Week 3 Binomial Tree

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1 Introduction

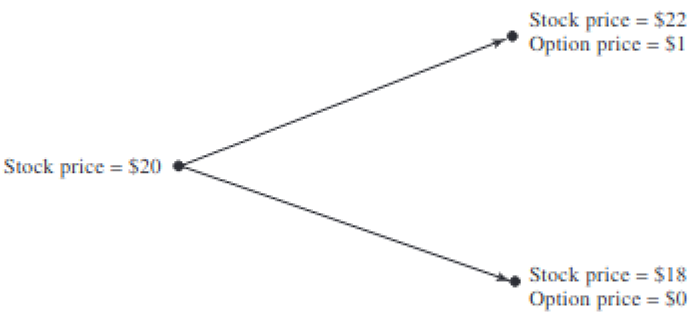
A useful and very popular technique for pricing an option involves constructing a binomial tree. This is a diagram representing different possible paths that might be followed by the stock price ove rthe life of an option. The underlying assumption is that the stock price follow a random walk.

In each time step . it has a certain probability of moving up by a certain percentage amount and certain prob of moving down by a certain percentage. In the limit, as the time step becomes smaller, the model is the same as the black scholes merton model refer to .

Consider the following

- Stock price is £ 20
- after 3 months it will be either £ 22 or £ 18
- Evaluating, Eu call option, after 3 months to buy this at £21
  - Option 1 : At the end or nearing maturity the strike pirce will be £22 and then the option would be that of £1
  - Option 2 : If the stock price turns out to be £18, the valyue of the option will be zero.

Figure 13.1 Stock price movements for numerical example in Section 13.1.



Note *Only one assumption required: Is that the arbitrage opportunities do not exist. We set up a portfolio of the stock and option in such a way that there is no uncertainty.* The reason we state that this portfolio has no risk, the return it earns must be equal the risk free interest rate.

- Allows us to work out the cost of setting up the portfolio
- This allows you to figure out what the option price is.

Consider a portfolio that consists of a long position, with Δ Shares, of the stock We can say the following:

Δ that makes the portfolio reiskless, if the stock price moves up from £20 to £22 - the value of the shares is $22\Delta - 1$ We say 22 Shares - 1.

Compared to if the stock prices move down from £20 to £18 . The value of the shares is 18Δ and the value of the option is zero. Such that the total value of the portfolio is 18Δ and the option is zero. The porfolio is riskless if the value of Δ is chocen so that the final value of the portfoliobI

Such that, for Δ You want the following

$$22\Delta - 1 = 18\Delta$$

or

A riskless portfolio is there for a

- Long : 0.25 Shares
- Short : 1 Option

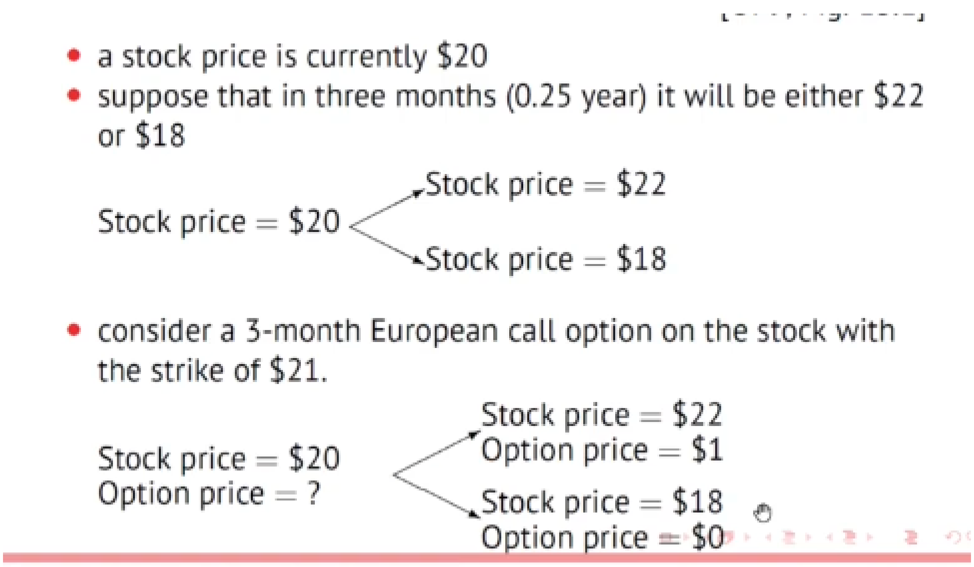
If the stock price moves up to £ 22 then the value of the porfolio is

$$22 \cdot 0.25 - 1 = 4.5$$

if the stock price moves down to £18 the value of the portfolio is also

$$18 \cdot 0.25 = 4.5$$

regardless of whether the stock price moves up or down the value of the portfolio is always 4.5 in the end of the option. This is why we call this riskless.



Riskless portfolios must, in the absence of arbitrage opportunities. Earn the risk free rate of interest.

Suppose that in thise case the risk free rate is 12 % er annumn. It follows that the value of the portfolio today must be the present value of 4.5.

$$4.5e^{\frac{-.0.12\cdot 3}{12}} = 4.367$$

Here if you know the value of the portfolio today, you can calculate the present value of the portfolio. Such that, riskless protfolios, do not need arbitrage opportunities, as they can rely on their risk free stature.

The value of the stock price todayu is known to be £20, such that suppose the option price is denoted by f . the value of the portfolio would be the following

$$20 \cdot 0.25 = f = 5 - f$$

$$s - f = 4.367$$

$$f = 0.633$$

- Current value of the option must be 0.633 if the value of the option were more than 0.633, the port would cost less than 4.367 to setup and would earn more than the risk free rate.
- If the value of the portfolio is reduced. such that would be less that 0.644, then *shorting* the portfolio would provide a way of borrowing money at a less than the risk free rate.

In general it is required to buy Δ shares for each option sold to from a riskless portfolio . The parameter $\Delta, isimportantinthehedgingofoptions.$

- We have essentially used the no arbitrage argument; If anyone disagrees with our valiuation of the option. They value the portfolio differently.
  - If they undervalue *We buy it from them*
  - If they Overvlaue *We sell it to them*
  - and make riskless profit at their expense

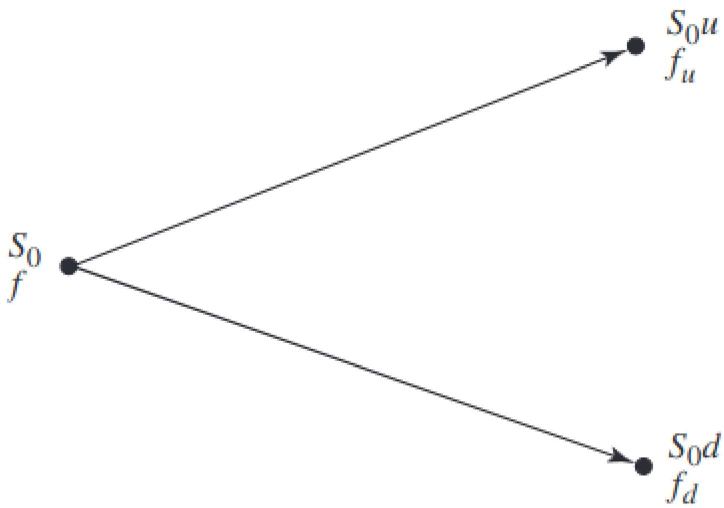
1.1 Generalization

We can generalize the no arbitrage argument- using the principle of stock. Consider the stock price S_0 *Stock price of the current time* such that there is an option on the stock - whose current price is f. We suppose that the option last for time T which we will then call the stock price will move from S_0 to S_0u where $u > 1$. Or down from $S_0 \to S_0d$ where $d < 1$.

One way of understanding how generalization would work is through the following.

- A derivative last for some time T and is dependent on a stock price.
- the sotck pirce can go up from $S_0 \to S_0u$ or it can do gown from $S_0 \to S_0d$ where you have the following condition $(u > 1, 1 < d)$
- the derivative initial cost f and can go to either f_u or f_d given the stock prices. we can evaluate f_u and f_ud using a tree like structure.

Figure 13.2 Stock and option prices in a general one-step tree.



(1)

here you can consider the following principle: regarding how the portfolio is split between two iotions. Recall we have this Δ Value which much always evaluate to be the same for both x and y values.

We calculate the value of Δ that makes the portfolio riskless. If there is an upmovement in the stock price the value of the portfolio at the time end will be

$$S_0u\Delta - f_u$$

If there is a down movement in the stock price, the value would become

$$S_0d\Delta - f_d$$

Which then must evaluate to the following

$$S_0u\Delta - f_u = S_0d\Delta - f_d$$

For this to be the case, you must have the following Delta, which is derived from the following

$$\Delta = \frac{f_u - f_d}{S_0u - S_0u}$$

Such that the following would evaluate to :

$S_0u \cdot \frac{f_u - f_d}{S_0u - S_0u} - f_u = S_0d\frac{f_u - f_d}{S_0u - S_0u} - f_d$ You must know the formulation for both f_u and f_d

In this case the portfolio is completely riskless, and for there to be no arbitrage opportunitiesl, it must earn the risk free interest rate. The above: Shows that Δ is the ratio of the change in option price to the change in stock price. Such that you can then further denote the risk free interest rate given r with the following

$$(S_0u\Delta - f_u) \cdot e^{-rT}$$

This is because we are using continious interest rate here for the portfolio where r is the interest rate, and T is our given time range.

The cost of setting up the portfolio is simple

$$S_0\Delta - f$$

such that

$$S_0\Delta - f = (S_0 u\Delta - f_u) \cdot e^{-rT}$$

This can be further redefined as the following

$$f = S_0\Delta(1 - ue^{-rT})f_u e^{-rT}$$