Confidence Set for Group Membership*

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Abstract

We develop new procedures to quantify the statistical uncertainty of data-driven clustering algorithms. In our panel setting, each unit belongs to one of a finite number of latent groups with group-specific regression curves. We propose methods for computing unit-wise and joint confidence sets for group membership. The unit-wise sets give possible group memberships for a given unit and the joint sets give possible vectors of group memberships for all units. We also propose an algorithm that can improve the power of our procedures by detecting units that are easy to classify. The confidence sets invert a test for group membership that is based on a characterization of the true group memberships by a system of moment inequalities. To construct the joint confidence, we solve a high-dimensional testing problem that tests group membership simultaneously for all units. We justify this procedure under $N, T \to \infty$ asymptotics where we allow T to be much smaller than N. As part of our theoretical arguments, we develop new simultaneous anti-concentration inequalities for the MAX and the QLR statistics. Monte Carlo results indicate that our confidence sets have adequate coverage and are informative. We illustrate the practical relevance of our confidence sets in two applications.

Keywords: Panel data, grouped heterogeneity, clustering, confidence set, machine learning, moment inequalities, joint one-sided tests, self-normalized sums, high-dimensional CLT, anti-concentration for QLR

JEL codes: C23, C33, C38

1. Introduction

Panel data models with grouped heterogeneity have emerged as useful modeling tools to learn about heterogeneous regression curves (cf. Bonhomme and Manresa 2015; Su, Shi, and Phillips 2016; Vogt and Linton 2017). In these models, it is assumed that the population is partitioned into a finite set of "groups." All members of a group share the same regression curve. Each unit's group membership is unobserved and has to be inferred from its behavior over time. The existing literature has focused on inference with respect to the group-specific regression curves (Bonhomme and Manresa 2015; Su, Shi, and Phillips 2016; Vogt and Linton 2017; Wang, Phillips, and Su 2016).

In the present paper, we focus on the clustering problem and study inference with respect to the group memberships. In particular, we construct joint and unit-wise confidence sets for group membership. For a panel of N units, an element of a joint confidence set is an N-dimensional vector giving a possible group assignment for every unit. Our construction guarantees that the joint confidence set contains the N-vector of true group memberships with a pre-specified probability, say 90%. The joint confidence set quantifies uncertainty about the estimated group structure. For a specific unit, a unit-wise confidence set is a collection of possible group memberships. Its construction ensures that it contains the unit's true group membership at least with a pre-specified probability. The unit-wise confidence set quantifies uncertainty about the estimated group membership of one specific unit.

Whenever we discuss properties of units or compare different units based on clustering results, it is important to keep in mind that the clustering is obtained by a statistical procedure and may suffer from statistical error. Our confidence sets for group membership are the first contribution in the econometric and statistical literature to quantify this error. There are several ways to use our confidence sets to quantify the statistical uncertainty of statements based on data-driven clustering.

For example, we may call a unit's group membership estimate "significant" if it is the only group membership reported by the unit-wise confidence set. If the unit's confidence set is not a singleton then its membership estimate is "insignificant" and other membership assignments than the estimated one are possible. A p-value for this notion of significance can be computed by considering the highest confidence level at which we compute a singleton confidence set. In case we do not want to determine a particular unit of interest in advance, we can use the joint confidence set to find a collection of units whose group memberships are clear from the data. We select all units for which the membership configurations in the joint confidence set agree on a unique group assignment. These units can be considered jointly significant.

Since a unit's group membership is associated with its regression curve, we can identify, at least with large probability, the true behaviors of all units with jointly significant group memberships. In an empirical applications following Wang, Phillips, and Su (2016), we use this strategy to pick out US states for which we can confidently say that unemployment reacts positively or negatively to changes in the minimum wage.

¹The group structure can be interpreted structurally or as an approximation to some underlying finer pattern of heterogeneity, as in Bonhomme, Lamadon, and Manresa (2016).

In an application following Acemoglu et al. (2008) and Bonhomme and Manresa (2015) we cluster countries according to their respective trajectories towards democratization. We use our joint confidence set to identify the countries in the "high democracy" cluster that are statistically separated from the "low democracy" group. For these countries, we can be confident that they are not "low democracy" countries that have been erroneously labeled as "high democracy". In this application, our confidence set allows us to rule out some, but not all possible misclassifications. This suggests that the observed clustering is affected by estimation error and that, e.g., t-tests of between cluster differences may not be valid comparisons of the corresponding population groups. In settings where our joint confidence set establishes joint significance of all units, such t-tests can be justified as a statistically sound approach to test for differences between the population groups.

Existing theoretical results in the clustering literature predict that group memberships are always estimated precisely. It is well understood that this theoretical prediction is at odds with the finite sample behavior of clustering algorithms. In their seminal paper, Bonhomme and Manresa (2015) document substantial misclassification rates in a simulation design calibrated to their empirical application (see Table S.III in their supplemental appendix). We provide additional simulation evidence illustrating that the probability of estimating the true group structure can be close to zero. In contrast to previous research, our asymptotic analysis does not impose a uniform bound on unit-specific variances of the idiosyncratic error terms and allows uncertainty about group memberships for some units to persist even in the limit.² This allows our asymptotic framework to replicate the classification errors observed in finite samples.

Our unit-wise confidence sets are computed by inverting a test for group membership. The test is based on the observation that a unit's group membership is identified from a system of moment inequalities. We exploit the specific structure of these inequalities to recenter them so that they are binding under the null hypothesis. It follows that testing group membership is equivalent to testing a one-sided hypothesis for a vector of moments. Since we can use mean-adjusted inequalities we avoid the overhead of a data-driven procedure to delete or recenter slack inequalities (Andrews and Barwick 2012; Romano, Shaikh, and Wolf 2014; Chernozhukov, Chetverikov, and Kato 2018).

We construct a joint confidence set by combining unit-wise confidence sets. This approach is computationally attractive. The alternative approach of inverting a joint test for the entire group membership structure is infeasible. If there are G groups then the inversion of such a joint test requires testing G^N possible membership configurations; an intractable task even in small panels. Instead, we exploit the natural grouping of the moment inequalities imposed by the panel setting and combine unit-wise confidence sets. A Bonferroni correction is used to control the correlation of moment inequalities between units and renders our procedure robust to any kind of cross-sectional

²For example, Bonhomme and Manresa (2015) and Vogt and Linton (2017) assume that the variance of the idiosyncratic shocks is bounded and show that, under regularity conditions, this implies that group memberships can be estimated at an exponential rate. To justify their estimators, it is sufficient but not necessary to identify all group memberships correctly. It is expected that their arguments can be extended to the case where group memberships are correctly estimated "on average". In Supplemental Appendix G, we verify this conjecture for a simple model and show that the group–specific coefficients can be estimated consistently in the absence of an uniformly consistent estimator of group membership.

dependence. We show that, under cross-sectional independence, the power loss from the Bonferroni correction is minute.

We propose three procedures for constructing unit-wise confidence sets, corresponding to three flavors of the underlying test of group membership. We consider two test statistics, MAX and QLR, from the literature on testing moment inequalities (cf. Rosen 2008; Andrews and Soares 2010; Romano, Shaikh, and Wolf 2014). The MAX statistic looks at the largest element of the tested vector of moments. We suggest two ways to compute critical values for the MAX statistic. The QLR statistic minimizes a quadratic form and can be derived as the quasi-likelihood ratio test statistic of our one-sided hypothesis.

The first procedure is based on the MAX test statistic and a critical value common to all units and groups. We call it the SNS procedure. This procedure is robust to correlations between within-unit moments but possibly conservative. SNS stands for "self-normalized sum", referring to the theory of self-normalized sums (de la Pena, Lai, and Shao 2009) which allows us to justify this procedure under much weaker moment conditions than our other procedures. The idea of using self-normalized sums in moment inequality testing is due to Chernozhukov, Chetverikov, and Kato (2018). Our critical value is defined differently from theirs and admits a finite-sample justification under an additional normality assumption.

The other two procedures are designed to increase power by taking the correlation of the withinunit moments into account. Within-unit inequalities can be highly correlated since they are based on the same time series of observations. We suggest critical values that adapt to this correlation and that are still cheap to compute. The critical values are unit-specific, allowing for unrestricted heteroscedasticity.

Our second procedure combines the MAX statistic with unit-specific critical values computed from a multivariate t-distribution. We call it the MAX procedure. Our third procedure combines the QLR statistic with unit-specific critical value computed from a mixture of F-distributions. We call it the QLR procedure.

Our unit-specific critical values for the MAX and the QLR procedures are easy to compute, requiring only the evaluation of standard distribution functions. To control the size of the joint test, we have to set a high nominal level for the marginal tests. In alternative testing approaches based on bootstrap methods (Romano, Shaikh, and Wolf 2014; Chernozhukov, Chetverikov, and Kato 2018), computing unit-wise critical values at high nominal levels requires approximating large quantiles of the bootstrap distribution. This is computationally challenging using the usual Monte Carlo approach.

For all three procedures, we allow for estimated group-specific coefficients. We are agnostic about the specific choice of estimator for the coefficients. For example, the estimator may be based on an auxiliary training data set where group memberships are observed. Alternatively, coefficients can be estimated without information about the true group memberships using approaches based on *kmeans* clustering (Bonhomme and Manresa 2015; Vogt and Linton 2017) or penalization (Su, Shi, and Phillips 2016; Wang, Phillips, and Su 2016). Recently, Okui and Wang (2018) apply our

method to compute confidence sets for group membership based on their proposed estimator which combines kmeans clustering with break detection by penalization. If the estimator satisfies a weak rate condition, its effect on the distribution of the unit-wise test statistics can be ignored when computing critical values.

We propose a variation of our procedure that can increase the power of the joint confidence set in settings with substantial heteroscedasticity. We call this approach unit selection. Intuitively, we identify units whose groups memberships are "obvious" from the data and ignore the randomness of the group assignment for these units when computing the confidence set. To motivate our approach, suppose that the panel is split into units with low noise for which the group membership is "obvious" and units with noisier measurements. We suggest an algorithm that learns the identities of the "obvious" units and excludes them from the computation of the Bonferroni adjustment. Unit selection is different from moment selection, a popular approach for increasing the power of a test in the literature of moment inequalities.³ Moment selection detects inequalities that are "obviously" slack, but is not applicable in our setting with re-centered moments. Unit selection detects units for which the estimated group memberships are "obviously" the true ones.

The justification of our procedures is based on a double asymptotic framework that sends both the number of units N and the number of time periods T to infinity. We show that the confidence sets have correct coverage uniformly over a broad class of data generating processes (DGPs). This class contains DGPs under which T is very small compared to N. For example, our result for the SNS procedure requires only $T^{-1/3}(\log N) \to 0$ under some regularity conditions.

Our testing problem is a high-dimensional problem since the number of simultaneously tested inequalities (G-1)N is large compared to the number of time periods T that determine the quality of the marginal Gaussian approximations. The theoretical analysis of high-dimensional problems is challenging since many standard arguments, such as Slutzky's lemma, are only available for settings with fixed dimensions. For some of our arguments we follow Chernozhukov, Chetverikov, and Kato (2018). Other arguments are completely new and advance the theory of high-dimensional one-sided testing in a way that may be of independent interest.

The most notable difference between Chernozhukov, Chetverikov, and Kato (2018) and our setting is that they combine many moment inequalities into one test statistic, whereas our tests are based on many marginal tests that each compare a unit-wise test statistic against a unit-specific critical value. Key ingredients to accommodate our setting are new simultaneous anti-concentration inequalities for the MAX and the QLR statistics. These inequalities are needed to prove a high-dimensional Slutzky-type result. They bound the probability that small perturbations in any of the marginal test statistics change the outcome of the joint test.

To the best of our knowledge, we are the first to study the QLR test statistic in a high-dimensional setting. Our high-dimensional analysis complements the classical results for the finite-dimensional

³Both moment selection and moment recentering address possible slackness of moment inequalities. For a comparison of the two approaches, see Allen (2017). These methods are developed in Andrews and Soares (2010), Bugni (2010), Andrews and Barwick (2012), Chernozhukov, Chetverikov, and Kato (2018), and Romano, Shaikh, and Wolf (2014).

setting in Wolak (1991) and Rosen (2008). We justify the joint Gaussian approximation of the unit-wise QLR statistics using a high-dimensional central limit theorem (CLT) for sparse-convex sets (Chernozhukov, Chetverikov, and Kato 2017).

We complement our asymptotic results by Monte Carlo experiments to study the performance of our procedures in finite samples. In our simulation designs, our confidence sets have good coverage. We also demonstrate that neither the MAX nor the QLR procedure dominates the other in terms of power, and that the SNS procedure is conservative. In a design with substantial heteroscedasticity, we illustrate the usefulness of our unit selection procedure.

We demonstrate the empirical relevance of our confidence sets for group membership in two applications. First, we follow Wang, Phillips, and Su (2016) and study heterogeneous relationships between a minimum wage and unemployment in a US state panel. We compute the set of states with jointly significant membership estimates and interpret it as the set of states for which we can confidently infer the sign of the relationship between a minimum wage and unemployment. Secondly, we study the country panel data on income and democracy from Acemoglu et al. (2008). We consider the specification with group-specific trends from Bonhomme and Manresa (2015). In a specification with four groups our joint confidence sets separate the two most extreme groups.

The rest of the paper is organized as follows: Section 2 discusses related literature and Section 3 introduces our panel model with a group structure. Section 4 motivates and describes our method for computing joint and unit-wise confidence sets. Section 5 gives an asymptotic justification of our procedures and Section 6 reports our simulation results. Section 7 discusses the two applications.

2. Related Literature

Classifying units into discrete groups is one of the oldest problems in statistics and statistical decision theory (Pearson 1896). Popular modeling tools are finite mixture models (McLachlan and Peel 2004). These models offer a random-effect approach to modeling discrete heterogeneity (Bonhomme, Lamadon, and Manresa 2016). In computer science, classification and clustering problems are often tackled using machine learning (Friedman, Hastie, and Tibshirani 2009). We have not been able to find any research on how to conduct joint inference on the population group structure in the machine learning literature.

Some algorithms in machine learning compute posterior probabilities of group membership (Murphy 2012, Chapter 5.7.2).⁴ In principle, it is possible to compute unit-wise or even joint Bayesian credible sets from the posterior distribution. However, in particular for a joint credible set we expect this approach to be computationally challenging, perhaps prohibitively so. An advantage of our frequentist approach over Bayesian credible sets is that it guarantees good coverage for all DGPs contained in a broad class.

We follow the recent econometric literature and treat the unobserved group memberships as a structural parameter. Inference in panel models with a latent group structure has been studied

⁴For example, in the case of finite mixture models, posterior probabilities can be computed in the E-step of the EM algorithm (Dempster, Laird, and Rubin 1977).

in Lin and Ng (2012), Bonhomme and Manresa (2015), Sarafidis and Weber (2015), Ando and Bai (2016), Vogt and Linton (2017), Wang, Phillips, and Su (2016), Lu and Su (2017), Vogt and Schmid (2017), and Gu and Volgushev (2018).⁵ Previous studies address inference with respect to the group-specific regression curves. We are the first to address inference on group membership.

In addition to formal clustering algorithms, empirical researchers often use informal ad-hoc methods to identify clusters of units that behave similarly. For example, Kneeland (2015) uses experimental data to cluster test subjects by orders of rationality, and Silveira (2017) uses data on court sentences to classify judges as either "lenient" or "harsh". The interpretation of such results hinges on whether the observed sample separation reflects a separation at the population level. This requires quantifying the degree of misclassification. Depending on the nature of the ad-hoc approach, it may be possible to leverage our confidence sets to this end.

Our theoretical analysis relies on the theory of self-normalized sums (de la Pena, Lai, and Shao 2009) and recent results in high-dimensional statistics, particularly the CLTs in Chernozhukov, Chetverikov, and Kato (2017) and the anti-concentration result in Chernozhukov, Chetverikov, and Kato (2015). We contribute new theoretical results for high-dimensional testing problems.

Our confidence set is based on a characterization of the true group memberships by a system of moment inequalities. A recent review of confidence sets constructed from moment inequalities is given in Canay and Shaikh (2016). Most of the previous literature focuses on finite systems of moment inequalities. Chernozhukov, Chetverikov, and Kato (2018) provide a framework for testing high-dimensional systems of moment inequalities.⁶ Our approach builds on and extends their results.

To compute our joint confidence set, we solve a multiple one-sided testing problem. We provide a theoretical argument for the validity of our procedure for a diverging number of simultaneously tested hypotheses. Romano and Wolf (2018) study a multiple one-sided testing problem in a simulation experiment, but do not provide an asymptotic analysis of their approach.

3. Setting

We observe panel data (y_{it}, x_{it}) , i = 1, ..., N and t = 1, ..., T, where y_{it} is a scalar dependent variable and x_{it} is a covariate vector. We assume that units are partitioned into a finite set of groups $\mathbb{G} = \{1, ..., G\}$, where the number of groups G is assumed to be known.⁷ Group membership is unobserved. The relationship between y_{it} and x_{it} is described by a linear model. Units within the same group share the same coefficient value. Between groups, coefficient values may vary. Let $\beta_{g,t}$ denote the vector of coefficients that applies to units in group $g \in \mathbb{G}$ at time t = 1, ..., T. Unit i's

⁵Models with a latent group structure have also been proposed for data other than panel data (Shao and Wu 2005). ⁶Estimation with many moment inequalities is examined by Menzel (2014).

⁷There are several ways to choose G in practice. Bonhomme and Manresa (2015) discuss information criteria and Lu and Su (2017) propose a test for the value of G.

true group membership is denoted g_i^0 . In period t, unit i's outcome is generated according to

$$y_{it} = x'_{it}\beta_{q_i^0, t} + u_{it}, (1)$$

where u_{it} is an error term.

This paper addresses inference with respect to the vector of latent group memberships $\{g_i^0\}_{1 \leq i \leq N}$. We assume that an estimator $\hat{\beta}_{g,t}$ of $\beta_{g,t}$ is available. For example, estimators based on the *kmeans* algorithm (Bonhomme and Manresa 2015) or on penalization (Su, Shi, and Phillips 2016, Wang, Phillips, and Su 2016) may be used. Under a weak rate condition, our procedure controls for uncertainty from parameter estimation.

In applications, two special cases of model (1) are of particular interest.

Example 1 (Random coefficient model with a group structure). The coefficient vector is assumed to be constant over time. The model is

$$y_{it} = x'_{it}\beta_{g_i^0} + u_{it}.$$

Estimation of this model is considered in Su, Shi, and Phillips (2016) and Wang, Phillips, and Su (2016). In Section D of the Supplemental Appendix, we discuss how to apply our procedures if unit fixed effects are added to this specification. Following Wang, Phillips, and Su (2016), we apply the random coefficient model to the analysis of heterogeneous effects of a minimum wage.

Example 2 (The group fixed effect model). The set of regressors contains a constant term. The coefficient on the constant term is group-specific and varies over time. It is called the group fixed effect (Bonhomme and Manresa 2015). The values of the coefficients on the time-varying regressors are constant over groups and time periods. The model is

$$y_{it} = w'_{it}\theta + \alpha_{g_i^0,t} + u_{it},$$

where w_{it} is a vector of time-varying regressors, θ is a common slope coefficient and $\alpha_{g_i^0,t}$ is the group fixed effect. As Bonhomme and Manresa (2015), we apply it to the clustering of countries according to their respective trajectories of democratization.

4. Procedure

This section describes our procedures for constructing confidence sets for group membership.

4.1. Definition of confidence set for group membership

We consider joint confidence sets for the entire group structure as well as unit-wise confidence sets for each unit i.

A joint confidence set quantifies uncertainty about the true group structure $\{g_i^0\}_{1 \leq i \leq N}$. It is a random subset of the set of all possible group configurations \mathbb{G}^N that contains the true group

structure with a pre-specified probability. Formally, for $0 < \alpha < 1$, the joint confidence set \widehat{C}_{α} with confidence level $1 - \alpha$ is a random set $\widehat{C}_{\alpha} \subset \mathbb{G}^N$ that satisfies

$$\lim_{N,T\to\infty} \inf_{P\in\mathbb{P}_N} P\left(\{g_i^0\}_{1\leq i\leq N} \in \widehat{C}_\alpha\right) \geq 1-\alpha,\tag{2}$$

where \mathbb{P}_N is a set of DGPs that satisfy certain regularity conditions. A typical element of \widehat{C}_{α} is $\{g_i\}_{1\leq i\leq N}$ with $g_i\in \mathbb{G}$. If $\{g_i\}_{1\leq i\leq N}\in \widehat{C}_{\alpha}$, then we cannot exclude the possibility that $\{g_i^0\}_{1\leq i\leq N}=\{g_i\}_{1\leq i\leq N}$ at a confidence level of at least $1-\alpha$. The infimum over \mathbb{P}_N in (2) ensures that the asymptotic coverage provides a good approximation to the finite sample counterpart uniformly over a wide range of underlying DGPs.

A unit-wise confidence set for unit i is a non-empty random subset $\widehat{C}_{\alpha,1,i}$ of the set of possible group memberships $\mathbb G$ that contains i's true group membership g_i^0 with a pre-specified probability. At confidence level $1-\alpha$

$$\liminf_{T \to \infty} \inf_{P \in \mathbb{P}} P\left(g_i^0 \in \widehat{C}_{\alpha,1,i}\right) \ge 1 - \alpha,$$

where \mathbb{P} is a set of DGPs.

A unit-wise confidence interval quantifies the uncertainty about the group membership of one specific unit. For example, if $\widehat{C}_{\alpha,1,i}$ is a singleton, say $\widehat{C}_{\alpha,1,i} = \{1\}$, then we may conclude at confidence level $1 - \alpha$ that unit i belongs to group 1. On the other hand, if $\widehat{C}_{\alpha,1,i} = \mathbb{G}$ then the data is uninformative about i's group membership at the designated confidence level.

4.2. Motivation of our approach

The key insight of our approach is that each unit's group membership can be characterized by a system of moment inequalities that can be used for a statistical test of the hypothesis $H_0: g_i^0 = g$. Our confidence sets are constructed by inverting such a test. To focus on the main idea, we assume in this section that group-specific parameters are known.

The null hypothesis $H_0: g_i^0 = g$ is equivalent to

$$\mathbb{E}\left[\left(y_{it} - x'_{it}\beta_{g,t}\right)^{2}\right] \leq \mathbb{E}\left[\left(y_{it} - x'_{it}\beta_{h,t}\right)^{2}\right]$$
(3)

for all $h \in \mathbb{G}$ and t = 1, ..., T. This inequality is justified under $\mathbb{E}[u_{it} \mid x_{it}] = 0$, which guarantees that the true group membership minimizes a least-squares criterion.

To test the inequalities (3), we introduce a mean-adjusted difference between squared residuals. Let

$$d_{it}(g,h) = \frac{1}{2} \left(\left(y_{it} - x'_{it}\beta_{g,t} \right)^2 - \left(y_{it} - x'_{it}\beta_{h,t} \right)^2 + \left(x'_{it} \left(\beta_{g,t} - \beta_{h,t} \right) \right)^2 \right).$$

The first two terms on the right-hand side are squared residuals. The third term recenters moments

and ensures that $d_{it}(g,h)$ has mean zero under the null hypothesis. This can be seen by writing

$$d_{it}(g,h) = -u_{it}x'_{it}(\beta_{g,t} - \beta_{h,t}) + \left(\beta_{g,t} - \beta_{g_i^0,t}\right)' x_{it}x'_{it}(\beta_{g,t} - \beta_{h,t}). \tag{4}$$

Here, the first term on the right-hand side has mean zero under $\mathbb{E}[u_{it} \mid x_{it}] = 0$. Under $g_i^0 = g$, the second term vanishes for $g = g_i^0$ and we have $\mathbb{E}[d_{it}(g,h)] = 0$ for all $h \in \mathbb{G} \setminus \{g\}$ and $t = 1, \ldots, T$. If $g_i^0 \neq g$ then there is $h \in \mathbb{G} \setminus \{g\}$ such that $\mathbb{E}[d_{it}(g,h)] > 0$ for all $t = 1, \ldots, T$. To see this, note that choosing $h = g_i^0 \in \mathbb{G} \setminus \{g\}$ guarantees that $d_{it}(g,h)$ has a strictly positive mean if $\mathbb{E}[x_{it}x'_{it}]$ is positive definite. Averaging along the time dimension⁹ this establishes that the null hypothesis $H_0: g_i^0 = g$ is equivalent to

$$\left(\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[d_{it}(g,h)]\right)_{h\in\mathbb{G}\setminus\{g\}} = 0$$
(5)

and the alternative $H_1: g_i^0 \neq g$ is equivalent to the vector on the left-hand side of (5) having at least one strictly positive component.

This shows that testing the group membership of a unit is equivalent to a one-sided significance test for a vector of moments.

Remark 1. The explicit mean adjustment is our solution to the problem of possibly slack moment inequalities in (3). It exploits the specific structure of our problem and ensures that we test inequalities that are binding under the null hypothesis. This turns the problem of testing the moment inequalities (3) into a one-sided testing problem for a vector of moments. In other testing problems with moment inequalities, a similar mean adjustment is not feasible and possible slackness of the tested inequalities has to be addressed in another way (Andrews and Soares 2010; Andrews and Barwick 2012; Romano, Shaikh, and Wolf 2014).

4.3. Procedures for computing confidence sets

4.3.1. Unit-wise confidence sets

Unit-wise confidence sets are computed by inverting a test for group membership. Let $\hat{T}_i(g)$ denote a test statistic. The test statistics that we propose below are based on a sample analogue of the left-hand side of (5) and measure its positive deviation from zero.

For a pre-specified probability α , let $c_{\alpha,1,i}(g)$ denote a critical value. Moreover, let \hat{g}_i denote an

⁸The assumption $\mathbb{E}[u_{it} \mid x_{it}] = 0$ implies $\mathbb{E}[d_{it}(g_i^0, h) \mid x_{it}] = 0$. The conditional version can yield a more powerful test if there is a specific alternative and a function f such that the moment $\mathbb{E}[d_{it}(g_i^0, h)f(x_{it})]$ reveals more evidence against the null hypothesis than the moment $\mathbb{E}[d_{it}(g_i^0, h)]$. In our setting, relevant alternatives are detected by large positive values of the quadratic form in (4). Therefore, we do not expect that the power of the test can be improved by using a function f to look in another direction.

⁹Some degree of averaging is required in order to express the null hypothesis in terms of an estimable quantity.

estimator of $g_i^{0.10}$ A unit-wise confidence set for unit i is given by

$$\widehat{C}_{\alpha,1,i} = \left\{ g \in \mathbb{G} : \widehat{T}_i(g) \le c_{\alpha,1,i}(g) \right\} \cup \left\{ \widehat{g}_i \right\}.$$

Adding the estimated group membership guarantees that the confidence set is never empty.¹¹

4.3.2. Joint confidence set

A joint confidence set for all units is constructed by stringing together Bonferroni-corrected unitwise confidence sets. Let $c_{\alpha,N,i}(g) = c_{\alpha/N,1,i}(g)$ be a Bonferroni-corrected critical value. Our joint confidence set is given by

$$\widehat{C}_{\alpha} = \underset{1 \leq i \leq N}{\times} \widehat{C}_{\alpha,N,i} = \underset{1 \leq i \leq N}{\times} \left\{ g \in \mathbb{G} : \widehat{T}_{i}(g) \leq c_{\alpha,N,i}(g) \right\} \cup \left\{ \widehat{g}_{i} \right\}.$$

Note that, in principle, a joint confidence set can be obtained by inverting a joint test for the group memberships for all units. However, such an approach is numerically infeasible in our setting with a large discrete null space. To compute the joint confidence set, we would have to invert G^N joint tests. Therefore, the computational cost of such a confidence set grows exponentially in N and is prohibitive even in small panels.

In contrast, the complexity of computing our joint confidence set scales linearly in N. Our joint confidence set achieves this computational advantage without sacrificing too much power by exploiting the panel structure of the moment inequalities.

In particular, our procedure can fully adapt to the correlation of the within-unit moment inequalities. This is important since the within-unit inequalities are all based on the same time series and are therefore expected to be strongly correlated. Correlations of the moment inequalities between units are controlled using a Bonferroni correction. The following result establishes that this Bonferroni correction renders our joint confidence set only minimally conservative if units are independent ("cross-sectional independence") and if N is not very small.

Theorem 1. Let $0 < \alpha < 1$ and suppose that the unit-wise confidence sets satisfy

$$P\left(g_i^0 \notin \widehat{C}_{\alpha,N,i}\right) = \alpha/N$$

and that units are independent. Then

$$\alpha - \frac{\alpha^2}{2} \le P\left(\{g_i^0\}_{1 \le i \le N} \notin \widehat{C}_\alpha\right) \le \alpha - \frac{\alpha^2}{2} \left(1 - \frac{\alpha}{3} + \frac{1}{N} \left(1 - \frac{\alpha}{N}\right)^{-2}\right).$$

For example, for $N \ge 8$ independent units and nominal coverage level $1 - \alpha = 0.9$, the theorem predicts that the Bonferroni correction inflates the coverage probability of the joint confidence set

¹⁰Typically, such an estimator is available as part of the procedure that estimates the group-specific parameters. If not, then such an estimator can be based on inequality (3) (cf. Bonhomme and Manresa 2015).

¹¹This is also required for our algorithm for unit selection to work.

by only between 0.5-0.55% if the empirical coverage probability of all unit-wise confidence sets is equal to the Bonferroni-corrected nominal level of $1 - \alpha/N$.

4.3.3. Test statistics and critical values

We consider different choices for the test statistic \hat{T}_i and the critical value $c_{\alpha,N,i}$. For $g \in \mathbb{G}$ and $t = 1, \ldots, T$, let $\hat{\beta}_{g,t}$ denote an estimator of $\beta_{g,t}$. Define

$$\hat{d}_{it}(g,h) = \frac{1}{2} \left(\left(y_{it} - x'_{it} \hat{\beta}_{g,t} \right)^2 - \left(y_{it} - x'_{it} \hat{\beta}_{h,t} \right)^2 + \left(x'_{it} \left(\hat{\beta}_{g,t} - \hat{\beta}_{h,t} \right) \right)^2 \right).$$

The test for group membership is based on the studentized statistic

$$\hat{D}_{i}(g,h) = \frac{\sum_{t=1}^{T} \hat{d}_{it}(g,h)}{\sqrt{\sum_{t=1}^{T} \left(\hat{d}_{it}(g,h) - \bar{\hat{d}}_{it}(g,h)\right)^{2}}},$$

where $\bar{d}_{it}(g,h) = \sum_{t=1}^{T} \hat{d}_{it}(g,h)/T$. Let $\hat{D}_{i}(g) = \{\hat{D}_{i}(g,h)\}_{h \in \mathbb{G} \setminus \{g\}}$. We consider two test statistics to measure the one-sided distance of $\hat{D}_{it}(g)$ from zero: the MAX and the QLR statistic defined, respectively, as

$$\hat{T}_i^{\text{MAX}}(g) = \max_{h \in \mathbb{G} \setminus \{g\}} \hat{D}_i(g, h),$$

$$\hat{T}_i^{\text{QLR}}(g) = \min_{t \le 0} \left(\hat{D}_i(g) - t \right)' \hat{\Omega}_i^{-1}(g) \left(\hat{D}_i(g) - t \right),$$

with $\widehat{\Omega}_i(g) = \widehat{\Omega}_i^*(g) + \max\{\epsilon - \det(\widehat{\Omega}_i^*(g)), 0\}I_{G-1}$, where I_{G-1} is the identity matrix in \mathcal{R}^{G-1} , $\widehat{\Omega}_i^*(g)$ is the $(G-1) \times (G-1)$ sample correlation matrix with entries

$$\left(\widehat{\Omega}_{i}^{*}(g)\right)_{h,h'} = \frac{\sum_{t=1}^{T} \left(\hat{d}_{it}(g,h) - \bar{\hat{d}}_{it}(g,h)\right) \left(\hat{d}_{it}(g,h') - \bar{\hat{d}}_{it}(g,h')\right)}{\sqrt{\sum_{t=1}^{T} \left(\hat{d}_{it}(g,h) - \bar{\hat{d}}_{it}(g,h)\right)^{2} \sum_{t=1}^{T} \left(\hat{d}_{it}(g,h') - \bar{\hat{d}}_{it}(g,h')\right)^{2}}},$$

and ϵ is a positive parameter that controls the regularization of the sample correlation matrix (cf. Andrews and Barwick 2012).¹²

For the MAX test statistic we offer two different strategies for computing critical values. The SNS critical value is given by

$$c_{\alpha,N,i}^{\text{SNS}}(g) = c_{\alpha,N}^{\text{SNS}} = \sqrt{\frac{T}{T-1}} t_{T-1}^{-1} \left(1 - \frac{\alpha}{(G-1)N} \right),$$

where $t_{T-1}^{-1}(p)$ denotes the p quantile of a t-distribution with T-1 degrees of freedom. This critical value is motivated by the exact finite sample behavior of $\hat{D}_i(g,h)$ under a normality assumption

¹²We do not study the choice of ϵ . In the simulations in Section 6 and the applications in Section 7 we follow Andrews and Barwick (2012) and set $\epsilon = 0.012$.

and known coefficients. We refer to the combination of the MAX statistic and SNS critical value as the SNS procedure and denote this joint confidence set by $\widehat{C}_{\alpha}^{\text{SNS}}$.

Our second strategy for computing critical values adapts to the correlation of the within-unit moments. The SNS critical value is robust against this correlation but can be conservative in the presence of strongly correlated within-unit moments. Our adaptive critical value is given by

$$c_{\alpha,N,i}^{\text{MAX}} = \! c_{\alpha,N}^{\text{MAX}} \left(\widehat{\Omega}_i(g) \right) = \sqrt{\frac{T}{T-1}} \left(t_{\max,\widehat{\Omega}_i(g),T-1} \right)^{-1} \left(1 - \frac{\alpha}{N} \right),$$

where $t_{\max,V,T-1}$ denotes the distribution function of the maximal entry of a centered random vector with multivariate t-distribution with scale matrix V and T-1 degrees of freedom. This critical value is straightforward to evaluate in modern statistical software. Asymptotically, the distribution $t_{\max,V,T-1}$ is equal to the distribution of the maximum of a multivariate normal vector. Compared to its asymptotic limit, $t_{\max,V,T-1}$ has better finite sample properties. In particular, it reduces undercoverage in short panels. We refer to the combination of the MAX statistic and the adaptive critical values as the MAX procedure and denote this joint confidence set by $\widehat{C}_{\alpha}^{\text{MAX}}$.

A popular approach for taking into account the correlation of moment inequalities is to compute critical values from a bootstrap distribution that replicates the correlation (Romano, Shaikh, and Wolf 2014; Chernozhukov, Chetverikov, and Kato 2018). It is difficult, if not infeasible, to apply this approach in our setting with unit-specific critical values. The unit-wise critical values are large quantiles of a bootstrap distribution and cannot be approximated precisely by unsophisticated Monte Carlo methods.¹⁴ In contrast, our critical values are easy to implement and compute fast.

To define the critical value for the QLR test statistic, let $w(\cdot,\cdot,\cdot)$ denote the weight function defined in Kudo (1963). For a $(G-1) \times (G-1)$ covariance matrix V, define the distribution function $F_{\text{QLR},V,T-1}$ by

$$F_{\text{QLR},V,T-1}(t) = 1 - \sum_{j=1}^{G-1} w \left(G - 1, G - 1 - j, V\right) P\left(F_{j,T-1} > t/j\right), \tag{6}$$

where $F_{j,T-1}$ has an F-distribution with j and T-1 degrees of freedom. The distribution $F_{\text{QLR},V,T-1}$ was first discussed by Wolak (1987) in the context of finite sample inference in a one-sided testing problem. Asymptotically, $F_{\text{QLR},V,T-1}$ is equivalent to the mixture of χ^2 distributions discussed in Rosen (2008) and Wolak (1991). Compared to its limit it provides better finite sample coverage. The critical value for the QLR statistic is given by

$$c_{\alpha,N,i}^{\mathrm{QLR}}(g) = c_{\alpha,N}^{\mathrm{QLR}}\left(\widehat{\Omega}_i(g)\right) = F_{\mathrm{QLR},\widehat{\Omega}_i(g),T-1}^{-1}\left(1 - \frac{\alpha}{N}\right).$$

We call the approach based on the QLR statistic and this critical value the QLR procedure and denote this joint confidence set by $\hat{C}_{\alpha}^{\text{QLR}}$.

¹³ The distribution function of the multivariate t-distribution can be efficiently approximated by modern algorithms (Genz 1992). Implementations exist for Stata (Grayling and Mander 2016) and R (Azzalini and Genz 2016).

¹⁴For example, if $\alpha = 0.1$ and N = 50, the upper 0.002 quantile needs to be simulated.

Remark 2. For G = 2 groups all three procedures compute the same joint and unit-wise confidence sets. It is easy to see that the two critical values for the MAX statistic are identical if there are only two groups. Moreover, in this case the QLR statistic and critical value can be obtained by squaring the MAX statistic and critical value.

4.4. Unit selection

In settings with substantial heteroscedasticity the discreteness of our null space can lead to conservative behavior of our confidence set. We propose an algorithm for unit selection that can alleviate this problem.

Our algorithm detects units that are easy to classify. These units can be ignored when computing the critical values for the other units. To illustrate our idea, consider the following hypothetical scenario. Suppose that a half of the units in the sample have very low error variances, making it easy to identify their group memberships. Because there is (almost) no uncertainty about the group memberships of these units, we can ignore them when constructing a joint confidence set. This decreases the effective number of tested moment inequalities from N(G-1) to N(G-1)/2. Reducing the number of tested inequalities allows us to construct a more powerful test.

Our algorithm identifies units that are easy to classify and does so in a way that controls the statistical error of misclassifying units as easy to identify even though they are not. This error control is achieved by slightly increasing the nominal level of the unit-wise confidence sets.

The algorithm examines two conditions to identify easily classified units. First, a test statistic that measures the difference between the left- and the right-hand side of (3) for $g = \hat{g}_i$ and $h \neq \hat{g}_i$ takes a large negative value. We call this moment selection. Second, all alternative group memberships $h \neq \hat{g}_i$ are rejected. We call this hypothesis selection.

The algorithm can be combined with any of the test statistics and critical values discussed above. For $i=1,\ldots,N$, let \hat{T}_i^{type} denote a unit-wise test statistic and $c_{\alpha,N,i}^{\text{type}}$ denote a corresponding critical value, where type = SNS, MAX or QLR. Our algorithm is parameterized by β , $0 \le \beta < \alpha/3$. The larger β , the more unit selection is carried out. Setting $\beta = 0$ switches off unit selection.

We need additional notation to describe the moment selection part of the algorithm. Let

$$\hat{D}_{i}^{U}(g,h) = \frac{\sum_{t=1}^{T} \hat{d}_{it}^{U}(g,h)}{\sqrt{\sum_{t=1}^{T} \left(\hat{d}_{it}^{U}(g,h) - \bar{d}_{i}^{U}(g,h)\right)^{2}}},$$

where

$$\hat{d}_{it}^{U}(g,h) = (y_{it} - x'_{it}\hat{\beta}_{g,t})^{2} - (y_{it} - x'_{it}\hat{\beta}_{h,t})^{2}$$

and $\bar{d}_i^U(g,h) = \sum_{t=1}^T \hat{d}_{it}^U(g,h)/T$. This is a counterpart to $\hat{D}_i(g,h)$ that does not adjust for the

mean under the null hypothesis. For $g \in \mathbb{G}$ and i = 1, ..., N, let

$$\widehat{M}_i(g) = \left\{ h \in \mathbb{G} \setminus \{g\} \mid \widehat{D}_i^U(g, h) > -2c_{\beta, N}^{\text{SNS}} \right\}.$$

This set gives the selected inequalities for the hypothesis $H_0: g_i^0 = g$. An empty $\widehat{M}_i(g)$ provides a strong evidence for $H_0: g_i^0 = g$ and is corresponds to moment selection. Here we assume that \hat{g}_i satisfies $\hat{D}_i^U(\hat{g}_i, h) \leq 0$ for any $h \in \mathbb{G}$. This is satisfied, for example, in the case of *kmeans* clustering.

Our algorithm proceeds as follows:

- 1. Set s = 0 and $H_i(0) = \mathbb{G}$.
- 2. Set $\hat{N}(s) = \sum_{i=1}^{N} \max_{g \in H_i(s)} \mathbf{1} \{ \# \widehat{M}_i(g) \neq 0 \}.$
- 3. Set

$$H_i(s+1) = \left\{ g \in \mathbb{G} \mid \hat{T}_i^{\text{type}}(g) \le c_{\alpha - 2\beta, \hat{N}(s), i}^{\text{type}}(g) \right\} \cup \left\{ \hat{g}_i \right\}.$$

If $H_i(s+1) = H_i(s)$ for all i then go to Step 5.

- 4. Set s = s + 1. Go to Step 2.
- 5. The confidence set with unit selection is given by $\widehat{C}_{\mathrm{sel},\alpha,\beta}^{\mathrm{type}} = X_{1 \leq i \leq N} H_i(s+1)$.

Step 2 of the algorithm counts the number $\hat{N}(s)$ of units that are not easy to classify. Step 3 carries out hypothesis selection with critical value associated with $\hat{N}(s)$. For each unit i, group memberships $g \in H_i(s+1)$ are not rejected under the critical value that accounts for $\hat{N}(s)$ simultaneously tested units. We iterate hypothesis selection (Step 3) while updating $\hat{N}(s)$ (Step 2) until convergence. Note that $\widehat{M}_i(g)$ becomes empty only if $g = \hat{g}_i$. Thus, unit i is not counted for $\hat{N}(s)$ only if $H_i(s) = \{\hat{g}_i\}$ and $\widehat{M}_i(\hat{g}_i)$ is empty. Because $H_i(s)$ depends on $\hat{N}(s-1)$, we iterate the process. Both $\hat{N}(s)$ and the cardinality of $H_i(s)$ are decreasing in s and the iteration always converges.

If there is a sufficient number of units that are easy to classify then $\widehat{C}_{\mathrm{sel},\alpha,\beta}^{\mathrm{type}}$ is more powerful ("smaller") than the confidence set $\widehat{C}_{\alpha}^{\mathrm{type}}$ without unit selection. However, there is a cost of unit selection. In the formulas for the critical values we replace α by $\alpha - 2\beta$. This adjustment controls two possible errors that each occur with probability β . The first error is estimating an incorrect group membership for a unit whose group membership is obvious "in population". The second error is classifying a non-obvious unit as obvious. Because of this cost of unit selection, confidence sets with unit selection can be more conservative ("larger") than those without if an insufficient number of units is eliminated.

Remark 3. Unit selection can be considered as a data-driven way to allocate error probability to each unit. Let α_i denote the probability that the marginal confidence set for unit i does not include i's true group membership. In principle, we may distribute the total error probability α arbitrarily

among the N units as long as $\sum_{i=1}^{N} \alpha_i = \alpha$. Without unit selection our procedures allocate the error probability evenly so that $\alpha_i = \alpha/N$. In our discrete testing problem, this even allocation can render the joint confidence set overly conservative. Each unit's marginal confidence set contains at least one group. For units that are very easy to classify, the probability that a singleton set containing only the estimated group membership does not cover the truth is less than the error probability α/N . This can render our joint confidence set conservative. Our algorithm for unit selection reshuffles allocated error probability from units that are easy to classify to units that are difficult to classify.

5. Asymptotic results

In this section, we establish theoretically that our procedures yield joint confidence sets that asymptotically cover the truth with a pre-specified probability, i.e., we show that (2) holds. All proofs are in the Appendix.

For the justification of the unit-wise confidence sets, we refer to existing results for confidence sets for finite-dimensional parameters defined by moment inequalities (Rosen 2008; Romano, Shaikh, and Wolf 2014).

5.1. Asymptotic framework and assumptions

Our asymptotic framework is of the long-panel variety and takes both the number of units N and the number of time periods T to infinity. In most panel data sets, the number of units far outstrips the number of time periods. We replicate this feature along the asymptotic sequence by allowing N to diverge at a much faster rate than T.

We introduce some assumptions. For a probability measure P, let \mathbb{E}_P denote the expectation operator that integrates with respect to the measure P.

- **Assumption 1.** (i) The set of latent groups is enumerated as $\mathbb{G} = \{1, \ldots, G\}$. For $g, h \in \mathbb{G}$ and $g \neq h$, $\max_{1 \leq t \leq T} \|\beta_{g,t} \beta_{h,t}\| > 0$. There exists K_{β} such that $\max_{g \in \mathbb{G}} \max_{1 \leq t \leq T} \|\beta_{g,t}\| \leq K_{\beta}$.
 - (ii) P is a probability measure such that, for each unit i = 1, ..., N, $(u_{it})_{1 \le t \le T}$ is an independent sequence such that, for t = 1, ..., T, $\mathbb{E}_P[u_{it} \mid x_{it}] = 0$, $\mathbb{E}_P[u_{it}^2] = \sigma_i^2$, $\mathbb{E}_P[x_{it}x'_{it}]$ is of full rank, and there exists $\underline{\sigma} > 0$ such that $\mathbb{E}_P[(u_{it}/\sigma_i)^2 \mid x_{it}] \ge \underline{\sigma}^2$.
- (iii) There exists a sequence $\gamma_{N,T,8}$ and estimators $\hat{\beta}_{g,t}$ of $\beta_{g,t}$ for all $g \in \mathbb{G}$ and t = 1, ..., T such that

$$P\left(\max_{g \in \mathbb{G}} \left(\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{\beta}_{g,t} - \beta_{g,t} \right\|^{8} \right)^{1/8} > \gamma_{N,T,8} \right) \leq \xi_{N,T}$$

for a vanishing sequence $\xi_{N,T}$.

(iv) Along the asymptotic sequence $T \leq N$ and $T^{-1/2}(\log N) \leq 1$. For i = 1, ..., N and t = 1, ..., T the moment $\mathbb{E}_P\left[|u_{it}/\sigma_i|^8 \|x_{it}\|^8 + \|x_{it}\|^{16}/\sigma_i\right]$ exists.

Part (i) restricts the group structure. The set of latent groups is assumed to be finite with known cardinality. Groups are unique, i.e., there are no groups that share the same coefficient values. We also assume that group-specific coefficients take values in a bounded set. This is a technical assumption to simplify statements of our theorems.

Part (ii) imposes assumptions on the error term. Most importantly, we assume that the innovations are independent. This rules out serial correlation. Our proofs build on recent advances in the theory of asymptotic approximations in high-dimensional settings that are currently only available for independent innovations.¹⁵ In the future, as new results become available, it may be possible to justify our procedure in settings with weakly dependent observations.

Part (iii) requires existence of an estimator $\beta_{g,t}$ that is consistent for $\beta_{g,t}$ at a certain rate. If the group-specific coefficients are estimated from a training set with observed group memberships for N_{aux} units then we can take $\gamma_{N,T,8} = O\left(N_{\text{aux}}^{-1/2}\right)$ under some regularity assumptions. In settings without training data, rate calculations can be based on the results in Bonhomme and Manresa (2015), Su, Shi, and Phillips (2016), and Wang, Phillips, and Su (2016). Their methods provide \sqrt{NT} consistent estimators for time invariant coefficients (i.e., $\beta_{g,t} = \beta_g$).

Part (iv) is a technical assumption that guarantees the existence of all moments that enter the statements of the theorems below.

Define

$$s_{i,T}^{2}(g,h) = \frac{1}{\sigma_{i}^{2}T} \sum_{t=1}^{T} \mathbb{E} \left(d_{it}(g,h) - \mathbb{E}[d_{it}(g,h)] \right)^{2}$$

and let P denote a probability measure that satisfies Assumption 1. For a matrix A write $\lambda_1(A)$ for its smallest eigenvalue. Assumptions 1(i) and (ii) imply

$$s_{i,T}^{2}(g_{i}^{0},h) \geq \underline{\sigma}^{2} \min_{1 \leq i \leq N} \min_{h \in \mathbb{G} \setminus \{g_{i}^{0}\}} \frac{1}{T} \sum_{t=1}^{T} \lambda_{1}(\mathbb{E}_{P}(x_{it}x_{it}')) \|\beta_{g_{i}^{0},t} - \beta_{h,t}\|^{2} =: \underline{s}_{N,T}^{2}(P) > 0.$$

The theorems below define a class \mathbb{P}_N of probability measures. This class satisfies a number of moment conditions that are defined in terms of

$$B_{N,T,p}(P) = \max_{1 \le t \le T} \left(\mathbb{E}_{P} \left[\max_{1 \le i \le N} \left(|u_{it}/\sigma_{i}|^{p} \|x_{it}\|^{p} + \|x_{it}\|^{2p}/\sigma_{i} \right) \right] / \underline{s}_{N,T}^{p}(P) \right)^{1/p},$$

$$D_{N,T,p}(P) = \max_{1 \le i \le N} \left(\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{P} \left[|u_{it}/\sigma_{i}|^{p} \|x_{it}\|^{p} + \|x_{it}\|^{2p}/\sigma_{i} \right] / \underline{s}_{N,T}^{p}(P) \right)^{1/p}.$$

In the following, for all quantities that depend on the probability measure P, this dependence is kept implicit.

¹⁵A high-dimensional CLT for possibly dependent data is proved in Zhang and Cheng (2018) for the MAX statistic. There exist some attempts to extend the SNS theory to dependent data (see, e.g., Chen et al. 2016). We are not aware of a high-dimensional anti-concentration inequality for dependent data.

5.2. The SNS procedure

In this section, we establish validity of the joint confidence set based on the MAX test statistic with SNS critical values.

Theorem 2. Let \mathbb{P}_N denote a sequence of classes of probability measures that satisfy Assumption 1 for constants depending only on \mathbb{P}_N , and let

$$\begin{split} \epsilon_{1,N} &= \sup_{P \in \mathbb{P}_N} \gamma_{N,T,8} (\log N) \big(T^{-5/24} B_{N,T,8}^2 \sqrt{\log N} + D_{N,T,4} \big), \\ \epsilon_{2,N} &= \sup_{P \in \mathbb{P}_N} \gamma_{N,T,8} \sqrt{T \log N} D_{N,T,2}, \\ \epsilon_{3,N} &= \sup_{P \in \mathbb{P}_N} T^{-1/6} D_{N,T,3} \sqrt{\log N}. \end{split}$$

and $\epsilon_N = \epsilon_{1,N} + \epsilon_{2,N} + \epsilon_{3,N} + \xi_{N,T}$. Suppose that $\epsilon_N \to 0$ and

$$\max_{P \in \mathbb{P}_N} T^{-5/24} B_{N,T,4} \sqrt{\log N} \le 1. \tag{7}$$

Then, for each $0 < \alpha < 1$, there is a constant C depending only on α , G, K_{β} and the sequence ϵ_N such that

$$\inf_{P \in \mathbb{P}_N} P\left(\left\{g_i^0\right\}_{1 \leq i \leq N} \in \widehat{C}_{\alpha}^{\text{SNS}}\right) \geq 1 - \alpha - C\epsilon_N.$$

This theorem states that the SNS confidence set contains the true group membership structure at least with probability $1 - \alpha - C\epsilon_N$. Note that the rate of convergence ϵ_N does not depend on P. Hence, convergence is uniform over \mathbb{P}_N .

Our result establishes that the SNS confidence set is valid even if T is very small compared to N. For example, if $D_{N,T,3}$ is bounded along the asymptotic sequence then $\epsilon_{3,N}$ vanishes if $T^{-1/3}(\log N) \to 0$, allowing T to diverge to infinity at a much slower rate than N. We therefore expect that the confidence set performs well even if the panel is rather short.

The rates $\epsilon_{1,N}$ and $\epsilon_{2,N}$ bound the effect of parameter estimation on the coverage probability of the confidence set. Approximating the behavior of the (G-1)N estimated moment inequalities jointly by scaled t-distributions contributes the rate $\epsilon_{3,N}$.¹⁶ Condition (7) is imposed to simplify the statement of the theorem and can be relaxed.

Our distributional approximation relies on a Cramér-type moderate deviation inequality for self-normalized sums (Jing, Shao, and Wang 2003). Chernozhukov, Chetverikov, and Kato (2018, Theorem 4.1) were the first to use this kind of argument in the context of testing many moment inequalities. Our result differs from theirs in several respects.

First, our critical value based on the t-distribution is always computable. Their critical value is a transformation of normal quantiles that is undefined for small T.¹⁷

¹⁶If the $\hat{d}_{it}(q_i^0, h)$ are normally distributed, then the t-distribution describes the exact finite sample distribution.

¹⁷In our setting, the critical value in Chernozhukov, Chetverikov, and Kato (2018) is given by $\Phi^{-1}(1-\alpha/((G-1)^{-1}))$

Second, Chernozhukov, Chetverikov, and Kato (2018) do not consider parameter uncertainty, whereas our results quantify the effect of estimating the group-specific parameters under low-level assumptions that are easy to interpret.¹⁸ In our proof, we reduce the problem with estimated parameters to a problem with known parameters with modified critical value corresponding to a nominal level of $1 - \alpha_N$. Based on a careful analysis of the tail of the t-distribution, we can prove $\alpha_N \to \alpha$.

5.3. The MAX procedure

In this section, we establish that the MAX procedure produces an asymptotically valid confidence set.

We allow for strong correlation of the within-unit moment inequalities. Let $\Omega_i(g_i^0)$ denote the $(G-1)\times (G-1)$ correlation matrix with entries

$$(\Omega_i(g_i^0))_{h,h'} = \frac{\sum_{t=1}^T \mathbb{E} \left[d_{it}(g_i^0, h) d_{it}(g_i^0, h') \right]}{\sqrt{\sum_{t=1}^T \mathbb{E} \left[d_{it}^2(g_i^0, h) \right] \sum_{t=1}^T \mathbb{E} \left[d_{it}^2(g_i^0, h') \right]}}.$$

For our theoretical result below, we assume that $\Omega_i(g_i^0)$ is nonsingular, ruling out that pairs of moment inequalities are perfectly correlated. To model strong correlation of the inequalities, we allow the matrix to approach singularity at a controlled rate.

Theorem 3. Suppose that there is a sequence $\omega_N > 0$ such that $\lambda_1(\Omega_i(g_i^0)) \geq \omega_N^{-1}$ for $i = 1, \ldots, N$. Let \mathbb{P}_N denote a sequence of classes of probability measures that satisfy Assumption 1 for constants depending only on \mathbb{P}_N , and let

$$\begin{split} \epsilon_{1,N} &= \sup_{P \in \mathbb{P}_N} \gamma_{N,T,8} (\log N) \big(T^{-3/14} B_{N,T,8}^2 \sqrt{\log N} + D_{N,T,4} \big), \\ \epsilon_{2,N} &= \sup_{P \in \mathbb{P}_N} \gamma_{N,T,8} \sqrt{T \log N} D_{N,T,2}, \\ \epsilon_{3,N} &= \sup_{P \in \mathbb{P}_N} T^{-1/7} B_{N,T,4} \log N \end{split}$$

and

$$\epsilon_N = (\epsilon_{1,N} + \epsilon_{3,N})(\omega_N^2 \vee 1) + \epsilon_{2,N} + \xi_{N,T}.$$

Suppose that $\epsilon_N \to 0$ and $T^{-1/7}(\log N) \to 0$. Then, for each $0 < \alpha < 1$ there is a constant C depending only on α , G and K_β and the sequence ϵ_N such that

$$\inf_{P \in \mathbb{P}_N} P\left(\left\{g_i^0\right\}_{1 \le i \le N} \in \widehat{C}_{\alpha}^{\text{MAX}}\right) \ge 1 - \alpha - C\epsilon_N.$$

 $¹⁾N))/\sqrt{1-\Phi^{-1}(1-\alpha/((G-1)N))^2/T}$. If T is small, then the term inside of the square root can be negative.
¹⁸Chernozhukov, Chetverikov, and Kato (2018) consider parameter uncertainty for their bootstrap procedures in their online supplement B.2, but not for their SNS procedures. For their bootstrap procedures they give a high-level assumption under which parameter uncertainty can be ignored.

This result establishes that, for all DGPs in \mathbb{P}_N , the empirical coverage probability of the MAX confidence set is at least $1 - \alpha - C\epsilon_N$.

The theorem requires slightly stronger assumptions than Theorem 2. For example, the conditions under which the relative magnitudes of N and T allow $\epsilon_{3,N}$ to vanish are more restrictive than in Theorem 2 and require at least $T^{-1/7}(\log N) \to 0$. Stronger assumptions are needed because, in contrast to the proof of Theorem 2, we eliminate the randomness of the denominator in $\hat{D}_i(g,h)$ before deriving a distributional approximation.

Our proof approach differs in substantial ways from the theoretical analysis of a bootstrap test for many moment inequalities in Chernozhukov, Chetverikov, and Kato (2018, Theorem 4.3). While they consider a single test statistic for many moment inequalities, we conduct many simultaneous unit-wise tests with different unit-wise MAX critical values. This requires new arguments.

To bound the effect of "small" estimation errors, we derive a new simultaneous anti-concentration inequality. With this inequality we can evaluate the effect of perturbations caused by parameter estimation and thus obtain a high-dimensional analogue of Slutsky's lemma. The anti-concentration inequality is *simultaneous* because it considers the effect of perturbations close to the unit-wise critical value for any of the unit-wise tests. It is therefore different from the inequality in Chernozhukov, Chetverikov, and Kato (2015) that examines concentration around a single critical value.

5.4. The QLR procedure

We now establish that the QLR confidence set has asymptotically the correct coverage.

Theorem 4. Suppose that there is λ_1 such that $\lambda_1(\Omega_i) \geq \lambda_1 > 0$ for i = 1, ..., N. Let \mathbb{P}_N denote a sequence of classes of probability measures that satisfy Assumption 1 for constants depending only on \mathbb{P}_N , and let

$$\begin{split} \epsilon_{1,N} &= \sup_{P \in \mathbb{P}_N} \gamma_{N,T,8} (\log N) \big(T^{-3/14} B_{N,T,8}^2 \sqrt{\log N} + D_{N,T,4} \big), \\ \epsilon_{2,N} &= \sup_{P \in \mathbb{P}_N} \gamma_{N,T,8} \sqrt{T \log N} D_{N,T,2}, \\ \epsilon_{3,N} &= \sup_{P \in \mathbb{P}_N} T^{-1/7} B_{N,T,4} \log N, \end{split}$$

and $\epsilon_N = \epsilon_{1,N} + \epsilon_{2,N} + \epsilon_{3,N} + \xi_{N,T} + N^{-1}$. Suppose that $\epsilon_N \to 0$ and $T^{-1/7}(\log N) \to 0$. and that all $P \in \mathbb{P}_N$ impose cross-sectional independence. Then, for each $0 < \alpha < 1$ there is a constant C depending only on α , λ_1 , G, K_β and the sequence ϵ_N such that

$$\inf_{P\in\mathbb{P}_N} P\left(\left\{g_i^0\right\}_{1\leq i\leq N}\in \widehat{C}_{\alpha}^{\mathrm{QLR}}\right) \geq 1-\alpha-C\epsilon_N.$$

The theorem establishes the uniform validity of the QLR approach under similar assumptions as those imposed in Theorem 3. Unlike in Theorem 3, we now require a uniform lower bound on the smallest eigenvalue of Ω_i . This bound is needed to verify the assumptions of a high-dimensional CLT (Chernozhukov, Chetverikov, and Kato 2017, Proposition 3.2).

To the best of our knowledge, Theorem 4 represents the first theoretical result for the QLR statistic in a high-dimensional setting. The result rests on two pillars: a distributional approximation via a high-dimensional CLT and a simultaneous anti-concentration inequality.

To apply a high-dimensional CLT we show that, for each unit i, the vectors $\hat{D}_i(g_i^0)$ for which the unit-wise test rejects form a convex subset of \mathcal{R}^{G-1} . This observation allows us to employ the CLT for sparse-convex sets in Chernozhukov, Chetverikov, and Kato (2017, Proposition 3.2). We conclude that the unit-wise test statistics converge *jointly* in distribution, where for each unit the marginal limit distribution is described by the mixture of χ^2 variables described in Kudo (1963), Nüesch (1966), Wolak (1991), and Rosen (2008).

For this collection of marginal distributions, we derive a simultaneous anti-concentration inequality that allows us to bound the probability that a small perturbation in any of the unit-wise test statistics causes the joint test to reject. This anti-concentration result is new and mathematically interesting since it illustrates that we can derive simultaneous anti-concentration results also for non-Gaussian random variables. Under cross-sectional independence, we can derive the anti-concentration result by exploiting the mixture representation of the marginals. This argument cannot be extended to the case of cross-sectional correlations. The cross-sectional independence assumption is therefore essential for our anti-concentration result. For the purposes of applying a high-dimensional CLT, cross-sectional independence is non-essential and can be relaxed at the expense of more stringent regularity conditions.

5.5. Unit selection

In this section, we provide an asymptotic justification of our algorithm for unit selection. We show that applying unit selection to any of our three procedures generates valid confidence sets. The following theorem gives conditions under which the coverage probability of the confidence set after unit selection converges to the nominal level. The convergence is uniform over DGPs.

Theorem 5. Let $\widehat{C}_{\mathrm{sel},\alpha,\beta}^{\mathrm{type}}$ denote a joint confidence set, where type = SNS, MAX or QLR. Suppose that $\{\widehat{g}_i\}_{1\leq i\leq N}$ satisfies $\widehat{D}_i^U(\widehat{g}_i,h)\leq 0$ for any $h\in\mathbb{G}$ and $i=1,\ldots,N$. Let \mathbb{P}_N denote a sequence of classes of probability measures that satisfy the conditions in Theorem 2 if type = SNS, Theorem 3 if type = MAX and Theorem 4 if type = QLR. In addition, suppose that

$$\max_{P \in \mathbb{P}_N} T^{-5/36} D_{N,T,3} \sqrt{\log(N/\beta)} \le 1,\tag{8}$$

$$\max_{P \in \mathbb{P}_N} T^{-5/24} B_{N,T,4} \log(N/\beta) \le 1, \tag{9}$$

$$\max_{P \in \mathbb{P}_N} T^{2/3} \gamma_{N,T,8} \left(T^{-5/24} B_{N,T,4} \sqrt{\log N} + D_{N,T,2} \right) \sqrt{\log(N/\beta)} \le 1, \tag{10}$$

$$\max_{P \in \mathbb{P}_N} T^{1/6} \gamma_{N,T,8}^2 \left(T^{-5/12} (\log N) B_{N,T,8}^4 + D_{N,T,4}^2 \right)$$

$$\times \left(D_{N,T,1} + \sqrt{\log N} + T^{-1/4} B_{N,T,4} \log N \right) \sqrt{\log(N/\beta)} \le 1.$$
 (11)

Then, for each $0 < \alpha < 1$, there is a constant C depending only on α , G, K_{β} and the sequence ϵ_N ,

defined in the theorem corresponding to the value of type, such that

$$\inf_{P \in \mathbb{P}_N} P\left(\left\{g_i^0\right\}_{1 \le i \le N} \in \widehat{C}^{\text{type}}_{\text{sel},\alpha,\beta}\right) \ge 1 - \alpha - C\epsilon_N - CT^{-1/6}.$$

The conditions assumed here are slightly stronger versions of the conditions required in the previous theorems. This is partly because we use an auxiliary test statistic based on moment inequalities that have not been mean-adjusted.

Even though our theoretical justification of Theorem 5 is similar to proof strategies found in the literature on moment selection (cf. Chernozhukov, Chetverikov, and Kato 2018), the underlying principles are different. The idea of moment selection is that obviously satisfied inequalities are unlikely to be binding and do not affect the asymptotic null distribution. The idea of unit selection is that, if the group membership for a unit is obvious, then the true membership should be the estimated one and therefore be included in the confidence set. Note that the idea of moment selection does not directly apply in our setting since our moment recentering ensures that there are no moment inequalities that are slack under the null hypothesis.

The assumption $\hat{D}_i^U(\hat{g}_i, h) \leq 0$ means that the estimator of group memberships is based on an empirical version of inequality (3). This assumption will be automatically satisfied for estimators in the *kmeans* family such as the estimator in Bonhomme and Manresa (2015). If it holds then estimation strategy and our inference follow the same principles and we can show that the probability that we estimate an incorrect group membership for a unit that is "easy to classify" is less than β .

6. Monte Carlo simulations

In this section, we study the finite-sample behavior of our procedures in Monte Carlo simulations. We consider both homoscedastic and heteroscedastic designs. For all our designs, we simulate panels of N=50 units that are observed over T=10,20,30,40 time periods. We assume that the group-specific parameters are known and compute joint confidence sets with nominal coverage probability $1-\alpha=0.9$. All simulation results are based on 1000 replications.

6.1. Homoscedastic design with three groups

For our first design, we consider a model with group fixed effects and G = 3 groups. For unit i = 1, ..., N, the outcome in period t is given by

$$y_{it} = \alpha_{g_i^0, t} + u_{it}. \tag{12}$$

The group fixed effects $\{\alpha_{g,t}\}_{1 \leq t \leq T}$ for the three groups are defined as follows. Let $\varphi_T(t) = -1/2 + 2|t - T/2|/T$. For $t = 1, \ldots, T$, $\alpha_{1,t} = 0$, $\alpha_{2,t} = \varphi_T(t) + 1$, $\alpha_{3,t} = \varphi_{T/2}(t \text{ mod } \lceil T/2 \rceil) - 1$. The time profile for the group fixed effects is plotted in Figure B.1 in the Appendix. Note that

 $^{19 \}lceil T/2 \rceil$ is the smallest integer larger than T/2.

			empirical coverage				cardinality of CS			
g^0	σ	T	\hat{g}_i	SNS	MAX	QLR	\hat{g}_i	SNS	MAX	QLR
1	0.25	10	0.12	0.96	0.96	0.96	1.00	2.40	2.21	2.09
1	0.25	20	0.15	0.92	0.93	0.95	1.00	1.74	1.59	1.53
1	0.25	30	0.14	0.92	0.91	0.95	1.00	1.54	1.42	1.39
1	0.25	40	0.15	0.92	0.92	0.94	1.00	1.45	1.35	1.33
1	0.50	10	0.00	0.94	0.93	0.93	1.00	2.91	2.87	2.84
1	0.50	20	0.00	0.92	0.93	0.92	1.00	2.82	2.75	2.73
1	0.50	30	0.00	0.90	0.92	0.93	1.00	2.77	2.70	2.68
1	0.50	40	0.00	0.92	0.92	0.94	1.00	2.75	2.67	2.65
2	0.25	10	0.39	0.97	0.95	0.93	1.00	1.84	1.81	1.85
2	0.25	20	0.38	0.96	0.93	0.90	1.00	1.42	1.41	1.51
2	0.25	30	0.39	0.94	0.92	0.92	1.00	1.30	1.30	1.39
2	0.25	40	0.39	0.96	0.91	0.92	1.00	1.25	1.25	1.33
2	0.50	10	0.00	0.95	0.92	0.89	1.00	2.63	2.53	2.47
2	0.50	20	0.00	0.95	0.92	0.91	1.00	2.28	2.20	2.20
2	0.50	30	0.00	0.95	0.91	0.91	1.00	2.17	2.11	2.13
2	0.50	40	0.00	0.95	0.92	0.90	1.00	2.12	2.07	2.10
3	0.25	10	0.38	0.97	0.95	0.94	1.00	1.84	1.81	1.85
3	0.25	20	0.41	0.96	0.91	0.92	1.00	1.42	1.42	1.51
3	0.25	30	0.38	0.94	0.91	0.91	1.00	1.30	1.30	1.38
3	0.25	40	0.36	0.95	0.92	0.90	1.00	1.25	1.25	1.32
3	0.50	10	0.00	0.97	0.93	0.91	1.00	2.62	2.53	2.47
3	0.50	20	0.00	0.95	0.92	0.90	1.00	2.28	2.20	2.20
3	0.50	30	0.00	0.94	0.90	0.89	1.00	2.17	2.11	2.12
3	0.50	40	0.00	0.94	0.91	0.90	1.00	2.12	2.07	2.09

Table 1: Homoscedastic design with G=3 groups. Results based on 1000 simulated joint confidence sets with $1-\alpha=0.9$. "Empirical coverage" gives the simulated coverage probability of the joint confidence set. "Cardinality of CS" gives the simulated expected average cardinality of a marginal (unit-wise) confidence set.

the groups can be ordered. The group fixed effect of group 2 is large in all time periods, and that of group 2 is small in all time periods. The group fixed effect of group 1 is straddled between the effects of the other two groups. All units are assigned to the same group $g^0 = 1, 2, 3$. Our specification induces strong correlation of the moment inequalities.²⁰

The error terms u_{it} are i.i.d. draws from $\mathcal{N}(0, \sigma^2 T)$ for $\sigma = 0.25, 0.5$. Note that the variance of the error term is scaled in a way that keeps the difficulty of the classification problem constant as we increase the number of observed time periods. This makes our simulation results for different values of T informative about the accuracy of the asymptotic approximation in finite-samples.²¹

We simulate our three joint confidence sets (SNS, MAX and QLR) as well as a naïve joint confidence set that reports only the vector of estimated group memberships $\{\hat{g}_i\}_{1\leq i\leq N}$. For this homoscedastic design, we turn off unit selection ($\beta=0$). Following Andrews and Barwick (2012), we set the parameter for regularizing $\hat{\Omega}_i$ to $\epsilon=0.012.^{22}$ The simulation results are summarized in

²⁰For example, for T=40 and $g^0=1$, our simulations indicate that $(\mathbb{E}\,\widehat{\Omega}_i(1))_{1,2}=-0.93$ and $(\mathbb{E}\,\widehat{\Omega}_i(2))_{1,2}=0.98$. For T=40 and $g^0=2$, $(\mathbb{E}\,\widehat{\Omega}_i(1))_{1,2}=-0.90$ and $(\mathbb{E}\,\widehat{\Omega}_i(2))_{1,2}=0.98$.

²¹Without rescaling the error variance, increasing T renders the classification problem trivial eventually.

²²The results are robust to different choices of ϵ .

Table 1, where we report simulated coverage probabilities and average cardinality of the marginal unit-wise confidence sets. For group assignments g^0 to the two "outer" groups (groups 2 and 3), the simulation results are almost identical. This is expected, since these two groups are symmetric by construction. Therefore, we only discuss results for $g^0 = 1, 2$.

All three procedures construct valid confidence sets in all designs, with the empirical coverage probability close to or exceeding the nominal coverage probability. In contrast, the coverage probability of the naïve confidence set $\{\{\hat{g}_i\}_{1\leq i\leq N}\}$ is substantially below the nominal level. In some designs it is close to zero. This illustrates that theoretical results of consistent group membership estimation cannot be leveraged for inference in finite samples. A more sophisticated approach to account for the statistical uncertainty in clustering is needed and is provided by our procedures.

We now discuss the power properties of our three procedures. First, the SNS procedure always yields a more conservative confidence set than the MAX procedure. This is because the SNS critical value is an upper bound to the MAX critical value. Therefore, the SNS confidence set, while easier to implement, is not as powerful as the MAX confidence set.

Second, how the MAX and the QLR procedures rank depends on the setting. For $g^0 = 1$, the QLR procedure provides narrower confidence sets than the MAX procedure, despite also being more conservative. For $g^0 = 2$, the result is reversed. The MAX procedure is more powerful than the QLR procedure, despite also being more conservative. This comparison demonstrates that neither of our two test statistics dominates the other.

To confirm our asymptotic results, we also simulate the QLR and MAX confidence sets with different critical values based directly on the limit distribution. These critical values can be obtained by sending T in the definitions of $c_{\alpha,N}^{\text{MAX}}$ and $c_{\alpha,N}^{\text{QLR}}$ to infinity. The simulation results are given in Table B.1 in the Appendix. As expected, with critical values corresponding to infinite T the confidence sets are undersized in short panels. In line with our asymptotic prediction, the size distortion vanishes as the number of time periods T increases.

Our design induces highly correlated within-unit moments. In the Supplemental Appendix, we report simulation evidence for an alternative design in which moment inequalities are not as strongly correlated. Our procedures perform well in this alternative design.

6.2. Heteroscedastic design with two groups

We now study the finite-sample properties of our algorithm for unit selection. To make unit selection meaningful we introduce heteroscedasticity.

Again, outcomes are generated from the linear model with group fixed effects (12). There are G=2 groups with time-constant group fixed effects. For all $t=1,\ldots,T$, the group fixed effects are given by $\alpha_{1,t}=0.5$ and $\alpha_{2,t}=-0.5$. We only simulate units with $g_i^0=1$. Due to the symmetry of the design this is without loss of generality.

There are two "types" of units that face different degrees of statistical noise. Set $\sigma = 0.25, 0.5$. For the "high noise" type the error terms $\{u_{it}\}_{1 \leq i \leq N}$ are i.i.d. draws from $\mathcal{N}(0, \sigma^2 T)$. For the "low noise" type, $\{u_{it}\}_{1 \leq i \leq N}$ are i.i.d. draws from $\mathcal{N}(0, (\sigma/5)^2 T)$. The type of a unit is randomized

			no unit selection		with unit selection		
σ	type ratio	T	coverage	power	coverage	\hat{N}/N	power
0.25	1:1	10	0.95	0.59	0.95	0.52	0.67
0.25	1:1	20	0.95	0.75	0.94	0.51	0.81
0.25	1:1	30	0.95	0.80	0.92	0.51	0.85
0.25	1:1	40	0.95	0.82	0.94	0.51	0.87
0.25	1:3	10	0.98	0.59	0.95	0.28	0.78
0.25	1:3	20	0.96	0.76	0.93	0.26	0.89
0.25	1:3	30	0.97	0.80	0.92	0.26	0.90
0.25	1:3	40	0.98	0.82	0.93	0.26	0.92
0.50	1:1	10	0.96	0.10	0.96	0.90	0.09
0.50	1:1	20	0.94	0.14	0.94	0.94	0.13
0.50	1:1	30	0.95	0.15	0.97	0.96	0.14
0.50	1:1	40	0.94	0.17	0.96	0.97	0.15
0.50	1:3	10	0.97	0.10	0.97	0.85	0.09
0.50	1:3	20	0.97	0.14	0.97	0.92	0.13
0.50	1:3	30	0.98	0.15	0.98	0.94	0.14
0.50	1:3	40	0.98	0.16	0.98	0.95	0.15

Table 2: Heteroscedastic design with two groups. Results based on 1000 simulated joint confidence sets (SNS) with $1 - \alpha = 0.9$. "Coverage" gives the simulated coverage probability of the joint confidence set. "Power" gives the simulated probability of reporting a singleton marginal (unit-wise) confidence set for the "high noise" type. \hat{N}/N gives the simulated expected proportion of selected units.

independently of everything else. Unit i is assigned to the "high noise" type with either probability 0.5 (1:1 type ratio) or with probability 0.25 (1:3 type ratio).

We only simulate SNS confidence sets. With G=2 groups, the QLR and MAX confidence sets yield numerically identical confidence sets. We set either $\beta=0$ (no unit selection) or $\beta=0.01$ (unit selection).

The simulation results are reported in Table 2. In the designs with $\sigma = 0.25$, the unit selection algorithm identifies units of the "low noise" type as easy to classify and ignores them when computing the Bonferroni adjustment of the critical values. Relative to the case of no unit selection, this lowers the critical values for units of the "high noise" type. Consequently, the unit-wise confidence sets for "high noise" units become more powerful and a higher proportion of singletons is reported. This effect is more pronounced in the setting with a higher proportion of "low noise" units (1:3 type ratio).

In the designs with $\sigma=0.50$, the unit selection algorithm identifies only a small proportion of the "low noise" types as easy to classify. Relative to the case of no unit selection, the unit-wise confidence sets for the "high noise" units become less powerful and a smaller proportion of singletons is reported. This provides a numerical illustration of the theoretical argument in Section 4.4: unit selection improves the power of the joint confidence set if many units are deleted, but may reduce the power if an insufficient number of units are deleted.

7. Applications

We apply the proposed confidence sets to two empirical applications. The first studies the effect of a minimum wage, and the second studies heterogeneous trajectories of democratization.

7.1. Minimum wage and unemployment

The first application studies heterogeneity in the effect of a minimum wage on unemployment. We examine panel data of states in the US and cluster them into two groups. The effect of a minimum wage is positive in one group and negative in the other. Our confidence sets quantify the uncertainty from using a data-driven method to sort states into one of the two groups.

To estimate the group-specific effects, we replicate results from Wang, Phillips, and Su (2016). Using US panel data, they follow an approach pioneered by Neumark and Wascher (1992) and identify the effect of a minimum wage from cross-state variation. Recently, Dube, Lester, and Reich (2010) argued that the way that a local economy reacts to a minimum wage may be affected by unobserved spatial heterogeneity. Wang, Phillips, and Su (2016) address this concern by proposing a linear panel model with a group structure. They estimate the following model for state i in time period t

$$\mathtt{ue}_{it} = \beta_{g_i^0,1} \mathtt{ue}_{i(t-1)} + \beta_{g_i^0,2} \mathtt{gr}_{i(t-1)} + \beta_{g_i^0,3} \mathtt{mw}_{i(t-1)} + \mu_i + u_{it},$$

where ue_{it} is the unemployment rate, gr_{it} is the growth rate of GDP, mw_{it} is the real state minimum wage, μ_i is a state fixed effect and u_{it} is an error term. The coefficients that describe the linear relationship may depend on the latent group membership of state i. We estimate the grouped panel model and compute unit-wise and joint confidence sets for group membership. The presence of the individual fixed effect μ_i renders this regression model different from our canonical model (1). In Section D of the Supplemental Appendix, we explain how to apply our methods to a linear panel data model after individual fixed effects have been differenced out.

We obtain all data from the online portal of the Federal Reserve Bank of St Louis.²³ We use yearly data for all 50 states (N = 50) from the period 1988 to 2014 (T = 26). For states in which state law does not specify a minimum wage, we use instead the federally mandated minimum wage. The data is standardized so that, for each state, the time series has standard deviation one.

Our estimation strategy is different from that employed in Wang, Phillips, and Su (2016), but our estimates are very similar.²⁴ We use the CLasso estimator from Su, Shi, and Phillips (2016) to estimate the group structure. Then, we estimate the group-specific parameters by post-Lasso least squares and perform a bias correction by half-panel Jackknifing (Dhaene and Jochmans 2015).

We detect G=2 groups with 26 and 24 members, respectively. Like Wang, Phillips, and Su

²³The GDP data is from the US Bureau of Economic Analysis, the minimum wage and unemployment data is from the US Department of Labor, and the CPI data is from the OECD Main Economic Indicators table.

²⁴Their procedure includes a post-processing step using a hierarchical clustering algorithm. The results of the procedure are sensitive to the choice of the regularization parameter that controls the intensity of the post-processing step. For our choice of estimator this post-processing step is not needed.

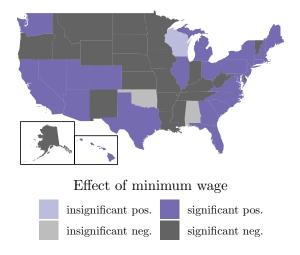


Figure 1: Estimated group memberships. The indicated significance is computed at level $\alpha = 0.1$.

(2016), we find that one group has a positive coefficient on the lagged minimum wage ("positive effect group"), whereas the other has a negative coefficient ("negative effect group"). The estimated coefficients are reported in Table B.2. The map in Figure 1 depicts the estimated group memberships. The indicated significance is based on unit-wise confidence sets at level $1 - \alpha = 0.9$. We call a state's estimated group membership significant at level α if the state's unit-wise confidence at level $1 - \alpha$ is a singleton containing only the estimated group membership.²⁵ We can compute the p-value corresponding to this notion of significance by finding the smallest possible value of α such that the observed unit-wise confidence set at level $1 - \alpha$ contains only the estimated group membership. These p-values are reported in Table 3.

It is of interest to compile a list of states for which our analysis gives conclusive evidence about their response to the minimum wage. Such a list is useful if, based on our empirical analysis, we want to make targeted policy recommendations to state legislators. It is also useful to applied economists who want to take a more detailed look at states that exhibit a certain behavior to investigate underlying mechanisms. For both purposes, it is important to control the probability of misclassification. When making policy recommendations we want to be confident that we give state legislators correct information about how their state reacts to the minimum wage. When analyzing the determinants of certain behaviors, applied economists need to be confident that the states they are studying exhibit these behaviors. Using estimated group memberships without controlling for estimation error can be misleading (see our simulation results in Section 6).

An approach based on our joint confidence set guarantees proper error control. From the observed confidence set at level $1-\alpha$, we identify all units for which the confidence set reports an unambiguous group assignment. These are the units i such that all vectors in the joint confidence set assign the same group membership to i. By construction of our confidence set, if i's group assignment is unambiguous then it is equal to its estimated group membership \hat{g}_i .

²⁵In this setting with G=2 groups, all our procedures compute identical confidence sets.

positive-effect group				
Arizona $(0.01, 0.65)$ Connecticut $(0.00, 0.11)$ Hawaii $(0.03, \ge 1)$ Maryland $(0.06, \ge 1)$ Nevada $(0.09, \ge 1)$ New York $(0.08, \ge 1)$ Pennsylvania $(0.06, \ge 1)$ Texas $(0.05, \ge 1)$ Washington $(0.05, \ge 1)$	California $(0.00, 0.16)$ Florida $(0.04, \geq 1)$ Illinois $(0.08, \geq 1)$ Massachusetts $(0.00, 0.06)$ New Hampshire $(0.03, \geq 1)$ North Carolina $(0.04, \geq 1)$ Rhode Island $(0.02, \geq 1)$ Utah $(0.08, \geq 1)$ Wisconsin $(0.11, \geq 1)$ negative-effect group	Colorado $(0.01, 0.34)$ Georgia $(0.05, \geq 1)$ Maine $(0.03, \geq 1)$ Michigan $(0.08, \geq 1)$ New Jersey $(0.06, \geq 1)$ Ohio $(0.03, \geq 1)$ South Carolina $(0.07, \geq 1)$ Virginia $(0.08, \geq 1)$		
Alabama $(0.19, \geq 1)$ Delaware $(0.02, \geq 1)$ Iowa $(0.06, \geq 1)$ Louisiana $(0.03, \geq 1)$ Missouri $(0.01, 0.55)$ New Mexico $(0.01, 0.40)$ Oregon $(0.03, \geq 1)$ Vermont $(0.00, 0.17)$	Alaska $(0.04, \ge 1)$ Idaho $(0.02, \ge 1)$ Kansas $(0.00, 0.07)$ Minnesota $(0.00, 0.14)$ Montana $(0.01, 0.53)$ North Dakota $(0.00, 0.24)$ South Dakota $(0.01, 0.30)$ West Virginia $(0.00, 0.00)$	Arkansas $(0.03, \geq 1)$ Indiana $(0.06, \geq 1)$ Kentucky $(0.04, \geq 1)$ Mississippi $(0.01, 0.30)$ Nebraska $(0.00, 0.19)$ Oklahoma $(0.14, \geq 1)$ Tennessee $(0.02, \geq 1)$ Wyoming $(0.00, 0.08)$		

Table 3: Estimated group memberships and p-values for significance of the estimate. The first number in parentheses gives the p-value of unit-wise significance, the second number gives the p-value of joint significance.

We call the membership estimates for these states jointly significant at level α . For the states with jointly significant membership estimates, we can be confident that the behavior of the group to which the clustering algorithm assigns them reflects their true behavior. The probability of misclassifying one or more of these states is at most α . Table 3 reports p-values for joint significance.²⁶ For each state i, this p-value gives the largest values of α at which state i is included in the set of states with jointly significant membership estimates. A formal definition is given in Appendix C. We find the states with jointly significant estimates at level α by selecting all states with a p-value of joint significance of less than α . For $\alpha = 0.1$ we select Kansas, Massachusetts, West Virginia and Wyoming. For $\alpha = 0.2$ we select California, Connecticut, Kansas, Massachusetts, Minnesota, Nebraska, Vermont, West Virginia and Wyoming.

7.2. Paths to democracy

Our second application addresses the classification of countries based on heterogeneous trajectories of democratization. We build on the group fixed effects model proposed in Bonhomme and Manresa (2015).

Acemoglu et al. (2008) use country panel data to estimate the relationship between income (measured by GDP per capita) and democracy (measured by the Freedom House democracy index). Bonhomme and Manresa (2015) expand on this seminal study and estimate an augmented

²⁶We compute a joint confidence set without unit selection.

specification with group fixed effects. For country i and time period t they estimate the model

$$\mathtt{democracy}_{it} = \theta_1 \mathtt{democracy}_{i(t-1)} + \theta_2 \log(\mathtt{gdp_pc}_{i(t-1)}) + \alpha_{q_i^0,t} + u_{it},$$

where $democracy_{it}$ is the level of democracy measured by the Freedom House indicator, gdp_pc is GDP per capita and u_{it} is an error term. The inclusion of the group fixed effect $\alpha_{g,t}$ lends credibility to the exogeneity assumption of the linear panel model. In particular, the group fixed effect can pick up exogenous events, such as the process of decolonization, that unfold over time and impact both democratization and income growth.

We use the replication data set provided by Bonhomme and Manresa (2015). It is based on the balanced subsample from Acemoglu et al. (2008) and contains observations for N = 90 countries. Each country is observed every five years over the period 1970 - 2000 (T = 7). Details on the estimation procedure and estimates can be found in Bonhomme and Manresa (2015). Here, we focus on the pattern of grouped heterogeneity.

Bonhomme and Manresa (2015) detect G=4 groups. Estimated time profiles for the group fixed effects are plotted in Figure B.2, which is a reproduction of the left-bottom panel of Figure 2 in Bonhomme and Manresa (2015). There are two groups for which the group fixed effects are stable over time, one with low values and the other with high values. These are called the "low democracy" and "high democracy" groups, respectively. Then, there are two transitioning groups for which the group fixed effect starts out at a low level and then transitions to a higher level. There is an early transitioning group for which the transition starts in 1975, and a late transitioning group for which the transition starts in 1990. Estimated group memberships are shown in the top panel of Figure 2 in Bonhomme and Manresa (2015).

Our procedures compute large, yet informative confidence sets. The fact that the confidence sets are large indicates that group memberships are estimated imprecisely. This confirms simulation evidence in Bonhomme and Manresa (2015) based on a design calibrated to this application.²⁷ Our joint confidence set provides a more formal way of assessing the variability of the group membership estimates that does not rely on parametric assumptions about the error term. For economists who want to interpret the estimated group memberships, the uncertainty detected by the confidence set begs caution.

However, our confidence sets also show that some aspects of the estimated group structure are significant. For example, we can reject the hypotheses that all countries are "low democracy" countries or that all countries are "high democracy" countries.

We compute the $1-\alpha=0.9$ joint confidence set based on the SNS, MAX and QLR procedures without unit selection. The cardinality of the marginal unit-wise confidence sets is reported in Table 4. All procedures generate an informative confidence set that rules out some group membership for some countries. For the MAX test statistic, we observe a power gain from taking into account the correlation of the within-unit moments. In particular, the MAX confidence set is uninformative

²⁷They report simulated misclassification probabilities in Table S.III of their supplemental appendix.

critical value	$ \widehat{C}_{\alpha,i} = 1$	$ \widehat{C}_{\alpha,i} = 2$	$ \widehat{C}_{\alpha,i} = 3$	$ \widehat{C}_{\alpha,i} = 4$
SNS	0	0	35	55
MAX	0	0	37	53
$_{ m QLR}$	0	0	42	48

Table 4: Cardinality of the marginal unit-wise confidence sets for a joint confidence set at level $1 - \alpha = 0.9$.

low democracy						
Algeria(*)	Burundi	Cameroon(*)				
Chad	China(*)	Congo, Rep.				
Cote d'Ivoire(*)	Dem. Rep. Congo	Egypt, Arab Rep.				
Gabon	Guinea(*)	Indonesia				
Iran(*)	Jordan	Kenya(*)				
Mauritania(*)	Morocco(*)	Nigeria				
Paraguay(*)	Rwanda(*)	Sierra Leone				
Singapore(*)	Syrian Arab Republic(*)	Togo				
Tunisia(*)	Uganda					
	high democracy					
Australia(*)	Austria(*)	Belgium(*)				
Canada(*)	Colombia	Costa Rica(*)				
Cyprus	Denmark(*)	Dominican Republic				
El Salvador	Finland(*)	France(*)				
Guatemala	Iceland(*)	India(*)				
Ireland(*)	Israel(*)	Italy(*)				
Jamaica	Japan(*)	Luxembourg(*)				
Malaysia	Netherlands(*)	New Zealand(*)				
Norway(*)	RB Venezuela	Sri Lanka				
Sweden(*)	Switzerland(*)	Trinidad and Tobago(*)				
Turkey	United Kingdom(*)	United States(*)				

Table 5: Estimated member countries for the "low democracy" and "high democracy" groups. The indicated significance of the estimated group assignments is based on a joint confidence set at level $1 - \alpha = 0.9$ (MAX procedure). Estimated "low democracy" countries with a (*) are not "high democracy" countries, and vice versa.

about the group membership of 53 countries, compared to 55 countries for the SNS confidence set.

We now discuss the MAX joint confidence set.²⁸ We focus on the clusters of units that are estimated to be "low democracy" or "high democracy" countries. These constitute 59 out of a total of 90 units. For the "low democracy" countries, we check whether their marginal confidence set contains the "high democracy" group. This divides the "low democracy" countries into countries that are statistically separated from the group at the opposite side of the political spectrum, and countries that are not. Vice versa, we check which "high democracy" countries we can rule out as members of the "low democracy" group. This characterization of the joint confidence set is reported in Table 5. For both groups, a vast majority of their estimated member countries are statistically different from the other group.

²⁸We observe a moderate degree of regularization of the estimated correlation matrix. This may potentially affect the performance of the QLR statistic.

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Appendix

A. Proofs of mains results

In the proofs, we drop the g argument for ease of notation and write, e.g., $d_{it}(h)$ instead of $d_{it}(g,h)$ (or $d_{it}(g_i^0,h)$). The g argument is made explicit in the statements of the lemmas. Here, we provide proofs of Theorem 1 – Theorem 4. All supporting lemmas and the proof of Theorem 5 are given in the Supplementary Appendix.

For our proof of the QLR procedure we analyze the limiting distribution of the QLR statistic, which we call the $\tilde{\chi}^2$ -distribution. Let V denote a $(G-1)\times(G-1)$ nonsingular covariance matrix, and let $X \sim \mathcal{N}(0, V)$. The $\tilde{\chi}^2(V)$ distribution is given by the distribution of the random variable

$$W = \min_{t \le 0} (X - t)' V^{-1} (X - t).$$

The $\tilde{\chi}^2(V)$ -distribution can be characterized as a mixture of χ^2 -distributions (Rosen 2008) that is closely related to the $\bar{\chi}^2$ -distribution (Kudo 1963, Nüesch 1966). Let $w(\cdot,\cdot,\cdot)$ denote the weight function defined in Kudo (1963). The cumulative distribution function of $\tilde{\chi}^2(V)$ is given by

$$F_{\text{QLR},V}^{*}(t) = 1 - \sum_{i=1}^{G-1} w(G-1, G-1-j, V) P(\chi_{j}^{2} > t), \qquad (13)$$

where χ_j^2 has a χ^2 -distribution with j degrees of freedom. Lemma E.14 in the Supplemental Appendix summarizes more properties of the $\tilde{\chi}^2$ -distribution.

For a non-singular covariance matrix V, let $\Phi_{\max,V}$ denote the cumulative distribution function of the maximum of a centered multivariate normal vector with covariance matrix V.

Define

$$\hat{S}_{i,T}^{2}(g,h) = \frac{1}{\sigma_{i}^{2}T} \sum_{t=1}^{T} \left(\hat{d}_{it}(g,h) - \bar{\hat{d}}_{it}(g,h) \right)^{2},$$

and different oracle versions of $\hat{D}_i(g_i^0, h)$

$$\tilde{D}_{i}\left(g_{i}^{0},h\right) = \frac{\sum_{t=1}^{T} d_{it}(g_{i}^{0},h)}{\sum_{t=1}^{T} \left(d_{it}(g_{i}^{0},h) - \bar{d}_{it}(g_{i}^{0},h)\right)^{2}},$$

$$D_{i}(g_{i}^{0},h) = \frac{T^{-1/2} \sum_{t=1}^{T} d_{it}(g_{i}^{0},h)/\sigma_{i}}{s_{i,T}(g_{i}^{0},h)},$$

and an oracle version of $\hat{T}_i^{\text{QLR}}(g_i^0)$

$$T_i^{\text{QLR}}(g_i^0) = \max_{t \le 0} (D_i(g_i^0) - t)' \Omega_i^{-1}(g_i^0) (D_i(g_i^0) - t).$$

Proof of Theorem 1. The result follows directly from Lemma E.1.

Proof of Theorem 2. We first evaluate the effect of estimation error from estimating the groupspecific coefficients. Let C_1 denote the constant from Lemma E.9 and let $\zeta_{N,T}$ as defined in Lemma E.9. Let

$$a_{N,T} = C_1 \sqrt{T} \gamma_{N,T,8} \left(T^{-5/24} B_{N,T,4} \sqrt{\log N} + D_{N,T,2} \right) + C_1 \zeta_{N,T} \sqrt{\log N} \left(1 + T^{-1/4} B_{N,T,4} \sqrt{\log N} \right).$$

Define the event

$$\mathcal{E}_{N,T,1} = \left\{ \max_{1 \le i \le N} \max_{h \in \mathbb{G} \setminus \{g_i^0\}} \left| \hat{D}_i(h) - \tilde{D}_i(h) \right| \le a_{N,T} \right\}.$$

Applying Lemma E.9 with c = 1/6 yields

$$1 - P\left(\mathcal{E}_{N,T,1}\right) \le N^{-1} + C_1 T^{-1/6} + C_1 \left(T^{-1/4} B_{N,T,4} / (\log N)\right)^4 \le N^{-1} + C T^{-1/6}.$$

Note that under the assumptions of the lemma, $\zeta_{N,T} \leq 3\epsilon_{N,1}$. On $\mathcal{E}_{N,T,1}$, for i = 1, ..., N and $h \in \mathbb{G} \setminus \{g_i^0\}$,

$$\left|\hat{D}_i(h) - \tilde{D}_i(h)\right| \le C \left(\epsilon_{1,N} + \epsilon_{2,N}\right) / \sqrt{\log N} =: b_N.$$

Next, we discuss the contribution of the estimation error to the coverage level. Define α_N implicitly by

$$c_{\alpha_N,N}^{\text{SNS}} = c_{\alpha,N}^{\text{SNS}} - b_N.$$

To see that α_N is well-defined, note that since $c_{\alpha,N}^{\rm SNS} \to \infty$ and $b_N \to 0$ the right-hand side of the equation is diverging, and therefore positive for large N. Moreover, $c_{p,N}^{\rm SNS} \downarrow 0$ as $p \uparrow N/2$. This establishes the existence of α_N . Uniqueness follows from the strict monotonicity of the distribution function of the t-distribution. Let t_{T-1} denote the distribution function of a t-distributed random variable with T-1 degrees of freedom, and let f_{T-1}^t denote its density function. Let $c(\alpha) = t_{T-1}^{-1}(1-\alpha/((G-1)N))$ and $b_N^* = \sqrt{(T-1)/T}b_N$. By the mean-value theorem

$$\frac{\alpha_N}{(G-1)N} - \frac{\alpha}{(G-1)N} = t_{T-1}(c(\alpha)) - t_{T-1}(c(\alpha_N))$$
$$= t_{T-1}(c(\alpha)) - t_{T-1}(c(\alpha) - b_N) = f_{T-1}^t(c^*)b_N^*,$$

where c^* is a value between $c(\alpha_N)$ and $c(\alpha)$. Noting that $c(\alpha_N) < c(\alpha)$ and that f_{T-1}^t is decreasing on the positive axis, rearranging this equality yields

$$|\alpha_N - \alpha| \le f_{T-1}^t (c(\alpha_N)) (G - 1) N b_N^* \le 2c(\alpha_N) (1 - t_{T-1}(c(\alpha_N)) (G - 1) N b_N^*$$

$$\leq 4b_N^* \alpha_N \sqrt{\log((G-1)N/\alpha_N)}$$

$$\leq 4b_N \alpha \sqrt{\log((G-1)N/\alpha)} + 4b_N |\alpha_N - \alpha| \sqrt{\log((G-1)N/\alpha)}$$

$$\leq 4b_N \sqrt{\log((G-1)N/\alpha)} + o(|\alpha_N - \alpha|),$$

where the second inequality follows from Lemma E.12, the third inequality follows from Lemma E.11 (with $\epsilon = 1$), and the fourth inequality follows from $b_N \sqrt{\log N} \to 0$. This recursion implies

$$|\alpha_N - \alpha| \le 5b_N \sqrt{\log((G-1)N/\alpha)}$$

for N large enough.

We now derive an approximation based on the theory of self-normalized sums, i.e., Lemma E.13. Let $g_T: x \to x/\sqrt{1+x^2/T}$ and

$$\tilde{D}_{i,T,3}(h) = \left(\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left| d_{it}(h) / (\sigma_i s_{i,T}(h)) \right|^3 \right)^{1/3}.$$

We apply Lemma E.13 with $\xi_t = d_{it}(h)/(\sigma_i s_{i,T}(h)), \nu = 1$, and $x = g_T(c_{\alpha_N,N}^{SNS})$. The lemma requires

$$g_T\left(c_{\alpha_N,N}^{\text{SNS}}\right) \le T^{1/6}/\tilde{D}_{i,T,3}(h),$$
 (14)

for N large enough, for all $i=1,\ldots,N$ and $h\in\mathbb{G}\setminus\{g_i^0\}$. To prove this inequality, note that under Assumption 1 there is a constant C such that

$$\sup_{1 \le i \le N} \sup_{h \in \mathbb{G} \setminus \{g_i^0\}} \tilde{D}_{i,T,3}(h) / D_{N,T,3} \le C,$$

so that it is sufficient to show $T^{-1/6}c_{\alpha,N}^{\rm SNS}D_{N,T,3}\to 0$. Setting $\epsilon=1$ in Lemma E.11 gives

$$T^{-1/6}c_{\alpha,N}^{SNS}D_{N,T,3} \le \sqrt{T/(T-1)}T^{-1/6}2\sqrt{\log((G-1)N/\alpha)}D_{N,T,3}.$$

Under our assumptions the right-hand side vanishes and condition (14) is verified. Applying Lemma E.13 yields

$$\left| P\left(\tilde{D}_{i}(h) > c_{\alpha_{N},N}^{\text{SNS}} \right) - \left(1 - \Phi\left(g_{T}(c_{\alpha_{N},N}^{\text{SNS}}) \right) \right| \\
= \left| P\left(\frac{\sum_{t=1}^{T} d_{it}(h) / (\sigma_{i}s_{i,T}(h))}{\sqrt{\sum_{t=1}^{T} d_{it}^{2}(h) / (\sigma_{i}s_{i,T}(h))^{2}}} > g_{T}(c_{\alpha_{N},N}^{\text{SNS}}) \right) - \left(1 - \Phi\left(g_{T}\left(c_{\alpha_{N},N}^{\text{SNS}} \right) \right) \right) \right| \\
\leq KT^{-1/2} \tilde{D}_{i,T,3}^{3} \left(1 + g_{T}(c_{\alpha_{N},N}^{\text{SNS}}) \right)^{3} \left(1 - \Phi\left(g_{T}(c_{\alpha_{N},N}^{\text{SNS}}) \right) \right), \tag{15}$$

where K is the constant from Lemma E.13. For standard normal $d_{it}(h)$, we can take $\tilde{D}_{i,T,3}(h) = 2^{3/2}/\sqrt{\pi}$, and (14) is easily verified provided that $T^{-1/3}(\log N) \to 0$. As $D_{N,T,3} \ge 1$, the assumption $\epsilon_{3,N} \to 0$ requires $T^{-1/3}(\log N) \to 0$. Evaluating (15) for the special case of standard normal $d_{it}(h)$ gives

$$\left| \frac{\alpha_N}{(G-1)N} - \left(1 - \Phi \left(g_T \left(c_{\alpha_N,N}^{\text{SNS}} \right) \right) \right) \right|$$

$$\leq KT^{-1/2} 2^{9/2} \pi^{-3/2} \left(1 + g_T \left(c_{\alpha_N,N}^{\text{SNS}} \right) \right)^3 \left(1 - \Phi \left(g_T \left(c_{\alpha_N,N}^{\text{SNS}} \right) \right). \tag{16}$$

Under $T^{-1/3}(\log N) \to 0$, the right-hand side vanishes and therefore the recursive nature of the inequality implies $1 - \Phi(g_T(c_{\alpha,N}^{\rm SNS})) = \alpha_N/((G-1)N) + o(\alpha_N/((G-1)N))$. Combining inequalities (15) and (16) gives

$$\begin{split} & \left| P\left(\tilde{D}_{i}(h) > c_{\alpha_{N},N}^{\mathrm{SNS}} \right) - \frac{\alpha_{N}}{(G-1)N} \right| \\ \leq & \left| P\left(\tilde{D}_{i}(h) > c_{\alpha_{N},N}^{\mathrm{SNS}} \right) - \left(1 - \Phi\left(g_{T}\left(c_{\alpha_{N},N}^{\mathrm{SNS}} \right) \right) \right) \right| \\ & + \left| \frac{\alpha_{N}}{(G-1)N} - \left(1 - \Phi\left(g_{T}\left(c_{\alpha_{N},N}^{\mathrm{SNS}} \right) \right) \right) \right| \\ \leq & KT^{-1/2} \left(\tilde{D}_{i,T,3}^{3} + 2^{9/2}\pi^{-3/2} \right) \left(1 + g_{T}(c_{\alpha_{N},N}^{\mathrm{SNS}}) \right)^{3} \left(1 - \Phi\left(g_{T}(c_{\alpha_{N},N}^{\mathrm{SNS}}) \right) \right) \\ \leq & CT^{-1/2} D_{N,T,3}^{3} \left(1 + g_{T}(c_{\alpha_{N},N}^{\mathrm{SNS}}) \right)^{3} \left(1 - \Phi\left(g_{T}(c_{\alpha_{N},N}^{\mathrm{SNS}}) \right) \right) \\ \leq & C \left(2T^{-1/6} D_{N,T,3} \sqrt{\log\left(\frac{(G-1)N}{\alpha} \right)} \right)^{3} \left(\frac{\alpha_{N}}{(G-1)N} + o\left(\frac{\alpha_{N}}{(G-1)N} \right) \right) . \end{split}$$

Summing up, we have

$$P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\backslash\{g_{i}^{0}\}}\hat{D}_{i}(h)>c_{\alpha,N}^{\mathrm{SNS}}\right)$$

$$\leq P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\backslash\{g_{i}^{0}\}}\tilde{D}_{i}(h)>c_{\alpha,N}^{\mathrm{SNS}}-b_{N}\right)+P\left(\mathcal{E}_{N,T,1}^{c}\right)$$

$$=P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\backslash\{g_{i}^{0}\}}\tilde{D}_{i}(h)>c_{\alpha,N}^{\mathrm{SNS}}\right)+P\left(\mathcal{E}_{N,T,1}^{c}\right)$$

$$\leq \sum_{i=1}^{N}\sum_{h\in\mathbb{G}\backslash\{g_{i}^{0}\}}P\left(\tilde{D}_{i}(h)>c_{\alpha,N}^{\mathrm{SNS}}\right)+P\left(\mathcal{E}_{N,T,1}^{c}\right)$$

$$\leq \alpha_{N}+C\left(2T^{-1/6}D_{N,T,3}\sqrt{\log\left(\frac{(G-1)N}{\alpha}\right)}\right)^{3}+1-P\left(\mathcal{E}_{N,T,1}\right)$$

$$\leq \alpha+C\left(b_{N}\sqrt{\log N}+\epsilon_{3,N}+T^{-1/6}+N^{-1}\right).$$

Proof of Theorem 3. Throughout the proof let C denote a generic constant depending only on G and K_{β} .

For a nonsingular covariance matrix V, write $c_{\alpha,N}(V) = \Phi_{\max,V}^{-1} \left(1 - \frac{\alpha}{N}\right)$. We first show that for N large enough and independent of V, $c_{\alpha,N}(V) \leq c_{\alpha,N}^{\text{MAX}}(V)$. We can argue similar as in Lemma E.3 (see equation (23)) that $c_{\alpha,N}(V) > \sqrt{\log(N/\alpha)} \to \infty$ so that we can assume $c_{\alpha,N} > t^*$ for the t^* defined in Lemma E.18. Since $t_{\max,V,T-1}$ is strictly increasing, we can establish $c_{\alpha,N}(V) \leq c_{\alpha,N}^{\max}(V)$, or equivalently $\sqrt{(T-1)/T}c_{\alpha,N}(V) \leq t_{\max,V,T-1}^{-1}(1-\alpha/N)$ by showing

$$t_{\max,V,T-1}\left(\sqrt{\frac{T-1}{T}}c_{\alpha,N}(V)\right) \le 1 - \alpha/N = \Phi_{\max,V}(c_{\alpha,N}(V)).$$

This inequality follows from setting $t = c_{\alpha,N}(V)$ in Lemma E.18. For N large enough we can thus bound

$$P\left(\{g_i^0\}_{1\leq i\leq N}\in\widehat{C}_{\alpha}^{\text{MAX}}\right)\geq 1-P\left(\exists i\in\{1,\ldots,N\}:\widehat{T}_i^{\text{MAX}}(g_i^0)>c_{\alpha,N}\left(\widehat{\Omega}_i(g_i^0)\right)\right).$$

We bound the right-hand side by applying Lemma E.2. Define the events

$$\mathcal{E}_{N,T,1} = \left\{ \max_{g \in \mathbb{G}} \left(\frac{1}{T} \sum_{t=1}^{T} \| \hat{\beta}_{g,t} - \beta_{g,t} \|^{8} \right)^{1/8} \le \gamma_{N,T,8} \right\},$$

$$\mathcal{E}_{N,T,2} = \left\{ \max_{1 \le i \le N} \max_{h,h' \in \mathbb{G} \setminus \{g_{i}^{0}\}} \left| (\hat{\Omega}_{i})_{h,h'} - (\Omega_{i})_{h,h'} \right| \le C_{1} (3\epsilon_{1,N} + \epsilon_{3,N}) \right\},$$

where C_1 is the maximum of the constants from Lemma E.8 and Lemma E.9. By Assumption 1((iii)), $P(\mathcal{E}_{N,T,1}) \geq 1 - \epsilon_N$. Lemma E.8 and Lemma E.9 imply $P(\mathcal{E}_{N,T,2}) \geq 1 - CT^{-1/7} \geq 1 - C\epsilon_N$. To see this, let Ω_i^* as defined in Lemma E.8 and decompose

$$\left| (\hat{\Omega}_i)_{h,h'} - (\Omega_i)_{h,h'} \right| \le \left| (\hat{\Omega}_i)_{h,h'} - (\Omega_i^*)_{h,h'} \right| + \left| (\Omega_i^*)_{h,h'} - (\Omega_i)_{h,h'} \right|.$$

The first term on the right-hand side is bounded by $C_1\zeta_{N,T}$ with probability more than $1-CT^{-1/7}$, where $\zeta_{N,T}$ is defined in Lemma E.9. This can be shown by applying Lemma E.9 with c=1/7. Under the assumptions of the theorem we have

$$\zeta_{N,T} \le \epsilon_{1,N}/(\log N)(1+\epsilon_{3,N}) + (\epsilon_{1,N}/\log N)^2 \le 3\epsilon_{1,N}/\log N.$$

For c = 1/7, Lemma E.8 controls the rate of $|(\Omega_i^*)_{h,h'} - (\Omega_i)_{h,h'}|$ and gives the upper bound

$$C_1 T^{-3/7} B_{N,T,4}^2(\log N) = C_1 T^{-1/7} \epsilon_{3,N}^2 / \log N \le C_1 \epsilon_{3,N} / \log N.$$

On $\mathcal{E}_{N,T,1} \cap \mathcal{E}_{N,T,2}$

$$\|\hat{\Omega}_i - \Omega_i\|_2 \le \|\hat{\Omega}_i - \Omega_i\|_F = \sqrt{\sum_{h,h'} |(\hat{\Omega}_i)_{h,h'} - (\Omega)_{h,h'}|^2} \le C(\epsilon_{1,N} + \epsilon_{3,N})/\log N.$$

Since Ω_i is a correlation matrix we have $\|\Omega_i\|_2 \leq \operatorname{tr}(\Omega_i) \leq G - 1$ and therefore

$$\|\Omega_{i}^{-1}\|_{2} (1 \vee \|\Omega_{i}\|_{2} \|\Omega_{i}^{-1}\|_{2}) \|\hat{\Omega}_{i}^{-1} - \Omega_{i}^{-1}\|_{2}$$

$$\leq C\omega_{N} (1 \vee \omega_{N} (G - 1)) (\epsilon_{1,N} + \epsilon_{2,N}) / \log N$$

$$\leq C \|\hat{\Omega}_{i} - \Omega_{i}\|_{2} (\omega_{N}^{2} \vee 1) \leq C_{1}^{*} (\omega_{N}^{2} \vee 1) (\epsilon_{1,N} + \epsilon_{3,N}) / \log N$$

where C_1^* depends only on G and K_{β} . Lemma E.9 with c = 1/7 gives a lower bound on the probability of the set on which

$$\left| \hat{D}_i(h) - D_i(h) \right| \le C_1 \left[\sqrt{T} \gamma_{N,T,8} \left(\epsilon_{3,N} + D_{N,T,2} \right) + \left(\zeta_{N,T} + \left(\epsilon_{3,N} T^{-1/7} / \sqrt{\log N} \right)^2 \right) (1 + \epsilon_{3,N}) \sqrt{\log N} \right]$$

$$\le C_2^* \left(\epsilon_{1,N} + \epsilon_{2,N} + \epsilon_{3,N} \right) / \sqrt{\log N},$$

where C_2^* depends only on G and K_{β} . Define the event

$$\mathcal{E}_{N,T,3} = \left\{ \max_{1 \le i \le N} \max_{h \in \mathbb{G} \setminus \{g_i^0\}} \left| \hat{D}_i(h) - D_i(h) \right| \le C_2^* \left(\epsilon_{1,N} + \epsilon_{2,N} + \epsilon_{3,N} \right) / \sqrt{\log N} \right\}.$$

By Lemma E.9,

$$P(\mathcal{E}_{N,T,3}) \ge 1 - N^{-1} - C_1 \left(T^{-1/7} + \left(T^{-(1/4 - 1/7)} \epsilon_{N,3} \right)^4 \right) \ge 1 - N^{-1} - CT^{-1/7}.$$

By Lemma E.4, there are random variables $(X_i)_{1 \leq i \leq N}$ with $X_i \sim \mathcal{N}(0, \Omega_i)$ such that

$$\sup_{(r_1, \dots, r_N) \in \mathcal{R}_{++}^N} \left| P\left(\max_{1 \le i \le N} \left(T_i^{\text{MAX}} - r_i \right) > 0 \right) - P\left(\max_{1 \le i \le N} \left(\max_{1 \le h \le G - 1} X_{i,h} - r_i \right) > 0 \right) \right|$$

$$\leq C \left(T^{-1/6} B_{N,T,4} \log^{7/6} N + T^{-1/6} B_{N,T,4}^2 \log N \right)$$

$$\leq C \left(\epsilon_{3,N} \left(T^{-1/7} (\log N) \right)^6 + \epsilon_{3,N}^2 \right) \log N \leq C_3^* \epsilon_{3,N} (\log N),$$

where C_3^* depends only on G and K_β . To avoid ambiguity, denote the quantities in the statement of Lemma E.2 with a \dagger superscript. The conclusion of the theorem follows from applying Lemma E.2 with $\epsilon_N^{\dagger} = (C_1^* \vee C_2^* \vee C_3^*) \epsilon_N / (\log N)$, $\hat{D}_i^{\dagger} = \hat{D}_i$, $D_i^{\dagger} = D_i$, $\hat{\Omega}_i^{\dagger} = \hat{\Omega}_i$ and $\Omega_i^{\dagger} = \Omega_i$ on the event $\mathcal{E}_{N,T,1} \cap \mathcal{E}_{N,T,2} \cap \mathcal{E}_{N,T,3}$.

Proof of Theorem 4. Throughout the proof, let C denote a generic constant depending only on G and K_{β} . For a covariance matrix V, define $F_{\text{QLR},V}^*$ as in (13). Lemma E.19 implies that for N large enough

$$P\left(\{g_i^0\}_{1\leq i\leq N}\in\widehat{C}_{\alpha}^{\mathrm{QLR}}\right)\geq 1-P\left(\exists i\in\{1,\ldots,N\}:\widehat{T}_i^{\mathrm{QLR}}(g_i^0)>F_{\mathrm{QLR},\widehat{\Omega}_i(g_i^0)}\left(1-\frac{\alpha}{N}\right)\right).$$

We bound the right-hand side by applying Lemma E.3.

Let \hat{V}_i denote the diagonal matrix with entries $(\hat{V}_i)_{h,h} = \hat{S}_{i,T}(h)/(\sigma_i s_{i,T}(h))$ and let

$$\Delta_i^D(h) = \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i s_{i,T}(h)},$$
$$\hat{\Omega}_i^V = \hat{V}_i(g) \hat{\Omega}_i(g) \hat{V}_i(g),$$

 $\Delta_i^D = (\Delta_i^D(h))_{h \in \mathbb{G} \setminus \{g_i^0\}}$. Using these definitions, we may rewrite the unit-specific test statistics in a way that eliminates all random denominators

$$\hat{T}_i^{\text{QLR}} = \min_{t < 0} \left(\Delta_i^D + D_i - t \right)' \left[\hat{\Omega}_i^V \right]^{-1} \left(\Delta_i^D + D_i - t \right).$$

Define the events $\mathcal{E}_{N,T,1}$, $\mathcal{E}_{N,T,2}$ and $\mathcal{E}_{N,T,3}$ as in the proof of Theorem 3. Recall that on $\bigcap_{\ell=1}^{3} \mathcal{E}_{N,T,\ell}$ we have

$$\|\hat{\Omega}_i - \Omega_i\|_2 \le C\epsilon_N / \log N$$
$$\|\hat{D}_i - D_i\| \le C\epsilon_N / \sqrt{\log N}$$

and $P\left(\bigcap_{\ell=1}^3 \mathcal{E}_{N,T,\ell}\right) \ge 1 - C\epsilon_N$. Work conditionally on $\bigcap_{\ell=1}^3 \mathcal{E}_{N,T,\ell}$. By the inequality $|\sqrt{a} - 1| \le |a - 1|$,

$$\begin{split} \left| (\hat{V}_i)_{h,h} - 1 \right| = & \left| \sqrt{\hat{S}_{N,T}^2(g_i^0, h) / (\sigma_i^2 s_{i,T}^2(g_i^0, h))} - 1 \right| \\ \leq & \left| \hat{S}_{N,T}^2(g_i^0, h) / (\sigma_i^2 s_{i,T}^2(g_i^0, h)) - 1 \right| \\ \leq & \left| (\hat{\Omega}_i)_{h,h} - (\Omega)_{h,h} \right| \leq \|\hat{\Omega}_i - \Omega_i\|_2 \leq C \epsilon_N / \log N. \end{split}$$

and therefore, $\|\hat{V}_i - I_{G-1}\|_2 \le C\epsilon_N/\log N$, where I_{G-1} is the (G-1) dimensional identity matrix. Write $V_i = I_{G-1}$ and decompose

$$\hat{\Omega}_{i}^{V} - \Omega_{i} = (\hat{V}_{i} - V_{i})(\hat{\Omega}_{i} - \Omega_{i})(\hat{V}_{i} - V_{i}) + 2V_{i}(\hat{\Omega}_{i} - \Omega_{i})(\hat{V}_{i} - V_{i}) + V_{i}(\hat{\Omega}_{i} - \Omega_{i})V_{i} + (\hat{V}_{i} - V_{i})\Omega_{i}(\hat{V}_{i} - V_{i}) + 2V\Omega_{i}(\hat{V}_{i} - V_{i}).$$

Noting that $\|\Omega_i\| \leq \operatorname{tr}(\Omega_i) \leq G - 1$, this decomposition implies

$$\|\hat{\Omega}_{i}^{V} - \Omega_{i}\|_{2} \le C \left(\epsilon_{N}^{3} + (2 + \|\Omega_{i}\|_{2})\epsilon_{N}^{2} + (1 + 2\|\Omega_{i}\|_{2})\epsilon_{N}\right) \le C\epsilon_{N},$$

where the second-to-last inequality follows from $\epsilon_N \leq 1$ for N large enough. Therefore,

$$(1 \vee \|\Omega_i^{-1}\|)(\|\Omega_i\| \vee \|\Omega_i^{-1}\|)\|\hat{\Omega}_i^V - \Omega_i\|_2$$

$$\leq C(1 \vee \lambda_1^{-1})(\lambda_1^{-1} \vee (G-1))\epsilon_N/\log N \leq C_1^* \epsilon_N/\log N,$$

where C_1^* depends only on λ_1 , G and K_{β} . Define the event

$$\mathcal{E}_{N,T,4} = \max_{1 \le i \le N} \left\{ ||D_i|| \le 2C_1 \sqrt{\log N} \right\}.$$

Taking N large enough that $\epsilon_{3,N} \leq 1$, Lemma E.8 with c = 1/7 yields

$$1 - P\left(\mathcal{E}_{N,T,4}\right) \leq \sum_{\ell=1}^{G-1} P\left(\max_{1 \leq i \leq N} \|D_{i,\ell}\| > 2C_1 \sqrt{\log N}\right)$$

$$\leq \sum_{\ell=1}^{G-1} P\left(\max_{1 \leq i \leq N} \|D_{i,\ell}\| > C_1 \sqrt{\log N} \left(1 + T^{-1/4} B_{N,T,4} \sqrt{\log N}\right)\right)$$

$$\leq (G-1) \left(N^{-1} + \left(T^{-(1/4-1/7)} \epsilon_{3,N}\right)^4\right) \leq C\epsilon_N,$$

where $D_{i,\ell}$ is the ℓ -th element of D_i . On $\bigcap_{\ell=1}^4 \mathcal{E}_{N,T,\ell}$,

$$(\|D_i\| \vee 1)\|\Omega_i^{-1}\|_2 \|\hat{D}_i - D_i\| \le (C\sqrt{\log N} \vee 1)\lambda_1^{-1}C(\epsilon_{1,N} + \epsilon_{2,N} + \epsilon_{3,N})/\sqrt{\log N}$$

$$\le C_2^*(\epsilon_{1,N} + \epsilon_{2,N} + \epsilon_{3,N}),$$

where C_2^* is a constant that depends only on λ_1 , G and K_β . By Lemma E.4, there are independent random variables $(U_i)_{1 \leq i \leq N}$ with $U_i \sim \tilde{\chi}^2(\Omega_i)$ such that

$$\sup_{(r_1,\dots,r_N)\in\mathcal{R}_{++}^N} \left| P\left(\max_{1\leq i\leq N} \left(T_i^{\text{QLR}} - r_i \right) > 0 \right) - P\left(\max_{1\leq i\leq N} \left(U_i - r_i \right) > 0 \right) \right|$$

$$\leq C \left(T^{-1/6} B_{N,T,4} \log^{7/6} N + T^{-1/6} B_{N,T,4}^2 \log N \right)$$

$$\leq C \left(\epsilon_{3,N} \left(T^{-1/7} (\log N) \right)^6 + \epsilon_{3,N}^2 \right) \log N \leq C_3^* \epsilon_{3,N} (\log N),$$

where C_3^* is a constant that depends only on λ_1 , G and K_{β} . To avoid ambiguity, denote the quantities in the statement of Lemma E.3 with a \dagger superscript. We apply Lemma E.3 with $\epsilon_N^{\dagger} = (C_1^* \vee C_2^* \vee C_3^*)(\epsilon_N - N^{-1})/\log N$, $\hat{D}_i^{\dagger} = D_i + \Delta_i^D$, $D_i^{\dagger} = D_i$, $\hat{\Omega}_i^{\dagger} = \hat{\Omega}_i^V$ and $\Omega_i^{\dagger} = \Omega_i$ on the event $\mathcal{E}_{N,T,1} \cap \mathcal{E}_{N,T,2} \cap \mathcal{E}_{N,T,3} \cap \mathcal{E}_{N,T,4}$. For C large enough this establishes

$$P\left(\exists i \in \{1, \dots, N\} : \hat{T}_i^{\text{QLR}}(g_i^0) > \left(F_{\text{QLR},\widehat{\Omega}_i(g)}^*\right)^{-1} \left(1 - \frac{\alpha}{N}\right)\right) \le \alpha + C\epsilon_N.$$

B. Figures and tables

			empirical coverage		cardinality of CS	
g^0	σ	T	MAX	QLR	MAX	QLR
1	0.25	10	0.64	0.71	1.37	1.36
1	0.25	20	0.75	0.83	1.28	1.28
1	0.25	30	0.81	0.88	1.26	1.26
1	0.25	40	0.83	0.90	1.24	1.24
1	0.50	10	0.47	0.49	2.46	2.46
1	0.50	20	0.71	0.77	2.52	2.52
1	0.50	30	0.81	0.84	2.54	2.54
1	0.50	40	0.82	0.83	2.55	2.55
2	0.25	10	0.75	0.76	1.28	1.37
2	0.25	20	0.79	0.78	1.20	1.28
2	0.25	30	0.83	0.83	1.18	1.25
2	0.25	40	0.85	0.83	1.17	1.23
2	0.50	10	0.59	0.62	1.96	2.00
2	0.50	20	0.79	0.76	1.95	2.01
2	0.50	30	0.81	0.82	1.96	2.01
2	0.50	40	0.84	0.84	1.95	2.00
3	0.25	10	0.72	0.74	1.28	1.36
3	0.25	20	0.81	0.80	1.20	1.27
3	0.25	30	0.83	0.83	1.18	1.24
3	0.25	40	0.87	0.85	1.17	1.23
3	0.50	10	0.58	0.58	1.96	1.98
3	0.50	20	0.76	0.76	1.96	1.98
3	0.50	30	0.82	0.82	1.95	1.99
3	0.50	40	0.85	0.86	1.96	1.99

Table B.1: Homoscedastic design with G=3 groups. Results based on 1000 simulated joint confidence sets with $1-\alpha=0.9$. Critical values for MAX and QLR procedures are based on the Gaussian limit. "Empirical coverage" gives the simulated coverage probability of the joint confidence set. "Cardinality of CS" gives the simulated expected average cardinality of a marginal (unit-wise) confidence set.

	$\log(\text{uerage})$	$\log(gr)$	$\log(\text{rminwg})$
Positive-effect group	0.62	-0.43	0.06
Negative-effect group	0.86	-0.18	-0.07

Table B.2: Estimated group-specific coefficients.

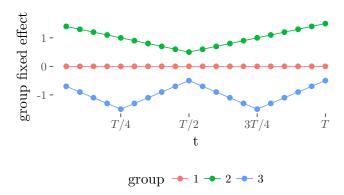


Figure B.1: Time profile of the group fixed effect for the simulation design from Section 6.1.

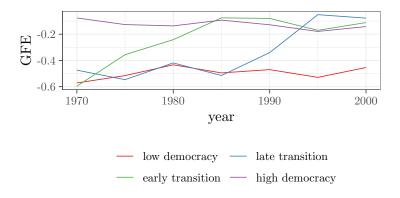


Figure B.2: Estimated time profiles for the group fixed effects.

C. P-values for joint significance

Let $\hat{N} = \hat{N}(\infty)$ denote the number of units selected by the unit selection procedure. If no unit selection is carried out (i.e., $\beta = 0$) then $\hat{N} = N$. For unit i, the p-value for joint significance is given by $\hat{\alpha}_i^{\text{type}}$, where type = SNS, MAX, QLR corresponds to the procedure used to compute the confidence set and

$$\begin{split} \hat{\alpha}_i^{\text{SNS}} = & (G-1)\hat{N} \max_{g \in \mathbb{G} \backslash \{\hat{g}_i\}} \left(1 - t_{T-1} \left(\sqrt{\frac{T-1}{T}} \hat{T}_i^{\text{SNS}}(g)\right)\right), \\ \hat{\alpha}_i^{\text{MAX}} = & \hat{N} \max_{g \in \mathbb{G} \backslash \{\hat{g}_i\}} \left(1 - t_{\max,\widehat{\Omega}(g),T-1} \left(\sqrt{\frac{T-1}{T}} \hat{T}_i^{\text{MAX}}(g)\right)\right), \\ \hat{\alpha}_i^{\text{QLR}} = & \hat{N} \max_{g \in \mathbb{G} \backslash \{\hat{g}_i\}} \left(1 - F_{\text{QLR},\widehat{\Omega}(g),T-1} \left(\hat{T}_i^{\text{QLR}}(g)\right)\right). \end{split}$$

Clearly, $\hat{\alpha}_i^{\text{type}}$ is not restricted to lie in the unit interval, but $\hat{\alpha}_i^{\text{type}} \geq 1$ implies a jointly insignificant membership estimates at any nominal level. The set of units with jointly significant membership

estimates at level α is given by

$$\left\{i=1,\ldots,N:\hat{\alpha}_i^{\mathrm{type}}<\alpha\right\}.$$

Supplemental Appendix

for Confidence Set for Group Membership

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D. Extension to model with individual fixed effects

In this section, we discuss an extension of the grouped random coefficient model from Example 1 that adds individual fixed effects. As in Example 1, the group-specific coefficients are assumed to be time-invariant. We argue that our procedures can be used after applying a fixed effect transformation.

Suppose that unit i's outcome is generated from

$$y_{it} = x'_{it}\beta_g + \mu_i + u_{it},$$

where μ_i is i's fixed effect and all other quantities are defined as before. The individual fixed effect can be removed by the fixed effect transformation

$$y_{it} - \bar{y}_i = (x_{it} - \bar{x}_i)' \beta_g + u_{it} - \bar{u}_i,$$

where $\bar{y}_i = \sum_{t=1}^{T} y_{it}/T$, $\bar{x}_i = \sum_{t=1}^{T} x_{it}/T$ and $\bar{u}_i = \sum_{t=1}^{T} u_{it}/T$.

We work with the transformed data to construct confidence sets. The natural counterpart to $\hat{d}_{it}(g,h)$ is given by

$$\hat{d}_{it}^{\text{FE}}(g,h) = \frac{1}{2} \left(\left(y_{it} - \bar{y}_i - (x_{it} - \bar{x}_i)' \hat{\beta}_g \right)^2 - \left(y_{it} - \bar{y}_i - (x_{it} - \bar{x}_i)' \hat{\beta}_h \right)^2 + \left((x_{it} - \bar{x}_i)' (\hat{\beta}_g - \hat{\beta}_h) \right)^2 \right).$$

Replacing $\hat{D}_i(g,h)$ by

$$\hat{D}_{i}^{\text{FE}}(g,h) = \frac{\sum_{t=1}^{T} \hat{d}_{it}^{\text{FE}}(g,h)}{\sum_{t=1}^{T} \left(\hat{d}_{it}^{\text{FE}}(g,h) - \bar{d}_{it}^{\text{FE}}(g,h)\right)^{2}},$$
(17)

we can follow the recipes for constructing confidence sets in Section 4.3. Strictly speaking, our asymptotic results from Section 5 do not apply here, since $\{(y_{it} - \bar{y}_i, x_{it} - \bar{x}_i)\}_{1 \leq t \leq T}$ is not an *i.i.d.* sequence. Heuristically, our approach is still expected to work well since

$$x_{it} - \bar{x}_i = x_{it} - \sum_{t=1}^{T} \mathbb{E}[x_{it}]/T + O_p(T^{-1/2}),$$

$$u_{it} - \bar{u}_i = u_{it} + O_p(T^{-1/2}),$$

so that, asymptotically, the correlation between time periods becomes negligible.

We now discuss bias in $\hat{D}_i^{\text{FE}}(g_i^0, h)$. To this end, suppose that the group-specific parameters are estimated accurately, i.e. $\hat{\beta}_g = \beta_g$ for $g \in \mathbb{G}$.²⁹ In the case of strictly exogenous regressors, the numerator in (17) has mean zero. With predetermined regressors, such as lagged outcomes, it may be biased. In this case, we can use the half-panel Jackknife from Dhaene and Jochmans (2015) for bias correction.³⁰

To explain the adjustment based on the half-panel Jackknife, define the split sample means

$$\bar{w}_{i,1,t_0} = \sum_{t=1}^{t_0} w_{it}/t_0, \quad \bar{w}_{i,2,t_0} = \sum_{t=t_0+1}^T w_{it}/(T-t_0)$$

for random vectors $(w_{it})_{1 \le t \le T}$. For $j = \{1, 2\}$, let

$$\hat{d}_{it,j,t_{0}}^{\text{FE}}(g,h) = \frac{1}{2} \left(\left(y_{it} - \bar{y}_{i,j,t_{0}} - (x_{it} - \bar{x}_{i,j,t_{0}})' \hat{\beta}_{g} \right)^{2} - \left(y_{it} - \bar{y}_{i,j,t_{0}} - (x_{it} - \bar{x}_{i,j,t_{0}})' \hat{\beta}_{h} \right)^{2} + \left((x_{it} - \bar{x}_{i})' (\hat{\beta}_{g} - \hat{\beta}_{h}) \right)^{2} \right),$$

$$\hat{d}_{it,1+2}^{\text{FE}}(g,h) = \left(\hat{d}_{it,(t-1) \bmod \lfloor T/2 \rfloor + 1, \lfloor T/2 \rfloor}^{\text{FE}}(g,h) + \hat{d}_{it,(t-1) \bmod \lceil T/2 \rceil + 1, \lceil T/2 \rceil}^{\text{FE}}(g,h) \right) / 2.$$

The Jackknifed version of (17) is given by

$$\tilde{D}_{i}^{\text{FE}}(g,h) = \frac{2\sum_{t=1}^{T} \hat{d}_{it}^{\text{FE}}(g,h) - \sum_{t=1}^{T} \hat{d}_{it,1+2}^{\text{FE}}(g,h)}{\sum_{t=1}^{T} \left(\hat{d}_{it}^{\text{FE}}(g,h) - \bar{d}_{it}^{\text{FE}}(g,h)\right)^{2}}.$$

 $\tilde{D}_i^{\mathrm{FE}}$ replaces \hat{D}_i in the test statistics described in Section 4.3.

²⁹Our asymptotic results in Section 5 provide conditions under which parameter estimation affects only higher-order terms.

 $^{^{30}}$ An alternative approach is analytical bias correction as in Hahn and Kuersteiner (2002).

E. Lemmas

Lemma E.1. Let $(\phi_i)_{i=1}^n$ denote a collection of independent, non-randomized tests and suppose that

$$\alpha_i = n\mathbb{P}\left(\phi_i > 0\right)$$

with $\alpha_{\max} := \max_{i=1,\dots,n} \alpha_i < 1$. Then

$$\alpha_{\min} - \frac{\alpha_{\min}^2}{2} \le \mathbb{P}\left(\max_{i=1,\dots,n} \phi_i > 0\right) \le \alpha_{\max} - \frac{\alpha_{\max}^2}{2} \left(1 - \frac{\alpha_{\max}}{3} + \frac{1}{n} \left(1 - \frac{\alpha_{\max}}{n}\right)^{-2}\right),$$

where $\alpha_{\min} := \min_{i=1,\dots,n} \alpha_i$.

Proof. For fixed 0 < x < 1, let \bar{x} denote a generic intermediate value between zero and x. By a Taylor expansion around x = 0,

$$\exp(-x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}\exp(-\bar{x})x^3$$

$$\geq 1 - x + x^2 \left(\frac{1}{2} - \frac{x}{6}\right).$$
(18)

Moreover,

$$\log(1-x) = 0 - x - \frac{x^2}{2(1-\bar{x})^2} \ge -x - \frac{x^2}{2(1-x)^2}.$$
 (19)

Now, for $0 < \alpha < 1$,

$$\left(1 - \frac{\alpha}{n}\right)^n = \exp\left(n\log\left(1 - \frac{\alpha}{n}\right)\right)$$

$$\geq \exp(-\alpha)\exp\left(-\frac{\alpha^2}{2n}\left(1 - \frac{\alpha}{n}\right)^{-2}\right)$$

$$\geq \left(1 - \alpha + \alpha^2\left(\frac{1}{2} - \frac{\alpha}{6}\right)\right)\left(1 - \frac{\alpha^2}{2n}\left(1 - \frac{\alpha}{n}\right)^{-2}\right)$$

$$\geq 1 - \alpha + \frac{\alpha^2}{2}\left(1 - \frac{\alpha}{3}\right) - \frac{\alpha^2}{2n}\left(1 - \frac{\alpha}{n}\right)^{-2},$$

where the first inequality uses (19), the second inequality uses (18) and the last inequality uses

$$1 - \alpha + \alpha^2 \left(\frac{1}{2} - \frac{\alpha}{6} \right) \le 1.$$

We conclude that

$$\mathbb{P}\left(\max_{i=1,\dots,n}\phi_i>0\right)=1-\mathbb{P}\left(\max_{i=1,\dots,n}\phi_i=0\right)$$

$$\leq 1 - \left(1 - \frac{\alpha_{\max}}{n}\right)^n \leq \alpha_{\max} - \frac{\alpha_{\max}^2}{2} \left(1 - \frac{\alpha_{\max}}{3}\right) + \frac{\alpha_{\max}^2}{2n} \left(1 - \frac{\alpha_{\max}}{n}\right)^{-2}.$$

Next, note that

$$\left(1 - \frac{\alpha}{n}\right)^n \le \exp(-\alpha) \le 1 - \alpha + \frac{\alpha^2}{2}$$

and therefore

$$\mathbb{P}\left(\max_{i=1,\dots,n} \phi_i > 0\right) = 1 - \mathbb{P}\left(\max_{i=1,\dots,n} \phi_i = 0\right)$$
$$\geq 1 - \left(1 - \frac{\alpha_{\min}}{n}\right)^n \geq \alpha_{\min} - \frac{\alpha_{\min}^2}{2}.$$

Lemma E.2 (Slutsky-type result for MAX statistic). Let α denote a constant $0 < \alpha < 1$. Let $\epsilon_N \geq N^{-1}$ such that

$$\max_{1 \le i \le N} \max_{h \in \mathbb{G} \setminus \{g_i^0\}} |\hat{D}_i(g_i^0, h) - D_i(g_i^0, h)| \le \epsilon_N \sqrt{\log N},$$

$$\max_{1 \le i \le N} \|\Omega_i^{-1}(g_i^0)\|_2 (1 \vee \|\Omega_i^{-1}(g_i^0)\|_2 \|\Omega_i(g_i^0)\|_2) \|\hat{\Omega}_i(g_i^0) - \Omega_i(g_i^0)\|_2 \le \epsilon_N.$$

Let $(X_i)_{1 \leq i \leq N}$ denote a collection of random vectors such that $X_i \sim N(0,\Omega_i)$ and suppose that

$$\sup_{(r_1,\dots,r_N)\in\mathbb{R}^N} \left| P\Big(\max_{1\leq i\leq N} T_i^{\text{MAX}}(g_i^0) \leq r_i \Big) - P\Big(\max_{1\leq i\leq N} \max_{1\leq h\leq G-1} X_{i,h} \leq r_i \Big) \right| \leq \epsilon_N(\log N).$$

Also, suppose that

$$64(G-1)^2 \log(N/\alpha)\epsilon_N \le 1.$$

Then, there is a threshold N_0 and a constant C depending only on G and α such that for $N \geq N_0$

$$P\left(\exists i \in \{1, \dots, N\} : \hat{T}_i^{\text{MAX}}(g_i^0) > \Phi_{\max, \widehat{\Omega}_i(g)}^{-1} \left(1 - \frac{\alpha}{N}\right)\right) \le \alpha + C\epsilon_N(\log N).$$

Proof. For nonsingular covariance matrix V, write $c_{\alpha,N}(V) = \Phi_{\max,V}^{-1}\left(1 - \frac{\alpha}{N}\right)$. Take N large enough such that

$$\log(N/\alpha) \ge \max\left\{1, \frac{\alpha^2(G-1)}{2\pi}, \frac{2^{G-1}-1}{8(G-1)^2}, \log(2(G-1))\right\},\,$$

If we choose N large enough, then the assumptions of the lemma imply $\epsilon_N \leq 1/2$ and thus

$$2 \max_{1 \le i \le N} \|\hat{\Omega}_i - \Omega_i\|_2 \|\Omega_i^{-1}\|_2 \le 1.$$

Therefore, we can employ Lemma E.15 to bound

$$\|\hat{\Omega}_i^{-1} - \Omega_i^{-1}\|_2 \le 2\|\Omega_i^{-1}\|_2^2 \|\hat{\Omega}_i - \Omega_i\|_2$$

for all $i = 1, \ldots, N$, and

$$\|\hat{\Omega}_{i}^{-1} - \Omega_{i}^{-1}\|_{2} \left(\|\Omega_{i}^{-1}\|_{2} \vee \|\hat{\Omega}_{i}^{-1}\|_{2}\right)$$

$$\leq 2\|\Omega_{i}^{-1}\|_{2}^{2}\|\Omega_{i}\|_{2}\|\hat{\Omega}_{i} - \Omega_{i}\|_{2} + 2\left(\|\Omega_{i}^{-1}\|_{2}\|\Omega_{i}^{-1}\|_{2}^{2}\|\hat{\Omega}_{i} - \Omega_{i}\|_{2}\right)^{2} \leq 4\epsilon_{N}.$$

$$(20)$$

Define

$$\alpha_N = \alpha \left(1 + 16(G - 1)^2 \log(N/\alpha) \max_{1 \le i \le N} \|\hat{\Omega}_i^{-1} - \Omega_i^{-1}\|_2 (\|\Omega_i\|_2 \vee \|\hat{\Omega}_i\|_2) \vee N^{-1} \right).$$

Note that (20) implies $\alpha \leq \alpha_N \leq 2\alpha$. First, we show that

$$c_{\alpha_N,N}(\Omega_i) \le c_{\alpha,N}(\hat{\Omega}_i). \tag{21}$$

Let $a_N^2 = 4(G-1)\|\Omega_i\|_2 \log(N/\alpha)$. Note that

$$\begin{split} &\|\hat{\Omega}_{i}^{-1} - \Omega_{i}^{-1}\|_{2} \left(\|\Omega_{i}^{-1}\|_{2} \vee \|\hat{\Omega}_{i}^{-1}\|_{2} \vee (G - 1)a_{N}^{2}\right) \\ \leq &8(G - 1)^{2}\|\Omega_{i}^{-1}\|_{2}^{2}\|\Omega_{i}\|_{2}\|\hat{\Omega}_{i} - \Omega_{i}\|_{2}\log\left(N/\alpha\right) + 2\left(\|\Omega_{i}^{-1}\|_{2}\|\hat{\Omega}_{i} - \Omega_{i}\|_{2}\right)^{2} \leq 1. \end{split}$$

This verifies the required assumption for the application of Lemma E.21 below. For $X \sim \mathcal{N}(0, \Omega_i)$ and $\hat{X} \sim \mathcal{N}(0, \hat{\Omega}_i)$ we have

$$\begin{split} &P\Big(\max_{j=1,\dots,G-1} X_j > c_{\alpha,N}(\hat{\Omega}_i)\Big) \\ \leq &P\Big(\max_{j=1,\dots,G-1} X_j > c_{\alpha,N}(\hat{\Omega}_i) \wedge \|X\|_{\max} \leq a_N\Big) + P(\|X\|_{\max} > a_N) \\ \leq &\frac{P\Big(\max_{j=1,\dots,G-1} X_j > c_{\alpha,N}(\hat{\Omega}_i) \wedge \|X\|_{\max} \leq a_N\Big)}{P\Big(\max_{j=1,\dots,G-1} \hat{X}_j > c_{\alpha,N}(\hat{\Omega}_i) \wedge \|\hat{X}\|_{\max} \leq a_N\Big)} P\Big(\max_{j=1,\dots,G-1} \hat{X}_j > c_{\alpha,N}(\hat{\Omega}_i)\Big) \\ &+ P(\|X\|_{\max} > a_N) \\ \leq &\Big(1 + (2^{G-1} - 1)\|\hat{\Omega}_i^{-1} - \Omega_i^{-1}\|_2 \|\hat{\Omega}_i\|_2 + 2(G-1)a_N^2 \|\hat{\Omega}_i^{-1} - \Omega_i^{-1}\|_2\Big)(\alpha/N) \\ &+ P(\|X\|_{\max} > a_N) \\ \leq &\Big(1 + \|\hat{\Omega}_i^{-1} - \Omega_i^{-1}\|_2 (\|\Omega_i\|_2 \vee \|\hat{\Omega}_i\|_2)\Big)\Big((2^{G-1} - 1) + \log(N/\alpha)8(G-1)^2\Big) \\ &+ \frac{(\alpha/N)\sqrt{G-1}}{\sqrt{2\pi \log(N/\alpha)}}\Big)\frac{\alpha}{N} \leq \frac{\alpha_N}{N}. \end{split}$$

The third inequality above follows from Lemma E.21 noting that, under the assumptions of the

lemma, we can take

$$(2^{G-1}-1)\|\hat{\Omega}_i^{-1}-\Omega_i^{-1}\|_2\|\hat{\Omega}_i\|_2 \le (2^{G-1}-1)\epsilon_N \le 1.$$

The fourth inequality follows from Lemma E.20. This establishes (21). Let $(X_{i,1}, \ldots, X_{i,G-1})$ denote a centered normal random vector with covariance matrix Ω_i . Next, we show that for a universal constant \tilde{C} and a threshold N_0 that is independent of $(\Omega_i)_{1 \leq i \leq N}$, for all $b_N > 0$,

$$P\left(\left|\max_{1\leq i\leq N} \left(\max_{1\leq h\leq G-1} X_{i,h} - c_{\alpha_N,N}(\Omega_i)\right)\right| \leq b_N\right)$$

$$\leq \tilde{C}(b_N \vee N^{-1})\sqrt{2\log(N\sqrt{G-1})}$$
(22)

for $N \geq N_0$. There exists N_0 , independent of Ω_i , such that for $N \geq N_0$

$$\sqrt{\log(N/\alpha_N)} < c_{\alpha_N,N}(\Omega_i) \le \sqrt{2\log(G-1)} + \sqrt{2\log(N/\alpha_N)}.$$
 (23)

The lower bound follows from the fact that $T_i^{\text{MAX}} \geq Z$ for standard normal Z in conjunction with a bound on the tail probability of a standard normal random variable (e.g., the argument in the proof of Lemma E.23 with a=2). The upper bound follows from Lemma D.4 in Chernozhukov, Chetverikov, and Kato (2018). The inequality

$$\log(N/\alpha_N) \ge \log(N/(2\alpha)) \ge \log(G-1)$$

implies that the right-hand side of (23) can be bounded by

$$\sqrt{2\log(N/\alpha_N)} + \sqrt{2\log(N/\alpha_N)} \le \sqrt{8\log(N/\alpha_N)}$$
.

Therefore, to prove (22) it suffices to show

$$\max_{\substack{(a_i)_{1 \leq i \leq N} \\ 1 \leq a_i \leq 2\sqrt{2}}} P\left(\left| \max_{i=1,\dots,N} \left(\max_{1 \leq h \leq G-1} X_{i,h} - a_i \sqrt{\log(N/\alpha_N)} \right) \right| \leq b_N \right) \\
\leq \tilde{C}(b_N \vee N^{-1}) \sqrt{2\log(N\sqrt{G-1})}.$$

For $N \geq 2$ we write

$$\begin{split} &\max_{\substack{(a_i)_{1\leq i\leq N}\\1\leq a_i\leq 8}}P\left(\Big|\max_{1\leq i\leq N}\left(\max_{1\leq h\leq G-1}X_{i,h}-a_i\sqrt{\log(N/\alpha_N)}\right)\Big|\leq b_N\right)\\ &\leq \max_{\substack{(a_i)_{1\leq i\leq N\\1\leq a_i\leq 8}}}\sup_{x\in\mathcal{R}}P\left(\Big|\max_{1\leq i\leq N}\max_{1\leq h\leq G-1}\frac{X_{i,h}}{a_i}-x\Big|\leq b_N\vee N^{-1}\right)\\ &\leq \tilde{C}(b_N\vee N^{-1})\sqrt{1\vee\log\left(\frac{N(G-1)}{b_N\vee N^{-1}}\right)}\leq \tilde{C}(b_N\vee N^{-1})\sqrt{2\log(N\sqrt{G-1})}. \end{split}$$

The second inequality follows from Corollary 1 in Chernozhukov, Chetverikov, and Kato (2015). Collecting the results from above yields

$$\begin{split} &P\Big(\max_{1\leq i\leq N}\Big(\hat{T}_i^{\text{MAX}} - c_{\alpha,N}(\hat{\Omega}_i)\Big) > 0\Big) \leq P\Big(\max_{1\leq i\leq N}\Big(\hat{T}_i^{\text{MAX}} - c_{\alpha_N,N}(\Omega_i)\Big) > 0\Big) \\ \leq &P\Big(\max_{1\leq i\leq N}\Big(\max_{1\leq h\leq G-1}X_{i,h} - c_{\alpha_N,N}(\Omega_i)\Big) + \epsilon_N\sqrt{\log N} > 0\Big) \\ &+ \Big|P\Big(\max_{1\leq i\leq N}\Big(\max_{h\leq \leq G-1}D_i(h) - c_{\alpha_N,N}(\Omega_i)\Big) + \epsilon_N\sqrt{\log N} > 0\Big) \\ &- P\Big(\max_{1\leq i\leq N}\Big(\max_{1\leq h\leq G-1}X_{i,h} - c_{\alpha_N,N}(\Omega_i)\Big) + \epsilon_N\sqrt{\log N} > 0\Big)\Big| \\ \leq &P\Big(\max_{1\leq i\leq N}\Big(\max_{1\leq h\leq G-1}X_{i,h} - c_{\alpha_N,N}(\Omega_i)\Big) > 0\Big) \\ &+ P\left(\Big|\max_{1\leq i\leq N}\Big(\max_{1\leq h\leq G-1}X_{i,h} - c_{\alpha_N,N}(\Omega_i)\Big) + \epsilon_N\sqrt{\log N}\Big) + \epsilon_N(\log N) \\ \leq &\sum_{i=1}^N \frac{\alpha_N}{N} + \tilde{C}\epsilon_N\sqrt{\log N}\sqrt{2\log(N\sqrt{G-1})} + \epsilon_N(\log N) \\ \leq &\alpha\left(1 + C\epsilon_N(\log N) + N^{-1}\right) + C\epsilon_N(\log N). \end{split}$$

The first inequality holds due to (21) and the fourth inequality holds due to (22). The last inequality holds due to

$$\alpha_N \le \alpha + \alpha \left(64(G-1)^2 \log (N/\alpha) \, \epsilon_N \vee N^{-1} \right)$$

$$\le \alpha + 64 (G-1)^2 (\log N) \epsilon_N \left(1 - \frac{\log \alpha}{\log N} \right).$$

Lemma E.3 (Slutzky-type result for QLR). Let α denote a constant $0 < \alpha < 1$. Suppose that there is a sequence ϵ_N such that

$$\begin{split} \max_{1 \leq i \leq N} (1 \vee \|\Omega_i^{-1}(g_i^0)\|_2^2) (\|\Omega_i(g_i^0)\|_2 \vee \|\Omega_i^{-1}(g_i^0)\|_2) \|\hat{\Omega}_i(g_i^0) - \Omega_i(g_i^0)\|_2 \leq & \epsilon_N, \\ \max_{1 \leq i \leq N} (\|D_i(g_i^0)\| \vee 1) \|\Omega_i^{-1}(g_i^0)\|_2 \|\hat{D}_i(g_i^0) - D_i(g_i^0)\| \leq & \epsilon_N (\log N), \end{split}$$

 $\epsilon_N \leq 1/48$ and

$$32(G-1)^2 \epsilon_N \log (N/\alpha) \le 1.$$

In addition, suppose that $\max_{1 \leq i \leq N} \|\hat{D}_i(g_i^0) - D_i(g_i^0)\| \leq 1$. Let $(U_i)_{1 \leq i \leq N}$ denote independent

random variables with $U_i \sim \tilde{\chi}^2(\Omega_i)$ such that

$$\sup_{(r_1,\dots,r_N)\in\mathcal{R}_{++}^N} \left| P\left(\max_{1\leq i\leq N} (U_i - r_i) > 0 \right) - P\left(\max_{1\leq i\leq N} (T_i^{\text{QLR}}(g_i^0) - r_i) > 0 \right) \right| \leq \epsilon_N(\log N).$$

Then, there is a constant C and a threshold N_0 depending only on α , G and the sequence ϵ_N such that for $N \geq N_0$

$$P\left(\exists i \in \{1,\ldots,N\} : \hat{T}_i^{\text{QLR}}(g_i^0) > \left(F_{\text{QLR},\widehat{\Omega}_i(g)}^*\right)^{-1} \left(1 - \frac{\alpha}{N}\right)\right) \le \alpha + C\left(\epsilon_N(\log N) + N^{-1}\right),$$

where $F_{\text{QLR},\widehat{\Omega}_i(g)}^*$ is defined in (13).

Proof. To simplify notation, we fix the null hypothesis and drop the g_i^0 argument. For nonsingular covariance matrix V, we write $c_{\alpha,N}(V) = \left(F_{\text{QLR},V}^*\right)^{-1} \left(1 - \frac{\alpha}{N}\right)$. Define

$$\alpha_N = \alpha \left(1 + 96\epsilon_N \log \left(N/\alpha\right)\right)$$

and

$$b_{1,N} = \max_{1 \le i \le N} 2 \|\Omega_i\|_2 \|\Omega_i^{-1}\|_2^2 \|\hat{\Omega}_i - \Omega_i\|_2,$$

$$b_{N,2} = \max_{1 \le i \le N} (2\|D_i\| + 3) \|\Omega_i^{-1}\|_2 \|\hat{D}_i - D_i\|.$$

Choose N large enough so that

$$32(G-1)^{2} \max_{1 \le i \le N} \|\hat{\Omega}_{i} - \Omega_{i}\|_{2} \left(1 \lor 2\|\Omega^{-1}\|_{2}^{2}\right) \left(\|\Omega_{i}\|_{2} \lor \|\Omega_{i}^{-1}\|_{2}\right) \le 1.$$

Then, Lemma E.22 gives $c_{\alpha_N,N}(\Omega_i) \leq c_{\alpha,N}(\hat{\Omega}_i)$ for N large enough. Next, we show that we can choose N_0 , depending only on α , such that, for $N \geq N_0$,

$$P\left(\left|\max_{1\leq i\leq N} \left(U_{i} - c_{\alpha_{N},N}(\Omega_{i})\right)\right| \leq b_{1,N} \max_{1\leq i\leq N} \left(c_{\alpha_{N},N}(\Omega_{i})\right) + b_{2,N}\right\}\right)$$

$$\leq C_{1}\left(b_{1,N} \max_{1\leq i\leq N} \left(c_{\alpha_{N},N}(\Omega_{i})\right) + b_{2,N} + N^{-1}\right) + N^{-1}$$
(24)

for a constant C_1 depending only on α and p. This follows from an application of Lemma E.10. First, we bound $c_{\alpha_N,N}(\Omega_i)$. We choose N large enough such that

$$\log N \ge \max\{-\log \alpha, 2\log \alpha\}.$$

The upper bound from Lemma E.23 implies that, for all N exceeding a threshold that depends

only on α ,

$$c_{\alpha_N,N}(\Omega_i) \le c_{\alpha,N}(\Omega_i) \le 4\log(N/\alpha) \le 4\log N \left(1 - (\log \alpha)/(\log N)\right) \le 8(\log N).$$

The lower bound from Lemma E.23 gives that for all N exceeding a threshold that depends only on α

$$c_{\alpha_N,N}(\Omega_i) \ge c_{2\alpha,N}(\Omega_i) \ge \log(N/\alpha) \ge \log N \left(1 - (\log \alpha)/(\log N)\right) \ge (1/2)(\log N).$$

These results imply that, when applying Lemma E.10, we can choose $\underline{a} = (1/2) \log N$ and $\bar{a} = 8 \log N$. Next, we choose N large enough that we can take $\epsilon_N \leq 1/48$. Then,

$$b_{1,N} + b_{2,N}/(6\log N) \le 3\epsilon_N \le 1/16$$

and

$$b_{1,N} \max_{1 \le i \le N} c_{\alpha_N,N}(\Omega_i) + b_{2,N} \le 8(\log N) (b_{1,N} + b_{2,N}/(6\log N)) \le \underline{a}/2\log N$$

and we can take

$$\left(b_{1,N} \max_{1 \le i \le N} c_{\alpha_N,N}(\Omega_i) + b_{2,N}\right) \vee N^{-1} \le \underline{a}/2\log N$$

for N large enough. Therefore, we may set $\tau = 1$ and

$$\epsilon = \left(b_{1,N} \max_{1 \le i \le N} c_{\alpha_N,N}(\Omega_i) + b_{2,N}\right) \vee N^{-1}$$

in Lemma E.10. This proves (24). Choose N_0 such that $16p\epsilon_N(\log N) \leq 1$ for $N \geq N_0$. This is sufficient to guarantee that the assumptions of Lemma E.22 are satisfied, and therefore, $c_{\alpha,N}(\hat{\Omega}_i) \geq c_{\alpha_N,N}(\Omega_i)$. By Lemma E.6 and Lemma E.7

$$\hat{T}_i^{\text{QLR}} \le (T_i^{\text{QLR}} + b_{2,N})(1 + b_{1,N})$$

for all i = 1, ..., N, and therefore

$$\begin{split} \left\{ \hat{T}_i^{\text{QLR}} > c_{\alpha,N}(\hat{\Omega}_i) \right\} \subset & \left\{ \hat{T}_i^{\text{QLR}} > c_{\alpha_N,N}(\Omega_i) \right\} \\ & \subset \left\{ T_i^{\text{QLR}} > c_{\alpha_N,N} - b_{1,N} c_{\alpha_N,N}(\Omega_i) / (1 + b_{1,N}) - b_{2,N} \right\} \\ & \subset \left\{ T_i^{\text{QLR}} > c_{\alpha_N,N} - b_{1,N} c_{\alpha_N,N}(\Omega_i) - b_{2,N} \right\}. \end{split}$$

Write $c_{N,i} = c_{\alpha_N,N}(\Omega_i) - b_{1,N}c_{\alpha_N,N}(\Omega_i) - b_{2,N}$. Collecting the results from above yields

$$P\left(\exists i \in \{1, \dots, N\} : \hat{T}_i^{\text{QLR}} > c_{\alpha, N}(\hat{\Omega}_i)\right)$$

$$\begin{split} &\leq P\left(\max_{1\leq i\leq N}\left(T_i^{\text{QLR}}-c_{\alpha_N,N}(\Omega_i)+b_{1,N}c_{\alpha_N,N}(\Omega_i)+b_{2,N}\right)>0\right)\\ &\leq P\left(\max_{1\leq i\leq N}\left(U_i-c_{N,i}\right)>0\right)\\ &+\left|P\left(\max_{1\leq i\leq N}\left(U_i-c_{N,i}\right)>0\right)-P\left(\max_{1\leq i\leq N}\left(T_i^{\text{QLR}}-c_{N,i}\right)>0\right)\right|\\ &\leq P\left(\max_{1\leq i\leq N}\left(U_i-c_{\alpha_N,N}(\Omega_i)\right)>0\right)\\ &+P\left(\left|\max_{1\leq i\leq N}U_i-c_{\alpha_N,N}(\Omega_i)\right|\leq \left(b_{1,N}\max_{1\leq i\leq N}c_{\alpha_N,N}(\Omega_i)+b_{2,N}\right)\vee N^{-1}\right)\\ &+\epsilon_N(\log N)\\ &\leq \sum_{i=1}^N P\left(U_i>c_{\alpha_N,N}(\Omega_i)\right)+C_1\left(b_{1,N}\max_{1\leq i\leq N}c_{\alpha_N,N}(\Omega_i)+b_{2,N}\right)\vee C_1N^{-1}+N^{-1}\\ &+\epsilon_N(\log N)\\ &\leq \alpha_N+C\left(\epsilon_N(\log N)+N^{-1}\right). \end{split}$$

where C is a constant that can be chosen to depend only on C_1 and α . The fourth inequality follows from the union bound and the anti-concentration inequality (24). The conclusion follows upon noting that

$$\alpha_N \le \alpha + 96\epsilon_N \log(N/\alpha) \le \alpha + 96\epsilon_N \log(N) \left(1 - \frac{\log \alpha}{\log N}\right)$$

 $\le \alpha + 192\epsilon_N \log(N)$.

Lemma E.4 (Large CLT for QLR statistic). Let P denote a probability measure that satisfies Assumption 1 and imposes cross-sectional independence. Let $\lambda_1 = \min_{i=1}^N \min_{g \in \mathbb{G}} \lambda_1(\Omega_i(g_i^0))$ and suppose that $\lambda_1 > 0$. Then, there are random variables $(U_i)_{1 \leq i \leq N}$ with $U_i \sim \tilde{\chi}^2(\Omega_i(g_i^0))$ such that

$$\begin{split} \sup_{(r_1, \dots, r_N) \in \mathcal{R}_{++}^N} \left| P\bigg(\max_{1 \leq i \leq N} \Big(T_i^{\text{QLR}}(g_i^0) - r_i \Big) > 0 \bigg) - P\bigg(\max_{1 \leq i \leq N} (U_i - r_i) > 0 \bigg) \right| \\ \leq C \left\{ \left(\frac{GB_{N,T,4}^6 \log^7((G-1)NT)}{T} \right)^{1/6} + \left(\frac{GB_{N,T,4}^6 \log^3((G-1)NT)}{\sqrt{T}} \right)^{1/3} \right\}, \end{split}$$

where C is a constant that depends only on λ_1 and G.

Proof. Let $t_i(x) = t_i(x_1, \dots, x_N) = \inf_{s \leq 0} (x_i - s)' \Omega_i^{-1}(x_i - s)$. We first show that, for all r > 0, the set $\{x \in \mathcal{R}^{N(G-1)} : t_i(x) \leq r\}$ is a convex set. Let $S = \{y \in \mathcal{R}^{G-1} : y \leq 0\}$. For $y \in \mathcal{R}^{G-1}$, define $\|y\|_{\Omega^{-1}} = \sqrt{y'\Omega^{-1}y}$ and $d_i(y, S) = \inf_{z \in S} \|y - z\|_{\Omega_i^{-1}}$. Convexity of S and positive definiteness of Ω_i imply that there is a unique \hat{y} such that $d(y, S) = \|y - \hat{y}\|_{\Omega_i^{-1}}$. For $y_1, y_2 \in \mathcal{R}^{G-1}$ and $\lambda \in [0, 1]$

define $y_{\lambda} = \lambda y_1 + (1 - \lambda)y_2$. Define also $y_{\lambda}^* = \lambda \hat{y}_1 + (1 - \lambda)\hat{y}_2$. Then, $y_{\lambda}^* \in S$ and therefore, by the triangle inequality,

$$d(y_{\lambda}, S) \leq \|y_{\lambda} - y_{\lambda}^*\|_{\Omega_i^{-1}} \leq \lambda \|y_1 - \hat{y}_1\|_{\Omega_i^{-1}} + (1 - \lambda)\|y_2 - \hat{y}_2\|_{\Omega_i^{-1}}$$
$$= \lambda d(y_1, S) + (1 - \lambda)d(y_2, S).$$

This proves that, for $r_i \in \mathcal{R}^{(G-1)}$, the set

$$\{x \in \mathcal{R}^{N(G-1)} : t_i(x) \le r_i\} = \{x \in \mathcal{R}^{N(G-1)} : d(x_i, S) \le \sqrt{r_i}\}$$

is convex. For $r_1, \ldots, r_N \in \mathcal{R}_{++}^N$ the set

$$\bigcap_{i=1}^{N} \left\{ x \in \mathcal{R}^{N(G-1)} : t_i(x) \le r_i \right\}$$

is therefore a sparse-convex set, as defined in Chernozhukov, Chetverikov, and Kato (2017). Let

$$Z_{it}(h) = d_{it}(h)/(\sigma_i s_{i,T}(h)).$$

and $Z_{it} = (Z_{it}(h))_{h \in \mathbb{G}\setminus\{g_i^0\}}$. Let $\tilde{X}_t = (\tilde{X}_{1t}, \dots, \tilde{X}_{Nt})'$ with $\dim(\tilde{X}_{it}) = G-1$ for $i=1,\dots,N$, $t=1,\dots,T$ denote a centered normal random vector with the property that \tilde{X}_t and \tilde{X}_s are independent for $t \neq s$ and $\mathbb{E}_P[\tilde{X}_t(\tilde{X}_t)'] = \mathbb{E}_P[Z_t(Z_t)']$ for $i=1,\dots,N,\ t=1,\dots,T$. Condition (M.1") in Chernozhukov, Chetverikov, and Kato (2017) is satisfied with $b=\lambda_1$. Let v denote a vector $v=(v_j)_{j=1}^{N(G-1)}$ with $||v_j||=1$ and $||v_j||_0 \leq (G-1)$. Also, let j(i,h)=(i-1)(G-1)+(h-1) and $v^{(i)}=(v_{j(i,j)})_{h\in\mathbb{G}\setminus\{g_i^0\}}$. Because of cross-sectional independence, we obtain

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{P} \left(\sum_{i=1}^{N} \sum_{h \in \mathbb{G} \setminus \{g_{i}^{0}\}} v_{j(i,h)} Z_{it}(h) \right)^{2}$$

$$= \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{h \in \mathbb{G} \setminus \{g_{i}^{0}\}} \sum_{h' \in \mathbb{G} \setminus \{g_{i}^{0}\}} v_{j(i,h)} v_{j(i,h')} \mathbb{E}_{P} \left[Z_{it}(h) Z_{it}(h') \right]$$

$$= \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} (v^{(i)})' \left[\Omega_{i,t} \right] v^{(i)}$$

$$= \sum_{i=1}^{N} (v^{(i)})' \left[\frac{1}{T} \sum_{t=1}^{T} \Omega_{i,t} \right] v^{(i)} = \sum_{i=1}^{N} (v^{(i)})' \left[\Omega_{i}(g_{i}^{0}) \right] v^{(i)}$$

$$\geq \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \lambda_{1} \left[\Omega_{i} \right] \|v^{(i)}\|^{2} \geq \lambda_{1} \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \|v^{(i)}\|^{2} = \lambda_{1}.$$

This verifies assumption (M.1") in Chernozhukov, Chetverikov, and Kato (2017). Next, by Hölder's

inequality there is a constant $C_1 \geq 1$ depending only on K_{β} such that

$$\frac{1}{T} \sum_{i=1}^{T} \mathbb{E}_{P}[|Z_{it}|^{3}] \leq C (B_{N,T,4}^{4})^{3/4} \leq C_{1} G^{1/2} B_{N,T,4}^{3},$$

$$\frac{1}{T} \sum_{i=1}^{T} \mathbb{E}_{P}[|Z_{it}|^{4}] \leq C B_{N,T,4}^{4} \leq (C_{1} G^{1/2} B_{N,T,4}^{3})^{2}.$$

This allows us to choose $C_1G^{1/2}B_{N,T,4}^3$ as the sequence of constants in assumption (M.2) in Chernozhukov, Chetverikov, and Kato (2017). Lastly, we verify assumption (E.2) in Chernozhukov, Chetverikov, and Kato (2017). To this end, note that

$$\mathbb{E}_{P}\left[\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g_{i}^{0}\}}\left|Z_{it}(h)/(G^{1/4}B_{N,T,4}^{3})\right|^{4}\right]\leq \sum_{h\in\mathbb{G}}\mathbb{E}_{P}\left[\max_{1\leq i\leq N}\left|Z_{it}(h)/(G^{1/4}B_{N,T,4}^{3})\right|^{4}\right]$$

$$\leq G^{2}C_{1}^{2}B_{N,T,4}^{6}/(G^{2}C_{1}^{4}B_{N,T,4}^{12})\leq 1\leq 2,$$

where we used that $B_{N,T,4} \ge 1$. We may now apply Proposition 3.2 in Chernozhukov, Chetverikov, and Kato 2017 to deduce

$$\sup_{(r_1, \dots, r_N) \in \mathcal{R}_{++}^N} \left| P\left(\max_{1 \le i \le N} \left(t_i(D_i) - r_i \right) > 0 \right) - P\left(\max_{1 \le i \le N} (U_i - r_i) > 0 \right) \right| \\
\leq \sup_{(r_1, \dots, r_N) \in \mathcal{R}_{++}^N} \left| P\left(\bigcap_{i=1}^N \left\{ t_i(D_i) \le r_i \right\} \right) - P\left(\bigcap_{i=1}^N \left\{ t_i \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \tilde{X}_{it} \right) \le r_i \right\} \right) \right| \\
\leq C \left\{ \left(\frac{GB_{N,T,4}^6 \log^7((G-1)NT)}{T} \right)^{1/6} + \left(\frac{GB_{N,T,4}^6 \log^3((G-1)NT)}{\sqrt{T}} \right)^{1/3} \right\},$$

where C is a constant that depends only on λ_1 , G and K_{β} . Next, note that

$$t_i \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{X}_{it} \right) = \inf_{t \ge 0} \left(-\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{X}_{it} - t \right)' \Omega_i^{-1} \left(-\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{X}_{it} - t \right).$$

Since $-\sum_{t=1}^{T} \tilde{X}_{it}/\sqrt{T}$ is a zero-mean normal random vector with covariance matrix Ω_i the right-hand side follows a $\tilde{\chi}^2(\Omega_i)$ -distribution.

Lemma E.5 (Large CLT for MAX statistic). Let P denote a probability measure satisfying Assumption 1. For i = 1, ..., N, there are centered normal random vectors X_i with $\mathbb{E}_P[X_i X_i'] = \Omega_i(g_i^0)$ such that

$$\sup_{(r_1,\dots,r_N)\in\mathcal{R}_{++}^N} \left| P\left(\max_{1\leq i\leq N} \left(\max_{h\in\mathbb{G}\setminus\{g_i^0\}} D_i(g_i^0,h) - r_i \right) > 0 \right) - P\left(\max_{1\leq i\leq N} \left(\max_{1\leq h\leq G-1} X_{i,h} - r_i \right) > 0 \right) \right|$$

$$\leq C \left\{ \left(\frac{GB_{N,T,4}^6 \log^7((G-1)NT)}{T} \right)^{1/6} + \left(\frac{GB_{N,T,4}^6 \log^3((G-1)NT)}{\sqrt{T}} \right)^{1/3} \right\},\,$$

where C is a constant depending only on G.

Let
$$Z_{it}(h) = d_{it}(g_i^0, h)/s_{i,T}(g_i^0, h)$$
 and

$$Z_t = ((Z_{1t}(g_1^0, h))_{h \in \mathbb{G} \setminus \{g_1^0\}}, \dots, (Z_{Nt}(g_N^0, h))_{h \in \mathbb{G} \setminus \{g_N^0\}})'.$$

Let $\tilde{X}_t = (\tilde{X}_{1t}, \dots, \tilde{X}_{Nt})'$ with $\dim(\tilde{X}_{it}) = G - 1$ for $i = 1, \dots, N$, $t = 1, \dots, T$ denote a normal random vector with the property that \tilde{X}_t and \tilde{X}_s are independent for $t \neq s$ and $\mathbb{E}_P[\tilde{X}_t(\tilde{X}_t)'] = \mathbb{E}_P[Z_t(Z_t)']$ for $i = 1, \dots, N$, $t = 1, \dots, T$. Define $X_i = \sum_{t=1}^T \tilde{X}_{it}/\sqrt{T}$. Clearly, X_i is a normal random vector with covariance matrix Ω_i . Let $a_i = -\infty$ and $b_i = r_i$. Then we may write

$$\begin{split} \sup_{(r_1,\dots,r_N)\in\mathcal{R}_{++}^N} & \left| P\bigg(\max_{1\leq i\leq N} \bigg(\max_{h\in\mathbb{G}\backslash \{g_i^0\}} D_i(g_i^0,h) - r_i \bigg) > 0 \bigg) \right. \\ & - P\bigg(\max_{1\leq i\leq N} \bigg(\max_{1\leq h\leq G-1} X_{i,h} - r_i \bigg) > 0 \bigg) \bigg| \\ & \leq \sup_{(r_1,\dots,r_N)\in\mathcal{R}_{++}^N} \left| P\bigg(\bigcap_{i=1}^N \bigcap_{h\in\mathbb{G}\backslash \{g_i^0\}} \left\{ a_i < D_i(g_i^0,h) \leq b_i \right\} \right) \right. \\ & - P\bigg(\bigcap_{i=1}^N \bigcap_{h=1}^{G-1} \left\{ a_i < X_{i,h} \leq b_i \right\} \bigg) \bigg| \\ & \leq \sup_{(r_1,\dots,r_N)\in\mathcal{R}_{++}^N} \left| P\bigg(\bigcap_{i=1}^N \bigcap_{h\in\mathbb{G}\backslash \{g_i^0\}} \left\{ a_i < \frac{1}{\sqrt{T}} \sum_{t=1}^T Z_{it}(g_i^0,h) \leq b_i \right\} \right) \\ & - P\bigg(\bigcap_{i=1}^N \bigcap_{h=1}^{G-1} \left\{ a_i < \frac{1}{\sqrt{T}} \sum_{t=1}^T \tilde{X}_{it,h} \leq b_i \right\} \bigg) \bigg| \\ & \leq C \left\{ \left(\frac{GB_{N,T,4}^6 \log((G-1)NT)}{T} \right)^{1/6} + \left(\frac{GB_{N,T,4}^6 \log((G-1)NT)}{\sqrt{T}} \right)^{1/3} \right\}. \end{split}$$

The last inequality holds by Proposition 2.1 in Chernozhukov, Chetverikov, and Kato (2017). Their assumption (M.1) holds trivially with b = 1. As in the proof of Lemma E.4, their assumption (M.2) can be verified for the deterministic sequence $G^{1/4}B_{N,T,4}^3$. Then, their assumption (E.2) holds with q = 4.

Lemma E.6. Suppose that $\Omega_i^{-1}(g)$ is symmetric and positive definite and

$$2\|\Omega_i^{-1}(g)\|_2\|\hat{\Omega}_i(g) - \Omega_i(g)\|_2 \le 1.$$

Then,

$$\hat{T}_{i}^{\text{QLR}}(g) \leq \left(1 + 2\|\Omega(g)\|_{2}\|\Omega_{i}^{-1}(g)\|_{2}^{2}\|\hat{\Omega}_{i}(g) - \Omega_{i}(g)\|_{2}\right) \times \min_{t \leq 0} \left(\hat{D}_{i}(g) - t\right)' \Omega_{i}^{-1}(g) \left(\hat{D}_{i}(g) - t\right).$$

Proof. For brevity, we write $\hat{D} = \hat{D}_i(g)$, $\hat{\Omega} = \hat{\Omega}_i(g)$ and $\Omega = \Omega_i(g)$ and

$$\hat{T}_i^{\Omega} = \min_{t \le 0} (\hat{D} - t)' \Omega^{-1} (\hat{D} - t).$$

Let $t^* \in \mathcal{R}^p$ such that $t^* \leq 0$ and $\hat{T}_i^{\Omega} = (\hat{D} - t^*)'\Omega^{-1}(\hat{D} - t^*)$. By definition

$$\hat{T}_{i}^{\text{QLR}} \leq (\hat{D} - t^{*})' \hat{\Omega}^{-1} (\hat{D} - t^{*})$$

$$\leq \hat{T}_{i}^{\Omega} + |(\hat{D} - t^{*})'[\hat{\Omega}^{-1} - \Omega^{-1}](\hat{D} - t^{*})|.$$

Let $0 < \lambda_{i,1} \le \cdots \le \lambda_{i,p}$ denote the eigenvalues of Ω and note that $\lambda_{i,p}^{-1} = \|\Omega\|_2^{-1}$. Let $\Omega^{-1} = P\Lambda P'$, where P is an orthogonal matrix and Λ is a diagonal matrix with diagonal entries $(\lambda_{i,j}^{-1})_{j=1}^p$. Then,

$$\begin{split} (\hat{D} - t^*)' \Omega^{-1} (\hat{D} - t^*) &= \sum_{j=1}^p \lambda_{i,j}^{-1} \left(P' (\hat{D} - t^*) \right)^2 \\ &\geq \min_{1 < j < p} \{ \lambda_{i,j}^{-1} \} \| P' (\hat{D} - t^*) \|^2 = \| \Omega \|_2^{-1} \| \hat{D} - t^* \|^2. \end{split}$$

Therefore,

$$\left| (\hat{D} - t^*)' \big[\hat{\Omega}^{-1} - \Omega^{-1} \big] (\hat{D} - t^*) \right| \leq \|\hat{D} - t^*\|^2 \|\hat{\Omega}^{-1} - \Omega^{-1}\|_2 \leq \|\Omega\|_2 \|\hat{\Omega}^{-1} - \Omega^{-1}\|_2 \hat{T}_i^{\Omega}.$$

By combining inequalities, we obtain

$$\hat{T}_i^{\text{QLR}} \leq \hat{T}_i^{\Omega} \left(1 + \|\Omega\|_2 \|\hat{\Omega}^{-1} - \Omega^{-1}\|_2 \right) \leq \hat{T}_i^{\Omega} \left(1 + 2\|\Omega\|_2 \|\Omega^{-1}\|_2^2 \|\hat{\Omega} - \Omega\|_2 \right),$$

where the last inequality holds by Lemma E.15.

Lemma E.7. Suppose that $\|\hat{D}_i(g) - D_i(g)\| \le 1$. Then

$$|T_i^{\text{QLR}}(g) - \min_{t \le 0} (\hat{D}_i(g) - t)' \Omega_i^{-1}(g) (\hat{D}_i(g) - t)|$$

$$\le (2||D_i(g)|| + 3) ||\Omega_i^{-1}(g)||_2 ||\hat{D}_i(g) - D_i(g)||.$$

Proof. For brevity, we write $D = D_i(g)$, $\hat{D} = \hat{D}_i(g)$ and $\Omega = \Omega_i(g)$ and define

$$\hat{T}_i^{\Omega} = \min_{t \le 0} \left(\hat{D}_i(g) - t \right)' \Omega_i^{-1}(g) \left(\hat{D}_i(g) - t \right).$$

Write $||v||_{\Omega^{-1}} = \sqrt{v'\Omega^{-1}v}$ and note that $||\cdot||_{\Omega^{-1}}$ defines a vector norm and $||v||_{\Omega^{-1}} \le ||v|| \cdot ||\Omega^{-1}||_2^{1/2}$. By the triangle inequality,

$$\begin{split} \sqrt{\hat{T}_i^{\Omega}} &= \min_{t \leq 0} \lVert \hat{D} - t \rVert_{\Omega^{-1}} \leq \lVert \hat{D} - D \rVert_{\Omega^{-1}} + \min_{t \leq 0} \lVert D - t \rVert_{\Omega^{-1}} \\ &= \lVert \hat{D} - D \rVert_{\Omega^{-1}} + \sqrt{T_i^{\text{QLR}}} \leq \lVert \hat{D} - D \rVert \sqrt{\lVert \Omega^{-1} \rVert_2} + \sqrt{T_i^{\text{QLR}}}. \end{split}$$

Taking squares and using $T_i^{\text{QLR}} \leq \|\Omega^{-1}\|_2 \|D\|^2$ and $\|\hat{D} - D\| \leq 1$ gives

$$\hat{T}_i^{\Omega} \le T_i^{\text{QLR}} + (2\|D\| + 1)\|\Omega^{-1}\|_2 \|\hat{D} - D\|.$$

Reversing the roles of \hat{D} and D gives

$$\begin{split} T_i^{\text{QLR}} \leq & \hat{T}_i^{\Omega} + (2\|\hat{D}\| + 1)\|\Omega^{-1}\|_2 \|\hat{D} - D\| \\ \leq & \hat{T}_i^{\Omega} + (2\|D\| + 2\|\hat{D} - D\| + 1)\|\Omega^{-1}\|_2 \|\hat{D} - D\| \\ \leq & \hat{T}_i^{\Omega} + (2\|D\| + 3)\|\Omega^{-1}\|_2 \|\hat{D} - D\|, \end{split}$$

where the third inequality follows by $\|\hat{D} - D\| \le 1$. The assertion follows by combining the inequalities.

Lemma E.8. Suppose that the probability measure P satisfies Assumption 1. For $h, h' \in \mathbb{G} \setminus \{g_i^0\}$ let

$$(\Omega_i^*)_{h,h'} = \frac{T^{-1} \sum_{t=1}^T (d_{it}(g_i^0, h) - \bar{d}_i(g_i^0, h))(d_{it}(g_i^0, h') - \bar{d}_i(g_i^0, h'))}{\sigma_i^2 s_{i,T}(g_i^0, h) s_{i,T}(g_i^0, h')}.$$

There is a constant C depending only on K_{β} and G such that for 0 < c < 1

(i)
$$P\left(\max_{1\leq i\leq N} \max_{h,h'\in\mathbb{G}\backslash\{g_i^0\}} \left| (\Omega_i^*)_{h,h'} - (\Omega_i(g_i^0))_{h,h'} \right| \right. \\ \left. > CT^{-(1-c)/2} (\log N) B_{N,T,4}^2 \right) \leq CT^{-c},$$
(ii)
$$P\left(T^{-1/2} \max_{1\leq i\leq N} \left| D_i(g_i^0,h) \right| > C\left(T^{-1/2} \sqrt{\log N} + T^{-3/4} B_{N,T,4} \log N\right) \right) \\ \leq N^{-1} + C\left(T^{-1/4} B_{N,T,4} / \log(N)\right)^4.$$

Proof.

Proof of (i) Decompose

$$\frac{T^{-1} \sum_{t=1}^{T} (d_{it}(h) - \bar{d}_{i}(h))(d_{it}(h') - \bar{d}_{i}(h'))}{\sigma_{i}^{2} s_{i,T}(h) s_{i,T}(h')} - (\Omega_{i})_{h,h'}$$

$$= \frac{T^{-1} \sum_{t=1}^{T} \left(d_{it}(h) d_{it}(h') - \mathbb{E}_{P}[d_{it}(h) d_{it}(h')] \right)}{\sigma_{i}^{2} s_{i,T}(h) s_{i,T}(h')} - \left(\frac{\bar{d}_{i}(h)}{\sigma_{i} s_{i,T}(h)} \right) \left(\frac{\bar{d}_{i}(h')}{\sigma_{i} s_{i,T}(h')} \right).$$

Below, we show that

$$P\left(\max_{1\leq i\leq N} \left| \frac{T^{-1} \sum_{t=1}^{T} (d_{it}(h)d_{it}(h') - \mathbb{E}_{P}[d_{it}(h)d_{it}(h')])}{\sigma_{i}^{2} s_{i,T}(h) s_{i,T}(h')} \right| > C_{1} B_{N,T,4}^{2} T^{-(1-c)/2} \log N \right) \leq 2T^{-c},$$
(25)

$$P\left(\max_{1\leq i\leq N} \left| \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_{i} s_{i,T}(h)} \right| > C_{2} \left(T^{-1/4} \sqrt{\log N} + T^{-3/4} \log(N) \right) B_{N,T,4} \right) \leq 2T^{-2}$$
 (26)

where C_1 and C_2 are constants that depend only on K_{β} . For

$$x = (2C_1 \vee C_2)T^{-(1-c)/2}(\log N)B_{N,T,4}^2(P) + C_2T^{-3/2}(\log^2 N)B_{N,T,4}^2(P)$$

and a constant C_3 depending only on K_β and G

$$P\left(\left|\frac{T^{-1}\sum_{t=1}^{T}(d_{it}(h)d_{it}(h') - \mathbb{E}_{P}[d_{it}(h)d_{it}(h')])}{\sigma_{i}^{2}s_{i,T}(h)s_{i,T}(h')} - \left(\frac{\bar{d}_{i}(h)}{\sigma_{i}s_{i,T}(h)}\right)\left(\frac{\bar{d}_{i}(h')}{\sigma_{i}s_{i,T}(h')}\right)\right| > 2x^{2}\right)$$

$$\leq P\left(\left|\frac{T^{-1}\sum_{t=1}^{T}(d_{it}(h)d_{it}(h') - \mathbb{E}_{P}[d_{it}(h)d_{it}(h')])}{\sigma_{i}^{2}s_{i,T}(h)s_{i,T}(h')}\right| > x^{2}\right) + \sum_{h \in \mathbb{G}} P\left(\left|\frac{\bar{d}_{i}(h)}{\sigma_{i}s_{i,T}(h)}\right| > x\right)$$

$$\leq C_{3}T^{-c},$$

where the last inequality follows from (25) and (26). The assertion of the lemma follows. It remains to establish the inequalities (25) and (26). Write

$$U_{it}(h, h') = (d_{it}(h)d_{it}(h') - \mathbb{E}_P[d_{it}(h)d_{it}(h')]) / (\sigma_i^2 s_{i,T}(h)s_{i,T}(h')).$$

By

$$\mathbb{E}_{P}[U_{it}(h,h')^{2}] \leq \max_{1 \leq t \leq T} \mathbb{E}_{P} \left[\max_{1 \leq i \leq N} |d_{it}(h)d_{it}(h')|^{2}/\sigma_{i}^{2} \right] / \left(s_{i,T}^{2}(h)s_{i,T}^{2}(h') \right)$$

$$\leq \max_{1 \leq t \leq T} \mathbb{E}_{P} \left[\max_{1 \leq i \leq N} \left(|u_{it}/\sigma_{i}|^{4} \|x_{it}\|^{4} \|\delta_{t}(g_{i}^{0},h)\|^{2} \|\delta_{t}(g_{i}^{0},h)\|^{2} \right) \right] / \underline{s}_{N,T}^{4}(P)$$

$$\leq 16K_{\beta}^{2} B_{N,T,4}^{4}(P)$$

we have $\mathbb{E}[\max_{1 \leq i \leq N} \max_{1 \leq i \leq N} |U_{it}(h, h')|^2] \leq 16K_{\beta}^2 T B_{N,T,4}^4$ and $\mathbb{E}[U_{it}^2(h, h')] \leq 16K_{\beta}^2 B_{N,T,4}^4$. By Lemma D.3 in Chernozhukov, Chetverikov, and Kato (2018) there is a universal constant K such

that for $C_4 = 32K_\beta^2 K$

$$\mathbb{E}_{P}\left[\max_{1\leq i\leq N}\left|\frac{1}{T}\sum_{t=1}^{T}(U_{it}(h,h')-\mathbb{E}_{P}[U_{it}(h,h')])\right|\right]\leq C_{4}B_{N,T,4}^{2}(\log N)/\sqrt{T}.$$

Thus, by Lemma D.2 in Chernozhukov, Chetverikov, and Kato (2018) for every r > 0 and a universal constant K_2

$$P\left(\max_{1\leq i\leq N} \left| U_{it}(h,h') - \mathbb{E}_P[U_{it}(h,h')] \right| \geq 2CB_{N,T,4}^2(\log N)/\sqrt{T} + r\right)$$

$$\leq e^{-Tr^2/(48K_\beta^2 B_{N,T,4}^4)} + K_2 16K_\beta^2 r^{-2} T^{-1} B_{N,T,4}^4.$$

Taking $r = C_1 T^{-(1-c)/2} B_{N,T,4}^2$ for 0 < c < 1 and $C_1 = 4(\sqrt{K_2} + \sqrt{3}) K_\beta \vee C$ then yields

$$P\left(\frac{T^{-1}\sum_{t=1}^{T} \left(d_{it}(h)d_{it}(h') - \mathbb{E}_{P}[d_{it}(h)d_{it}(h')]\right)}{\sigma_{i}^{2}s_{i,T}(h)s_{i,T}(h')} > C_{1}B_{N,T,4}^{2}T^{-(1-c)/2}\log N\right) \leq 2T^{-c}.$$

By Hölder's inequality

$$\mathbb{E}_{P}\left[\max_{1\leq i\leq N}\max_{1\leq t\leq T}\left|\frac{d_{it}(h)}{\sigma_{i}s_{i,T}(h)}\right|^{2}\right]\leq \sqrt{\mathbb{E}_{P}\left(\max_{1\leq i\leq N}\max_{1\leq t\leq T}\left|\frac{d_{it}(h)}{\sigma_{i}s_{i,T}(h)}\right|^{4}\right)}\leq \sqrt{T}4K_{\beta}B_{N,T,4}^{2}.$$

Thus, by Lemma D.3 in Chernozhukov, Chetverikov, and Kato (2018) for a universal constant K

$$\mathbb{E}_P\left[\max_{1\leq i\leq N}\left|\frac{1}{T}\sum_{t=1}^T\frac{d_{it}(h)}{\sigma_i s_{i,T}(h)}\right|\right]\leq K\left(T^{-1/2}\sqrt{\log N}+2T^{-3/4}\sqrt{K_\beta}B_{N,T,4}\log N\right).$$

Then, by Lemma A.2 in Chernozhukov, Chetverikov, and Kato (2018) for all r > 0 and a universal constant K_4

$$P\left(\max_{1 \le i \le N} \left| \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_{i} s_{i,T}(h)} \right| > 2K \left(T^{-1/2} \sqrt{\log N} + 2T^{-3/4} \sqrt{K_{\beta}} B_{N,T,4} \log N \right) + r \right)$$

$$\leq e^{-Tr^{2}/3} + K_{4} r^{-4} T^{-3} B_{N,T,4}^{4}.$$
(27)

Now, taking $r=2\sqrt{K_{\beta}}K_4^{1/4}T^{-1/4}B_{N,T,4}$ and noting that $B_{N,T,4}\geq 1$ yields

$$P\left(\max_{1\leq i\leq N} \left| \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_{i} s_{i,T}(h)} \right| > C_{2} \left(T^{-1/2} \sqrt{\log N} + T^{-3/4} \log(N) \right) B_{N,T,4} \right) \leq 2T^{-2},$$

where C_2 is a constant that depends only on K_{β} .

Proof of (ii): Taking $r = 3T^{-1/2}\sqrt{\log N}$ in (27) gives

$$P\left(\max_{1 \le i \le N} \left| \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_i s_{i,T}(h)} \right| > C_5 \left(T^{-1/2} \sqrt{\log N} + T^{-3/4} (\log N) B_{N,T,4} \right) \right)$$

$$\le N^{-1} + \left(T^{-1/4} B_{N,T,4} / (\log N) \right)^4.$$

Lemma E.9. Suppose that the probability measure P satisfies Assumption 1. Then, there is a constant C depending only on K_{β} and G such that for 0 < c < 1 and

$$\zeta_{N,T} = \gamma_{N,T,8} \left(T^{-(1-c)/4} \sqrt{\log N} B_{N,T,8}^2 + D_{N,T,4} \right) \left(1 + T^{-(1-c)/4} \sqrt{\log N} B_{N,T,4} \right)$$
$$+ \gamma_{N,T,8}^2 \left(T^{-(1-c)/2} (\log N) B_{N,T,8}^4 + D_{N,T,4}^2 \right),$$

we have

$$(i) \qquad P\left(\max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^{T} \frac{(\hat{d}_{it}(g_{i}^{0}, h) - \bar{\hat{d}}_{i}(g_{i}^{0}, h))(\hat{d}_{it}(g_{i}^{0}, h') - \bar{\hat{d}}_{i}(g_{i}^{0}, h'))}{\sigma_{i}^{2} s_{i,T}(g_{i}^{0}, h) s_{i,T}(g_{i}^{0}, h')} - \frac{1}{T} \sum_{t=1}^{T} \frac{(d_{it}(g_{i}^{0}, h) - \bar{d}_{i}(g_{i}^{0}, h))(d_{it}(g_{i}^{0}, h') - \bar{d}_{i}(g_{i}^{0}, h'))}{\sigma_{i}^{2} s_{i,T}(g_{i}^{0}, h) s_{i,T}(g_{i}^{0}, h')} \right| \\ > C\zeta_{N,T}) \leq CT^{-c},$$

$$(ii) \qquad P\left(T^{-1/2} \max_{1 \leq i \leq N} \left| D_{i}(g_{i}^{0}, h) - \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\hat{d}_{i}(g_{i}^{0}, h)}{\sigma_{i} s_{i,T}(g_{i}^{0}, h)} \right| \\ > C\gamma_{N,T,8} \left(T^{-(1-c)/4} B_{N,T,4} \sqrt{\log N} + D_{N,T,2}\right) \right) \leq CT^{-c}.$$

Suppose that, additionally, $\zeta_{N,T} \vee T^{-(1-c)/4} \sqrt{\log N} B_{N,T,4} \leq 1$. Then,

(iii)
$$P\left(\max_{1\leq i\leq N} \left| \hat{D}_{i}(h) - \tilde{D}_{i}(h) \right| > C\gamma_{N,T,8}\sqrt{T} \left(T^{-(1-c)/4}B_{N,T,4}\sqrt{\log N} + D_{N,T,2} \right) + C\zeta_{N,T} \left(1 + T^{-1/4}B_{N,T,4}\sqrt{\log N} \right) \sqrt{\log N} \right)$$

$$\leq N^{-1} + CT^{-c} + C\left(T^{-1/4}B_{N,T,4}/\log(N) \right)^{4},$$
(iv)
$$P\left(\max_{1\leq i\leq N} \left| \hat{D}_{i}(h) - D_{i}(h) \right| > C\sqrt{T}\gamma_{N,T,8} \left(T^{-(1-c)/4}B_{N,T,4}\sqrt{\log N} + D_{N,T,2} \right) + C\left(\zeta_{N,T} + T^{-(1-c)/2}(\log N)B_{N,T,4}^{2} \right) \times \left(1 + T^{-1/4}B_{N,T,4}\sqrt{\log N} \right) \sqrt{\log N}$$

$$\leq N^{-1} + CT^{-c} + C\left(T^{-1/4}B_{N,T,4}/\log(N) \right)^{4}.$$

Proof. Throughout the proof, let C denote a generic constant that depends only on K_{β} and G.

Proof of (i): Bound as follows

$$\left| \frac{1}{T} \sum_{t=1}^{T} (\hat{d}_{it}(h) - \bar{d}_{i}(h))(\hat{d}_{it}(h') - \bar{d}_{i}(h')) - \frac{1}{T} \sum_{t=1}^{T} (d_{it}(h) - \bar{d}_{i}(h))(d_{it}(h') - \bar{d}_{i}(h')) \right| \\
\leq \left| \frac{1}{T} \sum_{t=1}^{T} (\hat{d}_{it}(h) - d_{it}(h) - (\bar{d}_{i}(h) - \bar{d}_{i}(h)))(\hat{d}_{it}(h') - d_{it}(h') - (\bar{d}_{i}(h') - \bar{d}_{i}(h'))) \right| \\
+ \left| \frac{1}{T} \sum_{t=1}^{T} (d_{it}(h) - \bar{d}_{i}(h))(\hat{d}_{it}(h') - d_{it}(h') - (\bar{d}_{i}(h') - \bar{d}_{i}(h'))) \right| \\
+ \left| \frac{1}{T} \sum_{t=1}^{T} (d_{it}(h') - \bar{d}_{i}(h'))(\hat{d}_{it}(h) - d_{it}(h) - (\bar{d}_{i}(h) - \bar{d}_{i}(h))) \right| \\
\leq \sqrt{\frac{1}{T} \sum_{t=1}^{T} (\hat{d}_{it}(h) - d_{it}(h))^{2} \sqrt{\frac{1}{T} \sum_{t=1}^{T} (\hat{d}_{it}(h') - d_{it}(h'))^{2}} \\
+ \sqrt{\frac{1}{T} \sum_{t=1}^{T} (d_{it}(h) - \bar{d}_{i}(h))^{2} \sqrt{\frac{1}{T} \sum_{t=1}^{T} (\hat{d}_{it}(h') - d_{it}(h'))^{2}} \\
+ \sqrt{\frac{1}{T} \sum_{t=1}^{T} (d_{it}(h') - \bar{d}_{i}(h'))^{2} \sqrt{\frac{1}{T} \sum_{t=1}^{T} (\hat{d}_{it}(h) - d_{it}(h))^{2}}}.$$

Therefore,

$$\left| \frac{1}{T} \sum_{t=1}^{T} \frac{(\hat{d}_{it}(h) - \bar{d}_{i}(h))(\hat{d}_{it}(h') - \bar{d}_{i}(h'))}{\sigma_{i}^{2} s_{i,T}(h) s_{i,T}(h')} - \frac{1}{T} \sum_{t=1}^{T} \frac{(d_{it}(h) - \bar{d}_{i}(h))(d_{it}(h') - \bar{d}_{i}(h'))}{\sigma_{i}^{2} s_{i,T}(h) s_{i,T}(h')} \right| \\
\leq \sqrt{\frac{1}{T} \sum_{t=1}^{T} \frac{(\hat{d}_{it}(h) - d_{it}(h))^{2}}{\sigma_{i}^{2} s_{N,T}^{2}}} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \frac{(\hat{d}_{it}(h') - d_{it}(h'))^{2}}{\sigma_{i}^{2} s_{N,T}^{2}}} \\
+ \max_{h,h' \in \mathbb{G}} 2\sqrt{\frac{1}{T} \sum_{t=1}^{T} \frac{(d_{it}(h) - \bar{d}_{i}(h))^{2}}{\sigma_{i}^{2} s_{i,T}^{2}(h)}} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \frac{(\hat{d}_{it}(h') - d_{it}(h'))^{2}}{\sigma_{i}^{2} s_{N,T}^{2}}} \\
\leq \gamma_{N,T,8} \left(T^{-(1-c)/4} \sqrt{\log N} B_{N,T,8}^{2} + D_{N,T,4}\right) \left(1 + T^{-(1-c)/4} \sqrt{\log N} B_{N,T,4}\right) \\
+ \gamma_{N,T,8}^{2} \left(T^{-(1-c)/2}(\log N) B_{N,T,8}^{4} + D_{N,T,4}^{2}\right).$$

where the last inequality follows from Lemma E.8 and

$$P\left(\frac{1}{T}\sum_{t=1}^{T} \left(\frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_{i}s_{i,T}(h)}\right)^{2} > C\gamma_{N,T,8}^{2} \left(T^{-(1-c)/2}B_{N,T,8}^{4}(\log N) + D_{N,T,4}^{2}\right)\right) \leq CT^{-c}.$$
(28)

We now prove (28). For the following calculations note that

$$\|\hat{\delta}(g_i^0, h) - \delta(g_i^0, h)\|^2 \le 2\left(\|\hat{\beta}_{g_i^0} - \beta_{g_i^0}\|^2 + \|\hat{\beta}_h - \beta_h\|^2\right) \le 4 \max_{g \in \mathbb{G}} \|\hat{\beta}_g - \beta_g\|^2$$

and, since the matrix norm $\|\cdot\|_2$ is an induced norm and $\|x_{it}\| = \sqrt{x'_{it}x_{it}}$,

$$||x_{it}x'_{it}||_2 = \sup_{||y||=1} ||x_{it}x'_{it}y|| \le \frac{||x_{it}x'_{it}x_{it}||}{||x_{it}||} = ||x_{it}||^2.$$

Decompose $\hat{d}_{it}(h) - d_{it}(h)$ as follows

$$\hat{d}_{it}(h) - d_{it}(h)
= -u_{it}x'_{it}(\hat{\delta}_t(g_i^0, h) - \delta_t(g_i^0, h))
+ (\hat{\beta}_{q_i^0, t} - \beta_{q_i^0, t})'(x_{it}x'_{it})(\hat{\delta}_t(g_i^0, h) - \delta_t(g_i^0, h)) + (\hat{\beta}_{q_i^0, t} - \beta_{q_i^0, t})'(x_{it}x'_{it})\delta_t(g_i^0, h).$$

By the inequality $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$,

$$\left(\frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_{i}}\right)^{2} \leq 3 \left|\frac{u_{it}}{\sigma_{i}}\right|^{2} \|x_{it}\|^{2} \|\hat{\delta}_{t}(g_{i}^{0}, h) - \delta_{t}(g_{i}^{0}, h)\|^{2}
+ 3\sigma_{i}^{-2} \|\hat{\beta}_{g_{i}^{0}, t} - \beta_{g_{i}^{0}, t}\|^{2} \|x_{it}\|^{4} \|\hat{\delta}_{t}(g_{i}^{0}, h) - \delta_{t}(g_{i}^{0}, h)\|^{2}
+ 3\sigma_{i}^{-2} \|\hat{\beta}_{g_{i}^{0}, t} - \beta_{g_{i}^{0}, t}\|^{2} \|x_{it}\|^{4} \|\delta_{t}(g_{i}^{0}, h)\|^{2}.$$

Let $V_{it} = \left(|u_{it}/\sigma_i|^2 ||x_{it}||^2 + ||x_{it}||^4 / \sigma_i^4 \right) / \underline{s}_{N,T}^2$. Below, we show that for 0 < c < 1

$$P\left(\max_{1\leq i\leq N} \left| \frac{1}{T} \sum_{t=1}^{T} \left(V_{it}^{2} - \mathbb{E}_{P}[V_{it}^{2}] \right) \right| > CT^{-(1-c)/2} B_{N,T,8}^{4}(\log N) \right) \leq CT^{-c}.$$
 (29)

Now, by the Cauchy-Schwarz inequality

$$\frac{1}{T} \sum_{t=1}^{T} \left(\frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_{i} s_{i,T}(h)} \right)^{2}$$

$$\leq C \left\{ \max_{g \in \mathbb{G}} \left(\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{\beta}_{g,t} - \beta_{g,t} \right\|^{4} \right)^{1/2} + \max_{g \in \mathbb{G}} \left(\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{\beta}_{g,t} - \beta_{g,t} \right\|^{8} \right)^{1/2} \right\}$$

$$\times \left(\frac{1}{T} \sum_{t=1}^{T} \left(\left| \frac{u_{it}}{\sigma_{i}} \right|^{4} \left\| x_{it} \right\|^{4} + \left\| x_{it} \right\|^{8} / \sigma_{i}^{4} \right) / \underline{s}_{N,T}^{4} \right)^{1/2}$$

$$\leq C (\gamma_{N,T,8}^{2} + \gamma_{N,T,8}^{4}) \left(\frac{1}{T} \sum_{t=1}^{T} \left(V_{it}^{2} - \mathbb{E}_{P} \left[V_{it}^{2} \right] \right) + \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{P} \left[V_{it}^{2} \right] \right)^{1/2}.$$

Together with (29) this implies (28). It remains to prove (29). Note that $\mathbb{E}_P[V_{it}^2] \leq B_{N,T,8}^8$ and $\mathbb{E}_P[\max_{1\leq i\leq T} \max_{1\leq i\leq N} V_{it}^2] \leq TB_{N,T,8}^8$. By Lemma D.3 in Chernozhukov, Chetverikov, and Kato (2018) there is a universal constant K such that

$$\mathbb{E}_{P}\left[\max_{1 \le i \le N} \left| \frac{1}{T} \sum_{t=1}^{T} (V_{it}^{2} - \mathbb{E}_{P}[V_{it}^{2}]) \right| \right] \le K B_{N,T,8}^{4} \frac{\log N}{\sqrt{T}}.$$

Then, by Lemma A.2 in Chernozhukov, Chetverikov, and Kato (2018) for every r > 0

$$P\left(\max_{1 \le i \le N} \left| \frac{1}{T} \sum_{t=1}^{T} (V_{it}^2 - \mathbb{E}_P[V_{it}^2]) \right| > 2KB_{N,T,4}^2 \log N / \sqrt{T} + r \right)$$

$$\le e^{-Tr^2/(3B_{N,T,8}^8)} + Kr^{-2}T^{-1}B_{N,T,8}^8.$$

Then, taking $r = T^{-(1-c)/2}B_{N,T,8}^4$ for 0 < c < 1 yields (29).

Proof of (ii): By slightly modifying the arguments above, we can prove

$$\left| \frac{1}{T} \sum_{t=1}^{T} \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_{i} s_{i,T}(h)} \right| \\
\leq C(\gamma_{N,T,4} + \gamma_{N,T,4}^{2}) \left(\frac{1}{T} \sum_{t=1}^{T} \left(V_{it} - \mathbb{E}_{P} [V_{it}] \right) + \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{P} [V_{it}] \right)^{1/2}.$$

In addition, for 0 < c < 1,

$$P\left(\max_{1 \le i \le N} \left| \frac{1}{T} \sum_{t=1}^{T} \left(V_{it} - \mathbb{E}_{P}[V_{it}] \right) \right| > CT^{-(1-c)/2} B_{N,T,4}^{2}(\log N) \right) \le CT^{-c}$$

from whence the conclusion follows.

Proof of (iii): Define

$$S_{i,T}^{\Delta}(h) = \left(\frac{\hat{S}_{i,T}(h) - S_{i,T}(h)}{\sigma_i s_{i,T}(h)}\right) \frac{S_{i,T}(h)}{\sigma_i s_{i,T}(h)}.$$

By the inequality $|a-b| \leq |a-b|/(\sqrt{a}+\sqrt{b}) \leq |a-b|/\sqrt{a}$ and (i) of the lemma we have

$$S_{i,T}^{\Delta}(h) \le \left| \left(\hat{S}_{i,T}^2(h) / (\sigma_i s_{i,T}(h)) \right)^2 - \left(S_{i,T}^2(h) / (\sigma_i s_{i,T}(h)) \right)^2 \right| \le C_2 \zeta_{N,T}$$

uniformly over $i=1,\ldots,N$ on a set of probability less than CT^{-c} . By the inequality $|\sqrt{a}-1| \le |a-1|$ and Lemma E.8 we have

$$\left| S_{i,T}^2(h)/(\sigma_i s_{i,T}(h)) - 1 \right| \le \left| \left(S_{i,T}^2(h)/(\sigma_i s_{i,T}(h)) \right)^2 - 1 \right| \le C_1 T^{-(1-c)/2} (\log N) B_{N,T,4}^2$$

uniformly over i = 1, ..., N on a set of probability less than CT^{-c} . By Lemma E.8(ii)

$$\left| D_i(g_i^0, h) \right| \le C \left(\sqrt{\log N} + T^{-1/4} B_{N,T,4} \log N \right)$$

uniformly over i = 1, ..., N on a set of probability less than $N^{-1} + C(T^{-1/4}B_{N,T,4}/\log(N))^4$. Now, decompose

$$\hat{D}_{i}(h) - \tilde{D}_{i}(h) = \frac{\sigma_{i}s_{i,T}(h)}{\hat{S}_{i,T}(h)} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\hat{d}_{i}(g_{i}^{0}, h)}{\sigma_{i}s_{i,T}(g_{i}^{0}, h)} - D_{i}(g_{i}^{0}, h) \right)$$

$$- \frac{\hat{S}_{i,T} - S_{i,T}}{S_{i,T}(h)\hat{S}_{i,T}(h)} \sigma_{i}s_{i,T}(h)D_{i}(h)$$

$$= \frac{S_{i,T}(h)/(\sigma_{i}s_{i,T}(h))}{S_{i,T}^{\Delta}(h) + S_{i,T}(h)/(\sigma_{i}s_{i,T}(h))} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\hat{d}_{i}(g_{i}^{0}, h)}{\sigma_{i}s_{i,T}(g_{i}^{0}, h)} - D_{i}(g_{i}^{0}, h) \right)$$

$$- \frac{S_{i,T}^{\Delta}}{\left(S_{i,T}^{\Delta} + S_{i,T}^{2}(h)/(\sigma_{i}s_{i,T}(h))^{2}\right) S_{i,T}^{2}(h)/(\sigma_{i}s_{i,T}(h))} D_{i}(h).$$

In conjunction with part (ii) of the lemma, this decomposition implies

$$\max_{1 \le i \le N} \left| \hat{D}_i(h) - \tilde{D}_i(h) \right| \le C\gamma_{N,T,8} \sqrt{T} \left(T^{-(1-c)/4} B_{N,T,4} \sqrt{\log N} + D_{N,T,4}^2 \right) + C\zeta_{N,T} \left(\sqrt{\log N} + T^{-1/4} B_{N,T,4} \log N \right)$$

with probability less than $CT^{-c} + N^{-1} + C(T^{-1/4}B_{N,T,4}/\log(N))^4$.

Proof of (iv): Write

$$\tilde{D}_i(h) - D_i(h) = -(S_{i,T}/s_{i,T} - 1)(S_{i,T}/s_{i,T})^{-1}D_i(h).$$

The bounds derived in the proof of part (iii) imply

$$\max_{1 \le i \le N} \left| \tilde{D}_i(h) - D_i(h) \right| \le C_1 T^{-(1-c)/2} (\log N) B_{N,T,4}^2 \left(\sqrt{\log N} + T^{-1/4} B_{N,T,4} \log N \right)$$

with probability less than $CT^{-c} + N^{-1} + C(T^{-1/4}B_{N,T,4}/\log(N))^4$. The conclusion now follows from the triangle inequality and part (iii) of the lemma.

Lemma E.10. (Simultaneous anti-concentration) Let $\{V_i\}_{i=1}^N$ denote a collection of nonsingular $(p \times p)$ -variance matrices, and let $\{W_i\}_{i=1}^N$ denote a collection of independent random variables with marginal distribution $W_i \sim \tilde{\chi}^2(V_i)$. For positive constants \underline{a} and \bar{a} , let $\mathbb{S} = [\underline{a} \log N, \bar{a} \log N]$. Then, for each $\tau > 0$ there are constants C and N_0 that depend only on \underline{a} , \bar{a} , τ and p such that for all ϵ

with $N^{-\tau} < \epsilon < \underline{a}/2(\log N)$ we have

$$\sup_{(s_1,\dots,s_N)\in\mathbb{S}^N} P\left(|\max_{1\leq i\leq N} W_i - s_i| \leq \epsilon\right) \leq C\epsilon + N^{-1}.$$

Proof. For a collection of nonsingular $p \times p$ covariance matrices $(V_i)_{i=1}^N$ let $W_i \sim \tilde{\chi}^2(V_i)$. Let $(U_{i,j})_{j=1}^k$ denote a collection of chi-squared random variables with $U_{i,j} \sim \chi_j^2$ and $U_{i,j} \perp U_{i,k}$ for $j \neq k$. Let \bar{W}_i denote the random function $\bar{W}_i(d) = \sum_{j=1}^p 1\{d=j\}U_{i,j}$. and let $(D_i)_{i=1}^N$ denote random variables that are supported on $\{0,\ldots,p\}$ and satisfy $P(D_i=d)=w(p,p-d,V_i)$ for $d=0,\ldots,p$. This construction ensures that $\mathcal{L}(W_i)=\mathcal{L}(\bar{W}_i(D_i))$. Let $(D_i^*)_{i=1}^N$ denote random variables that are supported on $\{1,\ldots,p\}$ and satisfy $P(D_i^*=1)=w(p,p,V_i)+w(p,p-1,V_i)$ and $P(D_i^*=d)=w(p,p-d,V_i)$ for $d=0,\ldots,p$ and define $W_i^*=\bar{W}_i(D_i^*)$. For $\epsilon<\underline{a}/2(\log N)$,

$$P\left(\left|\max_{1\leq i\leq N}\left(W_{i}-s_{i}\right)\right| \leq \epsilon\right) = P\left(\left|\max_{1\leq i\leq N}\left(\bar{W}_{i}(D_{i})-s_{i}\right)\right| \leq \epsilon\right)$$

$$\leq P\left(\left|\max_{1\leq i\leq N}\left(\bar{W}_{i}(D_{i}^{*})-s_{i}\right)\right| \leq \epsilon\right) = P\left(\left|\max_{1\leq i\leq N}\left(W_{i}^{*}-s_{i}\right)\right| \leq \epsilon\right),$$

where the inequality holds since the upper bound on ϵ implies $\bar{W}_i(0) - s_i = 0 - s_i < -\epsilon$ so that units i with $D_i = 0$ do not contribute any probability mass. Then, Lemma E.17 gives C and N_0 such that for $N^{-\tau} < \epsilon < \underline{a}/2(\log N)$ and $N \ge N_0$,

$$P\left(\left|\max_{1\leq i\leq N}\left(W_{i}-s_{i}\right)\right| \leq \epsilon\right) \leq P\left(\left|\max_{1\leq i\leq N}\left(W_{i}^{*}-s_{i}\right)\right| \leq \epsilon\right)$$

$$\leq \sum_{(d_{1},\dots,d_{N})\in\{1,\dots,p\}^{N}} P\left(D_{1}^{*}=d_{1},\dots,D_{N}^{*}=d_{N}\right)$$

$$\times P\left(\left|\max_{1\leq i\leq N}\left(U_{i,d_{i}}-s_{i}\right)\right| \leq \epsilon\right) \leq C\epsilon + 2N^{-1}.$$

Lemma E.11. Let $\nu(N) \geq 1$ denote a sequence that converges to infinity, and let $c_N(\alpha)$ denote the $(1-\alpha/N)$ -quantile of the t-distribution with $\nu(N)$ degrees of freedom. Suppose that $(\log N)/\nu(N) \rightarrow 0$. For each $\epsilon > 0$ and $0 < \underline{\alpha} < 1$, there is a threshold N_0 such that for $N \geq N_0$

$$\sup_{\underline{\alpha} \le \alpha < 1} c_N(\alpha) \le \sqrt{2(1+\epsilon)\log(N/\alpha)}.$$

Proof. For notational convenience, write $\nu = \nu(N)$. We prove the bound for $\alpha = \underline{\alpha}$ and write $c_N = c_N(\alpha)$. Then, the uniformity follows from the monotonicity of the distribution function. Clearly, $c_N \to \infty$ so that we can take $c_N \ge 1$, provided that N is large enough. The density function of the t-distribution with ν degrees of freedom is given by $f_{\nu}^t(x) = c(\nu) \left(1 + x^2/\nu\right)^{-\frac{\nu+1}{2}}$,

where

$$c(\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \to \frac{1}{\sqrt{2\pi}}$$

as $\nu \to \infty$. It follows that there is a universal constant C such that $c(\nu) \leq C$. We first show that $c_N^2/\nu = O(1)$. The proof is by contradiction. Suppose that $\limsup_{N\to\infty} c_N^2/\nu = \infty$. Applying Theorem 1 in Soms (1976) with n=1 yields

$$1 - t_{\nu}(c_N) \le f_{\nu}^t(c_N) \frac{1}{c_N} \left(1 + \frac{c_N^2}{\nu} \right). \tag{30}$$

This implies that

$$\frac{\alpha}{N} \le c(\nu) \left(1 + \frac{c_N^2}{\nu}\right)^{-\frac{\nu+1}{2}} \left(1 + \frac{c_N^2}{\nu}\right) \le C \left(1 + \frac{c_N^2}{\nu}\right)^{-\frac{\nu-1}{2}}.$$

Taking logs and rearranging gives

$$\frac{\log(N/\alpha)}{\nu} \ge \frac{1}{2} \frac{\nu - 1}{\nu} \left(\log \left(1 + \frac{c_N^2}{\nu} \right) - C \right).$$

The left-hand side of the inequality vanishes under the assumptions of the lemma, whereas a subsequence of the right-hand side diverges to infinity. This establishes that the inequality is impossible and therefore $c_N^2/\nu = O(1)$. This implies that there exists a constant b such that

$$1 < b \le \left(1 + \frac{c_N^2}{\nu}\right)^{\frac{\nu}{c_N^2}} \le e,$$

so that we can take

$$\left(\left(1 + \frac{c_N^2}{\nu} \right)^{\frac{\nu}{c_N^2}} \right)^{-1} \le e^{-\frac{\nu}{\nu+1}(1 + \epsilon^*/2)^{-1}}$$

for a positive ϵ^* . Then,

$$f_{\nu}^{t}(c_{N}) \leq C \left[\left(1 + \frac{c_{N}^{2}}{\nu} \right)^{\frac{\nu}{c_{N}^{2}}} \right]^{-\frac{c_{N}^{2}}{2} \left[\frac{\nu+1}{\nu} \right]} \leq C \exp \left(-\frac{c_{N}^{2}}{2} (1 + \epsilon^{*}/2)^{-1} \right).$$

Take N large enough that

$$\frac{1}{1 + \epsilon^*/2} - \frac{4\log c_N}{c_N^2} > \frac{1}{1 + \epsilon^*}.$$

Then, the right-hand side of (30) can be bounded by

$$C \exp\left(-\frac{c_N^2}{2}(1 - \epsilon^*/2)^{-1}\right) \left(1 + \frac{c_N^2}{\nu}\right) \le 2C \exp\left(-\frac{c_N^2}{2}\left((1 + \epsilon^*/2)^{-1} - \frac{4\log c_N}{c_N^2}\right)\right) \le 2C \exp\left(-\frac{c_N^2}{2}\left(1 + \epsilon^*/2\right)^{-1}\right).$$

Plugging in $1 - t_{\nu}(c_N) = \alpha/N$ and taking logs gives

$$\begin{aligned} c_N^2 \leq & (1+\epsilon^*) \log \left(N/\alpha\right) + \log(2C) \\ \leq & 2(1+\epsilon^*) \log \left(N/\alpha\right) \left(1 + \frac{1}{2(1+\epsilon^*)} \frac{\log(2C)}{\log(N/\alpha)}\right). \end{aligned}$$

Hence, there is a constant C such that $c_N^2 \leq C \log(N/\alpha)$. Using this inequality, we can now verify that $c_N^2/\nu \to 0$ so that

$$\left(1 + \frac{c_N^2}{\nu}\right)^{\frac{\nu}{c_N^2}} \to e,$$

allowing us to take $\epsilon^* = \epsilon/2$ for sufficiently large N. Taking N large enough that

$$(1 + \epsilon/2) \left(1 + \frac{1}{2(1 + \epsilon/2)} \frac{\log(2C)}{\log(N)} \right) \le 1 + \epsilon$$

yields $c_N^2 \le 2(1+\epsilon)\log(N/\alpha)$.

Lemma E.12. For $\nu \geq 1$, let t_{ν} and f_{ν}^{t} denote the distribution and density function of a t-distributed random variable with ν degrees of freedom. For $x^{2} > 2$

$$f_{\nu}^{t}(x) < 2x \left(1 - t_{\nu}(x)\right).$$

Proof. Applying Theorem 1 in Soms (1976) with n=2 yields the inequality

$$1 - t_{\nu}(x) \ge (1 + x^2/\nu) \left(1 - \frac{\nu}{(\nu + 2)x^2}\right) f_{\nu}^t(x)/x.$$

Now, $x^2 > 2$ implies

$$1 - t_{\nu}(x) > \left(1 - \frac{1}{2}\right) f_{\nu}^{t}(x)/x.$$

Lemma E.13. Let ξ_1, \ldots, ξ_T be independent centered random variables with $E(\xi_t^2) = 1$ and $E(|\xi_t|^{2+\nu}) < \infty$ for all $1 \le t \le T$ where $0 < \nu \le 1$. Let $S_T = \sum_{t=1}^T \xi_t$, $V_T^2 = \sum_{t=1}^T \xi_t^2$ and

 $D_{T,\nu} = (T^{-1} \sum_{t=1}^{T} E(|\xi_t|^{2+\nu}))^{1/(2+\nu)}$. Then uniformly in $0 \le x \le T^{\nu/(2(2+\nu))}/D_{T,\nu}$,

$$\left| \frac{\Pr(S_T/V_T \ge x)}{1 - \Phi(x)} - 1 \right| \le KT^{-\nu/2} D_{T,\nu}^{2+\nu} (1 + x)^{2+\nu}.$$

Proof. This lemma is first proved by Jing, Shao, and Wang (2003). Here we use the version by Chernozhukov, Chetverikov, and Kato (2018, Lemma D.1), which is based on de la Pena, Lai, and Shao (2009, Theorem 7.4).

Lemma E.14 (Properties of $\tilde{\chi}^2$ -distribution). Let W denote a random variable with $\tilde{\chi}^2(V)$ distribution for a nondegenerate $p \times p$ covariance matrix. For j = 0, ..., p let w(p, j, V) denote the weight function for the $\bar{\chi}^2$ -distribution defined by Kudo (1963) and Nüesch (1966).

1. (Weights define a probability distribution) For j = 0, ..., p, w(p, j, V) > 0 and

$$\sum_{j=1}^{p} w(p, j, V) = 1.$$

Moreover, $w(p, j, V) \leq 1/2$ for j = 1, ..., N.

2. (Tail probabilities) Let $(U_j)_{j=1}^p$ denote chi-squared random variables, $U_j \sim \chi_j^2$. For all $c \geq 0$

$$P(W \ge c) = \sum_{j=1}^{p} w(p, p - j, V) P(U_j \ge c).$$

3. (Mixture representation) Let $(U_j)_{j=1}^p$ denote independent chi-squared random variables such that $U_j \sim \chi_j^2$. Let D denote a random variable with support in $\{0,\ldots,p\}$ and P(D=d)=w(p,p-d,V). Define $\bar{W}(d)=\sum_{j=1}^p \{d=j\}U_j$. Then

$$\mathcal{L}(W) = \mathcal{L}(\bar{W}(D)).$$

4. (Calculation of weights) For subsets $M \subset \{1,\ldots,p\}$ let \bar{M} denote $\{1,\ldots,p\}\setminus M$. For $M_1,M_2\subset \{1,\ldots,p\}$ and a $(p\times p)$ -matrix A let A_{M_1,M_2} denote A with the rows with indices corresponding to entries in \bar{M}_1 and the columns with indices corresponding to entries in \bar{M}_2 deleted. For $M\neq\emptyset$ define the normal vector $Y_1(M)\sim N(0,V_{M,M}^{-1})$ and the probability $p_1(M)=P(Y_1(M)\leq 0)$. For $M=\emptyset$ set $p_1(M)=1$. For $M\neq\{1,\ldots,p\}$ define the normal vector $Y_2(M)\sim N(0,(V^{-1})_{\bar{M},\bar{M}}^{-1})$ and the probability $p_2(M)=P(Y_2(M)>0)$. For $M=\{1,\ldots,p\}$ set $p_2(M)=1$. The weights can be written as

$$w(p, p - j, V) = \sum_{\substack{M \subset \{1, \dots, p\} \\ |M| = j}} p_1(M) p_2(M).$$

Proof. (1) In the derivation of the weights (see e.g., Nüesch 1966) the weights correspond to probabilities of events that partition the sample space. To prove the asserted upper bounds use the representation from (4) and write

$$\begin{split} w(p,p,V) &= P(Y_2(\emptyset) > 0) \le 1 - P(\text{there is } j = 1, \dots, p \text{ such that } Y_{2,j}(\emptyset) \le 0) \\ &\le 1 - \max_{j=1,\dots,p} P(Y_{2,j}(\emptyset) \le 0) = \frac{1}{2}. \end{split}$$

For the other weights, the bound can be proved in a similar way.

- (2) This can be proved analogously to the derivation of the distribution of the $\bar{\chi}^2$ statistic (see Kudo 1963; Nüesch 1966).
- (3) This follows from (2) upon observing that $\tilde{\chi}^2(V)$ is supported only on the nonnegative reals and that $\{[c,\infty):c>0\}$ is a generating class.

Lemma E.15. Let \hat{A} and A denote nonsingular $p \times p$ matrices and suppose that

$$2\|\hat{A} - A\|_2 \|A^{-1}\|_2 \le 1.$$

Then,

$$\|\hat{A}^{-1} - A^{-1}\|_2 \le 2\|\hat{A} - A\|_2 \|A^{-1}\|_2^2$$
.

Proof. This approach is originally due to Lewis and Reinsel (1985). Like any induced norm, the $\|\cdot\|_2$ -norm obeys submultiplicativity so that

$$\|\hat{A}^{-1} - A^{-1}\|_2 = \|\hat{A}^{-1}(\hat{A} - A)A^{-1}\|_2 \leq \|\hat{A} - A\|_2 \|A^{-1}\|_2 (\|A^{-1}\|_2 + \|\hat{A}^{-1} - A^{-1}\|_2).$$

Rearranging yields

$$\|\hat{A}^{-1} - A^{-1}\|_{2} \le \frac{\|\hat{A}^{-1} - A^{-1}\|_{2} \|A^{-1}\|_{2}^{2}}{1 - \|\hat{A} - A\|_{2} \|A^{-1}\|_{2}} \le 2\|\hat{A} - A\|_{2} \|A^{-1}\|_{2}^{2}.$$

Lemma E.16. Let $(\phi_i)_{i=1}^N$ denote normal random variables such that $\phi_i \sim N(0, I_{p_i})$ with $p_i \leq \bar{p}$. Let $\underline{a}, \bar{a} > 0$ and let c_N denote a deterministic sequence. For each $\tau > 0$ and $\kappa > 0$ there exist positive constants \bar{C} and N_0 such that for $N \geq N_0$ and all $\epsilon > N^{-\tau}$ we have

$$\sup_{(a_1,\dots,a_N)\in [\underline{a},\bar{a}]^N} P\left(\left|\max_{1\leq i\leq N}\frac{\|\phi_i\|}{a_i}-c_N\right|\leq \epsilon\right)\leq \bar{C}_a\epsilon\sqrt{\log N}+N^{-\kappa}.$$

Proof. Let ϵ denote a generic constant satisfying $\epsilon > N^{-\tau}$. Let Γ_i denote a δ_N -covering of a sphere

in \mathbb{R}^{p_i} with radius a_i^{-1} , where

$$\delta_N = \frac{1}{4} N^{-\tau} ((\kappa + 1) \log N)^{-1/2}.$$

It is without loss of generality to assume that, for all $\gamma \in \Gamma_i$, $\|\gamma\| = a_i^{-1}$. An upper bound on $\operatorname{card}(\Gamma_i)$ is given by

$$\operatorname{card}(\Gamma_i) \leq b_1 N^{b_2},$$

where b_1 and b_2 depend only on κ , τ , \underline{a} and \bar{p} . As in Zhilova (2015), note that

$$\frac{\|\phi_i\|}{a_i} = \sup_{\gamma \in \mathbb{R}^{p_i}: \|\gamma\| = a_i^{-1}} \gamma' \phi_i.$$

We employ an approximation argument based on the inequality

$$P\left(\left|\max_{1\leq i\leq N}\frac{\|\phi_{i}\|}{a_{i}}-c_{N}\right|\leq\epsilon\right)\leq P\left(\left|\max_{1\leq i\leq N}\max_{\gamma_{j}\in\Gamma_{i}}\gamma_{j}'\phi_{i}-c_{N}\right|\leq2\epsilon\right)$$

$$+P\left(\max_{1\leq i\leq N}\sup_{\gamma\in\mathbb{R}^{p_{i}}:\|\gamma\|=a_{i}^{-1}}\min_{\gamma_{j}\in\Gamma_{i}}|(\gamma-\gamma_{j})'\phi_{i}|>\epsilon\right)$$

$$\equiv A_{1}+A_{2}.$$

To bound A_1 , note that each $\gamma'_j \phi_i$ is a normal random variable with standard deviation bounded between \bar{a}^{-1} and \underline{a}^{-1} . This follows from our assumptions about the covering Γ_i and

$$\mathbb{E}\left[\left(\gamma_j'\phi_i\right)^2\right] = \gamma_j' \mathbb{E}\left[\phi_i \phi_i'\right] \gamma_j = \|\gamma_j\|^2 = a_i^{-2}.$$

Then, $\max_{1 \leq i \leq N} \max_{\gamma_j \in \Gamma_i} |\gamma'_j \phi_i|$ is the maximum of $\sum_{i=1}^N \operatorname{card}(\Gamma_i)$ independent normal random variables and the results for Levy concentration bounds in Chernozhukov, Chetverikov, and Kato (2015) apply. For a constant C_a depending only on \underline{a} and \bar{a} , their Corollary 1 yields

$$A_1 \le C_a \epsilon_{\sqrt{1 \vee \log\left(\frac{\sum_{i=1}^N \operatorname{card}(\Gamma_i)}{2\epsilon}\right)}} \le C_a \epsilon_{\sqrt{1 \vee \log\left((1/2)b_1 N^{b_2 + \tau}\right)}} \le \bar{C} \epsilon_{\sqrt{\log N}}.$$

The last inequality holds for $N \geq N_{0,a}$ and sufficiently large \bar{C} , where the choice of $N_{0,a}$ and \bar{C} depends only on κ , τ , \underline{a} and \bar{p} . To bound A_2 let $N_{0,b}$ be large enough such that for $N \geq N_{0,b}$ and $t_N = \frac{1}{2}(N^{-\tau}/\delta_N)^2$ we have $t_N > \bar{p}$. For $N \geq N_{0,b}$, by the Cauchy-Schwarz inequality

$$A_2 \le P\left(\max_{1 \le i \le N} \|\phi_i\|^2 > \left(\frac{\epsilon}{\delta_N}\right)^2\right)$$

$$\le P\left(\max_{1 \le i \le N} \|\phi_i\|^2 - p_i > t_N\right) \le N \exp\left(-\frac{t_N}{8}\right) \le N^{-\kappa}.$$

The fourth inequality follows from the fact that $\|\phi_i\|^2$ obeys the subexponential condition

$$\mathbb{E}\left[e^{\alpha(\|\phi_i\|^2 - p_i)}\right] \le e^{4^2\alpha^2/2} \quad \text{for all } |\alpha| < \frac{1}{2\sqrt{p_i}}.$$

This implies the tail bound

$$P(\|\phi_i\|^2 - p_i > t_N) \le e^{-\frac{t_N}{8}}$$

for $t_N > p_i$ (see, e.g., Proposition 2.2 in Wainwright 2015). The conclusion of the lemma follows by setting $N_0 = \max\{1, N_{0,a}, N_{0,b}\}$.

Lemma E.17. Let $(\phi_i)_{i=1}^N$ denote normal random vectors such that $\phi_i \sim N(0, I_{p_i})$ with $p_i \leq \bar{p}$. For $\underline{a}, \bar{a}, \gamma > 0$, and a positive deterministic sequence c_N such that $c_N \leq N^{\gamma}$ let $\mathbb{S}_N = [c_N \underline{a}, c_N \bar{a}]$. For each $\tau > 0$ and $\kappa > 0$ there exist positive constants \bar{C} and N_0 such that for $N \geq N_0$ and all $\epsilon > N^{-\tau}$ we have

$$\sup_{(s_1, \dots, s_N) \in \mathbb{S}^N} P\left(\left| \max_{1 \le i \le N} \left(\|\phi_i\|^2 - s_i \right) \right| \le \epsilon \right) \le \bar{C} \epsilon \sqrt{\frac{\log N}{c_N}} + N^{-\kappa}.$$

If the random vectors ϕ_i are independent, then we also have

$$\sup_{(s_1,\dots,s_N)\in\mathbb{S}^N} P\left(\left|\max_{1\leq i\leq N}\left(\|\phi_i\|^2-s_i\right)\right|\leq \epsilon\right)\leq \bar{C}\epsilon \left(1+\sqrt{\frac{\bar{a}c_N}{\log N}}\right)^{-1}+N^{-\kappa}.$$

Proof. Fix $\epsilon > N^{-\tau}$ and $(s_1, \ldots, s_N) \in \mathbb{S}^N$. Let L_N denote a lower bound on $\max_{1 \leq i \leq N} \|\phi_i\|$. Suppose first that the ϕ_i are independent. Then,

$$\max_{1 \le i \le N} \|\phi_i\| \ge \max_{1 \le i \le N} |\phi_{i,1}| \ge \max_{1 \le i \le N} \phi_{i,1}.$$

By Example 3.5.5 in Embrechts, Klüppelberg, and Mikosch (2013)

$$\frac{\max_{1 \le i \le N} \phi_{i,1}}{\sqrt{2 \log N}} \to 1 \qquad P\text{-almost surely}.$$

Therefore, there exists a finite $N_{0,a}$ for which we may assume $N \ge N_{0,a} \Rightarrow \max_{1 \le i \le N} ||\phi_i|| \ge \sqrt{\log N}$. This implies that, for independent ϕ_i , we may take $L_N = \sqrt{\log N}$, otherwise take $L_N = 0$. For each i = 1, ..., N write $s_i = c_N a_i$. For $N \ge N_{0,a}$

$$P\left(\left|\max_{1\leq i\leq N}\left(\|\phi_i\|^2 - s_i\right)\right| \leq \epsilon\right)$$

$$\leq P\left(\left|\max_{1\leq i\leq N}\frac{\|\phi_i\|^2}{a_i} - c_N\right| \leq \underline{a}^{-1}\epsilon\right)$$

$$\leq P\left(\left(\bar{a}^{-1/2}\max_{1\leq i\leq N}\|\phi_i\| + \sqrt{c_N}\right)\left|\max_{1\leq i\leq N}\frac{\|\phi_i\|}{\sqrt{a_i}} - \sqrt{c_N}\right| \leq \underline{a}^{-1}\epsilon\right)$$

$$\leq P\left(\left|\max_{1\leq i\leq N}\frac{\|\phi_i\|}{\sqrt{a_i}} - \sqrt{c_N}\right| \leq \frac{\underline{a}^{-1}\epsilon}{\overline{a}^{-1/2}L_N + \sqrt{c_N}}\right).$$

Let

$$\epsilon' = \frac{\underline{a}^{-1}\epsilon}{\overline{a}^{-1/2}L_N + \sqrt{c_N}}.$$

Since $c_N < N^{\gamma}$, we can find $\tau' > 0$, depending only on \underline{a} , \bar{a} , τ and γ , such that $\epsilon' > N^{-\tau'}$. Applying Lemma E.16 with ϵ' and τ' we may now conclude that there are constants $N_{0,b}$ and \bar{C}_a such that for $N \geq N_{0,b}$

$$P\left(\left|\max_{1\leq i\leq N}\frac{\|\phi_i\|}{\sqrt{a_i}} - \sqrt{c_N}\right| \leq \frac{\underline{a}^{-1}\epsilon}{\bar{a}^{-1/2}\sqrt{\log N} + \sqrt{c_N}}\right) \leq \bar{C}_a\epsilon'\sqrt{\log N} + N^{-\kappa}$$
$$\leq C\left(\frac{\sqrt{\log N}}{L_N + \sqrt{\bar{a}c_N}}\right) + N^{-\kappa}.$$

The last inequality holds for conformant C. The assertion of the lemma follows by choosing $N_0 = \max\{N_{0,a}, N_{0,b}\}$ and plugging in the appropriate value of L_N .

Lemma E.18. Let V denote a correlation matrix and let $\Phi_{\max,V}$ denote the distribution function of the maximum of a vector of multivariate normal random vector with covariance matrix V. There is $t^* \in \mathbb{R}$ independent of T and V such that for all $t > t^*$

$$t_{\max,V,T-1}\left(\sqrt{\frac{T-1}{T}}t\right) \le \Phi_{\max,V}(t).$$

Proof. By the definitions of $\Phi_{\max,V}$ and $t_{\max,V,T-1}$, we have

$$\Phi_{\max,V}(t) = \int_{\mathbf{x} \le t} \phi_V(\mathbf{x}) \, d\mathbf{x}$$

and

$$t_{\max,V,T-1}\left(\sqrt{\frac{T-1}{T}}t\right) = \int_{\mathbf{x} \le \sqrt{\frac{T-1}{T}}t} f_{V,T-1}^t(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{x} \le t} f_{V,T-1}^{t,*}(\mathbf{x}) d\mathbf{x},$$

where \mathbf{x} is a G-1 dimensional vector, and the inequality $\mathbf{x} \leq t$ is understood in an element-wise way, i.e., it means $\max_{j=1,\dots,G-1} \mathbf{x}_j \leq t$,

$$\phi_V(\mathbf{x}) = (2\pi)^{-(G-1)/2} (\det(V))^{-1/2} \exp\left(-\frac{1}{2}\mathbf{x}'V^{-1}\mathbf{x}\right),$$

and

$$f_{V,T-1}^{t}(\mathbf{x}) = (\pi(T-1))^{-(G-1)/2} (\det(V))^{-1/2} \Gamma\left(\frac{T+G-2}{2}\right) \left(\Gamma\left(\frac{T-1}{2}\right)\right)^{-1}$$

$$\times \left(1 + \frac{1}{T-1}\mathbf{x}'V^{-1}\mathbf{x}\right)^{-(T+G-2)/2},$$

is the density of the multivariate t distribution with scale matrix V and T-1 degrees of freedom, and

$$f_{V,T-1}^{t,*}(\mathbf{x}) = (\pi T)^{-(G-1)/2} (\det(V))^{-1/2} \Gamma\left(\frac{T+G-2}{2}\right) \left(\Gamma\left(\frac{T-1}{2}\right)\right)^{-1} \times \left(1 + \frac{1}{T}\mathbf{x}'V^{-1}\mathbf{x}\right)^{-(T+G-2)/2}.$$

We now identify the region in which $f_{V,T-1}^{t,*}(\mathbf{x}) \leq \phi_V(\mathbf{x})$. We have

$$\log f_{V,T-1}(\mathbf{x}) - \log \phi_V(\mathbf{x}) = A_T - \frac{T+G-2}{2} \log \left(1 + \frac{1}{T}\mathbf{x}'V^{-1}\mathbf{x}\right) + \frac{1}{2}\mathbf{x}'V^{-1}\mathbf{x},$$

where

$$A_T = -\frac{G-1}{2}\log(T) + \log\Gamma\left(\frac{T+G-2}{2}\right) - \log\Gamma\left(\frac{T-1}{2}\right) + \frac{G-1}{2}\log(2).$$

By the property of the logarithm function and the linear function, there is a unique value, denoted by x_T^* , such that $f_{V,T-1}^{t,*}(\mathbf{x}) \leq \phi_V(\mathbf{x})$ implies $\mathbf{x}'V^{-1}\mathbf{x} \leq x_T^*$. To see this, we consider the two functions $\log(1+y)$ and ay+b, where a=T/(T+G-2) and $b=2A_T/(T+G-2)$. We want to find a value of y, say y', such that if $y \geq y'$ then $\log(1+y) \leq ay+b$. Because $\log(1+y)$ is increasing and concave and a>0 there are two possibilities: 1) $ay+b \geq \log(1+y)$ for any y and $ay+b > \log(1+y)$ almost always; 2) the curves $\log(1+y)$ and ay+b intersect with each other at two points, say y_1 and y_2 such that $\log(1+y) < ay+b$ for $y < y_1$, $\log(1+y) \geq ay+b$ for $y_1 \leq y \leq y_2$, and $\log(1+y) < ay+b$ for $y > y_2$. The first case does not apply to our situation, because if this was the case then $f_{V,T-1}^{t,*}(\mathbf{x}) > \phi_V(\mathbf{x})$ almost always, contradicting the fact that both curves integrate to one. Thus, the second case applies. The values of y_1 and y_2 can be obtained by solving $\log(1+y) = ay+b$. It holds $y_2 > 0$ because the slope of $\log(1+y)$ at y_2 must be smaller than a and 0 < a < 1.

Choose t large enough such that $\mathbf{x}'V^{-1}\mathbf{x} \leq x_T^*$ implies $\mathbf{x} \leq t$. This choice of t depend on T only through x_T^* . In particular, if $x_T^* = O(1)$ then t can be chosen independently on T. To prove this set $t = \sqrt{x_T^*(G-1)}$ and prove the contrapositive. Since V is a correlation matrix its largest eigenvalue is bounded by G-1 and $\mathbf{x}'V^{-1}\mathbf{x} \geq \|\mathbf{x}\|^2/(G-1)$. If $\mathbf{x} > t$ then $\|\mathbf{x}\| > \sqrt{x_T^*(G-1)}$ and hence $\mathbf{x}'V^{-1}\mathbf{x} > x^*$.

We have

$$\Phi_{\max,V}(t) - t_{\max,V,T-1}\left(\sqrt{\frac{T-1}{T}}t\right) = \int_{\mathbf{x} \le t} \left(\phi_V(\mathbf{x}) - f_{V,T-1}^{t,*}(\mathbf{x})\right) d\mathbf{x}$$

$$\begin{split} &= \int_{\mathbf{x}'V^{-1}\mathbf{x} \leq x_T^*} \left(\phi_V(\mathbf{x}) - f_{V,T-1}^{t,*}(\mathbf{x}) \right) d\mathbf{x} \\ &+ \int_{\mathbf{x} \leq t, \mathbf{x}'V^{-1}\mathbf{x} > x_T^*} \left(\phi_V(\mathbf{x}) - f_{V,T-1}^{t,*}(\mathbf{x}) \right) d\mathbf{x}, \end{split}$$

where the first integral on the right hand side of the equation is taken over $\mathbf{x}'V^{-1}\mathbf{x} \leq a$ because $\{\mathbf{x}: \mathbf{x}'V^{-1}\mathbf{x} \leq x_T^*, \mathbf{x} \leq t\} = \{\mathbf{x}: \mathbf{x}'V^{-1}\mathbf{x} \leq x_T^*\}$ by our choice of t. Because both $\phi_V(\mathbf{x})$ and $f_{V,T-1}^{t,*}(\mathbf{x})$ are densities and integrate to one, we have

$$\int_{\mathbf{x}'V^{-1}\mathbf{x} \le x_T^*} \left(\phi_V(\mathbf{x}) - f_{V,T-1}^{t,*}(\mathbf{x}) \right) d\mathbf{x} = -\int_{\mathbf{x}'V^{-1}\mathbf{x} > x_T^*} \left(\phi_V(\mathbf{x}) - f_{V,T-1}^{t,*}(\mathbf{x}) \right) d\mathbf{x},$$

Thus, for t large enough such that $\mathbf{x}'V^{-1}\mathbf{x} \leq x_T^*$ implies $\mathbf{x} \leq t$, we have

$$\Phi_{\max,V}(t) - F_{\max,V,T-1}^{f}(t) > 0.$$

Next, we evaluate the order of x_T^* . Note that x_T^* solves

$$\frac{1}{2}x_T^* + A_T = \frac{T + G - 2}{2}\log\left(1 + \frac{1}{T}x_T^*\right).$$

We first show that $A_T = O(1)$ where the order is taken with respect to T. To see this, we consider the cases of odd and even G separately. Suppose that G is odd (we may assume $G \ge 3$). Then we have

$$A_T = -\frac{G-1}{2}\log(T) + \sum_{j=0}^{(G-1)/2-1}\log\left(\frac{T-1}{2}+j\right) + \frac{G-1}{2}\log(2)$$

$$= -\frac{G-1}{2}\log(T) + \sum_{j=0}^{(G-1)/2-1}\log\left(T-1+2j\right) - \frac{G-1}{2}\log(2) + \frac{G-1}{2}\log(2)$$

$$= \sum_{j=0}^{(G-1)/2-1}\log\left(\frac{T-1+2j}{T}\right) = O(1)$$

as $T \to \infty$. Next, we consider cases in which G is even. For G = 2, $A_T = O(1)$ follows from

$$\sqrt{\frac{T}{2}} \frac{\Gamma\left(\frac{T-1}{2}\right)}{\Gamma\left(\frac{T}{2}\right)} \to 1. \tag{31}$$

For $G \geq 4$ we have

$$A_T = -\frac{G-1}{2}\log(T) + \sum_{j=0}^{(G-2)/2-1}\log\left(\frac{T}{2} + j\right) + \log\Gamma\left(\frac{T}{2}\right) - \log\Gamma\left(\frac{T-1}{2}\right) + \frac{G-1}{2}\log(2)$$

$$= \sum_{j=0}^{(G-2)/2-1} \log\left(\frac{T+2j}{T}\right) + \frac{1}{2}\log\left(\frac{2}{T}\right) + \log\Gamma\left(\frac{T}{2}\right) - \log\Gamma\left(\frac{T-1}{2}\right).$$

By (31)

$$\log \Gamma\left(\frac{T}{2}\right) - \log \left(\Gamma\left(\frac{T-1}{2}\right) \left(\frac{T}{2}\right)^{1/2}\right) = O(1).$$

We have now established that $A_T = O(1)$ for all $G \ge 2$. To prove the lemma it now suffices to prove $x_T^* = O(1)$. Suppose the opposite is true. Then, there is a subsequence T_1, \ldots, T_k, \ldots such that $x_{T_k}^*$ monotonically diverges to infinity. By the definition of x_T^* we have

$$x_T^* + A_T = (T + G - 2)\log\left(1 + \frac{1}{T}x_T^*\right).$$

For sufficiently large $y, y/2 \ge \log(1+y)$. Therefore, for sufficiently large k, we have

$$x_{T_k}^* + A_T < \frac{T + G - 2}{2T} x_{T_k}^*$$

Rearranging terms yields

$$\frac{T - G + 2}{2T} x_{T_k}^* + A_T < 0,$$

contradicting that $A_T = O(1)$ and $x_{T_k}^*$ diverging to infinity can both be true. This proves $x_T^* = O(1)$.

Lemma E.19. Let V denote a $(G-1)\times (G-1)$ covariance matrix V and suppose that $(\log N)/T \to 0$. Then there is N_0 depending only on α such that for $N \geq N_0$, $F_{\text{QLR},V}^*$ as defined in (13), and all $1 \leq i \leq N$

$$c_{\alpha,N}^{\text{QLR}}(V) \ge \left(F_{\text{QLR},V}^*\right)^{-1} \left(1 - \frac{\alpha}{N}\right).$$

Proof. Write $x^* = \left(F_{\text{QLR},V}^*\right)^{-1} (1 - \alpha/N)$. To prove the claim it suffices to show

$$F_{\text{QLR},V}^{*}(x^{*}) - F_{\text{QLR},V}(x^{*})$$

$$= \sum_{j=1}^{G-1} w (G-1, G-1-j, V) \left(P(\chi_{j}^{2} \leq x^{*}) - P(F_{j,T-1} \leq x^{*}/j) \right) \geq 0.$$

Since the weights are non-negative (see Lemma E.14) this holds if

$$P(\chi_i^2 \le x^*) - P(F_{i,T-1} \le x^*/j) \ge 0 \tag{32}$$

for $j=1 \le j \le G-1$. First we consider cases with j=1,2. Note that $F_{j,T-1}$ has the same distribution as $(U_1/j)/(U_2/(T-1))$ where $U_1 \sim \chi_j^2$, $U_2 \sim \chi_{T-1}^2$ and U_1 and U_2 are independent. Thus,

$$P(F_{j,T-1} \le x^*/j) = P\left(U_1 \le \frac{U_2}{T-1}x^*\right).$$
 (33)

Let $F_{\chi^2,j}$ be the distribution function of a χ^2 random variable with j degrees of freedom. $F_{\chi^2,j}$ is concave for j=1,2. To see this, note that the density of χ^2_j is given by

$$f_{\chi^2,j}(x) = \frac{x^{\frac{j}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{j}{2}}\Gamma(\frac{j}{2})},$$

where $\Gamma(\cdot)$ is the gamma function. Differentiating the density yields

$$f'_{\chi^2,j}(x) = \left(2^{\frac{j}{2}}\Gamma\left(\frac{j}{2}\right)\right)^{-1} \left(-\frac{x}{2} + \left(\frac{j}{2} - 1\right)\frac{1}{x}\right) \left(x^{\frac{j}{2} - 1}e^{-\frac{x}{2}}\right).$$

which, for j = 1, 2, is negative on x > 0 (i.e., the support of χ_j^2). By Jensen's inequality and the fact that $E(U_2) = T - 1$

$$P(\chi_j^2 \le x^*) - P(F_{j,T-1} \le x^*/j) = F_{\chi^2,j}(x^*) - \mathbb{E}\left[F_{\chi^2,j}\left(\frac{U_2}{T-1}x^*\right)\right]$$
$$\ge F_{\chi^2,j}(x^*) - F_{\chi^2,j}\left(\frac{\mathbb{E}[U_2]}{T-1}x^*\right) = 0.$$

This verifies (32) for j = 1, 2. We now turn to the case of j > 2. Let

$$\ell_{j,T-1} = \left(T - 1 + \frac{1}{2}(j-2)\right)\log\left(1 + \frac{x^*}{(T-1)}\right)$$

and let $u_{j,T-1}$ solve $P(F_{j,T-1} \le x^*/j) = P(\chi_j^2 \le u_{j,T-1})$. To prove (32) it suffices to establish that $x^* \ge u_{j,T-1}$. Fujikoshi and Mukaihata (1993, Theorem 4.1(iii)) show that

$$\ell_{j,T-1} \ge u_{j,T-1}.$$

Use Lemma E.23 to find a lower bound on N such that

$$\log(N/\alpha) < x^* < (3/2)\log(N/\alpha).$$

Choose N also large enough such that

$$2(G-1) \le \log(N/\alpha) \le T - 1.$$

The inequality $\log(1+\tau) \le \tau - \tau^2/2(1-\tau/3)$ can be proved by Taylor expanding $\log(1+\tau)$ around

 $\tau = 0$. The inequality implies

$$\ell_{j,T-1} \le x^* + \frac{(j-2)x^*}{2(T-1)} - \left(\frac{x^*}{(T-1)}\right)^2 \left(1 - \frac{x^*}{3(T-1)}\right) (T-1 + (j-2)/2)/2$$

$$\le x^* + \left(\frac{x^*}{2(T-1)}\right) \left(j - 2 - x^* \left(1 - \frac{x^*}{3(T-1)}\right)\right) \le x^*.$$

It follows that $x^* \ge \ell_{j,T-1} \ge u_{jT-1}$ which proves (32) for $3 \le j \le G-1$.

Lemma E.20 (Extremal bound for normal vector). Let X be a centered normal random vector of length p with covariance matrix V. Let a > 0. Then,

$$P(\|X\|_{\max} > \sqrt{2p\|V\|_2 \log(a)}) \le P(\|X\|_{\max} > \sqrt{2\operatorname{tr}(V)\log(a)}) \le \frac{\sqrt{p}}{a\sqrt{\pi \log a}}.$$

Proof. For the first inequality, let $0 \le \lambda_1 \le \cdots \le \lambda_p$ denote the eigenvalues of V. Then

$$p||V||_2 = p\lambda_p \ge \sum_{j=1}^p \lambda_j = \operatorname{tr}(V).$$

For the second inequality, write $c = \sqrt{2\operatorname{tr}(V)\log(a)}$. Then

$$P(\|X\|_{\max} > c) \leq \sum_{j=1}^{p} (|X_j| > c) = 2 \sum_{j=1}^{p} \left(1 - \Phi\left(\frac{c}{\sigma_j}\right) \right)$$

$$\leq 2 \sum_{j=1}^{p} \frac{\sigma_j}{c} \phi\left(\frac{c}{\sigma_j}\right)$$

$$= \sqrt{\frac{2}{\pi}} \sum_{j=1}^{p} \frac{\sigma_j}{c} \exp\left(-\frac{c^2}{2\sigma_j^2}\right)$$

$$\leq \sqrt{\frac{2}{\pi}} \exp\left(-\frac{c^2}{2\operatorname{tr}(V)}\right) \frac{1}{c} \sum_{j=1}^{p} \sigma_j$$

$$\leq \sqrt{\frac{2p}{\pi}} \exp\left(-\frac{c^2}{2\operatorname{tr}(V)}\right) \frac{\sqrt{\operatorname{tr}(V)}}{c}$$

$$= \sqrt{\frac{p}{\pi}} \frac{1}{a\sqrt{\log a}},$$

where the second inequality uses Gordon's inequality for standard normal probabilities (see, e.g., Duembgen (2010)), and the last inequality uses the inequality $||x||_1 \leq \sqrt{p}||x||_2$ for vectors x of length p where $||\cdot||_1$ is the L_1 norm.

Lemma E.21 (Perturbation bound for rectangular normal probabilities). Consider the centered normal p-vectors X and \hat{X} with respective positive-definite covariance matrices V and \hat{V} . Let $x_1 = (\tilde{x}_1, \ldots, \tilde{x}_p) \in \mathcal{R}^p$ and $x = (\tilde{x}_{p+1}, \ldots, \tilde{x}_{2p}) \in \mathcal{R}^p$. Let $x_{\max} = \max_{1 \leq j \leq 2p} |\tilde{x}|_j$ and suppose that

 $x_{\text{max}} > 0$. Moreover, assume

$$\|\hat{V}^{-1} - V^{-1}\|_2 (\|V\|_2 \vee \|\hat{V}\|_2 \vee px_{\max}^2) \le 1.$$

Then, for any measurable function $g: \mathbb{R}^p \to \mathbb{R}$

$$\ell_{N,T} \le \frac{\mathbb{E}[g(\hat{X})1\{x_1 \le \hat{X} \le x_2\}]}{\mathbb{E}[g(X)1\{x_1 \le X \le x_2\}]} \le u_{N,T},$$

where

$$\ell_{N,T} = \left(1 + (2^p - 1)\|\hat{V}^{-1} - V^{-1}\|_2 \|\hat{V}\|_2\right)^{-1} \left(1 + px_{\max}^2 \|\hat{V}^{-1} - V^{-1}\|_2\right)^{-1},$$

$$u_{N,T} = \left(1 + (2^p - 1)\|\hat{V}^{-1} - V^{-1}\|_2 \|V\|_2\right) \left(1 + px_{\max}^2 \|\hat{V}^{-1} - V^{-1}\|_2\right).$$

Suppose that, in addition,

$$(2^p - 1)\|\hat{V}^{-1} - V^{-1}\|_2(\|V\|_2 \vee \|\hat{V}\|_2) \le 1$$

then

$$\left| P(x_1 \le \hat{X} \le x_2) - P(x_1 \le X \le x_2) \right|
\le \|\hat{V}^{-1} - V^{-1}\|_2 ((2^p - 1)(\|V\|_2 \lor \|\hat{V}\|_2) + 2px_{\max}^2).$$

Proof. Let f_X and $f_{\hat{X}}$ denote the probability densities corresponding to X and \hat{X} . Then,

$$\mathbb{E}\left[g(\hat{X})1\{x_{1} \leq \hat{X} \leq x_{2}\}\right]$$

$$=\mathbb{E}\left[g(X)1\{x_{1} \leq X \leq x_{2}\}\frac{f_{\hat{X}}(X)}{f_{X}(X)}\right]$$

$$=\frac{\det(V)}{\det(\hat{V})}\mathbb{E}\left[g(X)1\{x_{1} \leq X \leq x_{2}\}\exp\left(-\frac{1}{2}X'\left(\hat{V}^{-1} - V^{-1}\right)X\right)\right]$$

$$\leq \det(\hat{V}^{-1}V)\mathbb{E}\left[g(X)1\{x_{1} \leq X \leq x_{2}\}\exp\left(\frac{1}{2}\|X\|^{2}\|\hat{V}^{-1} - V^{-1}\|_{2}\right)\right]$$

$$\leq \left(1 + (2^{p} - 1)\|\hat{V}^{-1} - V^{-1}\|_{2}\|V\|_{2}\right)$$

$$\times \mathbb{E}\left[g(X)1\{x_{1} \leq X \leq x_{2}\}\exp\left(\frac{px_{\max}^{2}}{2}\|\hat{V}^{-1} - V^{-1}\|_{2}\right)\right]$$

$$\leq \mathbb{E}\left[g(X)1\{x_{1} \leq X \leq x_{2}\}\right]\left(1 + (2^{p} - 1)\|\hat{V}^{-1} - V^{-1}\|_{2}\|V\|_{2}\right)$$

$$\times \left(1 + px_{\max}^{2}\|\hat{V}^{-1} - V^{-1}\|_{2}\right).$$

The last inequality uses the inequality $\exp(x) \le 1 + 2x$ for $x \le 1/2$.³¹ For the second inequality note that Hadamard's inequality implies (see e.g. Lemma 2.5 in Ipsen and Rehman (2008))

$$\begin{split} \det(\hat{V}^{-1}V) & \leq \|\hat{V}^{-1}V\|_2^p \leq \left(\|I_p\|_2 + \|(\hat{V}^{-1} - V^{-1})V\|_2\right)^p \\ & \leq 1 + (2^p - 1)\left(\|\hat{V}^{-1} - V^{-1}\|_2\|V\|_2\right). \end{split}$$

This holds since $||I_p||_2 = 1$ and, for $0 \le a \le 1$, we have

$$(1+a)^p \le 1 + \sum_{k=1}^p \binom{p}{k} a^k \le 1 + (2^p - 1)a.$$

To derive the lower bound reverse the roles of X and \hat{X} .

Lemma E.22 (Perturbation bound for large quantiles). Suppose that \hat{V} and V are positive definite $(p \times p)$ variance matrices for $p \geq 2$. Let $W \sim \tilde{\chi}^2(V)$ and $\widehat{W} \sim \tilde{\chi}^2(\widehat{V})$. Let $\hat{c}_{\alpha,N}$ and $c_{\alpha,N}$ denote the $(1 - \alpha/N)$ -quantile of \widehat{W} and W, respectively. Suppose that

$$32p^2\|\hat{V} - V\|_2(1 \vee 2\|V^{-1}\|_2^2)(\|V\|_2 \vee \|V^{-1}\|_2)\log(N/\alpha) \le 1.$$

There is a threshold N_0 depending only on α and p such that for $N \geq N_0$ and

$$\alpha_N = \alpha \left(1 + 96 \left(\|\hat{V} - V\|_2 (1 \vee \|V^{-1}\|_2^2) (\|V\|_2 \vee \|V^{-1}\|_2) \log (N/\alpha) + N^{-1} \right)$$

we have

$$\hat{c}_{\alpha,N} \geq c_{\alpha_N,N}$$
.

Proof. Throughout the proof, we take N large enough so that

$$2\|\hat{V} - V\|_2\|V^{-1}\|_2(1 \vee 2\|V^{-1}\|_2\|V\|_2) \le 1.$$

This proof is based on the mixture representation of the $\tilde{\chi}^2$ -distribution from Lemma E.14. For each $M = (m_1, \ldots, m_{|M|}) \subset \{1, \ldots, p\}$ with $m_1 < \cdots < m_{|M|}$ where |M| is the cardinality of M, let S_M denote a $|M| \times p$ matrix with ones in the cells (m_k, k) , $k = 1, \ldots, |M|$, and zeros in all other entries. For $M_1, M_2 \subset \{1, \ldots, p\}$ and a symmetric positive-definite matrix A, let $A_{M_1, M_2} = M'_1 A M_2$. Let $\bar{M} = \{1, \ldots, p\} \setminus M$. For $M \subset \{1, \ldots, |M|\}$ and a symmetric, positive definite matrix A, we are

$$\exp(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \le 1 + x + x \sum_{n=1}^{\infty} \frac{1}{(n+1)!}x^n = 1 + x + x \sum_{n=1}^{\infty} \frac{1}{(n+1)(n!)}x^n \le 1 + \frac{x}{2}(e^x - 1) \le 1 + 2x.$$

³¹By the series expansion of the exponential function for $0 \le x \le 1/2$

interested in the centered normal random vector $Y_1(A, M)$ with covariance matrix

$$\Sigma_1(A, M) = \Sigma_1(A) = (A_{M,M})^{-1}$$

and the centered normal random vector $Y_2(A, M)$ with covariance matrix

$$\Sigma_2(A, M) = \Sigma_2(A) = A_{\bar{M}, \bar{M}} - A_{\bar{M}, M} (A_{M, M})^{-1} A_{M, \bar{M}} = ((A^{-1})_{\bar{M}, \bar{M}})^{-1}.$$

We first establish some useful inequalities. By Lemma E.15

$$\|\hat{V}^{-1} - V^{-1}\|_{2} \le 2\|\hat{V} - V\|_{2}\|V^{-1}\|_{2}^{2}. \tag{34}$$

In the following, let A denote a generic nonsingular, symmetric $(p \times p)$ -matrix and let M denote a generic subset of $\{1, \ldots, p\}$. For any submatrix B of A, $||B||_2 \leq ||A||_2$. Let $\lambda_1(A)$ denote the smallest eigenvalue of A. By the interlacing property for eigenvalues of principal submatrices (see, e.g., Theorem 4.3.28 in Horn and Johnson (2013)) and the fact that permuting a matrix does not change its eigenvalues, we have

$$\lambda_1((A)_{M,M}) \ge \lambda_1(A)$$

and therefore

$$\|(A)_{MM}^{-1}\|_{2} = \lambda_{1}^{-1}((A)_{M,M}) \le \lambda_{1}^{-1}(A) = \|A^{-1}\|_{2}. \tag{35}$$

Applying this result with A = V yields $\|\Sigma_1(V)\|_2 \le \|V^{-1}\|_2$. Moreover,

$$2\|\hat{V}_{M,M}^{-1} - V_{M,M}\|_2 \|V_{M,M}^{-1}\|_2 \le 2\|\hat{V}_{M,M} - V_{M,M}\|_2 \|\Sigma_1(V)\|_2 \le 2\|\hat{V} - V\|_2 \|V^{-1}\|_2 \le 1$$

so that by Lemma E.15

$$\|\Sigma_1(\hat{V}) - \Sigma_1(V)\|_2 \le 2\|\hat{V}_{M,M} - V_{M,M}\|_2 \|\Sigma_1(V)\|_2^2 \le 2\|\hat{V} - V\|_2 \|V^{-1}\|_2^2 \le \|V^{-1}\|_2.$$

Then, by the triangle inequality,

$$\|\Sigma_1(\hat{V})\|_2 \le \|\Sigma_1(V)\|_2 + \|\Sigma_1(\hat{V}) - \Sigma_1(V)\|_2 \le 2\|V^{-1}\|_2.$$

Moreover,

$$\|\Sigma_1^{-1}(\hat{V}) - \Sigma_1^{-1}(V)\|_2 \le \|\hat{V}_{M,M} - V_{M,M}\|_2 \le \|\hat{V} - V\|_2.$$

By inequality (35), we have $\|\Sigma_2(V)\|_2 \leq \|V\|_2$ and therefore

$$\|(\hat{V}^{-1})_{\bar{M},\bar{M}} - (V^{-1})_{\bar{M},\bar{M}}\|_2 \|\Sigma_2(V)\|_2 \le \|\hat{V}^{-1} - V^{-1}\|_2 \|V\|_2 \le \frac{1}{2},$$

where the last line follows from inequality (34). Thus, by Lemma E.15

$$\begin{split} \|\Sigma_{2}(\hat{V}) - \Sigma_{2}(V)\|_{2} &\leq 2\|(\hat{V}^{-1})_{\bar{M},\bar{M}} - (V)_{\bar{M},\bar{M}}^{-1}\|_{2}\|\Sigma_{2}(V)\|_{2}^{2} \\ &\leq 4\|\hat{V}^{-1} - V^{-1}\|_{2}\|\Sigma_{2}(V)\|_{2}^{2} \leq 4\|\hat{V} - V\|_{2}\|V^{-1}\|_{2}^{2}\|V\|_{2}^{2} \leq \|V\|_{2}. \end{split}$$

By the triangle inequality,

$$\|\Sigma_2(\hat{V})\|_2 \le \|\Sigma_2(V)\|_2 + \|\Sigma_2(\hat{V}) - \Sigma_2(V)\|_2 \le 2\|V\|_2.$$

Moreover by inequality (34)

$$\begin{split} \|\Sigma_2^{-1}(\hat{V}) - \Sigma_2^{-1}(V)\|_2 \leq & \|(\hat{V}^{-1})_{\bar{M},\bar{M}} - (V^{-1})_{\bar{M},\bar{M}}\|_2 \\ \leq & \|\hat{V}^{-1} - V^{-1}\|_2 \leq 2\|\hat{V} - V\|_2 \|V^{-1}\|_2^2. \end{split}$$

For $(\Sigma, \hat{\Sigma}) \in \{(\Sigma_1, \hat{\Sigma}_1), (\Sigma_2, \hat{\Sigma}_2)\}$, let Y and \hat{Y} denote random variables such that $Y \sim N(0, \Sigma)$ and $\hat{Y} \sim N(0, \hat{\Sigma})$. By the calculations above

$$\begin{split} \|\hat{\Sigma}^{-1} - \Sigma^{-1}\|_2 \leq & \|\hat{V} - V\|_2 (1 \vee 2\|V^{-1}\|_2^2), \\ \|\hat{\Sigma}\|_2 \vee & \|\Sigma\|_2 \leq & 2(\|V\|_2 \vee \|V^{-1}\|_2). \end{split}$$

Let $a_N = \sqrt{8p(\|V\|_2 \vee \|V^{-1}\|_2)\log(N/\alpha)}$ and let N be large enough such that

$$\log(N/\alpha) \ge \max\left\{1, \frac{9(2^{2p-1})\alpha^2 p}{\pi}, \frac{2^p - 1}{8p^2}\right\}.$$

By Lemma E.20

$$P(\|Y\|_{\max} > a_N) \le \frac{\alpha}{N^2} \sqrt{\frac{\alpha^2 p}{2\pi \log(N/\alpha)}} \le \frac{\alpha}{3N^2 2^p}.$$

Define the probabilities

$$\begin{split} p(A, M) = & P\left(Y_1(A, M) \leq 0\right) P\left(Y_2(A, M) > 0\right), \\ p_N(A, M) = & P\left(Y_1(A, M) \leq 0 \land \|Y_1(A, M)\|_{\max} \leq a_N\right) \\ & \times & P\left(Y_2(A, M) > 0 \land \|Y_2(A, M)\|_{\max} \leq a_N\right). \end{split}$$

Note that by the characterization of the $\tilde{\chi}^2$ -distribution in Lemma E.14, it suffices to show that

$$\sum_{M\subset\{1,\dots,p\}} p(V,M)P\left(U_{|M|} > \hat{c}_{\alpha,N}\right) \le \frac{\alpha_N}{N}.$$
(36)

We have

$$\begin{aligned} p(V,M) &\leq p_N(V,M) + P\left(\|Y_1(V,M)\| > a_N\right) \\ &+ P\left(\|Y_2(V,M)\| > a_N\right) + P\left(\|Y_1(V,M)\| > a_N\right) P\left(\|Y_2(V,M)\| > a_N\right) \\ &\leq p_N(V,M) + \frac{\alpha}{2pN}. \end{aligned}$$

By the definition of $\hat{c}_{\alpha,N}$,

$$\begin{split} \frac{\alpha}{N} &= \sum_{M \subset \{1,\dots,p\}} p(\hat{V},M) P\left(U_{|M|} > \hat{c}_{\alpha,N}\right) \\ &\geq \sum_{M \subset \{1,\dots,p\}} p_N(\hat{V},M) P\left(U_{|M|} > \hat{c}_{\alpha,N}\right). \end{split}$$

Hence,

$$\sum_{M \subset \{1,...,p\}} p(V,M)P\left(U_{|M|} > \hat{c}_{\alpha,N}\right)
\leq \frac{\alpha}{N} + \sum_{M \subset \{1,...,p\}} \left(p_{N}(V,M) - p_{N}(\hat{V},M)\right) P\left(U_{|M|} > \hat{c}_{\alpha,N}\right) + \sum_{M \subset \{1,...,p\}} \frac{\alpha}{2^{p}N^{2}}
\leq \frac{\alpha}{N} + \sum_{M \subset \{1,...,p\}} \left(p_{N}(V,M) - p_{N}(\hat{V},M)\right) P\left(U_{|M|} > \hat{c}_{\alpha,N}\right) + \frac{\alpha}{N^{2}}
\leq \frac{\alpha}{N} \left(1 + N^{-1} + \sum_{M \subset \{1,...,p\}} \left(\frac{p_{N}(V,M)}{p_{N}(\hat{V},M)} - 1\right)\right).$$
(37)

Note that, for $M \subset \{1, \ldots, p\}$

$$(2^{p} - 1) \|\Sigma_{1}(\hat{V}, M) - \Sigma_{1}(V, M)\|_{2} \|\Sigma_{1}(\hat{V}, M)\|_{2}$$

$$\leq 2(2^{p} - 1) \|\hat{V} - V\|_{2} \left(1 \vee 2\|V^{-1}\|_{2}^{2}\right) \left(\|V\|_{2} \vee \|V^{-1}\|_{2}\right) \leq 1$$

and

$$\begin{split} & \left\| \Sigma_{1}(\hat{V}, M) - \Sigma_{1}(V, M) \right\|_{2} \left(\left\| \Sigma_{1}(V, M) \right\|_{2} \vee \left\| \Sigma_{1}(\hat{V}, M) \right\|_{2} \vee 2a_{N}^{2} \right) \\ \leq & 2 \|\hat{V} - V\|_{2} \left(1 \vee 2\|V^{-1}\|_{2}^{2} \right) \left(\|V\|_{2} \vee \|V^{-1}\|_{2} \right) \left(1 \vee 4p^{2} \log \left(N/\alpha \right) \right) \leq 1. \end{split}$$

Therefore, we can apply Lemma E.21 to argue that

$$\frac{P(Y_1(V, M) \le 0 \land ||Y_1(V, M)||_{\max} \le a_N)}{P(Y_1(\hat{V}, M) \le 0 \land ||Y_1(\hat{V}, M)||_{\max} \le a_N)}$$

$$\le 1 + 16p^2 ||\hat{V} - V||_2 \left(1 \lor 2||V^{-1}||_2^2\right) \left(||V||_2 \lor ||V^{-1}||_2\right) \left(1 + \frac{2(2^p - 1)}{16p^2 \log(N/\alpha)}\right) \log(N/\alpha)$$

$$\leq 1 + 32p^2 \|\hat{V} - V\|_2 (1 \vee 2\|V^{-1}\|_2^2) (\|V\|_2 \vee \|V^{-1}\|_2) \log(N/\alpha).$$

Similarly, we can show that

$$\frac{P(Y_2(V, M) \le 0 \land ||Y_2(V, M)||_{\max} \le a_N)}{P(Y_2(\hat{V}, M) \le 0 \land ||Y_2(\hat{V}, M)||_{\max} \le a_N)}
\le 1 + 32p^2 ||\hat{V} - V||_2 (1 \lor 2||V^{-1}||_2^2) (||V||_2 \lor ||V^{-1}||_2) \log(N/\alpha).$$

Under the assumptions of the lemma,

$$32p^2 \|\hat{V} - V\|_2 \left(1 \vee 2\|V^{-1}\|_2^2\right) \left(\|V\|_2 \vee \|V^{-1}\|_2\right) \log(N/\alpha) \le 1,$$

so that

$$\frac{p_N(V, M)}{p_N(\hat{V}, M)} - 1 \le 96p^2 \|\hat{V} - V\|_2 \left(1 \lor 2\|V^{-1}\|_2^2\right) \left(\|V\|_2 \lor \|V^{-1}\|_2\right) \log \left(N/\alpha\right).$$

Plugging this bound into the right-hand side of (37) verifies (36) and concludes the proof.

Lemma E.23 (Bounds on large quantiles). For a nonsingular $p \times p$ covariance matrix V, let $c_{\alpha,N}(V)$ denote the $(1-\alpha/N)$ -quantile of $\tilde{\chi}^2(V)$. For each a>1, there is N_0 depending only on α , a and p, such that, for $N \geq N_0$,

$$2a^{-1}\log(N/\alpha) < c_{\alpha,N}(V) < 2a\log(N/\alpha).$$

Proof. Let $U_j \sim \chi_j^2$, j = 1, ..., p. For notational convenience, write $c_N = c_{\alpha,N}(V)$. By Lemma E.14, c_N is bounded from above by the $(1-\alpha/N)$ -quantile of U_p . Lemma 1 in Laurent and Massart (2000) implies that, for each $x \geq 0$,

$$P(U_p - p \ge 2\sqrt{px} + 2x) \le \exp(-x).$$

Suppose that $N \geq N_0 \geq \alpha^{-1}$. Choosing $x = \log(N/\alpha)$ in the above inequality yields

$$P(U_p \ge p + 2\sqrt{\log(N/\alpha)}(\sqrt{p} + \sqrt{\log(N/\alpha)})) \le \frac{\alpha}{N}.$$

For N large enough,

$$p + 2\sqrt{\log(N/\alpha)} \left(\sqrt{p} + \sqrt{\log(N/\alpha)}\right) < 2a\log(N/\alpha).$$

This establishes the upper bound on c_N . Let $x = \sqrt{2a^{-1}\log(N/\alpha)}$, and let Φ denote the distribution function and ϕ the density function of a standard normal random variable. Komatu's lower bound

(see, e.g., Duembgen (2010)) is given by

$$1 - \Phi(x) > \frac{2\phi(x)}{\sqrt{4 + x^2} + x}.$$

By Lemma E.14, the distribution $\tilde{\chi}^2(V)$ has point mass $w(p, p, V) \leq 1/2$ at zero. Let $W \sim \tilde{\chi}^2(V)$ and $U_j \sim \chi_j^2$, $j = 1, \ldots, p$. Then,

$$P(W > x^{2}) = \sum_{j=1}^{p} w(p, p - j, V) P(U_{j} > x^{2})$$
$$> (1 - w(p, p, V)) P(U_{1} > x^{2}) \ge \frac{1}{2} P(U_{1} > x^{2}).$$

Suppose that N is large enough such that $\sqrt{4/x^2+1} \leq 2$ For a standard normal random variable Z, we have

$$\frac{1}{2}P(U_1 > x^2) = \frac{1}{2}P(|Z| > x) = 1 - \Phi(x)$$

$$> \frac{2\phi(x)}{x(1 + \sqrt{4/x^2 + 1})}$$

$$> \frac{\sqrt{2}\exp\left(-\frac{x^2}{2}\right)}{3\sqrt{\pi}x} = \frac{(\alpha/N)^{a^{-1}}}{3\sqrt{\pi}a^{-1}\log(N/\alpha)} \equiv p_0^N.$$

Clearly, $p_0^N/(\alpha/N) \to \infty$. For large N, this establishes x^2 as a lower bound on the $(1 - \alpha/N)$ -quantile of W.

F. Proofs for unit selection procedures

We first introduce some additional notation. Let

$$d_{it}^{U}(g,h) = \frac{1}{2} [(y_{it} - x'_{it}\beta_{g,t})^{2} - (y_{it} - x'_{it}\beta_{h,t})^{2}]$$

and

$$\tilde{D}_{i}^{U}(g,h) = \frac{\sum_{t=1}^{T} d_{it}^{U}(g,h)}{\sqrt{\sum_{t=1}^{T} (d_{it}^{U}(g,h) - \bar{d}_{i}^{U}(g,h))^{2}}},$$

where $\bar{d}_i^U(g,h) = \sum_{t=1}^T d_{it}^U(g,h)/T$. Let

$$(s_{i,T}^{U}(g,h))^{2} = \frac{1}{\sigma_{i}^{2}T} \sum_{t=1}^{T} Var(d_{it}^{U}(g,h)),$$

and

$$(S_{i,T}^U(g,h))^2 = \frac{1}{T} \sum_{t=1}^T (d_{it}^U(g,h) - \bar{d}_i^U(g,h))^2.$$

Next, we observe that moment of $d_{it}^U(g_i^0, h)$ can be bounded by terms defined for $d_{it}(g_i^0, h)$. Let

$$Z_{it}^{U}(h) = \frac{d_{it}^{U}(g_i^0, h) - \mathbb{E}_P(d_{it}^{U}(g_i^0, h))}{\sigma_i s_{i,T}^{U}(g, h)}.$$

Note that

$$d_{it}^{U}(g_{i}^{0}, h) - \mathbb{E}_{P}(d_{it}^{U}(g_{i}^{0}, h))$$

$$= \frac{1}{2} \left[u_{it}^{2} - (u_{it} + x_{it}'(\beta_{g_{i}^{0}, t} - \beta_{h, t}))^{2} \right] - \mathbb{E}_{P}(x_{it}'(\beta_{g_{i}^{0}, t} - \beta_{h, t}))^{2}$$

$$= u_{it}x_{it}'\delta_{t}(h, g_{i}^{0}) + \frac{1}{2}\delta_{t}(g_{i}^{0}, h)'(x_{it}x_{it}' - \mathbb{E}_{P}(x_{it}x_{it}')\delta_{t}(g_{i}^{0}, h).$$

This formula indicates that

$$\max_{1 \le i \le N} \max_{h \in \mathbb{G} \setminus \{g_i^0\}} \left(\mathbb{E}_P \left(\frac{1}{T} \sum_{t=1}^T \left| Z_{it}^U(h) \right|^p \right) \right)^{1/p} \le GD_{N,T,p},$$

and

$$\max_{1 \leq t \leq T} \max_{h \in \mathbb{G} \setminus \{q^0\}} \left(\mathbb{E}_P \left(\max_{1 \leq i \leq N} \left| Z_{it}^U(h) \right|^p \right) \right)^{1/p} \leq GB_{N,T,p}.$$

Moreover, we have

$$\max_{1 \le i \le N} \max_{h \in \mathbb{G} \setminus \{g_i^0\}} \left(\frac{1}{T} \sum_{t=1}^T \frac{\mathbb{E}_P\left(\left|d_{it}^U(h)\right|^p\right)}{\sigma_i^p s_{i,T}^p(h)} \right)^{1/p} \le GD_{N,T,p}.$$

Lemma F.1. Suppose that the probability measure P satisfies Assumption 1. Then, there is a constant C depending only on K_{β} and G such that for 0 < c < 1 and

$$\zeta_{N,T}^{U} = \gamma_{N,T,8} \left(T^{-(1-c)/4} \sqrt{\log N} B_{N,T,8}^2 + D_{N,T,4} \right) \left(D_{N,T,2} + T^{-(1-c)/4} \sqrt{\log N} B_{N,T,4} \right) + \gamma_{N,T,8}^2 \left(T^{-(1-c)/2} (\log N) B_{N,T,8}^4 + D_{N,T,4}^2 \right),$$

we have

$$P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g_{i}^{0}\}}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{d}_{it}^{U}(g_{i}^{0},h)-d_{it}^{U}(g_{i}^{0},h)}{\sigma_{i}s_{i,T}^{U}(h)}\right)^{2}\right)$$
(i)
$$>C\gamma_{N,T,8}^{2}\left(T^{-(1-c)/2}B_{N,T,8}^{4}(\log N)+D_{N,T,4}^{2}\right)\right)\leq CT^{-c},$$

$$P\left(T^{-1/2}\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g_{i}^{0}\}}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{d_{it}^{U}(g_{i}^{0},h)}{\sigma_{i}s_{i,T}^{U}(g_{i}^{0},h)}-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\hat{d}_{it}^{U}(g_{i}^{0},h)}{\sigma_{i}s_{i,T}^{U}(g_{i}^{0},h)}\right|$$

$$>C\gamma_{N,T,8}\left(T^{-(1-c)/4}B_{N,T,4}\sqrt{\log N}+D_{N,T,2}\right)\right)\leq CT^{-c},$$

$$P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g_{i}^{0}\}}\left|\frac{1}{T}\sum_{t=1}^{T}\frac{(\hat{d}_{it}^{U}(g_{i}^{0},h)-\bar{d}_{i}^{U}(g_{i}^{0},h))^{2}}{\sigma_{i}^{2}s_{i,T}^{2}(g_{i}^{0},h)}-\frac{1}{T}\sum_{t=1}^{T}\frac{(d_{it}^{U}(g_{i}^{0},h)-\bar{d}_{i}^{U}(g_{i}^{0},h))^{2}}{\sigma_{i}^{2}s_{i,T}^{2}(g_{i}^{0},h)}\right|$$

$$>C\zeta_{N,T}^{U}\right)\leq CT^{-c}.$$
(ii)

Suppose that, additionally, $\zeta_{N,T}^U \vee T^{-(1-c)/4} \sqrt{\log N} B_{N,T,4} \leq 1$. Then

$$P\left(\max_{1 \le i \le N} \left| \hat{D}_{i}(g_{i}^{0}, h) - \tilde{D}_{i}(g_{i}^{0}, h) \right| > C\gamma_{N,T,8}\sqrt{T} \left(T^{-(1-c)/4}B_{N,T,4}\sqrt{\log N} + D_{N,T,2} \right) + C\zeta_{N,T}^{U} \left(D_{N,T,1} + \sqrt{\log N} + T^{-1/4}B_{N,T,4}\log N \right) \right)$$

$$\leq N^{-1} + CT^{-c} + C\left(T^{-1/4}B_{N,T,4}/\log(N) \right)^{4}.$$
 (iv)

Proof of Lemma F.1. **Proof of (i):** Decompose $\hat{d}_{it}^U(h) - d_{it}^U(h)$ as follows

$$\begin{aligned}
\hat{d}_{it}^{U}(h) - d_{it}^{U}(h) \\
&= -u_{it}x'_{it}(\hat{\delta}_{t}(g_{i}^{0}, h) - \delta_{t}(g_{i}^{0}, h)) \\
&+ (x'_{it}(\beta_{g_{i}^{0}, t} - \hat{\beta}_{g_{i}^{0}, t}))^{2}/2 + (x'_{it}(\beta_{h, t} - \hat{\beta}_{h, t}))^{2}/2 - (\beta_{h, t} - \hat{\beta}_{h, t})(x_{it}x'_{it})\delta_{t}(g_{i}^{0}, h).
\end{aligned}$$

By the inequality $(a + b + c + d)^2 \le 4(a^2 + b^2 + c^2 + d^2)$,

$$\left(\frac{\hat{d}_{it}^{U}(h) - d_{it}^{U}(h)}{\sigma_{i}}\right)^{2} \leq 4 \left|\frac{u_{it}}{\sigma_{i}}\right|^{2} \|x_{it}\|^{2} \|\hat{\delta}_{t}(g_{i}^{0}, h) - \delta_{t}(g_{i}^{0}, h)\|^{2}
+ 2\sigma_{i}^{-2} \|\hat{\beta}_{g_{i}^{0}, t} - \beta_{g_{i}^{0}, t}\|^{4} \|x_{it}\|^{4}
+ 2\sigma_{i}^{-2} \|\hat{\beta}_{h, t} - \beta_{h, t}\|^{4} \|x_{it}\|^{4}
+ 2\sigma_{i}^{-2} \|\hat{\beta}_{h, t} - \beta_{h, t}\|^{2} \|x_{it}\|^{4} \|\delta_{t}(g_{i}^{0}, h)\|^{2}.$$

Let $V_{it} = \left(|u_{it}/\sigma_i|^2 ||x_{it}||^2 + ||x_{it}||^4 / \sigma_i^4 \right) / \underline{s}_{N,T}^2$. Now, by the Cauchy-Schwarz inequality

$$\frac{1}{T} \sum_{t=1}^{T} \left(\frac{\hat{d}_{it}^{U}(h) - d_{it}^{U}(h)}{\sigma_{i} s_{i,T}(h)} \right)^{2}$$

$$\leq C \left\{ \max_{g \in \mathbb{G}} \left(\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{\beta}_{g,t} - \beta_{g,t} \right\|^{4} \right)^{1/2} + \max_{g \in \mathbb{G}} \left(\frac{1}{T} \sum_{t=1}^{T} \left\| \hat{\beta}_{g,t} - \beta_{g,t} \right\|^{8} \right)^{1/2} \right\}$$

$$\times \left(\frac{1}{T} \sum_{t=1}^{T} \left(\left| \frac{u_{it}}{\sigma_{i}} \right|^{4} \left\| x_{it} \right\|^{4} + \left\| x_{it} \right\|^{8} / \sigma_{i}^{4} \right) / \underline{s}_{N,T}^{4} \right)^{1/2}$$

$$\leq C (\gamma_{N,T,8}^{2} + \gamma_{N,T,8}^{4}) \left(\frac{1}{T} \sum_{t=1}^{T} \left(V_{it}^{2} - \mathbb{E}_{P} [V_{it}^{2}] \right) + \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{P} [V_{it}^{2}] \right)^{1/2}.$$

Here, we note that $var(d_{it}^U(h)) \ge var(d_{it}(h))$ so that $s_{i,T}^U(h) \ge s_{i,T}(h) \ge \underline{s}_{N,T}$. Together with (29), this implies the desired result.

Proof of (ii): By slightly modifying the arguments above, we can prove

$$\left| \frac{1}{T} \sum_{t=1}^{T} \frac{\hat{d}_{it}^{U}(h) - d_{it}^{U}(h)}{\sigma_{i} s_{i,T}^{U}(h)} \right|$$

$$\leq C(\gamma_{N,T,4} + \gamma_{N,T,4}^{2}) \left(\frac{1}{T} \sum_{t=1}^{T} \left(V_{it} - \mathbb{E}_{P} \left[V_{it} \right] \right) + \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{P} \left[V_{it} \right] \right)^{1/2}.$$

And, for 0 < c < 1,

$$P\left(\max_{1 \le i \le N} \left| \frac{1}{T} \sum_{t=1}^{T} \left(V_{it} - \mathbb{E}_{P}[V_{it}] \right) \right| > CT^{-(1-c)/2} B_{N,T,4}^{2}(\log N) \right) \le CT^{-c}.$$

Thus it follows that

$$P\left(T^{-1/2} \max_{1 \le i \le N} \left| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{d_{it}^{U}(g_{i}^{0}, h)}{\sigma_{i} s_{i,T}^{U}(g_{i}^{0}, h)} - \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\hat{d}_{it}^{U}(g_{i}^{0}, h)}{\sigma_{i} s_{i,T}^{U}(g_{i}^{0}, h)} \right|$$

$$> C\gamma_{N,T,8} \left(T^{-(1-c)/4} B_{N,T,4} \sqrt{\log N} + D_{N,T,2} \right) \right) \le CT^{-c}.$$

Proof of (iii): We observe that

$$\left| \frac{1}{T} \sum_{t=1}^{T} (\hat{d}_{it}^{U}(h) - \bar{d}_{i}^{U}(h))^{2} - \frac{1}{T} \sum_{t=1}^{T} (d_{it}^{U}(h) - \bar{d}_{i}^{U}(h))^{2} \right| \\
\leq \frac{1}{T} \sum_{t=1}^{T} \left(\hat{d}_{it}^{U}(h) - d_{it}^{U}(h) - (\bar{d}_{i}^{U}(h) - \bar{d}_{i}^{U}(h)) \right)^{2}$$

$$+2\left|\frac{1}{T}\sum_{t=1}^{T} \left(d_{it}^{U}(h)\right) \left(\hat{d}_{it}^{U}(h) - d_{it}^{U}(h) - \left(\bar{d}_{i}^{U}(h) - \bar{d}_{i}^{U}(h)\right)\right)\right|$$

$$\leq \frac{1}{T}\sum_{t=1}^{T} \left(\hat{d}_{it}^{U}(h) - d_{it}^{U}(h)\right)^{2}$$

$$+2\sqrt{\frac{1}{T}\sum_{t=1}^{T} \left(d_{it}^{U}(h)\right)^{2}}\sqrt{\frac{1}{T}\sum_{t=1}^{T} \left(\hat{d}_{it}^{U}(h) - d_{it}^{U}(h)\right)^{2}}.$$

Let

$$U_{it}^{U}(h) = \frac{\left(d_{it}^{U}(h)\right)^{2} - \mathbb{E}_{P}(\left(d_{it}^{U}(h)\right)^{2})}{\sigma_{i}^{2}s_{i,T}^{2}(h)}.$$

Note that

$$\sum_{t=1}^{T} U_{it}^{U}(h) = \frac{1}{T} \sum_{t=1}^{T} \frac{\left(d_{it}^{U}(h)\right)^{2}}{\sigma_{i}^{2} s_{i,T}^{2}(h)} - \frac{1}{T} \sum_{t=1}^{T} \frac{\mathbb{E}_{P}(\left(d_{it}^{U}(h)\right)^{2})}{\sigma_{i}^{2} s_{i,T}^{2}(h)}$$

Because, $\mathbb{E}_P\left(\max_{1\leq i\leq N}\max_{1\leq t\leq T}(U_{it}^U(h))^2\right)\leq CTB_{N,T,4}^4$, following the same argument as the proof of Lemma E.8 part (i) gives

$$P\left(\max_{1 \le i \le n} \frac{1}{T} \sum_{t=1}^{T} U_{it}^{U}(h) > CB_{N,T,4} T^{-(1-c)/2} \log N\right) \le CT^{-c}.$$
 (38)

Therefore, with probability at least $1 - CT^{-c}$,

$$\begin{split} &\left| \frac{1}{T} \sum_{t=1}^{T} \frac{(\hat{d}_{it}^{U}(h) - \hat{d}_{i}^{U}(h))^{2}}{\sigma_{i}^{2} s_{i,T}^{2}(h)} - \frac{1}{T} \sum_{t=1}^{T} \frac{(d_{it}^{U}(h) - d_{i}^{U}(h))^{2}}{\sigma_{i}^{2} s_{i,T}^{2}(h)} \right| \\ &\leq \frac{1}{T} \sum_{t=1}^{T} \frac{(\hat{d}_{it}^{U}(h) - d_{it}^{U}(h))^{2}}{\sigma_{i}^{2} \underline{s}_{N,T}^{2}} \\ &+ \max_{h \in \mathbb{G}} 2 \sqrt{\frac{1}{T} \sum_{t=1}^{T} \frac{(d_{it}^{U}(h))^{2}}{\sigma_{i}^{2} s_{i,T}^{2}(h)}} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \frac{(\hat{d}_{it}^{U}(h) - d_{it}^{U}(h))^{2}}{\sigma_{i}^{2} \underline{s}_{N,T}^{2}}} \\ &\leq \gamma_{N,T,8} \left(T^{-(1-c)/4} \sqrt{\log N} B_{N,T,8}^{2} + D_{N,T,4}\right) \left(D_{N,T,2} + T^{-(1-c)/4} \sqrt{\log N} B_{N,T,4}\right) \\ &+ \gamma_{N,T,8}^{2} \left(T^{-(1-c)/2}(\log N) B_{N,T,8}^{4} + D_{N,T,4}^{2}\right), \end{split}$$

where the last inequality follows from Lemma F.1 part (i) and (38).

Proof of (iv): Define

$$S_{i,T}^{U\Delta}(h) = \left(\frac{\hat{S}_{i,T}^U(h) - S_{i,T}^U(h)}{\sigma_i s_{i,T}(h)}\right) \frac{S_{i,T}^U(h)}{\sigma_i s_{i,T}(h)}.$$

By the inequality $|a-b| \le |a-b|/(\sqrt{a}+\sqrt{b}) \le |a-b|/\sqrt{a}$ and part (iii) of the lemma, we have

$$S_{i,T}^{U\Delta}(h) \le \left| \left(\hat{S}_{i,T}^2(h) / (\sigma_i s_{i,T}(h)) \right)^2 - \left(S_{i,T}^2(h) / (\sigma_i s_{i,T}(h)) \right)^2 \right| \le C_2 \zeta_{N,T}$$

uniformly over $i=1,\ldots,N$ on a set of probability less than CT^{-c} . By the inequality $|\sqrt{a}-1| \le |a-1|$ and Lemma E.8 we have

$$\left| (S_{i,T}^U(h))^2 / (\sigma_i s_{i,T}(h)) - 1 \right| \le \left| \left((S_{i,T}^U(h))^2 / (\sigma_i s_{i,T}(h)) \right)^2 - 1 \right| \le C_1 T^{-(1-c)/2} (\log N) B_{N,T,4}^2$$

uniformly over i = 1, ..., N on a set of probability less than CT^{-c} . Note that

$$\begin{aligned} \left| D_i^U(g_i^0, h) \right| &\leq \left| \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{d_{it}^U(h) - E_P(d_{it}^U(h))}{\sigma_i s_{i,T}(h)} \right| + \left| \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{E_P(d_{it}^U(h))}{\sigma_i s_{i,T}(h)} \right| \\ &\leq \left| \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{d_{it}^U(h) - E_P(d_{it}^U(h))}{\sigma_i s_{i,T}(h)} \right| + D_{N,T,1} \end{aligned}$$

Thus, by following the same argument as that in the proof of Lemma E.8 part (ii), it holds that

$$|D_i^U(h)| \le D_{N,T,1} + C\left(\sqrt{\log N} + T^{-1/4}B_{N,T,4}\log N\right)$$

uniformly over i = 1, ..., N on a set of probability less than $N^{-1} + C(T^{-1/4}B_{N,T,4}/\log(N))^4$. Now, decompose

$$\begin{split} \hat{D}_{i}^{U}(h) - \tilde{D}_{i}^{U}(h) &= \frac{\sigma_{i}s_{i,T}(h)}{\hat{S}_{i,T}^{U}(h)} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\hat{d}_{it}^{U}(h)}{\sigma_{i}s_{i,T}(h)} - D_{i}^{U}(h) \right) \\ &- \frac{\hat{S}_{i,T}^{U}(h) - S_{i,T}^{U}(h)}{S_{i,T}^{U}(h)\hat{S}_{i,T}^{U}(h)} \sigma_{i}s_{i,T}(h) D_{i}^{U}(h) \\ &= \frac{S_{i,T}^{U}(h)/(\sigma_{i}s_{i,T}(h))}{S_{i,T}^{U}(h) + S_{i,T}^{U}(h)/(\sigma_{i}s_{i,T}(h))} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\hat{d}_{it}^{U}(h)}{\sigma_{i}s_{i,T}(h)} - D_{i}^{U}(h) \right) \\ &- \frac{S_{i,T}^{U\Delta}}{\left(S_{i,T}^{U\Delta} + (S_{i,T}^{U}(h))^{2}/(\sigma_{i}s_{i,T}(h))^{2} \right) (S_{i,T}^{U}(h))^{2}/(\sigma_{i}s_{i,T}(h))} D_{i}^{U}(h). \end{split}$$

In conjunction with part (ii) of the lemma this decomposition implies

$$\begin{split} \max_{1 \leq i \leq N} \left| \hat{D}_i^U(h) - \tilde{D}_i^U(h) \right| \leq & C \gamma_{N,T,8} \sqrt{T} \left(T^{-(1-c)/4} B_{N,T,4} \sqrt{\log N} + D_{N,T,4}^2 \right) \\ & + C \zeta_{N,T}^U \left(D_{N,T,1} + \sqrt{\log N} + T^{-1/4} B_{N,T,4} \log N \right) \end{split}$$

with probability less than $CT^{-c} + N^{-1} + C(T^{-1/4}B_{N,T,4}/\log(N))^4$.

Proof of Theorem 5. We note that the hypothesis selection part of the procedure does not affect the theoretical analysis. This is because, here, we focus on size and thus need to consider only the behavior of the test statistics under $\{g_i^0\}_{i=1}^N$.

Let

$$J_{1} = \left\{ (i,h) \mid i \in \{1,\dots,N\}, h \in \mathbb{G} \setminus \{g_{i}^{0}\}, \frac{\sqrt{T}\mathbb{E}_{P}(\bar{d}_{i}^{U}(g_{i}^{0},h))}{\sigma_{i}s_{i,T}^{U}(g,h)} > -c_{\beta,N}^{SNS} \right\}$$

Roughly speaking, J_1 is the set of pairs of units and groups that are difficult to distinguish from true group membership.

In this proof, we set c = 1/6.

Step 1: We first prove that $P\left(\max_{(i,h)\in J_1^c} \tilde{d}_i^U(g_i^0,h) \leq 0\right) > 1 - \beta - CT^{-c}$. Note that $\bar{d}_i^U(g_i^0,h) > 0$ for some $(i,h)\in J_1^c$ implies that

$$\max_{(i,h)\in J_1} \frac{\sqrt{T}(\hat{\bar{d}}_i^U(g_i^0,h)) - \mathbb{E}_P(\bar{d}_i^U(g_i^0,h)))}{\sigma_i s_{i,T}^U(g,h)} > c_{\beta,N}^{\mathrm{SNS}}.$$

Let

$$c_{SN}(\beta) = \frac{\Phi^{-1}(1 - \beta/((G - 1)N))}{\sqrt{1 - \Phi^{-1}(1 - \beta/((G - 1)N))^2/T}}.$$

Let

$$\epsilon_{N,T,1}^{U} = \sqrt{T}\gamma_{N,T,8} \left(T^{-(1-c)/4} B_{N,T,4} \sqrt{\log N} + D_{N,T,2} \right).$$

We have

$$\begin{split} P\left(\max_{(i,h)\in J_1} \frac{\sqrt{T}(\bar{d}_i^U(g_i^0,h)) - \mathbb{E}_P(\bar{d}_i^U(g_i^0,h)))}{\sigma_i s_{i,T}^U(g,h)} > c_{\beta,N}^{\mathrm{SNS}}\right) \\ \leq & P\left(\max_{(i,h)\in J_1} \frac{\sqrt{T}(\bar{d}_i^U(g_i^0,h)) - \mathbb{E}_P(\bar{d}_i^U(g_i^0,h)))}{\sigma_i s_{i,T}^U(g,h)} > c_{\beta,N}^{\mathrm{SNS}} - \epsilon_{N,T,1}^U\right) \\ & + P\left(\max_{(i,h)\in J_1} \left|\frac{\sqrt{T}(\bar{d}_i^U(g_i^0,h)) - \bar{d}_i^U(g_i^0,h))}{\sigma_i s_{i,T}^U(g,h)}\right| > \epsilon_{N,T,1}^U\right). \end{split}$$

The second term on the right-hand side is bounded by CT^{-c} by Lemma F.1 part (ii). Let β_N solve $c_{\beta_N,N}^{SNS} = c_{\beta,N}^{SNS} - \epsilon_{N,T,1}^U$. As in the proof of Theorem 2, we have

$$|\beta_N - \beta| \le 4\epsilon_{N,T,1}^U \sqrt{\log((G-1)N/\beta)}$$
.

Thus we have

$$\begin{split} P\left(\max_{(i,h)\in J_{1}} \frac{\sqrt{T}(\bar{d}_{i}^{U}(g_{i}^{0},h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(g_{i}^{0},h)))}{\sigma_{i}s_{i,T}^{U}(g,h)} > c_{\beta,N}^{\mathrm{SNS}}\right) \\ \leq & P\left(\max_{(i,h)\in J_{1}} \frac{\sqrt{T}(\bar{d}_{i}^{U}(g_{i}^{0},h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(g_{i}^{0},h)))}{\sigma_{i}s_{i,T}^{U}(g,h)} > c_{\beta_{N},N}^{\mathrm{SNS}}\right) + CT^{-c} \\ = & P\left(\max_{(i,h)\in J_{1}} \frac{\sqrt{T}(\bar{d}_{i}^{U}(g_{i}^{0},h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(g_{i}^{0},h)))}{\sigma_{i}s_{i,T}^{U}(g,h)} > c_{SN}(c_{SN}^{-1}(c_{\beta_{N},N}^{\mathrm{SNS}}))\right) + CT^{-c}. \end{split}$$

Following essentially the same argument as that in Step 1 of the proof of Theorem 4.2 of Chernozhukov, Chetverikov, and Kato (2018) shows that, under Assumptions (8) and (9),

$$P\left(\max_{(i.h)\in J_1} \frac{\sqrt{T}(\bar{d}_i^U(g_i^0, h)) - \mathbb{E}_P(\bar{d}_i^U(g_i^0, h)))}{\sigma_i s_{i,T}^U(g, h)} > c_{SN}(c_{SN}^{-1}(c_{SN}^{SNS}))\right) \le c_{SN}^{-1}(c_{\beta_N, N}^{SNS}) + CT^{-c}.$$

Note that here we replace $\hat{\sigma}_j$ and σ_j in the proof of Chernozhukov, Chetverikov, and Kato (2018) with $(T^{-1}\sum_{t=1}^T (d_{it}^U(g_i^0, h) - \mathbb{E}_P(d_{it}^U(g_i^0, h))))^{1/2}$ and $\sigma_i s_{i,T}^U(g, h)$. We have

$$c_{SN}^{-1}(c_{\beta_N,N}^{\text{SNS}}) = (G-1)N\left(1 - \Phi\left(\frac{c_{\beta_N,N}^{\text{SNS}}}{\sqrt{1 + (c_{\beta_N,N}^{\text{SNS}})^2/T}}\right)\right)$$
$$= \beta + O\left(\frac{(c_{\beta_N,N}^{\text{SNS}})^3}{\sqrt{T}}\right).$$

We thus have

$$P\left(\max_{(i,h)\in J_{1}} \frac{\sqrt{T}(\bar{d}_{i}^{U}(g_{i}^{0},h)) - \mathbb{E}_{P}(d_{it}^{U}(g_{i}^{0},h)))}{\sigma_{i}s_{i,T}^{U}(g,h)} > \sqrt{\frac{T}{T-1}}t_{T-1}^{-1}\left(1 - \frac{\beta_{N}}{(G-1)N}\right)\right)$$

$$\leq \beta_{N} + O\left(\frac{(c_{\beta_{N},N}^{SNS})^{3}}{\sqrt{T}}\right) + CT^{-c} \leq \beta + CT^{-c},$$

where $(c_{\beta_N,N}^{\text{SNS}})^3/\sqrt{T} \leq CT^{-c}$ by that $(\log(N))^6/T \leq CT^{-c}$ which is implied by (8) together with $D_{N,T,3} \geq 1$ and Lemma E.11, and $\epsilon_{N,T,1}^U \sqrt{\log((G-1)N/\beta)} \leq CT^{-c}$ by assumption (10).

An implication of Step 1 is as follows. Let

$$\mathbb{N} = \left\{ i \in \{1, \dots, N\} \mid \max_{h \in \mathbb{G} \setminus \{g_i^0\}} \frac{\sqrt{T} \mathbb{E}_P(\bar{d}_i^U(g_i^0, h))}{\sigma_i s_{i, T}^U(g, h)} > -c_{\beta, N}^{\text{SNS}} \right\}.$$

Then

$$P\left(\max_{i\in\mathbb{N}^c}\max_{h\in\mathbb{G}\setminus\{g_i^0\}}\bar{\hat{d}}_i^U(g_i^0,h)\leq 0\right)>1-\beta-CT^{-c}.$$

Step 2: Next, we prove that $P(\times_{i=1}^N \hat{M}_i(g_i^0) \supseteq J_1) \ge 1 - \beta - CT^{-c}$. Here, we drop the g argument for simplicity of notation when arguments are g_i^0 and h.

We note that

$$P\left(\underset{i=1}{\overset{N}{\times}} \hat{M}_{i}(g_{i}^{0}) \not\supseteq J_{1} \right)$$

$$=P\left(\exists (i,h); \hat{D}_{i}^{U}(h) \leq -2c_{\beta_{N},N}^{\mathrm{SNS}} \text{ and } \frac{\sqrt{T}\mathbb{E}_{P}(\bar{d}_{i}^{U}(h))}{\sigma_{i}s_{i,T}^{U}(h)} > -c_{\beta,N}^{\mathrm{SNS}} \right)$$

$$\leq P\left(\exists (i,h); \hat{D}_{i}^{U}(h) \leq -2c_{\beta,N}^{\mathrm{SNS}} + \epsilon_{N,T,2}^{U} \text{ and } \frac{\sqrt{T}\mathbb{E}_{P}(\bar{d}_{i}^{U}(h))}{\sigma_{i}s_{i,T}^{U}(h)} > -c_{\beta,N}^{\mathrm{SNS}} \right)$$

$$+ P\left(\max_{1 \leq i \leq N} \max_{h \in \mathbb{G} \setminus \{g_{i}^{0}\}} \left| \hat{D}_{i}^{U}(h) - \tilde{D}_{i}^{U}(h) \right| > \epsilon_{N,T,2}^{U} \right),$$

where

$$\epsilon_{N,T,2}^{U} = C\gamma_{N,T,8}\sqrt{T} \left(T^{-(1-c)/4}B_{N,T,4}\sqrt{\log N} + D_{N,T,2} \right) + C\zeta_{N,T}^{U} \left(D_{N,T,1} + \sqrt{\log N} + T^{-1/4}B_{N,T,4}\sqrt{\log N} \right).$$

By part (iv) of Lemma F.1, noting that its condition is satisfied by (9), (10) and (11),

$$P\left(\max_{1 \le i \le N} \max_{h \in \mathbb{G} \setminus \{g_i^0\}} \left| \hat{D}_i^U(h) - \tilde{D}_i^U(h) \right| > \epsilon_{N,T,2}^U \right) < N^{-1} + CT^{-c} + C\left(T^{-1/4}B_{N,T,4}/\log(N)\right)^4 < N^{-1} + CT^{-c}.$$

where the second inequality follows because (9) implies $\left(T^{-1/4}B_{N,T,4}/\log(N)\right)^4 \leq T^{-1/6}$ We observe

$$\begin{split} &P\left(\exists (i,h); \tilde{D}_i^U(h) \leq -2c_{\beta,N}^{\mathrm{SNS}} + \epsilon_{N,T,2}^U \text{ and } \frac{\sqrt{T}\mathbb{E}_P(\bar{d}_i^U(h))}{\sigma_i s_{i,T}^U(h)} > -c_{\beta,N}^{\mathrm{SNS}}\right) \\ &\leq &P\left(\max_{1\leq i\leq N} \max_{h\in \mathbb{G}\setminus \{g_i^0\}} \left[\sqrt{T}(E(\bar{d}_i^U(h)) - \bar{d}_i^U(h)) - (2S_{i,T}^U(h) - \sigma_i s_{i,T}^U(h))c_{\beta,N}^{\mathrm{SNS}} + 2S_{i,T}^U(h)\epsilon_{N,T,2}^U\right] > 0\right). \end{split}$$

Let

$$(\tilde{S}_{i,T}^{U}(h))^{2} = \frac{1}{T} \sum_{t=1}^{T} (d_{it}^{U}(h) - \mathbb{E}_{P}(d_{it}^{U}(h)))^{2} - \left(\frac{1}{T} \sum_{t=1}^{T} (d_{it}^{U}(h) - \mathbb{E}_{P}(d_{it}^{U}(h)))\right)^{2}.$$

We observe that

$$(S_{i,T}^U(h))^2 = \frac{1}{T} \sum_{t=1}^T (d_{it}^U(h) - \mathbb{E}_P(d_{it}^U(h)))^2$$

$$+ \frac{2}{T} \sum_{t=1}^{T} (d_{it}^{U}(h) - \mathbb{E}_{P}(d_{it}^{U}(h))) (\mathbb{E}_{P}(d_{it}^{U}(h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(h)))$$

$$- \left(\frac{1}{T} \sum_{t=1}^{T} (d_{it}^{U}(h) - \mathbb{E}_{P}(d_{it}^{U}(h)))\right)^{2}$$

$$+ \frac{1}{T} \sum_{t=1}^{T} (\mathbb{E}_{P}(d_{it}^{U}(h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(h)))^{2}$$

$$\geq (\tilde{S}_{i,T}(h))^{2} + \frac{2}{T} \sum_{t=1}^{T} (d_{it}^{U}(h) - \mathbb{E}_{P}(d_{it}^{U}(h))) (\mathbb{E}_{P}(d_{it}^{U}(h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(h))).$$

If $1 - \sigma_i s_{i,T}^U(h)/\tilde{S}_{i,T}^U(h) \geq -r/2$ and

$$\frac{2}{T} \sum_{t=1}^{T} (d_{it}^{U}(h) - \mathbb{E}_{P}(d_{it}^{U}(h))) (\mathbb{E}_{P}(d_{it}^{U}(h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(h))) \ge -(\tilde{S}_{i,T}(h))^{2} \left(\frac{r}{2} - \frac{r^{2}}{16}\right),$$

for some 0 < r < 1, we have

$$2S_{i,T}^{U}(h) - \sigma_i s_{i,T}^{U}(h) \ge (1-r)\tilde{S}_{i,T}^{U}(h)$$

because

$$2S_{i,T}^{U}(h) - \sigma_{i} s_{i,T}^{U}(h) \ge 2\tilde{S}_{i,T}^{U}(h) \left(1 - \frac{r}{2} - \frac{r^{2}}{16}\right)^{1/2} - \sigma_{i} s_{i,T}^{U}(h)$$
$$= \tilde{\sigma}_{i,h} \left(2\left(1 - \frac{r}{4}\right) - \frac{\sigma_{i} s_{i,T}^{U}(h)}{\tilde{S}_{i,T}^{U}(h)}\right) \ge (1 - r)\tilde{S}_{i,T}^{U}(h).$$

We thus have

$$P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g_{i}^{0}\}}\left[\sqrt{T}(E(\bar{d}_{i}^{U}(h))-\bar{d}_{i}^{U}(h))-(2S_{i,T}^{U}(h)-\sigma_{i}S_{i,T}^{U}(h))c_{\beta,N}^{\mathrm{SNS}}+2S_{i,T}^{U}(h)\epsilon_{N,T,2}^{U}\right]>0\right)$$

$$\leq P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g(i)\}}\frac{\sqrt{T}(E(\bar{d}_{i}^{U}(h))-\bar{d}_{i}^{U}(h))}{\tilde{S}_{i,T}^{U}(h)}>(1-r)c_{\beta,N}^{\mathrm{SNS}}-2\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g(i)\}}\frac{S_{i,T}^{U}(h)}{\tilde{S}_{i,T}^{U}(h)}\epsilon_{N,T,2}^{U}\right)$$

$$(39)$$

$$+P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g(i)\}}\left|\frac{2}{T}\sum_{t=1}^{T}\tilde{a}_{it}(h)\right|>\frac{r}{2}-\frac{r^2}{16}\right)$$
(40)

$$+ P\left(\max_{1 \le i \le N} \max_{h \in \mathbb{G} \setminus \{g(i)\}} \left| \frac{\sigma_i s_{i,T}^U(h)}{\tilde{S}_{i,T}^U(h)} - 1 \right| > \frac{r}{2} \right), \tag{41}$$

where

$$\tilde{a}_{it}(h) = 2(d_{it}^{U}(h) - \mathbb{E}_{P}(d_{it}^{U}(h)))(\mathbb{E}_{P}(d_{it}^{U}(h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(h)))/(\tilde{S}_{i,T}^{U}(h))^{2}.$$

We now take $r = T^{-(1-c)/2}B_{T,N,4}^2 \log((G-1)N)$. The first term of (39) is

$$\begin{split} &P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\backslash\{g(i)\}}\frac{\sqrt{T}(E(\bar{d}_i^U(h))-\bar{d}_i^U(h))}{\tilde{S}_{i,T}^U(h)}>(1-r)c_{\beta,N}^{\mathrm{SNS}}-2\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\backslash\{g(i)\}}\frac{S_{i,T}^U(h)}{\tilde{S}_{i,T}^U(h)}\epsilon_{N,T,2}^U\right)\\ \leq &P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\backslash\{g(i)\}}\frac{\sqrt{T}(E(\bar{d}_i^U(h))-\bar{d}_i^U(h))}{\tilde{S}_{i,T}^U(h)}>(1-r)c_{\beta,N}^{\mathrm{SNS}}-C\epsilon_{N,T,2}^U\right)\\ &+P\left(\left|\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\backslash\{g(i)\}}\frac{S_{i,T}^U(h)}{\tilde{S}_{i,T}^U(h)}\right|>\frac{1}{2}C\right). \end{split}$$

Note that we can take C > 2 and

$$P\left(\left|\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g(i)\}}\frac{S_{i,T}^U(h)}{\tilde{S}_{i,T}^U(h)}\right|>\frac{1}{2}C\right)< CT^{-c}$$

holds because

$$\frac{S_{i,T}^U(h)}{\tilde{S}_{i,T}^U(h)} = \frac{S_{i,T}^U(h)}{\sigma_i s_{i,T}^U(h)} \frac{\sigma_i s_{i,T}^U(h)}{\tilde{S}_{i,T}^U(h)},$$

Lemma F.1 part (iii) and following the same argument of Lemma D.5 of Chernozhukov, Chetverikov, and Kato (2018). Following the argument in the proof of Step 2 of Theorem 4.2 of Chernozhukov, Chetverikov, and Kato (2018) under (8), (9) and that (10) and (11) implies $\epsilon_{N,T,2}^U \sqrt{\log((G-1)N/\beta)} \leq CT^{-1/6}$, it holds that

$$P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\backslash\{g(i)\}}\frac{\sqrt{T}(E(\bar{d}_i^U(h))-\bar{d}_i^U(h))}{\tilde{S}_{i,T}^U(h)}>(1-r)c_{\beta,N}^{\mathrm{SNS}}-C\epsilon_{N,T,2}^U\right)\leq \beta+CT^{-c}.$$

For the second term (40), let $a_{it}(h) = 2(d_{it}^U(h) - \mathbb{E}_P(d_{it}^U(h)))(\mathbb{E}_P(d_{it}^U(h)) - \mathbb{E}_P(\bar{d}_i^U(h)))/(\sigma_i s_{i,T}^U(h))^2$. The second term is

$$P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g(i)\}}\left|\frac{1}{T}\sum_{t=1}^{T}\tilde{a}_{it}(h)\right|>\frac{r}{2}-\frac{r^{2}}{16}\right)$$

$$\leq P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g(i)\}}\left|\frac{1}{T}\sum_{t=1}^{T}a_{it}(h)\right|>\left(1-\frac{r}{2}\right)\left(\frac{r}{2}-\frac{r^{2}}{16}\right)\right)$$

$$+P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g(i)\}}\left|\frac{(\tilde{S}_{i,T}^{U}(h))^{2}}{(\sigma_{i}s_{i,T}^{U}(h))^{2}}-1\right|>\frac{r}{2}\right),$$

where the inequality holds because $(\tilde{S}_{i,T}^U(h))^2 \geq (1-r/2)(\sigma_i s_{i,T}^U(h))^2$ if $1-(\tilde{S}_{i,T}^U(h))^2/(\sigma_i s_{i,T}^U(h))^2 > -r/2$. The second term is bounded by CT^{-c} by Lemma A.5 of Chernozhukov, Chetverikov, and Kato (2018) (Note that the statement of Lemma A.5 of Chernozhukov, Chetverikov, and Kato (2018) is about $\hat{\sigma}_j/\sigma_j$ (in their notation) but their proof is based on $\hat{\sigma}_j^2/\sigma_j^2$). For the first term,

observe that

$$\sum_{t=1}^{T} \mathbb{E}_{P}((a_{it}(h)/T)^{2}) = \frac{1}{T^{2}} \sum_{t=1}^{T} \frac{var(d_{it}^{U}(h))}{(\sigma_{i}s_{i,T}^{U}(h))^{4}} (\mathbb{E}_{P}(d_{it}^{U}(h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(h)))^{2}$$

$$\leq \frac{1}{T^{2}} \sum_{t=1}^{T} \frac{(\mathbb{E}_{P}(d_{it}^{U}(h)) - \mathbb{E}_{P}(\bar{d}_{i}^{U}(h)))^{2}}{(\sigma_{i}s_{i,T}^{U}(h))^{2}},$$

and

$$\begin{split} & \sum_{t=1}^T E\left(\max_{1 \leq i \leq N} \max_{h \in \mathbb{G} \backslash \{g(i)\}} (a_{it}(h)/T)^2\right) \\ \leq & \frac{1}{T^2} \sum_{t=1}^T B_{T,N,4}^2 \max_{1 \leq i \leq N} \max_{h \in \mathbb{G} \backslash \{g(i)\}} (\mathbb{E}_P(d_{it}^U(g_i^0,h)) - \mathbb{E}_P(\bar{d}_i^U(g_i^0,h)))^2/(\sigma_i s_{i,T}^U(h))^2 \\ \leq & \frac{1}{T} G B_{T,N,4}^2 D_{N,T,2}^2. \end{split}$$

By Lemma D.3 of Chernozhukov, Chetverikov, and Kato (2018), we have

$$E\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g(i)\}}\left|\frac{1}{T}\sum_{t=1}^{T}a_{it}(h)\right|\right)\leq CD_{T,N,2}\left(\frac{\sqrt{\log((G-1)N)}}{\sqrt{T}}+B_{T,N,4}\frac{\log((G-1)N)}{T}\right).$$

By Lemma D.2 of Chernozhukov, Chetverikov, and Kato (2018), we thus have

$$\begin{split} & P\left(\max_{1 \leq i \leq N} \max_{h \in \mathbb{G} \setminus \{g(i)\}} \left| \frac{1}{T} \sum_{t=1}^{T} a_{it}(h) \right| \geq C D_{T,N,2} \left(\frac{\sqrt{\log((G-1)N)}}{\sqrt{T}} + B_{T,N,4} \frac{\log((G-1)N)}{T} \right) + t \right) \\ \leq & e^{-t^2/(3(D_{T,N,2}^2/T))} + \frac{K}{t^2} \frac{1}{T} B_{T,N,4}^2 D_{T,N,2}^2, \end{split}$$

for any t>0. Taking $t=T^{-(1-c)/2}D_{T,N,2}B_{T,N,4}$ and arranging the terms, we have

$$P\left(\max_{1 \le i \le N} \max_{h \in \mathbb{G} \setminus \{g(i)\}} \left| \frac{1}{T} \sum_{t=1}^{T} a_{it}(h) \right| \ge CD_{T,N,2}B_{T,N,4}T^{-(1-c)/2}\log((G-1)N) \right) \le CT^{-c}.$$

We thus have

$$P\left(\max_{1\leq i\leq N}\max_{h\in\mathbb{G}\setminus\{g(i)\}}\left|\frac{1}{T}\sum_{t=1}^{T}\tilde{a}_{it}(h)\right|>\frac{r}{2}-\frac{r^2}{16}\right)\leq CT^{-c},$$

by Assumption (9).

The third term (41) can also be analyzed by following the argument in the proof of Step 2 of Theorem 4.2 of Chernozhukov, Chetverikov, and Kato (2018) and is bounded by $\beta + CT^{-c}$ under Assumptions (8) and (9).

Summing up, we have

$$P\left(\bigotimes_{i=1}^{N} \hat{M}_{i}(g_{i}^{0}) \not\supseteq J_{1}\right) \leq \beta + CT^{-c} + N^{-1}.$$

An implication of Step 2 is as follows. Let

$$\hat{\mathbb{N}} = \left\{ i \in \{1, \dots, N\} \mid M_i(g_i^0) \neq \varnothing \right\}.$$

Then

$$P\left(\hat{\mathbb{N}} \supseteq \mathbb{N}\right) \ge 1 - \beta - CT^{-c} - N^{-1}.$$

Step 3: First, consider the case in which $J_1 = \emptyset$. In this case, the argument in Step 1 yields that

$$P(\hat{g}_i = g_i^0, \forall i) = P\left(\max_{1 \le i \le N} \max_{h \in \mathbb{G} \setminus \{g(i)\}} \hat{D}_i^U(g_i^0, h) \le 0\right) > 1 - \beta - CT^{-c}.$$

Because $\{\hat{g}(i)\}_{i=1}^N$ is always included in the confidence set, the probability of the confidence set not including $\{g_i^0\}_{i=1}^N$ is less than $\beta + CT^{-c} < \alpha + CT^{-c}$.

Next, consider the case in which $|J_1| \ge 1$. Here, we consider the case with type = SNS. The proofs for the other two cases are similar, and therefore omitted. Observe that

$$\begin{split} &P\left(\{g_i^0\}_{i=1}^N \notin \hat{C}_{\mathrm{Sel},\alpha,\beta}^{\mathrm{SNS}}\right) \\ &= P\left(\bigcup_{i=1}^N \left(\left\{\hat{T}_i^{\mathrm{MAX}}(g_i^0) > c_{\alpha-2\beta,\hat{N}}^{\mathrm{SNS}}\right\} \cap \left\{\max_{h \in \mathbb{G} \backslash \{g_i^0\}} \hat{D}_i^U(g_i^0,h) > 0\right\}\right)\right) \\ &\leq P\left(\bigcup_{i \in \mathbb{N}} \left\{\hat{T}_i^{\mathrm{MAX}}(g_i^0) > c_{\alpha-2\beta,\hat{N}}^{\mathrm{SNS}}\right\} \cup \bigcup_{i \in \mathbb{N}^c} \left\{\max_{h \in \mathbb{G} \backslash \{g_i^0\}} \hat{D}_i^U(g_i^0,h) > 0\right\}\right) \\ &\leq P\left(\bigcup_{i \in \mathbb{N}} \left\{\hat{T}_i^{\mathrm{MAX}}(g_i^0) > c_{\alpha-2\beta,\hat{N}}^{\mathrm{SNS}}\right\}\right) + P\left(\bigcup_{i \in \mathbb{N}^c} \left\{\max_{h \in \mathbb{G} \backslash \{g_i^0\}} \hat{D}_i^U(g_i^0,h) > 0\right\}\right). \end{split}$$

By Step 1, we have

$$P\left(\bigcup_{i\in\mathbb{N}^c}\left\{\max_{h\in\mathbb{G}\backslash\{g_i^0\}}\hat{D}_i^U(g_i^0,h)>0\right\}\right)\leq \beta+CT^{-c}.$$

By Step 2, we have

$$\begin{split} &P\left(\bigcup_{i\in\mathbb{N}}\left\{\hat{T}_i^{\text{MAX}}(g_i^0) > c_{\alpha-2\beta,\hat{N}}^{\text{SNS}}\right\}\right) \\ \leq &P\left(\{\hat{\mathbb{N}}\supseteq\mathbb{N}\}\cap\bigcup_{i\in\mathbb{N}}\left\{\hat{T}_i^{\text{MAX}}(g_i^0) > c_{\alpha-2\beta,\hat{N}}^{\text{SNS}}\right\}\right) + P(\{\hat{\mathbb{N}}\not\supseteq\mathbb{N}\}) \end{split}$$

$$\leq P\left(\bigcup_{i\in\mathbb{N}}\left\{\hat{T}_i^{\text{MAX}}(g_i^0) > c_{\alpha-2\beta,|\mathbb{N}|}^{\text{SNS}}\right\}\right) + \beta + CT^{-c} + N^{-1}.$$

Thus we have

$$P\left(\{g_i^0\}_{i=1}^N \notin \hat{C}_{\mathrm{Sel},\alpha,\beta}^{\mathrm{SNS}}\right) \leq P\left(\bigcup_{i \in \mathbb{N}} \left\{\hat{T}_i^{\mathrm{MAX}}(g_i^0) > c_{\alpha-2\beta,|\mathbb{N}|}^{\mathrm{SNS}}\right\}\right) + 2\beta + CT^{-c} + N^{-1}.$$

Theorem 2 implies that

$$P\left(\{g_i^0\}_{i=1}^N \notin \hat{C}_{\mathrm{Sel},\alpha,\beta}^{\mathrm{SNS}}\right) \le \alpha + C\epsilon_N + CT^{-c} + N^{-1}.$$

G. Convergence rate of group-specific coefficients estimators without uniform consistency of group membership

In this section, we show that the group-specific coefficients can be estimated consistently and the estimator admits a suitable rate of convergence even when group memberships are not uniformly consistently estimated. We consider a simple model that includes only group-specific means:

$$y_{it} = \mu_{g_i^0} + u_{it},$$

where μ_g is the group–specific mean for group g and u_{it} is an error term. Let Γ denote the space of possible group assignments and let \mathcal{M} denote the space of possible group-specific means. Write $\mathbf{g} = (g_1, \dots, g_N)$ for a generic element of Γ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_G)$ for generic element of \mathcal{M} . The setting is summarized as follows:

Assumption 2. $\{u_{it}\}_{t=1}^T$ is an independent sequence for all i. For all $i=1,2,\ldots, t=1,2,\ldots$ we have $\mathbb{E}[u_{it}]=0$, $\mathbb{E}[u_{it}^2]=\sigma_i<\infty$ and that for a vanishing sequence $a_{N,T}$,

$$\frac{1}{N} \sum_{i=1}^{N} \left(\frac{\sigma_i^2}{T} \right) < a_{N,T}.$$

 \mathcal{M} is bounded.

We consider the *kmeans*-type estimation method as considered in Bonhomme and Manresa (2015). The objective function for estimation is defined on $\Gamma \times \mathcal{M}$ and is given by

$$Q_{N,T}(\mathbf{g}, \boldsymbol{\mu}) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{it} - \mu_{g_i})^2.$$

The estimator is defined as $(\hat{\boldsymbol{\mu}}, \hat{\mathbf{g}}) = \arg\min_{\boldsymbol{\mu} \in \mathcal{M}, \boldsymbol{g} \in \Gamma} Q_{N,T}(\mathbf{g}, \boldsymbol{\mu}).$

We show that $\hat{\boldsymbol{\mu}}$ is \sqrt{NT} -consistent even when $\hat{\mathbf{g}}$ is not uniformly consistent. For that, we make the following assumptions:

Assumption 3 (Group separation). There is a positive constant M_G such that

$$\min_{g \in \mathbb{G}} \min_{h \in \mathbb{G} \setminus \{g\}} \left| \mu_g^0 - \mu_h^0 \right| > M_G.$$

In addition, it holds that

$$\min_{g \in \mathbb{G}} \sum_{\substack{i=1,\dots,N\\q^0(i)=q}} \frac{1}{N} \ge \pi_{\min}.$$

Assumption 4. There are sequences $a_{N,T}$ and $q_{N,T}$ and s>1 and $c_0>0$ such that $a_{N,T}\to 0$ and

$$\mathbb{E}\left[\max_{i=1,\dots,N}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\frac{u_{it}}{\sigma_{i}}\right)\right|\right] \leq q_{N,T},$$

$$T^{-\left(\frac{s-2}{2}\right)}\mathbb{E}\left[\max_{i=1,\dots,N}\left|\frac{u_{i1}}{\sigma_{i}}\right|^{s}\right] = O\left(a_{N,T}\left(q_{N,T}\right)^{s}\right),$$

$$c_{0}\sqrt{\log N} \leq q_{N,T}.$$

Define

$$I_{N,T} = \left\{ i \in \{0, \dots, N\} : \sqrt{T} \, \frac{M_G}{9 \, \sigma_i} > q_{N,T} \right\}.$$

 $I_{N,T}$ indicates units whose error variances are small enough and whose group memberships can be estimated precisely.

The following lemma shows that the group membership estimator is consistent for units in $I_{N,T}$.

Lemma G.1. Suppose that Assumptions 2, 3 and 4 hold. Then,

$$P\left(\sup_{i\in I_{N,T}}\left|g_i^0-\hat{g}_i\right|>0\right)\to 0.$$

This holds uniformly over all data generating processes that satisfy the assumptions.

Note that this lemma allows the group memberships of some units not to be estimated consistently.

The following theorem indicates that the group-specific coefficients can be estimated with rate of convergence \sqrt{NT} even when the entire group membership structure is not estimated consistently as in the case of Lemma G.1.

Theorem 6. Suppose that Assumptions 2, 3 and 4 hold. In addition, let $v_{it} = u_{it}/\sigma_i$

$$I_{N,T}(g) = \{i \in I_{N,T} : g_i^0 = g\}, \quad I_{N,T}^{\mathsf{c}} = \{i = 1, \dots, N : i \notin I_{N,T}\},$$

$$\tilde{N}_g = \sum_{i \in I_{N,T}(g)} 1, \quad N_g = \sum_{i=1}^{N} 1_{\{g_i^0 = g\}}, \quad \hat{N}_g = \sum_{i=1}^{N} 1_{\{\hat{g}_i = g\}},$$

and suppose that for each $g \in \mathbb{G}$ there are positive constants δ_g and π_g such that $N_g/N \to \pi_g$ and

$$\frac{1}{\tilde{N}_g} \sum_{i \in I_{N,T}(g)} \sigma_i^2 + \frac{1}{\tilde{N}_g} \sum_{\substack{i,j \in I_{N,T}(g) \\ i \neq j}} \sigma_i \sigma_j \operatorname{cov}(v_{i1} v_{j1}) \to \delta_g.$$

Suppose also that there is a finite constant M_4 such that $Ev_{it}^4 < M_4$ and

$$\sum_{\substack{i,j,k \in I_{N,T}(g) \\ \{i\} \cap \{j\} \cap \{k\} = \emptyset}} \sigma_i \sigma_j \sigma_k \mathbb{E}[v_{i1}^2 v_{j1} v_{k1}] < M_4\left(\tilde{N}_g^2 T\right),$$

$$\sum_{\substack{i,j,k,\ell \in I_{N,T}(g) \\ i\} \cap \{j\} \cap \{k\} \cap \{\ell\} = \emptyset}} \sigma_i \sigma_j \sigma_k \sigma_\ell \mathbb{E}[v_{i1} v_{j1} v_{k1} v_{\ell1}] < M_4\left(\tilde{N}_g^2 T\right).$$

Moreover, suppose that there is a vanishing sequence $b_{N,T}$ such that

$$\begin{split} &|I_{N,T}^{\text{c}}| < &b_{N,T} \sqrt{N/T}, \\ \frac{1}{|I_{N,T}^{\text{c}}|} \sum_{i \in I_{N,T}^{\text{c}}} \left(\frac{\sigma_i^2}{T}\right) < &b_{N,T} \frac{N}{T|I_{N,T}^{\text{c}}|^2}. \end{split}$$

Then, for $g \in \mathbb{G}$

$$\sqrt{NT} \left(\hat{\mu}_g - \mu_g^0 \right) \stackrel{d}{\longrightarrow} \mathcal{N}(0, v_g^2)$$

with $v_g^2 = \pi_g^{-1} \delta_g$.

The proofs of these results require several lemmas. First show that this can be replaced by

$$\tilde{Q}_{N,T}(\mathbf{g}, \boldsymbol{\mu}) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} u_{it}^{2} + \frac{1}{N} \sum_{i=1}^{N} \left(\mu_{g_{i}^{0}}^{0} - \mu_{g_{i}} \right)^{2}.$$

Lemma G.2. Suppose that Assumption 2 holds. Then

$$\sup_{\mathbf{g} \in \Gamma, \boldsymbol{\mu} \in \mathcal{M}} \left| Q_{N,T}(\mathbf{g}, \boldsymbol{\mu}) - \tilde{Q}_{N,T}(\mathbf{g}, \boldsymbol{\mu}) \right| = o_p(1).$$

This holds uniformly over all data generating processes that satisfy the assumptions.

Proof. This proof is very similar to the proof of Lemma A.1 in Bonhomme and Manresa (2015).

Expanding $Q_{N,T}$ gives

$$Q_{N,T}(\mathbf{g}, \boldsymbol{\mu}) = \tilde{Q}_{N,T}(\mathbf{g}, \boldsymbol{\mu}) + \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} u_{it} \left(\mu_{g_i^0}^0 - \mu_{g_i} \right).$$

Write $v_{it} = u_{it}/\sigma_i$. By Cauchy-Schwarz

$$\left| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mu_{g_i} u_{it} \right|^2 \le \left(\frac{1}{N} \sum_{i=1}^{N} \mu_{g_i}^2 \right) \frac{1}{N} \sum_{i=1}^{N} \left\{ \left(\frac{\sigma_i^2}{T} \right) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} \right)^2 \right\}.$$

Under the assumptions of the lemma

$$\frac{1}{N} \sum_{i=1}^{N} E\left\{ \left(\frac{\sigma_i^2}{T} \right) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} \right)^2 \right\} < a_{N,T} \to 0.$$

Therefore, by Markov's inequality,

$$\frac{1}{N} \sum_{i=1}^{N} \left\{ \left(\frac{\sigma_i^2}{T} \right) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} \right)^2 \right\} = o_p(1).$$

Uniform boundedness of $N^{-1} \sum_{i=1}^{N} \mu_{g_i}^2$ follows from boundedness of \mathcal{M} . This proves

$$\sup_{\mathbf{g} \in \Gamma, \boldsymbol{\mu} \in \mathcal{M}} \left| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mu_{g_i} u_{it} \right| = o_p(1)$$

and concludes the proof.

Lemma G.3. Suppose that Assumption 2 holds.

$$\frac{1}{N} \sum_{i=1}^{N} \left(\mu_{g_i^0}^0 - \hat{\mu}_{\hat{g}_i} \right)^2 = o_p(1).$$

Proof. By definition,

$$Q_{N,T}(\hat{\mathbf{g}}, \hat{\boldsymbol{\mu}}) \leq Q_{N,T}(\mathbf{g}^0, \boldsymbol{\mu}^0).$$

Then, Lemma G.2 implies that

$$\tilde{Q}_{N,T}(\hat{\mathbf{g}}, \hat{\boldsymbol{\mu}}) = Q_{N,T}(\hat{\mathbf{g}}, \hat{\boldsymbol{\mu}}) + o_p(1) \le Q_{N,T}(\mathbf{g}^0, \boldsymbol{\mu}^0) + o_p(1) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} u_{it}^2 + o_p(1).$$

The lemma now follows by plugging in the definition of $\tilde{Q}_{N,T}$ on the left-hand side of the inequality.

Lemma G.4. Suppose that Assumptions 2 and 3 are satisfied. Then, the Hausdorff distance between $\hat{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}^0$ vanishes in probability:

$$d_H(\hat{\boldsymbol{\mu}}, \boldsymbol{\mu}^0) = \max \left\{ \max_{g \in \mathbb{G}} \min_{h \in \mathbb{G}} \left| \hat{\mu}_h - \mu_g^0 \right|, \max_{h \in \mathbb{G}} \min_{g \in \mathbb{G}} \left| \hat{\mu}_h - \mu_g^0 \right| \right\} = o_p(1).$$

Proof. This proof is very similar to the proof of Lemma B.3 in Bonhomme and Manresa (2015). By Lemma G.3

$$\frac{1}{N} \sum_{i=1}^{N} \left(\mu_{g_i^0}^0 - \hat{\mu}_{\hat{g}_i} \right)^2 = o_p(1).$$

Suppose that there is a $g \in \mathbb{G}$, a constant $\epsilon > 0$, and a sequence of sets $A_{N,T}$ such that on $A_{N,T}$

$$\min_{h} \left| \hat{\mu}_h - \mu_g^0 \right| > \frac{\epsilon}{\pi_{\min}}$$

and $\limsup_{N,T\to\infty} P(A_{N,T}) > \epsilon$. Then,

$$\frac{1}{N} \sum_{i=1}^{N} \left(\mu_{g_i^0}^0 - \hat{\mu}_{\hat{g}_i} \right)^2 > \frac{1}{N} \sum_{\substack{i=1,\dots,N \\ g^0(i)=g}} \frac{\epsilon}{\pi_{\min}} \ge \epsilon.$$

This contradicts the result of Lemma G.3. This implies that $\lim_{N,T\to\infty} P\left(\min_{h\in\mathbb{G}} \left|\hat{\mu}_h - \mu_g^0\right| > \epsilon\right) = 0$ for any $\epsilon > 0$. Because $P\left(\max_{g\in\mathbb{G}} \min_{h\in\mathbb{G}} \left|\hat{\mu}_h - \mu_g^0\right| > \epsilon\right) \le \sum_{g\in\mathbb{G}} P\left(\min_{h\in\mathbb{G}} \left|\hat{\mu}_h - \mu_g^0\right| > \epsilon\right)$ and \mathbb{G} is finite, we have for any $\epsilon > 0$,

$$\lim_{N,T\to\infty} P\left(\max_{g\in\mathbb{G}} \min_{h\in\mathbb{G}} |\hat{\mu}_h - \mu_g^0| > \epsilon\right) = 0.$$

Next, suppose that there are constants ϵ and δ , $0 < \epsilon < M_G/2$, $\delta > 0$, a group $h \in \mathbb{G}$ and a sequence of sets $B_{N,T}$ such that $P(B_{N,T}) > \delta$ and

$$\min_{g \in \mathbb{G}} \left| \hat{\mu}_h - \mu_g^0 \right| > \epsilon \tag{42}$$

on $B_{N,T}$. By the first part of the proof, it is without loss of generality to assume that

$$\max_{g \in \mathbb{G}} \min_{h \in \mathbb{G}} \left| \hat{\mu}_h - \mu_g^0 \right| < \epsilon$$

on $B_{N,T}$. For a given $g \in \mathbb{G}$ let $h \in \mathbb{G}$ such that the minimum is achieved, i.e., $|\hat{\mu}_h - \mu_g^0| < \epsilon$. Now, note that

$$\max_{\tilde{g} \in \mathbb{G} \setminus \{g\}} |\hat{\mu}_h - \mu_{\tilde{g}}^0| = \max_{\tilde{g} \in \mathbb{G} \setminus \{g\}} |\mu_g^0 - \mu_{\tilde{g}}^0 + \hat{\mu}_h - \mu_g^0| > M_G - \epsilon > \epsilon.$$

This shows that to each $g \in \mathbb{G}$ there is a unique choice $h^*(g)$ such that $|\hat{\mu}_{h^*(g)} - \mu_g^0| < \epsilon$. Since the mapping h^* is a bijection it is invertible and we can find, for any given $h \in \mathbb{G}$, a group $g = (h^*)^{-1}(h)$ such that $|\hat{\mu}_h - \mu_{(h^*)^{-1}(h)}^0| < \epsilon$. This contradicts (42) and establishes

$$\lim_{N,T\to\infty} P\left(\max_{h\in\mathbb{G}} \min_{g\in\mathbb{G}} \left| \hat{\mu}_h - \mu_g^0 \right| \right) = 0.$$

The conclusion follows.

We now prove Lemma G.1.

Proof of Lemma G.1. Note that $|g_i^0 - \hat{g}_i| > 0$ only if there is a $g \in \mathbb{G} \setminus \{g_i^0\}$ such that

$$\sum_{t=1}^{T} \left(y_{it} - \hat{\mu}_{g_i^0} \right)^2 \ge \sum_{t=1}^{T} \left(y_{it} - \hat{\mu}_g \right)^2.$$

Rewriting this inequality gives

$$\operatorname{sign}(\hat{\mu}_g - \hat{\mu}_{g_i^0}) \sum_{t=1}^{T} (y_{it} - \hat{\mu}_{g_i^0}) \ge \frac{T}{2} \left| \hat{\mu}_g - \hat{\mu}_{g_i^0} \right|.$$

Plugging in $y_{it} = u_{it} + \mu_{g_i^0}^0$ and applying Lemma G.4 gives

$$\operatorname{sign}(\hat{\mu}_{g} - \hat{\mu}_{g_{i}^{0}}) \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(\frac{u_{it}}{\sigma_{i}} \right) \ge \frac{\sqrt{T}}{2\sigma_{i}} \left\{ \left| \mu_{g}^{0} - \mu_{g_{i}^{0}}^{0} \right| - \left| \hat{\mu}_{g}^{0} - \mu_{g}^{0} \right| - \left| \hat{\mu}_{g_{i}^{0}}^{0} - \mu_{g_{i}^{0}}^{0} \right| \right\}$$

$$- \frac{\sqrt{T}}{\sigma_{i}} \left| \hat{\mu}_{g_{i}^{0}} - \mu_{g_{i}^{0}}^{0} \right|$$

$$> \frac{1}{3} \frac{\sqrt{T}}{\sigma_{i}} M_{G}$$

for N, T large enough. Therefore,

$$\begin{split} &P\left(\sup_{i\in I_{N,T}}\left|g_i^0-\hat{g}_i\right|>0\right)\\ \leq &P\left(\sup_{i\in I_{N,T}}\operatorname{sign}(\mu_g-\hat{\mu}_{g_0(i)})\frac{1}{\sqrt{T}}\sum_{t=1}^T\left(\frac{u_{it}}{\sigma_i}\right)>\frac{1}{3}\frac{\sqrt{T}}{\sigma_i}M_G\right)\\ \leq &P\left(\sup_{i=1,\dots,N}-\frac{1}{\sqrt{T}}\sum_{t=1}^T\left(\frac{u_{it}}{\sigma_i}\right)>3q_{N,T}\right)+P\left(\sup_{i=1,\dots,N}\frac{1}{\sqrt{T}}\sum_{t=1}^T\left(\frac{u_{it}}{\sigma_i}\right)>3q_{N,T}\right). \end{split}$$

Let $v_{it} := u_{it}/\sigma_i$. As v_{it} and $-v_{it}$ satisfy the same (absolute) moment conditions it suffices to show

that the second term on the right-hand side vanishes. By assumption,

$$P\left(\sup_{i \in I_{N,T}} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} > 3q_{N,T}\right)$$

$$\leq P\left(\sup_{i=1,\dots,N} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} > 2\mathbb{E}\left[\max_{i=1,\dots,N} \left| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} \right| \right] + q_{N,T}\right).$$

Apply Lemma D.2 in Chernozhukov, Chetverikov, and Kato (2018) with $X_{ij} = T^{-\frac{1}{2}}v_{it}$ and $t = q_{N,T}$ to the right-hand side. This gives, for a universal constant K_s depending only on s,

$$\begin{split} &P\left(\sup_{i=1,\dots,N}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}v_{it}>2\mathbb{E}\left[\max_{i=1,\dots,N}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}v_{it}\right|\right]+q_{N,T}\right)\\ \leq &N^{-c_{0}^{2}/3}+\frac{K_{S}}{\left(q_{N,T}\right)^{s}T^{\frac{s-2}{2}}}\max_{t=1,\dots,T}\mathbb{E}\left[\max_{i=1,\dots,N}\left|v_{it}\right|^{s}\right]=O\left(N^{-c_{0}^{2}/3}+a_{N,T}\right). \end{split}$$

We now provide the proof of Theorem 6.

Proof of Theorem 6. Throughout the prove let C denote a generic constant. Fix $g \in \mathcal{G}$. First, note that

$$\frac{\tilde{N}_g}{N} = \frac{N_g}{N} + \frac{\tilde{N}_g - N_g}{N} \ge \pi_g + \frac{\left|I_{N,T}^{\mathsf{c}}\right|}{N} + o(1) \ge \pi_g + b_{N,T} \frac{1}{\sqrt{NT}} + o(1).$$

Now, write

$$\frac{1}{\hat{N}_g} \sum_{i \in I_{N,T}(g)} \mu_{g_i^0}^0 = \mu_g^0 + \frac{\tilde{N}_g - \hat{N}_g}{\hat{N}_g} \mu_g^0.$$

Now, note that $\hat{N}_g \geq \tilde{N}_g$ for large N, T. $\hat{N}_g \geq \tilde{N}_g$ and therefore

$$\frac{\left|\tilde{N}_g - \hat{N}_g\right|}{\hat{N}_g} \le \frac{\left|\tilde{N}_g - \hat{N}_g\right|}{\tilde{N}_g} \le \frac{N}{\tilde{N}_g} \frac{\left|I_{N,T}^{\mathsf{c}}\right|}{N} \le 2\pi_g^{-1} b_{N,T} \frac{1}{\sqrt{NT}}.$$

Since μ_g^0 is contained in a bounded set this implies

$$\frac{1}{\hat{N}_g} \sum_{i \in I_{N,T}(g)} \mu_{g_i^0}^0 = \mu_g^0 + o\left(\frac{1}{\sqrt{NT}}\right)$$

uniformly in the underlying DGP. Similarly,

$$\left|\frac{1}{\hat{N}_g}\sum_{i\in I_{N,T}^c}1_{\{\hat{g}_i=g\}}\mu_{g_i^0}^0\right|\leq C\left|\frac{\left|I_{N,T}^c\right|}{\tilde{N}_g}=o\left(\frac{1}{\sqrt{NT}}\right).$$

Write $v_{it} = u_{it}/\sigma_i$. By Cauchy-Schwarz

$$\left| \frac{1}{\hat{N}_g} \sum_{i \in I_{N,T}^c} 1_{\{\hat{g}_i = g\}} \left(\frac{1}{T} \sum_{t=1}^T u_{it} \right) \right| \le \frac{1}{\hat{N}_g \sqrt{T}} \left(\sum_{i \in I_{N,T}^c} \sigma_i^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I_{N,T}^c} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T v_{it} \right)^2 \right)^{\frac{1}{2}}.$$

Taking expectations and applying Markov's inequality gives

$$\frac{1}{|I_{N,T}^{\mathsf{c}}|} \sum_{i \in I_{N,T}^{\mathsf{c}}} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} \right)^{2} = O_{p}(1).$$

Hence, on a set with probability approaching one

$$\left| \frac{1}{\hat{N}_g} \sum_{i \in I_{N,T}^{\mathbf{c}}} 1_{\{\hat{g}_i = g\}} \left(\frac{1}{T} \sum_{t=1}^{T} u_{it} \right) \right|$$

$$\leq C \frac{1}{\sqrt{NT}} \left(\frac{N}{\tilde{N}_g} \right) \left(\left\{ \frac{T |I_{N,T}^{\mathbf{c}}|^2}{N} \right\} \frac{1}{|I_{N,T}^{\mathbf{c}}|} \sum_{i \in |I_{N,T}^{\mathbf{c}}|} \left(\frac{\sigma_i^2}{T} \right) \right)^{\frac{1}{2}} \leq C \sqrt{b_{N,T}}$$

as $N, T \to \infty$. Summarizing the results so far gives

$$\hat{\mu}_{g} = \frac{1}{\hat{N}_{g}} \sum_{i \in I_{N,T}^{c}} 1_{\{\hat{g}_{i} = g\}} \left\{ \left(\frac{1}{T} \sum_{t=1}^{T} u_{it} \right) + \mu_{g_{i}^{0}}^{0} \right\} + \frac{1}{\hat{N}_{g}} \sum_{i \in I_{N,T}(g)} \left\{ \left(\frac{1}{T} \sum_{t=1}^{T} u_{it} \right) + \mu_{g_{i}^{0}}^{0} \right\}$$

$$= \mu_{g_{i}^{0}}^{0} + \frac{1}{\hat{N}_{g}} \sum_{i \in I_{N,T}(g)} \left\{ \sigma_{i} \left(\frac{1}{T} \sum_{t=1}^{T} v_{it} \right) \right\} + o_{p} \left((NT)^{-\frac{1}{2}} \right).$$

We will now apply the Lindeberg-Feller CLT to show

$$\frac{1}{\sqrt{\tilde{N}_g}} \sum_{i \in I_{N,T}(g)} \left\{ \sigma_i \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T v_{it} \right) \right\} \stackrel{d}{\longrightarrow} \mathcal{N}(0, \delta_g). \tag{43}$$

The variance of the term is given by

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{\tilde{N}_g}}\sum_{i\in I_{N,T}(g)}\sigma_i v_{it}\right)^2\right]$$

$$= \frac{1}{\tilde{N}_g} \sum_{i \in I_{N,T}(g)} \sigma_i^2 + \frac{1}{\tilde{N}_g} \sum_{\substack{i,j \in I_{N,T}(g) \\ i \neq j}} \sigma_i \sigma_j \operatorname{cov}(v_{i1}, v_{j1}) \to \delta_g$$

To verify the Lindeberg condition it suffices to show that $\mathbb{E}\left[T^{-1/2}\sum_{t=1}^{T}z_{N,t}\right]^4 < K$ eventually, where

$$z_{N,t} = \frac{1}{\sqrt{\tilde{N}_g}} \sum_{i \in I_{N,T}(g)} \sigma_i v_{it}$$

and K is a constant that does not depend on N and T. By independence across time periods

$$\mathbb{E}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}z_{N,t}\right]^{4} = \frac{T(T-1)}{T^{2}}\left(\mathbb{E}[z_{N,1}^{2}]\right)^{2} + \frac{1}{T}\mathbb{E}[z_{N,1}^{4}] = \delta_{g}^{2} + \frac{1}{T}\mathbb{E}[z_{N,1}^{4}] + o\left(1\right).$$

To bound the right-hand side write

$$\begin{split} & \mathbb{E}\left[\sqrt{\tilde{N}_{g}}z_{N,1}\right]^{4} = \mathbb{E}\left[\sum_{i \in I_{N,T}(g)} \sigma_{i}v_{it}\right]^{4} \\ & = \sum_{i \in I_{N,T}(g)} \sigma_{i}^{4}\mathbb{E}[v_{i1}^{4}] + \sum_{\substack{i,j \in I_{N,T}(g) \\ i \neq j}} \sigma_{i}^{2}\sigma_{j}^{2}\mathbb{E}[v_{i1}^{2}v_{j1}^{2}] \\ & + \sum_{\substack{i,j,k \in I_{N,T} \\ \{i\} \cap \{j\} \cap \{k\} = \emptyset}} \sigma_{i}\sigma_{j}\sigma_{k}\mathbb{E}[v_{i1}v_{j1}v_{k1}] \\ & + \sum_{\substack{i,j,k,\ell \in I_{N,T} \\ \{i\} \cap \{j\} \cap \{k\} \cap \{\ell\} = \emptyset}} \sigma_{i}\sigma_{j}\sigma_{k}\sigma_{\ell}\mathbb{E}[v_{i1}v_{j1}v_{k1}v_{\ell1}] < 4M_{4}(\tilde{N}_{g}^{2}T). \end{split}$$

The last inequality follows directly from our assumptions and by noting that applying Cauchy-Schwarz to bound the expectation yields

$$\begin{split} \sum_{\substack{i,j \in I_{N,T}(g) \\ i \neq j}} \sigma_i^2 \sigma_j^2 \mathbb{E}[v_{i1}^2 v_{j1}^2] &\leq M_4 \left\{ \sum_{i \in I_{N,T}(g)} \sigma_i^2 \right\}^2 \leq M_4(\tilde{N}_g^2 T) \left\{ \frac{1}{\tilde{N}_g} \sum_{i \in I_{N,T}(g)} \frac{\sigma_i^2}{\sqrt{T}} \right\}^2 \\ &< M_4(\tilde{N}_g^2 T). \end{split}$$

The last inequality follows by noting that for $i \in I_{N,T}(g)$

$$\sigma_i < \sqrt{T} \frac{M_G}{9q_{NT}}$$

and $q_{N,T} \to \infty$. This proves that $\mathbb{E}[T^{-1/2} \sum_{t=1}^T z_{N,t}]^4 < K$ with $K = 2\delta_g^2 + 4M_4$ and establishes (43). The conclusion follows from an application of Slutzky's theorem using that $N/\tilde{N}_g = \pi_g + o(1)$. \square

H. More simulation results

			empirical coverage				cardinality of CS			
g^0	σ	T	\hat{g}_i	SNS	MAX	QLR	\hat{g}_i	SNS	MAX	QLR
1	0.25	10	0.43	0.99	0.99	0.99	1.00	2.99	2.99	2.96
1	0.25	20	0.43	0.96	0.97	0.96	1.00	2.23	2.17	2.05
1	0.25	30	0.39	0.95	0.95	0.94	1.00	1.63	1.58	1.53
1	0.25	40	0.38	0.94	0.94	0.94	1.00	1.42	1.39	1.36
1	0.50	10	0.00	0.98	0.98	0.94	1.00	2.99	2.99	2.97
1	0.50	20	0.00	0.95	0.96	0.93	1.00	2.90	2.88	2.85
1	0.50	30	0.00	0.94	0.94	0.94	1.00	2.80	2.78	2.76
1	0.50	40	0.00	0.92	0.93	0.92	1.00	2.74	2.73	2.70
2	0.25	10	0.65	1.00	1.00	1.00	1.00	2.99	3.00	2.99
2	0.25	20	0.61	0.98	0.97	0.97	1.00	2.48	2.43	2.38
2	0.25	30	0.60	0.96	0.96	0.95	1.00	1.69	1.62	1.59
2	0.25	40	0.61	0.97	0.94	0.93	1.00	1.32	1.29	1.29
2	0.50	10	0.00	0.98	0.98	0.96	1.00	2.99	2.99	2.99
2	0.50	20	0.00	0.97	0.96	0.95	1.00	2.89	2.88	2.85
2	0.50	30	0.00	0.95	0.93	0.94	1.00	2.75	2.72	2.70
2	0.50	40	0.00	0.95	0.94	0.92	1.00	2.62	2.59	2.58
3	0.25	10	0.65	1.00	0.99	0.99	1.00	3.00	3.00	2.99
3	0.25	20	0.65	0.98	0.98	0.97	1.00	2.51	2.48	2.44
3	0.25	30	0.59	0.97	0.97	0.94	1.00	1.70	1.63	1.61
3	0.25	40	0.60	0.96	0.96	0.96	1.00	1.32	1.29	1.30
3	0.50	10	0.00	0.98	0.99	0.97	1.00	2.99	2.99	2.99
3	0.50	20	0.00	0.97	0.96	0.94	1.00	2.90	2.89	2.86
3	0.50	30	0.00	0.96	0.96	0.93	1.00	2.75	2.73	2.70
3	0.50	40	0.00	0.94	0.93	0.93	1.00	2.63	2.60	2.58

Table H.1: Homoscedastic design with G=3 groups. Results based on 1000 simulated joint confidence sets with $1-\alpha=0.9$. Critical values for MAX and QLR procedures are adjusted for short panels. "Empirical coverage" gives the simulated coverage probability of the joint confidence set. "Cardinality of CS" gives the simulated expected average cardinality of a marginal (unit-wise) confidence set.

H.1. Another homoscedastic design with G=3 groups

This design is defined exactly as that from Section 6.1 with the exception of defining a different set of group-specific coefficients. Let $\varphi_T^{(2)}(t) = -2 + 8 |t - T/2|/T$. For t = 1, ..., T, $\alpha_{1,t} = 0$, $\alpha_{2,t} = \varphi_T^{(2)}(t)$, $\alpha_{3,t} = \varphi_{T/2}^{(2)}(t \text{ mod } \lceil T/2 \rceil)$. This specification implies moment inequalities that are less correlated than those for the design in Section 6.1. For example, for T = 40 and $g^0 = 1$, our simulations indicate that $(\mathbb{E} \widehat{\Omega}_i(1))_{1,2} = 0.00$ and $(\mathbb{E} \widehat{\Omega}_i(2))_{1,2} = 0.68$. For T = 40 and $g^0 = 2$, $(\mathbb{E} \widehat{\Omega}_i(1))_{1,2} = -0.00$ and $(\mathbb{E} \widehat{\Omega}_i(2))_{1,2} = 0.69$. We simulate 1000 joint confidence sets based on the SNS, MAX (with short-panel adjustment), and QLR (with short-panel adjustment) approach. The simulation results are reported in Table H.1.

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