A Theory of Retailer Price Promotion Using Economic Foundations: It's All Incremental¹

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Abstract: This paper presents the Theory of Retailer Price Promotions (TRPP) for Consumer Packaged Goods (CPGs). The theory states that under several conditions that completely apply to CPGs, incremental retail sales generated by promotional price discounts are entirely incremental to the promoting manufacturer, the promoting retailer and the category, overall. In general and as observed examining traditional retail point-of-sale data, this implies that there is no post-period reduction in sales (dip) either in the short or long-term, nor is there a reduction of sales for competing brands, nor is there a reduction of sales for the promoted item in competing retailers. It is a Complete Category Expansion Effect (CCEE). The paper discusses the underlying theories of consumer demand that support the CCEE and the lack of economic rationale offered by prior literature. A calibration-simulation example is offered to support the CCEE.

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1. Introduction

Up until 2007-08, few topics in the Marketing academic literature were, as analyzed, controversial and misunderstood as the effects of retail price promotions. These effects can occur on the promoting brand, competitive brands or promoting retailer. We refer to this research as Sales Decomposition Research (SDR) for Promotions.

While interest from academia has declined significantly in recent years to move on to other more "interesting" topics such as digital marketing, the fundamental issues surrounding the sales decomposition issue have yet to be resolved. This is a major oversight by researchers in that trade promotion still accounts for the largest portion of the typical Consumer Package Goods (CPG) brand marketing budget in the U.S. Gartner (Gartner Research (2013)) estimates this spending now exceeds 20% of revenue and is growing its share of marketing spending every year.

A retail price promotion is defined as a temporary price reduction (TPR) to consumers offered by retailers. There is no controversy in the literature that the vast majority of these events create a short-term spike in the sales of the promoting brand at the promoting retailer (Van Heerde, Leeflang, Wittink (2000) Blattberg et al (1995)). We can see an example of this short-term spike in Exhibit 1. Also, the work of Pauwels et al (2002) and DelVecchio et al (2006) have established a consensus that there are no long-term effects on the promoting brand – positive or negative – from price promotions. What is still unanswered is where the sales spike is sourcing its incremental sales in the short-term and intermediate term. That is, who does the incremental sales source from

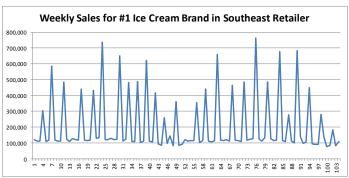


EXHIBIT 1: Evidence of Short-term Sales Spike from Retailer Price Promotion

(own brand, competitive brand or some other source)? When do the adjustment effects of the substitution occur (short, intermediate or long term)? Finally, where does the adjustment occur (promoting retailer or competing retailer)?

An answer to this question is even more important today than it was 10-15 years ago when most of this type of research was conducted, given the higher share of spending being allocated to trade promotions (Kantar retail (2012)). Indeed it is a recurring theme in the CPG (or fast Moving Consumer Goods (FMCG))⁵ industry that trade promotion is a necessary evil or an addiction that must be cured. In this sense, in August 2014 the CEOs of two public companies, Kraft Foods (Anthony Vernon) and Campbell Soup (Denise Morrison), cited lower response from trade promotions as a significant cause of their weak sales.

To understand the importance of measuring the degree to which promotional sales are incremental, one needs to look no further than the disastrous results of department store retailer J.C. Penney when they completely eliminated promotions in 2012.⁶

Up to these days, the majority of the SDR literature on retailer price promotions deals with the "who does the incremental sales source from (own brand, competitive brand or some other source)?" aspect. Gupta (1988) used household panel data of coffee purchasing. He estimated that 84% of the sales increase source from brand switching and 16% from own Brand. Bell et al (1999) claimed to have replicated these results using a much broader database of 13 categories and 174 brands. This study, however, yielded significantly different results in the coffee category, where Bell et al (1999) found 48% of the promoted sales where sourced from own brand vs. the 16% found by Gupta (1988).

Van Heerde et al (2003) offered a new measure for calculating sales decomposition that considers the critically important potential for *category expansion*. Previous studies had only looked at market share changes, which, by definition, leave no possibility of demand expansion. Just by redefining the sales elasticity from trade promotion to a unit-basis vs. share-basis, they found that only 33% of sales were sourced from brand switching, 33% from own brand and 33% came from category expansion. Around that same time Pauwels et al (2002) used a VAR modelling technique

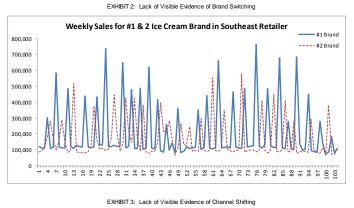
⁵ CPGs or FMCG are products with relative low price and high turnover. CPG/FMCG products include soft drinks, toiletries, OTC drugs, processed food, etc. They are typically sold in mass market retailers such as Grocery Stores, Super Centers, Hypermarkets, Drug-Pharmacy Stores, Mass Merchandisers, Value-Dollar Stores, or Club Stores.

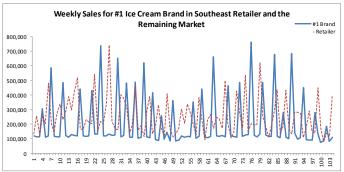
http://www.nytimes.com/2012/01/26/business/jc-penneys-chief-ron-johnson-announces-plans-to-revamp-stores.html? r=0

and a different data set to calculate a much higher category expansion (62% on average) with the remainder sourcing from brand switching (25%) and own brand cannibalization (13%).

As noted earlier, there is a consensus in the literature that there are no-long term effects from price promotion, nor is there controversy that the immediate effects are significant and positive on the promoting brand. This helps us to isolate most of the discussion of "When do the adjustment effects of substitution occur?" We define the short-term as the week of the promotion (week 0), and we will follow the definition for intermediate term offered by Pauwels et al (2002) to weeks 1 through 8. The interest lies in the potential for negative sales (i.e. sales dip) in that intermediate period. The negative sales can come from either pantry loading (taking a consumer out of the market for a repurchase during that time) or purchase acceleration (a consumer that purchased in the promotion week instead of their usual pattern that may have been a few weeks later).

This sales dip has been a great puzzle for numerous researchers such as Gupta (1988), Bell et al (1999), Pauwels et al (2002), Van Heerde et al (2003), Hendel and Nevo (2003). All of them expressed surprise that evidence of this sales dip did not exist when examining traditional retail point-of-sale data. Curiously though, several articles managed to use more exotic models and data sources to identify that the dip did, in fact, exist despite the lack of evidence in syndicated sales data, which is considered to be the standard of reality for measuring sales performance (Dekimpe, Hanssens, Nijs and Steenkamp (2005)). In Exhibits 2 & 3 we see examples of the types of retail sales data that puzzled so many researchers. In Exhibit 2 we see the strong promotion from Exhibit 1 have no noticeable effect the Own Brand or competing brands in the short or intermediate term. In Exhibit 3 we see no evidence of channel shifting for the #1 Ice Cream Brand in the market.





The issue of "where does the adjustment occur (promoting retailer or competing retailer)?" is the least researched element of SDR, and the few results available are inconclusive. Walters (1991) found some weak evidence for cross channel effects or channel shifting. Dawes (2004) concluded that the source of the sales spike was from competing retailers for competing brands in future weeks, a rather dubious conclusion that begs the question of why there were no obvious immediate effects on competing brands in the promoting retailer.

While there are highly variable conclusions emanating from this stream of literature, one thing in common with all of them is the minimal effort of offering a theoretical basis for evaluating the validity of results and conclusions. We would suggest that this lack of theoretical foundation is the primary reason for the continued controversy on this topic. Marketing literature, in general, has been content to rely on empirical generalization for the organization of knowledge (Bell, Chiang and Padmanhaban (1999), Hanssens (2010)) rather than on constructing or using a theoretical framework for the discipline. In particular there are few mentions of the theories or laws of economics. This is analogous to a situation where the body of engineering research ignores the laws of Physics. In this sense, Johnson (2006) notes this lack of cross-pollination between marketing and economics. Jetta (2008) notes that in Neslin's book, Trade Promotion (2002) —

which was a broad-reaching audit of extent literature on the topic, only 5% of the citations are from economics journals.

This paper demonstrates that a logical progression of three economic theories to support the Complete Category Expansion Effect (CCEE). Slutsky (1915) established the need for an empirically derived Substitution Effect in the Law of Demand; Hicks (1946) proved that substitution effect can be considered in the context of one product substituting with all discretionary income rather than just a specific product or category, and Cournot (1838) who showed that when substitution is considered in the context of all discretionary income the substitution effect for a low-priced product on any other specific product is immaterial. We join this background together with and two well-known utility functions to finally show that CCEE is completely feasible in real life. Thus, through that progress we build in the Theory of Retailer Price Promotion (TRPP) for CPGs which is supported with a robust mathematical tools and economic theory all supported by a calibration-simulation data analysis.

The remaining of the paper is organized as follows. Section 2 presents the economic foundation for the Theory of Retailer Price Promotion for CPGs. Section 3 describes the utility functions and the data used to verify that CCEE is technically feasible and that does not contradict any economic theory or law. A calibration example is presented in Section 4 and, Section 5 concludes and presents future venues of research.

2. Economic Foundations

We now look to the field of microeconomics to identify the source of the incremental sales. There is one law, Slutsky's (1915) Fundamental Value Theory or Slutsky equation, and two theorems, Hicks' (1946) Composite Goods Theorem and Cournot's (1846) Aggregation Condition of Demand. The rationale described in this paper is built by taking them in sequence. After we introduce these law and theorems, we present two utility functions that will help us to better understand these three theoretical components and to implement a calibration exercise to support our theoretical development.

2.1. Slutsky's Fundamental Value Theory (1915): The Slutsky equation was expanded upon by Hicks and Allen (1936) to become the Law of Demand. The Law of Demand states that the

quantity demanded for a good, Q^D, is a function of its price at a fixed point in time (t), subject to certain assumptions such as static tastes and preferences, static income, static information and static prices of competitive and substitute products. Q^D always has a negative slope as it responds inversely to price changes: quantity demanded increases with a price reduction and it decreases with a price increase.

The change in demand is a function of two effects: the *Substitution Effect* and the *Income Effect*. The intuition of the Substitution Effect is that when price is reduced (similar to what we see for retailer price promotions) a consumer will substitute their purchases of other goods (that in *relative terms* are more expensive) in order to purchase more of the promoted good (that in relative terms is cheaper). The intuition of the income effect is that for a sufficiently large reduction in price, the consumer can have an increase in *real purchasing power*, which accentuates the increase in Q^D even more.

Hicks (1946) stated in his discussion about the Income Effect that "It is therefore a consideration of great importance that this unreliable income effect will be of relatively little importance in all those cases where the commodity in question plays a fairly small part in the consumer's budget." This statement is of great importance in the discussion that follows in the next sections. Also, without the Income Effect we are left purely with the Substitution Effect for an understanding of the Complete Category Expansion Effect.

Past literature (Van Heerde et al (2002), Hendel and Nevo (2003)) presupposed that this substitution effect came from either brand switching or lower demand of the product (Q^D) in weeks t+1, t+2,...t+8. These authors, despite their acknowledgement that this substitution effect was not observable in scanning data, did not search for any explanation outside of the category construct used or with what Lancaster defined as "intrinsically similar goods". Henderson and Quandt (1980) make it clear that the substitution effect cannot be assumed, it must be empirically derived. Up to this point and even accepting the 33% category expansion estimate by Van Heerde et al (2003) or the 62% by Pauwels et al (2002), we are still left with the question: where does the substitution effect come from? Hicks (1946) provides a logical and appropriate explanation with his Composite Goods Theorem.

2.2. Hicks Composite Goods Theorem (1946): Hicks posited that the collection of remaining goods can be treated as a single unit so long as their prices remain constant. "A collection of physical things can always be treated as if they were divisible into units of a single commodity so long as their relative prices can be assumed to be unchanged...So long as the prices of other consumptions goods are assumed to be given, they can be lumped together into one commodity 'money' or 'purchasing power in general."

The Composite Goods Theorem is the reason why the Complete Category Expansion Effect (CCEE) can exist. The sales promotion spike would not only source from competitive items in the same category, but from *all* items for which the consumer can spend money: entertainment, clothes, fuel, home improvements, etc. Given that competitive, non-promoted products are part of the remaining discretionary income, we would still expect there to be some volume sourcing from these products, as well.

The only remaining issue, then, is why we have never been able to measure this effect. The Cournot Aggregation Condition provides the explanation.

2.3. Cournot's Aggregation Condition (1838): The Cournot's Aggregation equation shows the relationship between own and cross-price effects. It is obtained by differentiating the individual's budget constraint with respect to the price of a given good x. Recall that the most simple budget constraint for 2 bundles of goods, x and y, is given by:

$$p_{x}x + p_{y}y = I \tag{1}$$

Where, p_x and p_y represent the prices of bundles x and y, respectively. I represents the individual's income. Differentiating the individual's budget with respect to p_x and making this equal to zero to keep the individual's income unchanged:⁷

$$p_x \frac{\partial x}{\partial p_x} + x + p_y \frac{\partial y}{\partial p_x} = 0 \tag{2}$$

By some algebra manipulations that do not change the previous equation, we have:

⁷ We develop a more extensive mathematical development in the following section.

$$p_x \frac{\partial x}{\partial p_x} \frac{p_x}{l} \frac{x}{x} + x \frac{p_x}{l} \frac{x}{x} + p_y \frac{\partial y}{\partial p_x} \frac{p_x}{l} \frac{y}{y} = 0$$
 (3)

Finally, rearranging Equation (3),

$$S_{\chi}\varepsilon_{\chi,p_{\chi}} + S_{y}\varepsilon_{y,p_{\chi}=} - S_{\chi} \tag{4}$$

Where s_x and s_y represent the percentages of the individual's income spent on bundle x and y, respectively. The variable $\varepsilon_{x,px}$ and $\varepsilon_{y,py}$ represent the price elasticity of x and the cross price elasticity of y with respect to the price of x. The equations are provided below:

$$S_{x} = \frac{p_{x}x}{I}$$

$$S_{y} = \frac{p_{y}y}{I}$$
(5)

And,

$$\varepsilon_{x,p_x} = \frac{\partial x}{\partial p_x} \frac{p_x}{x}$$

$$\varepsilon_{y,p_x} = \frac{\partial y}{\partial p_x} \frac{p_x}{y}$$
(6)

Using Equation (4), Cournot (1838) showed that when substitution is considered in the context of *all discretionary income*, the substitution effect for a *low-priced* product on any other specific product is *immaterial*. To see this, note that if s_x (the percentage of the individual's income spent on bundle x) is small, a change on bundle x's price (p_x) has an insignificant effect on the quantity spent in bundle y.

With this result at hand together with the Hicks Composite Goods, we can construct a two dimensional space made of two bundles (say the CPGs bundle (x) and the all-the-other bundle (y)) and calculate the expected change in sales of both bundles given a percentage increase in the price of one of them (in our case a change in the price of bundle x, p_x).

Before moving to the calibration example we briefly introduce the concept of utility functions. Utility is understood as the perceived ability of a good to satisfy needs of the individuals. As soon as the utility that a good provides to individuals are is directly observed, economists have created mathematical ways of representing and measuring utility in terms of economic choices

that can be measured (Samuelson (1938)). In this sense, economists consider utility to be revealed in people's willingness to pay different amounts for different goods. Under this understanding, utility functions are simply mathematical functions that rank alternatives according to the perceived utility they provide to an individual. In the next section we present two of the most used utility functions. We start with the most basic one that we use as benchmark (the Cobb-Douglas utility function) and later we introduce the Constant elasticity of Substitution (CES) utility function that provides us with more flexibility to model what could be happening in real markets.

The following Sections present the main mathematical results. We refer the reader to the technical appendix for a detailed explanation.

2.4. The Cobb-Douglas Utility Function

From now on, our unit of analysis is a household (HH). The Cobb-Douglas utility function has the following mathematical representation:

$$U(x,y) = x^{\alpha} y^{\beta} \tag{7}$$

Where α and β are model parameters that represent the share of consumption of each bundle, by model definition $\alpha+\beta=1$. The variables x and y represent bundles of goods.⁸ A few statistics can help us to understand how this function works. The Marginal Rate of Substitution (MRS_{y,x}) measures the amount of y a consumer needs to get in order to give up a little of good x, keeping the same level of utility.⁹ Mathematically,

$$MRS_{y,x} = \frac{\Delta y}{\Delta x} = \frac{U_x(x,y)}{U_y(x,y)} = \frac{Marginal\ Utility\ of\ x}{Marginal\ Utility\ of\ y}$$
 (8)

Applying the formula for the Cobb-Douglas utility function, the MRS_{y,x} is given by:

$$MRS_{y,x} = \frac{\alpha}{\beta} \left(\frac{y}{x} \right) \tag{9}$$

The other important measure to understand the Cobb-Douglas in particular and any other utility function in general, is the *elasticity of substitution*. This elasticity measures the curvature of the indifference curve estimating the degree to which the consumer's valuation of good x

⁸ In our case x will represent the CPGs category and y will represent all the other goods in the consumer's basket.

⁹ Recall that if a consumer has a lot of x relative to y, then x is much less valuable than y, then MRS will be low.

depends on his holdings of x. Recall that utility functions are in general increasing at a decreasing rate. This means that utility provided by an extra unit of a good depends on how much someone already has. If some HH has very few of the good, having an additional one significantly increases the utility of that good. In the other hand, if some HH already has plenty of a good, the marginal utility of this additional units will not be as high as the previous case; thus his or her utility, even though increases, it does at a lower rate. This is the information that we get form the elasticity of substitution. The elasticity of substitution is measured as:

$$\varepsilon = \frac{\%\Delta(\frac{y}{x})}{\%\Delta MRS} \tag{10}$$

Where, $\%\Delta$ represents the percentage change of the variable in parenthesis. For the Cobb-Douglas utility function, the elasticity of substitution is (ϵ) equals 1. This result implies that the consumer's valuation of good x, regardless of his actual holdings of x, is constant and equal to 1.

Using the Cobb-Douglas utility function in order to determine the demand functions of bundles x and y, subject to a budget constraint, allows us to gain additional intuition about why it is perfectly feasible to have a Complete Category Expansion Effect (CCEE), even in this simple environment. The HH maximization problem is presented below and fully developed in the technical appendix:

$$\max_{\substack{x \ge 0, y \ge 0, \alpha \ge 0, \beta \ge 0, \alpha + \beta = 1 \\ s. t p_x x + p_y y = I}} U(x, y) = x^{\alpha} y^{\beta}$$

$$(11)$$

Where p_x and p_y represent the average prices of bundles x and y, respectively. The variable I represents the average consumer income.

Solving this constraint maximization problem, we obtain the optimal demanded quantity for x:

$$\chi = \left(\frac{\alpha}{\alpha + \beta}\right) \frac{I}{p_x} \tag{12}$$

Thus, the quantity of x demanded equal the share provided by the first term of the total unit that can be bought at the price p_x given a level of income I. The quantity demanded for y is given by:

$$y = \left(\frac{\beta}{\alpha + \beta}\right) \frac{I}{p_{y}} \tag{13}$$

From equations (12) and (13) we can obtain the log-linear demand curves:

$$\ln(x) = \ln\left(\frac{\alpha}{\alpha + \beta}\right) + \ln(I) - \ln(p_x) \tag{14}$$

$$\ln(y) = \ln\left(\frac{\beta}{\alpha + \beta}\right) + \ln(I) - \ln(p_y) \tag{15}$$

As expected, the quantities demanded are positively related to income and negatively related to the price of the bundle. Note that in this case, the quantity demanded of one of the bundles is independent of the demand of the other one.

Another property of this utility function is provided by its constant unit price elasticity, as can be seen below.

$$\varepsilon_{x,p_x} = \frac{\%\Delta x}{\%\Delta p_x} = \frac{\Delta x}{\Delta p_x} \frac{p_x}{x} = -\left(\frac{\alpha}{(\alpha+\beta)} \frac{I}{p_x^2}\right) \frac{p_x}{\left(\frac{\alpha}{\alpha+\beta}\right) \frac{I}{p_x}} = -1 \tag{16}$$

Moreover, the cross price elasticity equals to zero:

$$\varepsilon_{x,p_y} = \frac{\%\Delta x}{\%\Delta p_y} = \frac{\Delta x}{\Delta p_y} \frac{p_y}{x} = 0 \frac{p_y}{x} = 0 \tag{17}$$

This result is a direct consequence of not having p_y in the equation that corresponds to x (eq. (12)). This means that no matter what the price of y is, the demand of x will be independent of it. Finally, the income elasticity is also constant and equal to 1:

$$\varepsilon_{x,p_y} = \frac{\%\Delta x}{\%\Delta I} = \frac{\Delta x}{\Delta I} \frac{I}{x} = \left(\frac{\alpha}{\alpha - \beta}\right) \frac{1}{p_x} \frac{I}{\left(\frac{\alpha}{\alpha - \beta}\right) \frac{I}{p_x}} = 1$$
 (18)

Given these characteristics, the Cobb-Douglas utility function is one that we use as a benchmark one, to clearly show that the Complete Category Expansion Effect (CCEE) can exist.

A final point to note here. Later in the paper we interested in the percentage change of a HH's utility function when the price of the good x changes, keeping all the other variables constant. For the Cobb-Douglas, the percentage change in utility equals:

$$\frac{U_{p_{x1}}(x,y)}{U_{p_{x}}(x,y)} - 1 = \left(\frac{p_{x}}{p_{x1}}\right)^{\alpha} - 1 \tag{19}$$

Equation (19) is an important equation that tells us that the percentage change in the HH's utility depends on the ratio of actual prices to future prices and more importantly, depends on the share of income that product x has. As soon as α is defined in [0,1], the larger α the larger the percentage change in utility.¹⁰

As has been already noted, the main limitation of the Cobb-Douglas utility function is given by its zero cross-price elasticity. In real life, we expect that changes in price of bundle x has an impact on the quantity demanded of the other goods represented by bundle y and, vice versa. In order to provide a more flexible utility function that allows to captures cross-relationships between bundles, in the next section we briefly introduce the Constant Elasticity of Substitution (CES) utility function.¹¹

It is important to note that in the example presented in the next section we do not maximize Equation (11) but the natural logarithm version of it. We start by taking the natural logarithms of the Cobb-Douglas utility function and maximize the following:

$$\max_{x \ge 0, y \ge 0, \alpha \ge 0, \beta \ge 0, \alpha + \beta = 1} lnU(x, y) = \alpha ln(x) + \beta ln(y)$$

$$s. t p_x x + p_y y = I$$
(20)

Following a procedure similar to the one presented before, we obtain the following demands:

 $^{^{10}}$ For example, if we set P_x = 2 and P_{x1} =1.6 and a couple of values for α (0.07 and 0.36), the percentage changes in utility are1.57% and 8.4%, respectively. We provide more intuition later in the paper. Also note that an increase in prices causes more disutility the larger α is.

¹¹ It is important to note that the Cobb-Douglas is a special case of the CES utility functions.

$$x = \alpha \frac{I}{P_r} \tag{20a}$$

$$y = \beta \frac{I}{P_y} \tag{20b}$$

In this case, the demands depend only on their specific coefficients, income and own prices. As it is clear, the coefficients α and β simply represent the proportion of income used to buy CPGs (x) or the other categories of products that constitute the bundle others (y).

2.5. The Constant Elasticity of Substitution (CES)

The mathematical representation of the CES utility function is given by:

$$U(x, y) = (\alpha x^{\rho} + (1 - \alpha) y^{\rho})^{\frac{1}{\rho}}$$
 (21)

Again, α and β are the share parameters and ρ the parameter that controls for the elasticity of substitution. Following the sequence of the previous section, we proceed to define the Marginal rate of substitution (MRS_{y,x}) of this function. The MRS_{y,x} is given by:

$$MRS_{y,x} = \left(\frac{\alpha}{1-\alpha}\right) \left(\frac{y}{x}\right)^{1-\rho}$$
 (22)

The *elasticity of substitution* is measured is given by:

$$\varepsilon = \frac{1}{(1-\rho)} \tag{23}$$

Note that the parameter α controls the MRS_{y,x} and that ρ influences the elasticity of substitution that influences the slope of the demand curve. Again, using the CES utility function in order to determine the demand functions of bundles x and y, subject to a budget constraint, allows us to get the main intuition about why it is perfectly feasible to have a Complete Category Expansion Effect (CCEE). In this case, the maximization problem is presented below: ¹²

$$\max_{x \ge 0, y \ge 0} (\alpha x^{\rho} + (1 - \alpha) y^{\rho})^{\frac{1}{\rho}}$$

$$s. t p_{x} x + p_{y} y = I$$
(24)

¹² Again, for a detailed mathematical derivation we refer the reader to the technical appendix.

Where p_x and p_y represent the average prices of bundles x and y, respectively. The variable I represents the average consumer income. The quantity of bundle x demanded is given by:

$$x = \frac{I}{p_x + p_y \left(\frac{(1 - \alpha)p_x}{\alpha p_y}\right)^{\varepsilon}} \tag{25}$$

Using some algebraic manipulations, we get:

$$x = \frac{(p_x/\alpha)^{-\varepsilon}}{\alpha^{\varepsilon} p_x^{1-\varepsilon} + (1-\alpha)^{\varepsilon} p_y^{1-\varepsilon}} I$$
 (25a)

Finally, we obtain a similar equation for y:

$$y = \frac{(p_y/(1-\alpha))^{-\varepsilon}}{\alpha^{\varepsilon} p_x^{1-\varepsilon} + (1-\alpha)^{\varepsilon} p_y^{1-\varepsilon}} I$$
 (25b)

We use this model to obtain estimates that we compare with the benchmark provided by the Cobb-Douglas utility function.

3. A Simple Example

In this section we use US economic and demographic data to show that the Complete Category Expansion Effect (CCEE) is indeed feasible in the CPGs environment. As soon as data directly related to CPGs supply and demand are not completely known, we are going to base our analysis on a category that is highly tracked: food.

In order to model the importance and impact of price changes in food category, we rely on a couple of utility function widely used in economics. Specifically, we use income and expenditure related data for the U.S. we describe the data in the following sections.

3.1.Food Consumption Expenditure in the US

Figure 1 provides data from the US Bureau of Labor Statistics (BLS), Consumer Expenditure Survey of 2013, on In-Home Food spending by income quintile. We see that spending on Food increases measurably in the absolute for the higher income quintiles, from a low of \$3,655 to the lowest income quintile (Q1) to \$11,184 for the highest income quintile (Q5). More importantly, we see the proportion of Total Income spent on Food falls dramatically from Q1 (36.5%) to Q5 (only 6.9%). This information will be used to estimate utility functions for each quintile under promotional conditions.

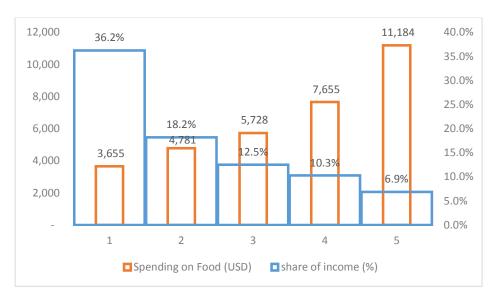


Figure 1: Food Consumption Expenditures in the US

A more complete dataset of quintile income and Food expenditures is provided in Table 1¹³.

Table 1: Food Expenditures in the US per Quintile

Quintile	Number of HH	Population	Food	Average	Per-capita	Aggregate mean
	(in thousands)	quintile	expenditures	income per	mean dollar	dollar spent on
		distribution	as share of	quintile (\$)	spent on	food per quintile
		(%)	income		food per	(\$)
			(%)		class quintile	
					(\$)	
lowest	25,090	20.0%	36.2%	10,092	3,655.00	91,703,950
second	25,219	20.1%	18.2%	26,275	4,781.00	120,572,039
middle	25,082	20.0%	12.5%	45,826	5,728.00	143,669,696
fourth	25,178	20.0%	10.3%	74,546	7,655.00	192,737,590
highest	25,101	20.0%	6.9%	162,720	11,184.00	280,729,584
total	125,670	100%				829,412,859

This table presents data on food expenditures in the US, per average income quintile for 2013 as reported by US Bureau of Labor Statistics. The per-capita mean dollar spent on food equals the average income per quintile, times food expenditures as share of income. The aggregate mean dollar spent on food per quintile equals per-capita mean dollar spent on food times the number of HH in each income quintile.

In this paper we use the expenditures in food as proxy for the expenditures in CPGs. From table 1 we can see that total Food expenditures are roughly \$829B. According to Nielsen,

¹³ These numbers come from the Bureau of Labor Statistics. http://www.bls.gov/cex/2013/combined/quintile.pdf.

Food sales tracked through scanners were \$390B in 2013. Projecting out another 20% for channels not tracked (e.g. Costco, Natural Food, Value Food, Convenience Stores, Specialty) brings the total CPG universe for Food to roughly \$488B, or about 59% of total food. Based on these numbers, we expect the CPG category to have a smaller impact on households' income than the one observed in food. In this sense our results should be considered conservative.

Finally we note the wide disparity in average income between Q1 and Q5, with Q5 average income 16 times higher than Q1. This observation has important implications in the later discussion.

3.2. Probabilities of buying on promotions

We now make use of the Food expenditure share of income by proposing that the likelihood of buying Food on promotion is directly proportional to the share of income spent on food. The rationale for the construct is that the higher the percentage of food expenditures with respect to the average income, the higher the likelihood of buying on promotions. We standardize all the shares of food expenditure relative to Q1 and obtain an estimate of the likelihood of buying on promotion for the other quintiles. This means that the likelihood of Q2 buying on promotion is 50.2% that of Q1 (18.2%/36.2%), and so on. The results of applying this metric are presented in Table 2. 15

Table 2: Probabilities of Buying on Promotions

		Food expenditures as	
	Number of HH (in	share of income	Probabilities of response
Quintile	thousands)	(%)	to a promotion
lowest	25,090	36.2%	100.0%
second	25,219	18.2%	50.2%
middle	25,082	12.5%	34.5%
fourth	25,178	10.3%	28.4%
highest	25,101	6.9%	19.0%
total	125,670		

This table presents the construct to explain the probabilities that a given HH has depending on its income quintile.

¹⁴ We are not arguing anything about what literature already discussed in terms of promotion effectiveness and related issues. We simply propose a logic idea that the higher the expenditure ratio with respect to income, assuming that the needs of the HH of the lower quintile are at least similar to the ones in higher quintiles, the higher the likelihood of actively looking for promotions and actually buying on promotions.

¹⁵ We also used set the lowest quintile to be a number in between 80% to 100% and, adjusted the other probabilities accordingly. Directionally the findings remain the same. Results are available upon request.

The basic idea underlying this construction is that the larger the expenditure to income ratio a category has, the more care and attention HH pays to ways of optimizing its consumption. Looking for promotions is a natural way of doing that, since this allows to buy the same quantity for less money, buy more for the same expenditure as before or buy even more sacrificing the consumption of the other categories and products in their basket.

An additional mathematical fact that helps us support this probability construct is given by noting that HHs' percentage change in their utilities, change accordingly not only to price changes but also by the actual share of income of a given category (food in our case). This can be clearly seen in Equation (19) that shows that the percentage change in the HH's utility depends on the ratio of actual prices (P_x) to discounted prices (P_{x1}) and more importantly, depends on the share of income that product x has (α) . As soon as α is defined in [0, 1], the larger α the larger the percentage change in utility. In terms of the column "Food expenditures as share of income" in Table 2, HHs in the lowest income quintile are the ones that experience the highest changes in utility even though their actual increases in units consumed is low with respect to the ones observed in the higher quintiles HHs. This, mathematical fact together with the economic intuition that HHs maximize utility functions can be interpreted as HHs in the lowest quintiles being those more prone to look and to actually buy on promotions.

With this last piece of information, we are able to present several numerical simulations and calibration exercises that help us prove that indeed a Complete Category Expansion Effect (CCEE) is feasible.¹⁶

3.3. Aggregate Demand

To show that a Complete Category Expansion Effect (CCEE) is feasible, we need to have an aggregate demand function to understand the dynamics that could be observed during promotion and non-promotion periods. For that we use as an example the log demand equations from (14) and (15) that we replicate below:

¹⁶ It is important to note that in the following example we assume a 20% aggregate price discount on Food process. In real life the aggregate price discounts are of the order of 2-3%. Also in this sense, our results allow us to present more conservative results that the actual ones that should be observed in reality.

$$ln(x) = ln(\alpha) + ln(I) - ln(p_x)$$
(26a)

$$ln(y) = ln(\beta) + ln(I) - ln(p_{\nu})$$
(26b)

These equations represent the individual demands for bundles x (CPGs in our case) and y (the other categories and products). If we know the individual demand functions of each HH, we can aggregate all the consumers (households in our case) to obtain the market demand function. We assume that in average the HH measure their utilities based on the Cobb-Douglas (later on the more general CES utility function). This aggregated demand function is simply achieved summing up equations (26a) and (26b). However, we know that these HH demand functions are heavily affected by the income level that determines the proportion of income assigned to each bundle, i.e. different share (α and β coefficients). Thus, the aggregation should happen first at the quintile level and after at the market level, i.e.

$$\ln(X_1) = \sum_{i=1}^{n_1} [\ln(\alpha_1) + \ln(I_1) - \ln(p_x)] = n_1(\ln(x_1))$$
...
$$\ln(X_5) = \sum_{i=1}^{n_5} [\ln(\alpha_5) + \ln(I_5) - \ln(p_x)] = n_5(\ln(x_5))$$
(27)

Where $\ln(X_q)$ is the aggregated demand for HHs in the income quintile q for q=1,...,5; α_q the share coefficient of bundle x corresponding to HHs in income quintile q. Finally, n_q represents the number of HHs in income quintile q. Finally, summing up each of the quintile demands, we obtain the market log-demand,

$$\ln(X) = \sum_{q=1}^{5} \ln(X_q) \tag{28}$$

The last issue regarding the demand aggregation is to find a way to separate the aggregate market demand into promotional and non-promotional periods. For this, we assume that households decide, care about or are aware of promotions, based on the proportion of their expenditure to income ratio that the bundle x represents. We use the probabilities construct presented in Table 3 and proceed as before to find the aggregated market demand for x. Let's define the demand of bundle x during promotional and non-promotional periods as:

$$\ln(x^{prom}) = \ln(\alpha) + \ln(I) - \ln(p_x^{promo})$$

$$\ln(x^{non-prom}) = \ln(\alpha) + \ln(I) - \ln(p_x^{non-promo})$$
(29)

We define the demand per HH belonging to a given quintile and, based on promotional or non-promotional periods as:

$$\ln(x_1) = \pi_1[\ln(x^{prom})] + (1 - \pi_1)[\ln(x^{non-prom})]$$
...
$$\ln(x_5) = \pi_5[\ln(x^{prom})] + (1 - \pi_5)[\ln(x^{non-prom})]$$
(30)

Where, π_q represents the probability that a HH in income quintile q (for q=1,...,5) buys the bundle x on promotion. The specific values of this probabilities are the ones presented in Table 3. Next, aggregating at the income quintile level:

$$\ln(X_1) = \pi_1[n_1\{\ln(\alpha_1) + \ln(I_1) - \ln(p_x^{promo})\}] + (1 - \pi_1)[n_1\{\ln(\alpha_1) + \ln(I_1) - \ln(p_x^{non-prom})\}]$$

$$\dots$$

$$\ln(X_5) = \pi_5[n_5\{\ln(\alpha_5) + \ln(I_5) - \ln(p_x^{promo})\}] + (1 - \pi_5)[n_5\{\ln(\alpha_5) + \ln(I_5) - \ln(p_x^{non-prom})\}]$$
(31)

Finally, the market log-demand is given by:

$$\ln(X) = \sum_{q=1}^{5} \ln(X_q) \tag{32}$$

Note that Equation (31) is an interesting one since it shows the impact of a promotion in the overall market demand. Given the way the probabilities have been set, where households in the lowest quintile react more to promotions than households in the highest income quintile, not all sales are done during promotional events. The demand in periods of non-promotional activity allows for consumer loyalty or simply for households buying different products within a category not motivated by pricing.¹⁷

3.4. Calibration and Simulation Exercise

In what follows we make the following assumptions: x refers to CPGs bundle; y refers to all the other products and categories that are consumed or bought by the HHs. We assume that food is a category that can be used as a proxy to determine the behavior of CPGs. Note however, that apparently CPGs constitute only a third of the food market in terms of dollars spent. However, we believe that using food is the most conservative approximation we can choose. We assume an average price for CPG products of \$2 and average price for all the other product of \$10.¹⁸

In this section we work with the log version of the Cobb-Douglas as presented in section 2.4. Recall, that the demands depend only on the share coefficients, income and own price. As it is clear from Equations (16e) and (16f), the coefficients α and β simply represent the proportion of income used to buy CPGs (x) or the other categories of products that constitute the bundle others (y).

¹⁷ There are other non-price related activities geared to increase demand. In this paper we just concentrate in the aggregate view in order to show that a complete category expansion is completely feasible and left this additional aspects for future research.

¹⁸ We have run many different simulations with different price ranges and the results appear to be robust. This information is available upon request.

Before proceeding with the results it is important to note that we calibrate the parameters of this utility function (α and β) in such a way that the ratio of expenditure over total income, approximately matches the one presented in the third column of Table 2. Table 3 presents the coefficients used for each case, the average income per quintile and the percentage allocated to buy the category according to the information presented before.

Table 3: Coefficients used for the Cobb-Douglas Utility Function

Coefficients	Lowest	Second	Middle	Fourth	Highest
α	0.362	0.182	0.125	0.103	0.069
β	0.638	0.818	0.875	0.897	0.931
Average Income (I)	10,092	26,275	45,826	74,546	162,720
% Income allocated according to data	36.2%	18.2%	12.5%	10.5%	6.9%

This table presents the coefficients used for the Cobb-Douglas utility function, the average income per quintile and the percentage allocated to buy the category according to the information presented in Table 2.

By model construction, when α decreases, β increases. In our case, α decreases moving from the lowest to the highest quintile. The following table presents the demand quantities resulting from these assumptions for individual HHs in each of the income quintiles considered.

Table 4: Results Obtained from the Cobb-Douglas Utility Function

	Lowe	est	Seco	nd	Midd	dle	Four	th	High	est
	Quant / Expend	%of total								
x (units) (Eq. (12))	1,827	74	2,391	53	2,864	42	3,828	36	5,592	27
y (units) (Eq. (13))	644	26	2,149	47	4,010	58	6,689	64	15,154	73
Spent on x (\$)	3,653	36	4,781	18	5,728	13	7,655	10	11,184	7
Spent on y (\$)	6,439	64	21,494	82	40,098	88	66,891	90	151,536	93
Utility0 (Eq. (11))	939		2,191		3,845		6,315		14,146	
		Consu	mption and	l expend	liture after	Promot	ion			
x_promo (units)	2,283	78	2,989	58	3,580	47	4,785	42	6,987	32
y (units)	644	22	2,149	42	4,010	53	6,689	58	15,154	68
Spent on x_promo (\$)	3,653	36	4,782	18	5,728	13	7,656	10	11,179	7

Spent on y (\$)	6,439	64	21,493	82	40,098	88	66,890	90	151,541	93
Utility1 (Eq. (11))	1,018		2,282		3,953		6,463		14,369	
Increase quant sales	457		598		716		957		1,397	
% increase utility (Eq. (19)	8.4%		4.1%		2.8%		2.3%		1.5%	

This table presents the demand results resulting from the assumptions for individual HHs shown in Table 3, in each of the income quintiles considered.

In Table 4 we can appreciate the main characteristics of the Cobb-Douglas utility function. First, the price promotion on bundle x does not affect the sales on the other bundle y. Thus, price promotions simply translate in more purchases of x using the same amount spent during not promotional weeks, i.e. the quantities demanded of bundle x increase but that the total spent in the category stays the same. Also as expected, the quantities bought increase more for the higher income quintiles HHs than for the lower income quintiles HHs. However, the marginal utility (or difference in utilities before versus after promotions) increases more for lower quintiles than with higher quintiles: a 20% price discount of bundle x, creates an additional (marginal) utility increase of 8.4% to a HH in the lowest income quintile vs 1.5% of a HH in the highest quintile.

Based on this information, we estimate the growth in aggregated demand of bundle x (proxy for CPGs) assuming full response to price promotions and, that household utility functions are all based on the Cobb-Douglas with different parameter values to account for what is observed in the marked (see Table 3).

Table 5: Complete Category Expansion Effect with Cobb-Douglas Utility Function

			Modeled Food		Modeled		
		Modeled Food	Per-capita	Modeled	Market		Modeled
		Per-capita	consumption	Market	consumption	Modeled	Market
	Number of	consumption	after price	consumption	after price	Market	consumption
	HH (in	at Regular	discount	non promo	discount	consumption	after promotion
Quintile	thousands)	Price (units)*	(units)*	(units)	(units)	non promo (\$)	(\$)
Lowest	25,090	1,827	2,283	45,830,699	57,288,373	91,661,397	91,661,397
Second	25,219	2,391	2,989	60,299,259	75,374,074	120,598,519	120,598,519
Middle	25,082	2,864	3,580	71,837,983	89,797,479	143,675,967	143,675,967
Fourth	25,178	3,828	4,785	96,368,795	120,474,750	192,737,590	192,759,601
Highest	25,101	5,589	6,987	140,300,333	175,375,416	280,600,665	280,600,665
total	125,670			414,637,069	518,310,093	829,274,138	829,296,149

This table presents the growth in aggregated demand of bundle x (proxy for CPGs) assuming full response to price promotions and, that household utility functions are all based on the Cobb-Douglas with different parameter values to account for what is observed in the marked (see Table 3).

From Table 5, we again observed what we mentioned before. Under this utility environment, unit sales of bundle x increase under a promotional setting but expenditure in the bundle remains the same. Note also that nothing happens in the demand of bundle y.

In table 6 we present the dynamics using the probabilities of buying on promotion and, the aggregated market demand presented in Sections 3.2 and 3.3, respectively.

Table 6: Complete Category Expansion Effect with Cobb-Douglas Utility Functions, Adjusted for Probability of Promotional Participation

Quintile	Number of HH (in thousands)	Modeled Market consumption non promo (units)	Prob. of response to a promotion	Modeled Purchases during non- promotions (units)	Modeled Purchases during promotions (units)	Modeled Market consumption non promo (\$)	Modeled Market consumption on promotion (\$)	Modeled total spent (\$)
Lowest	25,090	45,830,699	100%	0	57,288,373	-	91,661,397	91,661,397
Second	25,219	60,299,259	50%	30,003,778	37,869,352	60,007,556	60,590,963	120,598,519
Middle	25,082	71,837,983	35%	47,044,642	30,991,676	94,089,285	49,586,682	143,675,967
Fourth	25,178	96,368,795	28%	69,044,615	34,159,126	138,089,230	54,654,601	192,743,831
Highest	25,101	140,300,333	19%	113,674,403	33,282,412	227,348,805	53,251,860	280,600,665
Total	125,670	414,637,069		259,767,438	193,590,939	519,534,876	309,745,503	829,280,379

In this table we present the dynamics using the probabilities of buying on promotion and, the aggregated market demand presented in Sections 3.2 and 3.3, respectively.

This last table shows that even though the proportion of households buying purely on promotions decreases from 100% to 19%, depending on the income quintile, unit sales increase (519.5 million vs 414.6 million) but total expenditures remain constant (at 829 million). I.e. <u>in this case there is Complete Category Expansion Effect in terms of unit sales but not in terms of total expenditures</u>.

Next, we present the results of using the Constant Elasticity of Substitution (CES) utility function described in Section 2.5. In this case the demand of a given bundle depends also on the price of the other bundle, making the exercise a more realistic one.

In this section we only present the results using a single combination of the CES parameters (α and ρ) for all income quintiles. In this case, we assume that households across income

quintiles have the same elasticity of substitution that equals 1.25 (1/(1-0.2)). This implies that both bundles are assumed to be slightly substitutes, in the sense that households will be willing to marginally sacrifice consumption in one bundle when the price of the other one decreases.¹⁹ The CES parameter values, that allow us to obtain ratio of expenditure to income close to the ones observed in the economic data, is presented in Table 7.

Table 7: Coefficients used for the CES Utility Function with Same Elasticity of Substitution

Coefficients	Lowest	Second	Middle	Fourth	Highest
α	0.315	0.179	0.133	0.113	0.083
ρ	0.200	0.200	0.200	0.200	0.200
Average Income (I)	10,092	26,275	45,826	74,546	162,720
% Income allocated					
according to data	36.22%	18.20%	12.50%	10.27%	6.87%

This table presents the coefficients used for the Coefficients used for the CES Utility Function with Same Elasticity of Substitution, the average income per quintile and the percentage allocated to buy the category according to the information presented in Table 2.

Based on this parameter values and the assumptions stated before, the next table presents the main results obtained by the optimization described in Section 2.5. As before, we first estimate bundle x demand growth (proxy for CPGs) assuming full response to price promotions.

Table 8: Results Obtained from the CES Utility Function with Same Elasticity of Substitution

	Low	est	Seco	nd	Midd	dle	Four	th	High	est
	Quant /	%of	Quant /	%of	Quant /	%of	Quant /	%of	Quant /	%of
	Expend	total	Expend	total	Expend	total	Expend	total	Expend	total
x (units) (Eq. (25a))	1,827	74	2,391	53	2,864	42	3,828	36	5,588	27
Y (units) (Eq. (25b))	644	26	2,149	47	4,010	58	6,689	64	15,154	73
Spent on x (\$)	3,653	36	4,782	18	5,728	13	7,656	10	11,177	7
Spent on y (\$)	6,439	64	21,493	82	40,098	87	66,890	90	151,543	93
Utility0 (Eq. (24))	916		2,191		3,840		6,298		14,056	
		Consur	nption and	expend	iture after	Promoti	on			
x_promo (units) (Eq. (25a))	2,365	79	3,128	60	3,759	49	5,030	43	7,357	33
y (units) (Eq. (25b))	631	21	2,127	40	3,981	51	6,650	57	15,095	67
Spent on x_promo (\$)	3,784	38	5,004	19	6,014	13	8,047	11	11,771	7

¹⁹ We have changed this assumption from 1.10 to 1.50 to see the sensitivity of the model to these changes. Qualitatively the results stayed the same. These results are available upon request.

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Spent on y (\$)	6,308	63	21,271	81	39,812	87	66,499	89	150,949	93
Utility1_(Eq. (28))	995		2,284		3,951		6,447		14,278	
Increase quant sales	539		737		894		1,202		1,769	
% increase utility	8.57		4.24		2.90		2.38		1.59	

This table presents the demand results resulting from the assumptions for individual HHs shown in Table 7, in each of the income quintiles considered.

The first thing to note here is that a 20% price discount on bundle x, now has an impact on the sales of bundle y. The magnitude of this impact is governed by the elasticity of substitution (1.25 in our case). As soon as we have assumed same elasticity of substitution for all the households disregarded their income quintile, a 20% discount of the price of bundle x makes households to buy more of this good and slightly sacrifice consumption of the y bundle. For example, observe the income spent on y decreases from 64% to 63% for the lowest quintile. The same variation is observed across the board.

We estimate the growth in demand of bundle x (proxy for CPGs) assuming full response to price promotions and that household utility functions are all based on the CES with different parameter values to account for what is observed in the marked.

Table 9: Complete Category Expansion Effect with CES Utility Function with Same Elasticity of Substitution

Quintile	Number of HH (in thousands)	Modeled Food Per-capita consumption at Regular Price (units)*	Modeled Food Per-capita consumption after price discount (units)*	Modeled Market consumption non promo (units)	Modeled Market consumption after price discount (units)	Modeled Market consumption non promo (\$)	Modeled Market consumption after promotion (\$)
Lowest	25,090	1,827	2,365	45,831,364	59,343,473	91,662,729	94,949,557
Second	25,219	2,391	3,128	60,300,109	78,875,903	120,600,217	126,201,444
Middle	25,082	2,864	3,759	71,839,379	94,275,021	143,678,758	150,840,033
Fourth	25,178	3,828	5,030	96,377,074	126,636,809	192,754,148	202,618,894
Highest	25,101	5,588	7,357	140,270,853	184,670,254	280,541,706	295,472,407
Total	125,670			414,618,779	543,801,459	829,237,557	870,082,334

This table presents the growth in aggregated demand of bundle x (proxy for CPGs) assuming full response to price promotions and, that household utility functions are all based on the CES Utility Function with Same Elasticity of Substitution (see Table 7).

Now we can appreciate that indeed bundle x expansion appears not only in the quantity demanded but also in the amount spent on it (Complete Category Expansion Effect (CCEE)). Given a bundle x's 20% price discount, unit sales of this bundle increases by 31.16% and the amount spent increases by 4.93%.²⁰

Table 10 presents the dynamics under different probabilities of buying on promotion.

Table 10: Complete Category Expansion Effect with CES Utility Function with Same Elasticity of Substitution, Adjusted for Probability of Promotional Participation

Quintile	Number of HH (in thousands)	Modeled Market consumption non promo (units)	Prob. of response to a promotion	Modeled Purchases during non- promotions (units)	Modeled Purchases during promotions (units)	Modeled Market consumption non promo (\$)	Modeled Market consumption on promotion (\$)	Modeled total spent (\$)
Lowest	25,090	45,831,364	100%	0	59,343,473	-	94,949,557	94,949,557
Second	25,219	60,300,109	50%	30,004,201	39,628,736	60,008,401	63,405,978	123,414,379
Middle	25,082	71,839,379	35%	47,045,556	32,537,004	94,091,113	52,059,206	146,150,318
Fourth	25,178	96,377,074	28%	69,050,546	35,906,301	138,101,093	57,450,082	195,551,174
Highest	25,101	140,270,853	19%	113,650,518	35,046,369	227,301,036	56,074,191	283,375,226
Total	125,670	414,618,779		259,750,821	202,461,883	519,501,642	323,939,012	843,440,655

In this table we present the dynamics using the probabilities of buying on promotion and, the aggregated market demand presented in Sections 3.2 and 3.3, respectively.

With this last table we show that even though not everyone buys on promotion, Complete Category Expansion still is feasible. The amount spent in bundle x, increases by 1.7% (843,440,655/829,237,557).

The final scenario we present in this paper is one in which we allow the elasticity of substitution to change according to what is expected from someone from a given income quintile buying a given amount of bundle x. In order to determine who should have a larger elasticity of substitution (households from lower or higher quintiles), we recall that the elasticity of substitution determines how the relative expenditure between the bundles changes as relative prices change, i.e whether or not an increase in the relative price of y leads to an increase or

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²⁰ The difference in market consumption non-promo (in US\$) is due to rounding errors.

decrease in the relative expenditure on y depends on whether the elasticity of substitution is less than or greater than one. To see this, let S_{yx} represent the expenditure on y relative to the one on x as:

$$S_{yx} = \frac{p_y y}{p_x x} \tag{33}$$

One can show that as the relative price p_y/p_x changes, the relative expenditure changes according to:

$$\frac{dS_{yx}}{d(p_y/p_x)} = \frac{y}{x}(1-\varepsilon) \tag{34}$$

Where ε respresents the elasticity of substitution. From Equation (33) we can see that there are two effects of an increase of the relative price p_y/p_x (caused by a decreased of p_x , for example): the *direct effect* of an increase in relative prices is an increase expenditure on y since a given quantity of y is now more costly. However, there is also an *indirect effect*, the rise on the relative price p_y/p_x leads to a fall in relative demand for y, so that the quantity of y purchased falls, which reduces expenditure on this bundle. The question then is which of these two effects dominates.

The answer is provided in equation (34). From this equation one can see that the final relative expenditure depends on the magnitude of the elasticity of substitution (ϵ). When the elasticity of substitution is less than one, the first effect dominates and the relative demand for y falls, but by proportionally less than the rise in its relative price, so that relative expenditure rises. In this case, the goods are *gross complements*. In the other hand, when the elasticity of substitution is greater than one, the second effect dominates: the reduction in relative quantity of y exceeds the increase in the relative price, so that relative expenditure on y falls. In this case, the goods are *gross substitutes*.

With this in mind, we try to make educated guesses regarding the magnitude and direction of the elasticity of substitution per income quintile. Observing Figure 1, we see a striking fact: households in the lowest income quintile spend the less on food even though this expenditure represents 35.1% of their income. On the other hand, the highest quintile spent on average

\$13,708 but that only represents 7.5% of their income. From this, we can argue that the impact on relative prices on expenditure on bundle x should (relatively) decrease with the income level. A price promotion on bundle x, the rises the relative price p_y/p_x , should have a larger impact on lowest quintile households than higher quintile ones, i.e. lowest income quintile households should be more willing to buy on promotions.²¹

Thus, we have a reasonable assumption on the direction of the elasticity of substitution: it should be larger than 1 for households in the lowest quintile and should decrease towards 1 for the households in the highest quintiles.²² The next reasonable assumption that we can make is that the elasticity of substitution should not go below 1, since we do not see the reason why bundles x and y should be considered as complementary.

The educated guesses about the magnitude or actual value of the elasticity of substitution is trickier. However, if we think of the households in the lowest income quintile, we can argue that this elasticity of substitution cannot be much larger than 1, since a much larger number would imply that these households will be willing to make large sacrifices on their purchases of the y bundle to buy bundle x on promotions. At this point we can also argue other physical limitations these households could face and that have been widely mentioned in the literature: space limitations that prohibit them to buy as much as they want; monthly income limitations and volatility; other priorities encompassed on bundle y that could limit their desire to jump into the promoted x bundled; among others. Based on all these observations we have performed several simulations from which we present one in what follows.

As mentioned before, for this last case we assumed that households across income quintiles have different substitution elasticities ($\varepsilon = 1/(1-\rho)$). We have set the elasticity of substitution for the lowest income quintile close to 1.5, i.e. we still assume the bundles are slightly substitutes, in the sense that households will be willing to marginally sacrifice consumption in one bundle when the price of the other one decreases. However, in this case

²¹ This is related also to active seeking for promotions or price discounts in general.

²² This also goes in hand with the way we set up the probabilities in Section 3.2.

we allow for different reactions to relative price changes depending on the income quintile the household is located in.

Table 11: Coefficients used for the CES Utility Function with Different Elasticity of Substitution

Coefficients	Lowest	Second	Middle	Fourth	Highest
α	0.293	0.178	0.133	0.111	0.075
ρ	0.300	0.250	0.200	0.150	0.100
Average Income (I)	10,092	26,275	45,826	74,546	162,720
Elasticity of substitution (ε)	1.43	1.33	1.25	1.18	1.11
% Income allocated according to data	36.22%	18.20%	12.50%	10.27%	6.87%

This table presents the coefficients used for the Coefficients used for the CES Utility Function with Same Elasticity of Substitution, the average income per quintile and the percentage allocated to buy the category according to the information presented in Table 2.

Based on this parameter values and the assumptions stated before, the next table presents the main results obtained using the optimization described in Section 2.5. The following table presents the estimated growth in demand of bundle x (proxy for CPGs) assuming full response to price promotions.

Table 12: Results Obtained from the CES Utility Function with Different Elasticity of Substitution

	Lowest		Second		Middle		Fourth		Highest	
	Quant /	%of								
	Expend	total								
x (units) (Eq. (25a))	1,827	74	2,392	53	2,864	42	3,829	36	5,593	27
Y (units) (Eq. (25b))	644	26	2,149	47	4,010	58	6,689	64	15,153	73
Spent on x (\$)	3,654	36	4,783	18	5,728	13	7,658	10	11,185	7
Spent on y (\$)	6,438	64	21,492	82	40,098	87	66,888	90	151,535	93
Utility0 (Eq. (24))	906		2,191		3,840		6,302		14,104	
Consumption and expenditure after Promotion										
x_promo (units) (Eq. (25a))	2,425	80	3,176	60	3,759	49	4,958	43	7,154	32
y (units) (Eq. (25b))	621	20	2,119	40	3,981	51	6,661	57	15,127	68
Spent on x_promo (\$)	3,879	38	5,081	19	6,014	13	7,933	11	11,446	7
Spent on y (\$)	6,213	62	21,194	81	39,812	87	66,613	89	151,274	93
Utility1_(Eq. (24))	984		2,285		3,951		6,451		14,324	
Increase quant sales	598		784		894		1,129		1,561	
% increase utility	8.68		4.28		2.90		2.36		1.56	

This table presents the demand results resulting from the assumptions for individual HHs shown in Table 11, in each of the income quintiles considered.

Again, a 20% price discount on bundle x, now has an impact on the sales of bundle y and, this impact varies across the households in different income quintiles. The lowest quintile is willing to marginally sacrifice more of bundle y when there is a price promotion on bundle x. Their expenses on bundle y decreases by 2%. In the case of the higher income quintile households, the impact is much lower (they only sacrifice 0.16% of their income before used on bundle y). In table 13, we estimate bundle x market demand growth, assuming full response to price promotions.

Table 13: Complete Category Expansion Effect with CES Utility Function with Different Elasticity of Substitution

Quintile	Number of HH (in thousands)	Modeled Food Percapita consumption at Regular Price (units)*	Modeled Food Per-capita consumption after price discount (units)*	Modeled Market consumption non promo (units)	Modeled Market consumption after price discount (units)	Modeled Market consumption non promo (\$)	Modeled Market consumption after promotion (\$)
Lowest	25,090	1,827	2,425	45,835,166	60,833,491	91,670,333	97,333,586
Second	25,219	2,392	3,176	60,314,234	80,088,615	120,628,468	128,141,783
Middle	25,082	2,864	3,759	71,839,379	94,275,021	143,678,758	150,840,033
Fourth	25,178	3,829	4,958	96,404,863	124,831,030	192,809,726	199,729,647
Highest	25,101	5,593	7,154	140,378,589	179,568,394	280,757,177	287,309,430
Total	125,670			414,772,231	539,596,550	829,544,462	863,354,480

This table presents the growth in aggregated demand of bundle x (proxy for CPGs) assuming full response to price promotions and, that household utility functions are all based on the CES Utility Function with Different Elasticity of Substitution (see Table 11).

We can appreciate that bundle x expansion appears again not only in the quantity demanded but also in the amount spent on it (Complete Category Expansion Effect (CCEE)). Given a bundle x's 20% price discount, unit sales of this bundle increases by 30.09% and the amount spent on it increases by 4.08%. Table 14 shows the dynamics under different probabilities of buying on promotions.

Table 14: Complete Category Expansion Effect with CES Utility Function with Different Elasticity of Substitution, Adjusted for Probability of Promotional Participation

Quintile	Number of HH (in thousands)	Modeled Market consumption non promo (units)	Prob. of response to a promotion	Modeled Purchases during non- promotions (units)	Modeled Purchases during promotions (units)	Modeled Market consumption non promo (\$)	Modeled Market consumption on promotion (\$)	Modeled total spent (\$)
Lowest	25,090	45,835,166	100%	0	60,833,491	-	97,333,586	07 222 596
Second	25,219	60,314,234	50%	30,011,229	40,238,025	60,022,458	64,380,841	124,403,299
Middle	25,082	71,839,379	35%	47,045,556	32,537,004	94,091,113	52,059,206	146,150,318
Fourth	25,178	96,404,863	28%	69,070,456	35,394,295	138,140,913	56,630,872	194,771,784
Highest	25,101	140,378,589	19%	113,737,807	34,078,148	227,475,615	54,525,036	282,000,651
Total	125,670	414,772,231		259,865,049	203,080,963	519,730,099	324,929,540	844,659,639

In this table we present the dynamics using the probabilities of buying on promotion and, the aggregated market demand presented in Sections 3.2 and 3.3, respectively.

With this last table we show that even though not everyone buys on promotion, Complete Category Expansion still is feasible. The amount spent in bundle x, increases by 1.8%.

We have performed several other calibrations. The results are qualitatively similar are available upon request.

4. What Could Happen Within A Given Category?

As mentioned before, the majority of the SDR literature on retailer price promotions for CPGs deals with the "who does the incremental sales source from (own brand or competitive brand or any other source)? Even though, several researchers have estimated sales decomposition that considers the critically important component that accounts for the possibility of category expansion, there has not been a paper that offers an economic-based support of what we call a Complete Category Expansion Effect (CCEE).

If we now concentrate our efforts to explain what could happen within a given CPG category, we can easily see that all what we have previously mentioned holds (in a stronger way). As mentioned before, in the U.S., CGP expenditures are lower than those in food. Thus, we can safely argue that the percentage of CPG expenditures with respect to HHs' income is low. Based on this and on what we presented before, we can state the following:

- a. The expenditure in each product that is part of a given CPG category should represent a very small portion of the HH's income.
- b. The expenditure in a given product, with respect to the HH income should differ significantly across income quintiles (recall that HHs have severe income inequalities). However, even for the lowest quintile, the ratio of expenditure on an individual CPG product to income should be relatively small. This observation implies that HHs are able to consume more of a given product (under certain limitations) without significantly affecting the consumption of other products in other categories present in the HHs basket (the composite good).
- c. The severe income inequalities can influence the way HHs perceive and react in front of promotional activities, frequency and magnitude of them. This very simple reason could explain why, even though sales of a particular product increase due to promotional activities, a significant post-promotion dip is not observed. We can see this from tables 7 (Cobb-Douglas), 11 (CES with same elasticity of substitution) and 15 (CES with different elasticities of substitution). From these tables we can appreciate that is likely that even though HHs in the two first income quintiles make most of their purchases during promotions, the HHs in the other quintiles are the ones that support the sales during non-promotions (i.e. baseline sales, helping to avoid promotional dips).
- d. Also, we can mention (as many before us), that even though low income quintile HHs could be more prone to react to promotional activities in general, its reaction will also depend on the relative expenditure on the particular product with respect to the HHs income, magnitude of the price promotion and other physical restrictions that they can face (inventory space for instance).
- e. Finally, recall that even though the analysis presented in this paper is a static analysis one based on annual data, we can use educated guesses to make some inferences. We can argue that as soon as temporary price promotions happen less frequent during a year, the purchases of the HHs that use those promotions should be more concentrated that those HHs buying on non-promotions. To understand this, let's use the information provided in Table 14. Assuming that a year has exactly 52 weeks and that in 30% of those weeks there is a promotion (approximately 16 weeks), a possible sales dynamics that can be created is presented below.

In Figure 2, we have equally divided the total units purchased during non-promotional periods among the 52 weeks. Also we have equally divided the purchases during promotional periods

in 16 randomly selected weeks. Based on this, the baseline sales equal 7,218,474 units per week (259,865,049/36) and the promotional lifts are equal to 12,692,560 (203,080,963/16).²³

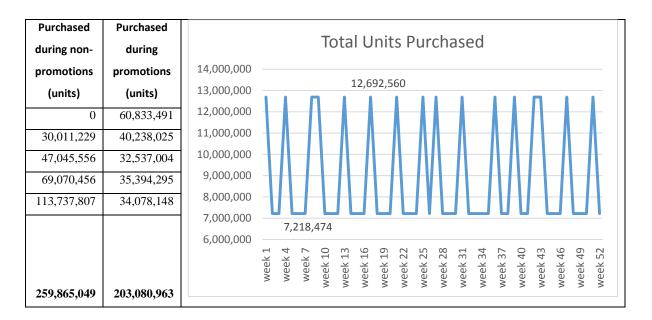


Figure 2: Unit Sales and the non-existence of the Promotion Dips

In this very simplified example, we are able to technically show that having no dips is perfectly feasible, at least for CPGs.

This example can be modified to include seasonal patterns and, it also provides a basis to analyze the impact of frequency and magnitude of promotions on sales. We leave all these extremely interesting research question for the future.

5. Permanent versus Temporary Price Reductions

The last interesting issue that we cover in this paper is whether the impact of a permanent price reduction (PPR) is equivalent to a temporary price reduction (TPR) in terms of units and revenue. To answer this, we assume that the HHs that do not buy on promotions are indifferent about price changes (at least no significant price changes). This assumption implies that if a CPG product reduces its price by, say 20%, either temporarily or permanently, the effect on their consumption

²³ Recall that the purchases during non-promotion weeks includes HHs from different income quintiles whose purchase probabilities are described in Table 2.

is negligible. However, we can assume that those HHs sensible to price reductions will increase their consumption whether prices are reduced temporarily or permanently.

The effects of these set of assumptions, allow us to explain in a very simple way why manufacturers prefer TPRs versus permanent ones. The explanation comes from the fact that during promotions not all buyers make their purchases, allowing manufacturers to ask for a higher price to this group of HHs.²⁴ Accordingly, applying a permanent price reduction will simply make some HHs to buy more of the good while keep the other HHs' (the indifferent ones) purchases constant. In the aggregate the revenues under permanent price reductions decreases by 12.31% according to the example developed in the paper.

Figure 3 presents the cumulative revenues obtained using the CES with different elasticities of substitution and two price reduction scenarios: permanent and temporary reductions.

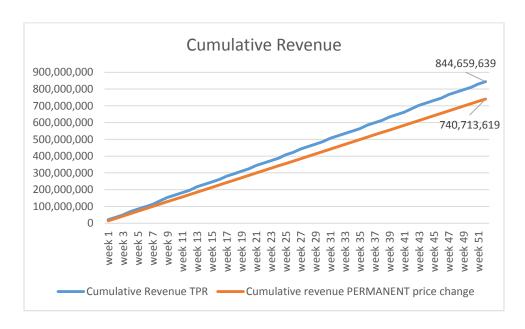


Figure 3: Permanent vs Temporary Price Discounts

The permanent price decrease depicted in this example, represents the highest revenue scenario. Something more realistic should assume that individuals that are responsive to changes in price, once they have learned that the change is a permanent one, can decrease their consumption to non-

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²⁴ In economic literature this is known as price discrimination.

promo levels, decreasing in this way the aggregate consumption of the good, decreasing even more the revenue obtained by the sales of this good under PPR.

6. Conclusions

Applying some fundamental principles of Consumer Demand Theory, with supporting empirical evidence, we have proven that economic theory that we name as the Theory of Retailer Price Promotions (TRPP) is the basis of what it is observed in real world. Using well known utility functions, largely used in economics, we have been able to proof that CPGs incremental sales from promotional spikes can create Complete Category Expansion Effects (CCEE) which sources from an extremely small marginal reduction on spending from every other good that would be considered for purchase in a consumer's discretionary income.

The ultimate validation comes with the ability of this framework to explain results in the real world. In the U.S. there have been four specific instances in the mass market which are explained by the Complete Category Expansion Effect, most notably the calamitous and immediate decline in sales and profitability of JC Penney when they eliminated price promotions in the first quarter of 2012. Other, less publicized, cases in the U.S. are Food Lion, Stop & Shop and Walgreens. In each of the four instances, major shifts in promotional strategy were cited by senior management as reasons for gains or declines in revenue, with the revenue changes moving in the same direction as changes in promotional depth and frequency. To a lesser extent, promotional activity has been identified as a likely cause of sluggish results in same-store sales for both Walmart and Target.

Obvious next steps for this research is to add more empirical evidence that tests the theory. A particular area of focus should be on intrinsically identical products as a source of the substitution effect. Specifically, there should be a broader based of products that are identical to the promoted product in every way except for package quantity (e.g. the effect on 6 pack Coca-Cola when 12 pack is promoted).

With this paper we intent to provide the economic basis for marketing researchers that can help them justify their results with well-established theoretical frameworks. Additionally, more care should be taken to ensure that results confirm to the laws and theories of microeconomics. In this sense, this paper is among the first ones in the Marketing literature to draw an explicit link between the empirical results and their consistency with microeconomic theory.

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Technical Appendix

1. The Cobb-Douglas Utility Function

The Cobb-Douglas utility function has the following mathematical representation:

$$U(x, y) = x^{\alpha} y^{\beta} \tag{35}$$

Where α and β are model parameters that represent the share of consumption of each bundle, by model definition $\alpha+\beta=1$. The variables x and y represent bundles of goods.

1.1. Marginal Rate of Substitution

Mathematically, the Marginal Rate of Substitution ($MRS_{y,x}$) is estimated as:

$$MRS_{y,x} = \frac{\Delta y}{\Delta x} = \frac{U_x(x,y)}{U_y(x,y)} = \frac{Marginal\ Utility\ of\ x}{Marginal\ Utility\ of\ y}$$
 (36)

Applying the formula for the Cobb-Douglas utility function:

$$U_x(x,y) = \alpha x^{\alpha-1} y^{\beta}$$

$$U_{\nu}(x,y) = \beta x^{\alpha} y^{\beta - 1}$$

Thus, the $MRS_{y,x}$ is given by:

$$MRS_{y,x} = \frac{\alpha x^{\alpha - 1} y^{\beta}}{\beta x^{\alpha} y^{\beta - 1}} = \frac{\alpha}{\beta} \left(\frac{y}{x} \right)$$
 (37)

1.2. The Elasticity of Substitution

The Elasticity of Substitution is measured as:

$$\varepsilon = \frac{\%\Delta(\frac{y}{x})}{\%\Delta MRS} \tag{38}$$

Where, $\%\Delta$ represents the percentage change of the variable in parenthesis. Rewriting Equation (38) as:

$$\varepsilon = \frac{\%\Delta(\frac{y}{x})}{\%\Delta MRS} = \frac{\Delta(\frac{y}{x})}{\frac{\Delta MRS}{MRS}} = \frac{\Delta(\frac{y}{x})}{\Delta MRS} \frac{MRS}{(\frac{y}{x})}$$
(39)

Differentiating Equation (37):

$$\Delta MRS_{y,x} = \Delta \left(\frac{\alpha}{\beta} \left(\frac{y}{x} \right) \right) = \frac{\alpha}{\beta} \Delta \left(\frac{y}{x} \right) \to \frac{\Delta MRS_{y,x}}{\Delta \left(\frac{y}{x} \right)} = \frac{\alpha}{\beta}$$
 (40)

Substituting (37) and (40) in (39),

$$\varepsilon = \frac{\Delta(\frac{y}{x})}{\Delta MRS} \frac{MRS}{(\frac{y}{x})} = \frac{\beta}{\alpha} \frac{\frac{\alpha}{\beta}(\frac{y}{x})}{(\frac{y}{x})} = 1$$
 (41)

This result implies that the consumer's valuation of good x, regardless of his actual holdings of x, is constant and equal to 1.

1.3. Demand Functions

Using the Cobb-Douglas utility function the HH maximization problem is presented below:

$$\max_{\substack{x \ge 0, y \ge 0, \alpha \ge 0, \beta \ge 0, \alpha + \beta = 1 \\ s. t p_x x + p_y y = I}} U(x, y) = x^{\alpha} y^{\beta}$$

$$(42)$$

Where p_x and p_y represent the average prices of bundles x and y, respectively. The variable I represents the average consumer income. In order to solve this maximization problem we first create the Lagrangean:

$$l(x, y, \lambda) = x^{\alpha} y^{\beta} + \lambda (I - p_x x - p_y y)$$
(43)

Differentiating to get the first order conditions,

$$l_x = \alpha x^{\alpha - 1} y^{\beta} - \lambda p_x = 0 \tag{44a}$$

$$l_{y} = \beta x^{\alpha} y^{\beta - 1} - \lambda p_{y} = 0 \tag{44b}$$

$$I - p_x x - p_y y = 0 \tag{44c}$$

Using (44a) and (44b), we get the following relationship:

$$\frac{\alpha x^{\alpha - 1} y^{\beta}}{p_{x}} = \frac{\beta x^{\alpha} y^{\beta - 1}}{p_{y}} \to y = \left(\frac{\beta}{\alpha}\right) \left(\frac{p_{x}}{p_{y}}\right) x \tag{44d}$$

Replacing (44d) into (44c):

$$p_x x + p_y \left(\frac{\beta}{\alpha}\right) \left(\frac{p_x}{p_y}\right) x = I \to x = \left(\frac{\alpha}{\alpha + \beta}\right) \frac{I}{p_x}$$
 (44e)

Thus, the quantity of x demanded equal the share provided by the first term of the total unit that can be bought at the price p_x given a level of income I. Finally, replacing (16e) into (16d),

$$y = \left(\frac{\beta}{\alpha}\right) \left(\frac{p_x}{p_y}\right) \left(\frac{\alpha}{\alpha + \beta}\right) \frac{I}{p_x} \to y = \left(\frac{\beta}{\alpha + \beta}\right) \frac{I}{p_y} \tag{44f}$$

From equations (44e) and (44d) we can obtain the log-linear demand curves:

$$\ln(x) = \ln\left(\frac{\alpha}{\alpha + \beta}\right) + \ln(I) - \ln(p_x) \tag{44g}$$

$$\ln(y) = \ln\left(\frac{\beta}{\alpha + \beta}\right) + \ln(I) - \ln(p_y) \tag{44h}$$

1.4. Percentage Change of a HH's Utility Function When the Price of a Good Changes

The percentage change of a HH's utility function when the price of the good x changes, keeping all the other variables constant, is derived mathematically in the following way. Let's start with equation (14) and replace the value of x by the formula presented in equation (16e):

$$U_{p_x}(x,y) = x^{\alpha} y^{\beta} = \left(\left(\frac{\alpha}{\alpha + \beta} \right) \frac{I}{p_x} \right)^{\alpha} y^{\beta}$$
 (45a)

Let's assume that the price of x (P_x) goes to P_{x1} . Using the same equation (20a):

$$U_{p_{x1}}(x,y) = \left(\left(\frac{\alpha}{\alpha+\beta}\right)\frac{l}{p_{x1}}\right)^{\alpha}y^{\beta}$$
 (45b)

Dividing equation (45a) by Equation (45b),

$$\frac{U_{p_{\chi_1}}(x,y)}{U_{p_{\chi}}(x,y)} = \frac{\left(\left(\frac{\alpha}{\alpha+\beta}\right)\frac{I}{p_{\chi_1}}\right)^{\alpha}y^{\beta}}{\left(\left(\frac{\alpha}{\alpha+\beta}\right)\frac{I}{p_{\chi}}\right)^{\alpha}y^{\beta}} = \frac{\left(\frac{1}{p_{\chi_1}}\right)^{\alpha}}{\left(\frac{1}{p_{\chi}}\right)^{\alpha}} = \left(\frac{p_{\chi}}{p_{\chi_1}}\right)^{\alpha}$$
(45c)

Finally, the percentage change in utility equals:

$$\frac{U_{p_{x1}}(x,y)}{U_{p_{x}}(x,y)} - 1 = \% \Delta U(x,y) = \left(\frac{p_{x}}{p_{x1}}\right)^{\alpha} - 1$$
 (45d)

2. The Constant Elasticity of Substitution Utility Function (CES)

The mathematical representation of the CES utility function is given by:

$$U(x,y) = (\alpha x^{\rho} + (1-\alpha)y^{\rho})^{\frac{1}{\rho}}$$
 (46)

Again, α and β are the share parameters and ρ the parameter that controls for the elasticity of substitution.

2.1. The Marginal rate of substitution

The Marginal utilities of the CES function are:

$$U_{x}(x,y) = \frac{1}{\rho} (\alpha x^{\rho} + (1-\alpha)y^{\rho})^{\frac{1}{\rho}-1} (\alpha \rho x^{\rho-1})$$

$$U_{y}(x,y) = \frac{1}{\rho} (\alpha x^{\rho} + (1-\alpha)y^{\rho})^{\frac{1}{\rho}-1} ((1-\alpha)\rho y^{\rho-1})$$

Thus, the $MRS_{y,x}$ is given by:

$$MRS_{y,x} = \frac{\frac{1}{\rho}(\alpha x^{\rho} + (1-\alpha)y^{\rho})^{\frac{1}{\rho}-1}(\alpha \rho x^{\rho-1})}{\frac{1}{\rho}(\alpha x^{\rho} + (1-\alpha)y^{\rho})^{\frac{1}{\rho}-1}((1-\alpha)\rho y^{\rho-1})} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{y}{x}\right)^{1-\rho}$$
(47)

2.2. The Elasticity of Substitution

As before, the *elasticity of substitution* is measured as:

$$\varepsilon = \frac{\%\Delta(\frac{y}{x})}{\%\Delta MRS} \tag{48}$$

Rewriting equation (48) in a more manageable form, using natural logs, we have:²⁵

$$\varepsilon = \frac{\%\Delta(\frac{y}{x})}{\%\Delta MRS} = \frac{\Delta(\frac{y}{x})}{\frac{\Delta MRS}{MRS}} = \frac{\partial \ln(\frac{y}{x})}{\partial \ln(MRS)}$$
(49)

Taking logs and differentiating Equation (47):

$$\partial lnMRS_{y,x} = \partial \left(ln \left(\frac{\alpha}{1-\alpha} \right) \left(\frac{y}{x} \right)^{1-\rho} \right) = \partial \left(ln \left(\frac{\alpha}{1-\alpha} \right) + (1-\rho) ln \left(\frac{y}{x} \right) \right) = (1-\rho) \partial ln \left(\frac{y}{x} \right)$$
(50)

Using Equation (50) in Equation (49):

$$\varepsilon = \frac{\partial \ln(\frac{y}{x})}{\partial \ln(MRS)} = \frac{\partial \ln(\frac{y}{x})}{(1-\rho)\partial \ln(\frac{y}{x})} = \frac{1}{(1-\rho)}$$
 (51)

2.3.Demand Function

In this case, the maximization problem is presented below:

$$\max_{x \ge 0, y \ge 0} (\alpha x^{\rho} + (1 - \alpha) y^{\rho})^{\frac{1}{\rho}}$$

$$s. t p_x x + p_y y = I$$
(52)

$$dlnf(x) = \frac{f'(x)}{f(x)}$$

²⁵ Recall that the derivative of ln(fx) equals the derivative of f(x) divided by the same f(x), i.e.:

Where p_x and p_y represent the average prices of bundles x and y, respectively. The variable I represents the average consumer income. Forming the Lagrangean,

$$l(x,y,\lambda) = (\alpha x^{\rho} + (1-\alpha)y^{\rho})^{\frac{1}{\rho}} + \lambda(I - p_x x - p_y y)$$
(53)

Finding the first order conditions,

$$l_x = \frac{1}{\rho} (\alpha x^{\rho} + (1 - \alpha) y^{\rho})^{\frac{1}{\rho} - 1} (\alpha \rho x^{\rho - 1}) - \lambda p_x = 0$$
 (53a)

$$l_{y} = \frac{1}{\rho} (\alpha x^{\rho} + (1 - \alpha) y^{\rho})^{\frac{1}{\rho} - 1} ((1 - \alpha) \rho y^{\rho - 1}) - \lambda p_{y} = 0$$
 (53b)

$$I - p_{x}x - p_{y}y = 0 \tag{53c}$$

Using (53a) and (53b) to get the following relationship:

$$\frac{\alpha \rho x^{\rho - 1}}{p_x} = \frac{(1 - \alpha)\rho y^{\rho - 1}}{p_y} \to y = \left(\frac{(1 - \alpha)p_x}{\alpha p_y}\right)^{\varepsilon} x \tag{53d}$$

Where ε is define in equation (49). Replacing (53d) into (53c),

$$p_{x}x - p_{y}\left(\frac{(1-\alpha)p_{x}}{\alpha p_{y}}\right)^{\varepsilon} x = I \to x = \frac{I}{p_{x} + p_{y}\left(\frac{(1-\alpha)p_{x}}{\alpha p_{y}}\right)^{\varepsilon}}$$
 (53e)

Using some algebra manipulations, we get:

$$x = \frac{(p_x/\alpha)^{-\varepsilon}}{\alpha^{\varepsilon} p_x^{1-\varepsilon} + (1-\alpha)^{\varepsilon} p_y^{1-\varepsilon}} I$$
 (53f)

Finally, we obtain a similar equation for y:

$$y = \frac{(p_y/(1-\alpha))^{-\varepsilon}}{\alpha^{\varepsilon} p_x^{1-\varepsilon} + (1-\alpha)^{\varepsilon} p_y^{1-\varepsilon}} I$$
 (53g)