6.4.5 Bremsstrahlung, Cyclotron and Synchrotron Radiation

To end our discussion, we derive the radiation due to some simple relativistic motion.

Power Radiated Again: Relativistic Larmor Formula

In Section 6.2.2, we derived the Larmor formula for the emitted power in the electric dipole approximation to radiation. In this section, we present the full, relativistic version of this formula.

We'll work with the expressions for the radiation fields $\mathbf{E}$ (6.42) and $\mathbf{B}$ (6.44). As previously, we consider only the radiative part of the electric and magnetic fields which drops off as $1/R$. The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0 c}\mathbf{E}\times(\hat{\mathbf{R}}\times\mathbf{E}) = \frac{1}{\mu_0 c}|\mathbf{E}|^2\hat{\mathbf{R}}$$

where all of these expressions are to be computed at the retarded time. The second equality follows from the relation (6.45), while the final equality follows because the radiative part of the electric field (6.42) is perpendicular to $\hat{\mathbf{R}}$. Using the expression (6.42), we have

$$\mathbf{S} = \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{|\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{a}]|^2}{\kappa^6 R^2}\hat{\mathbf{R}}$$

with $\kappa = 1 - \hat{\mathbf{R}}\cdot\mathbf{v}/c$.

Recall that everything in the formula above is evaluated at the retarded time t', defined by $t' + R(t')/c = t$. This means, that the coordinates are set up so that we can integrate $\mathbf{S}$ over a sphere of radius R that surrounds the particle at its retarded time. However, there is a subtlety in computing the emitted power, associated to the Doppler effect. The energy emitted per unit time t is not the same as the energy emitted per unit time t'. They differ by the factor $dt/dt' = \kappa$. The power emitted per unit time t', per solid angle $d\Omega$, is

$$\frac{d\mathcal{P}}{d\Omega} = \kappa R^2\,\mathbf{S}\cdot\hat{\mathbf{R}} = \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{|\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{a}]|^2}{\kappa^5} \tag{6.55}$$

To compute the emitted power, we must integrate this expression over the sphere. This is somewhat tedious. The result is given by

$$\mathcal{P} = \frac{q^2}{6\pi\epsilon_0 c^3}\gamma^4\left(a^2 + \frac{\gamma^2}{c^2}(\mathbf{v}\cdot\mathbf{a})^2\right) \tag{6.56}$$

This is the relativistic version of the Larmor formula (6.18). (There is a factor of 2 difference when compared to (6.20) because the former equation was time averaged). We now apply this to some simple examples.

Bremsstrahlung

Suppose a particle is travelling in a straight line, with velocity $\mathbf{v}$ parallel to acceleration $\mathbf{a}$. The most common situation of this type occurs when a particle decelerates. In this case, the emitted radiation is called *bremsstrahlung*, German for "braking radiation".

We'll sit at some point $\mathbf{x}$, at which the radiation reaches us from the retarded point on the particle's trajectory $\mathbf{r}(t')$. As before, we define $\mathbf{R}(t') = \mathbf{x} - \mathbf{r}(t')$. We introduce the angle θ, defined by

$$\hat{\mathbf{R}} \cdot \mathbf{v} = v \cos\theta$$

Because the $\mathbf{v} \times \mathbf{a}$ term in (6.55) vanishes, the angular dependence of the radiation is rather simple in this case. It is given by

$$\frac{d\mathcal{P}}{d\Omega} = \frac{q^2 a^2}{16\pi^2 \epsilon_0 c^3} \frac{\sin^2\theta}{(1 - (v/c)\cos\theta)^5}$$

For $v \ll c$, the radiation is largest in the direction $\theta \approx \pi/2$, perpendicular to the direction of travel. But, at relativistic speeds, $v \to c$, the radiation is beamed in the forward direction in two lobes, one on either side of the particle's trajectory. The total power emitted is (6.56) which, in this case, simplifies to

$$\mathcal{P} = \frac{q^2 \gamma^6 a^2}{6\pi\epsilon_0 c^3}$$

Cyclotron and Synchrotron Radiation

Suppose that the particle travels in a circle, with $\mathbf{v} \cdot \mathbf{a} = 0$. We'll pick axes so that $\mathbf{a}$ is aligned with the x-axis and $\mathbf{v}$ is aligned with the z-axis. Then we write

$$\hat{\mathbf{R}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$

After a little algebra, we find that the angular dependence of the emitted radiation is

$$\frac{d\mathcal{P}}{d\Omega} = \frac{q^2 a^2}{16\pi^2 \epsilon_0 c^3} \frac{1}{(1 - (v/c)\cos\theta)^3} \left(1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - (v/c)\cos\theta)^2}\right)$$

At non-relativistic speeds, $v \ll c$, the angular dependence takes the somewhat simpler form $(1 - \sin^2\theta\cos^2\phi)$. In this limit, the radiation is referred to as *cyclotron radiation*.

In contrast, in the relativistic limit $v \to c$, the radiation is again beamed mostly in the forwards direction. This limit is referred to as *synchrotron radiation*. The total emitted power (6.56) is this time given by

$$\mathcal{P} = \frac{q^2\gamma^4 a^2}{6\pi\epsilon_0 c^3}$$

Note that the factors of γ differ from the case of linear acceleration.