note that this system with vector field with $d + 1$ *components cannot have a fixed point, since one component cannot (the last) be zero. Still the stability can be discussed.* For linear non-autonomous systems: reference is `Adrianova` book.
For linear, time periodic [4]

$$A(t + T) = A(t) \forall t$$

Homogeneous vs Non-Homogeneous system of linear ordinary differential equation

$$\dot{x} = A(t)x$$

is homogeneous whereas

$$\dot{x} = A(t)x + f(t)$$

is non-homogeneous system.
If $A(t)$ is periodic in t with $A(t+T) = A(t), \quad \forall t \in I$ then **??** can have solution with another different from T.

$$\ddot{x} + \omega x = \sin(t)$$
$$\ddot{x} + \sin(kt)x = \sin(t)$$

may have solutions with period different from k or 1.

Arnold Cat map

The phase space X can be a manifold. Then map or vector field $f(x, t)$ will have to be re-defined (will be out of this course scope).
Except when the manifold is a torus
By Definition
1-dim torus: $S^1 \equiv \frac{\mathbb{R}}{\mathbb{Z}}$
d-dim torus: $\mathbb{T}^d \equiv S^1 \times S^1 \times \cdots S^1 S$
A map or vector field on d-torus: functions on $\mathbb{R}^d$ periodic in each co-ordinate with period 1.
Consider a map M as follows

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \bmod 1$$

This map has the following properties

- Clearly, $(0, 0)$ is the only fixed point
- $$f^n(x) = Ax$$
  where 'A' contains fibonacci numbers
- For periodic orbits
  $$p = 1 \implies x = (0, 0)$$
  Similarly for all the periods, there exists a solution. The set of all periodic orbits can be defined as
  $$S_p = x \in \mathbb{T}^2 \Big| f^n(x) = x$$
  for $n \in \mathbb{Z}$
  S_p is dense with zero area.

[4] This does not imply $x(t + T) = x(t)$

- Set of non-period orbits defined as

$$\tilde{S}_p = \mathbb{T}^2 - S_p = x \in \mathbb{T}^2 \Big| f^n(x) \neq x$$

for any $n \in \mathbb{Z}$.
$\tilde{S}_p$ is also dense with area 1.

- The eigenvector and eigenvalues of M are

$$\frac{1}{2}\left(3 \pm \sqrt{5}\right) \quad \left[\begin{array}{c} 1 \pm \sqrt{5} \\ 2 \end{array} \right]$$

One eigenvector corresponds to stable and other eigenvector corresponds to unstable.

- Any point on eigenvector $\in \tilde{S}_p$

$$M^h v_\pm = \lambda_\pm^h v_\pm$$

This map is also ergodic, mixing, positive lyapunov exponent , but not really chaotic...

- consider a image of NxN pixels, with mod N arnolds cat map will scramble the image. But for any N, there exists a period Π_N after which the image will return.
- This mapping is also called as *Toral Automorphism*
- determinant of the mapping is 1 implying that area is preserved.
- define a matrice $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ F_1 & F_2 \end{bmatrix}$ where F_n denotes nth Fibonacci number.
Then we can verify

$$F^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$$

The matrice F is called the *Golden cat map.*Because we have $F^2 = A$, implying

$$A^n = \begin{bmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{bmatrix}$$

and when we look for minimal period of Arnold's cat map we want smallest integer 'n' such that

$$A^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod N$$

i.e $F_{2n-1} = 1(\bmod N), F_{2n} = 1(\bmod N), F_{2n+1} = 1(\bmod N)$Hence the period of Arnold's cat map will have a direct connnection with *Pisano period.* In fact, for $N \geqslant 3$, the period of Arnold's cat map will be exactly half the Pisano period.

- The upper bound for the minimal period of Arnold's cat map is $3N$.
- define orbit of a point to be set of co-ordinates that an individual point will assume under iterations of dynamical system. The number of unique co-ordinates in the orbit is called orbit length. [5]
- Since the upper bound for the period of Arnold's cat map is $3N$, for all $N > 3$ no point can have an orbit that includes all the $N^2 - 1$ non-trivial points. therefore we conclude that there will be a number of disjoint orbits.
- Before reaching the minimal period, the image appears to be less chaotic than expected. Which we call *Miniatures and Ghosts*

[5]Points with orbit length 1 are nothing but fixed points

Lecture 2

Notes on
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Numerical methods
Most of the dynamical systems modeled to understand physical behaviour of systems cannot be solved analytically, only numerically. Therefore, one must be familiar with numerical methods. **Lorrenz system:**[1]

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(p - z) - y\\ \dot{z} &= xy - pz\\ X &= (x, y, z)\\ with \quad X(t_0) &= X_\circ\end{aligned}$$

we want $X(t)$ at $t = t_0, t_1, \ldots, t_n$ i.e consecutive time steps.
Commonly used numerical methods are

- Runge Kutta fourth order
- Other methods are available, depends on the type of system.

Solve for $X(t)$ by specifying the time we want. projection onto any plane, or in $\mathbb{R}^3$ shows the **Lorrenz attractor** This is the basic methodology of integrating the equation. Does this numerical method fail for some time step? Yes, one has to study the domain of time and domain of solution to avoid such instabilities in numerical methods. Both are different.

Consider the system

$$\dot{x} = \alpha x^2$$
$$x(0) = x_\circ$$

The solution for the above system is

$$x(t) = \left(\frac{1}{x_\circ - \alpha t}\right)^{-1}$$

when the denominator $\to 0$, the system diverges, hence the numerical method might fail. What is the domain of t in this function? $(\alpha = 1)$

$$t \neq \frac{1}{x_\circ}$$

Domain of solution

$$x_\circ > 0, \quad t < \frac{1}{x_\circ}$$

for

$$x_\circ < 0, \quad t > \frac{1}{x_\circ}$$

for

$$x_\circ = 0, \quad t \in \mathbb{R}$$

The domain of solution (by definition) is the region in which the solution is continuous and differentiable from its initial point. Since there is a discontinuity at $t = \frac{1}{x_\circ}$, once started from either side of it, it stays as shown in 1. It exists for all systems. One has to identify the region . When can you say a solution exists and when it is unique?

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[1]This was given by Lorrenz on 1963

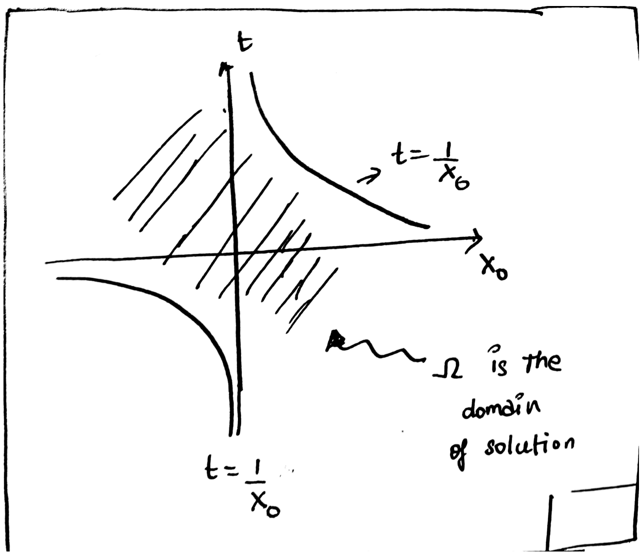

Figure 1: Domain in the $t\,vs\,x_o$

How about

$$\dot{x} = \sqrt{|x|}$$

and with

$$x_o = 0$$

$$x = \left(\frac{t + \sqrt{x_o}}{2}\right)^2$$

Any solution with negative parabola upto a point c_1 then stays zero upto c_2 then becoming a positive parabola is a valid solution as shown in 2.
What will you expect, in numerical solution ?

Going to a general problem

$$\dot{x} = f(x, t), \quad x(t_o) = x_o$$

This is called as **initial value problem.** A function defined with $f : D \to \mathbb{R}^d$ where $D \subset \mathbb{R}^{d+1}$ continuous on D acts on $x : I \to \mathbb{R}^d$ for $I \subset \mathbb{R}$ on interval called as maximum interval of existence[2] $(a, b), a < b$

PEANO THEOREM:
A solution to our equation exists if

- x is continuously differentiable [3]
- $(x(t), t) \in D \; \forall t \in I$
- if $t_o \in I$ and $x(t_o) = x_o$

It only tells the existence of solution, it may or may not be unique.

[2] Should be connected in $\mathbb{R}$, as it is time
[3] With usual definition of differentiability, refer `Lawrence Perko (1990)`

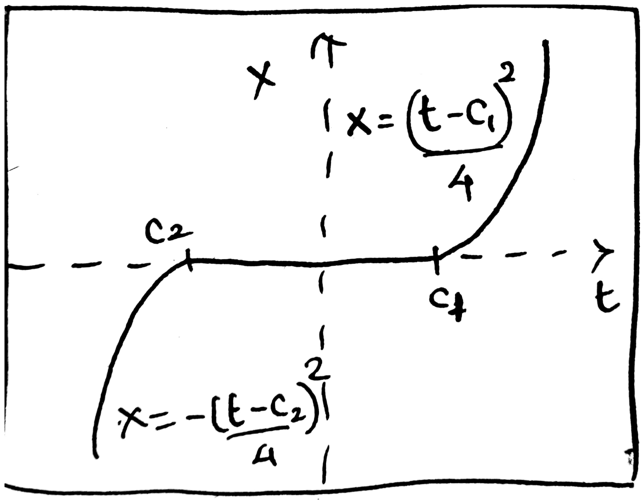

Figure 2: general solution

For any $(t_\circ, x_\circ) \in D$ there exists a solution of our equation. The proof uses fixed point theorem. But the idea is following, for "*appropriate*" functions $I \to \mathbb{R}^d$, let $\Psi = T\phi$ by

$$\Psi(t) = x_\circ + \int_{t_\circ}^{t} ds\, f(\phi(s), s)$$

If x is such that $Tx = x$ implies

$$x(t) = x_\circ + \int_{t_\circ}^{t} ds\, f(x(s), s)$$

This is infact a solution. If a map T exists then by fundamental theorem of calculus

$$\dot{x} = f(x(t), t)$$

For maps

$$x_{n+1} = f(x_n)$$

fixed point means $x = f(x)$ same trick was used in above derivation but in function space.
Proof requires finding a fixed point of the map T, actually we just need to prove the existence of a fixed point. This is called as **Schauder fixed point theorem**.

This is about existence of solution , but when it is unique.
DEFINITION:
f is called as *Lipschitz* if

$$|f(x,t) - f(y,t)| \leqslant K|x - y|$$

for $y \in (x - \varepsilon, x + \varepsilon)$ i.e y is in the neighbourhood of x. [4]

[4]normalization is $|x| = \sqrt{\sum_i x_i^2}$

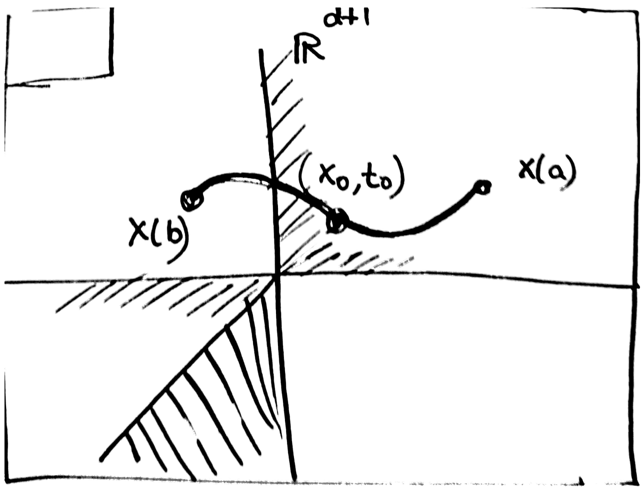

Figure 3: Existence of solution

This means, the graph of function should below the line of slope K, K depends on x but not on y. For e.g $\sqrt{|x|}$ is locally lipschitz everywhere except at $x = 0$. For x^2, it is locally lipschitz everywhere. By the way locally lipschitz implies continuous.

THEOREM: **Picard-Lindelöf**
If a f is locally lipschitz at $(t_\circ, x_\circ)$ then there exists a unique solution to equation on some interval.
Basically, it tells that a solution exists in a small neighbourhood, does not even tell what the neighbourhood is.
Going further if $(x(a), a)] \in D$ then we can solve

$$\dot{x} = f(x,t) \quad with \quad x(a) = x_1$$

And it will exist over a new interval (a_1, b_1)
This is called as continuation of solution or succesive approximation: $x_1 : I_1 \to \mathbb{R}^d$ is a continuation of $x : I \to \mathbb{R}^d \quad with\, I \subset I_1$ and $x(t) = x_1(t)$ for $t \in I_1$ with f being locally lipschitz at x_1 also.
To go further, we need the function is locally lipschitz everywhere.

THEOREM: If a f is locally lipschitz on D, then for any $(x_\circ, t_\circ) \in D$ there is a maximal interval of existence i.e there is no continuation outside that interval .Call that interval (a, b), they depend on $x_\circ, t_\circ$. As $t \to a_+$ and $t \to b_-$ then $(x(t), t)$ tends towards the boundary of D.
If the D is $\mathbb{R}$ for e.g

$$\dot{x} = x^2 \quad x(0) = x_\circ > 0$$

then as $t \to \frac{1}{x_\circ}$ then $x \to \infty$.
Another e.g

$$\dot{x} = -\frac{1}{x}$$

either $D = (x > 0) \times \mathbb{R}$ or $(x \neq 0) \times \mathbb{R}$
Check: if it is locally lipschitz?
The solution

$$x(t) = \sqrt{x_\circ^2 - 2t}$$

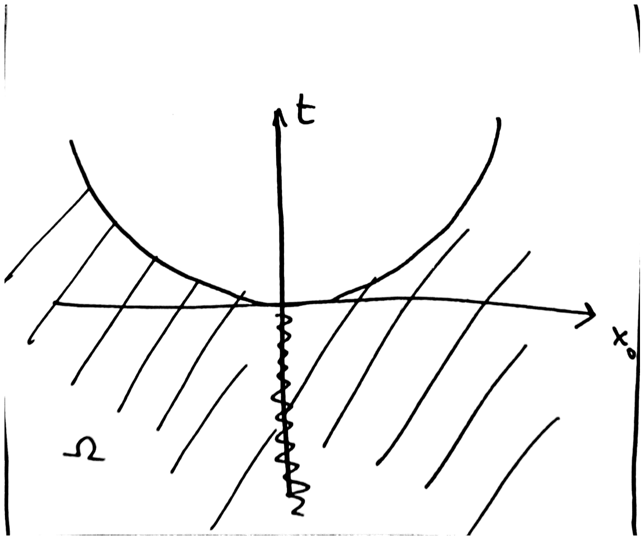

Figure 4: Domain of solution

defined for $t < \frac{1}{2}x_\circ^2$
The solution is defined in following domain Ω, as shown in 4

$$\Omega = \{(x_\circ, t)\} \quad with \quad t < \frac{1}{2}x_\circ^2, \quad x_\circ \neq 0$$

In general, the region Ω will be defined as

$$\Omega = \left\{ (x_\circ, t_\circ, t) \middle| t \in (a(x_\circ, t_\circ), b(x_\circ, t_\circ)) \, \forall x_\circ, t_\circ \in D \right\}$$

Flow: of a differential equation $\dot{x} = f(x)$ for autonomous differential equation is defined as the map

$$\phi : \Omega \to \mathbb{R}^d$$

that gives the solution of the differential equation.
In other words, given the initial time $t_\circ$ and position $x_\circ$ in the $I \times D$, we have a solution $x(t, x_\circ)$. Now for a particular initial value problem ,we have a unique trajectory given it is lipshitz. This is like a trajectory of a particle in a fluid flow. Which means, each particle at initial time corresponds to a unique initial position. Therefore, the flow of the differential equation is like the flow of the fluid for all possible initial conditions from the domain Ω.

Lecture 3

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`Quiz 1 : Conducted`

Autonomous systems

$$\dot{x} = f(x), \quad x(0) = x_0$$

Recap: if $f : D \to \mathbb{R}^d$ is Lipschitz continuous on D, then for every $x_0 \in D$, $\exists \alpha$ such that $x \in N_\alpha \subset \mathbb{R}^d$ is a unique solution .
[1] In particular, if for $|x - x_0| < \rho$

- $|f(x)| < M$
- f has a Lipschitz constant K then the unique solution exists for $\alpha = \min\left\{\frac{\rho}{M}, \frac{1}{K}\right\}$.
- For $0 < a < \alpha$, a unique solution exists for $t \in [-a, a]$
- *If* f *satisfies the following,* $|f(x)| < L|x|$, that is it does not grow faster than norm of x, then the maximal interval is $(-\infty, \infty)$. The solution exists for all time.

- If the maximal interval of existence is (a, b) and $b < \infty$ then, for every compact set $C \subset D$, there exists $t \in (a, b)$ such that $x(t) \notin C$
- As $t \to b$, the solution approaches to infinity.

As an example, consider the follow function: $\dot{\theta} = 1$ and $\dot{r} = -(r-1)r$,
In the r, θ co-ordinates, the solution approaches the boundary that is $r = 1$ *How is that a boundary ?.* for maximal interval $(-\infty, \infty)$.

In fact $r = 1$ is a limit cycle. For any initial condition, since the maximal interval is $\mathbb{R}$, thus the as $t \to \infty$ it reaches the boundary which is $r = 1$.

Consider $\Omega = \left\{(t, x_0) \middle| x_0 \in D, t \in I(x_0)\right\}$ where $I(x_0)$ is the maximal interval of existence and $f : D \to \mathbb{R}^d$

and for example the system

$$\dot{x} = -\frac{1}{x}, \quad x > 0$$

Here Ω is shown in figure 1.

[1] $N_\alpha$is the neighbourhood of x with radius α

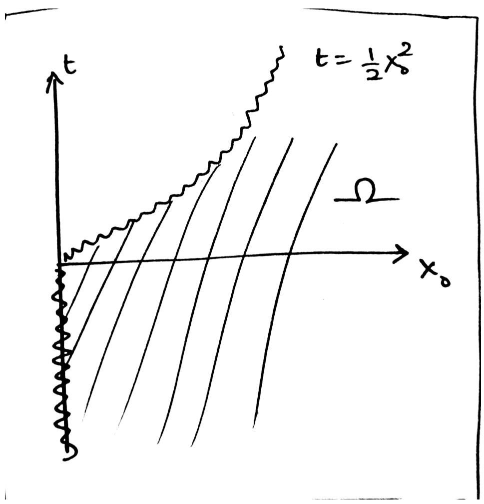

Figure 1: Region of existence

Flow of a differential equation

$$\phi : \Omega \to \mathbb{R}^d$$

such that $\phi(t, x_0) = x(t)$ or as $\phi_t(x_0)$. If $\phi(t, x_0) = y \in D$ then we can solve

$$\dot{x} = f(x), \quad x(0) = y$$

and if $s \in I(y)$, then $\phi(s, y)$ is solution after time s.
Lets say $\phi(s, y) = z$ then

$$\phi(t + s, x_0) = z$$

implies

$$\phi_{t+s}(x_0) = \phi_s(\phi_t(x_0))$$

or simply

$$\phi_{t+s} = \phi_s \circ \phi_t$$

whenever both sides are defined. This is called **Semi-group property**[2] of the flow of the differenial equation.

- ϕ_0 is identity mapping, that is $\phi(x_0, 0) = x_0$
- If $-t \in I(\phi_t(x_0))$ then clearly $\phi_{-t}(\phi_t(x_0)) = x_0$ then ϕ_{-t} is an inverse [3]
- If the interval of existence is $(-\infty, \infty)$ $\forall x_0 \in D$ then ϕ_t defines a group.

Once we know the conditions to check for existence and uniqueness, *which is a easy job to do ! ! !*. But one should be cautious with the interval of existence. Especially when solving the problem numerically.

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[2]sometimes the inverse maynot exists , thats why semigroup
[3]But in inverse , domain is a problem

Linearization

If $x(t)$ is a solution of $\dot{x} = f(x)$, with initial condition $x(0) = x_0$, then

- stability of the solution
- dependence on the initial condition

both can be studied using linearization. With a notation $F(x) = \frac{\partial f}{\partial x}(t)$ a matrice of derivatives F_{ij}
A linear non-automous ordinary differential equation of the form

$$\dot{z} = A(t)z$$

where $A(t) = F(x(t))$ is called the linearization around the solution $x(t)$

$$F = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_d}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_d} & \cdots & \frac{\partial f_d}{\partial x_d} \end{bmatrix}$$

If $x(t) = x_0$ is a fixed point, then $F(x(t)) = F(x_0)$ now the linearization is an autonomous linear ordinary differential equation.

Classification of fixed point

- If fixed point x_0 is called **hyperbolic** if $F(x_0)$ does not have any eigenvalue with 0 real part
- It is called as **Sink** if all eigenvalue have negative real parts (like converging to a fixed point)
- It is called as **Source** if all eigenvalue have positive real parts (like diverging from a fixed point)[4]
- It is called as **Saddle** if it is hyperbolic with atleast one eigenvalue with positive real and another eigenvalue with negative real part.
- It is called as **Center** if all the real parts of eigen values are 0.

`read for reference : Chapter about 2-D phase portrait in 1.Perko 2.Hirsch-Devaney`
Standard way to write is as shown in figure 2 showing $\mathrm{Tr}(A)$ *vs* $\mathrm{Det}(A)$

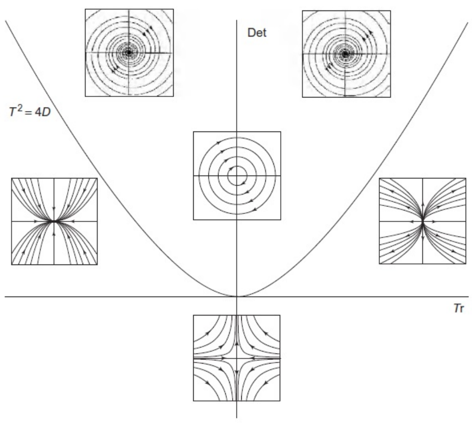

Figure 2: Classification of fixed points

[4] A sink, source, saddle all are could be hyperbolic

Lecture 4

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Dependence on initial conditions
For autonomous systems:

$$\dot{x} = f(x) \quad x(0) = x_0$$

Solution as

$$\phi(t, x_0) \equiv \phi_t(x_0) = x(t, x_0)$$

How does this change when x_0 is varied?

Denote

$$M = \frac{\partial \phi}{\partial x_0}$$

[1] Look for differential equation for M where $M \in \mathbb{R}^{d \times d}$ matrix

$$M_{ij} = \frac{\partial \phi_i}{\partial x_j}$$

$$\frac{\partial M}{\partial t} = \frac{\partial^2 \phi}{\partial x_0 \partial t} = \frac{\partial}{\partial x_0} f(\phi(t, x_0)) = Df\left(\phi(t, x_0)\right) \cdot \frac{\partial \phi}{\partial x_0}$$

[2]

$$= Df\left(\phi(t, x_0)\right) \cdot M$$

This is a linear equation for M

[3]

Looking at columns of M

$$M = [C_1, \cdots, C_d]$$

k^{th} column

$$(C_k)_j = \frac{\partial \phi_j}{\partial x_{0,k}} = M_{jk}$$

$$\frac{\partial}{\partial t}(C_k)_j = \frac{\partial}{\partial t} M_{jk} = \sum_i (Df)_{ji} M_{ik} = \sum_i (Df)_{ji} (C_k)_i$$

Each column satisfies the same linear differential equation. This is called linearization .

$$\dot{z} = Df\left(x(t)\right) \cdot z$$

What is the initial condition for M?

$$M(t, x_0) = \frac{\partial \phi}{\partial x_0}(t, x_0)$$

$$M(0, x_0) = \frac{\partial \phi(0, x_0)}{\partial x_0} = I_d$$

Hence $C_k\left(0, x_0\right) = e_k$ Standard basis vector for $\mathbb{R}^d$
This is the main reasoning behind the following theorem:

Essentially we need conditions for guarantee $\dot{z} = Df(x(t)) \cdot z$ has a unique solution. general equation was $\dot{z} = g(t, z)$, which has to be Lipschitz continuous in z.

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[1] Note $\frac{\partial \phi(t, x_0)}{\partial t} = f(\phi)$
[2] $Df(x) = \frac{\partial f}{\partial x}(x)$
[3] Dot product is basically matrice multiplication

For linearized systems, it is continuous for sure, but is it Lipschitz continuous?

Yes, as long as f is differentiable the matrice Df exists which is locally Lipschitz . that is we need g to be Lipschitz continuous in z with existence of Df.

THEOREM: Let f be a function $f : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable one. Then for $x_0 \in D$, there exists $a > 0$ and $\delta > 0$ such that the initial value problem $\dot{x} = f(x)$, $x(0) = y$ has a unique solution $\phi(t, y)$ for $t \in [-a, a]$ and $|x - y| < \delta$ and $\phi : [-a, a] \times N_\delta(x_0) \to \mathbb{R}^d$ is continuous differentiable in y and twice continuous differentiable in t and the derivative $M = \frac{\partial \phi}{\partial y}$ satisfies

$$\dot{M} = Df\left(\phi(t, y)\right) \cdot M$$

$$M(0) = I_d$$

and there exists $K > 0$ such that

$$|\phi\left(u, y + h\right) - \phi\left(t, y\right)| \leqslant |h| e^{Kt}$$

Note $\{a, \delta, K\}$ depend on x_0

The main steps in proof are as follows:

- Define a sequence of functions

  $$\phi_0(t, y) = y$$

  a constant function, then

  $$\phi_{k+1}(t, y) = y + \int_0^t ds\, f\left(\phi_k\left(s, y\right)\right)$$

  If the limit exists for these succesive functions, that is $k \to \infty$ exists: then it will be the solution. The solution will satisfy $\phi(t, y) = y + \displaystyle\int_0^t ds\, f\left(\phi\left(s, y\right)\right)$

- To prove that the limit exists, one way is to prove by induction:

  $$|\phi_{k+1}(t, y) - \phi_k(t, y)| \leqslant (Ka)^k \varepsilon$$

  for some ε, K, a

- So the Sequence converges if $a < \frac{1}{K}$, define

  $$g(t) = |\phi(t, y + h) - \phi(t, y)|$$

  Subracting we get

  $$= |h + \int_0^t ds\, f(\phi(t, y + h)) - f(\phi(t, y))|$$

  $$\leqslant |h| + \int_0^t ds |f_{y+h} - f_y|$$

  $$\leqslant |h| + K \int_0^t ds\, |\phi_{y+h} - \phi_y|$$

  $$g(t) \leqslant |h| + K \int_0^t ds\, g(s)$$

  (*Gronwall's Lemma*)

  $$g(t) \leqslant |h| e^{Kt}$$

  (part of theorem)

- Define another function

  $$M(t, y)$$

  satisfy $\dot{M} = Df\left(\phi(t, y)\right) \cdot M$ with $M(0) = I_d$

- Let

$$n(t) = |\phi(t, y) - \phi(t, y + h) + M(t, y) \cdot h|$$

If $g(x) = f'(x)$ then $\lim_{h \to 0} |f(x + h) - f(x) - g(x)h| = 0$

- Using *Gronwall inequality* for $n(t)$, we can show that M is the jacobian of ϕ.

How to understand the dependence on parameters? that is, if we have $\dot{x} = f(x, \nu)$ where ν is some parameter. Like in lorrenz system with ρ, σ, β. In general for p parameters, $\nu \in \mathbb{R}^p$, let $f : D \subseteq \mathbb{R}^{d+p} \to \mathbb{R}^d$ be continuous differentiable function of x and ν, then for $(x_0, \nu_0) \in D$ $\exists$ $a > 0$ and $\delta > 0$ such that the solution $\phi(t, y, \nu)$ is continuous differentiable in $[-a, a] \times N_\delta(x) \times N_\delta(\nu)$

Question : Whats the equation for $\frac{\partial \phi}{\partial \nu}$?

Answer: Consider $z = \begin{bmatrix} x \\ \nu \end{bmatrix}$ and $\dot{z} = \begin{bmatrix} f(x, \nu) \\ 0 \end{bmatrix} = F(z) \in \mathbb{R}^{d+p}$
Now use previous theorem

$$\phi(t; x_0, \nu_0) = \begin{bmatrix} \phi(t; x_0) \\ \nu_0 \end{bmatrix}$$

$$\frac{\partial \phi}{\partial (x_0, \nu_0)} = \begin{bmatrix} \frac{\partial \phi}{\partial x_0} & 0 \\ \frac{\partial \phi}{\partial \nu} & I_p \end{bmatrix}$$

In Summary: To understand the dependence on initial conditions, we need to study the solution of linear ordinary differential equations. 3 types as follows

1. If $\phi(t, x*) = x*$ is a fixed point then the linearization is a homogeneous autonomous linear ordinary differential equation that is $\dot{z} = Az$

$$Df\left(\phi\left(t, x*\right)\right) = Df\left(x*\right) = A$$

is independent of t

2. For periodic orbits, that is $\phi(t + T, x_0) = \phi(t, x_0) \forall t$ and some $T > 0$, which implies Df is also periodic function of t

$$A(t + T) = A(t)$$

3. ??

The linearization of linear O.D.E is the equation itself. Actually the linearization is same around every point, not only fixed point.

The solution of a linear autonomous O.D.E $\dot{z} = Az, A \in \mathbb{R}^{d \times d}$

$$z(t) = \exp\left[A(t)\right] z(0)$$

THEOREM: Jordan normal forms -If A is a $d \times d$ real matrice with real eigenvalue $\lambda_1, \cdots, \lambda_k$ (could be degenerate) and complex eigenvalue $\lambda_{k+1} = a_{k+1} + \imath b_{k+1}, \cdots, \lambda_n = a_n + \imath b_n$ with $k + 2n = d$ then $\exists$ a basis

$$P = [u_1, \cdots, u_k, \nu_{k+1}, u_{k+1}, \cdots, \nu_n, u_n]$$

of $\mathbb{R}^d$ of generalized eigenvector $u_1, \cdots, u_k$ and $u + \imath\nu$ for $k + 1$ to n.

$$A = P \begin{bmatrix} B_{real} & \\ & B_{complex} \end{bmatrix} P^{-1}$$

where for real eigenvalue $\lambda_i \quad i = 1, \cdots, k$, the jordan block is of form

$$B_r = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & \lambda_k \end{bmatrix}$$

for complex eigenvalue $\lambda_j = a_j + \iota b_j \quad j = k+1, \cdots, n$

$$B_c = \begin{bmatrix} D & I_2 & 0 \\ 0 & \ddots & I_2 \\ 0 & 0 & D \end{bmatrix}$$

where $D_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$.

Why is this useful? because $e^{At} = e^{PBP^{-1}}$ is just

$$Pe^{Bt}P^{-1}$$

, now that the exponent of a block diagonal matrice
The blocks just get exponentiated, even for different sizes.

$$e^{Bt} = \begin{bmatrix} e^{B_r t} & \\ & e^{B_c t} \end{bmatrix}$$

Then the steps are to keep track of real eigenvalue blocks and complex eigenvalue blocks. Seperate them as

$$B_r = \begin{bmatrix} \lambda_1 & \\ 0 & \lambda_k \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the second matrice is Nilpotent matrix of some order. Finally $\exp[N_c, t]$ will truncate after few terms.

Lecture 5

Notes on
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We saw the following

$$\dot{x} = f(x)$$

$$\dot{z} = A(t)z, \quad z \in \mathbb{R}^d$$

where $A(t) = Df(x(t))$, and $z(t)$ is linearization around $x(t)$. Note that to write $A(t)$ one needs to know $x(t)$ first. And also $z(t)$ is coupled to $x(t)$, not the other way around.

The system is

$$\dot{x} = y + \alpha x\left(1 - x^2 - y^2\right) \tag{1}$$
$$\dot{y} = x + \alpha y\left(1 - x^2 - y^2\right) \tag{2}$$
$$\tag{3}$$

In polar co-ordinates becomes

$$\dot{\theta} = 1 \tag{4}$$
$$\dot{r} = \alpha r\left(1 - r^2\right) \tag{5}$$
$$\tag{6}$$

depending on the sign of α, the solutions around $r = 1$ will look like in figure 1

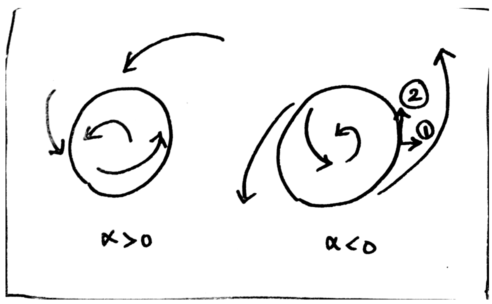

Figure 1: Trajectory for $\alpha > 0$ and $\alpha < 0$

Define

$$X = \begin{bmatrix} x \\ z \end{bmatrix}$$

and its evolution equation is

$$\dot{X} = F(X) = \begin{bmatrix} f(x) \\ A(t)z \end{bmatrix}$$

For a general solution, in $\mathbb{R}^d$ we can always choose d linear independent directions, in which we can cause perturbation around. The one along the trajectory, and $d - 1$ other direction which are perpendicular to the trajectory. depending on the direction of perturbation, the seperation can grow or decay or stay constant.

Recap: The general solution to autonomous system $\dot{x} = Ax, \quad x(0) = x_0$ with A being constant is

$$x(t) = e^{At}x_0$$

$$\exp(At) = P\exp(Bt)P^{-1}$$

wher B is block diagonal of the form

$$\begin{bmatrix} B_r & 0 \\ 0 & B_c \end{bmatrix}$$

where each

$$e^{B_r t} = e^{\lambda t} \begin{bmatrix} 1 & t \cdots & \frac{t^{m-1}}{(m-1)!} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

and

$$e^{B_c t} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \square$$

a similar matrice at the end.

- the solutions of $\dot{x} = Ax$ is a linear combination of the terms of the form $t^k e^{\lambda t}, t^k e^{at} \cos bt, t^k e^{at} \sin bt$ where λ are the real eigenvalues and $a + \iota b$ and $a - \iota b$ are the complex eigenvalues.

- That completely determines the solution for all times, that is $\forall t \in \mathbb{R}$
- If some $\lambda > 0$, then clearly the solution $\to \infty$ as $t \to \infty$ asymptotically.
- The eigenvalues of A completely determine the nature of the solutions as $t \to \pm\infty$

From now the reference will be `Adrianova: Linear systems of differential equations - Chapter 1`
A few general terms about linear systems: For nonautonomous case $\dot{x} = A(t)x$. Assume that $A(t)$ is continuous. [1]

- Any linear combination of solutions is a solution.
- Any set of d linear independent solutions $\{x_1(t), x_2(t), \cdots, x_d(t)\}$ is a basis of the space of the solutions and is called the fundamental system of solutions.
- The matrice formed $X(t) = [x_1(t), \cdots, x_d(t)]$ is called as fundamental matrice and satisfies the solution.

$$\dot{X} = A(t)X$$

- $X(t)$ is normalized at $t = t_0$ if $X(t_0) = X_0 = I_d$
- $X(t)$ is invertible.
- If C is a non-singular matrice which is a constant, then $X(t)C$ is also a fundamental matrice.

  Question: Given one fundamental matrice $X(t)$, are all fundamental matrices of the form $X(t)C$ for some C? Is $CX(t)$ a fundamental matrice?
  Why is this relation useful, because given a fundamental matrice $X(t)$: the solution of $\dot{x} = Ax, \quad x(0) = x_0$ is $x(t) = X(t)\,[X(0)]^{-1}\,x_0$.

- Then the solution of inhomogeneous equation $\dot{x} = A(t)x + f(t), \quad x(0) = x_0$ is given by variation of parameter method

$$x(t) = X(t)\,[X(0)]^{-1}\,x_0 + \int_0^t ds\, X(t)\,[X(s)]^{-1}\,f(s)$$

- The fundamental matrice normalized at t_0 has an explicit form (it is called matriciant of the system)

$$X(t) = \Omega_{t_0}^t A = I_d + \int_{t_0}^t dt_1 A(t_1) + \int_{t_0}^t dt_1 A(t_1) \int_{t_0}^{t_1} dt_2 A(t_2) + \cdots + \int_{t_0}^t dt_1 A(t_1) \int_{t_0}^{t_1} dt_2 A(t_2) \cdots \int_{t_0}^{t_{k-1}} dt_k A(t_k)$$

  Proof that the above series converges for any $A(t)$ is very similar to the proof of existence of $\dot{x} = f(x)$

- It satisfies the assumptions of the flow of ordinary differential equation, that is

$$\Omega_{t_0}^t A = \Omega_{t_1}^t A\, \Omega_{t_0}^{t_1}$$

[1] continuous need not mean, the solutions will exists for all time

- If $A(t)$ and $B(s)$ commute for all t, s then

$$\Omega_{t_0}^{t}(A+B) = \Omega_{t_0}^{t} A \Omega_{t_0}^{t} B$$

- If $A(t) = A$ then $\Omega_{t_0}^{t} A = e^{A(t-t_0)}$

How does one define logarithm of a matrice?

$$B = \ln A$$

For the time periodic case $A(t+w) = A(t), \quad \forall t$, then

- If $X(t)$ is fundamental matrice, then $X(t+\omega)$ is also a fundamental matrice.
- Since two fundamental matrices are always related by nonsingular matrice, hence

  $$X(t+\omega) = X(t)B$$

  for some nonsingular B

- Note that B depends on $X(t)$ For a different set of $X_1(t)$ there will be $B_1 \neq B$.
- The matrice B for the fundamental matrice normalized at $t = 0$ is called **Monodromy** matrice.

  $$B = X(\Omega) = \Omega_0^{\omega} A$$

  in terms of the matriciant of the system.

- The eigenvalues of the monodromy matrice B are called **Multipliers**.
- No multipliers is zero, if the fundamental matrice is nonsingular at some time t_0, then it is nonsingular $\forall t$. Then taking determinant of the equation

  $$X(t+\omega) = X(t)B$$

  will prove that the eigenvalues cannot be zero.

Lecture 6

Notes on
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`Did not do anything, new just clarified doubts.`
Discussed about discontinuous dynamical systems, and seen that there is no general theory to study it. Each case has to be studied seperately.

Lecture 7

Notes on
Dynamical Systems
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28 January 2019
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Definition 1 ***Stability:***
Let $\phi_t(x_0)$ be a solution of $\dot{x} = f(x,t), \forall t \geqslant 0$, then this solution is stable if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that [1]

$$|\phi_t(y) - \phi_t(x_0)| < \varepsilon |y - x| < \delta \tag{1}$$

- $\phi_t(x_0)$ is called *asymptotically* stable if in addition to being stable ,

$$|\phi_t(y) - \phi_t(x_0)| \to 0, \quad t \to \infty$$

  As an example ,

$$\dot{x} = y \tag{2}$$
$$\dot{y} = -x \tag{3}$$

  is not asymptotically stable, just stable.

- Similarly the definition goes for unstable , which is not stable.

Note that in particular, unstable does not mean stable as $t \to -\infty$, the direction ($t \to \infty$ only defines stablility)

Consider $\dot{x} = A(t)x$ where

$$A(t) = \left\{ \begin{array}{cc} A & t > 10 \\ f(t) & t \in [-10, 10] \\ -A & t < -10 \end{array} \right\} \tag{4}$$

where $f(t)$ is smoothly varying function. This system is unstable in both directions.

In Damped harmonic oscillator, origin is a stable solution for $t \to \infty$ and the solution $x(t) \to \infty$ as $t \to -\infty$ is not always the case. And other solutions to the system (damped) stable or unstable ? Given an ε, what is the corresponding δ?

Can we cook a system, which goes very far but eventually comes back?, Is such a system even stable ?

Note: For autonomous linear systems: all eigenvalues with negative real part implies asymptotically stablility. This is not sufficient to conclude its stable for a non autonomous linear order differential equation. Their eigenvalues do not determine stability.

$$A(t) = \left[\begin{array}{cc} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\cos t \sin t & -1 + \frac{3}{2}\sin^2 t \end{array} \right] \tag{5}$$

Both the eigenvalues are $\frac{-1 \pm \iota\sqrt{7}}{4}$, `Ref: Hale, Pg 121.`
The two linear independent solutions are

$$(\cos t, \sin t)\, e^{\frac{t}{2}}$$

which $\to \infty$ as $t \to \infty$

Then what determines stability?

[1] We need first the solution $\phi_t(y)$ to exists $\forall t \geqslant 0$

We will construct a new theory called *Floquet Theory*

$$\dot{x} = A(t)x$$

and $A(t+\omega) = A(t)$.
If $X(t)$ is a fundamental solution, then so is $X(t+\omega)$.

$$X(t+\omega) = X(t)B$$

for some non-singular B. If $X(t)$ is the matriciant $\Omega_0^t A$ then $B = \Omega_0^\omega A$ is called *Monodromy* matrice and its eigenvalues are called *Multipliers*. Now these multipliers will clearly tell us about the stability of the system, not the original eigenvalues of $A(t)$. Note that for different fundamental solutions the B's are similar to each other, which makes its eigenvalues unique.

Theorem 1 *ρ is a multiplier, iff $\exists$ a non-trivial solution (that is $x(t) \equiv 0$) such that*

$$x(t+\omega) = \rho x(t)$$

and such a solution is called as normal solution.

Proof 1.1 *If ρ is a multiplier, which is eigenvalue of $X(\omega)$, then*

$$X(\omega)v = \rho v$$

for some v, then let

$$x(t) = X(t)v$$

which is normal

$$\begin{aligned} x(t+\omega) &= X(t+\omega)v && (6)\\ &= X(t)X(\omega)v && (7)\\ &= \rho X(t)v && (8)\\ &= \rho x(t) && (9) \end{aligned}$$

which proves $a \Rightarrow b$, now for the $b \Rightarrow a$, now assume a normal solution

$$x(\omega) = \rho x(0)$$

but

$$\begin{aligned} x(t) &= X(t)x(0) && (10)\\ &= \rho x(0) && (11)\\ &= \rho x(0) \quad (t = \omega) && (12) \end{aligned}$$

Thus $x(0)$ is a eigenvector of $X(\omega)$ with eigenvalue ρ

In particular, $\exists$ a ω periodic solution, iff $\rho = 1$ is a multiplier. Also $\exists$ a anti-periodic solution, if

$$x(t+\omega) = -x(t)$$

iff $\rho = -1$ is a multiplier, also the same system is 2ω periodic. Is ω or 2ω periodic solutions has asymptotically stability of $x(t) = 0$ No, the periodic solution of such kind is not asymptotically stability solution of $x(t) = 0$. If $\exists \rho > 1$, implies $\exists x(t+\omega) = \rho^n x(t)$ which $\to \infty$ as $n \to \infty$, therefore origin is not stable.

What about ρ, is it always real?, what if complex ? For ρ being complex we have

$$X(\omega)v_\pm = \rho_\pm v_\pm$$

Qn 1: is a multiplier real always?
Qn 2: Are eigenvalues of all fundamental solutions real?

Definition 2 *Logarithm of a matrice:*
If Y *is such that* $e^Y = x$ *then*

$$Y \underset{\text{def}}{:=} \ln X$$

for a square matrice X*. Any non singular matrice has a logarithm.*

Proof 1.2 *(By construction) using Jordan canonical form, if*

$$X = J_l(x) = \begin{bmatrix} \lambda & 1 & - \\ - & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

with λ *being real. This can be written as*

$$X = \lambda \left[I + \frac{1}{\lambda} J_l(0) \right]$$

and also $J_l(0)^l = 0$ *thus the series expansion will terminate at* J^{l-1}*. Now define*

$$Y = \ln \lambda I + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k\lambda^k} J_l(0)^k$$

which clearly exists since the series terminates. Note that theses matrice commute and so we can take the exponential of Y*. Lets denote* $Y = A + B$

$$\exp Y = e^A e^B \tag{13}$$
$$= \lambda e^B \tag{14}$$

Since for $z \in \mathbb{C}$

$$\exp \left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k} \right) = 1 + z$$

for $|z| < 1$

$$\sum_{s=0}^{\infty} \left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{M^k}{k} \right)^s \frac{1}{s!} = I + M$$

for a matrice M *for which the series converge then*

$$\exp B = I + \frac{1}{\lambda} J_l(0)$$

For general case

$$X = S.\mathrm{diag}\left(J_{l_1}(\lambda_1), \cdots, J_{l_k}(\lambda_k)\right) \cdot S^{-1}$$

then

$$\ln X = S \cdot (\ln J_{l_1}, \cdots,) \cdot S^{-1}$$

Thus we proved the existence of logarithm, note that λ is not unique. All of this works even for λ being complex.

Theorem 2 ***Floquet:***
The matriciant of $\dot{x} = A(t)x$ *with periodic* $A(t)$ *can be written in the form*

$$X(t) = \Omega_0^t A = \phi(t) e^{\Lambda t}$$

where $\phi(t + \omega) = \phi(t)$ *and* $\Lambda = \frac{1}{\omega} \ln(\Omega_0^{\omega} A)$

Proof 2.1 *define,*

$$\rho\Phi(t) = \left(\Omega_0^t A\right) e^{-\Lambda t}$$

and verify.

Lecture 8

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Last class, we stated and proved Floquet theorem, which was

$$x(t) = \Omega_{-}^{t} A = \phi(t) e^{\Lambda t}$$

with $\Lambda = \frac{1}{\omega} \ln \Omega_0^{\omega} A$

This gives a non singular time dependent transformation from

$$\dot{x} = Ax$$

to an autonomous ordinary differential equation. In general if

$$y = P^{-1}(t)x \tag{1}$$
$$x = P(t)y \tag{2}$$

where $P(t)$ is non singular $\forall t$, then we have

$$\dot{y} = \left(P^{-1}AP - P^{-1}\dot{P}\right) y \tag{3}$$
$$\tag{4}$$

If we choose $P = \phi$ from floquet theor then we have

$$\dot{y} = \left(\phi^{-1}A\phi - \phi^{-1}\dot{\phi}\right) y \tag{5}$$
$$\phi = xe^{-\Lambda t} \tag{6}$$
$$\dot{\phi} = A\phi - \phi\Lambda \tag{7}$$
$$\dot{y} = \phi^{-1}(\phi\Lambda)y \tag{8}$$
$$= \Lambda y \tag{9}$$

Thus the eigenvalues of λ determine the asymptotical properties. The y system is stable only iff all eigenvalues $\Re\{\lambda_i\} \leqslant 0$ and the 0 real eigenvalues should not have a non-trivial Jordan block, in other words it should have simple divisors. *Why?*

Ans: because such Jordan blocks have 1's which will give rise to solutions of form t^n, which will go to infinity. *Read algebraic and geometric multiplicity.*

In addition, it is exponentially stable, if all eigenvalues strictly have real part $\Re\{\lambda\} < 0$. The solution of y is

$$y(t) = e^{\Lambda t} y_0 \tag{10}$$
$$X(t)x_0 = \phi(t)e^{\Lambda t}x_0 \tag{11}$$
$$= \phi(t)y(t) \tag{12}$$
$$\tag{13}$$

From which the solution of $x(t)$ is obtained.

The equation $\dot{x} = Ax$ is stable , iff the characteristic multipliers are all of modulus $|\rho| \leqslant 1$ and those with $|\rho| = 1$ have simple divisors.
The following theorem is about the existence of periodic solutions.
Before lets define adjoint system as

Theorem 1 *Sometimes the Adjoint equation is written as*

$$\dot{z} = -zA$$

Proof 1.1 *If* $X(t)$ *is a fundamental solution of* $\dot{x} = Ax$ *then* $z(t) = \left[X^{-1}(t)\right]^{\dagger}$ *is a fundamental solution to the adjoint equation. Since*

$$Z^{\dagger}X = 1 \tag{14}$$

$$\dot{Z}^{\dagger}X + Z^{\dagger}AX = 0 \tag{15}$$

$$\dot{Z}^{\dagger} = -Z^{\dagger}A \tag{16}$$

$$\dot{Z} = -A^{\dagger}Z \tag{17}$$

$$\text{if} \quad Y = X^{-1} \quad \text{then} \tag{18}$$

$$\dot{Y} = -YA \tag{19}$$

Theorem 2 *If* $\dot{x} = Ax$ *has* ω *periodic solutions* $\{\phi_1, \phi_2, \cdots, \phi_k\}$ *that are linear independent solutions, then*

1. $\dot{z} = -A^{\dagger}z$ *is the adjoint equation. Also has* k *linear independent solutions* $\{\psi_1, \psi_2, \cdots, \psi_k\}$
2. ***Fredholm Alternative:*** $\dot{x} = Ax + f$ *with* $f(t+\omega) = f(t)$ *has* ω *periodic solution iff*

$$\int_0^{\omega} dt \, \langle \psi(t) f(t) \rangle = 0$$

for $s = 1, 2, \cdots, k$ *and the* ω *periodic solutions are a k-parameter family. For this to be true,* $\rho = 1$ *has to exist.*[1]

Proof 2.1 1. *If* ϕ *is a* ω *periodic solution. But* $\phi(t) = X(t)\phi(0)$ *and* $\phi(\omega) = \phi(0) = X(\omega)\phi(0)$ *or* $(I - X(\omega))\,\phi(0) = 0$ *and* $\phi(0) \neq 0$.

This is true for k linear independent $\{\phi_s(0)\} : s = 1, 2, \cdots, k$ *implies the rank of* $I - X(\omega)$ *is* $n - k$ *So* $\left(X^{-1}(\omega)^{\dagger} - I\right)\psi(0) = 0$ *has k linear independent solutions exists. (just act with* X^{-1} *from left)*

2. *Solution of* $\dot{x} = Ax + f$ *is*

$$x(t) = X(t)x_0 + \int_0^t ds \, X(t,s)\, f(s)$$

where $X(t,s) = X(t)X^{-1}(s)$.

$$x(\omega) = x(0) = X(\omega)x_0 + \int_0^{\omega} ds \, X(\omega, s)\, f(s)$$

or

$$[I - X(\omega)]\, x_0 = b = \int_0^t ds \, X(t,s)\, f(s)$$

$$\left(X^{-1}(\omega)^{\dagger} - I\right)\psi_k(0) = 0 \tag{20}$$

$$(I - X(\omega))^{\dagger}\,\psi_k(0) = 0 \tag{21}$$

$$0 = \left\langle (I - X(\omega))^{\dagger}\,\psi_j(0), x_0 \right\rangle \tag{22}$$

$$= \langle \psi_j(0), (I - X(\omega))\, x_0 \rangle \tag{23}$$

$$= \left\langle \psi_j(0), \int_0^{\omega} ds \, X(t,s)\, f(s) \right\rangle \tag{24}$$

$$= \left\langle X(\omega)^{\dagger}\psi_j(0), \int_0^{\omega} ds \, X^{-1}(s) f(s) \right\rangle \tag{25}$$

$$= \left\langle \psi_j(0), \int_0^{\omega} ds \, X^{-1}(s) f(s) \right\rangle \tag{26}$$

$$= \int_0^{\omega} ds \, \langle X^{-1}(s)^{\dagger}\psi_j(0), f(s) \rangle \tag{27}$$

$$= \int_0^{\omega} ds \, \langle \psi_j(s), f(s) \rangle \tag{28}$$

for every ω *periodic solution.*

[1] This inner product essentially says, in the k-dimensional subspace, the function $f(t)$ is not in the subspace, or orthogonal to it.

3. *Let* $\sum_{\sigma} \alpha_s \psi_s(t) = y(t)$ *with* $\int ds\ \langle \psi(s), f(s) \rangle = 0, \quad \forall s = 1, 2, \cdots, k$ *now we have to show this is an* ω *periodic solution. Since they are linear combination of* ψ_j*, they must satisfy*

$$\left(X^{-1}(\omega)^{\dagger} - I\right) \psi(0) = 0 \tag{29}$$
$$y(t) = \left[X^{-1}(t)\right]^{\dagger} y_0 \tag{30}$$
$$= \Sigma_j \alpha_j \psi_s(t) \tag{31}$$
$$\tag{32}$$

thus the Fredholm alternative is satisfied by $y(t)$

$$0 = \int_0^{\omega} dt\ \left\langle (X^{-1}(t))^{\dagger} y_0, f(t) \right\rangle \qquad = \int_0^{\omega} dt\ \left\langle y_0, X^{-1}(t) f(t) \right\rangle \tag{33}$$
$$= \int_0^{\omega} dt\ \left\langle X(\omega)^{\dagger} y_0, X^{-1}(t) f(t) \right\rangle \tag{34}$$
$$= \int_0^{\omega} dt\ \left\langle y_0, X(\omega) X^{-1}(t) f(t) \right\rangle \tag{35}$$
$$= \left\langle y_0, \int_0^{\omega} dt\, X(\omega) X^{-1}(t) f(t) \right\rangle \tag{36}$$

Refer: Hill's equation, One special form of it called as Mathieu Eqn.

$$\ddot{x} + (a + \phi(t))\, x = 0, \quad \phi(t + \omega) = \phi(t)$$

This comes in the problem, where moon's periodic motion in the effect of both sun and earth.

Ref: Adrianova : Ch 1

Lecture 9

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Discussed H.W 1 and 2. Nothing done.

Lecture 10

Notes on
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`Continuation of previous lecture (8),`

For a inhomogeneous system, $\dot{y} = A(t)y + f(t)$ with both A, f being ω periodic. If there are k periodic solutions to homogeneous system, then the adjoint also has k linear independent solutions. And these solution have

$$\int_0^w \langle \psi(s), f(s) \rangle \, ds = 0 \tag{1}$$

Notice that $y(t)$ is a periodic solution to $\dot{x} = Ax + f$

$$y(\omega) = X(\omega)y(0) + \int_0^\omega ds \, X(\omega, s)f(s) \tag{2}$$
$$\mathtt{iff} \tag{3}$$
$$(I - X(\omega))\, y(0) = \int_0^\omega ds \, X(\omega, s)f(s) \tag{4}$$
$$= b(say) \tag{5}$$

If $\left(X(\omega)^\dagger - I\right) z_0 = 0$, then

$$z(t) = X^{-1}(t)^\dagger z_0 = \sum c_s \psi_s$$

$$0 = \int_0^\omega dt \ (z(t), f(t)) \tag{6}$$
$$= \int \left((X^{-1})^\dagger X^\dagger(\omega) z_0, f(t)\right) dt \tag{7}$$
$$= \left(z_0, \int_0^\omega dt \, X(\omega)X^{-1}(t)f(t)dt\right) \tag{8}$$

And a general periodic solution is given by

$$y(t) + \sum c_s \phi_s(t) \tag{9}$$

Stability of a genera non autonomous system, we need to define **"Stability"**

Hint: Compare the solution $x(t)$ with $e^{\alpha t}$. So find λ for which

$$\frac{x(t)}{e^{\alpha t}} \to 0, \ \alpha > \lambda :\to \infty, \ \alpha < \lambda$$

Definition 1 *For a complex valued function,* $f(t)$ *defined on* $[t_0, \infty]$ *the characteristic exponent is defined as*

$$\chi[f] = \text{limsup}_{t\to\infty} \frac{1}{t} \ln |f(t)| \tag{10}$$

example , for $f(t) = e^{t \cos t}$ then the exponent is $\chi[f] = 1$.

Definition 2 *For limit supremum: let*

$$g(t) = \sup_{\tau \geqslant t} f(\tau)$$

, $g(t)$ *is non-increasing. so define*

$$\text{limsup}_{t\to\infty} f(t) = \lim_{t\to\infty} g(t)$$

and similarly if $h(t) = \sup_{\tau \geqslant t}$ *then*

$$\text{liminf}_{t\to\infty} f(t) = \lim_{t\to\infty} h(t)$$

where we allow $\pm\infty$ *as limsup and liminf*

This exponent, cares only about how high it goes in the limit $t \to \infty$*, not its local peaks*
The other possible definition is that limsup is set of limits along a sequence $t_k \to \infty$,

This exponents is called *Lyapunov* exponents
Note that

$$\lim_{t\to\infty}$$

exists iff

$$\text{limsup} f(t) = \text{liminf} f(t)$$

Few examples,

$$\begin{aligned}
\chi[e^{\lambda t}] &= \lambda && (11)\\
\chi[t^n] &= \text{limsup}_{t\to\infty} \frac{1}{t} \ln t^n && (12)\\
&= 0,\ \forall n && (13)\\
\chi[\exp[t\cos(\omega t + \phi)]] &= 1 && (14)\\
\chi[0] &= -\infty && (15)\\
\chi[\exp(te^{\sin t})] &= e && (16)\\
\chi[\exp(-te^{\sin t})] &= -\frac{1}{e} && (17)\\
\chi[\exp(e^{t^2})] &= \text{limsup} \frac{1}{t} t^2 && (18)\\
&= \infty && (19)\\
\chi[t^t] &= \infty \chi[e^{-\lambda t^2}] \qquad = \frac{-\lambda t^2}{t} && (20)\\
&= -\infty && (21)
\end{aligned}$$

[1] Which of these are solutions of a linear ordinary differential equation?

$$e^{\lambda t}$$

is defnitely yes.
How about $x(t) = e^{t\cos(\omega t)}$,

$$\dot{x} = [\cos(\omega t) - \omega t \sin(\omega t)]\, x$$

so for this $A(t)$ we will get, but it is not bounded.

$$\dot{x} = 2tx, \Rightarrow x(t) = e^{t^2}$$

$$\frac{d}{dt} x(t) = \frac{d}{dt} e^{\ln x(t)}$$

$$(\frac{\dot{x}}{x}) x = A(t) x$$

once the solution $x(t)$ is differentiable.

---

[1] $\text{limsup}(-f(t)) = -\text{liminf}(f(t))$

Properties of lyapunov exponents

- $\chi[cf(t)] = \chi[f]$
- If there exists a τ such that $f(t) \leqslant g(t)$ for $t \geqslant \tau$ then $\chi[f] \leqslant \chi[g]$
- We see the function is growing almost as fast as its exponent.

Lemma 0.1 *For $\chi[f] = \alpha \neq \pm\infty$, iff for $\varepsilon > 0$*

$$\lim_{t\to\infty} \frac{|f(t)|}{e^{\alpha t + \varepsilon t}} = 0 \tag{22}$$

$$\text{limsup}_{t\to\infty} \frac{|f(t)|}{e^{(\alpha-\varepsilon)t}} = \infty \tag{23}$$

The following statements may not be correct ! Beware
But try to study it

$$\chi[f] = -\infty$$

iff

$$\lim_{t\to\infty} \frac{|f(t)|}{e^{\alpha t}} = 0,\ \forall \alpha \in \mathbb{R}$$

similarly

$$\chi[f] = \infty$$

iff

$$\text{limsup}_{t\to\infty} \frac{|f(t)|}{e^{\alpha t}} = \infty,\ \forall \alpha \in \mathbb{R}$$

The characteristic exponents of $\{f_1 + f_2 + \cdots + f_n\}$ is given as

$$\chi\left[f_1 + \cdots + f_n\right] \leqslant \max_k \chi[f_k] \tag{24}$$

And the equality hold if only one f_k has the maximum $\chi[f]$.

Lecture 11

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`Continuation of previous lecture (10),`
If $\chi[f] = \alpha \neq \pm\infty$, then

Theorem 1 $\forall \varepsilon > 0$ *we have*

- $$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t+\varepsilon t}} = 0, \Rightarrow \chi[f] \leqslant \alpha$$

- $$\text{limsup}_{t\to\infty} \frac{|f(t)|}{e^{\alpha t-\varepsilon t}} = \infty, \Rightarrow \chi[f] \geqslant \alpha$$

Proof 1.1 *1. Assume* $\chi[f] = \alpha$

- *For any* $\varepsilon/2 > 0$, $\exists T > 0, \forall t > T$ *we have*

$$\frac{1}{t} \ln |f(t)| < \alpha + \varepsilon/2$$

thus

$$\lim_{t\to\infty} \frac{|f(t)|}{e^{\alpha t+\varepsilon t}} < \lim_{t\to\infty} e^{-\frac{\varepsilon}{2}t} \tag{1}$$
$$= 0 \tag{2}$$

- *Let* $t_k \to \infty$ *as* $k \to \infty$ *be a sequence such that*

$$\lim_{k\to\infty} \frac{1}{t_k} \ln |f(t_k)| = \alpha \tag{3}$$
$$\forall \varepsilon > 0, \exists K, \quad k > K \tag{4}$$
$$\frac{1}{t_k} \ln |f(t_k)| > \left(\alpha - \frac{\varepsilon}{2}\right) \tag{5}$$
$$\lim_{k\to\infty} \frac{|f(t_k)|}{e^{(\alpha-\varepsilon)t_k}} \geqslant \lim_{k\to\infty} e^{\varepsilon t_k/2} \tag{6}$$
$$\tag{7}$$

2. Assume the above and prove $\chi[f] = \alpha$

- *Fix* ε $\exists T$ *such that* $\forall t > T$

$$|f(t)| < e^{(\alpha+\varepsilon)t}$$
$$\chi[f] \leqslant \alpha$$

- *Let* $t_k \to \infty$ *such that* $|f(t_k)| > e^{(\alpha-\varepsilon)t}$

$$\chi[f] \geqslant \lim_{k\to\infty} \frac{1}{t_k} \ln |f(t_k)| \geqslant \alpha - \varepsilon$$
$$\chi[f] \geqslant \varepsilon$$

thus $\chi[f] = \alpha$

To prove

Theorem 2

$$\chi\left[\sum_{k=1}^{N} f_k\right] \leqslant \max_{\{k\}}\chi[f_k] = \alpha$$

and the equality occurs iff only one of the functions have the maximum exponent.

Proof 2.1 *First we prove the inequality*

$$\lim_{t\to\infty} \frac{|\sum f_k(t)|}{e^{(\alpha+\varepsilon)t}} \leqslant \sum \lim_{t\to\infty} \frac{|f_k(t)|}{e^{(\alpha+\varepsilon)t}} \tag{8}$$

$$= 0 \tag{9}$$

$$\chi\left[\sum f_k\right] \leqslant \alpha \tag{10}$$

To prove the equality, assume that $\chi[f_1] = \alpha > \chi[f_k], k \neq 1$

$$\mathrm{limsup}_{t\to\infty} \frac{|f_1(t)|}{e^{(\alpha-\varepsilon)t}} = \infty \tag{11}$$

$$\mathrm{limsup}_{t\to\infty} \frac{|\Sigma f_k(t)|}{e^{(\alpha-\varepsilon)t}} \geqslant \mathrm{limsup}_{t\to\infty} \left(\frac{|f_1(t)|}{e^{(\alpha-\varepsilon)t}} + \Sigma \frac{|f_k(t)|}{e^{(\alpha_k+\varepsilon)t} e^{(\alpha-\alpha_k-2\varepsilon)t}} \right) \tag{12}$$

$$\mathsf{for} \quad 0 < \varepsilon < \alpha - \max_k \alpha_k \tag{13}$$

$$\geqslant \infty \tag{14}$$

Theorem 3

$$\chi\left[\prod_k f_k\right] \leqslant \Sigma_k \chi[f_k]$$

$$\mathrm{limsup}_{t\to\infty} \frac{1}{t} \ln |\prod_k f_k| \leqslant \Sigma_k \mathrm{limsup}_{t\to\infty} \frac{1}{t} \ln |f_k(t)|$$

equality if the limit exists

Do we need the condition that $\chi[f_k] \neq \pm\infty, \forall k$*, for the above?*

Now we have define these characteristic exponents for matrices and vectors. Before that we define something called as **Sharp exponent**.

Definition 1 *An characteristic exponent is called sharp if*

$$\lim_{t\to\infty} \frac{1}{t} \ln |f(t)| = \alpha$$

Theorem 4 *If* f *has a sharp exponent, iff*

$$\chi[f] + \chi[\frac{1}{f}] = 0$$

Proof 4.1 *1. Assume the existence of limit then*

$$-\chi[f] = -\lim_{t\to\infty} \frac{1}{t} \ln |f(t)| \tag{15}$$

$$= \chi[\frac{1}{f}] \tag{16}$$

We can bring the − sign inside the limit not if it is limsup.

2. To prove the converse

$$-\chi[f] = -\mathrm{limsup} \frac{1}{t} \ln |f(t)| \tag{17}$$

$$= \mathrm{liminf} \frac{1}{t} \ln |\frac{1}{f}| \tag{18}$$

$$= \mathrm{limsup} \frac{1}{t} \ln \frac{1}{|f|} \tag{19}$$

$$= \chi[\frac{1}{f}] \lim_{t\to\infty} \frac{1}{t} \ln |f(t)| \tag{20}$$

$$= \alpha \tag{21}$$

Definition 2 *If* $F(t)$ *is a* $n \times n$ *matrice then , define characteristic exponent of* F *as*

$$\chi[F] = \max_{i,j} \chi[F_{i,j}]$$

Theorem 5

$$\chi[F] = \chi[\| F \|]$$

for any matrice norm that satisfies

- $|F_{ij}| \leqslant \| F \|, \quad \forall i, j$
- $\| F \| \leqslant \sum_{i,j} |F_{ij}|$

Proof 5.1 *Proof is straightforward, use the first criteria to prove*

$$\chi[|F_{ij}|] \leqslant \chi[\| F \|]$$

and second to prove the reverse inequality. Thus they are equal

Definition 3 *Two norms are equivalent if*

$$k_2 \| F \|_2 \leqslant \| F \|_1 \leqslant k_1 \| F \|_2$$

for all F*.*

In finite dimensional spaces, all norms are equivalent.

Now we could define, for vectors $\chi[v] = \chi[|v|], \quad v \in \mathbb{R}^d$. The same result for the sum and product applies for this also, that is

$$\chi[\Sigma F_k] \leqslant \max \chi[F_k], \quad \chi\left[\prod F_k\right] \leqslant \Sigma \chi[F_k]$$

and similarly

$$\chi[\Sigma c_k v_k] \leqslant \max_k \chi[v_k], \quad \forall v \in \mathbb{R}^d$$

Lecture 13

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Regularity Coefficients

Let V be a vector space and V^* be its dual vector space. If $\{v_i\}$ and $\{v_i^*\}$ forms basis for V and V^* respectively then

$$\left\langle v_i, v_j^* \right\rangle = \delta_{ij}$$

$\forall i, j \in \{1, 2, \cdots, p\}$. Now we define the exponents for these dual basis as $\chi[v_i]$ and $\chi[v_i^*]$ respectively. Note that irrespective of the basis, the vector space which is finite dimensional has finite and different exponents. Let's denote the exponent of V as $\{\{\chi_1, \chi_2, \cdots, \chi_p\}\}$ and similarly for V^* as $\{\chi_1^*, \chi_2^*, \cdots, \chi_p^*\}$.

Given a dual basis $\{v_i\}, \{v_i^*\}$, we define the exponents to be dual $\forall 0 \leqslant i \leqslant p$

$$\chi[v_i] + \chi^*[v_i^*] \geqslant 0$$

which is denoted by $\chi \sim \chi^*$ for simplicity, assume that the exponents χ_i, χ_i^* are arranged in increasing and decreasing order respectively, that is

$$\chi_1 \leqslant \chi_2 \leqslant \cdots \chi_p$$
$$\chi_1^* \geqslant \chi_2^* \geqslant \cdots \chi_p^*$$

counted along with their multiplicities.

Definition 1 *The Regularity coefficient is defined as*

$$\gamma(\chi, \chi^*) = \min \quad \max\{\chi[v_i] + \chi^*[v_i^* : 1 \leqslant i \leqslant p\} \tag{1}$$

where the minimum is for dual basis and maximum is for all exponents within a basis. The Perron coefficient is defined as

$$\pi(\chi, \chi^*) = \max\{\chi_i + \chi_i^* : 1 \leqslant i \leqslant p\} \tag{2}$$

The regularity coefficient takes each pairs of basis, for which the exponents need not be in ordered and finds the maximum sum between the space and its dual and finally gets the minimum amongst all basis. If we think closely, it looks like the normal basis would win. Whereas the perron coefficient, takes the ordered exponents of the space and its dual and gets its maximum. We state two following statements regarding them

Theorem 1 *1.*

$$\pi(\chi, \chi^*) \leqslant \gamma(\chi, \chi^*)$$

2.

$$\chi \sim \chi^*,$$

then we have

$$0 \leqslant \pi(\chi, \chi^*) \leqslant \gamma(\chi, \chi^*) \leqslant p\pi(\chi, \chi^*)$$

Definition 2 *The pair of lyapunov exponents (χ, χ^*) is said to be regular, if $\chi \sim \chi^*$ and $\gamma(\chi, \chi^*) = 0$. Which implies that $\pi(\chi, \chi^*) = 0$ and also iff $\chi_i = -\chi_i^*$.*

Now the important theorem follows

Theorem 2 *If the pair* (χ, χ^*) *is regular then the filtrations* $\mathcal{V}_\chi, \mathcal{V}_{\chi^*}$ *are orthogonal, that is the number of subspaces formed are equal* $s = s^*$*,*

$$\mathrm{dim} V_i + \mathrm{dim} V^*_{s-i} = p$$

and the vectors from these subspaces are orthonormal

$$\langle v, v^* \rangle = 0, \quad \forall v \in V_i, v^* \in V^*_{s-i}$$

Consider the dynamical system $\dot{x} = A(t)x$ and its adjoint $\dot{z} = -A^\dagger(t)z$. Their exponents are dual to each other. Let $v(t)$ and $v^*(t)$ be the solutions to them, then

$$\frac{d}{dt}\langle v(t), v^*(t) \rangle = \langle A(t)v(t), v^*(t) \rangle + \left\langle v(t), -A^\dagger(t)v^*(t) \right\rangle \tag{3}$$

$$= \langle A(t)v(t), v^*(t) \rangle - \langle A(t)v(t), v^*(t) \rangle \tag{4}$$

$$= 0 \tag{5}$$

$$\langle v(t), v^*(t) \rangle = \langle v(0), v^*(0) \rangle \tag{6}$$

Now let $\{v_i\}, \{v_i^*\}$ be the dual basis then

$$\| v_i(t) \| \| v_i^*(t) \| \geqslant 1 \tag{7}$$

$$\chi[v_i] + \chi[v_i^*] \geqslant 0 \tag{8}$$

Thus exponents of the system and its adjoint are dual to each other.

Lecture 14

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`Recap (from Adrianova:`

$$\chi : V \Rightarrow \mathbb{R},\ \chi[0] = -\infty, \chi[\alpha v] = \chi[v], \chi[v+w] = \max\{\chi_v, \chi_w\}$$

- Distince lyapunov exponents implies linear independence amongst them.
- At most d lyapunov exponents in a d dimensional vector space.
- If χ is lyapunov exponent iff there exists adapted filtration $\chi_1 < \chi_2, \chi_3 < \cdots < \chi_s$ and the space $0 = V_0 \subsetneq V_1 \cdots \subsetneq V_s = V$ such that $\chi[v] = \chi_i$ for all $v \in V_i\ V_{i-1}$
- If $\{v_i\}$ is a normal basis then $\sum_i \chi[v_i] = \min \sum_i \chi[w_i]$ for anyother basis $\{w_i\}$
- The $\{v_i\}$ is a normal basis if $\{v_1, v_2, \cdots, v_{d_i}\}$ is a basis for V_{d_i}
- $\chi* : V* \Rightarrow \mathbb{R}$ is dual to χ if for any dual basis $\{v_i^*\}$ we have
$$\chi[v_i] + \chi[v_i^*] \geqslant 0, \forall i$$
- **Reg. coefficient**: $\gamma(\chi, \chi^*) = \min \max\{\chi[v_i] + \chi^*[v_i^*]\}$, the minimum is over all dual basis, and maximum is over index i
- **Perron. coefficient**. $\pi(\chi, \chi^*) = \max\{\chi_i, \chi_i^*\}$ where $\{\chi_i\}$ are ordered in increasing order and χ_i^* are ordered in decreasing fashion. Multiplicity accounted for
- χ, χ^* are regular if dual and $\gamma(\chi, \chi^*) = 0$ iff $\pi(\chi, \chi^*) = 0$ iff $\chi_i + \chi_i^* = 0$

For the solution of $\dot{x} = Ax$, the definition

$$\chi[x] = \overline{\lim}_{t\to\infty} \ln|x(t)|$$

is a lyapunov exponent and χ^* is dual for the solution of the adjoint system.

$$\dot{z} = -A^\dagger(t) z$$

.

- (χ, χ^*) are regular iff
$$\lim_{t\to\infty} \frac{1}{t} \ln(\Delta(t)) = \sum_{i=1}^{d} \chi_i$$
where $\Delta(t) = \exp\left[\int_0^t ds\, \mathrm{tr} A(s)\right]$
- For any normal ordered basis $\{v_i\}$ $L = \lim_{t\to\infty} \frac{1}{t} \ln(\Gamma_m(t))$ exists where $\Gamma_m(t) = \mathrm{Vol}(v_1(t), \cdots, v_m(t))$
- If (χ, χ^*) are regular then $L = \sum_{i=1}^{m} \chi[v_i]$

Proof will be done in tutorial

Lets look at for a two dimensional system

$$\Gamma_2(t) = |\nu_2(t)| \quad |\nu_1(t)| \sin\theta(t)$$
$$= e^{\chi_1 t} e^{\chi_2 t} \sin\theta(t)$$

but $\Gamma_2(t) = e^{(\chi_1+\chi_2)t}$ therefore the angle cannot decrease exponentially.

For $A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ check the angle goes to zero between two vectors.
Solution: The angle goes to zero as $\frac{1}{t}$

If ν_1, ν_2 have different lyapunov exponents then $\theta(t) \neq 0, t \to \infty$, **WHy?**

`Ref: Adrianova, Section 2.4`

Definition 1 *For a fundamental matrice* $X(t) = \{x_1, x_2, \cdots, x_d\}$ *we denote*

$$\sigma_x = \sum_{i=1}^{d} \chi[x_i]$$

For $\dot{x} = Ax$ with $A = \text{diag}\{1, 2, 3\}$

- $\sigma_{e_1,e_2,e_3} = 6$
- $\sigma_{e_1+e_3,e_2,e_3} = 8$

Definition 2 X *is normal fundamental matrice, if* σ_x *is minimal.*

By theorem 2.5 of `Pesin and Barera`, the normal is equal to subordinate.

Definition 3 $\{x_1, x_2, \cdots, x_k\}$ *is incompressible if*

$$\chi[\sum_i \alpha_i x_i] = \max(\chi[x_i], \alpha_i \neq 0)$$

Theorem 1 X *is normal iff incompressible.*

See the `Review of Modern Physics: Eckmann and Roulle, 1985`, for the definition of lyapunov exponents. Try to make the connection with that definition to ours.

Lecture 15

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For linear autonomous system $\dot{x} = Ax, x \in \mathbb{R}^d$ there are invariant subspaces.

The generalized eigenvectors of A being $u_k \pm \iota v_k$ and with eigenvalues $a_k \pm \iota b_k$.

1. The stable subspace defined as
$$\mathcal{E}^s = \text{span}\{u_k, v_k | a_k < 0\}$$
if $v \in \mathcal{E}^s$ then $\chi[v] < 0$
2. The unstable subspace is defined as
$$\mathcal{E}^u = \text{span}\{u_k, v_k | a_k > 0\}$$
if $v \in \mathcal{E}^u$ then $\chi[v] > 0$
3. The center is similarly defined with $a_k = 0$ and its exponent $\chi[v_{\text{center}}] = 0$

Now for nonlinear autonomous system $\dot{x} = f(x)$, with $x*$ being a fixed point. Then the linearization of the solution around $x*$ is given by
$$\dot{z} = Az, \quad A = Df(x*)$$
which is an autonomous linear system, which consequently has the invariant subspaces $\mathcal{E}^s, \mathcal{E}^u, \mathcal{E}^c$ respectively.

What is the behaviour of solutions of $\dot{x} = f(x)$ for $x(0) = x * + \delta x$?

let $\{d_s, d_u, d_c\}$ be the dimensions of those invariant subspaces. If $d_c = 0$ the there exists a manifold of initial conditions that behave like those on $\mathcal{E}^u$ or $\mathcal{E}^s$. This is the rough idea of the **Stable manifold theorem**.

Theorem 1 *Let $\dot{x} = f(x)$, where $x \in \mathbb{R}^d$, $f \in C^2$ on an open neighbourhood of $x = 0$ and $f(0) = 0$ and the flow is defined $\forall t \in \mathbb{R}$ in the neighbourhood of 0. Then the* ***Stable manifold theorem*** *says the following:*

Suppose $Df(0)$ has k eigenvalues with negative real part and $d - k$ eigenvalues with positive real part. Then there exists a smooth mapping $\Psi : \mathbb{R}^k \to \mathbb{R}^{d-k}$ such that the set defined by $\alpha\,(y, \Psi(y)) \in \mathbb{R}^d, \quad y \in \mathbb{R}^k$ is a k dimension manifold. α is a homeomorphism $\alpha : \mathbb{R}^d \to \mathbb{R}^d$. This manifold is tangent to $\mathcal{E}^s$ and this set is called as the ***Stable manifold*** *of this nonlinear system.*

Ψ is defined only in some neighbourhood of $N \subset \mathbb{R}^k$ of origin O. If $x_0 = (y, \Psi(y))$ for $y \in N$ then

1. *$x(t)$ belongs to stable manifold.*
2. *$x(t) \to 0, \ t \to \infty$*

In addition, there exists a mapping $\chi : \mathbb{R}^{d-k} \to \mathbb{R}^k$ such that the ***Unstable manifold*** *is defined by a mapping $\beta\,(y, \chi(y)) \in \mathbb{R}^d$ for $y \in \mathbb{R}^{d-k}$ is tangent to $\mathcal{E}^u$ and*

1. *$x(t)$ belongs to unstable manifold for $x(0)$ in unstable manifold.*
2. *$x(t) \to 0, \quad t \to -\infty$*

The proof is constructive in nature.

Note that the invariant subspaces of Linearized system $\mathcal{E}^s, \mathcal{E}^u$ are not orthogonal. Thus the stable and unstable manifold will not intersect orthogonally

Consider in $d = 3$, and $\Psi : \mathbb{R} \to \mathbb{R}^2$. Then every mapping will give a produce a curve parameterized by (x, Ψ_1, Ψ_2). Now if the mapping Ψ is defined as $\Psi : \mathbb{R}^2 \to \mathbb{R}$. Then the mapping produces a surface $\Psi(x, y)$. The dimension of set in the former case is 1 and in the latter is 2. In general, the dimension of the set defined is k.

`Ref: Perko`

Consider the example

$$f(x) = \begin{bmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{bmatrix}$$

The $Df(x)$ will be

$$Df = \begin{bmatrix} -1 & 0 & 0 \\ 2x_1 & -1 & 0 \\ 2x_1 & 0 & 1 \end{bmatrix}$$

Origin is the only fixed point. For which the

$$Df = \text{diag}\{-1, -1, 1\}$$

for which invariant subspaces are

- $\mathcal{E}^s$ is $x - y$ plane
- $\mathcal{E}^u$ is z axis

The solutions are

$$x_1 = c_1 e^{-t}$$
$$x_2 = c_2 e^{-t} + c_1^2 \left(e^{-t} - e^{-2t}\right)$$
$$x_3 = c_3 e^t + \frac{c_1^2}{3} \left(e^t - e^{-2t}\right)$$

Thus the stable and unstable manifold are

1. $S = \{c \in \mathbb{R}^3 | c_3 = -\frac{c_1^2}{3}\}$
2. $U = \{c \in \mathbb{R}^3 | c_1 = c_2 = 0\}$

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Stable manifold theorem, says if E^s, E^u are linearization subspaces around the hyperbolic fixed point x_0, then there exists a manifolds S, U corresponding to stable and unstable manifolds tangent to E^s, E^u respectively. These manifolds are forward and backward time invariant, respectively.

A very similar theorem to that is **Hartman Grobman theorem**

Theorem 1 *Assuming same conditions as for stable manifold theorem, with origin being the fixed point* $f(0) = 0$*, with no eigenvalue with zero real part for the linearized system. Let* $\phi^t_{t_0}$ *be the flow for the system* $\dot{x} = f(x)$*. And the linearization is given by* $\dot{z} = Az$*, with* $A = Df(0)$*. Then there exists a neighbourhood* N *of* 0 *and an interval* $I_0\mathbb{R}$ *containing* 0 *and a homeomorphism (continuous, one-one ,invertible)* H *such that* $\forall x_0 \in N$*,* $\forall t \in I_0$

$$H \circ \phi^t_{t_0}(x_0) = e^{At}\left(H \circ x_0\right)$$

Stable manifold theorem does not imply Hartman Grobman theorem, and viceversa. Because, there is no condition for asymptotically convergence in Hartman grobman, and there is no condition given for initial condition which is not in U *or* S *in Stable manifold*

The similarities are that they are both local in phase space, Hartman is local in time, stable manifold is asymptotical in time.

In general, two different flows ϕ, ψ are equivalent, with their domains $\Omega_\phi, \Omega_\psi \in \mathbb{R}^{d+1}$ and flow defined as $\phi : \Omega_\phi \to \mathbb{R}^d, \psi : \Omega_\psi \to \mathbb{R}^d$ if there is a homeomorphism h such that

$$h : \Omega_\phi \to \Omega_\psi$$

which involves mapping of the phase space and reparameterize the time such that

$$h \circ \phi(t, x_0) = \psi\left(h \circ (t, x_0)\right)$$

Hartman Grobman theorem does not involve a reparameterization of time

Theorem 2 $\dot{x} = f(x)$ *with* f *being* C^2 *in* $\mathbb{R}^n$ *with a flow* ϕ *(need not be defined for* $\forall t \in \mathbb{R}$*. Then the initial value problem,* $\dot{z} = \frac{f(z)}{1+|f(z)|}$ *with* $z(0) = x_0$ *has a unique solution* $\forall t \in \mathbb{R}$*,* $\forall x_0 \in \mathbb{R}^n$*, and the flow is topologically equivalent to* ϕ

Proof 2.1 *Consider a reparameterization of time given by*

$$\tau = \int_0^t ds\ \left[1 + |f\left(x(s)\right)|\right]$$

with the mapping $t \to \tau$ *should be one-one and differentiable. This is defined for* $t \in (\alpha, \beta)$ *which is the maximum interval of existence.*

$$\frac{dz}{d\tau} = \frac{dz}{dt}\frac{dt}{d\tau}$$

$$\frac{d\tau}{dt} = 1 + |f\left(x(t)\right)|$$

When $\tau \to ?$ *when* $t \to \alpha, \beta$

1. *$\beta = \infty$, then $\tau \to \infty, t \to \infty$*

2. *$\beta < \infty$, then we know that the solution will cannot remain inside any compact set as $t \to \beta$. Hence $\tau \to \infty$. and similarly it holds for α also.*

This reparameterization is not unique. All it needs is to map any interval of existence in the original system to whole of $\mathbb{R}$ in the new time

What type of dynamical systems are Topologically equivalent?

H.G theorem is local topologically equivalence between any hyperbolic non linear system and linear system. In general, the time will run differently for different initial condition, if we reparameterize it.

Numerical computation of unstable manifold

$\forall$distinct eigenvalues $a_k + \iota b_k,\ a_k > 0, b_k = 0$. Consider a initial condition $x_{0i} = \alpha_i v_k, 0 < \alpha_i < \varepsilon$ for small ε then evolve the system, and define a smooth curve passing through $x_i(t_*), i = 1, \cdots, N$ for small enough t_*. The existence of smooth curve is given by stable manifold theorem. Now increase the number of points on this smooth curve.

Consider a simple pendulum with a perturbation, that is

$$\mathcal{H} = \frac{p^2}{2} + \cos\theta + f(t)$$

, the perturbation is time dependent.

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Proof for Stable manifold theorem.

Proof 0.1 *Suppose* $\dot{y} = \begin{bmatrix} P_{k\times k} & 0 \\ 0 & Q_{n-k\times n-k} \end{bmatrix} y + g(y)$ *with all eigenvalues of* $P < 0$ *and* $Q > 0$ *with* $g(y)$ *being nonlinear and continuously differentiable, clearly the stable subspace* $E^s = \text{span}(e_1, e_2, \cdots, e_k)$ *and the unstable subspace* $E^u = \text{span}(e_{k+1}, \cdots, e_n$*. We want a* k *dimensional surface in* n *dimensional space which is tangent to* $x_1, x_2, \cdots, x_n$ *space.*

$$y_j = \psi_j\left(y_1, \cdots, y_k\right); j = k+1, \cdots, n$$

where $n-k$ *equations in* n *variables define* k *dimension surface. In general we can have a set of functions* $\chi_j\left(y_1, \cdots, y_n\right),\ j = 1, 2, \cdots, n-k$ *with some conditions for invertibility (**Implicit Function theorem**) can be used to the get the* ψ_j *mentioned earlier. The fact that this surface is tangent to our fixed point gives us the condition* $\psi_j(0) = 0, \nabla\psi_j(0) = 0$.
Let $B = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ *then*

$$e^{Bt} = U(t) + V(t)$$

where $U(t) = \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix}, V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix}$ *then* $\exists\alpha > 0$*such that* $\Re(\lambda_j) < -\alpha < 0$ *so that* $U \to 0, t \to \infty$ *and* $V \to 0, t \to -\infty$ *with* $\lambda_j^r = \Re(\lambda_j)$ *we have*

$$e^{\lambda_j^r} < e^{-\alpha t},\ \forall j, \forall t > 0$$

$$\| U(t) \| < Ke^{-(\alpha+\sigma)t}$$

for sufficiently small $\sigma > 0, t > 0$ *and*

$$\| V(t) \| < Ke^{\sigma t}, t < 0$$

1. *Step 1: Show that*

$$u(t,a) = U(t)a + \int_0^t ds\, U(t-s)G\left(u(s,a)\right) - \int_t^\infty ds\, V(t-s)G\left(u(s,a)\right)$$

   we need to show, that this equation solves

$$\dot{y} = Ay + G(y)$$

   Recall; solution of $\dot{x} = Ax + f(t)$ is $e^{At}x_0 + \int_0^t ds\, f(s)e^{A(t-s)}$
   So verify that the above equation is solution

2. *Step 2:* $\psi_j(a_1, \cdots, a_k) = U_j(0, a)$ *where* $a = (a_1, \cdots, a_k, 0, \cdots, 0)$ *gives us the required functions to define the stable manifold, that is* $\psi_j(0) = 0, \nabla\psi_j(0) = 0$

Existence of $u(t,a)$ *by induction, define a sequence of functions* $u^{(0)}(t,a) = 0$ *and*

$$u^{(j+1)}(t,a) = U(t)a + \int_0^t ds\, U(t-s)G\left(u^{(j)}(t,s)\right) - \int_t^\infty ds\, V(t-s)G\left(u^{(j)}(s,a)\right)$$

need to prove the sequence converges by showing

$$|u^j - u^{j-1}|(t,a) < K|a|e^{-\alpha t}\frac{1}{2^{j-1}}$$

which is

$$|G(u^j) - G(u^{j-1})| < C|u^j - u^{j-1}|$$

by noting that G *is lipshitz continuous, and use the induction assumption to prove*

$$|u(t,a)| < 2K|a|e^{-\alpha t}$$

This can be used as an algorithm to calculate u. *Note that the integral with* $U(t-s)$ *has last* $n-k$*components being zero, and the other has first* k *components being zero. Let* $a = (a_1, a_2, \cdots, a_k, 0, \cdots, 0)$ *so for* $j = k+1, \cdots, n$

$$u_j(0,a) = \left[-\int_0^\infty ds\, V(-s) G\left(u(s,a)\right) \right]_j$$

define $\psi_j(a_1, \cdots, a_k)$ *and* $u(t,a)$ *satisfying* $\dot{y} = By + G(y), \quad G(0) = 0$ *verify that* $u(t,0) = 0$ *and then* $\psi(0) = 0$

$$\frac{\partial \psi}{\partial a} = -\int_0^\infty ds\, V(-s) Dg\left(u(s,a)\right) \cdot \frac{\partial u(s,a)}{\partial a}$$

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Proof for Stable manifold theorem. (continued . . .)

We saw for the system, $\dot{y} = By + G(y)$ with $B = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$, where P has k negative eigenvalues and Q has $n-k$ positive eigenvalues. Then we proposed a solution as

$$u(t,a) = U(t)a + \int_{-}^{t} ds\, U(t-s)G\left(u(s,a)\right) ds - \int_{t}^{\infty} ds\, V(t-s)G\left(u(s,a)\right) \tag{1}$$

$$\tag{2}$$

Then define a function $\psi_j\left(\{a_1, a_2, \cdots, a_k\}\right) := u_j\left(0, (\{a_1, a_2, \cdots, a_k\}, 0, \cdots, 0)\right)$ for $j = k+1, \cdots, n$. Then $y_j = \psi_j\left(\{y_1, y_2, \cdots, y_k\}\right)$ defines the stable manifold S which is invariant and tangent to E^s.

This is only when B *is diagonal.*

For a general case with $\dot{x} = Ax + f(x)$ with k eigenvalues of A having negative real part, and remaining being positive real part. Let C be such that $B = C^{-1}AC$. If we write $y = C^{-1}x$ then

$$\dot{y} = C^{-1}ACy + C^{-1}f(Cy) = By + G(y)$$

, thus the stable manifold is given by

$$\begin{aligned} S_x &= \{x \in \mathbb{R}^n | x = Cy, \forall y \in S\} && (3) \\ &= \{x\mathbb{R}^n | x = Cy, y = \{y_1, y_2, \cdots, y_k, \psi_{k+1}, \psi_{k+2}, \cdots, \psi_n\}\} && (4) \end{aligned}$$

At most, we could always find a Jordan blocks P, Q *for the* B *having required signs of eigenvalue real parts*

To prove that these manifolds defined by ψ_j are tangent to the linearized subspaces. That is to prove $\left.\frac{\partial \psi_j}{\partial y}\right|_{y=0} = 0$

`We will do this proof in Tutorials`

Proof for Hartman Grobman theorem.

Proof 0.1 *The proof essentially contains the same initial part as for Stable manifold theorem, that is* $\dot{y} = By + G(y)$*, and eventually we want to show that* $\exists$ *a continuous one -one invertible mapping (homeomorphism) H such that*

$$H \circ \psi_t(y_0) = e^{Bt}H(y_0)$$

Change of co-ordinates followed by flow is same as flow first and then change of co-ordinates.

Let $x_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$ *where* $y_0 \in \mathbb{R}^k, z_0 \in \mathbb{R}^{n-k}$ *and* $\psi_t(x_0) = \begin{bmatrix} y(t, y_0, z_0) \\ z(t, y_0, z_0) \end{bmatrix}$*. But the linear system* $\dot{x} = Bx$ *will be*

$$x(t, x_0) = \begin{bmatrix} e^{Pt}y_0 & \\ & e^{Qt}z_0 \end{bmatrix}$$

Note that the decoupling between y_0 and z_0 is only in the linearized system, not in the non linear system.

Define new function $\tilde{Y}, \tilde{Z}$ as follows

$$\tilde{Y}(y_0, z_0) = y(1, y_0, z_0) - e^P y_0$$
$$\tilde{Z}(y_0, z_0) = z(1, y_0, z_0) - e^Q z_0$$

`We need to use that` $\| e^P \| < 1, \| e^{-Q} \| < 1$ `But not convincingly proved`
Since 0 is a fixed point, we have $\tilde{Y}(0) = 0$, and $\tilde{Z}(0) = 0$. From the way we defined $\tilde{Y}$ and $\tilde{Z}$ we have

$$D_{y_0}\tilde{Y} = \frac{\partial y}{\partial y_0} - e^P$$
$$D\tilde{Y}(0) = 0$$
$$D\tilde{Z}(0) = 0$$

So the taylor expansion for $\tilde{Y}, \tilde{Z}$ starts with the second order terms. So $\tilde{Y}, \tilde{Z}$ can be made as small as we wish near 0, for any a for small enough ε ,

$$\| D\tilde{Y} \| < a, \| D\tilde{Z} \| < a$$

See in 1D, we have something like a parabola or cubic polynomial, near origin it remains small in the value of function and its derivative

We can define $Y(y_0, z_0), Z(y_0, z_0)$ such that $y = \tilde{Y}, z = \tilde{Z}$ in a neighbourhod of radius ε near 0 and $y = 0, z = 0$ outside the neighbourhood of radius 2ε. because of the continuity, we can make the function smoothly go to zero after 2ε.
Let $L(x) = e^A x = \begin{bmatrix} e^P & 0 \\ 0 & e^Q \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} e^P y \\ e^Q z \end{bmatrix}$

$$T(x) = \begin{bmatrix} e^P y + Y(x) \\ e^Q z + Z(x) \end{bmatrix} \tag{5}$$
$$= L(x) + \begin{bmatrix} Y(x) \\ Z(x) \end{bmatrix} \tag{6}$$

Lemma 0.1 *There exists a one-one continuous invertible mapping H of a neighbourood of 0 such that $H \circ T = L \circ H$ that is L.H.S is*

$$H \begin{bmatrix} e^P y + Y(x) \\ e^Q z + Z(x) \end{bmatrix}$$

and the R.H.S is

$$\begin{bmatrix} e^P h_1(x) \\ e^Q h_2(x) \end{bmatrix}$$

with $H(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}$ *and we have to show the equality between them.*

Construct h_1, h_2 by an iterative defn as follows

$$\psi_0(y, z) = z$$

and

$$\psi_{k+1}(y, z) = e^{-Q}\psi_k \left(e^P y + Y(y, z), e^Q z + Z(y, z)\right)$$

if this converges as $k \to \infty$ then $\psi_k \to h_2$ and similarly for h_1. For the convergence , we have to prove by induction that

$$|\psi_k(x) - \psi_{k-1}(x)| \leqslant Cr^\delta$$

for some $r < 1$ and $\delta \in (0, 1)$ The convergence of this map proves the existence of H, that is the homeomorphism for $t = 1$. Now define

$$\tilde{H} = \int_0^1 ds\, e^{-As} H(\psi_s(x_0))$$

and prove that $\tilde{H} \circ \psi_t = e^{At} \circ \tilde{H}$ using the semi group propert of e^{At} and ψ_t which is $\psi_t \circ \psi_s = \psi_{t+s}$

In summary, construct H *: Define* $\tilde{\mathsf{Y}}, \tilde{\mathsf{Z}}$ *(after knowing the solutions) then* Y, Z
Now define $\mathsf{h}_1, \mathsf{h}_2$ *using the iterative scheme given above.*
Finally define $\tilde{\mathsf{H}} \circ \mathsf{L}$ *(which also requires the solution)*
Which shows the topologically equivalence of the flow of linearized equation to that of the non linear system.

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Assume that for $\dot{x} = f(x)$ the flow $\phi(t, x)$ is defined for some interval of existence $I \neq \mathbb{R}$. Then we can always look at a topologically equivalent system,

$$\dot{y} = \frac{f(y)}{1 + |f(y)|}$$

for which the interval of existence is $\mathbb{R}$.
But the domain is same for both the systems

So from now on, whatever system we describe, it will have interval of existence as $\mathbb{R}$.

This redefinition from $x \to y$ is not the only way, with interval of existence being $\mathbb{R}$

Definition 1 *An **Orbit** or a **Trajectory** of an initial condition x_0 is the set*

$$\Gamma = \{\phi(t, x_0) | t \in \mathbb{R}\}$$

The orbit of x_0 is same as the orbit of $\phi(t, x_0)$ because

$$\tilde{\Gamma} = \{\phi(s, \phi(t, x_0)) : s \in \mathbb{R}\} = \Gamma$$

Definition 2 ***Limit point**: p is a ω (or α) limit point of a trajectory Γ if $\exists$ a sequence $t_n \to \infty$ such that $\phi(t_n, x_0) \to p$ as $n \to \infty$ (or $t_n \to -\infty$).* [1]

Definition 3 *The set of all ω (or α) limit points is called **limit set** of $\Gamma : \omega(\Gamma) \text{or} \alpha(\Gamma)$.*

1. For the system $\dot{x} = y, \quad \dot{y} = -x$, for which the trajectory is a circle, every point on the circle is a limit point. Since if we define the sequence of times as $t, t + 2\pi, t + 4\pi, \cdots$, then as $t_n \to \infty$ we have $\phi(t_n, x_0) = x_0$ so if we vary x_0 the set becomes the circle itself.

2. Similarly, consider a pendulum, and see the phase space on the cylinder as shown in figure 1 (wrap it up around it, so that all trajectories become closed ones). The orbits $1, 2$ have both limit points as the whole orbit itself. For the orbits U_+, U_- the limit point is A. For $4, 5$ also the limit points are the orbits themselves. For point O, limit point is itself, and for A too.

[1] Notation is ω is for $t \to \infty$, and α is for $t \to -\infty$

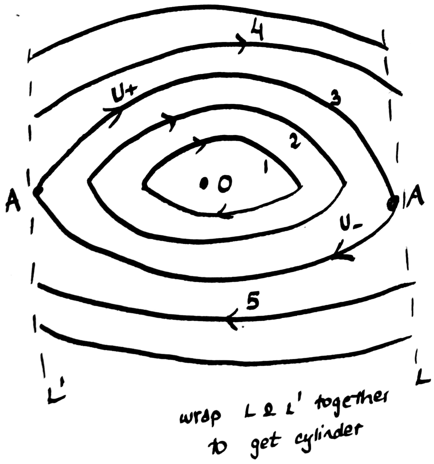

Figure 1: Phase space of Pendulum

3. Now take a perturbation of the pendulum,

$$\dot{x} = y, \quad \dot{y} = -\sin\theta + f(t)$$

with $f(t)$ being a time periodic perturbation $f(t+T) = f(t)$. Now we can take this to be an extended system with t being the third co-ordinate with evolution of it $\dot{t} = 1$.
Now the system, is very complicated

4. For Lorrenz system, the fixed pts are the limit sets
   - For the periodic lorrenz system. All the periodic orbits are limit points
   - For the chaotic lorrenz system. There are other limit sets which are neither periodic nor fixed points

5. *But if a system goes to infinity , then what are the limit Sets?*
For $\dot{x} = x$, there are three orbits
   - $\Gamma_0 : \quad x_0 = 0$
   - $\Gamma_+ : \quad x_0 > 0$
   - $\Gamma_- : \quad x_0 < 0$

   For all three orbits, the $\alpha = 0$, for the Γ_0 $\omega = 0$ too, but for Γ_+, γ_- the $\omega = \phi$ empty.

Theorem 1 *If Γ is contained in a compact subset of $\mathbb{R}^n$ then $\omega(\Gamma), \alpha(\Gamma)$ are each non-empty, closed, connected and compact subsets of $\mathbb{R}^n$*

Proof 1.1 *If $\{p_1, p_2, \cdots, p_n, \cdots,\} \in \omega(\Gamma)$ and $p_n \to p$ then $p \in \omega(\Gamma)$. For each $p_k, \exists$ say $t_n^{(k)} \to \infty$ such that $\phi(t_n^{(k)}, x_0) \to p_k$. Order the time sequence, $t_n^{(k+1)} > t_n^{(k)}$. Now choose $\varepsilon \to 0$ such that*

$$|\phi(t_s^{(n)}) - p_k| < \frac{\varepsilon}{2}, \quad |p_n - p| < \frac{\varepsilon}{2} \quad \forall s > N(n)$$

and using triangle inequality for $\phi(t_n, x_0) - p_n + p_n - p$

$$|\phi(t_n, x_0) - p| \leqslant |\psi(t_n, x_0) - p_n|$$

Let $t_n = t_s^{(n)} + |p_n - p|$ so that $t_n \to \infty$ and $\phi(t_n, x_0) \to p$. Now if $\omega(\Gamma) =$ null, then $\nexists t_n \to \infty$ such that $\phi(t_n)$ is contained in a compact set.
To show that $\omega(\Gamma)$ is connected. Suppose the limit set is two disconnected sets say, A and B. So there will be two sequences $\{t\} \in A$ and $\{s\} \in B$ then we construct

$$t_k < s_k < t_{k+1} < s_{k+1}, \cdots,$$

then for this sequence $\phi(t, x_0)$ doesnot belong to either of A, B. Which will be a contradiction.

Theorem 2 $\omega(\Gamma)$ *or* $\alpha(\Gamma)$ *are invariant with respect to the flow that is if* $p \in \omega(\Gamma)$ *then* $\phi(t, p) \in \omega(\Gamma) \forall t$ *that is if* p *is a* ω *limit point then* $\phi(t, p)$ *is also a limit point. If* $\phi(t_n, x_0) \to p$ *then* $\exists \{s_1, s_2, \cdots, s_n\}$ *such that* $\phi(s_n, x_0) \to \phi(t, p)$

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We had the system of $\dot{y} = By + G(y)$ with $G(0) = 0$.

$$B(y) = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

with P having k negative real part of eigenvalues and Q having $n-k$ positive real part of eigenvalues.
We had

1. Stable manifold theorem stating that $\exists$ a k dimensional surface (manifold) tangent to linearized stable subspace E_s

2. Hartman Grobman theorem stating that the y system is topologically equivalent to $\dot{z} = Bz$.

Now both these theorem, do not take zero eigenvalues. Suppose we have eigenvalues with zero real part, then we have a *Center manifold theorem.*

Theorem 1 *Consider a system*

$$\begin{aligned} \dot{x} &= Cx + F(x, y, z) \\ \dot{y} &= Py + G(x, y, z) \\ \dot{z} &= Qz + H(x, y, z) \end{aligned}$$

with P having negative real part of dimension s and Q having positive real part of dimension u and C have zero real part with dimension c.

$$u + s + c = n$$

The theorem says, that $\exists \delta > 0$ and a manifold W^c

$$W^c = \{(x, y, z) | y = h_1(x), z = h_2(x), |x| < \delta\}$$

which is invariant under the flow with

$$h_1(0) = h_c(0) = 0, \quad Dh_1(0) = 0 = Dh_2(0)$$

and is tangent to $E^c = \{(x, 0, 0) \in \mathbb{R}^n\}$ Thus requiring for $y = h_1(x)$ we need $\dot{y} = Dh_1(x)\dot{x}$ which gives

$$Dh_1(x) \left[Cx + F\left(x, h_1(x), h_2(x)\right)\right] = Ph_1(x) + G\left(x, h_1(x), h_2(x)\right)$$

The number of equations above is s. Similarly there will be u equations for $\dot{z} = Dh_2(x)\dot{x}$.

There is no Hartman grobman theorem equivalent to this. In other words, we cannot find a diffeomorphism between the center manifold and linearized center subspace.

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This class, we will see what definitions that we saw for continuous time system, are applicable to the discrete time systems?

Overview relating to the discrete time systems

From

$$\dot{x} = f(x), \quad \Rightarrow \mathrm{C} \tag{1}$$
$$x_{n+1} = F(x_n), \quad \Rightarrow \mathrm{D} \tag{2}$$

1. The theorem for existence and uniqueness of solutions for **C** require lipshitz continuous $f(x)$, within the interval of existence.
   For **D**, if F is invertible, then an orbit exists $\forall n \in \mathbb{Z}$, if not then starting with x_0 the Orbit exists $\forall n \in \mathbb{N}$.

2. Linearization of **C**:
$$\dot{z} = Df(x(t))z = A(t)z$$
   is non autonomous linear ordinary differential equation.

   Linearization of **D**:
$$z_{n+1} = DF(x_n)z_n = A_n z_n$$
   for which the solution is
$$z_n = A_n A_{n-1} \cdots A_0 z_0$$
   The role played by the fundamental matrice solution of $\dot{z} = Az$ is similar to the product $A_n \cdots A_0$ the derivative with respect to initial conditions also satisfies the linearized equations.

3. The lyapunov exponents quantify the stability of the solutions in both cases.

   - For autonomous linear discrete time systems: $z_{n+1} = Az_n$, the eigenvalues of A outside the unit circle implies unstable. All eigenvalues strictly inside the unit circle implies stability.
   - For **C**, the lyapunov exponent is
$$\chi[x] = \overline{\lim}_{t\to\infty} \frac{1}{t} \ln |x(t)|$$
   - For **D**,
$$\chi\left[\{x_n\}\right] = \overline{\lim}_{n\to\infty} \frac{1}{n} \ln x_n$$
     for especially the linear systems,
$$\chi\left[\{x_n\}\right] = \overline{\lim}_{n\to\infty} \frac{1}{n} \ln\left[|A_n A_{n-1} A_{n-2} \cdots A_0 x_0|\right]$$
     but all properties of lyapunov exponents are valid
     - At most d different exponents
     - existence of filtration
     - definition of regularity

     But the adjoint equation,
     - For **C**,
$$\dot{y} = -A^{\dagger} y$$
     - For **D**,
$$y_{n+1} = (A^{\dagger})^{-1} y_n$$

- For periodic solutions $x_{n+T} = x_n, \forall n$, the stability can be studied by simply looking at the T^{th} iteratoe fo the map.
  Define $G = F \circ F \circ$ T times $\circ\, F$ then x_n is the fixed point of G map. No floquet theory in this case.

4. The stable manifold theorem remains the same.
   If x^* is a fixed point and no eigenvalues of $DF(x^*)$ has absolute value 1, and if k eigenvalues are outside the unit circle, then there is a k dimensional invariant manifold that is tangent to the k- dimensional unstable space E^u and similarly for the stable manifold.

5. The center manifold theorem is not stated in the same way.

6. The definition of ω, α limit points and limit sets is the same. But the limit sets are not connected. Other properties are the same.

7. Regarding the Hartman Grobman theorem, we say the maps F and G are topologically conjugate if $\exists$ an invertible continuous transformation ψ such that ψ(orbit of F) is equal to orbit of G.
