Econ 241 C.

1. (40p). *Conjugate priors. Normal-Normal model.* Consider the normal linear regression model

$$Y_i = X_i'\theta_0 + \epsilon_i, \quad i = 1, 2, \ldots, n, \quad \epsilon_i \sim N(0,1) \quad \text{i.i.d} \tag{1}$$

Suppose $\theta_0 \sim N(0, \tau^2 I_p)$.

   (a) (5p) State the conditional likelihood function for the data $(X_i, Y_i)_{i=1}^n$:

   $$\prod_{i=1}^n f_{Y_i|X_i=x_i}(y_i \mid \theta_0) = \ldots.$$

   Recall that the prior weighting function is

   $$\pi(\theta) = \frac{1}{\sqrt{(2\pi\tau^2)^p}} \exp^{-\theta'\theta/2\tau^2}.$$

   (b) (5p) Invoking the Bayes rule, derive the posterior distribution $\pi(\theta \mid (X_i, Y_i)_{i=1}^n)$ for θ.

   (c) (5p) Discuss the mean posterior limit when

      i. $\tau = \infty, n$ finite

      ii. when $n \to \infty, \tau$ finite. Explain the statement “prior vanishes asymptotically”.

   (d) (5p). Derive the marginal distribution of $(X_i, Y_i)_{i=1}^n$ as a function of τ.

   (e) (5p). Propose an empirical Bayes approach to dealing with unknown τ.

   (f) (15p.) Summarize your answers for the questions above for a special case of regression on a constant:

   $$p = \dim(\theta) = 1, X_i = 1 \text{ a.s. }, \text{data} = (Y_i)_{i=1}^n$$

2. (36p.) *Beta-Binomial model.* Suppose there are K different groups of patients, where each group has n patients. Each group is given a different treatment for the same illness, and in the k th group, we count $X_k, k = 1, 2, \ldots, K$, the number of successful

treatments out of n. Since the groups receive different treatments, we expect different success rates; however, since we are treating the same illness, these rates should be somewhat related to each other. These considerations suggest the hierarchy

$$\begin{aligned} X_k &\sim B(p_k, n), \\ p_k &\sim \text{beta}(\alpha, \beta), \quad k = 1, 2, \ldots, K, \end{aligned}$$

where the K groups are tied together by the common prior distribution.

- (18p.) Derive the posterior distribution for p_k.
- (18p.) Sketch an empirical Bayes estimator of p_k.