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1 The Number Systems

1.1 The Naturals

Theorem I (Existence of $\mathbb{N}$). $\exists$ a set $\mathbb{N}$ satisfying the following Peano Axioms:

(PA1) $0 \in \mathbb{N}$

(PA2) $\exists S : \mathbb{N} \to \mathbb{N}$

(PA3) $\forall n \in \mathbb{N}, S(n) \neq 0$

(PA4) $S(n) = S(m) \implies n = m$

(PA5) Let $\mathsf{P}(n)$ be a property associated to each $n \in \mathbb{N}$. If
- $\mathsf{P}(0)$ is true, and
- $\mathsf{P}(n)$ is true $\implies \mathsf{P}(S(n))$,

then $\mathsf{P}(n)$ is true $\forall n \in \mathbb{N}$.

Proof. The existence of $\mathbb{N}$ follows directly from the Zermelo-Frankel axioms of set theory. ■

Definition (Addition). For any $m \in \mathbb{N}$, we define $0 + m = m$. Then, if $n + m$ is defined for $n \in \mathbb{N}$, we set $S(n) + m = S(n + m)$.

Proposition (Properties of Addition).

(A1) $\forall n \in \mathbb{N}, n + 0 = n$

(A2) $\forall m, n \in \mathbb{N}, n + S(m) = S(n + m)$

(A3) Commutativity. $\forall m, n \in \mathbb{N}, m + n = n + m$

(A4) Associativity. $\forall k, m, n \in \mathbb{N}, k + (m + n) = (k + m) + n$

(A5) Cancellation. $\forall k, m, n \in \mathbb{N}, n + k = n + m \implies k = m$

Remarks:

- $\forall n \in \mathbb{N}, 0 + n \overset{(A3)}{=} n + 0 \overset{(A1)}{=} n$
- $\forall n \in \mathbb{N}, S(n) \overset{(A1)}{=} S(n + 0) \overset{(A2)}{=} n + S(0) \overset{\text{def}}{=} n + 1$

Definition (Positivity). We say $n \in \mathbb{N}$ is *positive* iff $n \neq 0$.

Proposition (Properties of Positivity).

(P1) $\forall m, n \in \mathbb{N}, m \text{ positive} \implies m + n \text{ positive}$

(P2) $\forall m, n \in \mathbb{N}, m + n = 0 \implies m = n = 0$

(P3) $\forall n \in \mathbb{N}, n \text{ positive} \implies \exists! m \in \mathbb{N} \text{ s.t. } n = S(m)$

Proof.

(P1) Fix m and go by induction on n.

(P2) Suppose not, and either m or n positive. But then by (P1), $m+n$ positive – contradiction.

(P3) Uniqueness follows from (PA4). For existence go by induction on n and show that either $n = 0$ or $\exists m \in \mathbb{N}$ s.t. $S(m) = n$.

□

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Definition (Order). Fix $m, n \in \mathbb{N}$. We say $m \leq n$ or $n \geq m \iff n = m + p$ for some $p \in \mathbb{N}$. We say $m < n$ or $n > m \iff m \leq n$ and $m \neq n$.

Proposition (Properties of Order). Let $j, k, m, n \in \mathbb{N}$.

(O1) $n \geq n$

(O2) $m \leq n, k \leq m \implies k \leq n$

(O3) $m \geq n, m \leq n \implies m = n$

(O4) $j \leq k, m \leq n \implies j + m \leq k + n$

(O5) $j \leq k, m \leq n \implies j + m \leq k + n$

(O6) $m \leq n \iff S(m) \leq n$

(O7) $m < n \iff n = M + p$ for some positive $p \in \mathbb{N}$

(O8) $n \geq m \iff S(n) > m$

(O9) Either $n = 0$ or $n > 0$.

(O10) $n \in \mathbb{N}$ is positive $\iff 1 \leq n$

Theorem (Trichotomy of order on $\mathbb{N}$). Let $m, n \in \mathbb{N}$. Exactly one of $\{m < n, m = n, m > n\}$ is true.

Definition (Multiplication). Fix $m \in \mathbb{N}$. Define $0 \cdot m = 0$. Now, if $n \cdot m$ is defined for some $n \in \mathbb{N}$, we define $S(n) \cdot m = n \cdot m + m$.

Proposition (Properties of Multiplication).

(M1) $\forall m, n \in \mathbb{N}, m \cdot n = n \cdot m$

(M2) $\forall m, n \in \mathbb{N}, m, n \text{ positive} \implies m \cdot n \text{ positive}$

(M3) $\forall m, n \in \mathbb{N}, m \cdot n = 0 \iff m = 0 \text{ or } n = 0$

(M4) $\forall k, m, n \in \mathbb{N}, k \cdot (m \cdot n) = (k \cdot m) \cdot n$

(M5) $\forall k, m, n \in \mathbb{N}, k \text{ positive}, k \cdot m = k \cdot n \implies m = n$

(M6) $\forall k, m, n \in \mathbb{N}, k \cdot (m + n) = (m + n) \cdot k = k \cdot m + k \cdot n$

(M7) $\forall k, l, m, n \in \mathbb{N}, m < n, k \leq l, \text{and } k, l \text{ positive} \implies m \cdot k < n \cdot l$

Proof. By induction using properties of order/addition and their definitions. $\square$

1.2 The Integers

Consider the following relation on $\mathbb{N} \times \mathbb{N} = \{(m, n) \mid m, n \in \mathbb{N}\}$:

$$(m, n) \simeq (m', n') \iff m + n' = m' + n$$

Lemma. $\simeq$ is an equivalence relation: $[(m, n)] = \{(p, q) \mid (p, q) \simeq (m, n)\}$

Definition. $\mathbb{Z} = \{[(m, n)]\}$.

Definition. Let $[(n, m)], [(p, q)] \in \mathbb{Z}$. Then

1) $[(m, n)] + [(p, q)] = [(m + p, n + q)]$
2) $[(m, n)] \cdot [(p, q)] = [(mp + nq, mq + np)]$

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Remark. Notice that $\forall m, n \in \mathbb{N}$:

i) $[(m,0)] = [(n,0)] \iff m + 0 = n + 0 \iff m = n$
ii) $[(m,0)] + [(n,0)] = [(m+n,0)]$
iii) $[(m,0)] \cdot [(n,0)] = [(mn,0)]$

Definition.

1) For $n \in \mathbb{N}$ we set $n \in \mathbb{Z}$ to be $[(n,0)]$.
2) For $x = [(m,n)] \in \mathbb{Z}$ we define $-x = [(n,m)]$.

Theorem. Every $x \in \mathbb{Z}$ satisfies exactly one of $\{x = n, x = 0, x = -n\}$ where $n \in \mathbb{N}$ and n positive.

Proof. Write $x = [(p,q)]$ for some $p, q \in \mathbb{N}$ and use trichotomy of order on $\mathbb{N}$ for p, q. ■

Corollary. $\mathbb{Z} = \{0, 1, 2, \ldots\} \cup \{-1, -2, -3, \ldots\}$

Proposition (Algebra in $\mathbb{Z}$).

(AZ1) $x + y = y + x$
(AZ2) $x + (y + z) = (x + y) + z$
(AZ3) $x + 0 = 0 + x = x$
(AZ4) $x + (-x) = (-x) + x = 0$
(AZ5) $x \cdot y = y \cdot x$
(AZ6) $(x \cdot y) \cdot z = (x \cdot y) \cdot z$
(AZ7) $x \cdot 1 = 1 \cdot x = x$
(AZ8) $x \cdot (y + z) = x \cdot y + x \cdot z$

Proof. Write $x = [(m,n)], y = [(p,q)], z = [(k,l)]$ and expand using definitions. The results then follow from the corresponding results on $\mathbb{N}$. □

Definition. We define $x - y = x + (-y)$. The usual properties hold.

Definition. For $x, y \in \mathbb{Z}$:

- $x \le y \iff y \ge x \iff y - x = n$ for some $n \in \mathbb{N}$
- $x < y \iff y > x \iff x \le y$ and $x \ne y$.

Proposition (Properties of Order on $\mathbb{Z}$).

(OZ1) $x > y \iff x = y + p$ for some $p \in \mathbb{N}$ positive.
(OZ2) $x > y, z \ge w \implies x + z > y + w$
(OZ3) $x > y, z$ positive $\implies x \cdot z > y \cdot z$
(OZ4) $x > y \implies -y > -x$
(OZ5) $x > y, y > z \implies x > z$
(OZ6) Exactly one of $\{x < y, x = y, x > y\}$ holds.

Proof. Prove (OZ1). Everything else follows from Order on $\mathbb{N}$. □

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1.3 The Rationals and Ordered Fields

Consider the following relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$:

$$(m, n) \simeq (m', n') \iff mn' = m'n$$

Lemma. $\simeq$ is an equivalence relation.

Definition. $\mathbb{Q} = \{[(m, n)]\}$.

1) $[(m, n)] + [(p, q)] = [(mq + np, nq)]$
2) $[(m, n)] \cdot [(p, q)] = [(mp, nq)]$
3) If $m \neq 0$ we set $[(m, n)]^{-1} = [(n, m)]$.

Remark. Notice that $\forall m, n \in \mathbb{Z}$:

i) $[(m, 1)] = [(n, 1)] \iff m = n$
ii) $[(m, 1)] + [(n, 1)] = [(m + n, 1)]$
iii) $[(m, 1)] \cdot [(n, 1)] = [(m \cdot n, 1)]$

Definition.

1) If $m \in \mathbb{Z}$ we write $m = [(m, 1)] \in \mathbb{Q}$. In this way $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.
2) For $x, y \in \mathbb{Q}$, $x - y = x + (-y) \in \mathbb{Q}$.
3) For $x, y \in \mathbb{Q}, y \neq 0$, we define $x/y = x \cdot y^{-1}$.
   This is well-defined because $y \neq 0 \iff y = [(m, n)]$ with $m \neq 0$.

Proposition. $\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z}, n \neq 0\}$.

Proof. $x \in \mathbb{Q} \iff x = [(m, n)]$ for some $m, n \in \mathbb{Z}, n \neq 0$. But

$$x = [(m, n)] \overset{\text{def.}}{=} [(m, 1)] \cdot [(1, n)] = [(m, 1)] \cdot [(n, 1)]^{-1} = m \cdot n^{-1} = m/n$$

$\square$

Definition.

1) $\mathbb{Q}^{+} = \{m/n \in \mathbb{Q} \mid m, n \in \mathbb{N} \text{ are positive}\}$ is the positive rationals.
2) $\mathbb{Q}^{-} = \{-m/n \in \mathbb{Q} \mid m, n \in \mathbb{N} \text{ are positive}\}$ is the negative rationals.
3) For $x, y \in \mathbb{Q}$, we say $x < y$ or $y > x$ if and only if $y - x \in \mathbb{Q}^{+}$.
   We say $x \leq y$ or $y \geq x$ if and only if $x < y$ or $x = y$.

Proposition (Trichotomy of order on $\mathbb{Q}$). Let $x, y \in \mathbb{Q}$.
Exactly one of $\{x < y, x = y, x > y\}$ is true.

Proof. Follows directly from Trichotomy of order on $\mathbb{Z}$ (OZ6).

Definition. A **field** is a set $\mathbb{F}$ endowed with binary operations $(+, \cdot)$ satisfying the following axioms:

(A1) $\forall x, y \in \mathbb{F}, x + y \in \mathbb{F}$

(A2) $\forall x, y \in \mathbb{F}, x + y = y + x$

(A3) $\forall x, y, z \in \mathbb{F}, x + (y + z) = (x + y) + z$

(A4) $\exists 0 \in \mathbb{F}$ s.t. $\forall x \in \mathbb{F}, 0 + x = x + 0 = x$

(A5) $\forall x \in \mathbb{F}, \exists -x$ s.t. $x + (-x) = 0$

(M1) $\forall x, y \in \mathbb{F}, x \cdot y \in \mathbb{F}$

(M2) $\forall x, y \in \mathbb{F}, x \cdot y = y \cdot x$

(M3) $\forall x, y, z \in \mathbb{F}, x(yz) = (xy)z$

(M4) $\exists 1 \in \mathbb{F}$ s.t. $\forall x \in \mathbb{F}, 1 \cdot x = x \cdot 1 = x$

(M5) $\forall x \in \mathbb{F} \setminus \{0\}, \exists x^{-1} \in \mathbb{F}$ s.t. $x \cdot x^{-1} = 1$

(D1) $\forall x, y, z \in \mathbb{F}, x(y + z) = xy + xz$

Definition. Let E be a set. An order on E is a relation $<$ satisfying the following:

(1) $\forall x, y \in E$ exactly one of $\{x < y, x = y, x > y\}$ is true.

(2) $\forall x, y, z \in E$, $x < y$ and $y < z \implies x < z$.

Definition. An ordered field is a field $\mathbb{F}$ equipped with an order such that:

(1) $\forall x, y, z \in \mathbb{F}, y < z \implies x + y < x + z$

(2) $\forall x, y \in \mathbb{F}, x > 0, > 0 \implies x \cdot y > 0$

Theorem (Properties of an ordered field). Let $\mathbb{F}$ be a field. For $x, y, z \in \mathbb{F}$:

(F1) $x + z = y + z \iff x = y$

(F2) $x + y = 0 \implies y = -x$

(F3) $-(-x) = x$

(F4) $y \neq 0, xy = zy \implies x = z$

(F5) $x \neq 0, xy = 1 \implies y = x^{-1}$

(F6) $\left(x^{-1}\right)^{-1} = x$

(F7) $0 \cdot x = 0 \cdot x = 0$

(F8) $x \cdot y = 0 \implies x = 0$ or $y = 0$

(F9) $(-x)y = -(xy) = x(-y)$

(F10) $(-x)(-y) = xy$

If $\mathbb{F}$ is an ordered field, the following are also true.

(F11) $0 < x, z < y \implies xz < xy$

(F12) $x < 0, z < y \implies xy < xz$

(F13) $x > 0 \implies -x < 0$
$x < 0 \implies -x > 0$

(F14) $0 < y < x \implies 0 < x^{-1} < y^{-1}$

(F15) $x \neq 0 \implies x \cdot x > 0$

(F16) $0 < x < y \implies 0 < x \cdot x < y \cdot y$

Definition. Let $\mathbb{F}$ be a field. We define $x - y = x + (-y)$ and $x/y = xy^{-1}$ when $y \neq 0$.

Theorem. $\mathbb{Q}$ is an ordered field with order $<$.

1.4 Problems with $\mathbb{Q}$

Theorem. There does not exist a $q \in \mathbb{Q}$ such that $q^2 = 2$.

We view this result as informally saying that $\mathbb{Q}$ has "holes".

Definition. Let E be an ordered set with order $<$.

1) We say $A \subseteq E$ is bounded above $\iff \exists x \in E$ s.t. $a \leq x \; \forall a \in A$.
We say such an x is an upper bound of A.

2) We say $A \subseteq E$ is bounded below $\iff \exists x \in E$ s.t. $x \leq a \; \forall a \in A$.
   We say such an x is a lower bound of A.
3) We say $A \subseteq E$ is bounded $\iff$ it's bounded above and below.
4) x is a minimum of $A \iff x \in A$ and x is a lower bound of A.
5) x is a maximum of $A \iff x \in A$ and x is an upper bound of A.

Definition. Let E be an ordered set and $A \subseteq E$.

(1) $x \in E$ is the *supremum* (least upper bound) of A, written $x = \sup A$, if and only if:
- x is an upper bound of A
- if $y \in E$ is an upper bound of A, then $x \leq y$

(2) $x \in E$ is the *infimum* (greatest lower bound) of A, written $x = \inf A$, if and only if:
- x is a lower bound of A
- if $y \in E$ is a lower bound of A, then $y \leq x$

Definition. Let E be an ordered set. We say that E has the *least-upper-bound property* if and only if every $\emptyset \neq A \subseteq E$ that is bounded from above has a least-upper-bound.

Theorem. $\mathbb{Q}$ does not satisfy the least-upper-bound property.

1.5 The Real Numbers

Our goal now is to use $\mathbb{Q}$ to construct an ordered field satisfying the least-upper-bound property.

Definition. We say $\mathbb{Q}$ is **Archimedean** if and only if $\forall x \in \mathbb{Q}, x > 0 \implies \exists n \in \mathbb{N}$ s.t. $x < n$.

Lemma. If $\mathbb{Q}$ is Archimedean, then $\forall p, q \in \mathbb{Q}, p < q, \exists r \in \mathbb{Q}$ s.t. $p < r < q$.

We'll now construct the real numbers $\mathbb{R}$, proceeding through several steps.

1.5.1 Definitions

Recall that $\mathcal{P}(E)$ is the *power set* of a given set E.

Definition. We say that $\mathcal{C} \in \mathcal{P}(\mathbb{Q})$ is a **Dedekind cut** if and only if the following properties hold:

(C1) $e \neq \emptyset, e \neq \mathbb{Q}$
(C2) If $p \in \mathcal{C}$ and $q \in \mathbb{Q}$ with $q < p$, then $q \in \mathcal{C}$.
(C3) If $p \in \mathcal{C}$ then $\exists r \in \mathbb{Q}$ with $p < r$ s.t. $r \in \mathcal{C}$.

Lemma. Suppose $\mathcal{C}$ is a cut. Then

1) $p \in \mathcal{C}, q \notin \mathcal{C} \implies p < q$
2) $r \notin \mathcal{C}, r < s \implies s \notin \mathcal{C}$
3) $\mathcal{C}$ is bounded above.

Lemma. Suppose $q \in \mathbb{Q}$. Then $\mathcal{C} = \{p \in \mathbb{Q} \mid p < q\}$ is a cut.

Proof. We verify properties (C1-3):

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(C1) $q - 1 \in \mathcal{C} \implies \mathcal{C} \neq \emptyset$

(C2) If $p \in \mathcal{C}$ and $r \in \mathbb{Q}$ s.t. $r < p$, then $r < p < q \implies r < q \implies r \in \mathcal{C}$.

(C3) Let $p \in \mathcal{C}$ s.t. $p < q$. Since $\mathbb{Q}$ is Archimedean, $\exists r \in \mathbb{Q}$ s.t. $p < r < q \implies r \in \mathcal{C}$.

Definition. Given $q \in \mathbb{Q}$ we write $\mathcal{C}_q = \{p \in \mathbb{Q} \mid p < q\}$. By the above, $\mathcal{C}_q$ is a cut.

Definition. We write $\mathbb{R} = \{\mathcal{C} \in \mathcal{P}(\mathbb{Q}) \mid \mathcal{C} \text{ is a cut}\} \neq \emptyset$

Notation. Let A, B be two sets. We say $A \subseteq B \iff \forall x \in A, x \in B$.
We say $A = B \iff A \subseteq B, B \subseteq A$. We say $A \subset B \iff A \subseteq B, A \neq B$.

Lemma. The following hold:

- $\forall \mathcal{A}, \mathcal{B} \in \mathbb{R}$, exactly one of $\{\mathcal{A} \subset \mathcal{B}, \mathcal{A} = \mathcal{B}, \mathcal{B} \subset \mathcal{A}\}$ holds.
- $\forall \mathcal{A}, \mathcal{B} \in \mathbb{R}, \mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{C} \implies \mathcal{A} \subset \mathcal{C}$

Definition. If $\mathcal{A}, \mathcal{B} \in \mathbb{R}$, we say $\mathcal{A} < \mathcal{B} \iff \mathcal{A} \subset \mathcal{B}$.
This defines an order on $\mathbb{R}$ by the above lemma. We'll write $\mathcal{A} \leq \mathcal{B} \iff \mathcal{A} \subseteq \mathcal{B}$.

1.5.2 The least-upper-bound property

Lemma. Suppose $\emptyset \neq E \subseteq \mathbb{R}$ is bounded above. Then $\mathcal{B} = \bigcup_{\mathcal{A} \in E} \mathcal{A} \in \mathbb{R}$.

Theorem. $\mathbb{R}$ satisfies the least-upper-bound property.

1.5.3 Addition

Lemma. If $\mathcal{A}, \mathcal{B} \in \mathbb{R}$ then $\mathcal{A} + \mathcal{B} \in \mathbb{R}$.

Theorem. $\mathbb{R}, +, 0_{\mathbb{R}} = \mathcal{C}_0 = \{p \in \mathbb{Q} \mid p < 0\}$ satisfy the field axioms (A1-5) if we define

$$-\mathcal{A} = \{q \in \mathbb{Q} \mid \exists p > q \text{ s.t. } -p \notin \mathcal{A}\}$$

Claim. $\exists n \in \mathbb{Z}$ s.t. $n \cdot t \in \mathcal{A}$ and $(n+1) \cdot t \notin \mathcal{A}$.

Proof. Fix $p \in \mathcal{A}$. We first use the Archimedean property to produce $m \in \mathbb{N}$ s.t. $-p < mt$ and then $-mt < p \implies -mt \in \mathcal{A}$. Now consider the set $E = \{k \in \mathbb{N} \mid (-m+k) + \notin \mathcal{A}\}$. Notice that $0 \notin E$. Let $q \notin \mathcal{A}$. Then we may use the Archimedean property again to choose $k \in \mathbb{N}$ s.t. $m + q/t < k \implies mt + q < kt \implies q < (-m+k)t \implies (-m+k)t \notin \mathcal{A}$, and so $k \in E$. By the Well-ordering principle $\exists! e \in E$ s.t. $l = \min E$, and $l \geq 1$. Since $l \in E$ we know $(-m+l)t \notin \mathcal{A}$. Since $l - 1 \notin E$ and $l - 1 \in \mathbb{N}$, we know that $(-m+l-1)t \in \mathcal{A}$. Set $n = -m + l - 1 \in \mathbb{Z}$ and we're done. □

Theorem. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}$. Then $\mathcal{A} < \mathcal{B} \implies \mathcal{A} + \mathcal{C} < \mathcal{B} + \mathcal{C}$.

Proof. Trivially, $\mathcal{A} \subseteq \mathcal{B} \implies \mathcal{A} + \mathcal{C} \subseteq \mathcal{B} + \mathcal{C} \implies \mathcal{A} + \mathcal{C} \leq \mathcal{B} + \mathcal{C}$. If $\mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C}$ then we can add $-\mathcal{C}$ to both sides and use the last theorem to see that $\mathcal{A} = \mathcal{B}$ — contradiction. □

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1.5.4 Multiplication

Lemma. Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}$ satisfy $\mathcal{A}, \mathcal{B} > 0_{\mathbb{R}}$. Then

$$\mathcal{C} = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{ab \mid a \in \mathcal{A}, b \in \mathcal{B}, a > 0, b > 0\} \in \mathbb{R}$$

Proof.

(C1) $0 \in \mathcal{C} \implies \mathcal{C} \neq 0$. $\mathcal{A}$ and $\mathcal{B}$ are bounded above by some M_1 and M_2 respectively, so $M_1 \cdot M_2 + 1$ is not in $\mathcal{C}$, so $\mathcal{C} \neq \mathbb{Q}$.

(C2) Let $p \in \mathcal{C}$ and $q < p$. $q \leq 0 \overset{\text{def.}}{\implies} q \in \mathcal{C}$. If $q > 0$ then $0 < q < p$, but then $0 < p \implies p = a \cdot b$ for $a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0$. Then $0 < q < a \cdot b \implies q/a < b \implies 0 < q/a \in \mathcal{B}$. But then $q = a(q/a) \in \mathcal{C}$.

(C3) Let $p \in \mathcal{C}$. If $p \leq 0$ then any $a \cdot b$ with $a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0$ satisfies $p < a \cdot b \in \mathcal{C}$, so $r = a \cdot b$ is the desired element of rC. OTOH, if $p > 0$ then $p = a \cdot b$ for $a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0$. Choose $s \in \mathcal{A}$ s.t. $a < s$, $t \in \mathcal{B}$ s.t. $t > b$. Then $p = a \cdot b < s \cdot t \in \mathcal{C}$, so $r = s \cdot t$ does the job.

Definition. Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}$.

1) If $\mathcal{A} > 0_{\mathbb{R}}, \mathcal{B} > 0_{\mathbb{R}}$ we set $\mathcal{A} \cdot \mathcal{B} = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{a \cdot b \mid a \in \mathcal{A}, a > 0, b \in \mathcal{B}, b > 0\} \in \mathbb{R}$
2) If $\mathcal{A} = 0_{\mathbb{R}}$ or $\mathcal{B} = 0_{\mathbb{R}}$, we set $\mathcal{A} \cdot \mathcal{B} = 0 \in \mathbb{R}$.
3) If $\mathcal{A} > 0_{\mathbb{R}}$ and $\mathcal{B} < 0_{\mathbb{R}}$ we set $\mathcal{A} \cdot \mathcal{B} = -(\mathcal{A} \cdot (-\mathcal{B})) \in \mathbb{R}$.
4) If $\mathcal{A} < 0$ and $\mathcal{B} > 0$ we set $\mathcal{A} \cdot \mathcal{B} = -((-\mathcal{A}) \cdot \mathcal{B}) \in \mathbb{R}$.
5) If $\mathcal{A} < 0$ and $\mathcal{B} < 0$ we set $\mathcal{A} \cdot \mathcal{B} = (-\mathcal{A}) \cdot (-\mathcal{B}) \in \mathbb{R}$.

Theorem. $\{\mathbb{R}, \cdot\}$ satisifes (M1-5) with $1_{\mathbb{R}} = \mathcal{C}$, and:

- $\mathcal{A} > 0_{\mathbb{R}} \implies \mathcal{A}^{-1} = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{q \in \mathbb{Q} \mid q > 0, \exists p > q \text{ s.t. } p^{-1} \notin \mathcal{A}\} \in \mathbb{R}$
- $\mathcal{A} < 0 \implies \mathcal{A}^{-1} = -(-\mathcal{A})^{-1}$

Theorem. If $\mathcal{A}, \mathcal{B} > 0$ then $\mathcal{A} \cdot \mathcal{B} > 0$.

Proof. By definition $\mathcal{C}_0 \subseteq \mathcal{A} \cdot \mathcal{B} \implies 0_{\mathbb{R}} \leq \mathcal{A} \cdot \mathcal{B}$. It's clear that equality is impossible since $\mathcal{A} > 0, \mathcal{B} > 0$. □

1.5.5 Distributivity

Theorem. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}$. Then $\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$

Proof. We prove only the case $\mathcal{A}, \mathcal{B}, \mathcal{C} > 0$:
Let $p \in \mathcal{A} \cdot (\mathcal{B} + \mathcal{C})$. If $p \leq 0$ then $p \in \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$ is trivial. If $p > 0$ then $p = a \cdot (b + c)$ for $a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}$ with $a > 0, b + c > 0$. Regardless of sign of b or c, $a \cdot b \in \mathcal{A} \cdot \mathcal{B}$, $a \cdot c \in \mathcal{A} \cdot \mathcal{C}$. Hence $p = a \cdot (b + c) = a \cdot b + a \cdot c \in \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$. So, $\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) \subseteq \mathcal{A} \cdot \mathcal{C} + \mathcal{A} \cdot \mathcal{C}$.
We claim the opposite inclusion is true. Let $p \in \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C} \implies p = r + s$ for $r \in \mathcal{A} \cdot \mathcal{B}, s \in \mathcal{A} \cdot \mathcal{C}$. If $p \leq 0$ we know $p \in \mathcal{A} \cdot (\mathcal{B} + \mathcal{C})$ by definition, so it suffices to assume $p > 0$. Then $0 < p = r + s \implies$ at least one of r, s is ≥ 0.

Case 1: $r > 0, s \leq 0$. Then $r = a \cdot b$ for $a \in \mathcal{A}, b \in \mathcal{B}, a, b > 0$. Then $p = r + s = a \cdot b + s \leq a(b + 0)$ and $b + 0 \in \mathcal{B} + \mathcal{C}$ since $\mathcal{C} > 0$. So, $p \in \mathcal{A}(\mathcal{B} + \mathcal{C})$

Case 2: $s > 0, r \leq 0$ A similar argument shows $p \in \mathcal{A}(\mathcal{B}+\mathcal{C})$

Case 3: $r > 0, s > 0$ Then $r = ab,\ s = \hat{a}c, a\hat{a} \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}, a, \hat{a}, b, c > 0$

$$a \geq \hat{a} \implies p = r + s = ab + \hat{a}c \leq ab + ac = a(b+c) \in \mathcal{A}(\mathcal{B}+\mathcal{C})$$
$$a < \hat{a} \implies p = r + s = ab + \hat{a}c \leq \hat{a}b + \hat{c} = \hat{a}(b+c) \in \mathcal{A}(\mathcal{B}+\mathcal{C})$$

In either case, $p \in \mathcal{A}(\mathcal{B}+\mathcal{C})$ Therefore $p \in \mathcal{A}(\mathcal{B}+\mathcal{C})$ and hence $\mathcal{AB}+\mathcal{AC} \subseteq \mathcal{A}(\mathcal{B}+\mathcal{C})$

1.5.6 $\mathbb{Q} \subseteq \mathbb{R}$

Theorem. For $p, q \in \mathbb{Q}$ the following are true:

(1) $\mathcal{C}_{p+q} = \mathcal{C}_p + \mathcal{C}_q$

(2) $\mathcal{C}_{-p} = -\mathcal{C}_p$

(3) $\mathcal{C}_{pq} = \mathcal{C}_p\mathcal{C}_q$

(4) If $p \neq 0$ then $\mathcal{C}_{p^{-1}} = \mathcal{C}_p^{-1}$

(5) $p < q$ in $\mathbb{Q} \iff \mathcal{C}_p < \mathcal{C}_q$ in $\mathbb{R}$

Definition. For $q \in \mathbb{Q}$ we write $q = \mathcal{C}_q \in \mathbb{R}$. This allows us to say $\mathbb{Q} \subseteq \mathbb{R}$.

1.5.7 $\mathbb{R}$ is Archimedean

Theorem. $\mathbb{R}$ is Archimedean.

Proof. HW. □

Now we combine steps to finish.

Theorem. There exists an ordered field $\mathbb{R}$ that satisfies the least-upper-bound property. Moreover,

(1) $\mathbb{Q} \subseteq \mathbb{R}$

(2) $\mathbb{R}$ is unique: if $\mathbb{F}$ is another ordered field satisfying the least-upper-bound property, then $\mathbb{F} = \mathbb{R}$ (up to isomorphism).

(3) $\mathbb{R}$ is Archimedean.

Proof. The basic assertion is steps 0-4. Step 5 is (1), step 6 is (3). The proof of (2) is in this week's reading. □

1.6 Properties of $\mathbb{R}$

We know that $\mathbb{R}$ is Archimedean, and from homework we know the following.

Proposition. $\mathbb{R}$ satisfies the following.

(1) $\mathbb{R}$ is Archimedean: $\forall x \in \mathbb{R}, x > 0, \exists n \in \mathbb{N}$ s.t. $x < n$.

(2) $\mathbb{N} \subset \mathbb{R}$ is not bounded above.

(3) $\inf(\{1/n \mid n \in \mathbb{N}, n \geq 1\}) = 0$

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(4) $\forall x \in \mathbb{R}$ the set $B(x) = \{m \in \mathbb{Z} \mid x < m\}$ has a minimum $\in \mathbb{Z}$.

(5) $\forall x, y \in \mathbb{R}, x < y, \exists q \in \mathbb{Q}$ s.t. $x < q < y$.

Remarks.

1) (5) is interpreted as "the density of $\mathbb{Q} \in \mathbb{R}$". That is to say, any element $x \in \mathbb{R}$ can be approximated to arbitrary accuracy by elements of $\mathbb{Q}$. Indeed, for $n \geq 1$ choose q s.t.

$$x - \frac{1}{n} < q < x + \frac{1}{n} \implies 1 - \frac{1}{n} < x - 1 < \frac{1}{n}$$

By extension, $\mathbb{Q}_b$ is also dense in $\mathbb{R}$ for all $b \in \mathbb{N}, b \geq 2$

2) (5) allows us to define the "integer part" of any $x \in \mathbb{R}$. Indeed for any $x \in \mathbb{R}$ we set $\lfloor x \rfloor = \min(B(x)) - 1 \in \mathbb{Z}$. Then $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ and $0 \geq x - \lfloor x \rfloor < 1$.

Theorem. Let $x \in \mathbb{R}$ satisfy $x > 0$ and $n \in \mathbb{N}$ s.t. $n \geq 1$. Then $\exists! y \in \mathbb{R}$ s.t. $y > 0$ and $y^n = x$.

Proof. The case $n = 1$ is trivial, so assume that $n \geq 2$. Set $E = \{z \in \mathbb{R} \mid z > 0 \text{ and } z^n < x\}$. We want to show that $E \neq \emptyset$ and is bounded above. Set $t = x/(x+x)$ so $0 < t < 1$ and $t < x$. Hence $0 < t^n < t < x$, and so $t \in E$, which means $E \neq \emptyset$.
Set $s = 1 + x$. Then $1 < s$ and $x < s$. By a similar argument $x < s < s^n$, so if $z \in E$ then $z^n < x < s^n \implies z < s$. Hence s is an upper bound of E. By the least-upper-bound property $\exists y \in \mathbb{R}$ s.t. $y = \sup E$.
Notice that $t \in E \implies 0 < t < y$, so $y > 0$.
We claim that $y^n < x$ and $y^n > x$ are both impossible.

Claim: For $b, a \in \mathbb{R}$ we have $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})$.
(*) Then if $0 < a < b$ we may estimate $0 < b^n - a^n < (b-a)nb^{n-1}$.

1) Suppose $y^n < x$. Choose $h \in \mathbb{R}$ s.t. $0 < h < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-1}}$. Then we use (*) with $b = y + h, a = y$ to see that

$$0 < (y+h)^n - y^n < hn(y+h)^{n-1} < x - y^n \implies (y+h)^n < x, y > 0 \implies y + h \in E$$

But $y < y + h$ and so y is not an upper bound of E — contradiction.

2) Suppose $x < y^n$. Choose $0 < k < \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = y/n < y$, so $0 < y - k$. Set $b = y$ and $a = y - k$ in (*) to see

$$y^n - (y-k)^n < nky^{n-1} < y^n - x \implies x < (y-k)^n$$

So if $z \in E$ then $0 < z^n < (y-k)^n \implies z < y - k$, so $y - k$ is an upper bound of E, but $y - k < y$, contradicting the fact that y is the *least* upper bound.

Thus, by trichotomy, $y^n = x$.
Uniqueness is easy: if $y_1^n = y_2^n$ and $y_1 < y_2$, we get $y_1^n < y_2^n$ — contradiction. □

Definition. Let $n \geq 1$. For $x \in \mathbb{R}, x > 0$ we write $x^{\frac{1}{n}} = y$ where $y^n = x$. We set $0^{\frac{1}{n}} = 0$. Then we define the function $(\cdot)^{\frac{1}{n}} : \{x \in \mathbb{R} \mid x \geq 0\} \to \{x \in \mathbb{R} \mid x \geq 0\}$ via $x^{\frac{1}{n}} = y$ s.t. $y^n = x$.

Corollary. If $x, y \in \mathbb{R}, x, y \geq 0$ and $n \in \mathbb{N}$ s.t. $n \geq 1$, then $x^{\frac{1}{n}} y^{\frac{1}{n}} = (xy)^{\frac{1}{n}}$.

Proof. $(x^{\frac{1}{n}})^n = x, (y^{\frac{1}{n}})^n = y \implies xy = (x^{\frac{1}{n}})^n (y^{\frac{1}{n}})^n = (x^{\frac{1}{n}} y^{\frac{1}{n}})^n$. But then $(xy)^{\frac{1}{n}} = x^{\frac{1}{n}} y^{\frac{1}{n}}$ by definition.

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Corollary. Let $n \in \mathbb{N}$ with $n \geq 1$, $k \in \mathbb{N}$ with $k \geq 2$. Suppose $x_y \in \mathbb{R}$ with $x_i \geq 0$ for $i = 1, \ldots, k$. Then $(x_1 \cdot x_2 \cdots x_k)^{\frac{1}{n}} = x_1^{\frac{1}{n}} \cdot x_2^{\frac{1}{n}} \cdots x_k^{\frac{1}{n}}$.

Proof. Exercise: use induction on k.

Lemma. Let $x \in \mathbb{R}$ satisfy $x > 0$ and let $m, n, p, q \in \mathbb{Z}$ with $n, q > 0$.
If $\frac{m}{n} = \frac{p}{q}$, then $(x^m)^{\frac{1}{n}} = (x^p)^{\frac{1}{q}}$.

Proof. Since $mq = np \in \mathbb{Z}$ we know that $x^{mq} = x^{np}$. Then by the uniqueness of q^{th} roots we have

$$x^{mq} = (x^m)^q = x^{np} \implies x^m = (x^{np})^{\frac{1}{q}} = (\underbrace{x^p \cdot x^p \cdots x^p}_{\text{n times}})^{\frac{1}{q}} = \underbrace{(x^p)^{\frac{1}{q}}(x^p)^{\frac{1}{q}} \cdot x(x^p)^{\frac{1}{q}}}_{\text{n times}} = \left[(x^p)^{\frac{1}{q}}\right]^n$$

Again by uniqueness of roots, $(x^p)^{\frac{1}{q}} = (x^m)^{\frac{1}{n}}$. $\square$

Definition. For $r \in \mathbb{Q}$ with $r = m/n, n > 0$ we set $x^r = (x^m)^{\frac{1}{n}}$ whenever $x > 0$. The lemma guarantees that this is well-defined: if if $r = m/n = p/q, n, q > 0$, then $(x^m)^{\frac{1}{n}} = (x^p)^{\frac{1}{q}}$.

Definition. For $x \in \mathbb{R}$ we define

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

We may view $|\cdot| : \mathbb{R} \to \{x \in \mathbb{R} \mid x \geq 0\}$ as a function. We call $|x|$ the absolute value of x.

Proposition (Properties of $|\cdot|$).

1. $|x| \geq 0 \; \forall x \in \mathbb{R}$, and $|x| = 0 \iff 0 = x$
2. $\forall x, y \in \mathbb{R}, |x| < y \iff -y < x < y$
3. $\forall x, y \in \mathbb{R}, |xy| = |x||y|$
4. $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$
5. $\forall x, y \in \mathbb{R}, ||x| - |y|| \leq |x - y|$

2 Sequences

Definition. Let E be a set and $l \in \mathbb{Z}$. We say that a function $a : \{n \in \mathbb{Z} \mid n \geq l\} \to E$ is a sequence in E. We write $a_n = a(n) \in E$. Typically $l = 0$ or $l = 1$. We will also usually write $\{a_n\}_{n=l}^{\infty} \subseteq E$ to denote the sequence and the set in which it takes values.

Examples.

(1) $a_n = \frac{1}{n}$ for $n \geq 1$ $(E = \mathbb{Q} \subseteq \mathbb{R}, l = 1)$

(2) Let $\mathbb{F}$ be a field and $x \in \mathbb{F}$. Let $a_n = x^n \in \mathbb{F} \; \forall n \geq 0$. $(E = \mathbb{F}, l = 0)$

(3) Let $a_n = 3^{\frac{1}{n}}$ for $n \geq 1$. Here $(E = \mathbb{R}, l = 1)$

For the rest of this section we set $E = \mathbb{R}$.

2.1 Convergence

Definition. Let $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$. We say that $\{a_n\}$ converges to $a \in \mathbb{R}$ if and only if $\forall \epsilon \in \mathbb{R}, \epsilon > 0$ $\exists N \in \mathbb{Z}, N \geq l$ s.t. if $n \geq N$, then $|a_n - a| < \epsilon$.

We write $a = \lim_{n\to\infty} a_n$ or $a_n \to a$ as $n \to \infty$ when $\{a_n\}_{n=l}^{\infty}$ converges to a.

Examples.

(1) $a_n = \frac{1}{n}, n \geq 1$. We claim $a_n \to 0$ as $n \to \infty$. Let $\epsilon > 0$. We want to find $N \in \mathbb{N}$ s.t. $n \geq N \implies |a_n - a| < \epsilon$. Notice that $|a_n - 0| = |\frac{1}{n}| = \frac{1}{n}$, so it suffices to show that $\frac{1}{n} < \epsilon$ whenever $n \geq N$. $\mathbb{R}$ satisfies the Archimedean property. Hence $\exists N \in \mathbb{N}$ s.t. $\frac{1}{\epsilon} < N$. Then $n \geq N \implies \frac{1}{\epsilon} < N \leq n \implies \frac{1}{n} < \epsilon$. Since $\epsilon > 0$ was arbitrary, we deduce that $a_n \to 0$.

(2) $a_n = \frac{1}{n^2}, n \geq 1$. We claim $a_n \to 0$. Let $\epsilon > 0$ and choose the same N as in (1). Then if $n \geq N$ we know $\frac{1}{\epsilon} < N \leq n \leq n^2 \implies |a_n - 0| = \frac{1}{n^2} < \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that $a_n \to 0$.

(3) $a_n = n, n \geq 0$. We claim that a_n does not converge. If not, then $\{a_n\}$ does converge, and so $\exists a \in \mathbb{R}$ s.t. $a_n \to a$ as $n \to \infty$. Choose $\epsilon = 1$. Then $\exists N \in \mathbb{N}$ s.t. $n \geq N \implies |a_n - a| < \epsilon = 1$. Then $n \leq N \implies n = |a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|$, which contradicts the Arcihmedean property.

Lemma. If $a_n \to a$ and $a_n \to b$ as $n]to]infty$, then $a = b$. That is to say, limits are unique.

Proof. If not, then $|a - b| > 0$. Set $\epsilon = \frac{|a-b|}{4} > 0$. Since $a_n \to a$ $\exists N_1, N_2$ s.t.

$$\begin{cases} n \geq N_1 \implies |a_n - a| < \epsilon = \frac{a-b}{4} \\ n \geq N_2 \implies |a_n - b| < \epsilon = \frac{a-b}{4} \end{cases}$$

Se $N = \max\{N_1, N_2\}$, Then $n \geq N \implies |a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |b - a_n| < \frac{a-b}{4} + \frac{a-b}{4} = \frac{a-b}{2} \implies |a - b| < 0 \implies |a - b| - \frac{a-b}{2} < 0 \implies |a - b| < 0$, contradiction.

Definition. We say that a sequence $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$ is bounded if and only if $\exists M \in \mathbb{R} with M > 0$ s.t. $|a_n| < M$ $\forall n \geq l$.

Lemma. If $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$ converges, then $\{a_n\}$ is bounded.

Proof. Since $\{a_n\}$ is convergent, we know $\exists a \in \mathbb{R}$ s.t. $a_n \to a$ as $n \to \infty$. Choose $\epsilon = 1$. Then $\exists N$ s.t. $n \geq N \implies |a_n - a| < \epsilon = 1$. In particular, $n \geq N \implies |a_n| \implies |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|$. Let $K = \max\{|a_l|, |a_{l+1}|, \ldots, |a_{N-1}|\} \in \mathbb{R}$. Then $M = \max\{K, 1 + |a|\}$ satisfies $|a_n| < M$ $\forall n \geq l$. Hence $\{a_n\}$ is bounded. $\square$

Definition. Given $\{a_n\}, \{b_n\} \subseteq \mathbb{R}$, we define $\{a_n + b_n\} \subseteq \mathbb{R}$ to be the sequence whose elements are $a_n + b_n$. We similarly define $\{c \cdot a_n\}$ for fixed $c \in \mathbb{R}$, $\{a_n \cdot b_n\}$, and $\{a_n / b_n\}$, provided that $b_n \neq 0$, $n \geq l$.

Theorem (Algebra of convergence). Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$, $c \in \mathbb{R}$, and assume that $a_n \to a$, $b_n \to b$ as $n \to \infty$. Then the following hold:

(1) $a_n + b_n \to a + b$ as $n \to \infty$

(2) $c \cdot a_n \to c \cdot a$ as $n \to \infty$

(3) $a_n \cdot b_n \to a \cdot b$ as $n \to \infty$

(4) If $b_n \neq 0$ and $b \neq 0$ then $\frac{a_n}{b_n} \to \frac{a}{b}$ as $n \to \infty$

Proof. (1), (2) are in next week's HW.

(3) Notice first that

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n b_n - ab_n| + |ab_n - ab| = |b_n||a_n - a| + |a||b_n - b|$$

Since $b_n \to b$ we know that $\exists M > 0$ s.t. $|b_n| \leq M$ $\forall n \geq l$. Let $\epsilon > 0$. Then

$$\text{Since} \begin{cases} a_n \to a \text{ we may choose } N_1 \text{ s.t. } n \geq N_1, |a_n - a| < \frac{\epsilon}{2M} \\ b_n \to b \text{ we may choose } N_2 \text{ s.t. } n \geq N_2, |b_n - b| < \frac{\epsilon}{2(1+|a|)} \end{cases}$$

Then set $N = \max\{N_1, N_2\}$. So if $n \geq N$ we know

$$|a_n b_n - ab| \leq |b_n||a_n - a| + |a||b_n - b| < M|a_n - a| + |a||b_n - b| < \frac{M \cdot \epsilon}{2M} + \frac{|a|\epsilon}{2(1+|a|)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since ϵ was arbitrary, we deduce that $a_n b_n \to ab$.

(4) First notice that

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - ab_n}{b_n b}\right| = \left|\frac{a_n b - ab + ab - ab_n}{b_n b}\right| \leq \frac{|a_n b - ab|}{|b_n||b|} + \frac{|ab - ab_n|}{|b||b_n|} = \frac{|a_n - a|}{|b_n|} + \frac{|a|}{|b||b_n|}|b_n - b|$$

Let $\epsilon > 0$. Since $b_n \to b \neq 0$ we know that $\exists N_1$ s.t. $n \geq N_1 \implies |b_n - b| < \frac{|b|}{2}$. So

$$\begin{aligned} n \geq N &\implies 0 < |b| = |b - b_n + b_n| \leq |b - b_n| + |b_n| < \frac{|b|}{2} + |b_n| \\ &\implies 0 < \frac{|b|}{2} \leq |b_n| \implies 0 < \frac{1}{|b_n|} < \frac{2}{|b|} \end{aligned}$$

$$\text{OTOH} \begin{cases} a_n \to a \implies \exists N_2 \text{ s.t. } \left(n \geq N_2 \implies |a_n - a| < \frac{\epsilon}{4}|b|\right) \\ b_n \to b \implies \exists N_3 \text{ s.t. } \left(n \geq N_3 \implies |b_n - b| < \frac{\epsilon}{1+|a|}\frac{|b|^2}{4}\right) \end{cases}$$

Set $N = \max\{N_1, N_2, N_3\}$. Then for $n \geq N$ we know

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq \frac{|a_n - a|}{|b_n|} + \frac{|a|}{|b_n||b|}|b_n - b| < \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2}\left(\frac{|a|}{1+|a|}\right) < \epsilon$$

Since $\epsilon > 0$ was abritrary, we deduce that $\frac{a_n}{b_n} \to \frac{a}{b}$ as $n \to \infty$.

Lemma. Let $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$ converge to $a \in \mathbb{R}$. Then $\forall \epsilon > 0$ $\exists N$ s.t. $m, n \geq N \implies |a_n - a_m| < \epsilon$.

Proof. Let $\epsilon > 0$. Since $a_n \to a$ we can choose N s.t. $n \geq N \implies |a_n - a| < \frac{\epsilon}{2}$. Then

$$m, n \geq N \implies |a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_M - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Definition. We say $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$ is **Cauchy** if and only if $\forall \epsilon > 0$ $\exists N$ s.t. $m, n \geq N \implies |a_n - a_m| < \epsilon$.

Lemma. If $\{a_n\}$ is Cauchy, then it's bounded.

Proof. Let $\epsilon = 1$. Then $\exists N$ s.t. $m,n \geq N \implies |a_m - a_n| < 1$. In particular, $n \geq N \implies |a_n - a_N| < 1 \implies |a_n| < |a_n - a_N| + |a_N| < 1 + |a_N|$. Set $M = \max\{1 + |a_N|, K\}$, where $K = \max\{|a_l|, \ldots, |a_{N-1}|\}$. Then $|a_n| < M\ \forall n \geq l$. Hence $\{a_n\}$ is bounded. $\square$

Theorem. Let $\{a_n\} \subseteq \mathbb{R}$. Then $\{a_n\}$ converges $\iff$ $\{a_n\}$ is Cauchy.

Proof.

$\implies$**:** is the 2^{nd} to last lemma.

$\impliedby$**:** Suppose $\{a_n\}$ is Cauchy. This means $|a_n| < M\ \forall n \geq l$ by the last lemma. Set $E = \{x \in \mathbb{R} \mid \exists N \text{ s.t. } n \geq N \implies x < a_n\}$. Note that $-M < a_n\ \forall n \geq l$ and so $-M \in E$, so $E \neq 0$. OTOH, $x \in E \implies \exists N_x$ s.t. $n \geq N_x \implies x < a_n < M$, and so M is an upper bound of E. $\mathbb{R}$ satisfies the l.u.b. property, so we know that $a = \sup E \in \mathbb{R}$. We claim that $a_n \to a$ as $n \to \infty$.
Let $\epsilon > 0$. Then since $\{a_n\}$ is Cauchy $\exists N$ s.t. $m,n \geq N \implies |a_n - a_m| < \frac{\epsilon}{2}$. In particular, $|a_n - a_N| < \frac{\epsilon}{2}$ when $n \geq N$. Then

$$n \geq N \implies a_N - \frac{\epsilon}{2} < a_n \implies a_N - \frac{\epsilon}{2} \in E \implies a_N - \frac{\epsilon}{2} \leq a$$

OTOH

$$x \in E \implies \exists N_x \text{ s.t. } \left(n \geq N_x \implies x < a_n < a_N + \frac{\epsilon}{2}\right)$$

Hence $a_N + \frac{\epsilon}{2}$ is an upper bound of $E \implies a \leq a_N + \frac{\epsilon}{2}$. Combining, we see that $|a - a_N < \frac{\epsilon}{2}$. But then $n \geq N$ then $|a_n - a| \leq |a_n - a_N| + |a_N - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Lemma (Squeeze lemma): Let $\{a_n\}_{n=l}^{\infty}, \{b_n\}_{n=l}^{\infty}, \{c_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$ and suppose that $a_n \to a, c_n \to a$ as $n \to \infty$. If $\exists K \geq l$ s.t. $a_n \leq b_n \leq c_n\ \forall n \geq K$ then $b_n \to a$ as $n \to \infty$.

Proof. Let $\epsilon > 0$. Since $a_n \to a$, $c_n \to a$ $\exists N_1, N_2$ s.t.

$$\begin{cases} n \geq N_1 \implies |a_n - a| < \epsilon & (-\epsilon < a_n - a < \epsilon) \\ n \geq N_2 \implies |c_n - a| < \epsilon & (-\epsilon < c_n - a < \epsilon) \end{cases}$$

Set $N = \max\{N_1, N_2, K\}$. Then $n \geq N \implies -\epsilon < a_n - a \leq b_n - a \leq c_n - a < \epsilon \implies |b_n - a| < \epsilon$. Since ϵ was arbitrary, we deduce that $b_n \to a$ as $n \to \infty$. $\square$

Examples:

1) Suppose $a_n \to 0$ and $\{b_n\}$ is bounded, i.e. $|b_n| \leq M\ \forall n \geq l$. Then $|a_n b_n| = |a_n||b_n| \leq |a_n| M$. From HW $c_n \to 0 \iff |c_n| \to 0$. Then $\overset{\to 0}{0} \leq |a_n b_n| \leq \overset{\to 0}{|a_n|} M$, and by the Squeeze lemma $|a_n b_n| \to 0 \implies a_n b_n \to 0$.
2) Fix $k \in \mathbb{N}$ with $k \geq 1$. Set $a_n = \frac{1}{n^k}, n \geq 1$. Then $\overset{\to 0}{0} \leq \frac{1}{n^k} \leq \overset{\to 0}{\frac{1}{n}} \overset{(SL)}{\implies} \frac{1}{n^k} \to 0$
3) Fix $k \in \mathbb{N}$ with $k \geq 2$. Let $a_n = \frac{1}{k^n}, n \geq 0$. Claim: $n \leq k^n\ \forall n \in \mathbb{N}$. Proof is by induction on n. Then we know $0 \leq \frac{1}{k^n} \leq \frac{1}{n}$, so by the Squeeze lemma $\frac{1}{k^n} \to 0$ as $n \to \infty$.

2.2 Monotonicity and limsup, liminf

Definition. Let $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$. We say that $\{a_n\}$ is

1) increasing iff $a_n < a_{n+1}\ \forall n \geq l$

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2) non-decreasing iff $a_n \leq a_{n+1}$ $\forall n \geq l$
3) decreasing iff $a_{n+1} < a_n$ $\forall n \geq l$
4) non-increasing iff $a_{n+1} \leq a_n$ $\forall n \geq l$

We say $\{a_n\}$ is monotone iff it is either non-increasing or non-decreasing.
Remark: increasing $\implies$ non-decreasing; decreasing $\implies$ non-increasing.

Theorem. Suppose that $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$ is monotone.
Then $\{a_n\}_{n=0}^{\infty}$ is bounded $\iff$ $\{a_n\}_{n=l}^{\infty}$ is convergent.

Proof. $\Leftarrow$ is done in a previous lemma. For the forwards direction we'll prove the result when the sequence is non-decreasing. The other case is handled by a similar argument. Set $E = \{a_n \mid n \geq l\} \subseteq \mathbb{R}$. Clearly $E \neq \emptyset$. Also, since $\{a_n\}_{n=l}^{\infty}$ is bounded, the set E is as well, and in particular it's bounded above. By the least-upper-bound property of $\mathbb{R}$ $\exists a = \sup E \in \mathbb{R}$. We claim that $a = \lim_{n\to\infty} a_n$. Let $\epsilon > 0$. Since $a = \sup E$ we know that $a - \epsilon$ is not an upper bound of E, and hence $\exists N \geq l$ s.t. $a - \epsilon < a_N$. But since the sequence is non-decreasing, $a_n \leq a_{n+1}$ $\forall n \geq l$, and so $n \geq N \implies a_N \leq a_n$. Then

$$n \geq N \implies a - \epsilon < a_N \leq a_n \leq a$$

because a is an upper bound of E. So

$$n \geq N \implies -\epsilon < a_N - a \leq 0 \implies |a_n - a| < \epsilon$$

Since $\epsilon > 0$ was arbitrary, we deduce that $a_n \to a$ as $n \to \infty$. $\square$

Lemma. Suppose that $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ is bounded. Set

$$S_m = \sup\{a_n \mid n \geq m\}$$
$$I_m = \inf\{a_n \mid n \geq m\}$$

Then $S_m, I_m \in \mathbb{R}$ are well-defined $\forall m \geq l$ and $\{S_m\}_{m=l}^{\infty}$ is non-increasing, and $\{I_m\}_{m=l}^{\infty}$ is non-decreasing. Both sequences are bounded.

Proof. Let $E_m = \{a_n \mid n \geq m\}$. The set E is bounded since the sequence is. As such, $\sup E_m = S_m \in \mathbb{R}$. Similarly, $\inf E_m = I_m \in \mathbb{R}$. Also, $E_{m+1} \subseteq E_m$, so

$$\begin{cases} S_{m+1} = \sup E_{m+1} \leq \sup E_m = S_m & \text{so } \{S_m\} \text{ is non-increasing} \\ I_m = \inf E_m \leq \inf E_{m+1} = I_{m+1} & \text{so } \{I_m\} \text{ is non-decreasing} \end{cases}$$

It's easy to see that if $|a_n| \leq M$ $\forall n \geq l$ then $|S_m| \leq M, |I_m| \leq M$ $\forall m \geq l$. $\square$

Definition. Suppose $\{a_n\}_{n=l}^{\infty} \subseteq \mathbb{R}$ is bounded. We set

$$\limsup_{n\to\infty} a_n = \lim_{m\to\infty} S_m \in \mathbb{R}$$
$$\liminf_{n\to\infty} a_N = \lim_{m\to\infty} I_m \in \mathbb{R}$$

Both limits exist by the lemma and the previous theorem.
It's also true that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$.

Examples.

1) $a_n = (-1)^n, n \geq 0$. $E_m = \{a_n \mid n \geq m\} = \{+1, -1\}\ \forall m \geq 0$. $S_m = 1, I_m = -1$, so $\limsup_{n\to\infty}(-1)^n = 1$, $\liminf_{n\to\infty}(-1)^n = -1$.
2) For $n \geq 0$:
$$a_n = \begin{cases} 3 & n \text{ even} \\ \frac{1}{n} & n \text{ odd} \end{cases}$$
$S_m = 3\ \forall m, I_m = 0\ \forall m$. Then $\limsup_{n\to\infty} a_n = 3, \liminf_{n\to\infty} a_n = 0$.
3) Fix $p \in \mathbb{N}$ with $p \geq 2$. For every $n \geq 1$ $\exists! q_n, r_n$ with $0 \leq r_n < p$ s.t. $n = pq_n + r_n$. Set $a_n = r_n\ \forall n \geq 1$. Then $\limsup_{n\to\infty} a_n = p-1, \liminf_{n\to\infty} a_n = 0$.

2.3 Subsequences

Definition. Let $\varphi : \{n \in \mathbb{Z} \mid n \geq l\} \to \{n \in \mathbb{Z} \mid n \geq l\}$ be order preserving (increasing), which is to say $m < n \implies \varphi(m) < \varphi(n)$. Let $\{a_n\}_{n=l}^\infty \subseteq \mathbb{R}$ be a sequence. We say the sequence $\{a_{\varphi(k)}\}_{k=l}^\infty$ is a subsequence of $\{a_n\}_{n=l}^\infty$.

Remarks.

1) $\varphi(k) = k$ is order preserving, so every sequence is a subsequence of itself.
2) Not every a_n has to be in the subsequence $\{a_{\varphi(k)}\}_{k=l}^\infty$. For example, if $l = 0$ then $\varphi(k) = 2k$ is order preserving. In this case a_n, n odd does not appear in the subsequence $\{a_{\varphi(k)}\}_{k=l}^\infty$.
3) We will often write $n_k = \varphi(k)$ to simplify notation. $\{a_{n_k}\}_{k=l}^\infty$ denotes a subsequence.
4) From HW1 we know that $k \leq \varphi(k)\ \forall k \geq l$.

Proposition. Suppose $\{a_n\}_{n=l}^\infty \subseteq \mathbb{R}$ satisfies $a_n \to a \in \mathbb{R}$ as $n \to \infty$.
Then any subsequence of $\{a_n\}_{n=l}^\infty$ also converges to a.

Proof. Let $\{a_{\varphi(k)}\}_{k=l}^\infty$ be a subsequence of $\{a_n\}_{n=l}^\infty$. Let $\epsilon > 0$. Since $a_n \to a$ as $n \to \infty$, $\exists N \geq l$ s.t. $n \geq N \implies |a_n - a| < \epsilon$. We claim $\exists K \geq l$ s.t. $k \geq K \implies \varphi(k) \geq N$. If not, then $\varphi(k) \leq N\ \forall k \geq l$, but $k \leq \varphi(k) < N\ \forall k \geq l$ is a contradiction. Hence, the claim is true. Then $k \geq K \implies \varphi(k) \geq N \implies |a_{\varphi(k)} - a| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we deduce that $a_{\varphi(k)} \to a$ as $k \to \infty$. □

Remark. The converse fails. For example: $a_n = (-1)^n$ does not converge, but a_{2n} and a_{2n+1} converge to 1 and -1 respectively.

Theorem (Limsup theorem). Let $\{a_n\}_{n=l}^\infty \subseteq \mathbb{R}$ be bounded. The following hold.

1) Every subsequence of $\{a_n\}_{n=l}^\infty$ is bounded.
2) If $\{a_{n_k}\}_{k=l}^\infty$ is a subsequence, then $\limsup_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$.
3) If $\{a_{n_k}\}_{k=l}^\infty$ is a subsequence, then $\liminf_{n\to\infty} a_{n_k} \leq \liminf_{k\to\infty} a_n$.
4) $\exists$ a subsequence $\{a_{n_k}\}_{k=l}^\infty$ s.t. $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$.
5) $\exists$ a subsequence $\{a_{n_k}\}_{k=l}^\infty$ s.t. $\lim_{k\to\infty} a_{n_k} = \liminf_{n\to\infty} a_n$.

Proof.

1) is trivial.

2) Since $k \le \varphi(k)$ we have that $\{a_{\varphi(n)} \mid n \ge k\} \subseteq \{a_n \mid a \ge k\}$ for every order preserving φ. Hence, $\sup\{a_{\varphi(n)} \mid n \ge k\} \subseteq \sup\{a_n \mid n \ge k\}$. But

$$\limsup_{n\to\infty} a_{\varphi(n)} = \lim_{k\to\infty} \sup\{a_{\varphi(n)} \mid n \ge k\} \le \lim_{k\to\infty} \sup\{a_n \mid n \ge k\} = \limsup_{n\to\infty} a_n$$

3) Similar to (2): left as an exercise.
4) For convenience let's assume $l = 0$. Otherwise we just study $b_n = a_{n+l}, n \ge 0$. Set $n_0 = 0$. Note that $\limsup_{n\to\infty} a_n = \lim_{k\to\infty} \sup\{a_n \mid n \ge k\} = \lim_{k\to\infty} S_k$. There must exists $n_1 > n_0$ s.t. $S_{n_0+1} - 1 < a_{n_1} \le S_{n_0+1}$ because $S_{n_0+1} = \sup\{a_n \mid n \ge n_0+1\}$. Suppose now that we have chosen $n_1 < n_2 < \cdots < n_k$ s.t. $S_{n_{j-1}+1} - \frac{1}{j} < a_{n_j} \le S_{n_{j-1}} + 1$ for $1 \le j \le k$. We may then choose $n_{k+1} > n_k$ s.t. $S_{n_k+1} - \frac{1}{k} < a_{n_{k+1}} \le S_{n_k+1}$ by the same reason as above. This yields a subsequence $\{a_{n_k}\}_{k=0}^\infty$ s.t. $S_{1+n_{k-1}} - \frac{1}{k} < a_{n_k} \le S_{1+n_{k-1}}$ $\forall k \ge 1$. Note that by the earlier proposition, $S_{n_{k-1}+1} \to \lim_{n\to\infty} S_n = \limsup_{n\to\infty} a_n$. So, $S_{1+n_{k-1}} - \frac{1}{k} \to \limsup_{n\to\infty} a_n$. By the squeeze lemma, $a_{n_k} \to \limsup_{n\to\infty} a_n$. □
5) Similar to (4): left as an exercise.

Theorem. Suppose $\{a_n\}_{n=l}^\infty \subseteq \mathbb{R}, a \in \mathbb{R}$. The following are equivalent:

1) $a_n \to a$ as $n \to \infty$.
2) $\{a_n\}$ is bounded, and every convergent subsequence converges to a.
3) $\{a_n\}$ is bounded, and $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

If any hold, then $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

Proof.

(1) $\implies$ (2) is done already.

(2) $\implies$ (3) Limsup theorem (4)/(5) $\implies$ $\exists$ subsequences $\{a_{\varphi(k)}\}_{k=l}^\infty, \{a_{\psi(k)}\}_{k=l}^\infty$ s.t.

$$a_{\varphi(k)} \to \limsup_{n\to\infty} a_n \qquad a_{\psi(k)} \to \liminf_{n\to\infty} a_n$$

as $k \to \infty$. By (2), the limits must agree.

(3) $\implies$ (1) We know

$$I_m = \inf\{a_n \mid n \ge m\} \le \sup\{a_n \mid n \ge m\} = S_m$$

By (3), $I_m \to \liminf_{n\to\infty} a_n, S_m \to \limsup_{n\to\infty} a_n$, so the squeeze lemma implies that $a_m \to \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_m$. □

Theorem (Bolzano-Weierstrass): If $\{a_n\}_{n=l}^\infty \subseteq \mathbb{R}$ is bounded, $\exists$ a convergent subsequence.

Proof. Item (4) or (5) of Limsup theorem. □

Theorem. Let $\{a_n\}_{n=l}^\infty \subseteq \mathbb{R}$ be bounded. $E = \{x \in \mathbb{R} \mid x \text{ is a limit of a subsequence of } \{a_n\}_{n=l}^\infty\}$. Then the following hold.

1) $E \ne 0$, and E is bounded.
2) $\max E = \limsup_{n\to\infty} a_N$, $\min E = \liminf_{n\to\infty} a_n$.

Proof.

1) follows from Bolzano-Weierstrass and the fact that $\{a_n\}_{n=l}^\infty$ is bounded.

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2) We'll prove only that $\max E = \limsup_{n\to\infty} a_n$. The other identity follows from a similar argument. If $x \in E$ then $x = \lim_{k\to\infty} a_{n_k}$ for some subsequence $\{a_{n_k}\}_{k=l}^{\infty}$. By Limsup theorem and the limsup/liminf characterization of convergence, we know that

$$x = \lim_{k\to\infty} a_{n_k} = \limsup_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$$

Hence $\limsup_{n\to\infty} a_n$ is an upper bound of E. But the Limsup theorem says that there is a subsequence $\{a_{n_k}\}_{k=l}^{\infty}$ s.t. $a_{n_k}\ k \overset{\to}{\to} \infty \limsup_{n\to\infty} a_n$. Hence $\limsup_{n\to\infty} a_n \in E$, and so $\max E = \limsup_{n\to\infty} a_n$. □

2.4 Some special sequences

Definition. Given $a_k \in \mathbb{R}$ for $0 \leq k \leq n, n \in \mathbb{N}$, we define

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n$$

Lemma (Binomial Theorem). Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y_{n-k} \qquad \text{where } \binom{n}{k} := \frac{n!}{k!(n-k)!} \in \mathbb{N}$$

Proof. By induction. □

Theorem. In the following assume that $n \geq 1$.

1) Let $x \in \mathbb{R}$ with $x > 0$. Then $a_n = \frac{1}{n^x} \to 0$ as $n \to \infty$.
2) Let $x \in \mathbb{R}$ with $x > 0$. Then $a_n = x^{\frac{1}{n}} \to 1$ as $n \to \infty$.
3) Let $x_n = n^{\frac{1}{n}}$. Then $a_n \to 1$ as $n \to \infty$.
4) Let $\alpha, x \in \mathbb{R}$ with $x > 0$. Then $a_n = \frac{n^\alpha}{(1+x)^n} \to 0$ as $n \to \infty$.
5) Let $x \in \mathbb{R}$ with $|x| < 1$. Then $a_n = x^n \to 0$ as $n \to \infty$.

Proof.

1) Choose $q \in \mathbb{Q}$ with $0 < q < x$. Then by definition $n^q < n^x$, and so $0 < \frac{1}{n^x} < \frac{1}{n^q}$. Also by HW, $\frac{1}{n^q} \to 0$ as $n \to \infty$ when $q \in \mathbb{Q}$ with $q > 0$. Hence by the Squeeze lemma, $\frac{1}{n^x} \to 0$.
2) Assume first that $x > 1$. Then $\left(x^{\frac{1}{n}}\right)^n = x > 1^n \iff x^{\frac{1}{n}} > 1$. Set $b_n = x^{\frac{1}{n}} - 1 > 0$. Then $(1+b_n)^n = \left(x^{\frac{1}{n}}\right)^n = x$. By the Binomial theorem,

$$x = (1+b_n)^n = \sum_{k=0}^{n} \binom{n}{k} b_n^k \underbrace{1^{n-k}}_{=1} \geq 1 + \binom{n}{1} b_n = 1 + nb_n$$

So, $0 < b_n \leq \frac{x-1}{n} \to 0$. By the Squeeze lemma, $b_n \to 0$ as $n \to \infty$. But $b_n = a_n - 1$, so $a_n \to 1$ as $n \to \infty$ when $x > 1$. If $x = 1$ then $a_n = 1 \to 1$ as $n \to \infty$. If $x < 1$ then $\frac{1}{x} > 1 \implies \frac{1}{x^{\frac{1}{n}}} = \left(\frac{1}{x}\right)^{\frac{1}{n}} \to 1$ and so $x^{\frac{1}{n}} \to 1$ as $n \to \infty$ as well.

3) Let $b_n = n^{\frac{1}{n}} - 1 > 0$. Then

$$n = (1+b_n)^n = \sum_{k=0}^{n} \binom{n}{k} b_n^k \geq 1 + \binom{n}{2} b_n^2 = 1 + \frac{n(n-1)}{2} b_n^2$$

So, if $n \geq 2$ then $\frac{n(n-1)}{2} b_n^2 \leq n-1 \implies 0 < b_n \leq \left(\frac{2}{n}\right)^{\frac{1}{2}} \to 0$, so again by the Squeeze lemma, $b_n \to 0$. Hence $a_n \to 1$ as $n \to \infty$.

4) Fix $k \in \mathbb{N}$ s.t. $k > \max\{1, \alpha\}$. Then if $n \geq 2k$ we have that

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j \geq \binom{n}{k} x^k = \frac{n(n-1)\cdots(n-k+1)}{k!} x^k \geq \left(\frac{n}{2}\right)^k \frac{x^k}{k!}$$

Then $n \geq 2k \implies 0 < \frac{n^\alpha}{(1+x)^n} \leq \frac{2^k k! n^\alpha}{x^k n^k} = \left(\frac{2^k k!}{x^k}\right) \frac{1}{n^{k-\alpha}} \to 0$ since $k = \alpha > 0$. Again by the Squeeze lemma, $\frac{n^\alpha}{(1+x)^n} \to 0$.

5) Since $|x| < 1$ we know $1 < \frac{1}{|x|} \implies z = \frac{1}{|x|} - 1 > 0$. By (4) with $\alpha = 0$, we know that $\frac{1}{(1+z)^n} \to 0$ as $n \to \infty$. But $\frac{1}{1+z} = \frac{1}{1/|x|} = |x|$, so $|x|^n \to 0$, but $|x|^n = |x^n|$, so $x^n \to 0$ (by HW).

3 Series

Definition. Let $\{a_n\}_{n=l}^\infty \subseteq \mathbb{R}$. For $p < q$ we write

$$\sum_{n=p}^{q} a_n = a_p + \cdots + a_q$$

1) We define, for each $n \geq l$, $S_n = \sum_{k=l}^n a_k \in \mathbb{R}$ to be the n^{th} partial sum of $\{a_n\}_{n=l}^\infty$.
2) If $\exists S \in \mathbb{R}$ such that $S_n \to S$ as $n \to \infty$ then we write $\sum_{n=l}^\infty a_n = S$. In this case we say the "infinite series" $\sum_{n=l}^\infty a_n$ converges.
3) If $\sum_{n=l}^\infty a_n$ does not converge, then we say it diverges.

Examples.

1) Let $a_n = x^n$ for some $x, n \in \mathbb{R}$ with $n \geq 0$. Then $S_n = \sum_{k=0}^n x^k$. Notice that

$$(1-x)S_n = \sum_{k=0}^{n} x^k - \sum_{k=0}^{n} x^{k+1} = \sum_{k=0}^{n} x^k - \sum_{k=1}^{n+1} x^k = x^0 - x^{n+1} = 1 - x^{n+1}$$

Hence $S_n = \frac{1-x^{n+1}}{1-x}$. By (5) of the previous theorem, if $|x| < 1$ then $S_n \to \frac{1}{1-x}$. So, $|x| < 1 \implies \sum_{n=0}^\infty x^n = \frac{1}{1-x}$. In particular $\sum_{n=0}^\infty \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$.

2) Suppose $\{b_n\}_{n=0}^\infty \subseteq \mathbb{R}$ s.t. $b_n \to b$ as $n \to \infty$. Set $a_n = b_{n+1} - b_n$ for $n \geq 0$. Then the series $\sum_{n=0}^\infty a_n$ converges and in particular $\sum_{n=0}^\infty a_n = b - b_0$.

$$S_n = \sum_{k=0}^{n} a_k = (b_{n+1} - b_n) + \cdots + (b_1 + b_0) = b_{n+1} - b_0$$

But $b_{n+1} - b_0 \to b - b_0$ as $n \to \infty$, so by definition $\sum_{n=0}^\infty a_n = b - b_0$.

3.1 Convergence results

Our goal here is to develop tools that will let us deduce the convergence of a series without actually knowing its value.

Theorem. Suppose $\sum_{n=l}^{\infty} a_n$ converges. Then $a_n \to 0$ as $n \to \infty$.

Proof. Notice that $a_n = S_n - S_{n-1}$ and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = S_S = 0$.

Corollary. $\sum_{n=0}^{\infty}(-1)^n$ and $\sum_{n=0}^{\infty} n$ both diverge.

Proof. $(-1)^n$ does not converge to 0, and neither does n.

Corollary. The series $\sum_{n=0}^{\infty} x^n$ converges $\iff |x| < 1$.

Proof. $\Leftarrow$ was done previously.
$\Rightarrow$ It suffices to note that $|x| \geq 1 \implies |x^n| = |x|^n \geq 1\ \forall n \in \mathbb{N}$.

Next we provide a characterization of convergence in terms of the size of the "tails" of a series.

Theorem. $\sum_{n=l}^{\infty} a_n$ converges $\iff \forall \epsilon > 0, \exists N \geq l$ s.t. $m \geq k \geq N \implies |\sum_{n=k}^{m} a_n| < \epsilon$.

Proof. $\sum_{n=l}^{\infty} a_n$ converges $\iff S_k = \sum_{n=l}^{k} a_n$ converges $\iff \{S_k\}$ is Cauchy. $\iff \forall \epsilon > 0$, $\exists N \geq l$ s.t. $m \geq k \geq N \implies |\sum_{n=k}^{m} a_n| < \epsilon$. □

Theorem.

1) Suppose $|a_n| \leq b_n\ \forall n \geq K$ for some $K \geq l$. If $\sum_{n=l}^{\infty} b_n$ converges, then $\sum_{n=l}^{\infty} a_n$ converges.
2) If $0 \leq a_n \leq b_n\ \forall n \geq K$ for some $K \geq l$, and $\sum_{n=l}^{\infty} a_n$ diverges, then $\sum_{n=l}^{\infty} b_n$ diverges.

Proof.

1) Since $\sum_{n=l}^{\infty} b_n$ converges we know that $\forall \epsilon > 0, \exists N \geq l$ s.t. $m \geq k \geq N \implies |\sum_{n=k}^{m} b_n| < \epsilon$. Let $\epsilon > 0$. Then if $m \geq k \geq \max\{N, K\}$ we have that
$$\left|\sum_{n=k}^{m} a_n\right| \leq \sum_{n=k}^{m} |a_n| \leq \sum_{n=k}^{m} b_n < \epsilon$$
Since $\epsilon > 0$ is arbitrary, by the previous theorem we deduce that $\sum_{n=l}^{\infty} a_n$ converges.
2) This follows from the contrapositive of (1).

Examples.

1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ converges because $\left|\frac{(-1)^n}{2^n}\right| = \frac{1}{2^n}$ and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges ($\frac{1}{2} < 1$).
2) Suppose $\sum_{n=0}^{\infty} a_n$ converges and $a_n \geq 0\ \forall n \geq 0$. Let $\{b_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ be bounded, i.e. $|b_n| \leq M\ \forall n$. Then $|a_n b_n| = |a_n||b_n| \leq M a_n$. Clearly $MS_n = M\sum_{k=0}^{n} a_n = \sum_{k=0}^{n} M a_n$, so $\sum_{n=0}^{\infty} M a_n = M \sum_{n=0}^{\infty} a_n$. Hence by the theorem, $\sum_{n=0}^{\infty} a_n b_n$ converges.
3) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \cdot \frac{n!}{n^n} \cdot \frac{3n^2}{4n^2+2}$ converges because $\frac{(-1)^n}{n^n} n! \frac{3n^2}{4n^2+2}$ is bounded.

Theorem. Suppose $a_n \geq 0\ \forall n \geq l$. $\sum_{n=l}^{\infty} a_n$ converges $\iff \{S_n\}_{n=l}^{\infty}$ is bounded.

Proof. Since $a_n \geq 0\ \forall n \geq l$, the sequence $S_n = \sum_{k=l}^{n} a_k$ is non-decreasing. $S_{n+1} = a_{n+1} + S_n \geq S_n$. We know that monotone sequences converge $\iff$ they are bounded.

Theorem (Cauchy criterion). Suppose that $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ satisfies $a_n \geq 0\ \forall n \geq 1$, and $a_{n+1} \leq a_n\ \forall n \geq 1$. Then $\sum_{n=1}^{\infty} a_n$ converges $\iff$ $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Proof. Let $\sum_{k=1}^{n} a_k$ and $T_m = \sum_{n=0}^{m} 2^n a_{2^n}$. Notice that if $m \leq 2^k$ then

$$\begin{aligned} S_m &\leq a_1 + \cdots + a_{2^k} \\ &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} = T_k \end{aligned}$$

On the other hand, if $m \geq 2^k$ then

$$\begin{aligned} S_m &\geq a_1 + \cdots + a_{2^k} \\ &= a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}-1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2}T_k \end{aligned}$$

Now, $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges, then $T_n \to T$ as $n \to \infty$, and so $S_m \leq \lim_{n\to\infty} T_n = T$, which means $\{S_m\}$ is bounded, and hence $\sum_{n=1}^{\infty} a_n$ converges. Similarly, if $\sum_{n=1}^{\infty} a_n$ converges, then $T_k \leq 2\lim_{n\to\infty} S_n \implies \{T_k\}$ is bounded $\implies \sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

⋮

5 Continuity

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5.3 Compactness and Continuity

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Theorem. Let $E \subseteq \mathbb{R}$ be compact and $n \in \mathbb{N}$ with $n \geq 2$. Then $f(x) = x^n$ is uniformly continuous on E.

Proof. Let $\varepsilon > 0, \delta = \varepsilon / nM^{n-1}$. Then

$$|x^n - y^n| = |x - y||x^{n-1} + x^{n-2}y + \cdots + y^{n-1}| \leq |x - y| nM^{n-1} < \varepsilon$$

Definition. We say $f : E \to \mathbb{R}$ is **Lipschitz** if $\forall x, y \in E\ |f(x) - f(y)| \leq k|x - y|$ for some $k > 0$.

Theorem. If f is Lipschitz, then f is uniformly continuous.

Theorem. If $K \subseteq \mathbb{R}$ compact and $f : K \to \mathbb{R}$ is continuous, then f is uniformly continuous on K.

Proof. Let $\varepsilon > 0$. Since f is continuous on K, we know that

$$\forall x \in K \quad \exists \delta_x > 0.\ y \in K \wedge |x-y| < \delta_x \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$$

Clearly $\{B(x, \delta_x/2)\}_{x \in K}$ is an open cover of K. Since K is compact, there exists a finite subcover where $K \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_{x_i}/2)$. Let $\delta = \min\{\delta_{x_i}/2 \mid i = 1, \ldots, n\} > 0$.

Suppose $x, y \in K$ and $|x - y| < \delta$. By construction of the finite subcover,

(1) $\exists i \in 1, \ldots, n.\ |x - x_i| < \delta_{x_i}/2 \implies |f(x) - f(x_i)| < \varepsilon/2$

(2) $\exists i \in 1, \ldots, n.\ |y - x_i| \leq |y - x| + |x - x_i| < \delta + \delta_{x_i}/2 \leq \delta_{x_i} \implies |f(y) - f(x_i)| < \varepsilon/2$

Hence $|f(x) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

5.4 Continuity and Connectedness

Theorem. Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be continuous on E.
If $X \subseteq E$ is connected then $f(X)$ is connected.

Theorem (Intermediate Value Theorem).
Let $a, b \in \mathbb{R}$ with $a < b$. Suppose $f : [a, b] \to \mathbb{R}$ is continuous.
If $f(a) < f(b)$ and $f(a) < c < f(b)$, then $\exists x \in (a, b).\ f(x) = c$.
If $f(b) < f(a)$ and $f(b) < c < f(a)$, then $\exists x \in (a, b).\ f(x) = c$.

Proof. Since $[a, b]$ is connected, we know that $f([a, b])$ is connected. Then $f(a), f(b) \inf([a, b])$ and $f(a) < c < f(b) \implies c \in f([a, b])$ via characterization of connected sets.

5.5 Discontinuities

Definition. Suppose $E \subseteq \mathbb{R}$, $f : E \to \mathbb{R}, p \in E$ is a limit point of E. Suppose that f is not continuous at p.

1. We say f has a simple discontinuity (or jump) at p if

$$\begin{cases} p \text{ is not a lim pt. of } E_p^+ \text{ and } \lim_{x \to p^-} f(x) \text{ exists (but } f(p) \neq \lim_{x \to p^-} f(x)) \\ p \text{ is not a lim pt. of } E_p^- \text{ and } \lim_{x \to p^+} f(x) \text{ exists (but } f(p) \neq \lim_{x \to p^+} f(x)) \\ p \text{ is a lim pt. of } E_p^+, E_p^- \text{ and } \lim_{x \to p^+} f(x) \text{ and } \lim_{x \to p^-} f(x) \text{ both exist.} \end{cases}$$

2. Otherwise we say f has an essential discontinuity at p.

Examples. Let $f : E \to \mathbb{R}$.

1. $E = R \quad f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$
   f has a simple discont. at $x = 0$, and is cont. on $\mathbb{R} \setminus \{0\}$.
2. $E = [0, 1] \quad f(x) = \begin{cases} 12 & x = 0 \\ x & x \in (0, 1] \end{cases}$
   f is cont. on $(0, 1]$ but has a simple discont. at $x = 0$.
3. $E = [0, 1] \quad f(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{2} & x \in (0, 1] \end{cases}$
   f is cont. on $(0, 1]$ but has a essential discont. at $x = 0$.

$\vdots$

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6 Differentiation

6.1 The Derivative

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6.2 Mean Value Theorems

$\vdots$

Theorem. Let $f : E \to \mathbb{R}$. Suppose further that f is differentiable at $x \in E$, and x is a limit point of both E_x^+ and E_x^-. If f has a local extremum at x then $f'(x) = 0$.

Proof. It suffices to assume that f has a local max at x.
Let $\delta > 0, t \in E$. $|x - t| < \delta \implies f(t) \le f(x)$. Then

$$t \in E,\ 0 < x - t < \delta \implies \frac{f(t) - f(x)}{t - x} \ge 0, \text{ so } f'(x) = \lim_{t \to x^-} \frac{f(t) - f(x)}{t - x} \ge 0$$
$$t \in E,\ 0 < t - x < \delta \implies \frac{f(t) - f(x)}{t - x} \le 0, \text{ so } f'(x) = \lim_{t \to x^+} \frac{f(t) - f(x)}{t - x} \le 0$$

Hence $f'(x) \ge 0 \wedge f'(x) \le 0 \implies f'(x) = 0$. $\square$

Remark. The result is false if x is not a limit point of either E_x^+ or E_x^-. Consider $f(x) = x$ on $E = [0, 1]$. f has a local min at $x = 0$ and local max at $x = 1$, but $f'(x) = 1$ for all $x \in [0, 1]$.

Theorem (Monotonicity, part 1). Let $f : E \to \mathbb{R}$, and assume that f is differentiable at $x \in E$.

1) If f is non-decreasing on E, then $f'(x) \ge 0$.
2) If f is non-increasing on E, then $f'(x) \le 0$.

Remark. f increasing $\implies f'(x) > 0$ is false. Consider $f(x) = \begin{cases} x^2 & x \ge 0 \\ -x^2 & x < 0 \end{cases}$.
$f : \mathbb{R} \to \mathbb{R}$ is increasing and differentiable, but $f'(0) = 0$.

Theorem (Cauchy's mean value theorem). Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Then $\exists x \in (a, b).\ (g(b) - g(a))f'(x) = (f(b) - f(a))g'(x)$.

Proof. Consider $h : [a, b] \to \mathbb{R}$ via

$$h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$$

Clearly h is continuous on $[a, b]$ and differentiable on (a, b). It suffices to find $x \in (a, b)$ such that $h'(x) = 0$. Notice that $h(a) = g(b)f(a) - g(a)f(b) = h(b)$. If h is constant then $h'(x) = 0$ for all $x \in (a, b)$. Assume, then, that h is non-constant. We have two cases:

$$\exists t \in (a, b).\ h(t) > h(a) \overset{\text{EVT}}{\implies} \exists x \in (a, b).\ h(x) = \max h([a, b]) \implies h'(x) = 0$$
$$\exists t \in (a, b).\ h(t) < h(a) \overset{\text{EVT}}{\implies} \exists x \in (a, b).\ h(x) = \min h([a, b]) \implies h'(x) = 0$$

$\square$

Corollary (MVT). Let $f : [a, b] \to \mathbb{R}$. If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists x \in (a, b).\ f(b) - f(a) = f'(x)(b - a)$.

Corollary (Monotonicity, part 2). Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Then

1) $f'(x) > 0\ \forall x \in (a, b) \implies f$ is increasing
2) $f'(x) \geq 0\ \forall x \in (a, b) \implies f$ is non-decreasing
3) $f'(x) = 0\ \forall x \in (a, b) \implies f$ is constant
4) $f'(x) \leq 0\ \forall x \in (a, b) \implies f$ is non-increasing
5) $f'(x) < 0\ \forall x \in (a, b) \implies f$ is decreasing

Proof. By MVT, if $x < x_1 < x_2 < b$, then $f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$ for some $x \in (x_1, x_2)$.

Remark. If $f : E \to \mathbb{R}$, where E is open but disconnected, then the result is false.
Consider $E = (0, 1) \cup (2, 3)$ and $f(x) = \begin{cases} x & x \in (0,1) \\ x - 5 & x \in (2,3) \end{cases}$
Then $f'(x) = 1$ for all $x \in E$ but f is not increasing.

6.3 Darboux's Theorem

Definition. We say that $g : \mathbb{R} \to \mathbb{R}$ is periodic with period $p > 0$ if $g(x + p) = g(x)\ \forall x \in \mathbb{R}$.

Theorem (Darboux). Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(a) < \gamma < f'(b)$. Then $\exists x \in (a, b).\ f'(x) = \gamma$.

Proof. Define $g : [a, b] \to \mathbb{R}$ via $g(x) = f(x) - \gamma x$, which is clearly differentiable on $[a, b]$. Since $g'(x) = f'(x) - \gamma$, it suffices to find $x \in (a, b)$ such that $g'(x) = 0$.
Note that $g'(a) = f'(a) - \gamma < 0$ and $g'(b) = f'(b) - \gamma > 0$.

The Newtonian approximation guarantees that $\exists \delta_a > 0$ such that $t \in [a, b]$ with

$$|t - a| < \delta_a \implies |g(t) - (g(a) + g'(a)(t - a))| < -g'(a)(t - a)$$

by choosing $\varepsilon = -g'(a) > 0$. In particular,

$$t \in [a, b] \wedge |t - a| < \delta_a \implies g(t) - g(a) - g'(a)(t - a) < -g'(a)(t - a) \implies g(t) < g(a)$$

A similar argument shows that $\exists \delta_b > 0$ such that $t \in [a, b] \wedge |t - b| < \delta_b \implies g(t) < g(b)$. By EVT $\exists x \in [a, b].\ g(x) = \min g([a, b])$. Then $x \in (a, b)$ and $g'(x) = 0$.

Corollary. If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, then f' has no simple discontinuities.

6.4 L'Hôpital's Rule

Theorem. Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$, differentiable on $[a, b]$, and $g'(x) \neq 0\ \forall x \in [a, b]$. If $f(a) = g(a) = 0$, then $L = \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Proof. We claim that $g(x) \neq 0$ for $x \in (a, b]$. Suppose not, for some $x \in (a, b]$). Then since $g(a) = 0$, $g'(z) = \frac{g(x) - g(a)}{x - a} = 0$ for some $z \in (a, x)$, a contradiction. So the function f/g is well-defined. Let $\{x_n\}_{n=l}^{\infty} \subseteq (a, b]$ satisfy $x_n \to a$ as $n \to \infty$. We claim $\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = L$.

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We apply Cauchy's MVT on $[a, x_n]$:

$$\exists y_n \in (a, x_n).\ f'(y_n)g(x_n) = f'(y_n)(g(x_n) - g(a)) = g'(y_n)(f(x_n) - f(a)) = g'(y_n)f(x_n)$$

Then $\frac{f(x_n)}{g(x_n)} = \frac{f'(y_n)}{g'(y_n)}\ \forall n \geq l$. Since $a < y_n < x_n$, the squeeze lemma implies that $y_n \to a$.