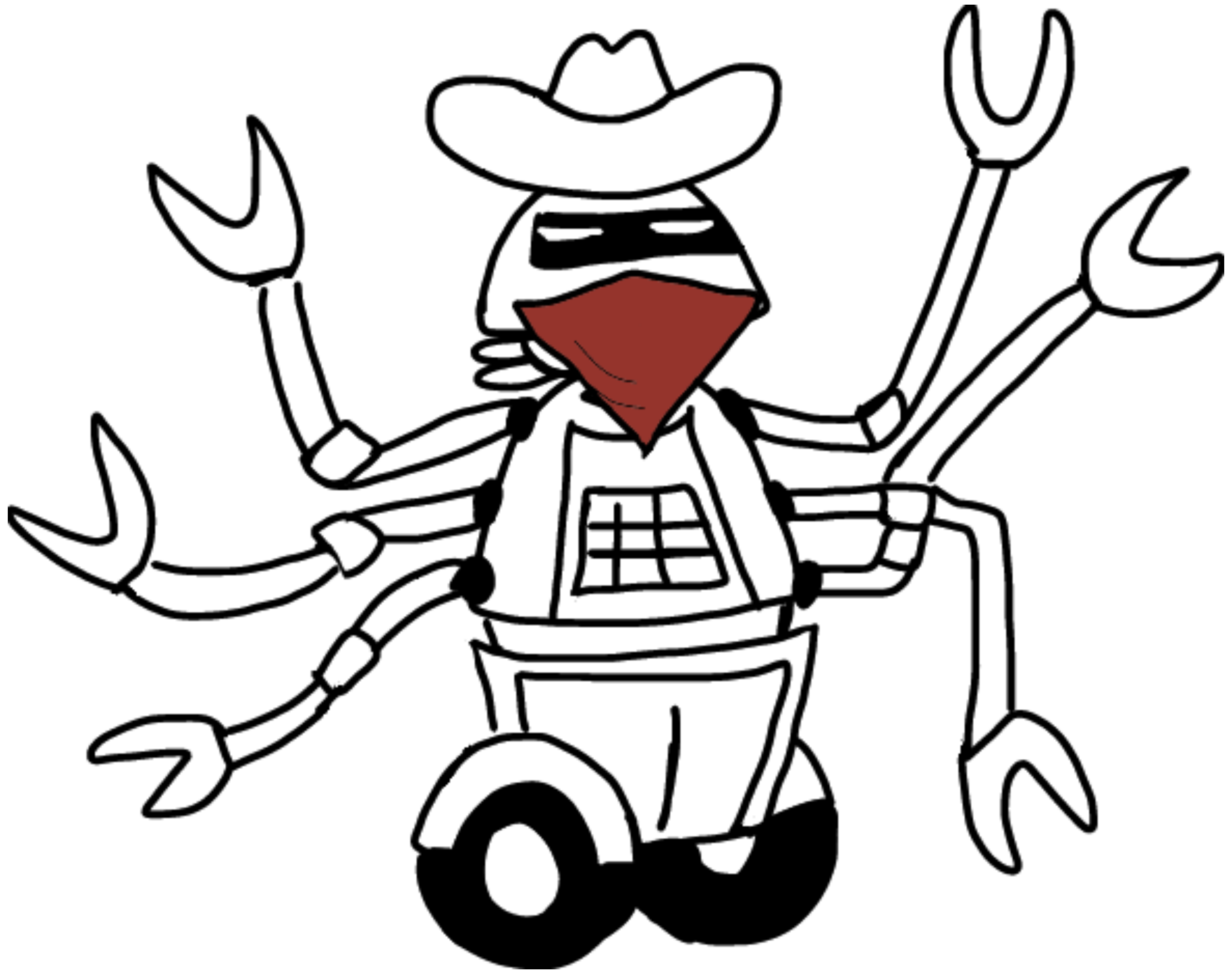
Reinforcement Learning

HSE, autumn - winter 2022

Lecture 7: Multi-armed Bandits



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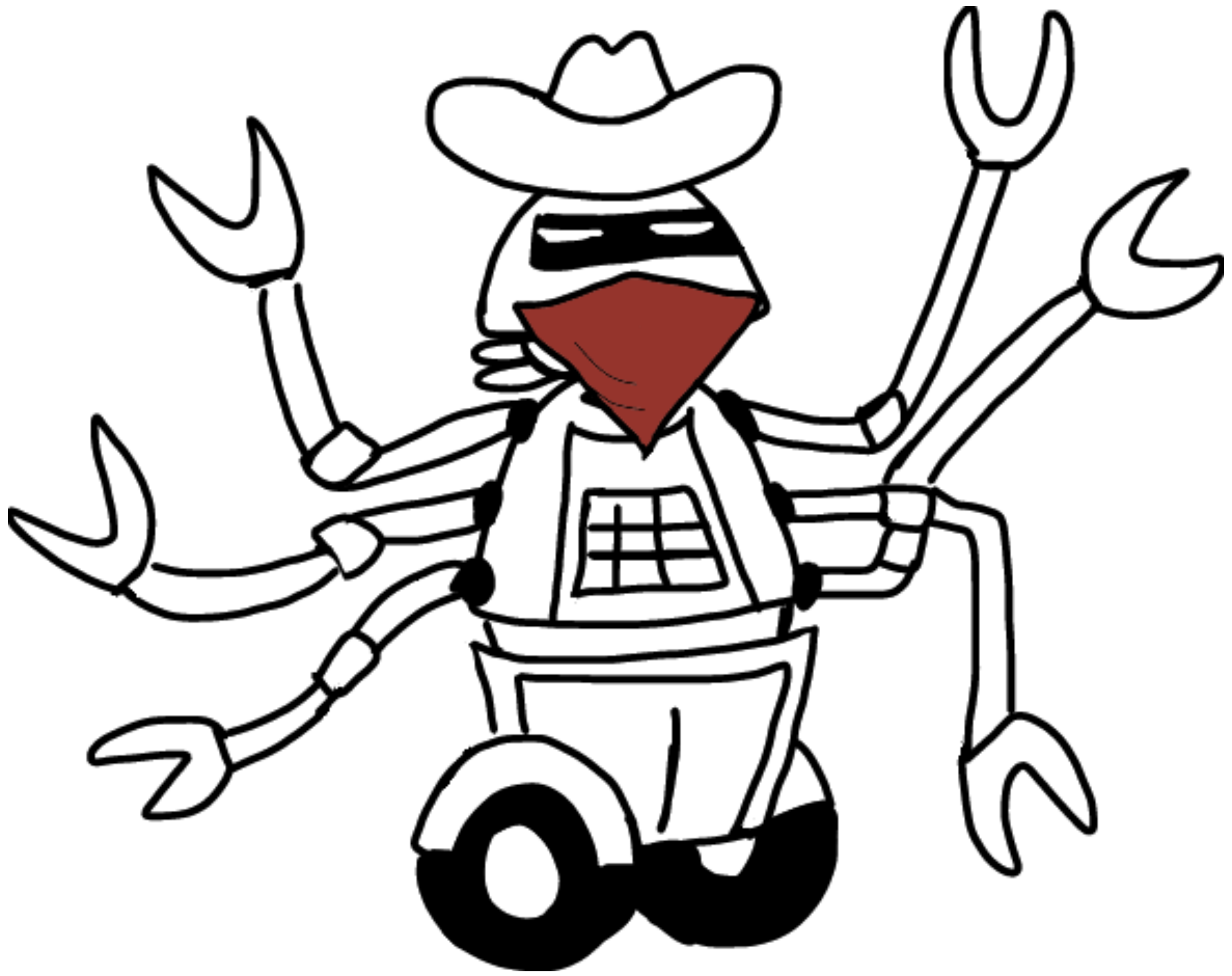
Background

1. Practical RL course by YSDA, week 5
2. Sutton & Barto, Chapter 2
3. DeepMind course, Lecture 2

Multi-armed Bandit Problem Statement

Assume that the episode ends after the first step so we have only one state in the environment. You as an agent are facing faced repeatedly with a choice among k different actions.



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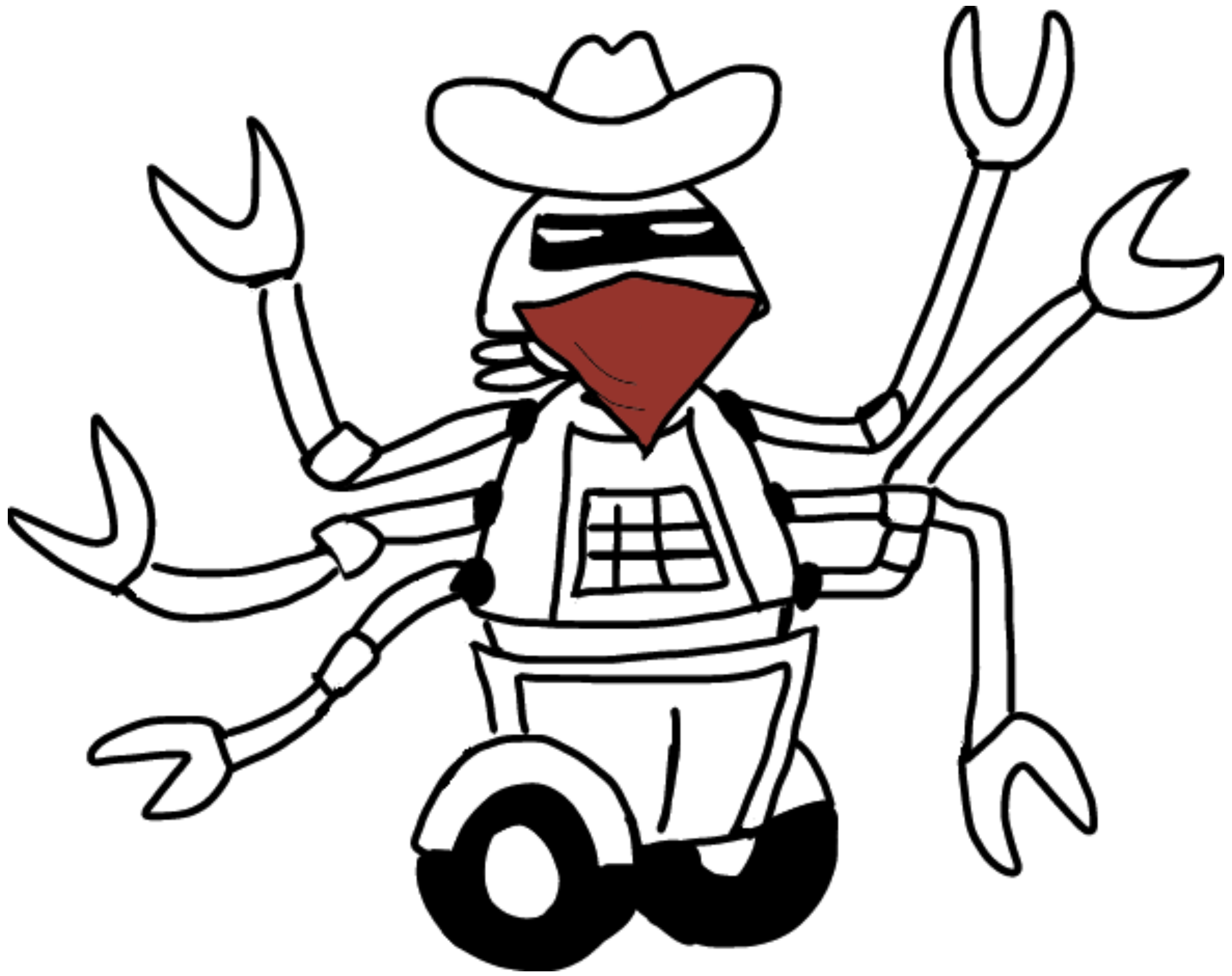
- A multi-armed bandit is a set of distributions $\{\mathcal{R}_a | a \in \mathcal{A}\}$
- On each step t an agent chooses A_t and get reward $R_t \sim \mathcal{R}_{A_t}$
- The goal is to maximise $\mathbb{E}_{p(r|a)}[\sum_{t=1}^{T} R_t]$ by choosing an action on each step



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Multi-armed Bandit Problem Statement

Exploration: find the best action which maximises expected reward

Action value function: $Q(a) = \mathbb{E}[R_t | A_t = a]$

Optimal value: $V* = \max_a Q(a)$

Regret: $V* - Q(a) \geq 0$

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Regret Minimisation

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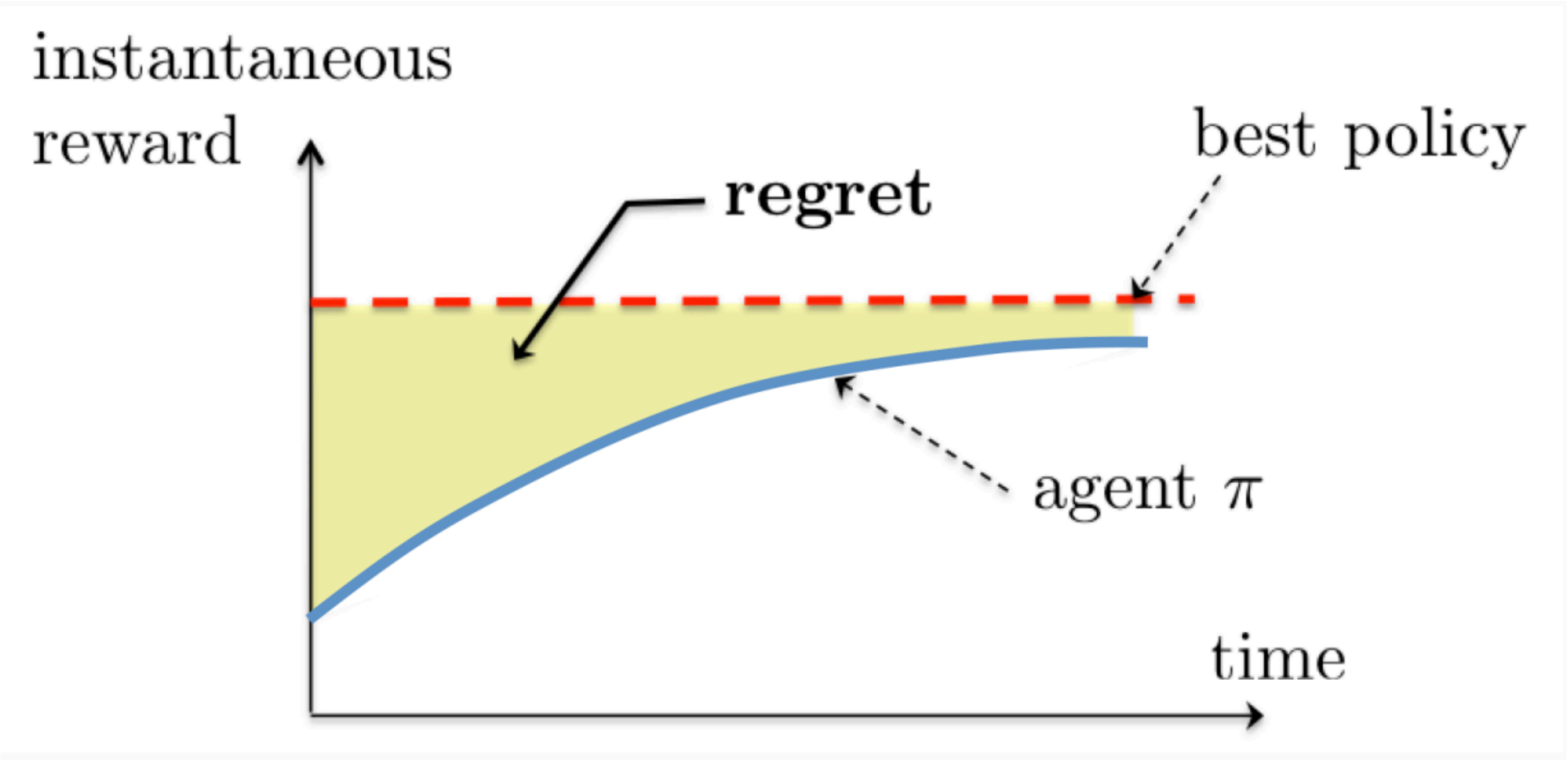

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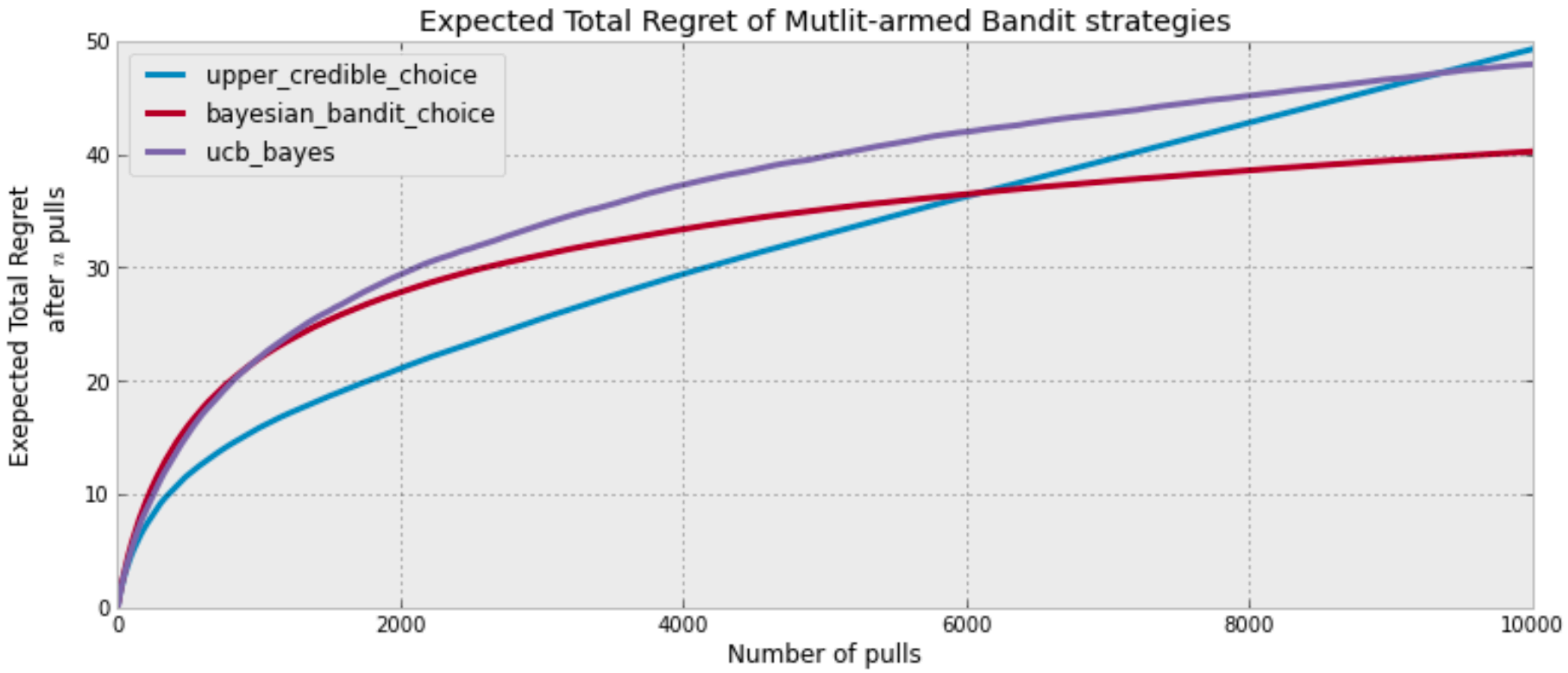

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Action Values

$$Q_t(a) = \frac{\sum_{n=1}^{t} \mathbb{I}(A_n = a) R_n}{\sum_{n=1}^{t} \mathbb{I}(A_n = a)} = \frac{\sum_{n=1}^{t} \mathbb{I}(A_n = a) R_n}{N_t(a)} \iff$$

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ε-greedy Policy

$$\pi_t(a) = \begin{cases} (1-\varepsilon) + \dfrac{\varepsilon}{|\mathscr{A}|}, \text{ if } Q_t(a) = max_b Q_t(b) \\ \dfrac{\varepsilon}{|\mathscr{A}|}, \text{ otherwise} \end{cases}$$

- Greedy can stuck on a suboptimal action forever
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ε-greedy policy has linear regret

Gradient Policy

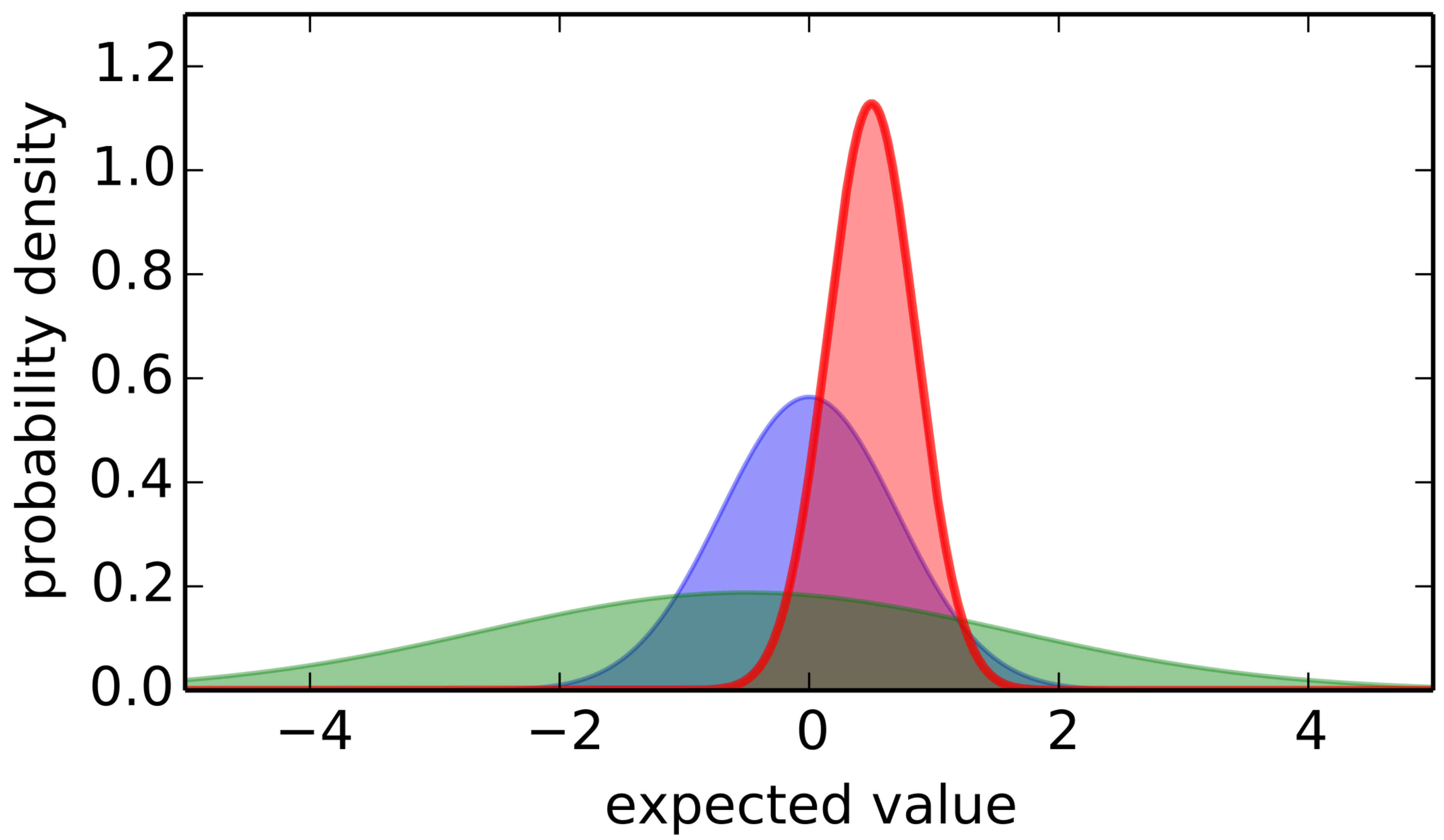
We can learn softmax policy using REINFORCE via gradient ascent, but there is no still guarantees for convergence to global optimum.

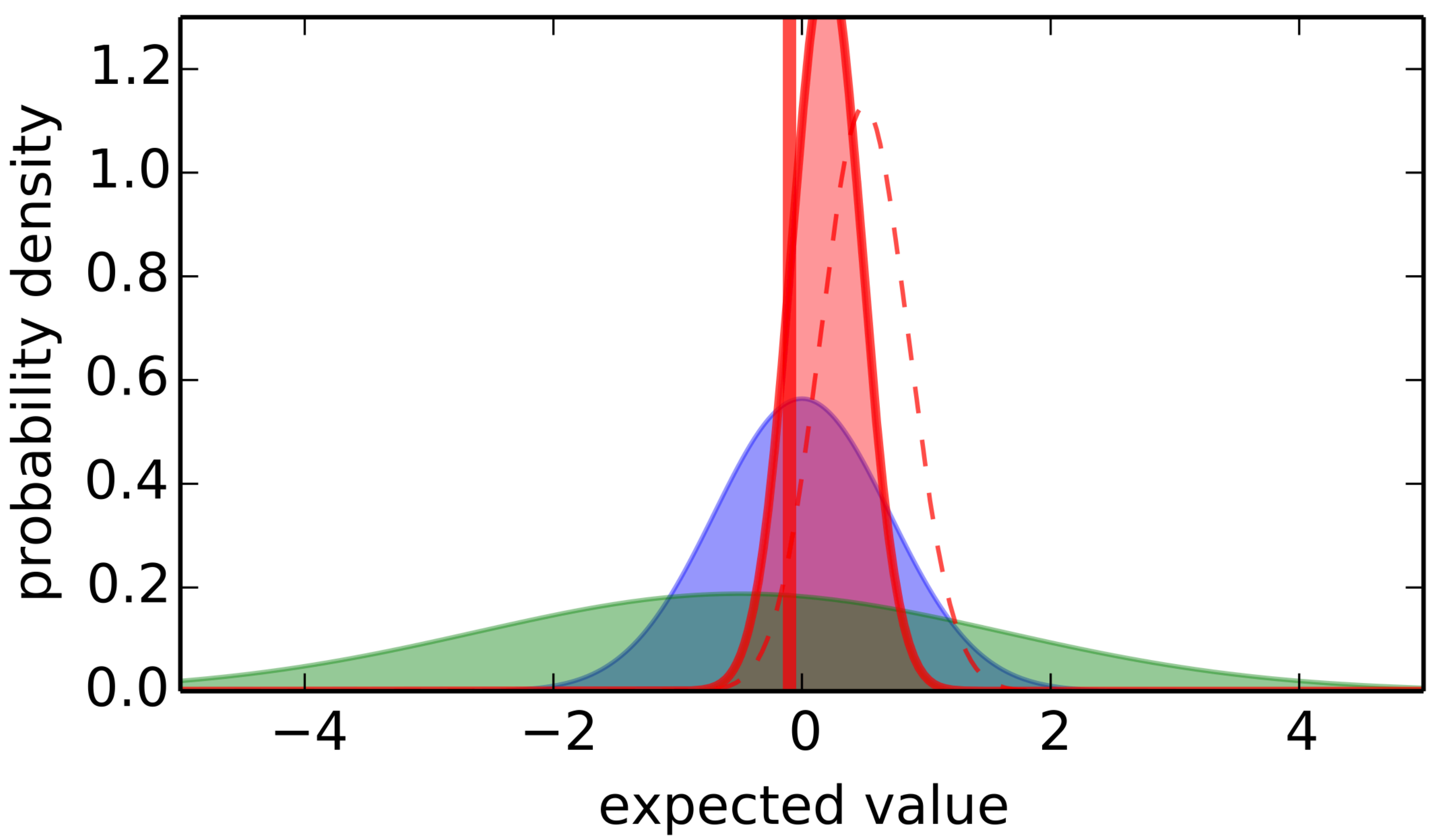
Regret Lower Bound

Theorem:

$$\sum_{t=1}^{T} [V^* - Q(a_t)] \geq \log T \sum_{a|V^*>Q(a)} \frac{V^* - Q(a)}{KL(\mathcal{R}_a || \mathcal{R}_{a^*})}$$

Optimism in the Face of Uncertainty





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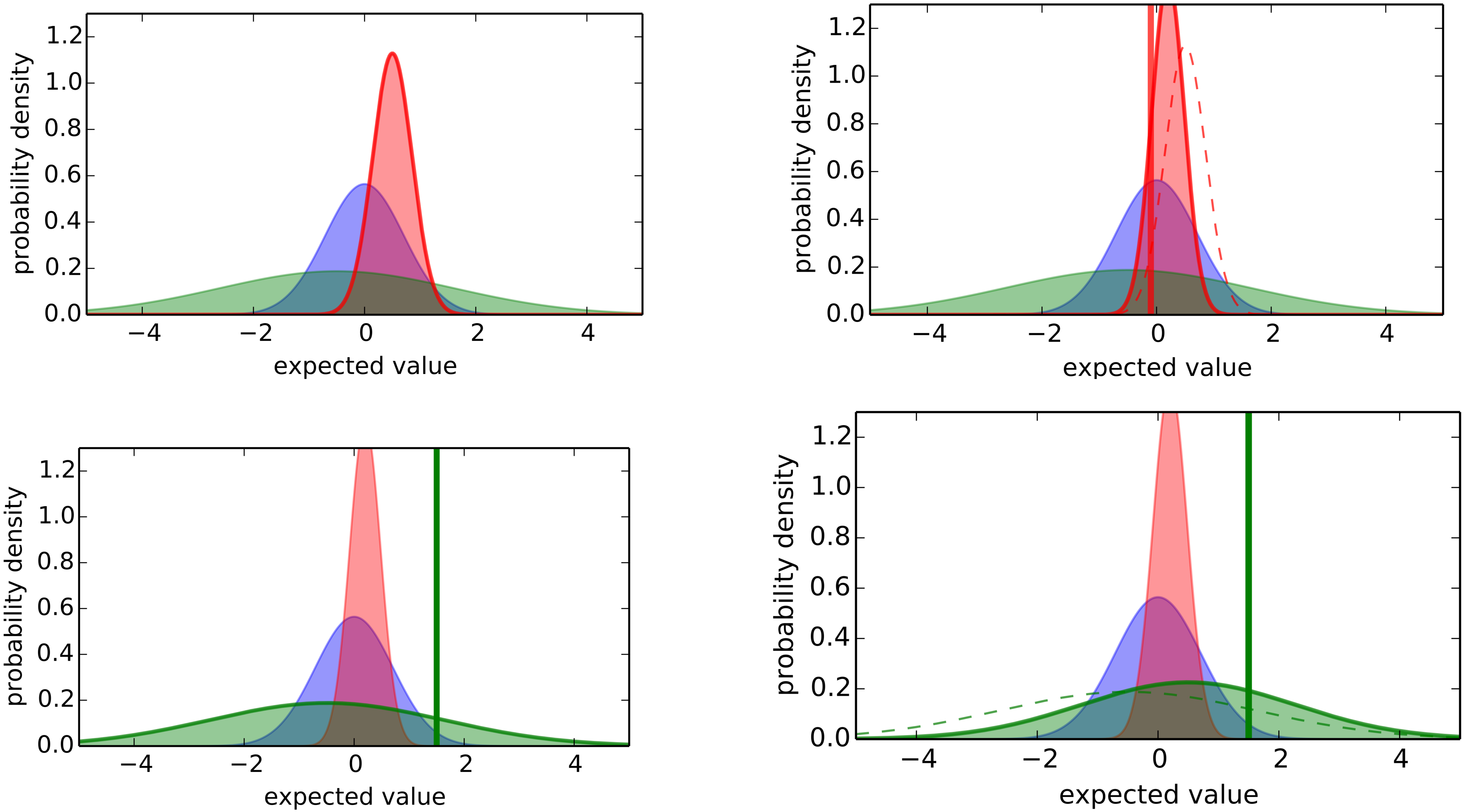

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Upper Confidence Bound

- Estimate an upper confidence $U_t(a)$ for each action value, such that $Q(a) \leq Q_t(a) + U_t(a)$ with high probability.
- Select action maximising upper confidence bound (UCB): $a_t = argmax_{a \in A}[Q_t(a) + U_t(a)]$

Optimality of UCB

Hoeffding's Inequality:

Let $X_1, \ldots, X_n$ be i.i.d. random variables in $[0,1]$ with true mean μ, and let $\bar{X}_n$ be the sample mean. Then $\mathbb{P}(\bar{X}_n + u \leq \mu) \leq e^{-2nu^2}$.

$$\mathbb{P}(Q_t(a) + U_t(a) \leq Q(a)) \leq e^{-2N_t(a)U_t(a)^2}$$

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If $U_t(a) = \sqrt{\frac{-\log p}{2N_t(a)}}$ then $e^{-2N_t(a)U_t(a)^2} = p$

Reduce p as we observe more rewards, e.g. $p = \frac{1}{t}$

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Reduce p as we observe more rewards, e.g. $p = \dfrac{1}{t}$

UCB

- Select action maximising upper confidence bound (UCB):

$$a_t = argmax_{a \in A}[Q_t(a) + c\sqrt{\frac{\log t}{2N_t(a)}}]$$

- Theorem: if $c = \sqrt{2}$ then UCB achieves logarithmic expected total regret

Bayesian Approach

- We could adopt Bayesian approach and model distributions over values $p(r \mid a) \approx p(r \mid \theta_a)$ and use model-based approach
- E.g., θ_a could contain the means and variances of Gaussian belief distributions
- Allows us to inject rich prior knowledge θ_a^0
- We can then use posterior belief to guide exploration

Thompson Sampling

- Priors $p(\theta_a), a \in \mathscr{A}$
- $p(\theta_a) \leftarrow p(\theta_a | r_t) \propto p(r_t | \theta_a)p(\theta_a)$ is a bayesian update
- We can choose an action with maximal expected reward under the known distributions: $a_{t+1} = argmax_a \mathbb{E}_{\theta_a \sim p(\theta_a)} \mathbb{E}_{p(r|\theta_a)} r$
- However there is a probability that the chosen action will be suboptimal: $\mathbb{E}_{p(r|\theta_b)} r > \mathbb{E}_{p(r|\theta_a)} r$
- Let's choose action with the probability of being optimal: $\pi(a) = \mathbb{P}(\mathbb{E}_{p(r|\theta_a)} r = \max_b \mathbb{E}_{p(r|\theta_b)} r)$
- We only have to sample $\theta_a \sim p(\theta_a), a \in \mathscr{A}$ and choose action with the maximal expected reward under the θ_a

Thank you for your attention!