PAC Bounds for Discounted MDPs

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Abstract

We study upper and lower bounds on the sample-complexity of learning near-optimal behaviour in finite-state discounted Markov Decision Processes (MDPs). For the upper bound we make the assumption that each action leads to at most two possible next-states and prove a new bound for a UCRL-style algorithm on the number of time-steps when it is not Probably Approximately Correct (PAC). The new lower bound strengthens previous work by being both more general (it applies to all policies) and tighter. The upper and lower bounds match up to logarithmic factors.

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Keywords

Reinforcement learning; sample-complexity; exploration exploitation; PAC-MDP; Markov decision processes.

1 Introduction

The goal of reinforcement learning is to construct algorithms that learn to act optimally, or nearly so, in unknown environments. In this paper we restrict our attention to finite state discounted MDPs with unknown transitions. The performance of reinforcement learning algorithms in this setting can be measured in a number of ways, for instance by using regret or PAC bounds (Kakade, 2003). We focus on the latter, which is a measure of the number of time-steps where an algorithm is not near-optimal with high probability. Many previous algorithms have been shown to be PAC with varying bounds (Kakade, 2003; Strehl and Littman, 2005; Strehl et al., 2006, 2009; Szita and Szepesvári, 2010; Auer, 2011).

We modify the Upper Confidence Reinforcement Learning (UCRL) algorithm of Auer et al. (2010); Auer (2011); Strehl and Littman (2008) and, under the assumption that there are at most two possible next-states for each state/action pair, prove a PAC bound of

$$\tilde{O}\left(\frac{|S \times A|}{\epsilon^2 (1-\gamma)^3} \log \frac{1}{\delta}\right).$$

This bound is an improvement¹ on the previous best (Auer, 2011) and published best (Szita and Szepesvári, 2010), which are

$$\tilde{O}\left(\frac{|S \times A|}{\epsilon^2 (1-\gamma)^4} \log \frac{1}{\delta}\right)$$
 and $\tilde{O}\left(\frac{|S \times A|}{\epsilon^2 (1-\gamma)^6} \log \frac{1}{\delta}\right)$

respectively. The additional assumption is unfortunate and is probably unnecessary as discussed in Section 6.

We also present a matching (up to logarithmic factors) lower bound that is both larger and more general than the previous best given by Strehl et al. (2009). The class of MDPs used in the counter-example satisfy the assumption used in the upper bound.

2 Notation

Unfortunately, we found it impossible to reduce the amount of notation and number of constants. While we have endeavoured to define everything before we use it, readers are encouraged to consult the tables of notation and constants found in the appendix.

General. $\mathbb{N} = \{0, 1, 2, \dots\}$ is the natural numbers. For the indicator function we write $\llbracket x = y \rrbracket = 1$ if x = y and 0 if $x \neq y$. We use \land and \lor for logical and/or respectively. If A is a set then |A| is its size and A^* is the set of all finite ordered subsets. Unless otherwise mentioned, log represents the natural logarithm. For random variable X we write $\mathbf{E}X$ and $\mathrm{Var}\,X$ for its expectation and variance respectively. We make frequent use of the progression $z_i = 2^i - 2$ for $i \geq 1$. Define a set $\mathcal{Z}(a) := \{z_i : 1 \leq i \leq \arg\min_i \{z_i \geq a\}\}$.

Markov Decision Process. An MDP is a tuple $M = (S, A, p, r, \gamma)$ where S and A are finite sets of states and actions respectively. $r: S \to [0, 1]$ is the reward function. $p: S \times A \times S \to [0, 1]$ is the transition function and $\gamma \in (0, 1)$ the discount rate. A

¹In this slightly restricted setting.

stationary policy π is a function $\pi:S\to A$ mapping a state to an action. We write $p_{s,a}^{s'}$ as the probability of moving from state s to s' when taking action a and $p_{s,\pi}^{s'}:=p_{s,\pi(s)}^{s'}$. The value of policy π in M and state s is $V_M^\pi(s):=r(s)+\gamma\sum_{s'\in S}p_{s,\pi}^{s'}V_M^\pi(s')$. We view V_M^π either as a function $V_M^\pi:S\to\mathbb{R}$ or a vector $V_M^\pi\in\mathbb{R}^{|S|}$ and similarly $p_{s,a}\in[0,1]^{|S|}$ is a vector. The optimal policy of M is defined $\pi_M^*:=\arg\max_\pi V_M^\pi$. Common MDPs are M, \widehat{M} and \widehat{M} , which represent the true MDP, the estimated MDP using empirical transition probabilities and a model. We write $V:=V_M$, $\widehat{V}:=V_{\widehat{M}}$ and $\widetilde{V}:=V_{\widehat{M}}$ for their values respectively. Similarly, $\widehat{\pi}^*:=\pi_{\widehat{M}}^*$ and in general, variables with an MDP as a subscript will be written with a hat, tilde or nothing as appropriate and the subscript omitted.

3 Estimation

In the next section we will introduce the new algorithm, but first we give an intuitive introduction to the type of parameter estimation required to prove sample-complexity bounds for MDPs. The general idea is to use concentration inequalities to show the empiric estimate of a transition probability approaches the true probability exponentially fast in the number of samples gathered. There are a wide variety of concentration inequalities, each catering to a slightly different purpose. We improve on previous work by using Bernstein's inequality, which takes variance into account (unlike Hoeffding). The following example demonstrates the need for Bernstein's inequality when estimating the value functions of MDPs. It also gives insight into the workings of the proof in the next two sections.

Consider the Markov reward process on the right with two states where rewards are shown inside the states and transition probabilities on the edges. Note this is not an MDP because there are no actions. We are only concerned with how well the value can be approximated. Assume $p > \gamma$, q arbitrarily large (but not 1) and let \hat{p} be the empiric estimate of p and consider the error in our estimated value and the true value while in state s_0 . One can show that

$$\begin{array}{c}
1 - p \\
r = 1 \\
1 - q \\
p
\end{array}$$

$$\left| V(s_0) - \widehat{V}(s_0) \right| \approx \frac{|\widehat{p} - p|}{(1 - \gamma)^2}. \tag{1}$$

Therefore if $V-\widehat{V}$ is to be estimated to within ϵ accuracy, we need $|\widehat{p}-p|<\epsilon(1-\gamma)^2$. Now suppose we bound $|\widehat{p}-p|$ via a standard Hoeffding bound, then with high probability $|\widehat{p}-p|\lesssim \sqrt{L/n}$ where n is the number of visits to state s_0 and $L=\log(1/\delta)$. Therefore to obtain an error less than $\epsilon(1-\gamma)^2$ we need $n>\frac{L}{\epsilon^2(1-\gamma)^4}$ visits to state s_0 , which is already too many for a bound in terms of $1/(1-\gamma)^3$. If Bernstein's inequality is used instead, then $|\widehat{p}-p|\lesssim \sqrt{Lp(1-p)/n}$ and so $n>\frac{Lp(1-p)}{\epsilon^2(1-\gamma)^4}$ is required, but Equation (1) depends on $p>\gamma$. Therefore $n>\frac{L}{\epsilon^2(1-\gamma)^3}$ visits are sufficient. If $p<\gamma$ then Equation (1) can be improved.

4 Upper Confidence Reinforcement Learning Algorithm

UCRL is based on the optimism principle for solving the exploration/exploitation dilemma. It is model-based in the sense that at each time-step the algorithm acts according to a

model (in this case an MDP, \widetilde{M}) chosen from a model class. The idea is to choose the smallest model class guaranteed to contain the true model with high probability and act according to the most optimistic model within this class. With a good choice of model class this guarantees a policy that biases its exploration towards unknown states that may yield good rewards while avoiding states that are known to be bad. The approach has been successful in obtaining uniform sample complexity (or regret) bounds in various domains where the exploration/exploitation problem is an issue (Lai and Robbins, 1985; Agrawal, 1995; Auer et al., 2002; Strehl and Littman, 2005; Auer and Ortner, 2007; Auer et al., 2010; Auer, 2011).

Unfortunately, to prove our new bound we needed to make an assumption about the transition probabilities of the true MDP. We do not believe this assumption is crucial, but it substantially eases the analysis by removing some dependencies in the more general problem. In Section 6 we present an approach to remove the assumption as well as some intuition into why this ought to be possible, but non-trivial.

Assumption 1. The true unknown MDP, M, satisfies $p_{s,a}^{s'} = 0$ for all but two $s' \in S$ denoted $\mathfrak{A}^+, \mathfrak{A}^- \in S$.

The pseudo-code of UCRL can be found below, but first we define a knownness index, κ . If n is the number of times a state/action pair has been visited then $\kappa(\iota,n)$ is the knownness of that state/action pair at level ι . The knownness of a state increases with the number of visits, is bounded by |S| and is always a natural number. The reason for defining these now is that UCRL will only perform an update when the knownness index of some states would be changed by an update. Unfortunately, the definition below is unlikely to be very intuitive. A more thorough explanation of knownness is given in Section 5.

Definition 2 (Knownness). Define constants

$$\begin{split} w_{\min} &:= \frac{\epsilon(1-\gamma)}{4|S|} & w_{\iota} := 2^{\iota} w_{\min} & \iota_{\max} := \left\lceil \frac{1}{\log 2} \log \frac{8|S|}{\epsilon(1-\gamma)^2} \right\rceil \\ \mathcal{I} &:= \left\{ 0, 1, \cdots, \iota_{\max} \right\} & \mathcal{K} := \mathcal{Z}(|S|). \end{split}$$

We define the knownness index, $\kappa: \mathcal{I} \times \mathbb{N} \to \mathcal{K}$ by

$$\kappa(\iota, n) := \max \left\{ z \in \mathcal{K} : z \le \frac{n}{w_{\iota} m} \right\},$$

where $m \in \tilde{O}\left(\frac{1}{\epsilon^2(1-\gamma)^2}\log\frac{|S\times A|}{\delta}\right)$ is defined in Appendix D.

Note that the existence of the function EXTENDEDVALUEITERATION is proven and an algorithm given by Strehl and Littman (2008).

²Note that sa^+ and sa^- are dependent on (s,a) and are known to the algorithm.

Algorithm 1 UCRL

```
1: t = 1, k = 1, n(s, a) = n(s, a, s') = 0 for all s, a, s' and s_1 is the start state.
 2: H := \frac{1}{1-\gamma} \log \frac{8|S|}{\epsilon(1-\gamma)}, L_1 := \log \frac{2}{\delta_1} and \delta_1 := \frac{\delta}{2|S \times A|^2|\mathcal{K} \times \mathcal{I}|}
           \hat{p}_{s,a}^{sa^+} := n(s,a)/\max\{1, n(s,a,sa^+)\} \text{ and } \hat{p}_{s,a}^{sa^-} := 1 - \hat{p}_{s,a}^{sa^+}
 4:
           \mathcal{M}_k := \left\{ \widetilde{M} : |\widetilde{p}_{s,a}^{sa^+} - \widehat{p}_{s,a}^{sa^+}| \le \text{ConfidenceInterval}(\widetilde{p}_{s,a}^{sa^+}, n(s,a)), \ \forall (s,a) \right\}
 5:
           M = \text{ExtendedValueIteration}(\mathcal{M}_k)
 6:
 7:
 8:
           v(s, a) = v(s, a, s') = 0 for all s, a, s'
           while \kappa(\iota, n(s, a) + v(s, a)) = \kappa(\iota, n(s, a)), \forall (s, a), \iota \in \mathcal{I} do
 9:
10:
           Delay and Update
11:
12: function Delay
13:
            for j = 1 \rightarrow H do
                 Act
14:
     function Update
15:
            n(s, a) = n(s, a) + v(s, a) and n(s, a, s') = n(s, a, s') + v(s, a, s') \ \forall s, a, s' and k = k + 1
16:
17: function Act
18:
            a_t = \pi_k(s_t)
                                                                                                                           ▷ Sample from MDP
19:
            s_{t+1} \sim p_{s_t,a_t}
            v(s_t, a_t) = v(s_t, a_t) + 1 and v(s_t, a_t, s_{t+1}) = v(s_t, a_t, s_{t+1}) + 1 and t = t + 1
20:
     function EXTENDED VALUE ITERATION (\mathcal{M})
           return optimistic \widetilde{M} \in \mathcal{M} such that V_{\widetilde{M}}^*(s) \geq V_{\widetilde{M}'}^*(s) for all s \in S and \widetilde{M}' \in \mathcal{M}.
22:
     \mathbf{function} \ \ \mathbf{ConfidenceInterval}(p,n)
23:
           return min \left\{\sqrt{\frac{2L_1p(1-p)}{n}} + \frac{2L_1}{3n}, \sqrt{\frac{L_1}{2n}}\right\}
24:
```

5 Upper PAC Bounds

We present two new PAC bounds. The first improves on all previous analysis, but relies on Assumption 1. The second is completely general, but gains an additional dependence on |S| leading to a PAC bound in terms of $|S|^2$ and $1/(1-\gamma)^3$. This bound is worse than the previous best in terms of |S|, but better in terms $1/(1-\gamma)$.

Theorem 3. Let M be the true MDP satisfying Assumption 1. Let π be the actual (non-stationary) policy of UCRL (Algorithm 1), then $V^*(s_t) - V^{\pi}(s_t) > \epsilon$ for at most

$$HU_{\max} + HE_{\max} \in \emptyset \frac{|S \times A|}{\epsilon^2 (1 - \gamma)^3} \log \frac{|S \times A|}{\delta \epsilon (1 - \gamma)} \log^2 |S| \log^2 \frac{|S|}{\epsilon (1 - \gamma)} \log^2 \log \frac{1}{1 - \gamma}$$

time-steps with probability at least $1 - \delta$. (U_{max} and E_{max} are defined in Appendix D.)

Note that although π_k is stationary, the global policy of UCRL is non-stationary. Despite this, we will abuse notation by allowing ourselves to write $V^{\pi}(s_t)$, whereas really V^{π} should depend on the entire history. Fortunately, when UCRL is not delaying, the policy π is nearly stationary in the sense that it will be so for the next H time-steps. This allows us to work almost entirely with stationary policies and so discard the cumbersome notation required for non-stationary policies.

Theorem 4. Let M be the true MDP (possibly not satisfying Assumption 1) then there exists a policy π such that $V^*(s_t) - V^{\pi}(s_t) > \epsilon$ for at most $|S| \log^3 |S| (E_{\max} H + U_{\max} H)$ time-steps with probability at least $1 - \delta$.

The proof of Theorem 4 is omitted, but follows easily by converting an arbitrary MDP with |S| states into a functionally equivalent MDP with $O(|S|^2)$ states that satisfies Assumption 1. This is done by adding a tree of 2|S| states for each state/action pair and rescaling γ .

Proof Overview. The proof of Theorem 3 borrows components from the work of Auer et al. (2010), Strehl and Littman (2008) and Szita and Szepesvári (2010).

- 1. Bound the number of updates by $|S \times A| \log \frac{|S \times A|}{|\mathcal{K} \times \mathcal{I}|}$, which follows from the algorithm and the definition of knownness. This bounds the number of delaying time-steps to $\tilde{O}(\frac{1}{1-\gamma}|S \times A|\log \frac{|S \times A|}{|\mathcal{K} \times \mathcal{I}|})$ time-steps, which is insignificant from the point of view of Theorem 3.
- 2. Show that the true MDP remains in the model class \mathcal{M}_k for all k.
- 3. Use the optimism principle to show that if $M \in \mathcal{M}_k$ and $V^* V^{\pi} > \epsilon$ then $|\widetilde{V}^{\pi_k} V^{\pi_k}| > \epsilon/2$. This key fact shows that if π is not nearly-optimal at some timestep t then the true value and model value of π_k differ and so some information is (probably) gained by following this policy.
- 4. The final component is to bound the number of time-steps when π is not nearly-optimal.

Episodes and phases. UCRL operates in episodes, which are blocks of time-steps ending when UPDATE is called. The length of each episode is not fixed, instead, an episode ends when the knownness of a state changes. We often refer to time-step t and episode k and unless there is ambiguity we will not define k and just assume it is the episode in which t resides. A delay phase is the period of H contiguous time-steps where UCRL is in the function DELAY, which happens immediately before an update. An exploration phase is a period of H time-steps starting at t where t is not in a delay phase and where $\tilde{V}^{\pi_k}(s_t) - V^{\pi_k}(s_t) \ge \epsilon/2$. Exploration phases do note overlap. More formally, the starts of exploration phases, t_1, t_2, \cdots , are defined inductively

$$\begin{split} t_1 &:= \min \left\{ t : \widetilde{V}^{\pi_k}(s_t) - V^{\pi_k}(s_t) \geq \epsilon/2 \wedge t \text{ is not in a delay phase} \right\} \\ t_i &:= \min \left\{ t : t \geq t_{i-1} + H \wedge \widetilde{V}^{\pi_k}(s_t) - V^{\pi_k}(s_t) \geq \epsilon/2 \wedge t \text{ is not in a delay phase} \right\}. \end{split}$$

Note there need not, and with high probability will not, be infinitely many such t_i . The exploration phases are only used in the analysis, they are not known to UCRL.

Weights and variances. We define the weight³ of state/action pair (s, a) as follows.

$$w^{\pi}(s, a|s') := [(s', \pi(s')) = (s, a)] + \gamma \sum_{s''} p_{s', \pi(s')}^{s''} w^{\pi}(s, a|s'') \quad w_t(s) := w^{\pi_k}(s, \pi_k(s)|s_t).$$

³Also called the discounted future state distribution in Kakade (2003).

As usual, \tilde{w} and \hat{w} are defined as above but with p replaced by \tilde{p} and \hat{p} respectively. Think of $w_t(s)$ as the expected number of discounted visits to state/action pair $(s, \pi_k(s))$ while following policy π_k starting in state s_t . The important point is that this value is approximately equal to the expected number of visits to state/action pair $(s, \pi_k(s))$ within the next H time-steps. We also define the local variance of the value function. These measure the variability of values while following policy π .

$$\sigma^{\pi}(s)^{2} := p_{s,\pi} \cdot V^{\pi^{2}} - [p_{s,\pi} \cdot V^{\pi}]^{2} \qquad \qquad \tilde{\sigma}^{\pi}(s)^{2} := \tilde{p}_{s,\pi} \cdot \tilde{V}^{\pi^{2}} - [\tilde{p}_{s,\pi} \cdot \tilde{V}^{\pi}]^{2}.$$

The active set. We will shortly see that states with small $w_t(s)$ cannot influence the differences in value functions. Thus we define an active set of states where $w_t(s)$ is not tiny. At each time-step t define the active set X_t by

$$X_t := \left\{ s : w_t(s) > \frac{\epsilon(1-\gamma)}{4|S|} =: w_{min} \right\}.$$

Knownness. We now expand on the concept of knownness and explain its purpose. We write $n_t(s,a)$ for the value of n(s,a) at time-step t and $n_t(s) := n_t(s,\pi_k(s))$ where k is the episode associated with time-step t. Let t be some non-delaying time-step and suppose sis active $(s \in X_t)$. Now let $\iota_t(s) := \arg\min_t w_t(s) > w_t$ and note that $\iota_t(s) \in \mathcal{I}$. We define a partition of the active set X_t by

$$K_t(\kappa,\iota) := \left\{ s \in X_t : \iota_t(s) = \iota \wedge \kappa_t(\iota_t(s), n_t(s)) = \kappa \right\}.$$

The set $K_t(\kappa, \iota)$ represents a set of states that have comparable weights and visit counts. We will show that if $|K_t(\kappa,\iota)| \leq \kappa$ for all κ,ι then the values V and V are reasonably close. This result forms a key stage in the proof of Theorem 3 because it shows that if π is not nearly-optimal at time-step t then there exists a $K_t(\kappa, \iota)$ that is quite large and where states have not been visited sufficiently. Furthermore, the weights $w_t(s)$ where $s \in K_t(\kappa, \iota)$ are large enough that some learning is expected to occur.

Analysis. The proof of Theorem 3 follows easily from three key lemmas.

Lemma 5. The following hold:

- 1. The total number of updates is bounded by $U_{\max} := |S \times A| \log \frac{|S \times A|}{|\mathcal{K} \times \mathcal{I}|}$. 2. If $M \in \mathcal{M}_k$ and t is not in a delay phase and $V^*(s_t) V^{\pi}(s_t) > \epsilon$ then

$$\widetilde{V}^{\pi_k}(s_t) - V^{\pi_k}(s) > \epsilon/2.$$

Lemma 6. $M \in \mathcal{M}_k$ for all k with probability at least $1 - \delta/2$.

Lemma 7. The number of exploration phases is bounded by E_{max} with probability at least

The proofs of the lemmas are delayed while we apply them to prove Theorem 3.

Proof of Theorem 3. By Lemma 6, $M \in M_k$ for all k with probability $1 - \delta/2$. By Lemma 7 we have that the number of exploration phases is bounded by E_{max} with probability $1 - \delta/2$. Now if t is not in a delaying or exploration phase and $M \in \mathcal{M}_k$ then by Lemma 5, π is nearly-optimal. Finally note that the number of updates is bounded by U_{max} and so the number of time-steps in delaying phases is at most HU_{max} . Therefore UCRL is nearly-optimal for all but $HU_{\text{max}} + HE_{\text{max}}$ time-steps with probability $1 - \delta$.

We now turn our attention to proving Lemmas 5, 6 and 7. Of these, only Lemma 7 presents a substantial challenge.

Proof of Lemma 5. For part 1 we note that for $\iota \in \mathcal{I}$ the knownness of a state/action pair at level ι satisfies $\kappa \in \mathcal{K}$. Since the knownness index for each ι is non-decreasing and an update only occurs when an index is increased, the total number of updates is bounded by $U_{\max} := |S \times A| |\mathcal{K} \times \mathcal{I}|$.

The proof of part 2 is closely related to the approach taken by Strehl and Littman (2008). Recall that \widetilde{M} is chosen optimistically by extended value iteration. This generates an MDP, \widetilde{M} , such that $V_{\widetilde{M}}^*(s) \geq V_{\widetilde{M}'}^*(s)$ for all $\widetilde{M}' \in \mathcal{M}_k$. Since we have assumed $M \in \mathcal{M}_k$ we have that $\widetilde{V}^{\pi_k}(s) \equiv V_{\widetilde{M}}^*(s) \geq V_M^*(s)$. Therefore $\widetilde{V}^{\pi_k}(s_t) - V^{\pi}(s_t) > \epsilon$. Finally note that t is a non-delaying time-step and so policy π will remain stationary and equal to π_k for at least H time-steps. Using the definition of the horizon, H, we have that $|V^{\pi}(s_t) - V^{\pi_k}(s_t)| < \epsilon/2$. Therefore $\widetilde{V}^{\pi_k}(s_t) - V^{\pi_k}(s_t) > \epsilon/2$ as required.

Proof of Lemma 6. In the previous lemma we showed that there are at most U_{\max} updates. Therefore we only need to check $M \in \mathcal{M}_k$ for each k up to U_{\max} . Fix an (s,a) pair and apply the best of either Bernstein or Hoeffding inequalities to show that $|\hat{p}_{s,a}^{ac^+} - p_{s,a}^{ac^+}| \leq \text{ConfidenceInterval}(\hat{p}_{s,a}^{ac^+} - p_{s,a}^{ac^+}, n(s,a)))$ with probability $1 - \delta_1$. Setting $\delta_1 := \frac{\delta}{2|S \times A|U_{\max}} \equiv \frac{\delta}{2|S \times A|^2|\mathcal{K} \times \mathcal{I}|}$ and applying the union bound completes the proof.

We are now ready to work on Lemma 7. The proof follows from two lemmas:

- 1. If t is the start of an exploration phase then there exists a (κ, ι) such that $|K_t(\kappa, \iota)| > \kappa$.
- 2. If $|K_t(\kappa, \iota)| > \kappa$ for sufficiently many t then sufficient information is gained that some state/action pair must have an increase in knownness.

Lemma 8. Let t be a non-delaying time-step and assume $M \in \mathcal{M}_k$. If $|K_t(\kappa, \iota)| \leq \kappa$ for all $\kappa, \iota \in \mathcal{K}$ then $|\widetilde{V}^{\pi_k}(s_t) - V^{\pi_k}(s_t)| \leq \epsilon/2$.

The full proof is long, technical and has been relegated to Appendix B. We provide a sketch, but first we need some useful results about MDPs and the differences in value functions.

Lemma 9. Let M and \widetilde{M} be two Markov decision processes differing only in transition probabilities and π be a stationary policy then

$$V^{\pi}(s_t) - \widetilde{V}^{\pi}(s_t) = \gamma \sum_{s} w_t(s) (p_{s,\pi} - \widetilde{p}_{s,\pi}) \cdot \widetilde{V}^{\pi}.$$

Proof sketch. Drop the π superscript and write $V(s_t) = r(s_t) + \gamma \sum_{s_{t+1}} p_{s_t,\pi}^{s_{t+1}} V(s_{t+1})$. Then $V(s_t) - \widetilde{V}(s_t) = \gamma [p_{s_t,\pi} - \widetilde{p}_{s_t,\pi}] \cdot \widetilde{V} + \gamma \sum_{s_{t+1}} p_{s_t,\pi}^{s_{t+1}} [V(s_{t+1}) - \widetilde{V}(s_{t+1})]$. The result is obtained by continuing to expand the second term of the right hand side.

Lemma 10. If $M \in \mathcal{M}_k$ at time-step t and $\widetilde{V} := \widetilde{V}^{\pi_k}$ then

$$|(p_{s,\pi_k} - \tilde{p}_{s,\pi_k}) \cdot \widetilde{V}| \le \sqrt{\frac{8L_1\tilde{\sigma}^{\pi_k}(s)^2}{n_t(s)}} + \frac{2}{1-\gamma} \left(\frac{L_1}{n_t(s)}\right)^{3/4} + \frac{4L_1}{3n_t(s)(1-\gamma)},$$

where
$$\tilde{\sigma}^{\pi_k}(s)^2 := \tilde{p}_{s,a} \cdot \widetilde{V}^2 - \left[\tilde{p}_{s,a} \cdot \widetilde{V}\right]^2$$
.

The idea is to note that M, \widetilde{M} are in \mathcal{M}_k and apply the definition of the confidence intervals. The full proof is subsumed in the proof of the more general Lemma 33 in Appendix C. The following lemma bounds the expected total discounted local variance.

Lemma 11. For any stationary π and \widetilde{M} , $\sum_{s \in S} \widetilde{w}_t(s) \widetilde{\sigma}^{\pi}(s)^2 \leq \frac{1}{\gamma^2 (1-\gamma)^2}$.

See the paper of Sobel (1982) for a proof.

Proof sketch of Lemma 8. For ease of notation we drop references to π_k . We approximate $w(s) \approx \tilde{w}(s)$ and $|(p_{s,\pi_k} - \tilde{p}_{s,\pi_k}) \cdot \tilde{V}| \lesssim \sqrt{\frac{L_1\tilde{\sigma}(s)^2}{n(s)}}$. Using Lemma 9

$$|\widetilde{V}(s_t) - V(s_t)| \equiv \left| \gamma \sum_{s \in S} w_t(s) (p_{s, \pi_k} - \widetilde{p}_{s, \pi_k}) \cdot \widetilde{V} \right| \lesssim \left| \sum_{s \in X} w_t(s) (p_{s, \pi_k} - \widetilde{p}_{s, \pi_k}) \cdot \widetilde{V} \right|$$
(2)

$$\lesssim \sum_{s \in X} w_t(s) \sqrt{\frac{L_1 \tilde{\sigma}(s)^2}{n(s)}} \lesssim \sum_{\kappa, \iota \in \mathcal{K} \times \mathcal{I}} \sum_{s \in K(\kappa, \iota)} \sqrt{\frac{L_1 \tilde{w}_t(s) \tilde{\sigma}(s)^2}{\kappa m}}$$
(3)

$$\leq \sum_{\kappa,\iota \in \mathcal{K} \times \mathcal{I}} \sqrt{\frac{L_1 |K(\kappa,\iota)|}{\kappa m} \sum_{s \in K(\kappa,\iota)} \tilde{w}_t(s) \tilde{\sigma}(s)^2} \leq \sqrt{\frac{L_1 |\mathcal{K} \times \mathcal{I}|}{m \gamma^2 (1-\gamma)^2}}, \quad (4)$$

where in Equation (2) we used Lemma 9 and the fact that states not in X are visited very infrequently. In Equation (3) we used the approximations for $(p - \tilde{p}) \cdot \tilde{V}$, the definition of $K(\kappa, \iota)$ and the approximation $w \approx \tilde{w}$. In Equation (4) we used the Cauchy-Schwartz inequality,⁴ the fact that $\kappa \geq |K(\kappa, \iota)|$ and Lemma 11. Substituting $m := \frac{20L_1|\mathcal{K} \times \mathcal{I}||\mathcal{D}|^2}{\epsilon^2(1-\gamma)^{2+2/\beta}}$ completes the proof. The extra terms in m are needed to cover the errors in the approximations made here.

The full proof requires formalising the approximations made at the start of the sketch above. The second approximation is comparatively easy while the showing that $w(s) \approx \tilde{w}(s)$ requires substantial work.

The following lemmas are used to show that $|K_t(\kappa, \iota)|$ cannot be larger than κ for too many time-steps with high probability. Combined with Lemma 8 above this will be sufficient to bound the number of exploration phases. Let t be the start of an exploration phase and define $\nu_t(s)$ to be the number of visits to state s within the next H time-steps. Formally, $\nu_t(s) := \sum_{i=t}^{t+H-1} \llbracket s_t = s \rrbracket$.

Lemma 12. Let t be the start of an exploration phase and $w_t(s) \ge w_{\min}$ then $\mathbf{E}\nu_t(s) \ge w_t(s)/2$.

 $^{||4|\}langle 1, v \rangle| \le ||1||_2 ||v||_2.$

Proof sketch. Use the definition of the horizon to show that $w_t(s)$ is not much larger than a bounded-horizon version. Compare $\mathbf{E}\nu_t(s, \pi_t(s))$ and the definition of $w_t(s)$.

Lemma 13. Let N be as in Appendix D. If $|K_{t_i}(\kappa,\iota)| > \kappa$ for 4N exploration phases t_1, t_2, \dots, t_{4N} then $\sum_{i=1}^{4N} \sum_{s \in K_{t_i}(\kappa,\iota)} \nu_{t_i}(s,\pi(s)) \geq N \kappa w_{\iota}$ with probability at least $1 - \delta_1$.

Proof. As in the previous proof we drop π superscripts and denote $K_i := K_{t_i}(\kappa, \iota)$. Define

$$\nu_i := \sum_{s,a \in K_i} \nu_{t_i}(s)$$

$$\mathbf{E}\nu_i = \sum_{s \in K_i} \mathbf{E}\nu_{t_i}(s).$$

Now $|K_i| > \kappa$ and so by Lemma 12 we have $\mathbf{E}\nu_i \ge \kappa w_\iota/2$. We now prepare to use Bernstein's inequality. Let $X_i = \nu_i - \mathbf{E}\nu_i$, $\mu := \frac{1}{4N} \sum_{i=1}^{4N} \mathbf{E}\nu_i$ and $\sigma^2 := \frac{1}{4N} \sum_{i=1}^{4N} \operatorname{Var} X_i$ then

$$P\left\{\sum_{i=1}^{4N} \nu_{i} \leq N w_{i} \kappa\right\} \leq P\left\{\sum_{i=1}^{4N} \nu_{i} \leq \sum_{i=1}^{4N} \mathbf{E} \nu_{i}/2\right\}$$

$$= P\left\{\sum_{i=1}^{4N} [\nu_{i} - \mathbf{E} \nu_{i}] \leq -\sum_{i=1}^{4N} \mathbf{E} \nu_{i}/2\right\} \leq 2 \exp\left(-\frac{4N \mu^{2}}{8\sigma^{2} + \frac{16\mu}{3(1-\gamma)}}\right).$$

Setting this equal to δ_1 and solving for 4N gives

$$4N \geq \frac{8\sigma^2 + \frac{16\mu}{3(1-\gamma)}}{\mu^2} \log \frac{2}{\delta_1} = \left[\frac{8\sigma^2}{\mu^2} + \frac{16}{3\mu(1-\gamma)} \right] \log \frac{2}{\delta_1}.$$

Naively bounding $\sigma^2/\mu^2 \leq 1/((1-\gamma)\mu)$ and noting that $\mu \geq w_{\min}/2$ leads to

$$4N \geq \frac{14|S \times A|}{\epsilon (1-\gamma)^2} \log \frac{2}{\delta_1}.$$

Since 4N satisfies this, the result is complete.

Proof of Lemma 7. We proceed in two stages. First we bound the total number of useful visits before $|K(\kappa, \iota)| \leq \kappa$. We then show this number of visits occurs after $\tilde{O}(m)$ exploration phases with high probability.

Bounding the number of useful visits. A visit to state/action pair (s, a) in timestep t is (κ, ι) -useful if $\kappa(\iota, n_t(s, a)) = \kappa$. Fixing a (κ, ι) we bound the number of (κ, ι) useful visits to state/action pair (s, a). Suppose $t_1 < t_2$ and $\kappa(\iota, n_{t_1}(s, a)) = \kappa$ and $n_{t_2}(s, a) - n_{t_1}(s, a) \ge mw_{\iota}(2\kappa + 2)$ then $\kappa(\iota, n_{t_3}(s, a)) > \kappa$ for all $t_3 \ge t_2$. Therefore for each (κ, ι) pair there at most $6|S \times A|mw_{\iota}\kappa$ visits that are (κ, ι) -useful.

Bounding the number of exploration phases. Let $N:=6|S\times A|m$ and t be the start of an exploration phase. Therefore $\widetilde{V}^{\pi_k}(s_t)-V^{\pi_k}(s_t)>\epsilon/2$ and so by Lemma 8 there exists a $(\kappa,\iota)\in\mathcal{K}$ such that $|S|\geq |K(\kappa,\iota)|>\kappa$. If $|K_{t_i}(\kappa,\iota)|>\kappa$ at the start of 4N exploration phases, t_1,t_2,\cdots,t_{4N} then by Lemma 13

$$P\left\{\sum_{i=1}^{4N} \sum_{s,a \in K_{t_i}(\kappa,\iota)} v_{t_i}(s,a) \le N w_{\iota} \kappa\right\} \le \delta_1.$$

Therefore by the union bound there are at most $E_{\max} := 4N|\mathcal{K} \times \mathcal{I}|$ exploration phases with probability $1 - \delta_1 |\mathcal{K} \times \mathcal{I}| \equiv 1 - |\mathcal{K} \times \mathcal{I}|_{\overline{2|S \times A|U_{\max}}} > 1 - \delta/2$.

6 Eliminating the Assumption

The upper bound in the previous section could only be proven using Assumption 1. In this section we describe a possible approach to generalising the proof and why this may be non-trivial. In the work above we used the assumption to bound $(p_{s,\pi} - \tilde{p}_{s,\pi}) \cdot \tilde{V}^* \lesssim \sqrt{L_1 \tilde{\sigma}^{\pi}(s)^2/n}$. A natural approach to generalising this comes from Bernstein's inequality (Theorem 30). If $V^{\pi} \in \mathbb{R}^{|S|}$ is a value function independent of \hat{p} then Bernstein's inequality can be used to show that $(p_{s,\pi} - \hat{p}_{s,\pi}) \cdot V^{\pi} \lesssim \sqrt{L_1 \sigma^{\pi}(s)^2/n}$. This suggests we adjust our model class by letting $\pi := \tilde{\pi}^*$ and changing the condition to $(\tilde{p}_{s,\pi} - \hat{p}_{s,\pi}) \cdot \tilde{V}^* \lesssim \sqrt{L_1 \tilde{\sigma}^{\pi}(s)^2/n}$. We might then bound $(p_{s,\pi} - \tilde{p}_{s,\pi}) \cdot \tilde{V}^* \equiv (p_{s,\pi} - \hat{p}_{s,\pi}) \cdot \tilde{V}^* + (\hat{p}_{s,\pi} - \tilde{p}_{s,\pi}) \cdot \tilde{V}^*$. The right term is then bounded by the conditions on the model class and the left term can perhaps be bounded by noting that $p_{s,\pi}$ is the true probability distribution. Unfortunately, there are a few problems with this approach:

- 1. Bounding $(p_{s,\pi} \hat{p}_{s,\pi}) \cdot \widetilde{V}^*$ does not result in a bound in terms of $\tilde{\sigma}^{\pi}(s)^2$. This issue can be solved by again applying Bernstein's inequality to bound $(p_{s,a} \hat{p}_{s,a}) \cdot \widetilde{V}^{*2}$.
- 2. The value \widetilde{V}^* is not in general independent of \widehat{p} . This is because \widetilde{M} must be chosen to satisfy the conditions on $(\widehat{p}_{s,\pi} \widetilde{p}_{s,\pi}) \cdot \widetilde{V}^*$, which depends on \widehat{p} . This dependence violates the conditions of Bernstein's inequality when trying to bound $(p_{s,\pi} \widehat{p}_{s,\pi}) \cdot \widetilde{V}^*$. The dependence is intuitively quite weak, but nevertheless presents problems for rigorous proof.
- 3. The last problem is that extended value iteration is no longer a trivial operation (even granting infinite computation). The problem is that the condition $(p_{s,a} \hat{p}_{s,a}) \cdot V^*$ is not local to (s,a), it also depends on the choice of $p_{s',a'}$ for $(s',a') \in S \times A$. This complication is probably resolvable, but the formal demonstration of extended value iteration is no longer so easy.

Progress. The first issue above can be solved, as remarked, by bounding $(p_{s,a} - \hat{p}_{s,a}) \cdot \tilde{V}^{*2}$ using another Bernstein inequality. The problem here is that this condition must now be added to the definition of the model class. The second issue is non-trivial and we cannot claim to have made progress there. We did manage to show that extended value iteration can be extended to the case where the only constraints take the form $(\tilde{p}_{s,\pi} - \hat{p}_{s,\pi}) \cdot \tilde{V}^* \lesssim \sqrt{L_1 \tilde{\sigma}^{\pi}(s)^2/n}$. In this case it can be shown the existence of a globally optimistic MDP. Unfortunately if you add constraints on higher moments, $(\tilde{p}_{s,\pi} - \hat{p}_{s,\pi}) \cdot \tilde{V}^2$ then results become substantially more complex. Note that in the complete proof of Lemma 8 we used higher moments still, but this is not required. Lemma 8 can be proven using only bounds on $(\tilde{p} - \hat{p}) \cdot \tilde{V}^*$ and $(\tilde{p} - \hat{p}) \cdot \tilde{V}^{*2}$.

7 Lower PAC Bound

We now turn our attention to proving a matching lower bound. The approach is similar to that of Strehl et al. (2009), but we make two refinements to improve the bound to depend on $1/(1-\gamma)^3$ and remove the policy restrictions. The first is to add a delaying state where no information can be gained, but where an algorithm may still fail to be PAC. The second is more subtle and will be described in the proof.

Definition 14. A non-stationary policy is a function $\pi: S^* \to A$.

Theorem 15. Let π be a (possibly non-stationary) policy depending on $S, A, r, \gamma, \epsilon$ and δ , then there exists a Markov decision process M_{hard} such that $V^*(s_t) - V^{\pi}(s_t) > \epsilon$ for at least N time-steps with probability at least δ where

$$N := \frac{c_1 |S \times A|}{\epsilon^2 (1 - \gamma)^3} \log \frac{c_2}{\delta}$$

and $c_1, c_2 > 0$ are independent of the policy π as well as all inputs $S, A, \epsilon, \delta, \gamma$.

The proof can found in Appendix A, but we give the counter-example MDP and intuition.

Counter Example. We prove Theorem 15 for a class of MDPs where $S = \{0, 1, \oplus, \ominus\}$ and $A = \{1, 2, \cdots, |A|\}$. The rewards and transitions for a single action are depicted in the diagram on the right where $\epsilon(a^*) = 16\epsilon(1-\gamma)$ for some $a^* \in A$ and $\epsilon(a) = 0$ for all other actions. Some remarks:

- 1. States \oplus and \ominus are almost completely absorbing and confer maximum/minimum rewards respectively.
- 2. The transitions are independent of actions for all states except state 1. From this state, actions lead uniformly to \oplus/\ominus except for one action, a^* , which has a slightly higher probability of transitioning to state \oplus . Thus a^* is the optimal action in state 1.
- 3. State 0 has an absorption rate such that, on average, a policy will stay there for $1/(1-\gamma)$ time-steps.

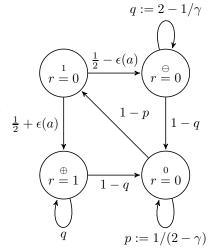


Figure 1: Hard MDP

Intuition. The MDP above is very bandit-like in the sense that once a policy reaches state 1 it should choose the action most likely to lead to state \oplus whereupon it will either be rewarded or punished (visit state \oplus or \ominus). Eventually it will return to state 1 when the whole process repeats. This suggests a PAC-MDP algorithm can be used to learn the bandit with $p(a) := p_{1,a}^{\oplus}$. We can then make use of a theorem of Mannor and Tsitsiklis (2004) on bandit sample-complexity to show that the number of times a^* is not selected is at least

$$\tilde{O}\left(\frac{1}{\epsilon^2(1-\gamma)^2}\log\frac{1}{\delta}\right). \tag{5}$$

Improving the bound to depend on $1/(1-\gamma)^3$ is intuitively easy, but technically somewhat annoying. The idea is to consider the value differences in state 0 as well as state 1. State 0 has the following properties:

1. The absorption rate is sufficiently large that any policy remains in state 0 for around $1/(1-\gamma)$ time-steps.

2. The absorption rate is sufficiently small that the difference in values due to bad actions planned in state 1 still matter while in state 0.

While in state 0 an agent cannot make an error in the sense that $V^*(0) - Q^*(0, a) = 0$ for all a. But we are measuring $V^*(0) - V^{\pi}(0)$ and so an agent can be penalised if its policy upon reaching state 1 is to make an error. Suppose the agent is in state 0 at some time-step before moving to state 1 and making a mistake. On average it will stay in state 0 for roughly $1/(1-\gamma)$ time-steps during which time it will plan a mistake upon reaching state 1. Thus the bound in Equation (5) can be multiplied by $1/(1-\gamma)$. The proof is harder because an agent need not plan to make a mistake in all future time-steps when reaching state 1 before eventually doing so in one time-step. Note that Strehl et al. (2009) proved their theorem for a specific class of policies while Theorem 15 holds for all policies.

8 Conclusion

Summary. We presented matching upper and lower bounds on the number of time-steps when a reinforcement learning algorithm can be nearly-optimal with high probability. While the lower bound is completely general, the upper bound depends on the assumption that there are at most two next-states for each state/action pair. This assumption aside, the new upper bound improves on the previously best known bound of Auer (2011). If the assumption is dropped then the new proof can be used to construct an algorithm that is better than the bound of Auer (2011) in terms of $1/(1-\gamma)$, but worse in |S|. The lower bound, which comes without assumptions, improves on the work of Strehl et al. (2009) by being both larger and more general. The class of MDPs used for the counter-example do satisfy Assumption 1 and so the upper and lower bounds now match in this restricted case.

Running Time. We did not analyze the running time of our version of UCRL, but expect analysis similar to that of Strehl and Littman (2008) can be used to show that UCRL can be approximated to run in polynomial time with no cost to sample-complexity.

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A Proof of Lower PAC Bound

The proof makes use of a simple form of bandit and Theorem 16, which lower bounds the sample-complexity of bandit algorithms. We need some new notation required for non-stationary policies and bandits.

History Sequences. We write $s_{1:t} = s_1, s_2, \dots, s_t$ for the history sequence of length t. Histories can be concatenated, so $s_{1:t} \oplus = s_1, s_2, \dots, s_t, \oplus$ where $\oplus \in S$.

Bandits. An A-armed bandit is a vector $p:A\to [0,1]$. A policy interacts with a bandit sequentially. In time-step t some arm a_t is played whereupon the policy receives reward 1 with probability p(a) and reward 0 otherwise. This is repeated over all time-steps. More formally, a bandit policy is a function $\pi:\{0,1\}^*\to A$. The optimal arm is defined $a^*:=\arg\max_a p(a)$. A policy dependent on ϵ,δ and A has sample-complexity $T:=T(A,\epsilon,\delta)$ if for all bandits the arm chosen on time-step T satisfies $p(a^*)-p(a_T)\leq \epsilon$ with probability at least $1-\delta$.

Theorem 16 (Mannor and Tsitsiklis, 2004). There exist positive constants c_1 , c_2 , ϵ_0 , and δ_0 , such that for every $A \geq 2$, $\epsilon \in (0, \epsilon_0)$ and $\delta \in (0, \delta_0)$ there exists a bandit $p \in [0, 1]^A$ such that

$$T(A, \epsilon, \delta) \ge c_1 \frac{|A|}{\epsilon^2} \log \frac{c_2}{\delta}$$

with probability at least δ .

Remark 17. The bandit used in the proof of Theorem 16 satisfies $p(a) = \frac{1}{2}$ for all a except a^* which has $p(a^*) := \frac{1}{2} + \epsilon$.

We now prepare to prove Theorem 15. For the remainder of this section let π be an arbitrary policy and $M_{\rm hard}$ be the MDP of Figure 2. As in previous work we write $V^{\pi} := V^{\pi}_{M_{\rm hard}}$. The idea of the proof will be to use Theorem 16 to show that π cannot be approximately correct in state 1 too often. Then use this to show that while in state 0 before-hand it is also not approximately correct.

Definition 18. Let $s_{1:\infty} \in S^{\infty}$ be the sequence of states seen by policy π and for arbitrary history $s_{1:t}$ let

$$\Delta(s_{1:t}) := V^*(s_{1:t}) - V^{\pi}(s_{1:t}).$$

Lemma 19. If $\gamma \in (0,1)$, $p := 1/(2-\gamma)$ and $q := 2-1/\gamma$ then

$$p^{\frac{1}{4(1-\gamma)}} > 3/4$$
 and $\sum_{t=0}^{\infty} p^t (1-p) \gamma^t = \frac{1}{2}$.

Proof sketch. Both results follow from the geometric series and easy calculus.

The following lemma lower-bounds $\Delta(s_{1:t})$ if sub-optimal action $a \neq a^*$ is taken in state 1.

Lemma 20. Let $s_{1:t}$ be a history such that $s_t = 1$ and $a := \pi(s_{1:t}) \neq a^*$ then

$$\Delta(s_{1:t}) \geq 8\epsilon$$
.

Proof. The result essentially follows from the definition of the value function.

$$\Delta(s_{1:t}) \equiv V^*(s_{1:t}) - V^{\pi}(s_{1:t})$$

$$= \gamma \left[p_{1,a^*}^{\oplus} V^*(s_{1:t} \oplus) + p_{1,a^*}^{\ominus} V^*(s_{1:t} \ominus) \right] - \gamma \left[p_{1,a}^{\oplus} V^{\pi}(s_{1:t} \oplus) + p_{1,a}^{\ominus} V^{\pi}(s_{1:t} \ominus) \right]$$

$$= \frac{\gamma}{2} \left[V^*(s_{1:t} \oplus) - V^{\pi}(s_{1:t} \oplus) + V^*(s_{1:t} \ominus) - V^{\pi}(s_{1:t} \ominus) \right] + \gamma \epsilon(a^*) V^*(s_{1:t} \oplus)$$

$$\geq 8\epsilon,$$

where we used the definition of the value function and MDP, M_{hard} .

We now define time-intervals where the policy is in state 0. Recall we chose the absorption in this state such that the expected number of time-steps a policy remains there is approximately $1/(1-\gamma)$. We define the intervals starting when a policy arrives in state 0 and ending when it leaves to state 1.

Definition 21. Define $t_1^0 := 1$ and

$$t_i^0 := \min \{ t : t > t_{i-1} \land s_t = 0 \land s_{t-1} \neq 0 \}$$
 $t_i^1 := \min \{ t - 1 : s_t = 1 \land t > t_i^0 \}$.

Define the intervals $I_i := [t_i^0, t_i^1] \subseteq \mathbb{N}$. We call interval I_i the *i*th *phase*.

Note the following facts:

- 1. Since all transition probabilities are non-zero, t_i^0 and t_i^1 exist for all $i \in \mathbb{N}$ with probability 1.
- 2. $|I_i|$ is the number of time-steps spent in state 0 before moving to state 1.
- 3. The values $|I_i|$ are independent of π and each other.

Definition 22. Suppose $t \in \mathbb{N}$ and $s_t = 0$ and define the weight of action $a, w_t(a)$ by

$$w_t(a) := \sum_{k=0}^{\infty} p^k (1-p) \gamma^k \llbracket \pi(s_{1:t} 0^k 1) = a \rrbracket.$$

Lemma 23. $\sum_{a \in A} w_t(a) = \frac{1}{2}$ for all t where $s_t = 0$.

Proof. We use Lemma 19.

$$\sum_{a \in A} w_i(a) \equiv \sum_{a \in A} \sum_{k=0}^{\infty} p^k (1-p) \gamma^k [\pi(s_{1:t}0^k 1) = a]$$

$$= \sum_{k=0}^{\infty} p^k (1-p) \gamma^k = \frac{1}{2}$$

as required.

Definition 24. Define random variables A_i and X_i by

$$A_i \ := \ [\![|I_i| \geq 1/[16(1-\gamma)] \wedge \sum_{a \neq a^*} w_{t_i^0}(a) \geq 1/4]\!] \qquad X_i \ := \ [\![|I_i| \geq 1/[4(1-\gamma)]]\!]$$

Intuitively, X_i is the event that the *i*th phase lasts at least $1/[4(1-\gamma)]$ time-steps. A_i is the event that the *i*th phase lasts at least $1/[16(1-\gamma)]$ time-steps and the combined weight of sub-optimal actions at the start of a phase is at least 1/4. The following lemma shows that at least two thirds of all phases have $X_i = 1$ with high probability.

Lemma 25. For all $n \in \mathbb{N}$, $P\left\{\sum_{i=1}^{n} X_i \leq \frac{2}{3}n\right\} \leq 2e^{-n/72}$.

Proof. Preparing to use Hoeffding's bound,

$$P\{X_i = 1\} := P\{|I_i| \ge 1/[4(1-\gamma)]\} = p^{1/[4(1-\gamma)]} > 3/4,$$

where we used the definitions of X_i , I_i and Lemma 19. Therefore $\mathbf{E}X_i > 3/4$.

$$P\left\{\sum_{i=1}^{n} X_{i} \leq \frac{2}{3}n\right\} \leq P\left\{\sum_{i=1}^{n} X_{i} \leq \frac{1}{12}n + n\mathbf{E}X_{i}\right\} = P\left\{\sum_{i=1}^{n} X_{i} - \mathbf{E}X_{i} \leq \frac{1}{12}n\right\} \leq 2e^{-n/72}$$

where we applied basic inequalities followed by Hoeffding's bound.

Lemma 26. If $\gamma > \frac{3}{4}$ and $\sum_{a \neq a^*} w_t(a) \ge \frac{1}{4}$ then $\sum_{a \neq a^*} w_{t+k}(a) \ge \frac{1}{8}$ for all $t \in \mathbb{N}$ and k satisfying $0 \le k \le 1/[16(1-\gamma)]$.

Proof. Working from the definitions.

$$\frac{1}{4} \leq \sum_{a \neq a^*} w_{t_i^0}(a) \equiv \sum_{j=0}^{\infty} p^j (1-p) \gamma^j \llbracket \pi(s_{1:t_i^0} 0^j) \neq a^* \rrbracket
= \sum_{j=0}^{k-1} p^j (1-p) \gamma^j \llbracket \pi(s_{1:t_i^0} 0^j) \neq a^* \rrbracket + p^k \gamma^k \sum_{a \neq a^*} w_a(s_{1:t_i^0} 0^k)
\leq (1-p) \sum_{j=0}^{k-1} p^j \gamma^j + p^k \gamma^k \sum_{a \neq a^*} w_a(s_{1:t_i^0} 0^k)$$

Rearranging, setting $0 \le k \le 1/[16(1-\gamma)]$ and using the geometric series completes the proof.

So far, none of our results have been especially surprising. Lemma 25 shows that at least two thirds of all phases have length exceeding $1/[4(1-\gamma)]$ with high probability. Lemma 26 shows that if at the start of a phase π assigns a high weight to the sub-optimal actions, then it does so throughout the entire phase. The following lemma is more fundamental. It shows that the number of phases where π assigns a high weight to the sub-optimal actions is of order $\frac{1}{\epsilon^2(1-\gamma)^2}\log\frac{1}{\delta}$ with high probability.

Lemma 27. Let $N := \frac{c_1 A}{\epsilon^2 (1-\gamma)^2} \log \frac{c_2}{\delta}$ with constants as in Theorem 16 then

$$\left| \left\{ i : \sum_{a \neq a^*} w_{t_i^0}(a) > \frac{1}{4} \land i < 2N + 1 \right\} \right| > N$$

with probability at least δ .

The idea is similar to that in (Strehl et al., 2009). Assume a policy exists that doesn't satisfy the condition above and then use it to learn the bandit defined by $p(a) := p_{1,a}^{\oplus}$.

Proof. Let $p(a) := p_{1,a}^{\oplus}$ be a bandit and use π to learn bandit p using Algorithm 2 below, which returns an action a_{best} defined as

$$a_{\text{best}} := \arg\max_{a} \sum_{i=1}^{2N} \bar{a}_i, \quad \bar{a}_i := \arg\max_{a'} w_{t_i^0}(a')$$

By Theorem 16, the strategy in Algorithm 2 must fail with probability at least δ . Therefore with probability at least δ , $a_{\text{best}} \neq a^*$. However a_{best} is defined as the majority action of all the \bar{a}_i and so for at least N time-steps $\bar{a}_i \neq a^*$. Suppose $w_{t_i^0}(a) > \frac{1}{4}$, then by Lemma 23, $\sum_{a \neq a^*} w_{t_i^0}(a) < \frac{1}{4}$ and $\bar{a}_i \equiv \arg\max_a w_{t_i^0}(a) = a^*$. This implies that with probability δ , for at least N time-steps $\sum_{a \neq a^*} w_{t_i^0}(a) > \frac{1}{4}$ as required.

Algorithm 2 Learn Bandit

```
t = 1, s_t = 0, k = 0
loop
a_t = \pi(s_{1:t})
if s_t = 1 \text{ then}
r \sim p(a_t) \qquad \qquad \triangleright \text{ sample from bandit}
if r = 1 \text{ then}
s_{t+1} = \oplus
else
s_{t+1} = \ominus
k = k + 1
if k = 2N \text{ then}
a_{\text{best}} = \arg\max_{a} \sum_{i=1}^{2N} \llbracket a = \arg\max_{a'} w_{t_i^0}(a') \rrbracket
exit
else
s_{t+1} \sim p_{st,at}
\triangleright \text{ sample from MDP}
```

Proof of Theorem 15. Suppose $A_i = 1$ and $0 \le k \le 1/[16(1-\gamma)]$ then $s_{1:t_i^0+k} = s_{1:t_i^0}0^k$ and

$$\Delta(s_{1:t_i+k}) = \sum_{t=0}^{\infty} p^t (1-p) \gamma^t \Delta(s_{1:t_i+k} 0^t 1)$$
(6)

$$\geq \sum_{t=0}^{\infty} p^{t} (1-p) \gamma^{t} \sum_{a \neq a^{*}} \llbracket \pi(s_{1:t_{i}^{0}+k} 0^{t} 1) = a \rrbracket 8\epsilon$$
 (7)

$$\geq \sum_{a \neq a^*} w_{t_i^0 + k}(a) 8\epsilon \tag{8}$$

$$>\epsilon$$
, (9)

where Equation (6) follows from the definition of M_{hard} and the value function. Equation (7) by Lemma 20. Equation (8) by the definition of $w_{t_i+k}(a)$ and Equation (8) by Lemma 26. Thus for each i where $A_i = 1$, policy π makes at least $1/[16(1-\gamma)]$ ϵ -errors. The proof is completed by showing that $A_i = 1$ for at least N/6 time-steps with probability at least δ , which follows easily from Lemma 27 and Lemma 25.

Dependence on S is added trivially by chaining arbitrarily many such Markov decision processes together.

Remark 28. Dependence on $S \log S$ can possibly be added by a similar technique used by Strehl et al. (2009), but details could be messy.

B Technical Results

Theorem 29 (Hoeffding Inequality). Let X_1, \dots, X_n be independent [0,1]-valued random variables with probability 1. Then

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}X_{i}-\mathbf{E}X_{i}\right|\geq\epsilon\right\} \leq 2e^{-2\epsilon^{2}n}.$$

Theorem 30 (Bernstein's Inequality (Bernstein, 1924)). Let X_1, \dots, X_n be independent real-valued random variables with zero mean and variance $\operatorname{Var} X_i = \sigma_i^2$. If $|X_k| < c$ with probability one then

$$P\left\{ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \ge \epsilon \right\} \le 2e^{-\frac{\epsilon^2 n}{2\sigma^2 + 2c\epsilon/3}},$$

where $\sigma^2 := \frac{1}{n} \sum_{i=1}^n \sigma_i^2$.

We can use Hoeffding and Bernstein to bound the gaps $|p-\hat{p}|$ and $|\hat{p}-\tilde{p}|$ we now want to combine these together in a nice way to bound $|p-\tilde{p}|$.

Lemma 31. Let $p, \hat{p}, \tilde{p} \in [0, 1]$ satisfy

$$|p - \hat{p}| \le \min\left\{CI_1, CI_2\right\},\,$$

where

$$CI_1 := \sqrt{\frac{2p(1-p)}{n}\log\frac{2}{\delta}} + \frac{2}{3n}\log\frac{2}{\delta}$$
 $CI_2 := \sqrt{\frac{1}{2n}\log\frac{2}{\delta}}.$

Then

$$|p - \tilde{p}| \leq \sqrt{\frac{8\tilde{p}(1 - \tilde{p})}{n}\log\frac{2}{\delta}} + 2\left(\frac{1}{n}\log\frac{2}{\delta}\right)^{\frac{3}{4}} + \frac{4}{3n}\log\frac{2}{\delta}$$

Proof. Using the first confidence interval

$$|p - \hat{p}| \le \sqrt{\frac{2p(1-p)}{n}\log\frac{2}{\delta}} + \frac{2}{3n}\log\frac{2}{\delta}$$

Assume without loss of generality that $1-p \geq 1-\tilde{p}$ (the case where $p \geq \tilde{p}$ is identical. Therefore

$$|p - \hat{p}| \le \sqrt{\frac{2\tilde{p}(1 - \tilde{p})}{n} \log \frac{2}{\delta}} + \sqrt{\frac{2(p - \tilde{p})(1 - \tilde{p})}{n} \log \frac{2}{\delta}} + \frac{2}{3n} \log \frac{2}{\delta}$$

$$\le \sqrt{\frac{2\tilde{p}(1 - \tilde{p})}{n} \log \frac{2}{\delta}} + \sqrt{\frac{4\sqrt{\frac{1}{2n} \log \frac{2}{\delta}}}{n} \log \frac{2}{\delta}} + \frac{2}{3n} \log \frac{2}{\delta}$$

$$= \sqrt{\frac{2\tilde{p}(1 - \tilde{p})}{n} \log \frac{2}{\delta}} + 8^{\frac{1}{4}} \left(\frac{1}{n} \log \frac{2}{\delta}\right)^{\frac{3}{4}} + \frac{2}{3n} \log \frac{2}{\delta},$$

where we used the second confidence interval and algebra. Bounding $|\hat{p} - \tilde{p}|$ by the first confidence interval leads to

$$|p - \tilde{p}| \leq \sqrt{\frac{8\tilde{p}(1 - \tilde{p})}{n}\log\frac{2}{\delta}} + 2\left(\frac{1}{n}\log\frac{2}{\delta}\right)^{\frac{3}{4}} + \frac{4}{3n}\log\frac{2}{\delta}$$

as required.

C Proof of Lemma 8

We need to define some higher "moments" of the value function. This is somewhat unfortunate as it complicates the proof, but may be unavoidable.

Definition 32. We define the space of bounded value/reward functions \mathcal{R} by

$$\mathcal{R}(i) := \left\{ v \in \left[0, \left(\frac{1}{1-\gamma} \right)^i \right]^{|S|} \right\} \subset \mathbb{R}^{|S|}.$$

Let π be some stationary policy. For $r_d \in \mathcal{R}(d)$ define values V_d^{π} by the Bellman equations

$$V_d^{\pi}(s) = r_d(s) + \gamma \sum_{s'} p_{s,\pi}^{s'} V_d^{\pi}(s').$$

Additionally,

$$\sigma_d^{\pi}(s)^2 \ := \ p_{s,\pi} \cdot V_d^{\pi 2} - [p_{s,\pi} \cdot V_d^{\pi}]^2 \,.$$

Note that $V_d \in \mathcal{R}(d+1)$ and $\sigma_d^2 \in \mathcal{R}(2d+2)$. Let $r_0 \in \mathcal{R}(0)$ be the true reward function $r_0(s) := r(s)$ and define a recurrence by $r_{2d+2}(s) := \sigma_d^{\pi}(s)^2$. We define \tilde{r}_d , \hat{r}_d , \tilde{V}_d^{π} , \tilde{V}_d^{π} and $\tilde{\sigma}_d^{\pi}$, $\hat{\sigma}_d^{\pi}$ similarly but where all parameters have hat/tilde.

The following lemma generalises Lemma 10.

Lemma 33. Let $M \in \mathcal{M}_k$ at time-step t then

$$|(p_{s,\pi} - \tilde{p}_{s,\pi}) \cdot \tilde{V}_d^{\pi}| \le \sqrt{\frac{8L_1\tilde{\sigma}_d^{\pi}(s)^2}{n_t(s)}} + 2\left(\frac{L_1}{n_t(s)}\right)^{\frac{3}{4}} \frac{1}{(1-\gamma)^{d+1}} + \frac{4L_1}{3n_t(s)(1-\gamma)^{d+1}}$$

Proof. Drop references to π and let $p := p_{s,\pi}^{sa^+}$, $\tilde{p} := \tilde{p}_{s,\pi}^{sa^+}$ and $n := n_t(s)$. Since $M, \widetilde{M} \in \mathcal{M}_k$ then apply Lemma 31 to obtain

$$|p - \tilde{p}| \le \sqrt{\frac{8L_1\tilde{p}(1-\tilde{p})}{n}} + 2\left(\frac{L_1}{n}\right)^{\frac{3}{4}} + \frac{4L_1}{3n}$$

Assume without loss of generality that $\widetilde{V}_d(sa^+) \geq \widetilde{V}_d(sa^-)$. Therefore we have

$$|(p_{s,\pi} - \tilde{p}_{s,\pi}) \cdot \tilde{V}_d| \le \sqrt{\frac{8L_1\tilde{p}(1-\tilde{p})}{n}} \left(\tilde{V}_d(sa^+) - \tilde{V}_d(sa^-) \right) + 2\left(\frac{L_1}{n}\right)^{\frac{3}{4}} \frac{1}{(1-\gamma)^{d+1}} + \frac{4L_1}{3n(1-\gamma)^{d+1}}, \tag{10}$$

where we used Assumption 1 and the fact that $V_d \in \mathcal{R}_{d+1}$.

$$\tilde{p}(1-\tilde{p})\left(\tilde{V}_d(\mathbf{s}a^+)-\tilde{V}_d(\mathbf{s}a^-)\right)^2 = \tilde{p}(1-\tilde{p})\left(\tilde{V}_d(\mathbf{s}a^+)^2 + \tilde{V}_d(\mathbf{s}a^-)^2 - 2\tilde{V}_d(\mathbf{s}a^+)\tilde{V}_d(\mathbf{s}a^-)\right) \\
= \tilde{p}\tilde{V}_d(\mathbf{s}a^+)^2 + (1-\tilde{p})\tilde{V}_d(\mathbf{s}a^-)^2 - \left(\tilde{p}\tilde{V}_d(\mathbf{s}a^+) + (1-\tilde{p})\tilde{V}_d(\mathbf{s}a^-)\right)^2 \\
= \tilde{\sigma}_d(s)^2.$$

Substituting into Equation (10) completes the proof.

Proof of Lemma 8. For ease of notation we drop π and t super/subscripts. Let

$$\Delta_d := \left| \sum_{s \in S} [w(s) - \tilde{w}(s)] r_d(s) \right| \equiv |\tilde{V}_d(s_t) - V_d(s_t)|.$$

Using Lemma 9

$$\Delta_d = \gamma \left| \sum_{s \in S} w(s) (p_s - \tilde{p}_s) \cdot \tilde{V}_d \right|$$

$$\leq \frac{\epsilon}{4(1 - \gamma)^d} + \left| \sum_{s \in X} w(s) (p - \tilde{p}) \cdot \tilde{V}_d \right|$$

$$\leq \frac{\epsilon}{4(1 - \gamma)^d} + A_d + B_d + C_d,$$

where

$$A_d := \sum_{s \in X} w(s) \sqrt{\frac{8L_1 \tilde{\sigma}_d^2}{n(s)}} \quad B_d := \sum_{s \in X} w(s) \frac{4L_1}{3n(s)(1-\gamma)^{d+1}} \quad C_d := \sum_{s \in X} w(s) 2\left(\frac{L_1}{n(s)}\right)^{3/4}.$$

The expressions B_d and C_d are substantially easier to bound than A_d . First we give a naive bound on A_d , which we use later.

$$A_{d} \leq \sum_{s \in X} \sqrt{\frac{8w(s)\tilde{\sigma}_{d}^{2}(s)L_{1}}{n(s)}} \equiv \sum_{\kappa,\iota \in \mathcal{K} \times \mathcal{I}} \sum_{s \in K(\kappa,\iota)} \sqrt{\frac{8w(s)\tilde{\sigma}_{d}^{2}(s)L_{1}}{n(s)}}$$

$$\leq \sum_{\kappa,\iota \in \mathcal{K} \times \mathcal{I}} \sqrt{\frac{8L_{1}|K(\kappa,\iota)|}{m\kappa}} \sum_{s \in K(\kappa,\iota)} w(s)\tilde{\sigma}_{d}^{2}(s) \leq \sum_{\kappa,\iota \in \mathcal{K} \times \mathcal{I}} \sqrt{\frac{8L_{1}}{m}} \sum_{s \in K(\kappa,\iota)} w(s)\tilde{\sigma}_{d}^{2}(s)$$

$$(11)$$

(12)

$$\leq \sqrt{\frac{8|\mathcal{K}\times\mathcal{I}|L_1}{m}\sum_{\kappa,\iota\in\mathcal{K}}\sum_{s\in K(\kappa,\iota)}w(s)\tilde{\sigma}_d^2(s)} \leq \sqrt{\frac{8|\mathcal{K}\times\mathcal{I}|L_1}{m}\sum_{s\in X}w(s)\tilde{\sigma}_d^2(s)} \tag{13}$$

$$\leq \sqrt{\frac{8|\mathcal{K}\times\mathcal{I}|L_1}{m(1-\gamma)^{2d+3}}},\tag{14}$$

where in Equation (11) we used the definitions of A_d and \mathcal{K} . In Equation (12) we applied Cauchy-Schwartz and the assumption that $|K(\kappa)| \leq \kappa$. In Equation (13) we used Cauchy-Schwartz again and the definition of \mathcal{K} . Finally we apply the trivial bound of $\sum w(s)\tilde{\sigma}_d^2(s) \leq 1/(1-\gamma)^{2d+3}$. Unfortunately this bound is not sufficient for our needs. The solution is approximate w(s) by $\tilde{w}(s)$ and use Lemma 11 to improve the last step

above.

$$A_d \leq \sqrt{\frac{8|\mathcal{K} \times \mathcal{I}|L_1}{m} \sum_{s \in S} w(s)\tilde{\sigma}_d^2(s)}$$
 (15)

$$\equiv \sqrt{\frac{8|\mathcal{K}\times\mathcal{I}|L_1}{m}} \sum_{s\in S} \tilde{w}(s)\tilde{\sigma}_d^2(s) + \frac{8|\mathcal{K}\times\mathcal{I}|L_1}{m} \sum_{s\in S} (w(s) - \tilde{w}(s))\tilde{\sigma}_d^2(s) \tag{16}$$

$$\leq \sqrt{\frac{8|\mathcal{K} \times \mathcal{I}|L_1}{m(1-\gamma)^{2d+2}}} + \frac{8|\mathcal{K} \times \mathcal{I}|L_1}{m} \Delta_{2d+2},\tag{17}$$

where Equation (15) is as in the naive bound. Equation (16) is substituting w(s) for $\tilde{w}(s)$ and Equation (16) uses the definition of Δ . Therefore

$$\Delta_d \leq \frac{\epsilon}{4(1-\gamma)^d} + B_d + C_d + \sqrt{\frac{8L_1|\mathcal{K} \times \mathcal{I}|}{m} \left[\frac{1}{(1-\gamma)^{2d+2}}\right]} + \sqrt{\frac{8|\mathcal{K} \times \mathcal{I}|L_1}{m}} \Delta_{2d+2}.$$

Expanding the recurrence up to β leads to

$$\Delta_{0} \leq 8 \sum_{d \in \mathcal{D} - \{\beta\}} \left(\frac{L_{1}|\mathcal{K} \times \mathcal{I}|}{m} \right)^{d/(d+2)} \left[\frac{\epsilon}{4(1-\gamma)^{d}} + B_{d} + C_{d} + \sqrt{\frac{L_{1}|\mathcal{K} \times \mathcal{I}|}{m}} \left[\frac{1}{(1-\gamma)^{2d+2}} \right] \right]^{2/(d+2)} + 8 \left(\frac{L_{1}|\mathcal{K} \times \mathcal{I}|}{m} \right)^{\beta/(\beta+2)} \left[2\sqrt{\frac{L_{1}|\mathcal{K} \times \mathcal{I}|}{m(1-\gamma)^{2\beta+3}}} + B_{\beta} + C_{\beta} \right]^{2/(\beta+2)},$$

$$(18)$$

where we used the naive bound to control A_{β} . The bounds on B_d and C_d are somewhat easier, and follow similar lines to the naive bound on A_d .

$$B_{d} \equiv \sum_{s \in X} w(s) \frac{4L_{1}}{3n(s)(1-\gamma)^{d+1}} = \frac{4L_{1}}{3(1-\gamma)^{d+1}} \sum_{\kappa,\iota \in \mathcal{K}} \frac{|K(\kappa,\iota)|}{m\kappa} \leq \frac{4|\mathcal{K} \times \mathcal{I}|L_{1}}{3m(1-\gamma)^{d+1}}$$

$$C_{d} \equiv 2 \sum_{s \in X} w(s) \left(\frac{L_{1}}{n(s)}\right)^{\frac{3}{4}} \frac{1}{(1-\gamma)^{d+1}} \leq \frac{2}{(1-\gamma)^{d+1+1/4}} \left(\frac{|\mathcal{K} \times \mathcal{I}|L_{1}}{m}\right)^{\frac{3}{4}}.$$

Letting
$$m := \frac{20L_1|\mathcal{K} \times \mathcal{I}||\mathcal{D}|^2}{\epsilon^2(1-\gamma)^{2+2/\beta}}$$
 completes the proof.

D Constants

The proof of Theorem 3 uses many constants, which can be hard to keep track of. For convenience we list them below, including approximate upper/lower bounds as appropriate.

 \mathbf{O}/Ω Constant $\iota_{\max} := \left\lceil \frac{1}{\log 2} \log \frac{8|S|}{\epsilon (1-\gamma)^2} \right\rceil \quad \emptyset \log \frac{|S|}{\epsilon (1-\gamma)}$ $\beta := \left\lceil \frac{1}{2\log 2} \log \frac{1}{1-\gamma} \right\rceil \qquad \quad \Omega\left(\log \frac{1}{1-\gamma}\right)$ $\emptyset \log \log \frac{1}{1-\alpha}$ $|\mathcal{D}| := |\mathcal{Z}(\beta)|$ $|\mathcal{K}| := |\mathcal{Z}(|S|)|$ $\emptyset \log |S|$ $\emptyset \log \frac{|S|}{\epsilon(1-\gamma)}$ $|\mathcal{I}| := \iota_{\max} + 1$ $|\mathcal{K} \times \mathcal{I}| := |\mathcal{K}||\mathcal{I}|$ $\emptyset \log |S| \log \frac{|S|}{\epsilon(1-\gamma)}$ $H := \frac{1}{1 - \gamma} \log \frac{8|S|}{\epsilon(1 - \gamma)} \qquad \emptyset \frac{1}{1 - \gamma} \log \frac{|S|}{\epsilon(1 - \gamma)}$ $\Omega\left(\frac{\epsilon(1-\gamma)}{|S|}\right)$ $w_{\min} := \frac{\epsilon(1-\gamma)}{4|S|}$ $\delta_1 := \frac{\delta}{2|S \times A|U_{\max}}$ $\emptyset \frac{\delta}{|S \times A|^2 \log |S| \log \frac{|S|}{\epsilon(1-\gamma)}}$ $\emptyset \log \frac{|S \times A|}{\delta \epsilon (1-\gamma)}$ $L_1 := \log \frac{2}{\delta_1}$ $m := \frac{20L_1|\mathcal{K} \times \mathcal{I}||\mathcal{D}|^2}{\epsilon^2 (1-\gamma)^{2+2/\beta}}$ $\emptyset_{\frac{1}{\epsilon^2(1-\gamma)^2}} \log \frac{|S \times A|}{\delta \epsilon (1-\gamma)} \log |S| \log \frac{|S|}{\epsilon (1-\gamma)} \log^2 \log \frac{1}{1-\gamma}$ $\emptyset \frac{|S \times A|}{\epsilon^2 (1-\gamma)^2} \log \frac{|S \times A|}{\delta \epsilon (1-\gamma)} \log |S| \log \frac{|S|}{\epsilon (1-\gamma)} \log^2 \log \frac{1}{1-\gamma}$ $N := 6|S \times A|m$ $E_{\max} := 4N|\mathcal{K} \times \mathcal{I}| \qquad \qquad \emptyset_{\frac{|S \times A|}{\epsilon^2(1-\gamma)^2}} \log \frac{|S \times A|}{\delta \epsilon(1-\gamma)} \log^2 |S| \log^2 \frac{|S|}{\epsilon(1-\gamma)} \log^2 \log \frac{1}{1-\gamma}$ $U_{\max} := |S \times A| |\mathcal{K} \times \mathcal{I}| \qquad \emptyset |S \times A| \log |S| \log \frac{|S|}{\epsilon(1-\gamma)}$

E Table of Notation

S, AFinite sets of states and actions respectively. The discount fact. Satisfies $\gamma \in (0,1)$. γ The required accuracy. ϵ δ The probability that an algorithm makes more mistakes than its sample-complexity. \mathbb{N} The natural numbers, starting at 0. The natural logarithm. log \wedge, \vee Logical and/or respectively. $\mathbf{E}X$, $\operatorname{Var}X$ The expectation and variance of random variable X respectively. $z_i := 2^i - 2$. z_i Defined as a set of all z_i up to and including a. $\mathcal{Z}(a)$ $\mathcal{Z}(a) := \{z_i : i \leq \arg\min_i \{z_i \geq a\}\}.$ Contains approximately $\log a$ elements. A policy. π The transition function, $p: S \times A \times S \rightarrow [0,1]$. We also write p $p_{s,a}^{s'} := p(s,a,s')$ for the probability of transitioning to state s'from state s when taking action a. $p_{s,\pi}^{s'} := p_{s,\pi(s)}^{s'}$. $p_{s,a} \in [0,1]^{|S|}$ is the vector of transition probabilities. \hat{p}, \tilde{p} Other transition probabilities, as above. The reward function $r: S \to A$. MThe true MDP. $M := (S, A, p, r, \gamma)$. \widehat{M} The MDP with empirically estimated transition probabilities. $\widehat{M} := (S, A, \widehat{p}, r, \gamma).$ \widetilde{M} An MDP in the model class, \mathcal{M} . $\widetilde{M} := (S, A, \tilde{p}, r, \gamma)$. The value function for policy π in MDP M. Can either be viewed V_M^{π} as a function $V_M^{\pi}: S \to \mathbb{R}$ or vector $V_M^{\pi} \in \mathbb{R}^{|S|}$. \widetilde{V}^{π} , \widehat{V}^{π} The values of policy π in MDPs \widetilde{M} and \widehat{M} respectively. $\pi^* \equiv \pi_M^*$ The optimal policy in MDP M. The optimal policy in \widetilde{M} . $\tilde{\pi}^* \equiv \pi_{\widetilde{M}}^*$ $\hat{\pi}^* \equiv \pi^*_{\widehat{M}}$ The optimal policy in \widehat{M} . The (stationary) policy at used in episode k. π_k $n_t(s,a)$ The number of visits to state/action pair (s, a) at time-step t.

 $n_t(s, a, s')$ The number of visits to state s' from state s when taking action a at time-step.

 $n_t(s)$ The number of visits to state/action pair $(s, \pi_t(s))$ at time-step t.

 $v_{t_k}(s, a)$ If t_k is the start of an exploration phase then this is the total number of visits to state (s, a) in that exploration phase.

 s_t, a_t The state and action in time-step t respectively.

 V_d^{π} A higher "moment" value function. See Definition 32.

 $\sigma_d^{\pi}(s)^2$ The variance of $V_d(s')$ when taking action $\pi(s)$ in state s'. Defined in Definition 32.

 L_1 Defined as $\log(2/\delta_1)$.

 \mathcal{D} Defined as $\mathcal{Z}(\beta)$.

 $w_t(s)$ The expected discounted number of visits to state $s, \pi_k(s)$ while following policy π_k .

 X_t The active set containing states s where $w(s) \geq w_{\min}$.

 \mathcal{K} A set if indices, $\mathcal{K} := \mathcal{Z}(|S|)$.

 \mathcal{I} A set of indices, $\mathcal{I} := \{0, 1, 2, \dots, \iota_{\max}\}.$

 $K_t(\kappa, \iota)$ A set of states that have

$$w_t(s) \in [w_t, 2w_t) \land n_t(s) \in m[\kappa w_t, (2\kappa + 2)w_t).$$

Note that $\bigcup_{\kappa,\iota} K_t(\kappa,\iota)$ contains all states with $w(s) \geq w_{\min}$.