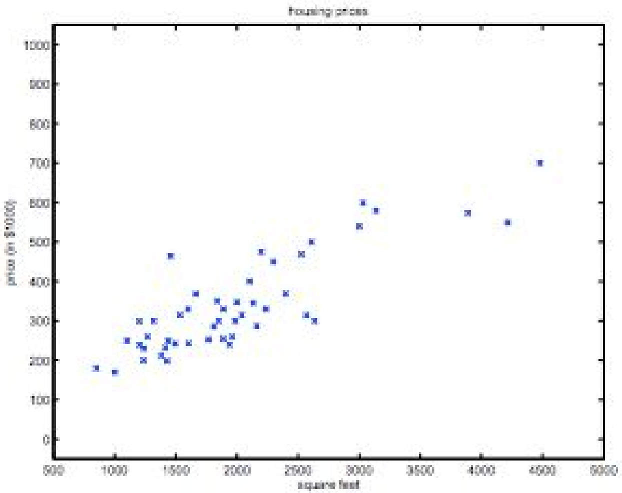
Linear Regression

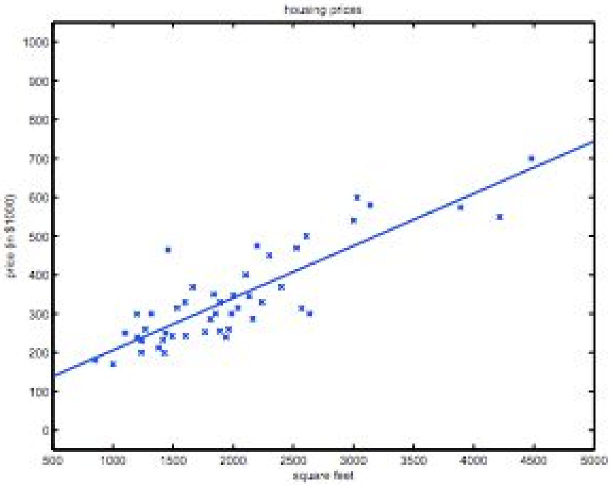
GA DAT3

Agenda

1. Overview
2. Cost Function
3. Gradient Descent
4. Normal Equation
5. Probabilistic Interpretation
6. Locally Weighted Linear Regression

Linear Regression Overview





Supervised Learning

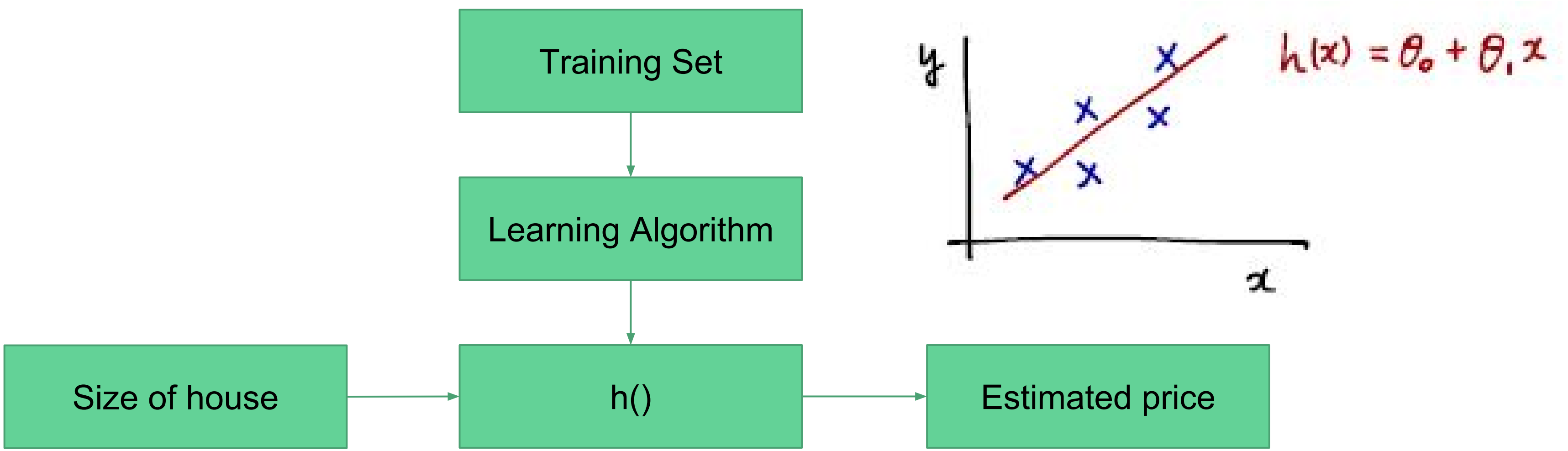
Regression: predict real-value output
Classification: predict discrete value output

Training set of housing prices (Portland, OR)

Size in feet^2 (x)	Price ($) in 1000's (y)
2104	460
1416	232
1534	315
852	178
...	...

Notation:

- **m** = Number of training examples
- **x**'s = "input" variable / features
- **y**'s = "output" variable / "target" variable

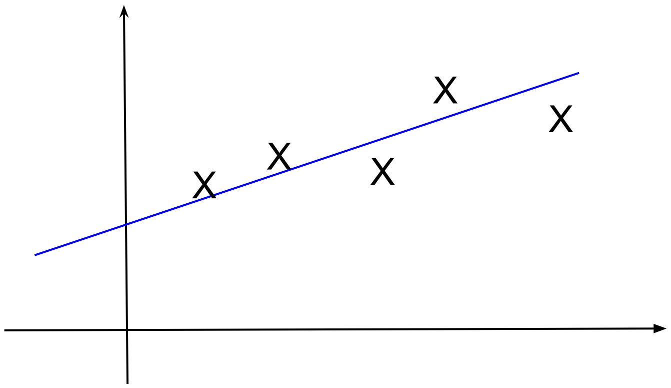


Linear regression with one variable :
Univariate linear regression

Cost Function : Squared Error Loss

Minimize: $$J(\theta) = \frac{1}{2}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2.$$

WHY?



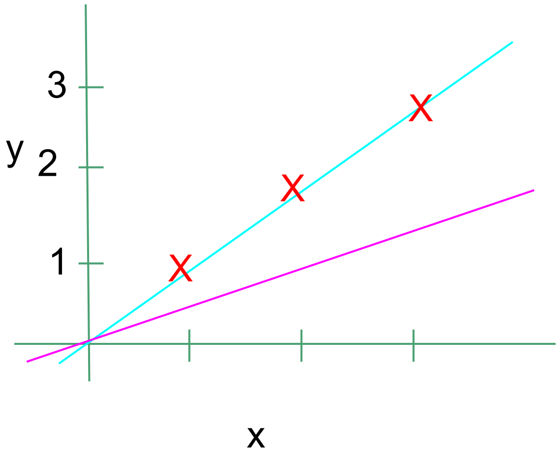
General Multivariate Form

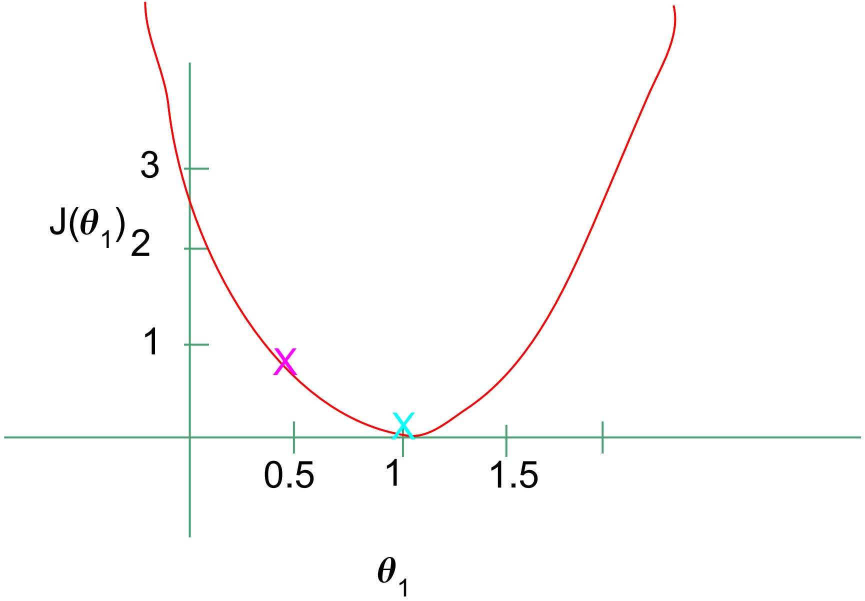
$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$$

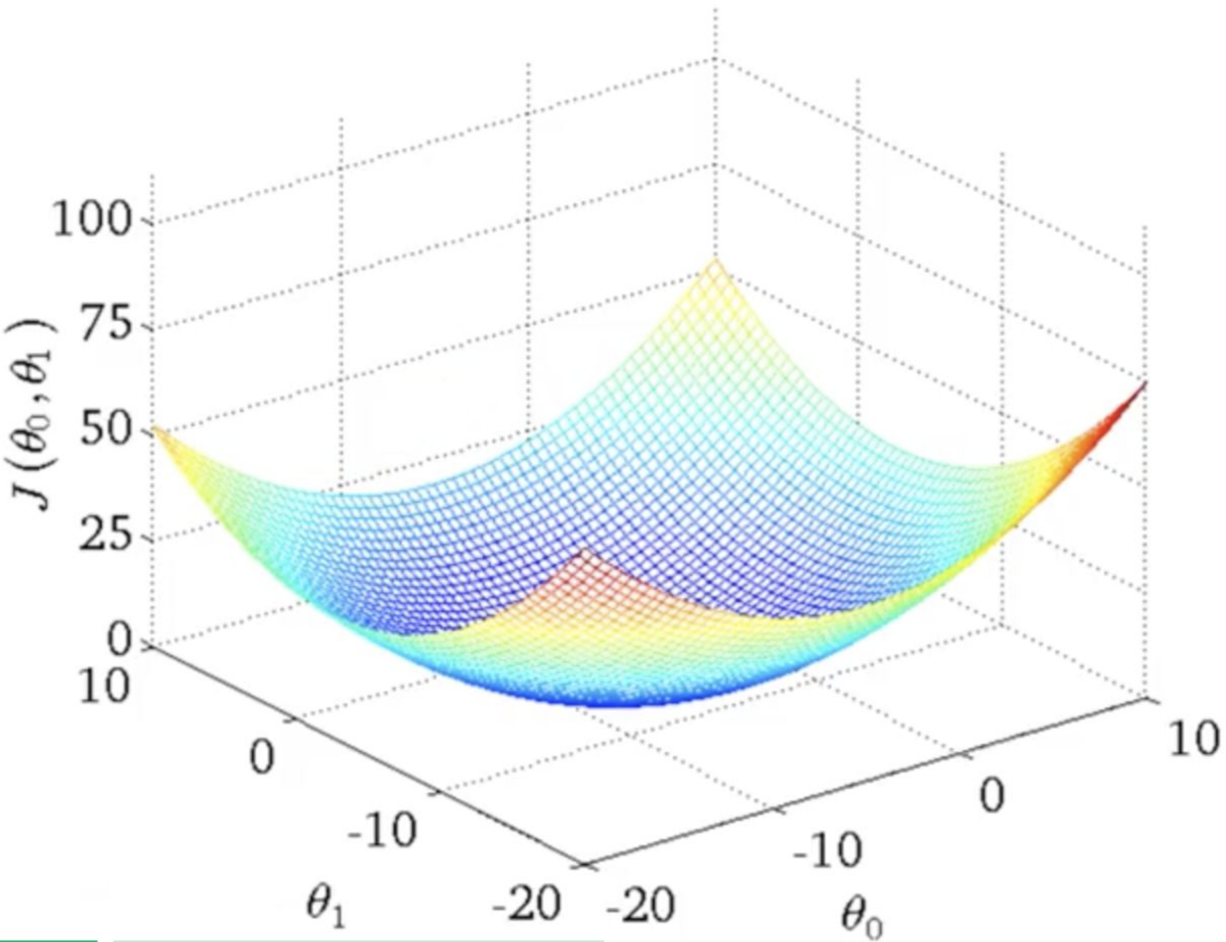
$$h(x) = \sum_{i=0}^{n} \theta_i x_i = \theta^T x,$$

Cost Function : Squared Error Loss

Minimize: $$J(\theta) = \frac{1}{2}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2.$$







100
75
50
25
0
$J(\theta_0, \theta_1)$
10
0
-10
-20
θ_1
-20
-10
0
10
θ_0

Gradient Descent

Update Rule:

Learning rate

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta).$$

$$\begin{aligned}
\frac{\partial}{\partial \theta_j} J(\theta) &= \frac{\partial}{\partial \theta_j} \frac{1}{2} \left(h_\theta(x) - y\right)^2 \\
&= 2 \cdot \frac{1}{2} \left(h_\theta(x) - y\right) \cdot \frac{\partial}{\partial \theta_j} \left(h_\theta(x) - y\right) \\
&= \left(h_\theta(x) - y\right) \cdot \frac{\partial}{\partial \theta_j} \left(\sum_{i=0}^{n} \theta_i x_i - y \right) \\
&= \left(h_\theta(x) - y\right) x_j
\end{aligned}$$

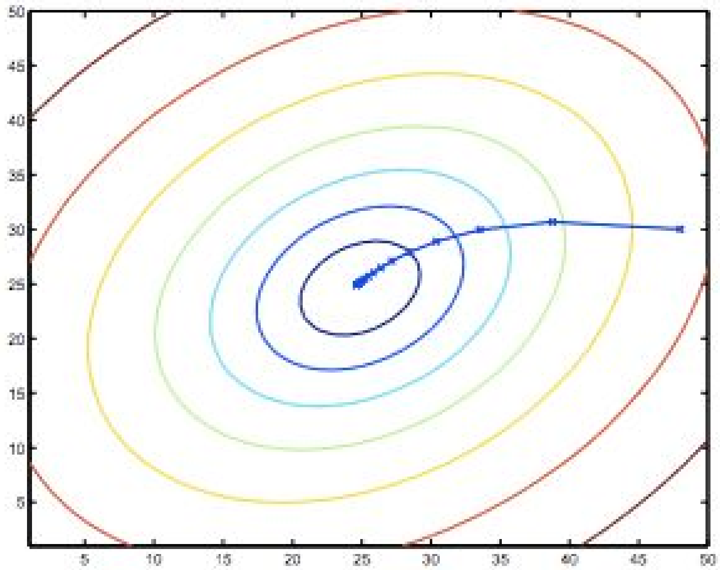
Repeat until convergence {

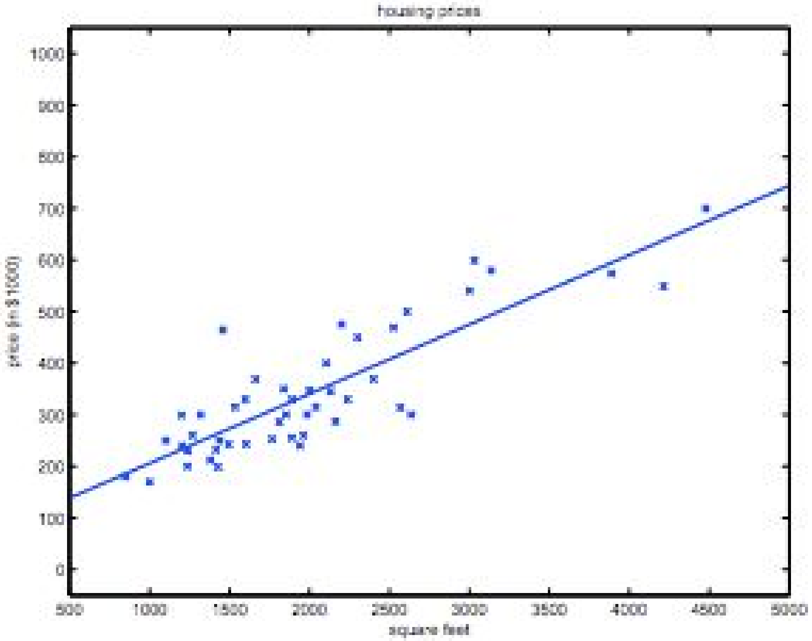
$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left(y^{(i)} - h_\theta(x^{(i)})\right) x_j^{(i)} \qquad \text{(for every } j\text{).}$$

}

For sufficiently small **α**, **θ** should decrease on every iteration.
But if **α** is too small, gradient descent can be slow to converge.'\

What are the axes?




housing prices
price (in $1000)
square feet
1000
900
800
700
600
500
400
300
200
100
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000

Batch vs Stochastic Descent

```
for theta:
    for each example:
        ...
for each example:
    for theta:
        ...
```

Normal Equations

$$
\begin{aligned}
\nabla_\theta J(\theta) &= \nabla_\theta \frac{1}{2}(X\theta - \vec{y})^T (X\theta - \vec{y}) \\
&= \frac{1}{2}\nabla_\theta \left(\theta^T X^T X\theta - \theta^T X^T \vec{y} - \vec{y}^T X\theta + \vec{y}^T \vec{y}\right) \\
&= \frac{1}{2}\nabla_\theta \operatorname{tr}\left(\theta^T X^T X\theta - \theta^T X^T \vec{y} - \vec{y}^T X\theta + \vec{y}^T \vec{y}\right) \\
&= \frac{1}{2}\nabla_\theta \left(\operatorname{tr}\theta^T X^T X\theta - 2\operatorname{tr}\vec{y}^T X\theta\right) \\
&= \frac{1}{2}\left(X^T X\theta + X^T X\theta - 2X^T \vec{y}\right) \\
&= X^T X\theta - X^T \vec{y}
\end{aligned}
$$

$$\theta = (X^T X)^{-1} X^T \vec{y}.$$

Gradient Descent vs Normal Equation

Gradient Descent	**Normal Equation**
Need to choose **α**	No need to choose **α**
Need many iterationa	Don’t need to iterate
Work well even with large n	Need to compute $(X^TX)^{-1}$
	Slow if n is large

Probabilistic Interpretation

$$y^{(i)} = \theta^T x^{(i)} + \epsilon^{(i)}$$

$$p(\epsilon^{(i)}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\epsilon^{(i)})^2}{2\sigma^2}\right).$$

We can minimise **cost**

or

maximise likelihood

What's the likelihood?

Likelihood

This is a **conditional probability density**

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{m} p(y^{(i)} \mid x^{(i)}; \theta) \\
&= \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right)
\end{aligned}
$$

Can you write this as a sum?

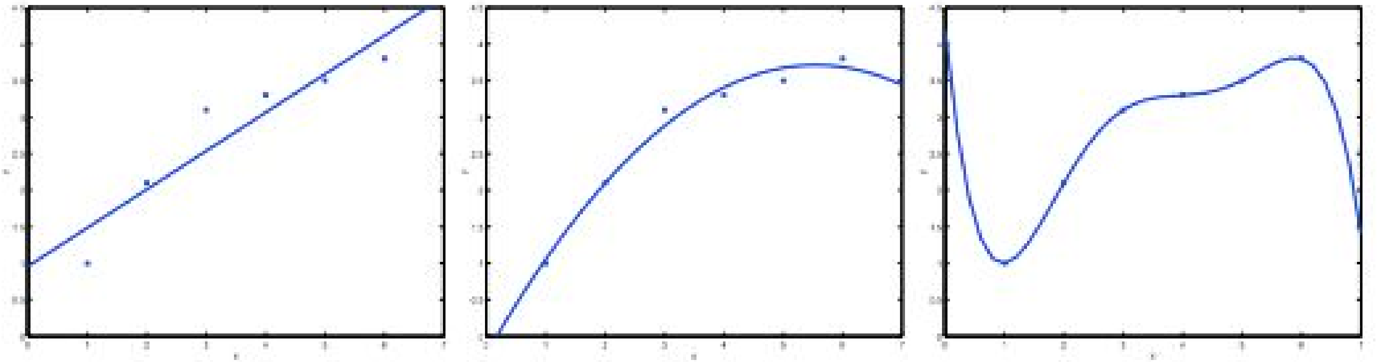
Likelihood - For you advanced folks

$$
\begin{aligned}
\ell(\theta) &= \log L(\theta) \\
&= \log \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right) \\
&= \sum_{i=1}^{m} \log \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right) \\
&= m \log \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{\sigma^2} \cdot \frac{1}{2} \sum_{i=1}^{m} (y^{(i)} - \theta^T x^{(i)})^2.
\end{aligned}
$$

Hence, maximizing $\ell(\theta)$ gives the same answer as minimizing

$$
\frac{1}{2} \sum_{i=1}^{m} (y^{(i)} - \theta^T x^{(i)})^2,
$$

Locally Weighted Linear Regression



classic tradeoff

Locally Weighted Linear Regression

Instead of doing:

1. Fit θ to minimize $\sum_i (y^{(i)} - \theta^T x^{(i)})^2$.
2. Output $\theta^T x$.

We do:

1. Fit θ to minimize $\sum_i w^{(i)} (y^{(i)} - \theta^T x^{(i)})^2$
2. Output $\theta^T x$.

Locally Weighted Linear Regression

Good examples of weights?

$$w^{(i)} = \exp\left(-\frac{(x^{(i)} - x)^2}{2\tau^2}\right)$$

Locally Weighted Linear Regression

Parametric vs Non-Parametric

What was k-Nearest Neighbours?

http://learning.cis.upenn.edu/cis520_fall2009/index.php?n=Lectures.LocalLearning#toc8

Q??