The inverse of a matrix

Introduction

In this leaflet we explain what is meant by an inverse matrix and how it is calculated.

1. The inverse of a matrix

The **inverse** of a square $n \times n$ matrix A, is another $n \times n$ matrix denoted by A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

where I is the $n \times n$ identity matrix. That is, multiplying a matrix by its inverse produces an identity matrix. Not all square matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be **singular**. Only non-singular matrices have inverses.

2. A formula for finding the inverse

Given any non-singular matrix A, its inverse can be found from the formula

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

where adj A is the **adjoint matrix** and $|A|$ is the determinant of A. The procedure for finding the adjoint matrix is given below.

3. Finding the adjoint matrix

The adjoint of a matrix A is found in stages:

(1) Find the transpose of A, which is denoted by A^T. The transpose is found by interchanging the rows and columns of A. So, for example, the first column of A is the first row of the transposed matrix; the second column of A is the second row of the transposed matrix, and so on.

(2) The **minor** of any element is found by covering up the elements in its row and column and finding the determinant of the remaining matrix. By replacing each element of A^T by its minor, we can write down a *matrix of minors* of A^T.

(3) The **cofactor** of any element is found by taking its minor and imposing a **place sign** according to the following rule

$$\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \dots & \dots & \dots & \ddots \end{pmatrix}$$

This means, for example, that to find the cofactor of an element in the first row, second column, the sign of the minor is changed. On the other hand to find the cofactor of an element in the second row, second column, the sign of the minor is unaltered. This is equivalent to multiplying the minor by '+1' or '−1' depending upon its position. In this way we can form a *matrix of cofactors* of A^T. This matrix is called the **adjoint** of A, denoted adj A.

> The matrix of cofactors of the transpose of A, is called the **adjoint** matrix, adj A

This procedure may seem rather cumbersome, so it is illustrated now by means of an example.

Example

Find the adjoint, and hence the inverse, of $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$.

Solution

Follow the stages outlined above. First find the transpose of A by taking the first column of A to be the first row of A^T, and so on:

$$A^T = \begin{pmatrix} 1 & 3 & -1 \\ -2 & 1 & 2 \\ 0 & 5 & 3 \end{pmatrix}$$

Now find the minor of each element in A^T. The minor of the element '1' in the first row, first column, is obtained by covering up the elements in its row and column to give $\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$ and finding the determinant of this, which is -7. The minor of the element '3' in the second column of the first row is found by covering up elements in its row and column to give $\begin{pmatrix} -2 & 2 \\ 0 & 3 \end{pmatrix}$ which has determinant -6. We continue in this fashion and form a new matrix by replacing every element of A^T by its minor. Check for yourself that this process gives

$$\text{matrix of minors of } A^T = \begin{pmatrix} -7 & -6 & -10 \\ 14 & 3 & 5 \\ 7 & 0 & 7 \end{pmatrix}$$

Then impose the place sign. This results in the matrix of cofactors, that is, the adjoint of A.

$$\text{adj } A = \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

Notice that to complete this last stage, each element in the matrix of minors has been multiplied by 1 or -1 according to its position.

It is a straightforward matter to show that the determinant of A is 21. Finally

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{21} \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

Exercise

1. Show that the inverse of $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ -1 & 3 & 0 \end{pmatrix}$ is $\frac{1}{4} \begin{pmatrix} -3 & 6 & -7 \\ -1 & 2 & -1 \\ 5 & -6 & 5 \end{pmatrix}$.