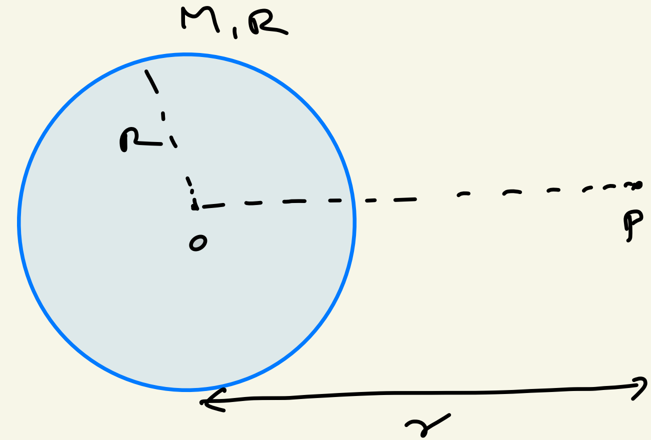
GRAVITATION - II

Solid sphere (or) spherical volume of mass Distribution

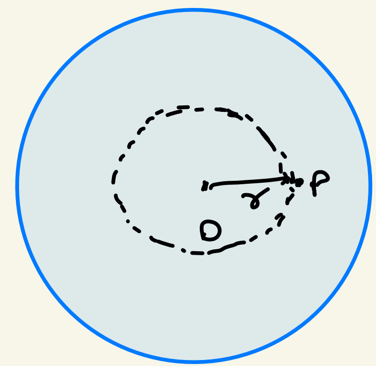
case i $\quad r \geq R$

$$\boxed{V_P = -\frac{GM}{r}}$$



*** **case ii** $\quad r \leq R$

$$\boxed{V_P = -\frac{GM}{2R^3}\left[3R^2 - r^2\right]}$$



NOTE:- At the centre of the sphere potential is

i.e $r = 0$

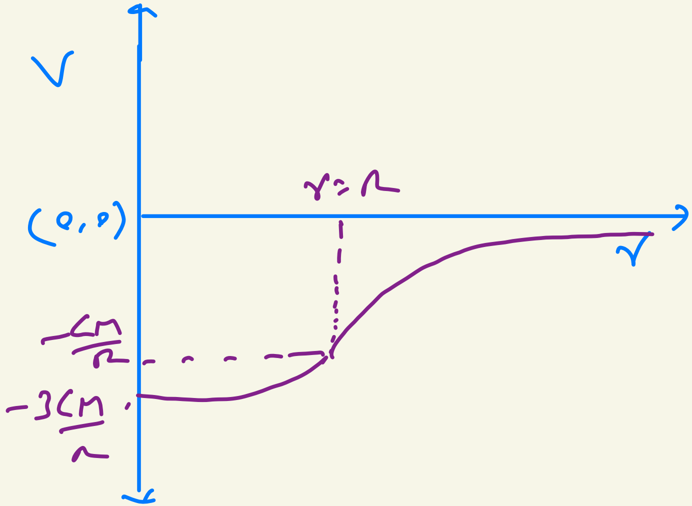
$$\boxed{V_0 = -\frac{3GM}{2R}}$$

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \Rightarrow M = \rho \frac{4}{3}\pi R^3$$

$$V_0 = -\frac{\cancel{3}G \times \rho \frac{\cancel{4}^{2}}{\cancel{3}}\pi R^{\cancel{3}^{2}}}{\cancel{2}\cancel{R}} = -2G\rho\pi R^2$$

$$\boxed{V_0 = -2G\rho\pi R^2}$$

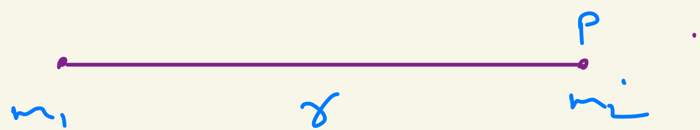
2



Gravitational potential energy

The amount of energy required to bring masses from infinity and to make a configuration is called Gravitational P.E.

For two particle system:



Work done to bring m_2 from infinity and place it at point P is given by

$$W = V_P m_2$$

$$W = \left(-\frac{Gm_1}{r}\right)m_2$$

*

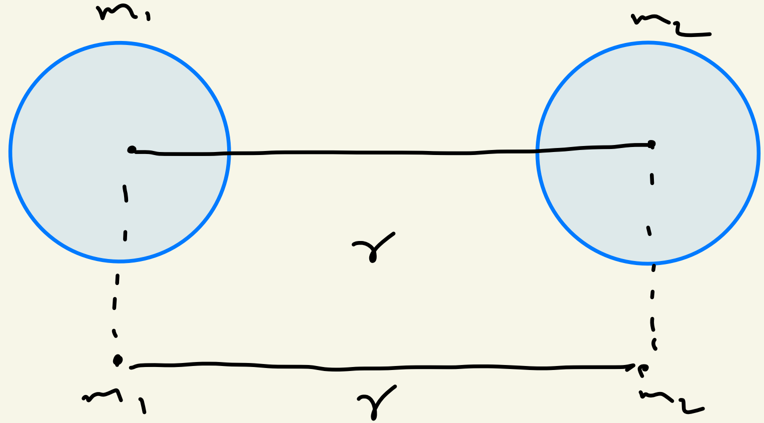
$$\boxed{U = -\frac{Gm_1m_2}{r}}$$

NOTE:- ① Gravitational potential can also be defined per unit mass i.e. gravitational potential energy per unit mass

or gravitational potential

$$\boxed{V = \frac{U}{m}}$$

② The above formula (G.P.E) can also be applied for spherical objects



$$\boxed{U = -\frac{Gm_1m_2}{r}}$$

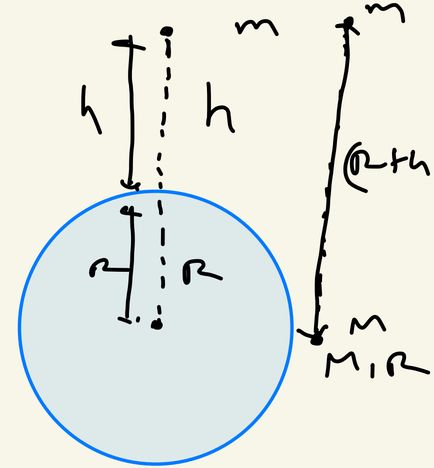
(3) we can also apply above formula (G.P.E) for particle and earth system

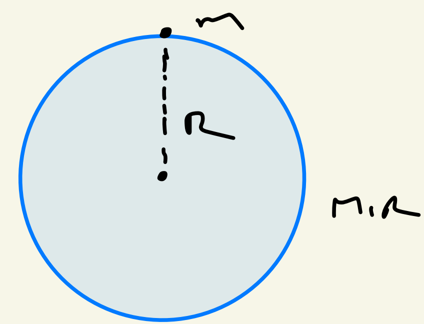
$$\boxed{U = -\frac{GMm}{R+h}}$$

If the particle is on the surface of the earth Then G.P.E is given

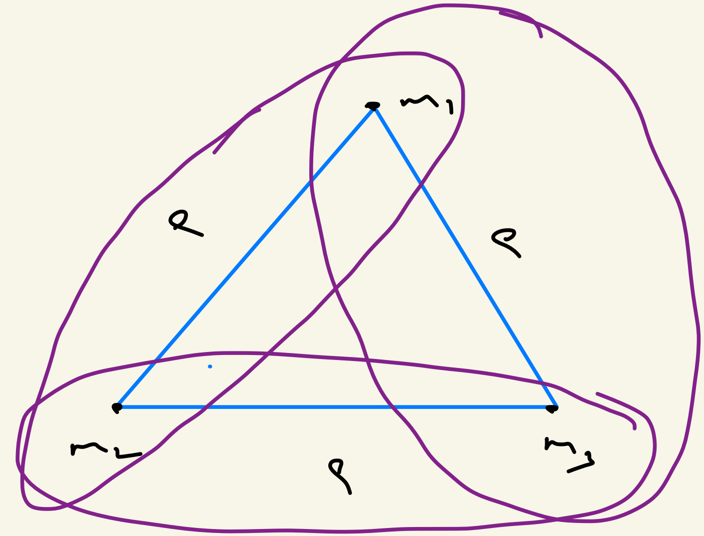
i.e $h = 0$

$$\boxed{U = -\frac{GMm}{R}}$$





For Three particle System:-



$$U_1 = -\frac{Gm_1m_2}{r}$$

$$U_2 = -\frac{Gm_2m_3}{r}$$

$$U_3 = -\frac{Gm_1m_3}{r}$$

$$U_{Total} = U_1 + U_2 + U_3$$

$$\boxed{U = -\frac{Gm_1m_2}{r} - \frac{Gm_2m_3}{r} - \frac{Gm_1m_3}{r}}$$

NOTE:- If there are N particles in the system then the no. of pairs of G.P.E is $\boxed{\frac{N(N-1)}{2}}$

problem:- A particle of mass m is released from a height h = R from the surface of the earth. Find with what speed particle strikes the surface of the earth? Given that mass of the earth is M and radius of the earth is R.

sol:- By applying COME we can find speed

$$\cancel{K_i} + U_i = K_f + U_f$$

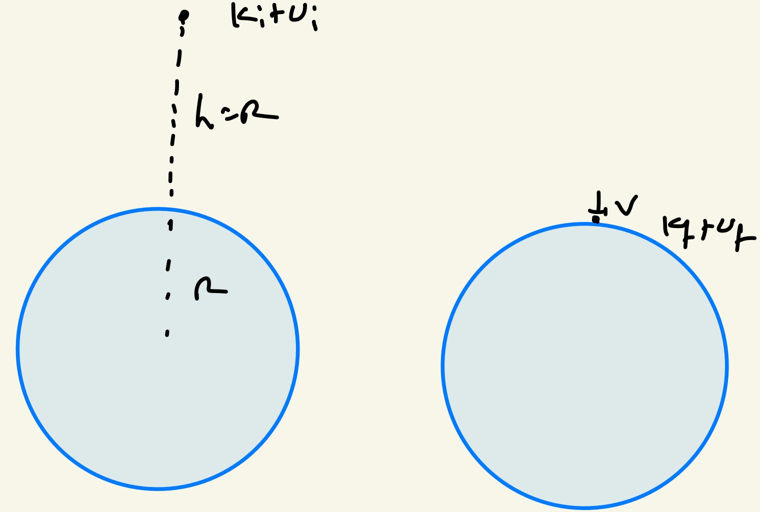
$$-\frac{GMm}{R+R} = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

$$-\frac{GMm}{2R} + \frac{GMm}{R} = \frac{1}{2}mv^2$$

$$\frac{1}{2}mv^2 = \frac{-GMm + 2GMm}{2R}$$

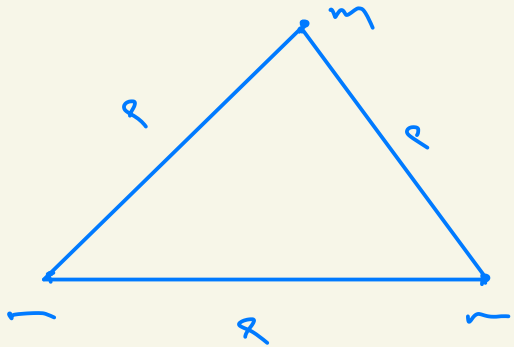
$$\frac{1}{\cancel{2}}\cancel{m}v^2 = \frac{GM\cancel{m}}{\cancel{2}R}$$

$$\boxed{v = \sqrt{\frac{GM}{R}}}$$

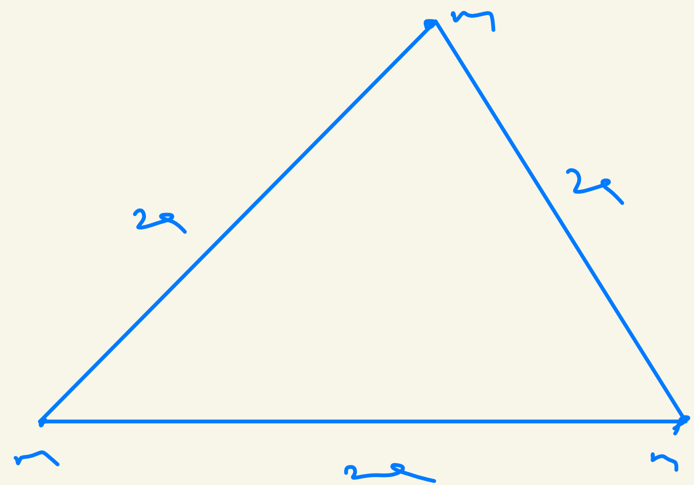


Eg: Three identical particles each of mass m are kept on vertices of equilateral triangle of side length 'a'. To move these particles and to place these particles on the vertices equilateral triangle of side length '2a' how much amount work is required ?

sol: -



configuration I



configuration - II

$$W_{ext} = \Delta U$$

$$U_i = -\frac{3Gm^2}{a} \qquad U_f = -\frac{3Gm^2}{2a}$$

$$W_{ext} = U_f - U_i = -\frac{3Gm^2}{2a} - \left(-\frac{3Gm^2}{a}\right)$$

$$W_{ext} = \frac{-3Gm^2 + 6Gm^2}{2a} = \frac{3Gm^2}{2a}$$

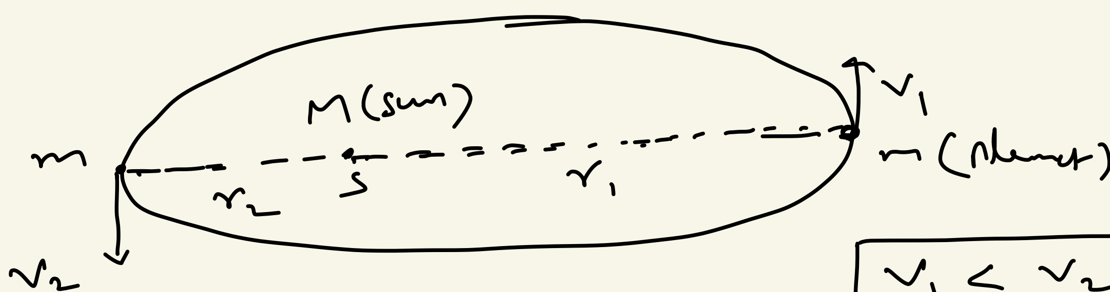
$$\boxed{W_{ext} = \frac{3Gm^2}{2a}}$$

Motion of planets and Satellites:-

Kepler's laws:-

Ist Law (Law of orbit)

All planets revolve around the sun in an elliptical orbit with sun at its one of foci

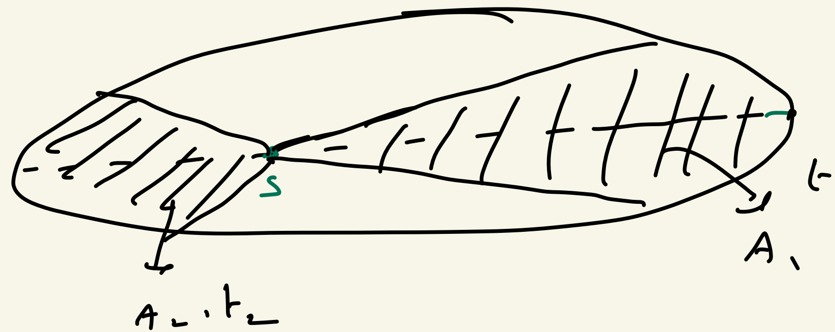


$$\boxed{v_1 < v_2}$$

IInd law :- (Law of Areal velocity)

$$\frac{dA}{dt} = \text{constant}$$

$$\boxed{\frac{dA}{dt} = \text{constant} = \frac{L}{2m}}$$



L - Angular momentum of the planet about sun

m - mass of the planet

$\frac{dA}{dt}$ - Areal velocity

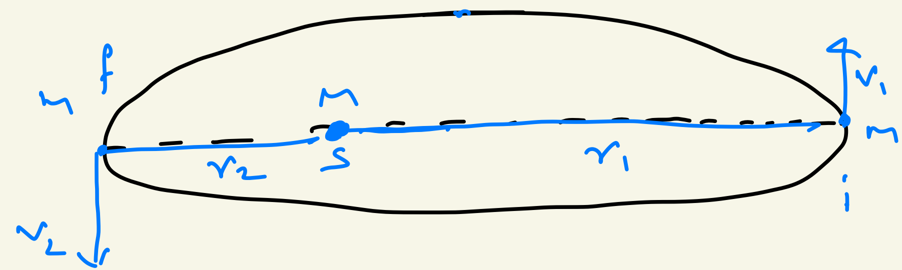
$$\boxed{\frac{A_1}{t_1} = \frac{A_2}{t_2}}$$

→ Since, external Torque acting on the system is equal to zero, Angular momentum of the system is conserved.

$$L_i = L_f$$

$$\cancel{m} v_1 r_1 = \cancel{m} v_2 r_2$$

$$\boxed{v_1 r_1 = v_2 r_2}$$



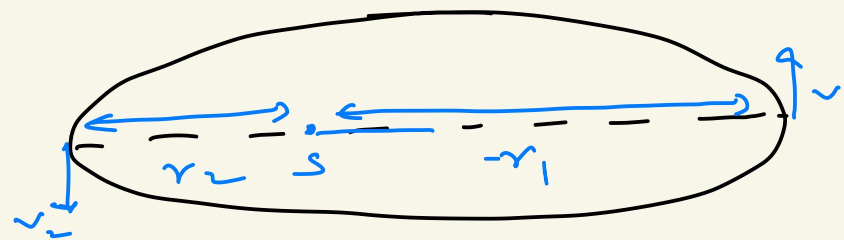
$$\boxed{r_1 > r_2 \Rightarrow v_1 < v_2}$$

<u>III^{rd} law</u> (Law of Timeperiod)

Square of the Timeperiod of the planet moving in an elliptical orbit is directly proportional to cube of semi-major axis

$$T^2 \propto \left(\frac{r_1 + r_2}{2}\right)^3$$

$$T \propto \left(\frac{r_1 + r_2}{2}\right)^{3/2}$$

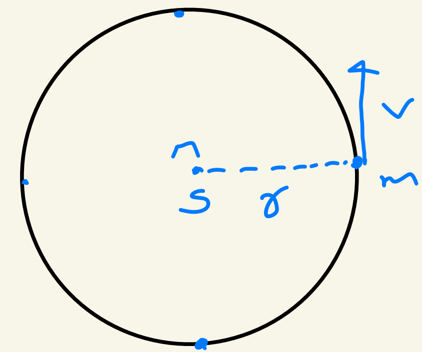


→ For problems we assume planets are moving in a circular path (unless it is mentioned in an elliptical orbit) Therefore square of the time period of the planet is directly proportional to cube of radius of the orbit.

$r_1 = r_2 = r$

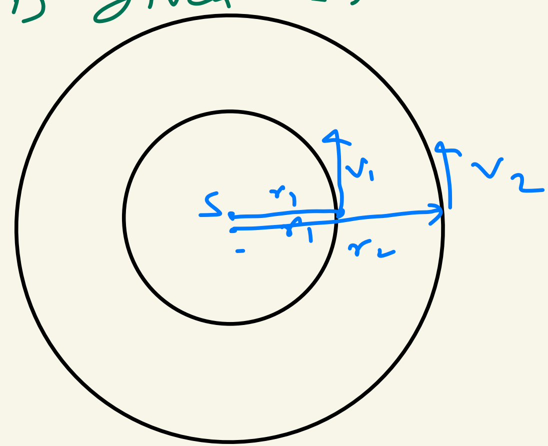
$$\boxed{T^2 \propto r^3}$$

$$\boxed{T \propto r^{3/2}}$$



→ Two planets are moving in two circular orbits of r_1 and r_2, Then the ratio of Time period of the Two planets is given by

$$\boxed{\begin{gathered}\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3} \\ \frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{3/2}\end{gathered}}$$



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Motion of planets

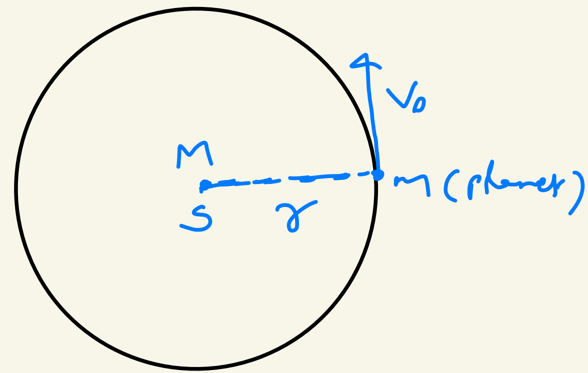
orbital velocity (V_o)

$$F = \frac{GMm}{r^2} \quad — (1)$$

$$F = \frac{mV_o^2}{r} \quad — (2)$$

$$\frac{\cancel{m}V_o^2}{\cancel{r}} = \frac{GM\cancel{m}}{r\cancel{r}}$$

$$\boxed{V_o = \sqrt{\frac{GM}{r}}}$$



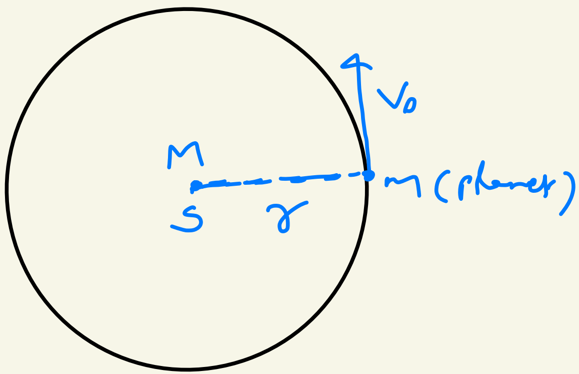
M - Mass of the sun

r - radius of the circular orbit.

→ orbital velocity does not depend on mass of the planet

Time period:-

→ Time taken by the planet to complete one revolution is called Time period.



$$v = r\omega$$

$$V_o = r\left(\frac{2\pi}{T}\right)$$

$$\sqrt{\frac{GM}{r}} = r\left(\frac{2\pi}{T}\right)$$

squaring on both sides

$$\frac{GM}{r} = r^2 \frac{4\pi^2}{T^2}$$

$$T^2 = \frac{4\pi^2}{GM} r^3$$

$$T^2 \propto r^3$$

$$\boxed{T = 2\pi\sqrt{\frac{r^3}{GM}}}$$

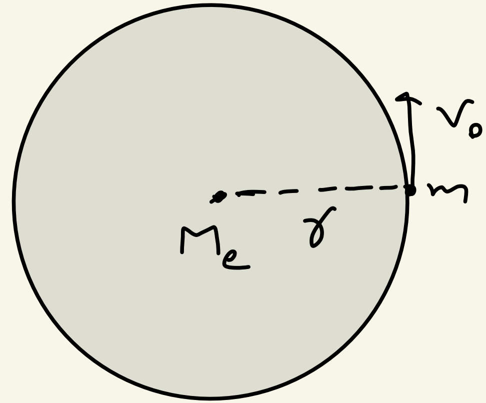
$$\boxed{\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}}$$

Satellites:-

→ Satellites revolve around the planets

orbital velocity (v_o)

$$\frac{\cancel{m} v_o^2}{\cancel{r}} = \frac{G M \cancel{m}}{r^{\cancel{2}}}$$



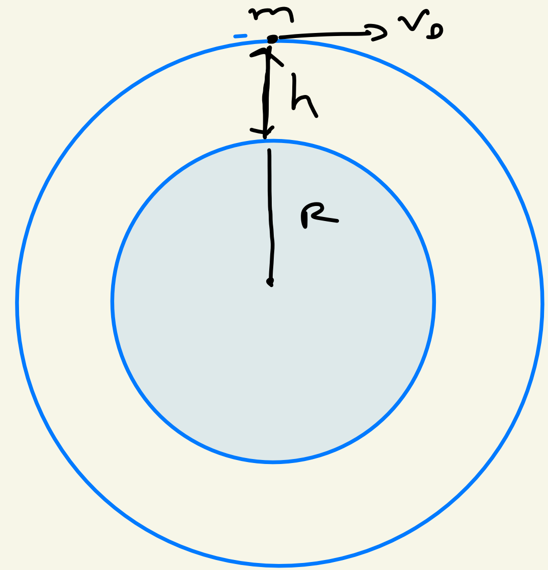
$$\boxed{v_o = \sqrt{\frac{G M_e}{r}}}$$

M_e - mass of the earth

r - radius of the orbit

$$\boxed{V_0 = \sqrt{\frac{GM}{R+h}}}$$

$r = R + h$



h - height of the satellite from surface of the earth

R - Radius of the earth

M - mass of the earth.

$$g = \frac{GM}{R^2}$$

$$\boxed{V_0 = \sqrt{\frac{gR^2}{(R+h)}}}$$

If $h << R$

$$\boxed{V_0 = \sqrt{gR}}$$

g - Acceleration due to gravity

Time period satellite:

$$\boxed{T = 2\pi\sqrt{\frac{(R+h)^3}{GM_e}}}$$

Potential energy:-

$$\boxed{U = -\frac{GMm}{r}}$$

Kinetic energy:-

$$K = \frac{1}{2} m v_0^2$$

$$K.E = \frac{1}{2} m \frac{GM}{r}$$

$$\boxed{K.E = \frac{GMm}{2r}}$$

Total energy

$$T.E = K + U$$

$$T.E = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$\boxed{T.E = -\frac{GMm}{2r}}$$

NOTE:-

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$$\boxed{K.E = -T.E = -\frac{P.E}{2}}$$

Eg:- A satellite is revolving around around. The K.E of the satellite is E. what is the P.E and T.E of the satellite?

sol:- $K.E = E$

$T.E = -K.E = -E$

$$\boxed{T.E = -E}$$

$$\boxed{P.E = -2K.E = -2E}$$

Geostationary satellite:-

Escape velocity :- (v_e)

The minimum velocity required for an object to make it free from the gravitational field of the planet i.e. to move to infinite distance from the point of projection.

The condition to escape an object from the planet gravitational field is

$$\boxed{\text{Total energy of the particle (T.E.)} \geq 0}$$

Escape velocity of object from surface of the earth

$v = 0$

$$K_i + U_i = K_f + U_f$$

since we want minimum velocity

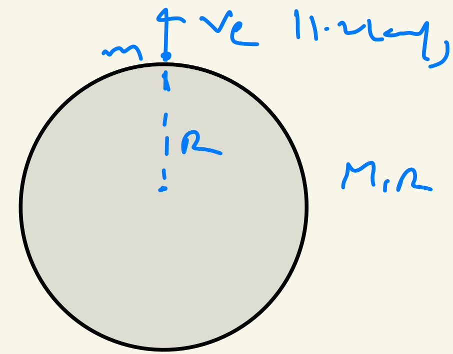
$$T.E = 0$$

$$K + U = 0$$

$$\frac{1}{2} m v_e^2 - \frac{GMm}{R} = 0$$

$$\frac{1}{2} \cancel{m} v_e^2 = \frac{GM\cancel{m}}{R}$$

$$\boxed{v_e = \sqrt{\frac{2GM}{R}}}$$



M - mass of the earth (or) planet

R - Radius of the earth (or) planet

→ If we substitute G, M and R value we get

$\boxed{V_e = 11.2\ km/s}$ → Escape velocity from the surface of the earth

Important points

→ Escape velocity depends on mass of the planet and radius planet

→ Escape velocity depends on point of projection

→ Escape velocity is Independent on mass of the body which we are projecting

→ Escape velocity is independent on angle of projection

→ The relation between orbital velocity and escape velocity is (Assume satellite is very close to the planet)

$$V_0 = \sqrt{gR}$$

$$V_0 = \sqrt{\frac{GM}{R}}$$

$$V_e = \sqrt{\frac{2GM}{R}}$$

* $\boxed{V_e = \sqrt{2}\, V_0}$ (It is not general formula)

Eg:- what is the escape velocity of the particle which is at a height $h = R$ from the surface of the earth? (Given that mass of the earth is M and radius of the earth is R)

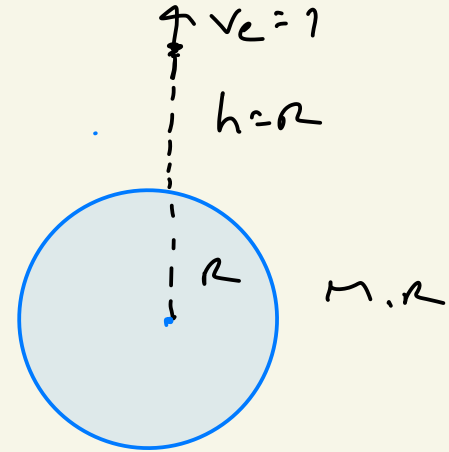
Sol:-

$$K_i + U_i = 0$$

$$\frac{1}{2}mv_e^2 - \frac{GMm}{2R} = 0$$

$$\frac{1}{\cancel{2}}\cancel{m}v_e^2 = \frac{GM\cancel{m}}{\cancel{2}R}$$

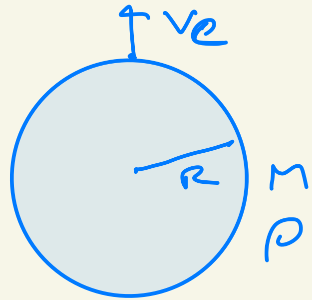
$$\boxed{v_e = \sqrt{\frac{GM}{R}}} \Rightarrow \boxed{\frac{11.2}{\sqrt{2}}\ km/s}$$



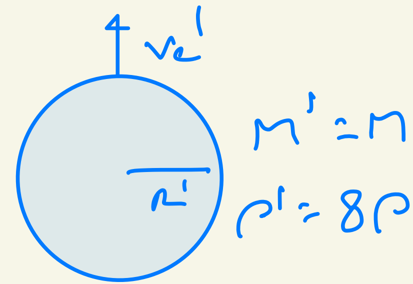
<u>Prob:-</u> The escape velocity of the particle from the surface of the earth is v_e. What is the escape velocity of the same particle from the surface of the other planet. whose mass is same as the mass of the earth and density is

8 times the density of the earth ?

sol :-



Earth



planet

$$v_e = \sqrt{\frac{2GM}{R}} \quad -(1)$$

$$v_e' = \sqrt{\frac{2GM'}{R'}} = \sqrt{\frac{2GM \times 2}{R}}$$

$$\frac{\cancel{M}}{\cancel{\frac{4}{3}\pi} R'^3} = \frac{8\cancel{M}}{\cancel{\frac{4}{3}\pi} R^3}$$

$$v_e' = \sqrt{2}\,\sqrt{\frac{2GM}{R}}$$

$$R^3 = 8R'^3 \Rightarrow R' = R/2$$

$$\boxed{v_e' = \sqrt{2}\, v_e}$$

1. A planet in a distant solar system is 10 times more massive than the earth and its radius is 10 times smaller. Given that the escape velocity from the earth is 11 kms^{-1}, the escape velocity from the surface of the planet would be

(*a*) 1·1 kms^{-1} (*b*) 11 kms^{-1}

(*c*) 110 kms^{-1} (*d*) 0·11 kms^{-1}

Option AIEEE)

sol :-

$$v_e = \sqrt{\frac{2GM}{R}} \qquad v_e' = \sqrt{\frac{2GM'}{R'}}$$

$$v_e' = \sqrt{\frac{2G \times 10 \times M \times 10}{R}}$$

$v_e' = 10\sqrt{\frac{2GM}{R}}$

$\boxed{v_e' = 10 \times 11 = 110\,km/s}$

4. The height at which the acceleration due to gravity becomes $g/9$ (where g = the acceleration due to gravity on the surface of the earth) in terms of R, (the radius of the earth), is

(a) $2R$ (b) $R/\sqrt{2}$

(c) $R/2$ (d) $\sqrt{2}R$

Sol :-

$g_h = g\left(\frac{R}{R+h}\right)^2$

$\frac{\cancel{g}}{9} = \cancel{g}\left(\frac{R}{R+h}\right)^2$

$\frac{R}{R+h} = \frac{1}{3}$

$3R = R+h$

$\boxed{h = 2R}$

12. What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of $2R$?

(a) $\frac{GmM}{3R}$ (b) $\frac{5GmM}{6R}$

(c) $\frac{2GmM}{3R}$ (d) $\frac{GmM}{2R}$

Two satellites A and B go around a planet P in circular orbit having radii 4 R and R respectively. If the speed of the satellite A is 3 V, the speed of the satellite B will be

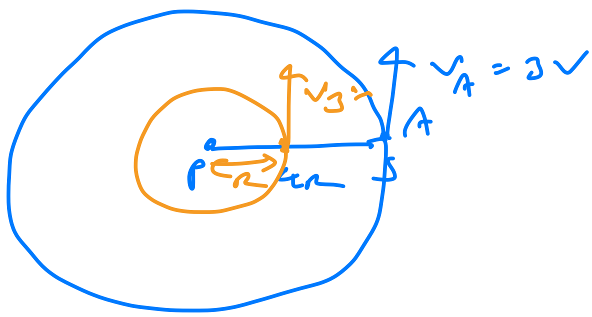
(a) 12 V (b) 6 V

(c) (4/3) V (d) (3/2) V

$$V_o = \sqrt{\frac{GM}{r}}$$

$$V_A = 3v = \sqrt{\frac{GM}{4R}} \quad — (1)$$

$$V_B = \sqrt{\frac{GM}{R}} \quad — (2)$$



$$\frac{3v}{V_B} = \sqrt{\frac{\cancel{GM}}{4\cancel{R}} \times \frac{\cancel{R}}{\cancel{GM}}}$$

$$\boxed{V_B = 6v}$$

47. A particle is thrown vertically upwards from the surface of earth and it reaches to a maximum height equal to the radius of earth. The ratio of the velocity of projection to the escape velocity on the surface of earth is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) $\frac{1}{2\sqrt{2}}$

48. A projectile is fired from the surface of earth of radius R with a velocity kv_e (where v_e is the escape velocity from surface of earth and $k < 1$). Neglecting air resistance, the maximum height of rise from the centre of earth is

(a) $\frac{R}{k^2 - 1}$ (b) $k^2 R$

(c) $\frac{R}{1 - k^2}$ (d) $k R$

20. The rotation of the earth having radius R about its axis speeds upto a value such that a man on equator feels weightless. The duration of the day in such case will be

(a) $2\pi\sqrt{\frac{R}{g}}$

(b) $\pi\sqrt{\frac{R}{g}}$

(c) $8\pi\sqrt{\frac{R}{g}}$

(d) $4\pi\sqrt{\frac{R}{g}}$

19. Potential (V) at a point in space is given by $V = x^2 + y^2 + z^2$. Gravitational field at a point (x, y, z) is

(a) $-2x\hat{i} - 2y\hat{j} - 2x\hat{k}$ (b) $2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

(c) $x\hat{i} + y\hat{j} + z\hat{k}$ (d) $-x\hat{i} - y\hat{j} - z\hat{k}$

54. An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the surface of earth. The height of the satellite above the surface of earth's surface will be (Radius of earth, $R = 6400$ km)

(*a*) 6000 km (*b*) 5800 km

(*c*) 7500 km (*d*) 6400 km

55. Geostationary satellite orbits around the earth in a circular orbit at a height of 36000 km from the earth's surface. Then, the time period of a spy satellite orbiting at a height of 1600 km above the earth's surface ($R_{earth} = 6400$ km) will approximately be

(*a*) 1/2 hr (*b*) 1 hr

(*c*) 2 hr (*d*) 4 hr