Quantum Mechanics from an Information Theory Perspective

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Abstract

Starting with the "non-collapse" axioms of quantum mechanics we explain how it can be developed in an entirely unitary manner, very much in the spirit of the many worlds interpretation. Starting from the principle of conservation of information, the no-cloning and no-deletion theorems, we develop an explicit procedure of "unitary measurement" which also satisfies repeatability of experiment. The Born rule as well as interference effects are shown to be derivable starting from our procedure. We finally explore the index set, which is the set of labels for the basis vector of the Hilbert space, and its properties.

Section 1

Introduction

The usual axioms of quantum mechanics give rise to two distinct kinds of processes by which quantum states evolve:

- unitary evolution, which is time reversible and smooth, and,
- wavefunction collapse, which is non reversible and abrupt.

Starting with the principle of conservation of information, we come up with a framework wherein wavefunction "collapse" can be circumvented and where only unitary evolution is required.

The axioms

Using the sources [1, 2, 3, 4, 5, 6, 7], we summarise the axioms of quantum mechanics.

Axiom 1.1. *The state of a quantum system is completely described by a ray $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. This means that it is normalised $\langle\psi|\psi\rangle = 1$ with an appropriately defined square inner product $\langle\ |\ \rangle$ and that overall phases do not matter so that $|\psi\rangle \sim e^{i\delta}\,|\psi\rangle$.*

Axiom 1.2. *The time-evolution of a quantum system is described by a unitary $\mathcal{U}$ on its Hilbert space $|\psi(t')\rangle = \mathcal{U}(t', t)\,|\psi(t)\rangle$.*

Axiom 1.3. *An observable is a Hermitian operator on the Hilbert space $\mathcal{M} = \mathcal{M}^\dagger$ so that it is diagonalisable and can be written as $\mathcal{M} = \sum_i M_i \mathbb{P}_i$, where $\mathbb{P}_i$ are projectors onto the subspaces of $\mathcal{M}$ with eigenvalues M_i. The* *outcomes of measurements in experiments can only be one of the eigenvalues of an observable. Immediately measuring the observable, so that the state has had no time to evolve away unitarily, on the same quantum state gives the same eigenvalue.*

Axiom 1.4. *The state of the system immediately after a measurement is one of the eigenstates of the related observable. Measurement takes place as,*

$$|\psi\rangle \xrightarrow{M_i} \frac{\mathbb{P}_i |\psi\rangle}{\langle\psi| \mathbb{P}_i |\psi\rangle} \tag{1.1}$$

given that M_i was the outcome of the measurement. The final state is properly normalised. The probability of this occurring is given by the Born rule,

$$\mathcal{P}(i) = \langle\psi| \mathbb{P}_i |\psi\rangle . \tag{1.2}$$

Axiom 1.5. *The combined state of two systems in $\mathcal{H}_\mathrm{a}$ and $\mathcal{H}_\mathrm{b}$ is in the tensor product space $\mathcal{H}_\mathrm{a} \otimes \mathcal{H}_\mathrm{b}$.*

In addition we postulate another axiom that states that physics is local so that two systems that are space-like separated cannot influence each other.

Axiom 1.6. *A quantum system only influences and is influenced by other systems within its light cone.*

The importance of this axiom is that quantum entanglement correlations exist even for space-like event separation. Therefore, it is important to postulate that any physics of a system cannot be influenced by other systems outside its light cone. This is slightly expanded on in the discussion section 6.

Axioms 1.2 and 1.4 indicate the two different ways quantum states evolve. In section 2 we start with the principle of information conservation (defined there in more detail) and come up with a measurement framework which is completely unitary. In essence this would lead to a redefinition of axiom 1.4 so that the state after measurement is different from equation 1.1. Section 4 explores if our framework is compatible with the more "mysterious" phenomenon of interference; starting with the double slit experiment, we move on to explain the delayed choice quantum experiment. In section 3 we explain, based on the work by Wojciech Zurek [8, 9] how the Born rule can be derived from the other principles, thereby completely eliminating the need for axiom 1.4. Finally, in section 5 we explore how one could come up with a description for the Hilbert space in axiom 1.1 in accordance with the principles of axiom 1.3. Concluding remarks follow in section 6.

Section 2

The Measurement Process

This section explores a unitary measurement process wherein there is no wavefunction collapse. Instead the system to be measured and the system performing the measurement get entangled with each other, and this is what we argue constitutes measurement. Measurement and entanglement are very closely related and the rest of this section explains in what way we believe this is true. This point of view is not novel and has been explored elsewhere [10, 11]; however, we define an explicit measurement protocol to achieve this.

2.1 Motivation

Consider a gedankenexperiment where the spin of an electron is being measured. For this sake we consider three participants; the system (the electron to be measured), the observer (whatever "looks" at this electron spin whether a human or a computer system or anything else) and the environment. The environment is basically the measurement apparatus that mediates the interaction between the system and the observer. But in a sense the environment could also mean the rest of the universe; anything else that could "disturb" this interaction by imprinting its presence on the interaction. For simplicity it is first assumed that the system, observer and environment form a closed system so that nothing else can disturb the interaction.

Guiding Principle. *A closed system evolves unitarily so that it is time reversible. This is what we interpret as information conservation so that the information cannot simply be lost. (This is related to the no-cloning [12] and* *no-deletion [13] theorems*[1]*.)*

At the end of the measurement process the system and observer are entangled and the environment is in the state that the observer was previously in. For the initial motivation, it is assumed that the system, observer and environment belong to the same Hilbert space,

$$\left|\ \right\rangle_{\mathrm{s}}, \left|\ \right\rangle_{\mathrm{o}}, \left|\ \right\rangle_{\mathrm{e}} \in \mathcal{H}.$$

For concreteness consider the qubit system

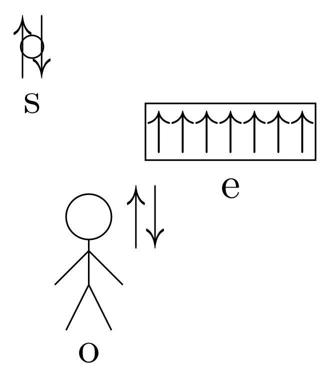
$$\left|\psi\right\rangle_{\mathrm{s}} = \psi_0 \left|0\right\rangle_{\mathrm{s}} + \psi_1 \left|1\right\rangle_{\mathrm{s}}. \tag{2.1}$$

The observer state lives in the same Hilbert space and is another arbitrary quantum state. Let

$$\left|\phi\right\rangle_{\mathrm{o}} = \phi_0 \left|0\right\rangle_{\mathrm{o}} + \phi_1 \left|1\right\rangle_{\mathrm{o}} \tag{2.2}$$

be this arbitrary state. The observer state can be imagined to be some mental representation of the system state.

The environment mediates the interaction between the system and the observer. The environment can be imagined to be a magnet system that changes the path of the electron so that it reaches one counter or another, in effect being "measured".



It is first assumed that the environment state is simple,

$$\left|\chi\right\rangle_{\mathrm{e}} = \left|0\right\rangle_{\mathrm{e}}. \tag{2.3}$$

[1]An important point to note here is that the derivations of these theorems only depend upon the linearity of quantum mechanics, axioms 1.1 and 1.2.

For the system to be effectively measured observer and system must have the same state after the measurement. That is, they should either be both $|0\rangle$ or both $|1\rangle$. In effect, this means that they should be entangled. Additionally, motivated by the guiding principle, the environment should absorb the state initially possessed by the observer. Overall,

$$\begin{aligned} |\psi\rangle_{\mathrm{s}} |\phi\rangle_{\mathrm{o}} |\chi\rangle_{\mathrm{e}} &= (\psi_0 |0\rangle + \psi_1 |1\rangle)_{\mathrm{s}} (\phi_0 |0\rangle + \phi_1 |1\rangle)_{\mathrm{o}} |0\rangle_{\mathrm{e}} \\ &\overset{!}{\to} (\psi_0 |00\rangle + \psi_1 |11\rangle)_{\mathrm{so}} (\phi_0 |0\rangle + \phi_1 |1\rangle)_{\mathrm{e}} \end{aligned} \tag{2.4}$$

which means

$$\begin{aligned} |000\rangle_{\mathrm{soe}} &\to |000\rangle_{\mathrm{soe}} \\ |010\rangle_{\mathrm{soe}} &\to |001\rangle_{\mathrm{soe}} \\ |100\rangle_{\mathrm{soe}} &\to |110\rangle_{\mathrm{soe}} \\ |110\rangle_{\mathrm{soe}} &\to |111\rangle_{\mathrm{soe}} \end{aligned} \tag{2.5}$$

Each basis state is such that system and observer are entangled and the environment is in the state the observer was previously in. In fact,

$$(\phi_0 |0\rangle + \phi_1 |1\rangle)_{\mathrm{e}} = |\phi\rangle_{\mathrm{e}} .^2$$

Additionally, indicating the entangled state as

$$(\psi_0 |00\rangle + \psi_1 |11\rangle)_{\mathrm{so}} =: |\Psi\rangle_{\mathrm{so}} ,$$

equation 2.4 can be succinctly written as

$$|\psi\rangle_{\mathrm{s}} |\phi\rangle_{\mathrm{o}} |0\rangle_{\mathrm{e}} \to |\Psi\rangle_{\mathrm{so}} |\phi\rangle_{\mathrm{e}} . \tag{2.6}$$

Extending the measurement procedure outlined above and in equation 2.5 unitarily to the rest of the basis vectors determines the action on the rest of the Hilbert space. This means that in addition to unitarity we request that the environment absorb the state previously in the observer. A natural extension is thus,

$$|001\rangle_{\mathrm{soe}} \to |010\rangle_{\mathrm{soe}}$$

[2]Note that $|\phi\rangle = \phi_0 |0\rangle + \phi_1 |1\rangle$ irrespective of the system. There are two sorts of labels being used here, the Latin lettered labels, $()_{\mathrm{s,o,e}}$, referring to the Hilbert space associated with the system, observer or environment and the Greek lettered labels, $|\psi, \phi, \chi\rangle$, indicating the amplitudes associated with the state irrespective of which Hilbert space it is in.

$$\begin{aligned}|011\rangle_{\mathrm{soe}} &\to |011\rangle_{\mathrm{soe}}\\ |101\rangle_{\mathrm{soe}} &\to |100\rangle_{\mathrm{soe}}\\ |111\rangle_{\mathrm{soe}} &\to |101\rangle_{\mathrm{soe}}\end{aligned}$$

further leading to

$$|\psi\rangle_{\mathrm{s}} |\phi\rangle_{\mathrm{o}} |1\rangle_{\mathrm{e}} \to (\psi_0 |01\rangle + \psi_1 |10\rangle)_{\mathrm{so}} |\phi\rangle_{\mathrm{e}} =: \left|\bar{\Psi}\right\rangle_{\mathrm{so}} |\phi\rangle_{\mathrm{e}} . \tag{2.7}$$

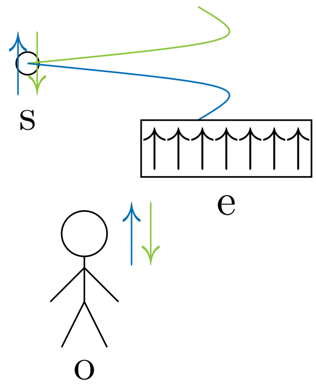
For these states, where the environment is initially in state $|1\rangle_{\mathrm{e}}$, the entanglement necessarily goes wrong in the sense that system and observer are anticorrelated in the $\{|0\rangle, |1\rangle\}$ basis. This is a direct consequence of the above requirements of unitarity, irrespective of the choice of extension.

Now consider a generic state for the environment,

$$\begin{aligned}|\psi\rangle_{\mathrm{s}} |\phi\rangle_{\mathrm{o}} |\chi\rangle_{\mathrm{e}} &= (\psi_0 |0\rangle + \psi_1 |1\rangle)_{\mathrm{s}} (\phi_0 |0\rangle + \phi_1 |1\rangle)_{\mathrm{o}} (\chi_0 |0\rangle + \chi_1 |1\rangle)_{\mathrm{e}}\\ &\to (\chi_0 |\Psi\rangle_{\mathrm{so}} + \chi_1 \left|\bar{\Psi}\right\rangle_{\mathrm{so}}) |\phi\rangle_{\mathrm{e}} ,\end{aligned} \tag{2.8}$$

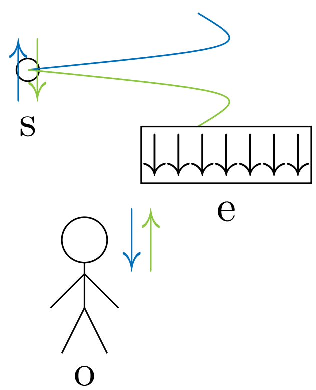
so that entangling goes right with probability $|\chi_0|^2$ and wrong with probability $|\chi_1|^2$, depending on the initial state of the environment. The closer $|\chi_0|^2$ is to 1, the more likely the entanglement goes right. Note that even though no wavefunction collapse takes place, the Born rule is nonetheless used to calculate probabilities. This is further explored in section 3.

It is instructive to draw a mental picture of the above process. Consider as part of the experimental setup a magnet with its North end pointing upwards. In this case the electron deflects a particular way depending on its spin. The described procedure entangles the direction of the electron deflection (which reflects its spin state) with the observer's mental state that notes the spin of the electron. As a result, the observer's prior state escapes to the environment in accordance with our guiding principle of unitary evolution (subsection 2.1) that prohibits the destruction of information. In this case the environment (magnet) is in state $|0\rangle_{\mathrm{e}}$ and indicates the case where the entanglement is correct.



North pointing up.

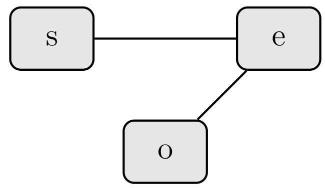
Consider now that the magnet has its North end pointing downwards. In this case the electron's direction of deflection is exactly the opposite of what it was before. The entanglement between electron spin and the observer's mental representation goes "wrong" and corresponds to the environment being in state $|1\rangle_e$. Of course, this can be corrected by "measuring" the environment state. This discussion is continued in section 2.3, where a further measurement of the environment is introduced.



North pointing down.

2.1.1 Local Operators

This subsection aims to address the question of whether the above process can be encoded by local operations between system, environment and observer. By local we mean that only two of the three interact in one instance. It will turn out that the system and the observer do not need to interact directly, but only indirectly through the environment. The environment interacts sequentially with both the system and the observer, thereby mediating the interaction between them.



Consider the operator, $\mathcal{I}_{\mathrm{a}\to\mathrm{b}}$ with

$$\begin{aligned} \mathcal{I}_{\mathrm{a}\to\mathrm{b}}\,|00\rangle_{\mathrm{ab}} &= |00\rangle_{\mathrm{ab}} & \qquad \mathcal{I}_{\mathrm{a}\to\mathrm{b}}\,|01\rangle_{\mathrm{ab}} &= |01\rangle_{\mathrm{ab}} \\ \mathcal{I}_{\mathrm{a}\to\mathrm{b}}\,|10\rangle_{\mathrm{ab}} &= |11\rangle_{\mathrm{ab}} & \qquad \mathcal{I}_{\mathrm{a}\to\mathrm{b}}\,|11\rangle_{\mathrm{ab}} &= |10\rangle_{\mathrm{ab}}\,, \end{aligned} \tag{2.9}$$

which is a kind of imprinting (or entangling) operator that imprints the contents of system a onto system b. It is, in fact, exactly the CNOT operator in quantum information theory.

Consider now the operator, $\mathcal{S}_{\mathrm{a}\leftrightarrow\mathrm{b}}$ with

$$\begin{aligned} \mathcal{S}_{\mathrm{a}\leftrightarrow\mathrm{b}}\,|00\rangle_{\mathrm{ab}} &= |00\rangle_{\mathrm{ab}} & \qquad \mathcal{S}_{\mathrm{a}\leftrightarrow\mathrm{b}}\,|01\rangle_{\mathrm{ab}} &= |10\rangle_{\mathrm{ab}} \\ \mathcal{S}_{\mathrm{a}\leftrightarrow\mathrm{b}}\,|10\rangle_{\mathrm{ab}} &= |01\rangle_{\mathrm{ab}} & \qquad \mathcal{S}_{\mathrm{a}\leftrightarrow\mathrm{b}}\,|11\rangle_{\mathrm{ab}} &= |11\rangle_{\mathrm{ab}}\,, \end{aligned} \tag{2.10}$$

which swaps the contents of the qubits on which it acts. This is the SWAP operation in quantum information theory.

All the operations defined above are unitary as they map elements of an orthonormal basis in the Hilbert space $(\mathcal{H}\otimes\mathcal{H})$ to other orthonormal basis elements; for example, $|0\rangle_{\mathrm{a}}\,|1\rangle_{\mathrm{b}} \to |1\rangle_{\mathrm{a}}\,|0\rangle_{\mathrm{b}}$ for $\mathcal{S}_{a\leftrightarrow b}$.

The entire measurement procedure described before can be performed using just the operators $\mathcal{I}_{\mathrm{a}\to\mathrm{b}}$ and $\mathcal{S}_{\mathrm{a}\leftrightarrow\mathrm{b}}$. Consider the overall system to begin in the state

$$|\psi\rangle_{\mathrm{s}}\,|\phi\rangle_{\mathrm{o}}\,|\chi\rangle_{\mathrm{e}}\,.$$

First the system imprints itself on the environment during its interaction with it, $\mathcal{I}_{s\to e}$. Then, the environment and observer swap their qubits during their interaction ($\mathcal{S}_{o\leftrightarrow e}$).

$$|000\rangle_{\mathrm{soe}} \quad \xrightarrow{\mathcal{I}_{s\to e}} \quad |000\rangle_{\mathrm{soe}} \quad \xrightarrow{\mathcal{S}_{o\leftrightarrow e}} \quad |000\rangle_{\mathrm{soe}}$$

$$
\begin{array}{ccccc}
|010\rangle_{\mathrm{soe}} & \longrightarrow & |010\rangle_{\mathrm{soe}} & \longrightarrow & |001\rangle_{\mathrm{soe}} \\
|100\rangle_{\mathrm{soe}} & \longrightarrow & |101\rangle_{\mathrm{soe}} & \longrightarrow & |110\rangle_{\mathrm{soe}} \\
|110\rangle_{\mathrm{soe}} & \longrightarrow & |111\rangle_{\mathrm{soe}} & \longrightarrow & |111\rangle_{\mathrm{soe}} \\
|001\rangle_{\mathrm{soe}} & \longrightarrow & |001\rangle_{\mathrm{soe}} & \longrightarrow & |010\rangle_{\mathrm{soe}} \\
|011\rangle_{\mathrm{soe}} & \longrightarrow & |011\rangle_{\mathrm{soe}} & \longrightarrow & |011\rangle_{\mathrm{soe}} \\
|101\rangle_{\mathrm{soe}} & \longrightarrow & |100\rangle_{\mathrm{soe}} & \longrightarrow & |100\rangle_{\mathrm{soe}} \\
|111\rangle_{\mathrm{soe}} & \longrightarrow & |110\rangle_{\mathrm{soe}} & \longrightarrow & |101\rangle_{\mathrm{soe}}
\end{array}
$$

This means that the system and observer are now entangled and the environment is in the state that was priorly in the observer, so that one obtains equation 2.8 again,

$$|\psi\rangle_{\mathrm{s}} |\phi\rangle_{\mathrm{o}} |\chi\rangle_{\mathrm{e}} \to (\chi_0 |\Psi\rangle_{\mathrm{so}} + \chi_1 |\bar{\Psi}\rangle_{\mathrm{so}}) |\phi\rangle_{\mathrm{e}} ,$$

which when expanded is

$$
\begin{aligned}
&(\psi_0 |0\rangle + \psi_1 |1\rangle)_{\mathrm{s}}(\phi_0 |0\rangle + \phi_1 |1\rangle)_{\mathrm{o}}(\chi_0 |0\rangle + \chi_1 |1\rangle)_{\mathrm{e}} \to \\
&\qquad (\chi_0(\psi_0 |00\rangle + \psi_1 |11\rangle)_{\mathrm{so}} + \chi_1(\psi_0 |01\rangle + \psi_1 |10\rangle)_{\mathrm{so}})(\phi_0 |0\rangle + \phi_1 |1\rangle)_{\mathrm{e}}.
\end{aligned}
$$

Using just local unitaries between system and environment and between observer and environment, the entanglement of system and observer has been accomplished. Note that in this case, the entanglement goes right in case the environment is initially in state 0, $\chi_0 |0\rangle_{\mathrm{e}}$, and wrong in case the environment is initially in state 1, $\chi_1 |1\rangle_{\mathrm{e}}$. Once again, the probabilities are, assuming the Born rule, $|\chi_0|^2$ and $|\chi_1|^2$, respectively.

2.2 Generalisation

This subsection aims to extend the analysis from a qubit system to a more generic d state quantum system, a qudit. Not only does this generalise our procedure but also simplifies notation. The operators $\mathcal{I}$ and $\mathcal{S}$ are generalised. Let the Hilbert spaces now be more generic,

$$\mathcal{H}_{\mathrm{s}} = \mathcal{H}_{\mathrm{o}} = \mathcal{H}_{\mathrm{e}} = \mathrm{span}\{|i\rangle\}$$

for some basis set $\{|i\rangle\} = \{|0\rangle , |1\rangle , ... |d-1\rangle\}$.

Define

$$\mathcal{I}_{\mathrm{a}\to\mathrm{b}} |i\rangle_{\mathrm{a}} |j\rangle_{\mathrm{b}} = |i\rangle_{\mathrm{a}} |j+i\rangle_{\mathrm{b}} , \tag{2.11}$$

where '+' is now defined appropriately; for the qubit system it was addition modulo 2 and for an qudit with states $\{|i\rangle\}, i \in \{0, ..d-1\}$ it would be addition modulo d.

'+' is an operator in the index set $\mathfrak{I}(\mathcal{H}) = \{i\}$ (of the Hilbert space $\mathcal{H} = \text{span}\{|i\rangle\}$) so that the action $\mathcal{I}_{\mathrm{a}\to\mathrm{b}}$ on the quantum states is unitary. The index set is explored in more detail in section 5 where it is argued that $\mathfrak{I}(\mathcal{H})$ has the structure of a semigroup and '+' would be the corresponding semigroup operator.

Similarly define

$$\mathcal{S}_{\mathrm{a}\leftrightarrow\mathrm{b}} \left|i\right\rangle_{\mathrm{a}} \left|j\right\rangle_{\mathrm{b}} = \left|j\right\rangle_{\mathrm{a}} \left|i\right\rangle_{\mathrm{b}} . \tag{2.12}$$

Consider starting at the initial state

$$\left|\zeta\right\rangle_{\mathrm{soe}} = \left|\psi\right\rangle_{\mathrm{s}} \left|\phi\right\rangle_{\mathrm{o}} \left|\chi\right\rangle_{\mathrm{e}} = \left(\sum_i \psi_i \left|i\right\rangle_{\mathrm{s}}\right) \left(\sum_j \phi_j \left|j\right\rangle_{\mathrm{o}}\right) \left(\sum_k \chi_k \left|k\right\rangle_{\mathrm{e}}\right),$$

which, after collecting terms can be written as

$$= \sum_{ijk} \psi_i \phi_j \chi_k \left|i\right\rangle_{\mathrm{s}} \left|j\right\rangle_{\mathrm{o}} \left|k\right\rangle_{\mathrm{e}} .$$

Firstly, the system imprints itself onto the environment,

$$\left|\zeta'\right\rangle_{\mathrm{soe}} = \mathcal{I}_{s\to e} \left|\zeta\right\rangle_{\mathrm{soe}} = \sum_{ijk} \psi_i \phi_j \chi_k \left|i\right\rangle_{\mathrm{s}} \left|j\right\rangle_{\mathrm{o}} \left|k+i\right\rangle_{\mathrm{e}} .$$

Secondly, the observer and environment exchange qudits so the "measurement" is performed. This involves the environment going to the state the observer was previously in and the system and observer being entangled,

$$\left|\zeta''\right\rangle_{\mathrm{soe}} = \mathcal{S}_{o\leftrightarrow e} \left|\zeta'\right\rangle_{\mathrm{soe}} = \sum_{ijk} \psi_i \phi_j \chi_k \left|i\right\rangle_{\mathrm{s}} \left|k+i\right\rangle_{\mathrm{o}} \left|j\right\rangle_{\mathrm{e}} ,$$

which when factored conveniently is

$$= \left[\sum_k \chi_k \left(\sum_i \psi_i \left|i\right\rangle \left|i+k\right\rangle\right)_{\mathrm{so}}\right] \left(\sum_j \phi_j \left|j\right\rangle_{\mathrm{e}}\right),$$

and can be written as

$$\left(\sum_k \chi_{\mathrm{k}} \left|\Psi_k\right\rangle_{\mathrm{so}}\right) \left|\phi\right\rangle_{\mathrm{e}} , \tag{2.13}$$

where

$$|\Psi_k\rangle_{\mathrm{so}} = \sum_i \psi_i \, |i\rangle_{\mathrm{s}} \, |i+k\rangle_{\mathrm{o}} \tag{2.14}$$

indicates the entanglement being correct up to extent k.

Ideally what is required is that the system and observer be in the same state $|i\rangle$; however, one can consider a degree of the entanglement going wrong so that the system and environment are wrongly entangled to extent k. This shall be addressed in section 5, where there is a discussion about the topology of the index set. Moreover, this can be corrected for and the correction is explained in section 2.3.

The probability for the system to be entangled as $|\Psi_k\rangle_{\mathrm{so}}$ is $|\chi_k|^2$ using the Born rule (section 3). This is justified as the $\{|\Psi_k\rangle_{\mathrm{so}}\}$ form an orthonormal set,

$$\begin{aligned}
\langle \Psi_l | \Psi_k \rangle_{\mathrm{so}} &= \sum_{ij} \psi_i^* \psi_j \, \langle i|j\rangle_{\mathrm{s}} \, \langle i+l|j+k\rangle_{\mathrm{o}} \\
&= \sum_i |\psi_i|^2 \, \langle i+l|i+k\rangle_{\mathrm{o}} = \delta_{kl}.
\end{aligned}$$

2.2.1 Extension to density matrices

The formalism can be extend to density matrices naturally, for example, imprinting is represented by

$$\rho \xrightarrow{\mathcal{I}_{\mathrm{a}\to\mathrm{b}}} \mathcal{I}_{\mathrm{a}\to\mathrm{b}} \rho \mathcal{I}^\dagger_{\mathrm{a}\to\mathrm{b}},$$

so that

$$\begin{aligned}
\tilde{\mathcal{I}}_{\mathrm{a}\to\mathrm{b}}(|i\rangle \langle k|_{\mathrm{a}} \, |j\rangle \langle l|_{\mathrm{b}}) &= (\mathcal{I}_{\mathrm{a}\to\mathrm{b}} \, |i\rangle_{\mathrm{a}} \, |j\rangle_{\mathrm{b}})(\langle k|_{\mathrm{a}} \, \langle l|_{\mathrm{b}} \, \mathcal{I}^\dagger_{\mathrm{a}\to\mathrm{b}}) \\
&= |i\rangle_{\mathrm{a}} \, |j+i\rangle_{\mathrm{b}} \, \langle k|_{\mathrm{a}} \, \langle l+k|_{\mathrm{b}} \\
&= |i\rangle \langle k|_{\mathrm{a}} \, |j+i\rangle \langle l+k|_{\mathrm{b}} \,,
\end{aligned}$$

and similarly

$$\tilde{\mathcal{S}}_{\mathrm{a}\leftrightarrow\mathrm{b}}(|i\rangle \langle k|_{\mathrm{a}} \, (|j\rangle \langle l|_{\mathrm{b}}) = |j\rangle \langle l|_{\mathrm{a}} \, |i\rangle \langle k|_{\mathrm{b}} \,,$$

where $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{S}}$ indicate the action on density matrices.

2.3 Learning the Environment

This subsection looks at how one can start with a "classical" environment and end up with something that resembles wavefunction collapse. In this section a special form for the environment is assumed[3],

$$|\chi\rangle_{\mathrm{e}} = \sum_k \chi_k \, |k\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} |k\rangle_{\mathrm{e}_3} \ldots |k\rangle_{\mathrm{e}_\mathrm{N}} \,, \tag{2.15}$$

so that the environment consists of a set of N qudits such that each qudit is entangled with every other (see appendix A for details).

The system and observer states are as before, $|\psi\rangle_{\mathrm{s}} = \sum_i \psi_i \, |i\rangle_{\mathrm{s}}$ and $|\phi\rangle_{\mathrm{o}}$, respectively.

Let the usual measurement procedure as described in section 2.2 take place between s, o and e_1 so that[4]

$$\begin{array}{ll} \mathcal{S}_{o\leftrightarrow e_1} \circ \mathcal{I}_{s\to e_1} & \begin{array}{l} \mathrm{s} : |i\rangle_{\mathrm{s}} \\ \mathrm{o} : |\phi\rangle_{\mathrm{o}} \\ \mathrm{e} : |k\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} \ldots |k\rangle_{\mathrm{e}_\mathrm{N}} \end{array} \\ \\ \hookrightarrow & \begin{array}{l} \mathrm{s} : |i\rangle_{\mathrm{s}} \\ \mathrm{o} : |k+i\rangle_{\mathrm{o}} \\ \mathrm{e} : |\phi\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} \ldots |k\rangle_{\mathrm{e}_\mathrm{N}} \,. \end{array} \end{array}$$

Therefore, by disentangling e_2 from o one obtains perfect entanglement between system and observer[5],

$$\begin{array}{ll} \mathcal{I}^{-1}_{e_2\to o} & \begin{array}{l} \mathrm{s} : |i\rangle_{\mathrm{s}} \\ \mathrm{o} : |k+i\rangle_{\mathrm{o}} \\ \mathrm{e} : |\phi\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} \ldots |k\rangle_{\mathrm{e}_\mathrm{N}} \end{array} \\ \\ & \mathrm{s} : |i\rangle_{\mathrm{s}} \end{array}$$

[3] We explain the procedure here using the minimal number of qudits possible.

[4] Summation indices and amplitudes are suppressed for brevity.

[5] Note that a key requirement for this procedure to work is that + is associative and commutative. This may not always be the case as in section 5.2 and in these cases one has to be careful of the order of operations.

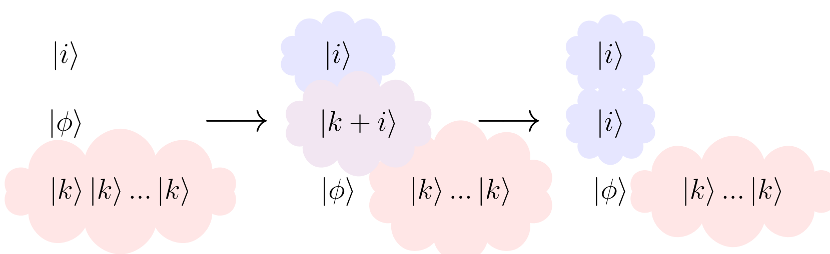
$$\hookrightarrow \qquad \begin{array}{l} \mathrm{o} : |k+i-k\rangle_{\mathrm{o}} \\ \mathrm{e} : |\phi\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} \dots |k\rangle_{\mathrm{e_N}} \end{array}$$

$$= \qquad \begin{array}{l} \mathrm{s} : |i\rangle_{\mathrm{s}} \\ \mathrm{o} : |i\rangle_{\mathrm{o}} \\ \mathrm{e} : |\phi\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} \dots |k\rangle_{\mathrm{e_N}} . \end{array}$$

Continuing the discussion in section 2.2, the observer now measures two entities, the system and the environment. By getting to know the state of the environment, any biases in the system's measurement can be corrected and perfect entanglement ensues. This is what the last disentangling operation does.

An important point to note is that the environment has to be internally entangled, prior to measurement, as described above where each qudit is entangled with each other. Therefore, such a "classical" system is required for the measurement procedure to work correctly (appendix A).

Also to be noted is that the measurement comes at a cost. As the system and observer are entangled, the environment loses some of its initial entanglement. In particular, the environment loses one qudit of entanglement while the system and observer become entangled.



2.4 Multiple Observations and Decoherence

The procedure explained in section 2.3 can be repeated by multiple observers, thereby creating a chain of observations of the quantum state $|\psi\rangle_{\mathrm{s}}$. Consider a chain of observers observing the state of system s,

$$\sum_i \psi_i \, |i\rangle_{\mathrm{s}} \, |\phi_1\rangle_{\mathrm{o}_1} \, |\phi_2\rangle_{\mathrm{o}_2} \, |\phi_3\rangle_{\mathrm{o}_3} \dots$$

$$\overset{\mathrm{o_1\ measures}}{\rightarrow} \sum_i \psi_i \left|i\right\rangle_{\mathrm{s}} \left|i\right\rangle_{\mathrm{o_1}} \left|\phi_2\right\rangle_{\mathrm{o_2}} \left|\phi_3\right\rangle_{\mathrm{o_3}} \ldots$$
$$\overset{\mathrm{o_2\ measures}}{\rightarrow} \sum_i \psi_i \left|i\right\rangle_{\mathrm{s}} \left|i\right\rangle_{\mathrm{o_1}} \left|i\right\rangle_{\mathrm{o_2}} \left|\phi_3\right\rangle_{\mathrm{o_3}} \ldots$$
$$\overset{\mathrm{o_3\ measures}}{\rightarrow} \sum_i \psi_i \left|i\right\rangle_{\mathrm{s}} \left|i\right\rangle_{\mathrm{o_1}} \left|i\right\rangle_{\mathrm{o_2}} \left|i\right\rangle_{\mathrm{o_3}} \ldots, \tag{2.16}$$

where at each stage the prior states $|\phi_i\rangle$ are dumped into corresponding environments. The environments might be different for the different observers so that environment

$$\left|{}^i\chi\right\rangle_{\mathrm{^ie}} = \sum_k {}^i\chi_k \left|k\right\rangle_{\mathrm{^ie_1}} \left|k\right\rangle_{\mathrm{^ie_2}} \left|k\right\rangle_{\mathrm{^ie_3}} \ldots \left|k\right\rangle_{\mathrm{^ie_{N_i}}}$$

is used for observer o_i. This means that different apparatuses may be used by the different observers to measure the system.

This also explains how different observers correlate observations leading to repeatability of experiments and a shared reality among different observers. Specifically in the quantum context, this refers to the fact that an immediate second measurement of a system produces the same results, which is what is required by axiom 1.3 in section 1. Therefore our measurement procedure is consistent with axiom 1.3 as required.

In equation 2.16 multiple observers measure the system and correct for the environment as in section 2.3 so that it corresponds to repeated measurement. However, even a single observer is usually strongly coupled to the outside environment and interacts with it. For each such interaction the observer, and thereby the system, gets entangled with an outside environment system, and this makes the whole procedure irreversible. Consider an outside environment that looks like

$$\left|{}^1\xi\right\rangle_{\mathrm{^1e}} = \sum_{k_1} {}^1\xi_{k_1} \left|k_1\right\rangle_{\mathrm{^1e}},$$

which as soon as it interacts with the observer turns into

$$= \sum_{k_1} {}^1\xi_{k_1} \left(\sum_i \psi_i \left|i\right\rangle_{\mathrm{s}} \left|i\right\rangle_{\mathrm{o}} \left|k_1 + i\right\rangle_{\mathrm{^1e}} \right).$$

This happens for each outside environment system that interacts with the observer. It is noted that the important thing here is the entanglement, and therefore there is no correction for the environment state as in equation 2.16. If a chain of such entanglements take place, what obtains is

$$= \sum_{k_1 k_2 \ldots} {}^1\xi_{k_1} {}^2\xi_{k_2} \ldots \left(\sum_i \psi_i \left|i\right\rangle_{\mathrm{s}} \left|i\right\rangle_{\mathrm{o}} \left|k_1 + i\right\rangle_{1_{\mathrm{e}}} \left|k_2 + i\right\rangle_{2_{\mathrm{e}}} \ldots \right). \tag{2.17}$$

This ensures that the system and observer are still entangled as before irrespective of which branch of $\left|k_1 + i\right\rangle_{1_{\mathrm{e}}} \left|k_2 + i\right\rangle_{2_{\mathrm{e}}} \ldots$ is chosen. Irreversibility and classicality begin to emerge here, as in order to reverse the process one would need to bring together all the different quantum systems $\left|k_1 + i\right\rangle_{1_{\mathrm{e}}} \left|k_2 + i\right\rangle_{2_{\mathrm{e}}} \ldots$ to disentangle them. This makes sure that classicality is a stable property of the system in that it is quite difficult to undo the emergence of classicality. Another key point here is that all the operations are local (axiom 1.6) and this locality plays a key role in maintaining stability. In case non-local unitaries were allowed this classicality could be erased even if the quantum systems $\left|k_1 + i\right\rangle_{1_{\mathrm{e}}} \left|k_2 + i\right\rangle_{2_{\mathrm{e}}} \ldots$ were far apart.

Combining the procedure on multiple observers and the fact that each observer interacts with the outside environment, one obtains in diagram,

$$\begin{array}{lll}
\mathrm{s}: & \left|i\right\rangle & \\
\mathrm{o}_1: & \left|i\right\rangle & \left|k_{11} + i\right\rangle \left|k_{12} + i\right\rangle \ldots \\
\mathrm{o}_2: & \left|i\right\rangle & \left|k_{21} + i\right\rangle \left|k_{22} + i\right\rangle \ldots \\
\mathrm{o}_3: & \left|i\right\rangle & \left|k_{31} + i\right\rangle \left|k_{32} + i\right\rangle \ldots \\
\vdots & &
\end{array}$$

where k_{ij} refers to the j'th outside environment system that gets entangled with observer o_i. Thus, what is represented is a set of classical observers o_i that all separately learn a system s.

2.4.1 Measurements in different bases

Here we discuss what would happen in case the measurements by the different observers are performed in different bases. For this we consider the simplified case of a qubit system but the idea should extend to larger systems. Consider once again a qubit system

$$\psi_0 \left|0\right\rangle_{\mathrm{s}} + \psi_1 \left|1\right\rangle_{\mathrm{s}},$$

which when measured by the first observer becomes

$$\to \psi_0 \left|0\right\rangle_{\mathrm{s}} \left|0\right\rangle_{\mathrm{o}_1} + \psi_1 \left|1\right\rangle_{\mathrm{s}} \left|1\right\rangle_{\mathrm{o}_1}.$$

If now the second observer o_2 measures in the

$$\left\{ \left|+\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle, \left|-\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle \right\}$$

basis, the previous state can be written as

$$= \left|+\right\rangle_{\mathrm{s}} \frac{1}{\sqrt{2}} \left(\psi_0 \left|0\right\rangle_{\mathrm{o}_1} + \psi_1 \left|1\right\rangle_{\mathrm{o}_1}\right) + \left|-\right\rangle_{\mathrm{s}} \frac{1}{\sqrt{2}} \left(\psi_0 \left|0\right\rangle_{\mathrm{o}_1} - \psi_1 \left|1\right\rangle_{\mathrm{o}_1}\right),$$

which when measured gives

$$\begin{aligned} &= \left|+\right\rangle_{\mathrm{s}} \frac{1}{\sqrt{2}} \left(\psi_0 \left|0\right\rangle_{\mathrm{o}_1} + \psi_1 \left|1\right\rangle_{\mathrm{o}_1}\right) \left|+\right\rangle_{\mathrm{o}_2} \\ &+ \left|-\right\rangle_{\mathrm{s}} \frac{1}{\sqrt{2}} \left(\psi_0 \left|0\right\rangle_{\mathrm{o}_1} - \psi_1 \left|1\right\rangle_{\mathrm{o}_1}\right) \left|-\right\rangle_{\mathrm{o}_2}. \end{aligned} \tag{2.18}$$

This means that the probability for o_2 to measure + (or -) is 1/2, as the density matrix corresponding to the above state is,

$$\rho_{\mathrm{o}_2} = \frac{1}{2}\left|+\right\rangle\left\langle+\right|_{\mathrm{o}_2} + \frac{1}{2}\left|-\right\rangle\left\langle-\right|_{\mathrm{o}_2}. \tag{2.19}$$

This makes sense as the qubit once measured in the $\{\left|0\right\rangle, \left|1\right\rangle\}$ basis "collapses" and thus is completely undetermined in the $\{\left|+\right\rangle, \left|-\right\rangle\}$ basis.

Now if qubit o_1 is measured in the $\{\left|+\right\rangle, \left|-\right\rangle\}$ basis by observer o_3'[6], surprising results are obtained. First of all it depends on whether observer

[6] A $'$ is used to denote that observer o_3' is observing o_1 instead of s.

o_1 is classical or not. Consider first the case when o_1 is well isolated from the outside environment so that it behaves quantumly. In this case, the resulting state, conveniently factorised, is

$$\begin{aligned}\overset{\text{quantum } o_1}{\rightarrow} & \frac{1}{2}\left[(\psi_0+\psi_1)\left|+\right\rangle_s\left|+\right\rangle_{o_2}+(\psi_0-\psi_1)\left|-\right\rangle_s\left|-\right\rangle_{o_2}\right]\left|+\right\rangle_{o_1}\left|+\right\rangle_{o_3'} \\ & \frac{1}{2}\left[(\psi_0-\psi_1)\left|+\right\rangle_s\left|+\right\rangle_{o_2}+(\psi_0+\psi_1)\left|-\right\rangle_s\left|-\right\rangle_{o_2}\right]\left|-\right\rangle_{o_1}\left|-\right\rangle_{o_3'}\end{aligned} \tag{2.20}$$

whose naïve density matrix, tracing out s, o_1 and o_3' is simply the maximally mixed one of equation 2.19. However, if only states with $\left|+\right\rangle_{o_1}\left|+\right\rangle_{o_3'}$ are chosen, only cases when observers o_1 and o_3' give +, an interference pattern is obtained! Now the density matrix is

$$\rho_{o_2|s=+}=\frac{\left|\psi_0+\psi_1\right|^2}{2}\left|+\right\rangle\left\langle+\right|_{o_2}+\frac{\left|\psi_0-\psi_1\right|^2}{2}\left|-\right\rangle\left\langle-\right|_{o_2} \tag{2.21}$$

where $\rho_{o_2|s=+}$ corresponds to the density matrix of o_2, tracing out s, o_1 and o_3', given that s is in state +. The factor of 1/2 from before vanishes because a post selection of states with just s = 1/2 is performed. The other 1/2 corresponds to s = −. The measurement of observer o_1 has been erased; this is explored in more detail in section 4.

Continuing onto the case of a “classical” observer o_1, we see how this case differs. In this case the observer o_1 forms a chain of secondary observers as in equation 2.17. The resulting state is

$$\left|+\right\rangle_s\frac{1}{\sqrt{2}}\begin{pmatrix}\psi_0|0\rangle_{o_1} + \psi_1|1\rangle_{o_1} \\ |0\rangle_{11_e} \qquad\quad |1\rangle_{11_e} \\ \vdots \qquad\qquad \vdots\end{pmatrix}\left|+\right\rangle_{o_2}+\left|-\right\rangle_s\frac{1}{\sqrt{2}}\begin{pmatrix}\psi_0|0\rangle_{o_1} - \psi_1|1\rangle_{o_1} \\ |0\rangle_{11_e} \qquad\quad |1\rangle_{11_e} \\ \vdots \qquad\qquad \vdots\end{pmatrix}\left|-\right\rangle_{o_2},$$

which after being observed by o_3' becomes

$$\overset{\text{classical } o_1}{\rightarrow}\frac{1}{2}\left[\begin{matrix}(\psi_0|0\rangle_{11_e}+\psi_1|1\rangle_{11_e})\left|+\right\rangle_s\left|+\right\rangle_{o_2} \\ \vdots \qquad\qquad \vdots \\ +\ (\psi_0|0\rangle_{11_e}-\psi_1|1\rangle_{11_e})\left|-\right\rangle_s\left|-\right\rangle_{o_2} \\ \vdots \qquad\qquad \vdots\end{matrix}\right]\left|+\right\rangle_{o_1}\left|+\right\rangle_{o_3'}$$

$$+\frac{1}{2}\left[\begin{array}{c}(\psi_0|0\rangle_{11_\mathrm{e}} - \psi_1|1\rangle_{11_\mathrm{e}})\,|+\rangle_\mathrm{s}\,|+\rangle_{\mathrm{o}_2}\\ \vdots \qquad\qquad \vdots \\ + \; (\psi_0|0\rangle_{11_\mathrm{e}} + \psi_1|1\rangle_{11_\mathrm{e}})\,|-\rangle_\mathrm{s}\,|-\rangle_{\mathrm{o}_2} \\ \vdots \qquad\qquad \vdots \end{array}\right] |-\rangle_{\mathrm{o}_1}\,|-\rangle_{\mathrm{o}'_3}, \tag{2.22}$$

so that whether the entire density matrix is chosen or the one given $|+\rangle_{\mathrm{o}_1}|+\rangle_{\mathrm{o}_3}$ the result is the maximally mixed state of 2.19,

$$\rho_{\mathrm{o}_2} = \rho_{\mathrm{o}_2|\mathrm{s}=+} = \frac{1}{2}\,|+\rangle\,\langle+|_{\mathrm{o}_2} + \frac{1}{2}\,|-\rangle\,\langle-|_{\mathrm{o}_2}\,. \tag{2.23}$$

A classical state, therefore, is stable as the basis cannot be easily changed from $\{|0\rangle\,,|1\rangle\}$ to $\{|+\rangle\,,|-\rangle\}$. This problem is explored in more detail in section 4.

2.5 Discussion (so far)

Thus, what the entire procedure accomplishes is perfect entanglement between the system and observer states using only local unitaries between system and environment and observer and environment. In order for the procedure to work properly one also requires a classical environment consisting of a larger number of states all entangled with each other. system imprints onto environment and then observer and environment swap, leading to entanglement between system and observer.

The prior entanglement of the environment turns out to be a necessary resource for measurement. A part of this entanglement is effectively transferred onto the system-observer subsystem. This is what we claim constitutes measurement. For the observer to have properly measured the system what is required is for the two of them to end up in the same state, that is,

$$\sum_i \psi_i\,|i\rangle_\mathrm{s}\,|i\rangle_\mathrm{o}\,. \tag{2.24}$$

This ensures that whatever state the system is in is also the state the observer is in. The combined state is now a superposition of such perfectly aligned states, which is what entanglement is. The quantum indeterminacy persists, the exact outcome is unknown; however, the system and observer are perfectly correlated and there is no wavefunction collapse. Each of these branches $\psi_i \left|i\right\rangle_{\mathrm{s}} \left|i\right\rangle_{\mathrm{o}}$ are separate from each other and unless a further disentangling operation is performed these branches do not influence each other.

Thus, the wavefunction does not only consist of the outcome that actually results but also those that could have resulted. However, the system or observer is stuck in one of these branches and cannot actually access the others unless the state is disentangled, which can only be done by bringing the systems back in contact and might be prohibitively difficult for a system that is entangled with many observers (or even a single "classical" observer as remarked in the previous section 2.4). Thus, decoherence[7]comes about as soon as a system interacts with even a single observer but irreversibility is a consequence of the system interacting with many observers (or a single classical observer)!

It can also be seen that multiple observers may observe consistent results if the same measurement procedure is performed multiple times with multiple observers,

$$\sum_i \psi_i \left|i\right\rangle_{\mathrm{s}} \left|i\right\rangle_{\mathrm{o}_1} \left|i\right\rangle_{\mathrm{o}_2} \left|i\right\rangle_{\mathrm{o}_3} \ldots,$$

where the environments (equation 2.15) need not be the same for different observers. This hints at the repeatability of experiments and in the specific quantum context refers to the fact that an immediate second measurement of a system produces the same results, as required by axiom 1.3 in section 1.

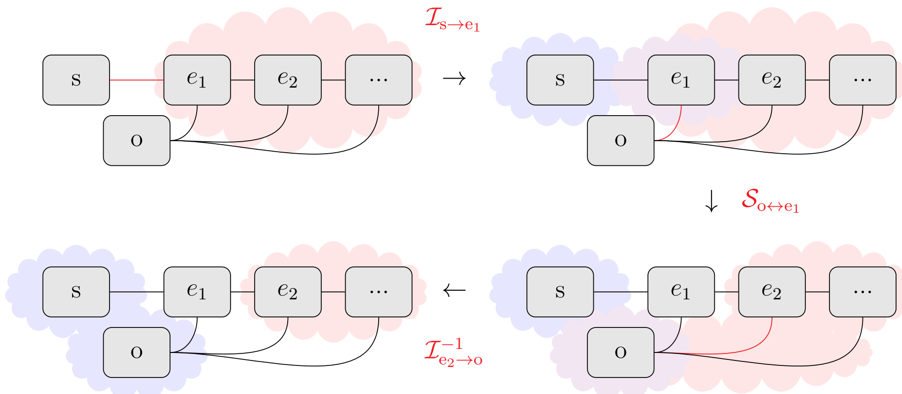
[7]Here by decoherence [14, 15, 16] is meant the conversion of a pure state to a mixed state as a result of entanglement with an observer. For example

$$\left|\psi\right\rangle_{\mathrm{s}} = \sum_i \psi_i \left|i\right\rangle_{\mathrm{s}} \to \sum_i \psi_i \left|i\right\rangle_{\mathrm{s}} \left|i\right\rangle_{\mathrm{o}}$$

leading to the density matrix

$$\rho_{\mathrm{s}} = \sum_i \left|\psi\right|^2 \left|i\right\rangle \left\langle i\right|_{\mathrm{s}}$$

without any phases and only the probabilites accoring to Born's rule 3.



The process of entangling system and observer to achieve measurement. Lines between subsystems indicate that they are local, that is, that they can interact; red lines indicate that an operation between the subsystems is taking place.

Section 3

Born's Rule

In this section we attempt to justify the Born rule that has already been used before. We use a symmetry, envariance, in order to derive the Born rule. This section follows very closely the papers [8, 9], simply adapting its notation to our framework.

3.1 Envariance

Envariance is a symmetry of quantum systems and it would form the basis of our argument in deriving the Born rule. A quantum system that exists in complete isolation from everything else without even interacting with a measurement apparatus would have no preferred basis for its description and there need be no probabilities to describe its evolution. As soon as a system interacts with an environment there is now a preferred basis for the system, and from this emerges the Born rule.

Consider therefore a system which has already interacted with an environment so that they are now entangled with each other

$$|\Psi\rangle_{\mathrm{se}} = \sum_i \Psi_i \, |e_i\rangle_{\mathrm{s}} \, |f_i\rangle_{\mathrm{e}}, \tag{3.1}$$

where $\{\Psi_i\}$ are the Schmidt coefficients (appendix A) and $\{|e_i\rangle_{\mathrm{s}}\}$ and $\{|f_i\rangle_{\mathrm{e}}\}$ are bases for the system and environment respectively. The uniqueness of these states is explored in appendix B.

The system is said to be envariant with respect to a unitary $\mathcal{U}_{\mathrm{s}}$ acting on the system alone if and only if there exists a unitary $\mathcal{U}_{\mathrm{e}}$ acting on the environment alone such that their combined application leaves unchanged the overall state of system and environment, $|\Psi\rangle_{\mathrm{se}}$; that is,

$$(\mathbb{1}_{\mathrm{s}} \otimes \mathcal{U}_{\mathrm{e}})(\mathcal{U}_{\mathrm{s}} \otimes \mathbb{1}_{\mathrm{e}}) |\Psi\rangle_{\mathrm{se}} = |\Psi\rangle_{\mathrm{se}} . \tag{3.2}$$

Notice that these operations commute, and thus it doesn't matter if the system unitary $\mathcal{U}_{\mathrm{s}}$ is applied before the environment $\mathcal{U}_{\mathrm{e}}$ or the other way around.

Envariance is called an assisted symmetry as something must change elsewhere (that is at the environment) in order to bring the system back to its original state. Notice that the environment may be far separated from the system in question and any changes to it must not be able to physically influence the system; here we are appealing to the locality of physics (axiom 1.6). And thus, the envariant unitary $\mathcal{U}_{\mathrm{s}}$ itself shouldn't change the physics as this could be reversed by applying $\mathcal{U}_{\mathrm{e}}$ to the environment, which may be non-causally separated from the system.

3.2 The Symmetry

As appendix B shows, the only possible unitaries $\mathcal{U}_{\mathrm{s}}$ are those that do not mix subspaces corresponding to states with different absolute values of Schmidt coefficients. Thus, any envariant unitary is block diagonal in the subspaces with constant values of Schmidt coefficients.

For simplicity, consider two basis vectors with the same absolute values of Schmidt coefficients. Any complex factors of $e^{i\delta}$ can be removed using envariant unitaries and are, therefore, not relevant to the following analysis. Consider a state

$$|\Psi\rangle_{\mathrm{se}} = ... + \alpha\, |e_i\rangle_{\mathrm{s}} |f_i\rangle_{\mathrm{e}} ... + \alpha\, |e_j\rangle_{\mathrm{s}} |f_j\rangle_{\mathrm{e}} ...$$

with $\alpha = |\alpha_i| = |\alpha_j|$. The unitary

$$\mathcal{U}_{\mathrm{s}} = |e_i\rangle \langle e_j|_{\mathrm{s}} + |e_j\rangle \langle e_i|_{\mathrm{s}} \tag{3.3}$$

is envariant as the prior state can be restored using

$$\mathcal{U}_{\mathrm{e}} = |f_i\rangle \langle f_j|_{\mathrm{e}} + |f_j\rangle \langle f_i|_{\mathrm{e}} , \tag{3.4}$$

noticing that the equality of $|\alpha_i|$ and $|\alpha_j|$ was crucial.

Therefore, the unitary $\mathcal{U}_{\mathrm{s}}$ in the previous paragraph cannot physically alter the system, and there is a symmetry between the states i and j. This means, in particular, that they must have the same probability of occurrence as there is no way of physically distinguishing them.

3.3 Different Magnitudes

Consider the case now where the Schmidt coefficients are of different magnitudes. The reference [8] makes use of magnitudes of the form $\{\sqrt{m}/\sqrt{m+n}, \sqrt{n}/\sqrt{m+n}\}$, but we attempt to extend it to a more generic case.

Consider the same entangled state of system and environment as before, $|\Psi\rangle_{\mathrm{se}} = \sum_i \Psi_i |e_i\rangle_{\mathrm{s}} |f_i\rangle_{\mathrm{e}}$. Consider a fine graining of the environment by assuming that the environment states can be written as,

$$|f_i\rangle_{\mathrm{e}} = \int_{\Omega_i} \frac{1}{\Psi_i} |ix_i\rangle_{\mathrm{e}}\, dx_i, \qquad \Omega_i = [0, |\Psi_i|^2]. \tag{3.5}$$

This is a normalised state in the Hilbert space as $\langle ix|jy\rangle = \delta_{ij}\delta(x-y)$ and

$$\begin{aligned}\langle f_i|f_j\rangle_{\mathrm{e}} &= \int_{\Omega_i} \frac{1}{\Psi_i^*} \langle ix| \int_{\Omega_j} \frac{1}{\Psi_j} |jy\rangle\, dy\, dx \\ &= \delta_{ij} \frac{1}{|\Psi_i|^2} \int_{\Omega_i} dx = \delta_{ij}.\end{aligned}$$

The state of the system and environment together is now

$$|\Psi\rangle_{\mathrm{se}} = \sum_i \Psi_i |i\rangle_{\mathrm{s}} |f_i\rangle_{\mathrm{e}} = \sum_i \int_{\Omega_i} |i\rangle_{\mathrm{s}} |ix_i\rangle_{\mathrm{e}}\, dx_i, \tag{3.6}$$

with a renaming $|e_i\rangle_{\mathrm{s}} \to |i\rangle_{\mathrm{s}}$.

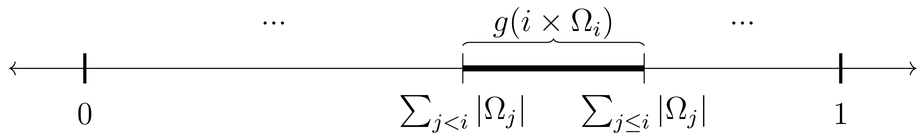
A second environment e$'$ can be added in so that the overall state becomes

$$|\Psi'\rangle_{\mathrm{e'se}} = \sum_i \int_{\Omega_i} |g(ix_i)\rangle_{\mathrm{e'}} |i\rangle_{\mathrm{s}} |ix_i\rangle_{\mathrm{e}}\, dx, \tag{3.7}$$

where g is a bijective map,

$$g : \bigcup_{i \in \mathfrak{I}} (i \times \Omega_i) \to [0,1] \subset \mathbb{R},$$

so that every pair of ix_i with $x_i \in \Omega_i$ maps to a real number between $[0,1]$. This is consistent with $\sum_i |\Omega_i| = 1$, which is enforced by the fact that the state $\Psi_{\rm se}$ is a normalised state in a Hilbert space.



Now consider the unitaries

$$\mathcal{U}_{\rm se} = |j\rangle\,|jy_j\rangle\,\langle i|\,\langle ix_i|_{\rm se} + |i\rangle\,|ix_i\rangle\,\langle j|\,\langle jy_j|_{\rm se} \tag{3.8}$$

$$\mathcal{U}_{\rm e'} = |g(jy_j)\rangle\,\langle g(ix_i)|_{\rm e'} + |g(ix_i)\rangle\,\langle g(jy_j)|_{\rm e'}\,, \tag{3.9}$$

such that

$$(\mathbb{1}_{\rm e'} \otimes \mathcal{U}_{\rm se})(\mathcal{U}_{\rm e'} \otimes \mathbb{1}_{\rm se})\,|\Psi'\rangle_{\rm e'se} = |\Psi'\rangle_{\rm e'se}\,,$$

which makes $\mathcal{U}_{se}$ an envariant operator for $|\Psi'\rangle_{\rm e'se}$.

The probability to be in state k is given by the sum of the probabilities for being in all states of equation 3.7 corresponding to $|k\rangle_{\rm s}$, that is,

$$\int_{\Omega_k} |g(kx_k)\rangle_{\rm e'}\,|k\rangle_{\rm s}\,|kx_k\rangle_{\rm e}\;dx,$$

a set of measure $\int_{\Omega_k} dx = |\Omega_k| = |\Psi_k|^2$ and one therefore has the Born rule!

Another way to see this is to compare the measures of states $|k\rangle_{\rm s}$ and $|k'\rangle_{\rm s}$. As the overall state

$$|\Psi'\rangle_{\rm e'se} = \sum_i \int_{\Omega_i} |g(ix_i)\rangle_{\rm e'}\,|i\rangle_{\rm s}\,|ix_i\rangle_{\rm e}$$

is symmetric, with equal absolute values of Schmidt coefficients, the relative measures between states corresponding to $|k\rangle_{\rm s}$ and $|k'\rangle_{\rm s}$ are in the ratio

$$|\Omega_k| : |\Omega_{k'}| = |\Psi_k|^2 : |\Psi_{k'}|^2. \tag{3.10}$$

Moreover, the probabilities all add to 1 as $\sum_i |\Omega_i| = 1$. As a result,

$$\mathcal{P}(i) = |\Omega_i| = |\Psi_i|^2 \tag{3.11}$$

gives the Born probabilities consistently!

The $|\quad|^2$ rule for probabilities comes about because of the symmetry arguments of the previous subsection as well as because the quantum states are defined on a Hilbert space with unit norm in the square inner product. In fact, requiring linearity for an inner product already implies the square norm.

Section 4

Double Slit Experiment and Quantum Eraser

Our analysis so far has concentrated on the measurement problem in quantum mechanics where we have developed a framework that avoids wavefunction collapse and instead explains measurement in a completely unitary manner. In this section we analyse the double slit experiment using our framework and thereby address what happens when there is interference. As opposed to an electron we would use a photon for this analysis but the physics remains the same. It shall be seen that merely finding out which slit the photon passes through destroys the interference pattern, as predicted by Feynman [17].

We first outline the essential ideas using a discrete version of the experiment and later extend it to the continuous case. Finally, we analyse the delayed choice quantum eraser experiment [18] in our framework.

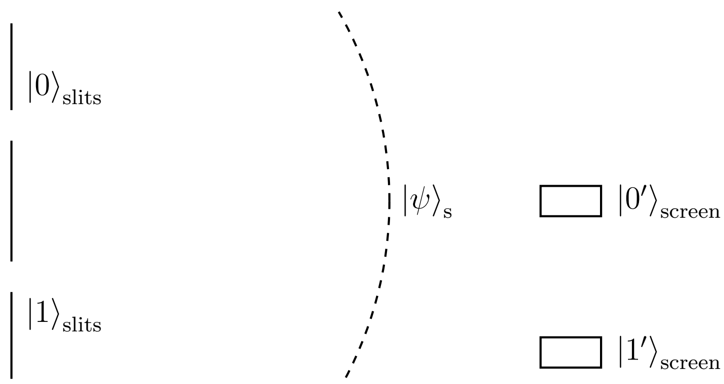
4.1 Discrete Version

We first consider a discrete version of the double slit experiment, restricting to qubits. One could imagine that the screen in the double slit experiment is just two detectors conveniently placed to observe interference effects. It could also be imagined as a screen where all points other than two special ones are ignored. The main ideas are developed using this simplified system and extension to a continuous screen is explored in the next subsection.

Consider the system, a photon, being measured,

$$|\psi\rangle_{\rm s} = \psi_0 \, |0\rangle_{\rm s} + \psi_1 \, |1\rangle_{\rm s} \, ,$$

where $|0\rangle_{\rm s}$ corresponds to the photon being at slit 0 and $|1\rangle_{\rm s}$ corresponds to the photon being at slit 1.



There are two separate observers, one corresponding to the slits

$$|\chi\rangle_{\rm slits} = \chi_0 \, |0\rangle_{\rm slits} + \chi_1 \, |1\rangle_{\rm slits} \, ,$$

and one to the screen

$$|\chi'\rangle_{\rm screen} = {\chi'}_{0'} \, |0'\rangle_{\rm screen} + {\chi'}_{1'} \, |1'\rangle_{\rm screen} \, .$$

The screen state is written in the $\{|0'\rangle \, , |1'\rangle\}$ basis to indicate that it should be distinguished from the basis for the slits. The slits and screen states must have environment states into which they can dispose off their prior states (as in section 2) but this will be suppressed in the following analysis as it is not relevant here.

Consider the evolution of the quantum state of the system from the slits to the screen,

$$\begin{aligned} |\psi\rangle_{\rm s} \to |\psi'\rangle_{\rm s} &= \mathcal{U}_{\rm s} \, |\psi\rangle_{\rm s} \\ &= {\psi'}_{0'} \, |0'\rangle_{\rm s} + {\psi'}_{1'} \, |1'\rangle_{\rm s} \\ &= \frac{\psi_0 + \psi_1}{\sqrt{2}} \, |0'\rangle_{\rm s} + \frac{\psi_0 - \psi_1}{\sqrt{2}} \, |1'\rangle_{\rm s} \, , \end{aligned}$$

reflecting what happens in the interference experiment. The wavefronts meet constructively at $0'$, therefore $\psi_0 + \psi_1$, and destructively at $1'$, therefore $\psi_0 - \psi_1$. Thus, the evolution unitary is given by

$$\begin{aligned}\mathcal{U}_{\mathrm{s}} =& \frac{1}{\sqrt{2}} |0'\rangle_{\mathrm{s}} \langle 0|_{\mathrm{s}} + \frac{1}{\sqrt{2}} |0'\rangle_{\mathrm{s}} \langle 1|_{\mathrm{s}} \\ &+ \frac{1}{\sqrt{2}} |1'\rangle_{\mathrm{s}} \langle 0|_{\mathrm{s}} - \frac{1}{\sqrt{2}} |1'\rangle_{\mathrm{s}} \langle 1|_{\mathrm{s}} \,. \end{aligned} \tag{4.1}$$

4.1.1 No collapse at the slits

The first case considered is when there is no collapse of the wavefunction at the slits so that it could be considered that there is no observer at the slits or that the observer there was unsuccessful at measuring the wavefunction of the slits. In this case the system simply passes through the slits and an interference pattern would be observed.

Starting at the initial product state the system passes through the slits, evolves according to the unitary $\mathcal{U}_{\mathrm{s}}$, and finally "collapses" at the screen[1],

$$\begin{aligned} & |\chi\rangle_{\mathrm{slits}} |\psi\rangle_{\mathrm{s}} |\chi'\rangle_{\mathrm{screen}} \\ \overset{\text{no collapse at slits}}{\longrightarrow} \quad & |\chi\rangle_{\mathrm{slits}} |\psi\rangle_{\mathrm{s}} |\chi'\rangle_{\mathrm{screen}} \\ \overset{\mathbb{1}_{\mathrm{slits}}\otimes\mathcal{U}_{\mathrm{s}}\otimes\mathbb{1}_{\mathrm{screen}}}{\longrightarrow} \quad & |\chi\rangle_{\mathrm{slits}} |\psi'\rangle_{\mathrm{s}} |\chi'\rangle_{\mathrm{screen}} \\ \overset{\text{collapse at screen}}{\longrightarrow} \quad & |\chi\rangle_{\mathrm{slits}} \left({\psi'}_{0'} |0'\rangle_{\mathrm{s}} |0'\rangle_{\mathrm{screen}} + {\psi'}_{1'} |1'\rangle_{\mathrm{s}} |1'\rangle_{\mathrm{screen}} \right) . \end{aligned} \tag{4.2}$$

The experimenter only has access to the screen observer and therefore the other states have to be traced out in order to see what happens there,

$$\begin{aligned}\rho_{\mathrm{screen}} &= |{\psi'}_{0'}|^2 |0'\rangle \langle 0'|_{\mathrm{screen}} + |{\psi'}_{1'}|^2 |1'\rangle \langle 1'|_{\mathrm{screen}} \\ &= \frac{|\psi_0 + \psi_1|^2}{2} |0'\rangle \langle 0'|_{\mathrm{screen}} + \frac{|\psi_0 - \psi_1|^2}{2} |1'\rangle \langle 1'|_{\mathrm{screen}} \,, \end{aligned} \tag{4.3}$$

and so interference emerges! The justification for using a density matrix can be found in section 3 where the Born rule is discussed.

[1]Note that there must be a screen environment state that absorbs the prior state of the screen as in section 2 but this is suppressed as it isn't relevant for the analysis. In reality, there is no collapse but only entanglement.

4.1.2 Collapse at the slits

In case the wavefunction does collapse at the slits, the evolution of the state is quite different.

The initial product state already gets entangled with the slits observer, the system then evolves according to the unitary $\mathcal{U}_\mathrm{s}$, and finally there is a collapse at the screen as before,

$$
\begin{aligned}
& \left|\chi\right\rangle_\mathrm{slits}\left|\psi\right\rangle_\mathrm{s}\left|\chi'\right\rangle_\mathrm{screen} \\
\xrightarrow{\text{collapse at slits}} \quad & \left(\psi_0\left|0\right\rangle_\mathrm{slits}\left|0\right\rangle_\mathrm{s}+\psi_1\left|1\right\rangle_\mathrm{slits}\left|1\right\rangle_\mathrm{s}\right)\left|\chi'\right\rangle_\mathrm{screen}, \\
\xrightarrow{\mathbb{1}_\mathrm{slits}\otimes\mathcal{U}_\mathrm{s}\otimes\mathbb{1}_\mathrm{screen}} \quad & \Bigg[\psi_0\left|0\right\rangle_\mathrm{slits}\left(\frac{1}{\sqrt{2}}\left|0'\right\rangle_\mathrm{s}+\frac{1}{\sqrt{2}}\left|1'\right\rangle_\mathrm{s}\right) \\
& +\psi_1\left|1\right\rangle_\mathrm{slits}\left(\frac{1}{\sqrt{2}}\left|0'\right\rangle_\mathrm{s}-\frac{1}{\sqrt{2}}\left|1'\right\rangle_\mathrm{s}\right)\Bigg]\left|\chi'\right\rangle_\mathrm{screen} \\
= & \Bigg[\left(\frac{\psi_0}{\sqrt{2}}\left|0\right\rangle_\mathrm{slits}+\frac{\psi_1}{\sqrt{2}}\left|1\right\rangle_\mathrm{slits}\right)\left|0'\right\rangle_\mathrm{s} \\
& +\left(\frac{\psi_0}{\sqrt{2}}\left|0\right\rangle_\mathrm{slits}-\frac{\psi_1}{\sqrt{2}}\left|1\right\rangle_\mathrm{slits}\right)\left|1'\right\rangle_\mathrm{s}\Bigg]\left|\chi'\right\rangle_\mathrm{screen} \\
\xrightarrow{\text{collapse at screen}} \quad & \left(\frac{\psi_0}{\sqrt{2}}\left|0\right\rangle_\mathrm{slits}+\frac{\psi_1}{\sqrt{2}}\left|1\right\rangle_\mathrm{slits}\right)\left|0'\right\rangle_\mathrm{s}\left|0'\right\rangle_\mathrm{screen} \\
& +\left(\frac{\psi_0}{\sqrt{2}}\left|0\right\rangle_\mathrm{slits}-\frac{\psi_1}{\sqrt{2}}\left|1\right\rangle_\mathrm{slits}\right)\left|1'\right\rangle_\mathrm{s}\left|1'\right\rangle_\mathrm{screen}.
\end{aligned} \tag{4.4}
$$

Performing the partial trace this times gives a different result

$$
\begin{aligned}
\rho_\mathrm{screen} &= \frac{\left|\psi_0\right|^2+\left|\psi_1\right|^2}{2}\left|0'\right\rangle\left\langle 0'\right|_\mathrm{screen}+\frac{\left|\psi_0\right|^2+\left|\psi_1\right|^2}{2}\left|1'\right\rangle\left\langle 1'\right|_\mathrm{screen} \\
&= \frac{1}{2}\left|0'\right\rangle\left\langle 0'\right|_\mathrm{screen}+\frac{1}{2}\left|1'\right\rangle\left\langle 1'\right|_\mathrm{screen},
\end{aligned} \tag{4.5}
$$

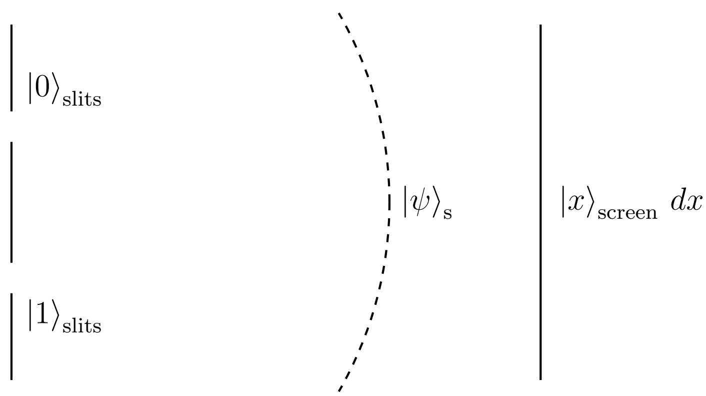
and so the interference pattern is destroyed! It is seen that the mere act of the slits observer observing the system causes the interference pattern to be destroyed.

4.2 Continuous Version

In this section we extend the analysis to a continuous screen. This means that the basis used to describe the photon state as it reaches the screen is now continuous; $|x\rangle_{\text{screen}}$ indicates position on the screen where the photon reaches. For example, the state of the screen before the experiment could be written as

$$|\chi'\rangle_{\text{screen}} = \int \chi'(x)\,|x\rangle_{\text{screen}}\;dx$$

so that an integral instead of a sum is used to write out the state of the screen.



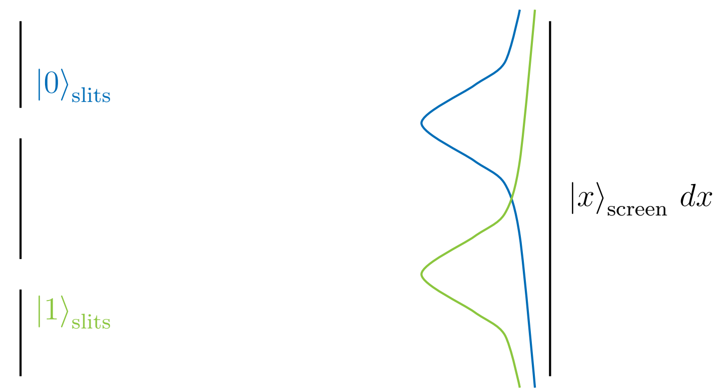
4.2.1 Evolution of the photon state

Consider the evolution of the photon wavefunction starting from slit 0,

$$|0\rangle_{\text{s}} \to |\text{u}_0\rangle_{\text{s}} = \mathcal{U}_{\text{s}}\,|0\rangle_{\text{s}} = \int \text{u}_0(x)\,|x\rangle_{\text{s}}\;dx,$$

indicating that the amplitude is distributed as $\text{u}_0(x)$ on the screen. Similarly, a photon starting at slit 1 evolves as

$$|1\rangle_{\text{s}} \to |\text{u}_1\rangle_{\text{s}} = \mathcal{U}_{\text{s}}\,|1\rangle_{\text{s}} = \int \text{u}_1(x)\,|x\rangle_{\text{s}}\;dx.$$



$|u_0(x)|^2$ *is indicated by blue whereas* $|u_1(x)|^2$ *is indicated by green.*

By linearity, an arbitrary quantum state evolves as

$$|\psi\rangle_s \to |\psi'\rangle_s = \mathcal{U}_s |\psi\rangle_s = \int (\psi_0 u_0(x) + \psi_1 u_1(x)) |x\rangle_s \, dx \tag{4.6}$$

and therefore, the unitary $\mathcal{U}_s$ can be expressed as

$$\mathcal{U}_s = |u_0\rangle_s \langle 0|_s + |u_1\rangle_s \langle 1|_s = \int (u_0(x) |x\rangle_s \langle 0|_s + u_1(x) |x\rangle_s \langle 1|_s) \, dx. \tag{4.7}$$

To be unitary it is required that $|u_0\rangle_s$ and $|u_1\rangle_s$ are orthogonal to each other. We attempt to argue this physically. Using the Born rule (section 3) it is seen that

$$\langle u_i|u_i\rangle_s = \int u_i^*(x) u_i(x) \, dx \stackrel{!}{=} 1,$$

simply indicating that the states must be normalised as per axiom 1.1 of section 1. If this equation is multiplied by number of particles or number of particles per second, one gets frequency and intensity respectively.

Thus, using intensity conservation it can be physically argued that the intensity at the slits and at the screen must be the same with the interpretation that the total intensity is redistributed along the screen. Therefore, we argue that

$$\begin{aligned} \langle\psi|\psi\rangle_s &\stackrel{!}{=} \langle\psi'|\psi'\rangle_s \\ \implies |\psi_0|^2 + |\psi_1|^2 &\stackrel{!}{=} (\psi_0^* \langle u_0| + \psi_1^* \langle u_1|)(\psi_0 |u_0\rangle + \psi_1 |u_1\rangle) \\ \implies 0 &\stackrel{!}{=} \psi_0^* \psi_1 \langle u_0|u_1\rangle + \psi_1^* \psi_0 \langle u_1|u_0\rangle \end{aligned}$$

$$= \mathrm{Re}\{\psi_0^* \psi_1 \langle \mathrm{u}_0 | \mathrm{u}_1 \rangle\}$$

for any $\{\psi_0, \psi_1\}$.

Again, motivated physically one can introduce $\psi_0 = 1$ and $\psi_1 = e^{i\delta}$ to indicate a phase shift of δ between the slits so that it is required that

$$\mathrm{Re}\Big\{e^{i\delta} \langle \mathrm{u}_0 | \mathrm{u}_1 \rangle\Big\} = 0$$

for arbitrary δ. This can only be true if $\langle \mathrm{u}_0 | \mathrm{u}_1 \rangle$ is identically 0 as $\langle \mathrm{u}_0 | \mathrm{u}_1 \rangle$ is a complex number and therefore $e^{i\delta}$ rotates it by the angle δ in the complex plane.

4.2.2 No collapse at slits

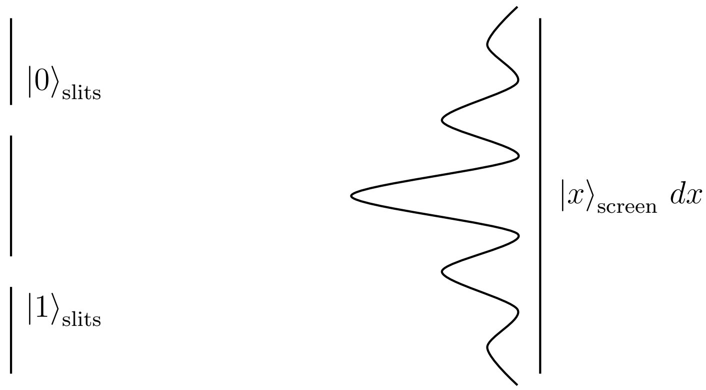
The first step is exactly the same as in the discrete case, the slits do not play a part in the analysis and the slits quantum state simply factorises. In the second step the evolution of the photon wavefunction is slightly different, and so is the collapse,

$$\begin{aligned}
& |\chi\rangle_{\mathrm{slits}} |\psi\rangle_{\mathrm{s}} |\chi'\rangle_{\mathrm{screen}} \\
\overset{\text{no collapse at slits}}{\longrightarrow} \quad & |\chi\rangle_{\mathrm{slits}} |\psi\rangle_{\mathrm{s}} |\chi'\rangle_{\mathrm{screen}} \\
\overset{\mathbb{1}_{\mathrm{slits}} \otimes \mathcal{U}_{\mathrm{s}} \otimes \mathbb{1}_{\mathrm{screen}}}{\longrightarrow} \quad & |\chi\rangle_{\mathrm{slits}} |\psi'\rangle_{\mathrm{s}} |\chi'\rangle_{\mathrm{screen}} \\
= \; & |\chi\rangle_{\mathrm{slits}} \left(\int (\psi_0 \mathrm{u}_0(x) + \psi_1 \mathrm{u}_1(x)) |x\rangle_{\mathrm{s}} \, dx \right) |\chi'\rangle_{\mathrm{screen}} \\
\overset{\text{collapse at screen}}{\longrightarrow} \quad & |\chi\rangle_{\mathrm{slits}} \int (\psi_0 \mathrm{u}_0(x) + \psi_1 \mathrm{u}_1(x)) |x\rangle_{\mathrm{s}} |x\rangle_{\mathrm{screen}} \, dx.
\end{aligned} \tag{4.8}$$

As before, the slits play no role here and the reduced density matrix reads

$$\rho_{\mathrm{screen}} = \int |\psi_0 \mathrm{u}_0(x) + \psi_1 \mathrm{u}_1(x)|^2 \, |x\rangle \langle x|_{\mathrm{screen}} \, dx; \tag{4.9}$$

interference is observed.



$|\psi_0 \mathrm{u}_0(x) + \psi_1 \mathrm{u}_1(x)|^2$ *gives rise to an interference pattern.*

4.2.3 Collapse at the slits

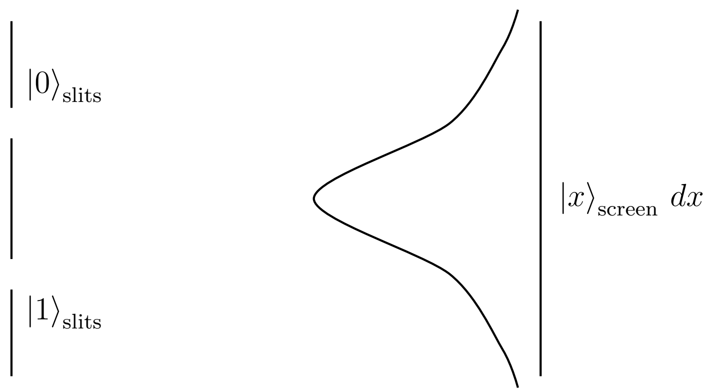
In this case there is already a collapse at the slits, the system then evolves according to the unitary $\mathcal{U}_{\text{s}}$, and there is a collapse at the screen as always,

$$\begin{aligned}
& |\chi\rangle_{\text{slits}} |\psi\rangle_{\text{s}} |\chi'\rangle_{\text{screen}} \\
\overset{\text{collapse at slits}}{\longrightarrow} \quad & (\psi_0 |0\rangle_{\text{slits}} |0\rangle_{\text{s}} + \psi_1 |1\rangle_{\text{slits}} |1\rangle_{\text{s}}) |\chi'\rangle_{\text{screen}} \\
\overset{\mathbb{1}_{\text{slits}} \otimes \mathcal{U}_{\text{s}} \otimes \mathbb{1}_{\text{screen}}}{\longrightarrow} \quad & (\psi_0 |0\rangle_{\text{slits}} |\mathrm{u}_0\rangle_{\text{s}} + \psi_1 |1\rangle_{\text{slits}} |\mathrm{u}_1\rangle_{\text{s}}) |\chi'\rangle_{\text{screen}} \\
= & \left(\int (\psi_0 \mathrm{u}_0(x) |0\rangle + \psi_1 \mathrm{u}_1(x) |1\rangle)_{\text{slits}} |x\rangle_{\text{s}} \, dx \right) |\chi'\rangle_{\text{screen}} \\
\overset{\text{collapse at screen}}{\longrightarrow} \quad & \int (\psi_0 \mathrm{u}_0(x) |0\rangle + \psi_1 \mathrm{u}_1(x) |1\rangle)_{\text{slits}} |x\rangle_{\text{s}} |x\rangle_{\text{screen}} \, dx. \qquad (4.10)
\end{aligned}$$

Performing the partial trace this times gives,

$$\rho_{\text{screen}} = \int \left(|\psi_0 \mathrm{u}_0(x)|^2 + |\psi_1 \mathrm{u}_1(x)|^2 \right) |x\rangle \langle x|_{\text{screen}} \, dx \qquad (4.11)$$

and so the interference pattern is destroyed once again!



$|\psi_0 \mathrm{u}_0(x)|^2 + |\psi_1 \mathrm{u}_1(x)|^2$ *gives no interference pattern but instead a washed out pattern.*

4.3 Quantum Eraser

As seen in the previous sections if a slits observer successfully observes the state of the photon as it passes through the slits, the interference pattern is destroyed. In a quantum eraser experiment one attempts to undo this observation and make available the interference pattern once again. In case the erasure is performed after the photon is incident on the screen one has the delayed choice quantum eraser experiment. Our discussion is influenced by the papers [18, 19, 20, 21, 22, 23, 24].

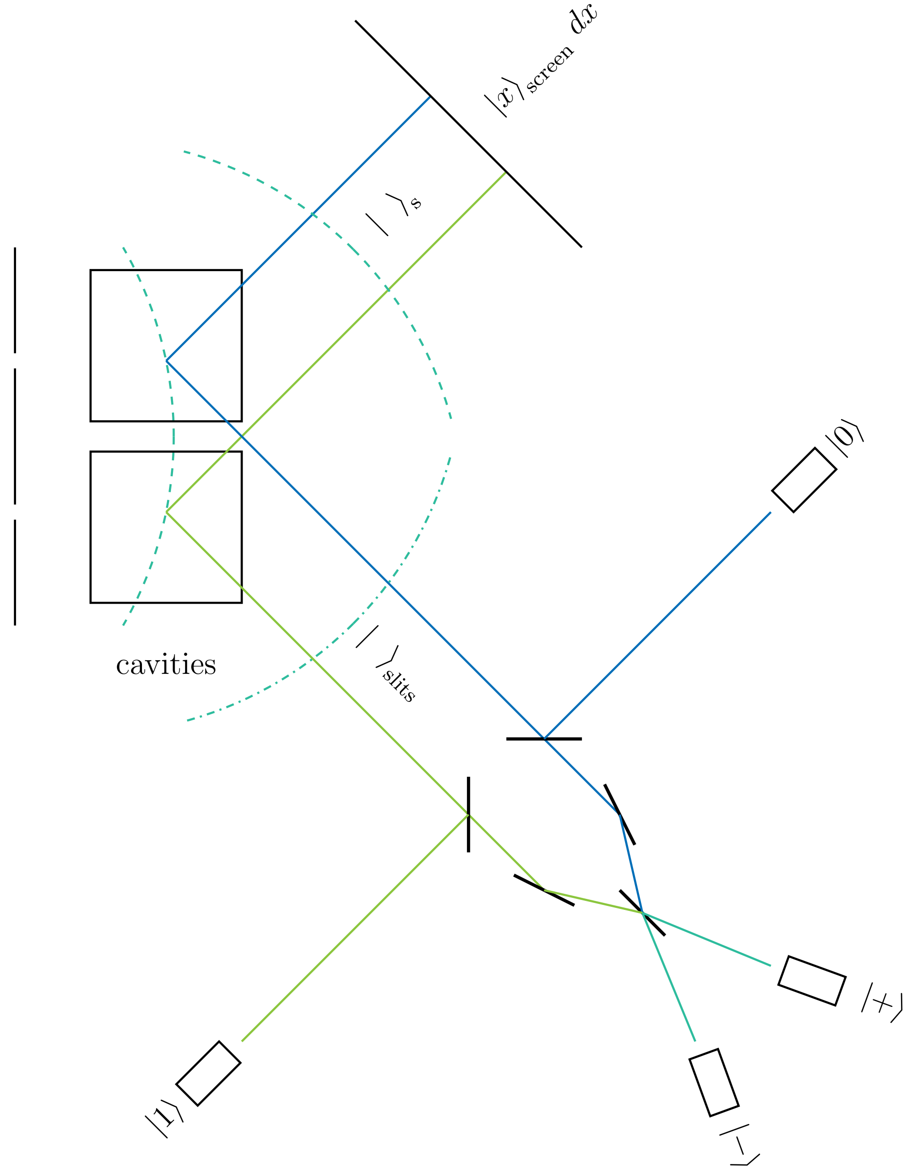
4.3.1 The experiment

In this subsection we explain the schematic of a quantum eraser experiment using photons passing through a double-slit. The same ideas apply to experiments performed with atoms, kaons or using entanglement (see [24] for an overview). We explain a version with a continuous screen but the experiment can also be performed with a Mach-Zehnder interferometer as explained in [23].

A photon passes through a double slit apparatus and is incident on a pair of cavities. This photon then creates a pair of entangled photons using some process; for example, a parametric-down convertor is used by [19]. The pair of entangled photons take separate paths onto different apparatuses; the "first" photon $|\ \rangle_{\mathrm{s}}$ (shown as moving diagonally up in the diagram below) is incident on a screen and produces a pattern while the "second" photon $|\ \rangle_{\text{slits}}$ (moving diagonally down in the diagram below) passes through a set of detectors. The cavities are localised and thereby provide “which-way” information about the entangled photons destroying the interference pattern created by the first photon and instead giving rise to a washed out pattern as seen in the previous subsections. More specifically, detectors for the second photon would be able to tell which of the cavities it originated from and as the first photon is entangled with the second photon, the origin of the first photon is also accessible.

The quantum eraser experiment tries to “erase” this which-way information and once again make available the interference pattern. This is achieved by using a different basis to measure the second photon, as we show in detail in following. The delayed choice experiment involves increasing the path length of the second photon to being larger than that of the first photon. In this case the first photon is already incident on the screen before the second photon has been measured by a detector (any of the four detectors in the bottom half of the diagram below).

A key point to note is that the interference pattern on the screen emerges only after filtering out some of the photons incident on the screen. If the photons corresponding to detectors $|0\rangle$ or $|1\rangle$ are kept, they would show no interference pattern but only a washed out pattern. However, in case the photons corresponding to detector $|+\rangle$ is kept, an interference pattern emerges! And this is true irrespective of the path length of the second photon, therefore seemingly indicating that the decision for the first photon to form an interference pattern was influenced retrocausally by the measurement in the $\{|+\rangle, |-\rangle\}$ basis. We aim to show in the following that an explanation can be reached without resorting to retrocausality. A measurement by the $|-\rangle$ detector would lead to formation of an “anti-interference” pattern which is also an interference pattern but which together with the interference pattern of $|+\rangle$ forms a washed out pattern.



The system $|\ \ \rangle_{\text{s}}$ moves through the double slits onto the cavities where there it gets entangled with the $|\ \ \rangle_{\text{slits}}$ state (slits collapse). The entangled system $|\ \ \rangle_{\text{s}}|\ \ \rangle_{\text{slits}}$ then gets entangled with the screen state (screen collapse). The slits state can be measured in the $\{|0\rangle, |1\rangle\}$ basis or the $\{|+\rangle, |-\rangle\}$ basis which lead to differing results.

The delayed choice quantum eraser experiment is a source of intense debate because of the fact that it seems to imply retrocausality of physics. As we show in the following subsections, a purely causal explanation for the experiment is possible and is the view espoused by the authors of [18, 19, 20]. However there are different interpretations proposed as in [25, 26].

4.3.2 Erasing the measurement

The photon passes through the slits and to the cavities producing an entangled pair. At this stage the which-way information has been recorded and as a result the first photon $|\ \rangle_{\mathrm{s}}$ would not show an interference pattern. The second photon $|\ \rangle_{\mathrm{slits}}$ travels towards detectors and it could be arranged to erase the which-way information and once again make available the interference pattern.

The entangling apparatus produces, just as in equation 4.10,

$$\begin{aligned} &|\chi\rangle_{\mathrm{slits}} |\psi\rangle_{\mathrm{s}} |\chi'\rangle_{\mathrm{screen}} \\ &\qquad \to \int \left(\psi_0 \mathrm{u}_0(x) |0\rangle_{\mathrm{slits}} + \psi_1 \mathrm{u}_1(x) |1\rangle_{\mathrm{slits}}\right) |x\rangle_{\mathrm{s}} |x\rangle_{\mathrm{screen}}\, dx, \end{aligned} \tag{4.12}$$

and exactly the exact density matrix (equation 4.11) as before is obtains. Notice that this corresponds to measuring $|\ \rangle_{\mathrm{slits}}$ in the $\{|0\rangle, |1\rangle\}$ basis.

The erasure is performed by observing in the $\{|+\rangle, |-\rangle\}$ basis. Writing equation 4.12 as

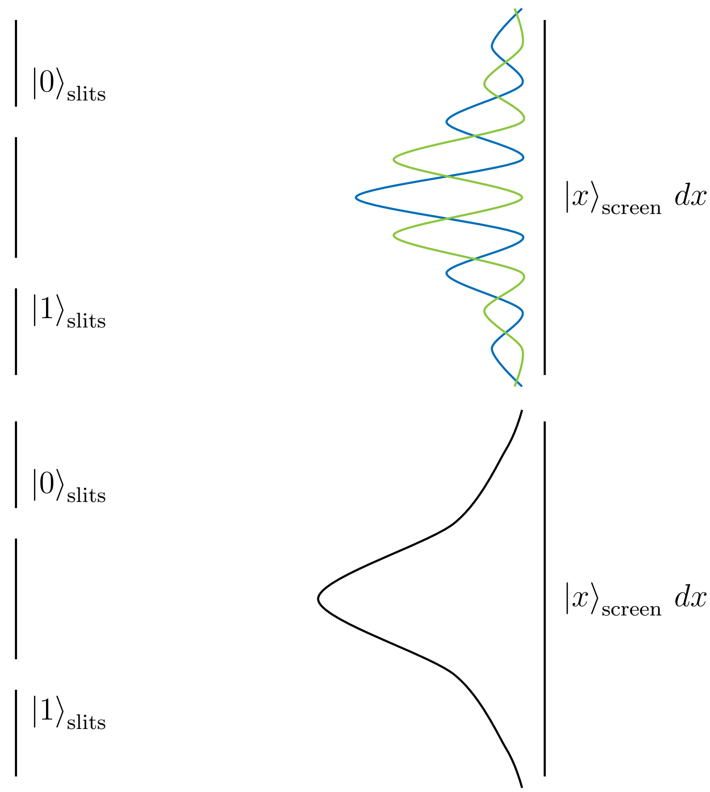
$$\begin{aligned} &= \int \left(\psi_0 \mathrm{u}_0(x) \frac{|+\rangle_{\mathrm{slits}} + |-\rangle_{\mathrm{slits}}}{\sqrt{2}} + \psi_1 \mathrm{u}_1(x) \frac{|+\rangle_{\mathrm{slits}} - |-\rangle_{\mathrm{slits}}}{\sqrt{2}}\right) |x\rangle_{\mathrm{s}} |x\rangle_{\mathrm{screen}}\, dx \\ &= \int \left(\frac{\psi_0 \mathrm{u}_0(x) + \psi_1 \mathrm{u}_1(x)}{\sqrt{2}} |+\rangle_{\mathrm{slits}} + \frac{\psi_0 \mathrm{u}_0(x) - \psi_1 \mathrm{u}_1(x)}{\sqrt{2}} |-\rangle_{\mathrm{slits}}\right) |x\rangle_{\mathrm{s}} |x\rangle_{\mathrm{screen}}\, dx, \end{aligned}$$

it is seen that there is a sign of the interference pattern emerging.

Note that nothing weird has happened yet, the density matrix is exactly the same as before. However, if only the photons corresponding to $|+\rangle_{\mathrm{slits}}$ (or equivalently $|-\rangle_{\mathrm{slits}}$) are observed, an interference (anti-interference) pattern is obtained,

$$\rho_{\mathrm{screen}} \sim \int \frac{|\psi_0 \mathrm{u}_0(x) + (-)\psi_1 \mathrm{u}_1(x)|^2}{2} |x\rangle \langle x|_{\mathrm{screen}}\, dx! \tag{4.13}$$

The $\frac{1}{2}$ makes sense as one is only observing one of the two branches of the universe; the properly normalised density matrix would not have this factor. One could alternatively say that the intensity is halved because half the photons are being rejected. Adding together the interference and anti-interference patterns one would get the washed out pattern.



The interference pattern in blue, $|\psi_0 \mathrm{u}_0(x) + \psi_1 \mathrm{u}_1(x)|^2/2$ *and anti-interference pattern in green,* $|\psi_0 \mathrm{u}_0(x) - \psi_1 \mathrm{u}_1(x)|^2/2$ *added together give rise to the washed out pattern,* $|\psi_0 \mathrm{u}_0(x)|^2 + |\psi_1 \mathrm{u}_1(x)|^2$.

Therefore, the quantum eraser experiment can be explained without any reference to retrocausality. The mathematics leads the way in coming up with an explanation.

Section 5

The Index Set

The index set, $\mathfrak{I}(\mathcal{H})$, was introduced in section 2.2 to extend the procedure of entanglement from 2-state qubit systems to d-state systems. We study the index set in more detail in this section with the aim of generalising it as much as possible and to observe what algebra is forced on it. The idea is motivated starting from a physical situation, then the algebra of this set is explored, mainly through example. The section ends with an exploration of a topology for the index set.

5.1 Labelling

Choosing a suitable labelling for the quantum states is a somewhat subtle problem. Consider the canonical electron spin system that has been considered before. In this case labelling the states as

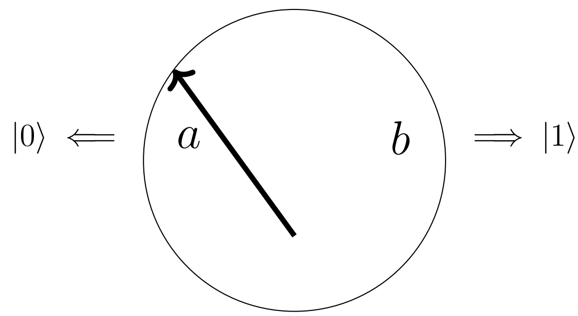
- spin up $\leftrightarrow |0\rangle$
- spin down $\leftrightarrow |1\rangle$,

is suitable as this is the relevant feature of the electron that is being measured. Even though the electron has other quantum numbers such as, for example, a lepton number, this is not a relevant feature considered in the electron spin analysis and is therefore not required to be considered in the labelling process.

However, for the labelling to be useful, more is required: distinguishability between quantum states belonging to different labels. Thus, there has to be an observable $\mathcal{M}$ on the relevant Hilbert space with the given labelled states as eigenvectors and with distinct eigenvalues, for example,

$$\mathcal{M} = a\,|0\rangle\,\langle 0| + b\,|1\rangle\,\langle 1|\,, \tag{5.1}$$

so that the two states $|0\rangle$ and $|1\rangle$ are distinguishable with some meter indicating a and b respectively for the two states.



The pointer landing left refers to the meter reading 'a' (meaning the state is $|0\rangle$) and it landing right corresponds to the meter reading 'b' (meaning the state is $|1\rangle$).

The choice of Hilbert space and set of relevant observables is interconnected. Observables on a Hilbert space are Hermitian operators, $\mathcal{M}^\dagger = \mathcal{M}$, so that they possess a spanning orthonormal eigenbasis with real eigenvalues, $\mathcal{H} = \text{span}\{|e_m\rangle\}$ with $\mathcal{M}\,|e_m\rangle = m\,|e_m\rangle$ and $\langle e_m|e_n\rangle = \delta_{mn}$.

In general, there may be eigenvalues of $\mathcal{M}$ whose eigenspaces are more than one dimensional. In this case, if the states are to be distinguishable there must be other observables that are able to distinguish between states in degenerate eigenspaces. Consider, therefore, a vector of commuting observables $(\mathcal{M}_\alpha) = (\mathcal{M}_0, \mathcal{M}_1, ...)$[1] that are able to distinguish between the relevant states of the Hilbert space to whatever granularity required,

$$(\mathcal{M}_\alpha)\,|e_m\rangle = (m_\alpha)\,|e_m\rangle\,. \tag{5.2}$$

The vectors of real numbers (m_α) and (n_α) corresponding to distinct states $|e_m\rangle$ and $|e_n\rangle$, respectively, are different from each other,

$$|e_m\rangle \neq |e_n\rangle \implies (m_\alpha) \neq (n_\alpha)$$

[1]Here the observables are assumed to commute with each other ($\forall \alpha, \beta, [\mathcal{M}_\alpha, \mathcal{M}_\beta] = 0$), this is because what is required is a set of observables all simultaneously observable that can distinguish between each of the basis states. The subtlety of non-commuting observables is not an issue here.

so that the vector of meter readings would help distinguish between the two states. In other words, if two states are indistinguishable within this context, they would have the same label. Contrarily, the Hilbert space itself could be defined with respect to the vector of observables. In this case, the set of observables is what is first defined and then the Hilbert space is obtained as the span of these basis vectors with distinct values of observables. Quantum mechanics says that the physical state of a system in such a Hilbert space could be in a superposition.

If the Hilbert space is finite, each of the vector of eigenvalues (m_α) could be uniquely mapped to natural numbers m so that the states can simply be labelled as, $|e_m\rangle = |m\rangle$; this has been implicitly assumed in the other sections. The index set is the set of these labels, $\mathfrak{I}(\mathcal{H}) = \{m\}$. There may be some additional structure on the index set such as an additive group structure induced by $+$ in section 2.2; this is explored further in the next subsections.

5.2 Algebra of the Index Set

In section 2.2 the entanglement was performed using a generalisation of the CNOT operator in quantum information theory and it was, therefore, assumed that the algebra of the index set was commutative with operation $+$. This section explores to what extent this can be generalised.

Restricting to a finite index set, $\mathfrak{I}$[2], equation 2.11 generalises to

$$\mathcal{I}_{\mathrm{a}\to\mathrm{b}} \left|i\right\rangle_{\mathrm{a}} \left|j\right\rangle_{\mathrm{b}} = \left|i\right\rangle_{\mathrm{a}} \left|j \cdot i\right\rangle_{\mathrm{b}} , \tag{5.3}$$

where $\cdot$ is a function

$$\cdot : \mathfrak{I} \times \mathfrak{I} \to \mathfrak{I}.$$

What are the requirements that $\cdot$ needs to satisfy? In order for $\mathcal{I}$ to be unitary, distinct j's need to map to distinct $j{\cdot}i$'s, for a fixed i. This can be justified by looking at equation 5.3. If $j_1 \neq j_2$,

$$\langle j_1 | j_2 \rangle \overset{!}{=} 0,$$

which means that

$$\langle j_1 \cdot i | j_2 \cdot i \rangle \overset{!}{=} 0$$

[2]Here $\mathfrak{I}(\mathcal{H})$ is abbreviated as $\mathfrak{I}$ for brevity; some Hilbert space $\mathcal{H}$ is implicit.

if $\mathcal{I}$ is to be unitary. Note that as i is a constant here, it doesn't play a role in determining the unitarity.

And thus, $\cdot i$ must be a bijection for each i. Note that this also indicates that the action $\mathcal{I}$ is invertible with

$$ {}_R\mathcal{I}^{-1}_{\mathrm{a}\to\mathrm{b}} \left|i\right\rangle_{\mathrm{a}} \left|j \cdot i\right\rangle_{\mathrm{b}} := \left|i\right\rangle_{\mathrm{a}} \left|j\right\rangle_{\mathrm{b}} \tag{5.4} $$

being well defined. Here the ${}_R$ is used to indicate that it is a right inverse, which is the relevant use case in our current discussion of invertibility. A left inverse can also be defined as

$$ {}_L\mathcal{I}^{-1}_{\mathrm{a}\to\mathrm{b}} \left|i\right\rangle_{\mathrm{a}} \left|i \cdot j\right\rangle_{\mathrm{b}} := \left|i\right\rangle_{\mathrm{a}} \left|j\right\rangle_{\mathrm{b}} . \tag{5.5} $$

This would be of use later while defining the disentanglement operator for generic action $\cdot$.

If $j \cdot$ is not bijective, unitarity can still be guaranteed, as $\langle i_1 | i_2 \rangle = 0$ before and after application of the unitary $\mathcal{I}$. However, in this case the entanglement is not proper. Say, i_1 and i_2 are such that $j \cdot i_1 = k = j \cdot i_2$, implying that after the imprinting procedure, the entangled state is

$$ \begin{aligned} ...\Psi_{i_1} \left|i_1\right\rangle_{\mathrm{a}} \left|j\right\rangle_{\mathrm{b}} + ...\Psi_{i_2} \left|i_2\right\rangle_{\mathrm{a}} \left|j\right\rangle_{\mathrm{b}} &\overset{\mathcal{I}_{\mathrm{a}\to\mathrm{b}}}{\to} ...\Psi_{i_1} \left|i_1\right\rangle_{\mathrm{a}} \left|k\right\rangle_{\mathrm{b}} + ...\Psi_{i_2} \left|i_2\right\rangle_{\mathrm{a}} \left|k\right\rangle_{\mathrm{b}} \\ &= \left(\Psi_{i_1} \left|i_1\right\rangle_{\mathrm{a}} + \Psi_{i_2} \left|i_2\right\rangle_{\mathrm{a}}\right) \left|k\right\rangle_{\mathrm{b}} + ..., \end{aligned} $$

further implying that the subspace corresponding to i_1 and i_2 is not properly entangled (appendix A). Thus, it is supposed that $j \cdot$ is also a bijection.

This means that for all i and j there exist unique x and y such that

$$ i \cdot y = j \qquad\qquad x \cdot i = j. $$

This is the structure of a quasigroup. In case the operation $\cdot$ is associative it would be an associative quasigroup and if additionally an identity exists, it would be a group. The $+$ operation is more structured in that it is also commutative.

5.2.1 An example

Consider a set of three elements, $\mathfrak{I} = \{a, b, c\}$ such that

$$
\begin{array}{ccc}
\mathbf{a}\cdot & \mathbf{b}\cdot & \mathbf{c}\cdot \\
\begin{tikzcd} a \arrow[r] & a \\ b \arrow[dr] & b \\ c \arrow[ur] & c \end{tikzcd} &
\begin{tikzcd} a \arrow[ddr] & a \\ b \arrow[r] & b \\ c \arrow[uur] & c \end{tikzcd} &
\begin{tikzcd} a \arrow[dr] & a \\ b \arrow[ur] & b \\ c \arrow[r] & c \end{tikzcd}
\end{array}, \tag{5.6}
$$

so that the multiplication table looks like

·	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

and it is a quasigroup. As there is no element,

$$
\begin{tikzcd} a \arrow[r] & a \\ b \arrow[r] & b \\ c \arrow[r] & c \end{tikzcd},
$$

there is no identity[3]. It can also be checked to be non associative,

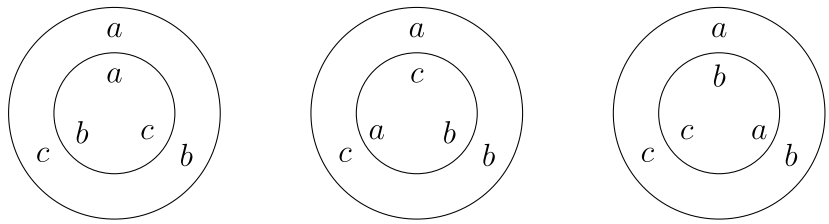
$$a \cdot (b{\cdot}c) = a{\cdot}a = a \neq c = c{\cdot}c = (a{\cdot}b){\cdot}c.$$

[3]In this case each element is its own "inverse" in the sense that $\cdot \circ \cdot$ maps each state to itself. However one can easily think of an action where this is missing,

·	a	b	c	d
a	a	c	d	b
b	c	b	a	d
c	b	d	a	c
d	d	a	b	c

.

A physical interpretation of this is possible. Consider a pinball machine like system with the three states on a circle. Consider a rotary inner disc with the states arranged with opposite circular orientation,



so that the a's are aligned in the first case and b's and c's in the next two respectively. This alignment is what corresponds to the "magnetic" states of North up and South up in section 2.1. So that when the magnet state is a,

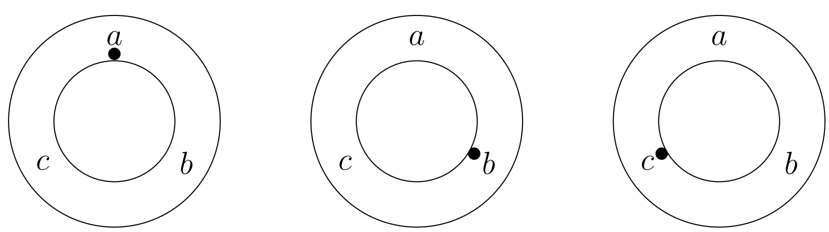
- outer $a \leftrightarrow$ inner a
- outer $b \leftrightarrow$ inner c
- outer $c \leftrightarrow$ inner b

and respectively for magnet states b and c.

Now the quantum mechanics is as follows, as the outcome of the pinball is probabilistic the state of the outer disc, assuming equal probability, is

$$|\psi\rangle_{\mathrm{o}} = \frac{1}{\sqrt{3}}\,|a\rangle_{\mathrm{o}} + \frac{1}{\sqrt{3}}\,|b\rangle_{\mathrm{o}} + \frac{1}{\sqrt{3}}\,|c\rangle_{\mathrm{o}} \tag{5.7}$$

so that there is an equal probability for the pinball to land on a, b or c. Landing on a, b or c respectively is shown below.



The state of the inner disc is initially some undetermined state, $|\phi\rangle_{\mathrm{i}}$. Say the initial state of the magnet, which simply indicates the alignment of the inner and outer discs is

$$|\chi\rangle_{\mathrm{e}} = |a\rangle_{\mathrm{e}}\,,$$

so that the outer and inner discs are aligned at a.

Therefore, one has to end up with an entangled state,

$$|\Psi\rangle_{\mathrm{oi}} = \frac{1}{\sqrt{3}} |a\rangle_{\mathrm{o}} |a\rangle_{\mathrm{i}} + \frac{1}{\sqrt{3}} |b\rangle_{\mathrm{o}} |c\rangle_{\mathrm{i}} + \frac{1}{\sqrt{3}} |c\rangle_{\mathrm{o}} |b\rangle_{\mathrm{i}}, \tag{5.8}$$

and similarly for the other states of the magnet/environment.

Thus, this is exactly the outcome as in section 2.2 replacing $+$ with the action $\cdot$ (equation 5.6). In detail, for magnetic state a,

$$\begin{aligned}
|\psi\rangle_{\mathrm{o}} |\phi\rangle_{\mathrm{i}} |\chi\rangle_{\mathrm{e}} &= \sum_{i\in\mathfrak{I}} \psi_i |i\rangle_{\mathrm{o}} |\phi\rangle_{\mathrm{i}} |a\rangle_{\mathrm{e}} \\
&\xrightarrow{\mathcal{S}_{\mathrm{i}\leftrightarrow\mathrm{e}}\circ\mathcal{I}_{\mathrm{o}\to\mathrm{e}}} \sum_{i\in\mathfrak{I}} \psi_i |i\rangle_{\mathrm{o}} |a\cdot i\rangle_{\mathrm{i}} |\phi\rangle_{\mathrm{e}} \\
&= \frac{1}{\sqrt{3}} \left(|a\rangle_{\mathrm{o}} |a\cdot a\rangle_{\mathrm{i}} + |b\rangle_{\mathrm{o}} |a\cdot b\rangle_{\mathrm{i}} + |c\rangle_{\mathrm{o}} |a\cdot c\rangle_{\mathrm{i}} \right) |\phi\rangle_{\mathrm{e}} \\
&= |\Psi\rangle_{oi} |\phi\rangle_{\mathrm{e}},
\end{aligned}$$

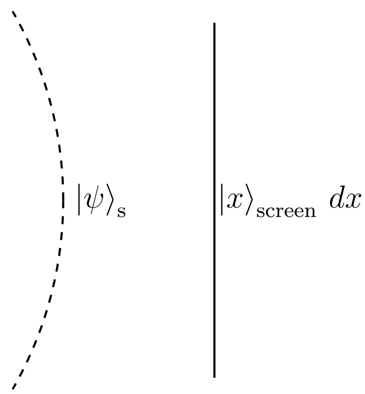
consistent with equation 5.8. This works similarly for magnet states b and c and is simply a replacement of $+$ with $\cdot$.

Some care is to be taken when defining the disentanglement operator required to produce proper entanglement of system and observer as in section 2.3. One has to use the left inverse (equation 5.5) in order to properly disentangle even though right action was used during entanglement,

$$\begin{array}{lcl}
\mathcal{S}_{o\leftrightarrow e_1} \circ \mathcal{I}_{s\to e_1} & & \begin{array}{l} \mathrm{s} : |i\rangle_{\mathrm{s}} \\ \mathrm{o} : |\phi\rangle_{\mathrm{o}} \\ \mathrm{e} : |k\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} \ldots |k\rangle_{\mathrm{e_N}} \end{array} \\[2em]
{}_L\mathcal{I}^{-1}_{e_2\to o} & \hookrightarrow & \begin{array}{l} \mathrm{s} : |i\rangle_{\mathrm{s}} \\ \mathrm{o} : |k\cdot i\rangle_{\mathrm{o}} \\ \mathrm{e} : |\phi\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} \ldots |k\rangle_{\mathrm{e_N}} \end{array} \\[2em]
 & \hookrightarrow & \begin{array}{l} \mathrm{s} : |i\rangle_{\mathrm{s}} \\ \mathrm{o} : |i\rangle_{\mathrm{o}} \\ \mathrm{e} : |\phi\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} \ldots |k\rangle_{\mathrm{e_N}}. \end{array}
\end{array}$$

5.3 Topology of the Index Set

In this section we discuss another structure that might exist on the index set which is its topology. The index set is the set of labels used to describe the basis vectors that span the Hilbert space on which the state of a quantum system lives. It is intimately connected to the kinds of measurements one might perform on the system and corresponding observables. It might be the case that these observables and meter readings have a natural topology which is then inherited by the index set. As an example consider the measurement of a system in the position basis, say, for example, the position on a screen where an electron might collapse.



Consider the state of an electron,

$$|\psi\rangle_{\mathrm{s}} = \int \psi(x)\, |x\rangle_{\mathrm{s}}\ dx$$

where an integral is used instead of a sum because the underlying index set is continuous. Here the observable is position on the screen so that

$$\mathcal{X} = \int x\, |x\rangle \langle x|\ dx.$$

The natural topology of the screen is then inherited by the index set so that one can define a distance between the basis vectors, $|x\rangle$ and $|y\rangle$ to be $|x-y|$, naturally inherited from the topology of the physical screen and thereby the observable $\mathcal{X}$. The importance of this topology is that the resulting quantum state $|\psi\rangle_{\mathrm{s}} = \int \psi(x)\, |x\rangle_{\mathrm{s}}\ dx$ has some correlation in its amplitude $\psi(x)$, say for example that it is continuous.

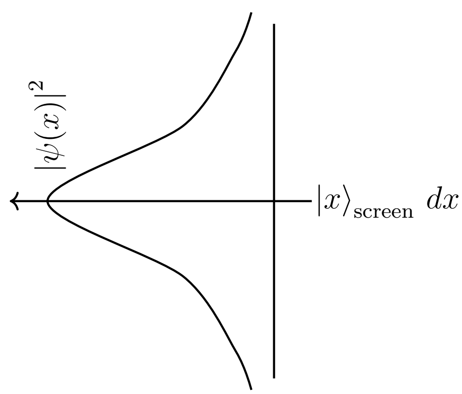
When the electron is measured by the screen, the resulting state is

$$\int \psi(x) \left|x\right\rangle_{\mathrm{s}} \left|x\right\rangle_{\mathrm{screen}} \, dx \tag{5.9}$$

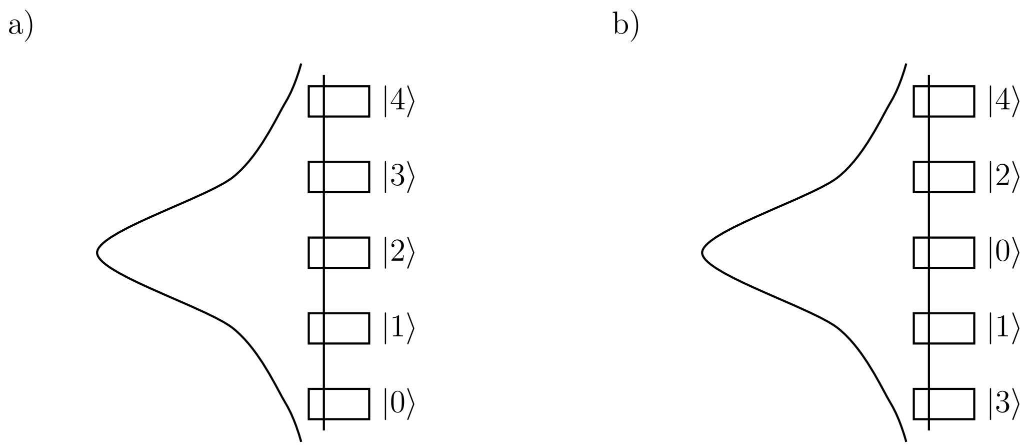
for the combined state of system and screen. The resulting density matrix of the electron is

$$\rho_{\mathrm{s}} = \int \left|\psi(x)\right|^2 \left|x\right\rangle \left\langle x\right|_{\mathrm{s}} \, dx \tag{5.10}$$

and in this case the topology might indicate that $\left|\psi(x)\right|^2$ is a continuous function which comes about because $\psi(x)$ is continuous. Continuity here implies that if $|x-y|$ is small, so is $|\psi(x)-\psi(y)|$ and therefore $\left|\left|\psi(x)\right|^2-\left|\psi(y)\right|^2\right|$.



If the electron was instead measured in the momentum basis, the index set would be different and so would be the topology as well as the definition of continuity. To illustrate this we consider a set of discrete detectors on the screen, two different schemes of enumeration create different sorts of metric structure.



a) and b) correspond to two different ways of enumerating the 5 detectors on the screen.

As can be seen from the figure above, two different methods of enumerating the different detectors would lead to different topologies on the index set. Whereas in case a, detectors 2 and 4 are two apart, they are adjacent in case b; meaning that $|\psi(2) - \psi(4)|$ is larger in case a than in case b. Similarly, 3 and 4 are adjacent in case a but are furthest apart in case b (even though $|\psi(3) - \psi(4)|$ is actually 0 in case b due to the symmetry of our chosen $\psi(x)$).

As another example, one could consider a set of detectors in two dimensions, that is a two-dimensional screen. In this case it would make sense to label the detectors (and therefore the labels in the index set) using a pair of integers allowing the natural topology of the underlying physical screen to be inherited.

(0,4)	(1,4)	(2,4)	(3,4)	(4,4)
(0,3)	(1,3)	(2,3)	(3,3)	(4,3)
(0,2)	(1,2)	(2,2)	(3,2)	(4,2)
(0,1)	(1,1)	(2,1)	(3,1)	(4,1)
(0,0)	(1,0)	(2,0)	(3,0)	(4,0)

A natural method of labelling the detectors.

This allows the natural topology of the two-dimensional screen to be inherited by the index set and therefore the distance between states (i, j) and

(i', j') is given by $|(i, j) - (i', j')|$. Using a single integer index would give rise to a contrived distance function in the index set. Therefore, consideration of the topology of the index set might be an important factor depending on the problem at hand.

Section 6

Discussion

Our main objective was to come up with a completely unitary dynamics for quantum mechanics avoiding wavefunction collapse. Here we conclude by summarising what we have done so far and by commenting on what could be done in the future.

"Collapsed" research

In section 2 we describe a unitary procedure for measurement motivated by the principle of conservation of information [12, 13] where it is argued that measurement is the same as entanglement of a system and an observer so that their states are correlated. In order for the procedure to work, a third system, the environment, is required. The environment influences the entanglement of system and observer and indeed in order to obtain perfect entanglement the existence of a "classical" environment (section 2.3) is required. Section 2 concludes with a discussion of decoherence and why classical states are so persistent (this is similar to the discussions in the talks [6, 27]). Our state of the world is the same as in the many worlds interpretation [28, 29, 30] where there is no requirement for wavefunction collapse and it is shown that the measurement axiom of 1.4 is half eliminated.

In order to completely eliminate the need for axiom 1.4 one would also need to derive the Born rule starting from the other axioms and we do this in section 3. We follow Zurek [8, 9] and derive the Born rule starting with the other axioms and with an assumption of locality (axiom 1.6). Having done so we have reduced the axioms of section 1 to the subset without the axiom 1.4, effectively replacing the axiom with the measurement procedure which is completely unitary. The axioms are repeated here for continuity.

Axiom 6.1. *The state of a quantum system is completely described by a ray* $|\psi\rangle$ *in a Hilbert space* $\mathcal{H}$*. This means that it is normalised* $\langle\psi|\psi\rangle = 1$ *with an appropriately defined square inner product* $\langle\ |\ \rangle$ *and that overall phases do not matter so that* $|\psi\rangle \sim e^{i\delta}|\psi\rangle$*.*

Axiom 6.2. *The time-evolution of a quantum system is described by a unitary* $\mathcal{U}$ *on its Hilbert space* $|\psi(t')\rangle = \mathcal{U}(t', t)|\psi(t)\rangle$*.*

Axiom 6.3. *An observable is a Hermitian operator on the Hilbert space* $\mathcal{M} = \mathcal{M}^\dagger$ *so that it is diagonalisable and can be written as* $\mathcal{M} = \sum_i M_i \mathbb{P}_i$ *where* $\mathbb{P}_i$ *are projectors onto the subspaces of* $\mathcal{M}$ *with eigenvalues* M_i*. The outcomes of measurements in experiments can only be one of the eigenvalues of an observable. Immediately measuring the observable, so that the state has had no time to evolve away unitarily, on the same quantum state gives the same eigenvalue.*

Axiom 6.4. *The combined state of two systems in* $\mathcal{H}_\mathrm{a}$ *and* $\mathcal{H}_\mathrm{b}$ *is in the tensor product space* $\mathcal{H}_\mathrm{a} \otimes \mathcal{H}_\mathrm{a}$*.*

Axiom 6.5. *A quantum system only influences and is influenced by other systems within its light cone.*

In section 4 we discuss the double slit experiment with our framework. This is an important test of consistency as any procedure developed would still have to allow for interference effects as required by usual quantum mechanics. The section ends with a discussion of the delayed choice quantum eraser experiment. Once again our results are in agreement with that reached by assuming the many worlds interpretation.

In section 5 we discuss details of the index set, which is the set of labels used for the basis vectors of the Hilbert space in which the quantum system evolves. After motivating the definition of the index set guided by axiom 1.3 we derive the most general (useful according to us) algebra that such an index set can possess. We also discuss how the topology of the index set might be an important factor to consider.

"Not "yet" collapsed" research

There are many directions in which this work could be expanded. In section 2.3 we merely postulate the existence of a classical environment but it would be very interesting to explore how such a system is formed starting with a highly disordered ("high temperature") state. It would be illuminating to use an Ising model or some other statistical mechanics model in order to explain how such a classical system forms in the first place. This seems to be related to the difficult problem of the emergence of classicality and might be related to the proliferation of self replicating entities. This "Darwinism" at the level of quantum states might be a highly fruitful direction of research [27]. The motivation is that the sorts of quantum states that survive the process of decoherence with the environment are the ones that proliferate.

Additionally it might be possible to postulate errors in the measurement procedure due to the fact that systems are not perfectly entangled as $|k\rangle \ldots |k\rangle \ldots |k\rangle$ but rather might have errors such as $|k\rangle \ldots |k+\epsilon\rangle \ldots |k\rangle$. Using a model for the amplitude distribution of such states it might be possible to come up with experiment tests of our models.

A second direction of expansion would be to use ideas from statistical mechanics and stochastic process to imagine the measurement procedure of equation 2.4 as a process with a certain rate and to derive the dynamics of decoherence from that standpoint. Of course, this would have to done carefully to ensure that while the procedure is deterministic for the entire wavefunction, an effective lack of knowledge as in statistical mechanics would give rise to statistical laws. This might be akin to using Newtonian mechanics to derive thermodynamics. Relatedly, the measurement procedure of equation 2.4 could be elevated to a second quantisation picture and cross sections for this process could be derived. This is in the spirit of quantum field theories and experimental tests in this direction could also be devised. Using either methodology as a seed, field theories for the measurement procedure could be explored.

In section 5 we discussed how the structure of the Hilbert space can be influenced using observables. However, using observables to define the Hilbert space which then determines the observables makes it a top-down approach attached to a bottom-up approach rendering the argument circular. One could further explore how this could be derived more transparently. This is related to the idea of choosing a basis to describe the Hilbert space and explaining why it is a good choice. The idea of a suitable basis is important in discussing the kinds of quantum states that survive the procedure of measurement 2.4 and also in deriving the Born rule 3 and thereby deserves much more thought.

It was claimed in 3 that the born rule basically originates from the inner product structure of the Hilbert space. It would be interesting to check if some other norm might be used to give rise to the same physics of quantum mechanics even if the underlying mathematics is different. It is to be noted that linearity of the inner product already implies the square inner product and therefore it might be that with a different norm one would have to redefine interference effects as they would no longer be linear.

Another direction to expand the discussion on section 3 would be to follow more closely the discussion of [8, 9] where the author also discusses how the wavefunction reflects reality and how to essentially derive probability theory from the objective reality of the universe wavefunction. It would also be very instructive to discuss the difference between this approach and that of other derivations of the Born rule. An overview of different approaches of deriving the Born rule can be found here [31, 32].

As remarked in 4, the quantum eraser experiment is sometimes explained using retrocausality. It would be interesting to approach our argument from a more philosophically satisfying viewpoint so that one could see if retrocausality is necessary, unnecessary or inconsistent. Another direction to extend the analysis of the quantum eraser experiment is to look at erasure experiments of quantum entanglement [33].

Finally, locality was introduced in a rather ad-hoc manner (axiom 1.6). This deserves more thought because locality in quantum mechanics is a subtle issue that needs to be addressed properly. A seed for a discussion on this topic could be found in the article [34].

Appendix A

What is Entanglement?

We describe entanglement for our purposes in this paper. The aim would be to provide some intuition for what constitutes entanglement.

Two systems are said to be decomposable in case the wavefunction of the combined system can be factored into a product of wavefunctions. Consider systems a and b and their combined system ab. The states of these systems live in different Hilbert spaces, $|\ \rangle_{\mathrm{a}} \in \mathcal{H}_{\mathrm{a}}, |\ \rangle_{\mathrm{b}} \in \mathcal{H}_{\mathrm{b}}$ and the combined state lives in the Hilbert space that is a tensor product of the two, $|\ \rangle_{\mathrm{ab}} \in \mathcal{H}_{\mathrm{ab}} = \mathcal{H}_{\mathrm{a}} \otimes \mathcal{H}_{\mathrm{b}}$. If the wavefunction of the combined system

$$|\eta\rangle_{\mathrm{ab}} = |\xi\rangle_{\mathrm{a}} |\zeta\rangle_{\mathrm{b}}\,, \tag{A.1}$$

can be written as a product, this means that the combined state is decomposable and not entangled.

On the other hand, as the resulting space is a tensor product space, there are states which are not factorisable as above, and are therefore entangled. For example, with $i, j \in \mathfrak{I}(\mathcal{H}_{\mathrm{a}}) = \mathfrak{I}(\mathcal{H}_{\mathrm{b}})$ and $i \neq j$, consider

$$|\eta\rangle_{\mathrm{ab}} = \frac{1}{\sqrt{2}} |i\rangle_{\mathrm{a}} |i\rangle_{\mathrm{b}} + \frac{1}{\sqrt{2}} |j\rangle_{\mathrm{a}} |j\rangle_{\mathrm{b}}\,, \tag{A.2}$$

which cannot be written as a product of two states. The above state is therefore entangled.

This can be proved by contradiction. Assume that there does exist a decomposition of equation A.2, say

$$|\eta\rangle_{\mathrm{ab}} = |\xi\rangle_{\mathrm{a}} |\zeta\rangle_{\mathrm{b}} = (\xi_0 |0\rangle_{\mathrm{a}} + \xi_1 |1\rangle_{\mathrm{a}})(\zeta_0 |0\rangle_{\mathrm{b}} + \zeta_1 |1\rangle_{\mathrm{b}}),$$

which means that

$$\xi_1\zeta_0 \stackrel{!}{=} 0 \stackrel{!}{=} \xi_0\zeta_1 \qquad\qquad \xi_0\zeta_0 \stackrel{!}{=} \frac{1}{\sqrt{2}} \stackrel{!}{=} \xi_1\zeta_1, \tag{A.3}$$

which is impossible as at least one of ξ_1 or ζ_0 is zero meaning that both the expressions on the right column of equation A.3 cannot be non-zero.

In general, whether a quantum state is entangled or not can be ascertained using the Schmidt decomposition of the state (see appendix B and [1]). Given any quantum state in a bipartite system $\mathcal{H}_\mathrm{a} \otimes \mathcal{H}_\mathrm{b}$ there exist orthonormal bases $|e_i\rangle_\mathrm{a}$ and $|f_i\rangle_\mathrm{b}$ for $\mathcal{H}_\mathrm{a}$ and $\mathcal{H}_\mathrm{b}$ respectively such that

$$|\eta\rangle = \sum_i \eta_i \, |e_i\rangle_\mathrm{a} \, |f_i\rangle_\mathrm{b},$$

where the coefficients η_i are positive real numbers. The set $\{\eta_i\}$ of Schmidt coefficients is unique. In case there exist more than one non zero η_i the state $|\eta\rangle$ is entangled and otherwise it is decomposable.

For this reason states such as, $(\Psi_{i_1} |i_1\rangle_\mathrm{a} + \Psi_{i_2} |i_2\rangle_\mathrm{a}) |k\rangle_\mathrm{b}$ are not properly entangled in the $\{|i_1\rangle_\mathrm{a}, |i_2\rangle_\mathrm{a}\}$ subspace as remarked in section 5.2.

In case of the classical state of section 2.3,

$$|\chi\rangle_\mathrm{e} = \sum_k \chi_k \, |k\rangle_{\mathrm{e}_1} |k\rangle_{\mathrm{e}_2} |k\rangle_{\mathrm{e}_3} \ldots |k\rangle_{\mathrm{e}_\mathrm{N}} \, ,$$

for any partition of the overall state into two, the two halves are entangled. For this reason it was remarked that every subsystem is entangled with every other. However a state such as

$$|\phi\rangle_{\mathrm{e}_1} \sum_k \chi_k \, |k\rangle_{\mathrm{e}_2} \ldots |k\rangle_{\mathrm{e}_\mathrm{N}}$$

can be separated into a product of $|\phi\rangle_{\mathrm{e}_1}$ and $\sum_k \chi_k \, |k\rangle_{\mathrm{e}_2} \ldots |k\rangle_{\mathrm{e}_\mathrm{N}}$ states and is therefore not completely entangled.

Appendix B

The Schmidt Decomposition

In this appendix we show that any state in a bipartite quantum system $|\Psi\rangle_{\mathrm{ab}} \in \mathcal{H}_{\mathrm{a}} \otimes \mathcal{H}_{\mathrm{b}}$ has a Schmidt decomposition,

$$|\Psi\rangle_{\mathrm{ab}} = \sum_i \Psi_i \, |e_i\rangle_{\mathrm{a}} \, |f_i\rangle_{\mathrm{b}},$$

with

$$\mathbb{R} \ni \Psi_i \geq 0$$

and $\{|e_i\rangle_{\mathrm{a}}\}$ and $\{|f_i\rangle_{\mathrm{b}}\}$ orthonormal bases for $\mathcal{H}_{\mathrm{a}}$ and $\mathcal{H}_{\mathrm{b}}$ respectively.

B.1 Schmidt Decomposition

Consider an operator

$$\mathcal{U} : \mathcal{H}_{\mathrm{a}} \to \mathcal{H}_{\mathrm{b}}$$

which can always be written as

$$\mathcal{U} = \sum_{ij'} \mathtt{U}_{ij'} \, |j'\rangle_{\mathrm{b}} \, \langle i|_{\mathrm{a}} \tag{B.1}$$

where $\{|i\rangle_{\mathrm{a}}\}$ and $\{|j'\rangle_{\mathrm{b}}\}$ are orthonormal bases for $\mathcal{H}_{\mathrm{a}}$ and $\mathcal{H}_{\mathrm{b}}$ respectively and $\mathtt{U}_{ij'} \in \mathbb{C}$.

There exists a bijection between operators between two Hilbert spaces and matrices in $\mathbb{C}^{m\times n}$. For $m = \dim \mathcal{H}_{\mathrm{a}}$ and $n = \dim \mathcal{H}_{\mathrm{b}}$ a matrix $\mathtt{U}$ on $\mathbb{C}^{m\times n}$ can be defined such that

$$\mathcal{U} \leftrightarrow \mathtt{U} = [\mathtt{U}_{ij'}],$$

a matrix $\mathtt{U}$ whose ij''th component is the same as the $|j'\rangle_{\mathrm{b}}\langle i|_{\mathrm{a}}$'th element of $\mathcal{U}$.

Consider now a matrix Ψ. Any $m \times n$ matrix Ψ has a singular value decomposition,

$$\underset{(m\times n)}{\Psi} = \underset{(m\times m)}{\mathtt{V}} \quad \underset{(m\times n)}{\Sigma} \quad \underset{(n\times n)}{\mathtt{U}}, \tag{B.2}$$

where $\mathtt{U}$ and $\mathtt{V}$ are unitaries in their respective spaces,

$$\mathtt{U}^\dagger\mathtt{U} = \mathbb{1}_n = \mathtt{U}\mathtt{U}^\dagger \qquad \mathtt{V}^\dagger\mathtt{V} = \mathbb{1}_m = \mathtt{V}\mathtt{V}^\dagger,$$

and Σ is a "quasi"-diagonal matrix

$$\Sigma = \begin{pmatrix} \Sigma_{00} & & & & & & \\ & \Sigma_{11} & & & & & \\ & & \ddots & & & & \\ & & & \Sigma_{r-1r-1} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \\ & & & & & & \\ & & & & & & \end{pmatrix},$$

where $\Sigma_{ij} = 0$ whenever $i \neq j$ and $\Sigma_{ii} = 0$ whenever $i \geq r$. Moreover all $\mathbb{R} \ni \Sigma_{ii} \geq 0$ and they are unique upto reordering(see [35, 36]).

This means that following B.1, $|\Psi\rangle_{\mathrm{ab}}$ can be expanded as[1]

$$\begin{aligned} |\Psi\rangle_{\mathrm{ab}} &= \sum_{ij'} \Psi_{\mathtt{i}\mathtt{j}'} |i\rangle_{\mathrm{a}} |j'\rangle_{\mathrm{b}} \\ &= \sum_{ij'} \sum_{k=0}^{r-1} \mathtt{U}_{ik}\Sigma_{kk}\mathtt{V}_{kj'} |i\rangle_{\mathrm{a}} |j'\rangle_{\mathrm{a}}, \end{aligned} \tag{B.3}$$

noting that all other $\Sigma_{lm} = 0$. By further defining

$$\Psi_k = \Sigma_{\mathtt{kk}} \qquad |e_k\rangle_{\mathrm{a}} = \sum_i \mathtt{U}_{ik} |i\rangle_{\mathrm{a}} \qquad |f_k\rangle_{\mathrm{b}} = \sum_{j'} \mathtt{V}_{kj'} |j'\rangle_{\mathrm{a}}{}^2, \tag{B.4}$$

[1] It is $|\ \rangle\langle\ |$ in one case and $|\ \rangle|\ \rangle$ in another but the same arguments work.

one obtains

$$|\Psi\rangle_{\mathrm{ab}} = \sum_{k=0}^{r-1} \Psi_k \, |e_k\rangle_{\mathrm{a}} \, |f_k\rangle_{\mathrm{b}},$$

which is the Schmidt decomposition!

There is a subtlety in that $|e_k\rangle_{\mathrm{a}}$ and $|f_k\rangle_{\mathrm{b}}$ have to be extended to orthonormal bases for $\mathcal{H}_{\mathrm{a}}$ and $\mathcal{H}_{\mathrm{b}}$ but this can always be done. Details can be checked in references [35, 36] where the singular value decomposition is explored. A discussion for the Schmidt decomposition from a quantum information perspective can also be found in [1].

B.2 Uniqueness of Schmidt Coefficients

In this subsection we explore how the spectrum of the operator determines the freedom in choosing unitaries $\mathtt{U}$ and $\mathtt{V}$. The reference [37] is used as a guide while going through the arguments.

Consider two different factorisations of B.2,

$$\mathtt{V}_1\Sigma\mathtt{U}_1 = \Psi = \mathtt{V}_2\Sigma\mathtt{U}_2,$$

so that

$$\begin{aligned} \Psi\Psi^\dagger &= \mathtt{V}_1\Sigma\Sigma^\dagger{\mathtt{V}_1}^\dagger = \mathtt{V}_2\Sigma\Sigma^\dagger{\mathtt{V}_2}^\dagger \\ \implies \Sigma\Sigma^\dagger{\mathtt{V}_1}^\dagger\mathtt{V}_2 &= {\mathtt{V}_1}^\dagger\mathtt{V}_2\Sigma\Sigma^\dagger \end{aligned} \tag{B.5}$$

and setting $\mathtt{D} = \Sigma\Sigma^\dagger$ and $\mathtt{V} = {\mathtt{V}_1}^\dagger\mathtt{V}_2 \implies \mathtt{V}_1\mathtt{V} = \mathtt{V}_2$, what obtains is

$$\mathtt{DV} = \mathtt{VD}. \tag{B.6}$$

Σ is a quasi-diagonal matrix with eigenvalues $\Sigma_{\mathtt{kk}}$. In this form all the eigenvalues $\{\Sigma_{\mathtt{kk}}\}$ are listed and the repeated eigenvalues are not accounted for. In order to keep track of the repeated eigenvalues they are enumerated without repetition, $\{\Sigma_\alpha\} = \{\Sigma_{\mathtt{kk}}\}$ but $\Sigma_\alpha \neq \Sigma_\beta$ if $\alpha \neq \beta$. 0 could be one of the singular values in the sets $\{\Sigma_\alpha\} = \{\Sigma_{\mathtt{kk}}\}$ in case Σ has a non trivial null

[2] The order of the indices for $\mathtt{V}$ and $\mathtt{U}$ are different.

space. Therefore $\mathtt{\Sigma}$ can be written as

$$\mathtt{\Sigma} = \begin{pmatrix} \ddots & & \\ & \mathtt{\Sigma}_\alpha \mathbb{1}_{\mathcal{H}_\alpha} & \\ & & \ddots \\ & & \end{pmatrix} = \sum_\alpha \mathtt{\Sigma}_\alpha \tilde{\mathbb{P}}_\alpha \tag{B.7}$$

where $\mathbb{1}_{\mathcal{H}_\alpha}$ is the identity matrix within the α eigenvalue subspace of $\mathtt{\Sigma}$ and $\tilde{\mathbb{P}}_\alpha$ is a projector onto this subspace,

$$\tilde{\mathbb{P}}_\alpha = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & \mathbb{1}_{\mathcal{H}_\alpha} & \\ & & & \ddots \\ & & & \end{pmatrix},$$

so that $\tilde{\mathbb{P}}_\alpha v \in \mathbb{1}_{\mathcal{H}_\alpha}$ for all v in the row space of $\mathtt{\Sigma}$. That is, $\tilde{\mathbb{P}}_\alpha$ projects all vectors in the row space of $\mathtt{\Sigma}$ onto the subspace of the column space with eigenvalue $\mathtt{\Sigma}_\alpha$. This means that

$$\mathtt{D} = \mathtt{\Sigma}\mathtt{\Sigma}^\dagger = \sum_\alpha \lambda_\alpha \mathbb{P}_\alpha \tag{B.8}$$

where λ_α correspond to the unique eigenvalues of $\mathtt{\Sigma}\mathtt{\Sigma}^\dagger$; that is, they are $\lambda_\alpha = {\mathtt{\Sigma}_\alpha}^2$ without any repetitions and with one of the λ_α possibly being 0. $\mathbb{P}_\alpha$ are quite similar to $\tilde{\mathbb{P}}_\alpha$ corresponding to projectors onto subspaces with eigenvalues λ_α with one subtle point that the 0 eigenvalue subspace might be larger due to the shape of $\mathtt{\Sigma}$ and $\mathtt{\Sigma}\mathtt{\Sigma}^\dagger$ being different. That is, in order to make the row space equal the column space, the null space with eigenvalue $\mathtt{\Sigma}_\alpha = 0$ might have to be extended. In case the row space is larger, the column space might have to be extended. In either case, the subspaces corresponding to non-zero eigenvalues are maintained as is. As the matrices are square, the usual projector equation holds, $\mathbb{P}_\alpha \mathbb{P}_\beta = \delta_{\alpha\beta}\mathbb{P}_\alpha$.

As $\mathtt{D}$ is diagonal it has a spanning eigenbasis and therefore the projectors add up to give unity,

$$\sum_\alpha \mathbb{P}_\alpha = \mathbb{1}.$$

We divide $\mathtt{V}$ into blocks compatible with $\mathtt{D}$ so that the blocks $\mathtt{D}_{\alpha\beta}$ and $\mathtt{V}_{\alpha\beta}$ are of dimensions $\dim \mathbb{1}_{\mathcal{H}_\alpha} \times \dim \mathbb{1}_{\mathcal{H}_\beta}$,

$$\mathtt{D} = \begin{pmatrix} \ddots & & \\ & \lambda_\alpha \mathbb{1}_{\mathcal{H}_\alpha} & \\ & & \ddots \end{pmatrix} \sim \begin{pmatrix} \ddots & & \\ & \ddots & \mathtt{V}_{\alpha\beta} \\ & & \ddots \end{pmatrix} = \mathtt{V}. \tag{B.9}$$

Equation B.6 then shows that,

$$\begin{aligned} &(\mathtt{DV})_{\alpha\beta} = (\mathtt{VD})_{\alpha\beta} \\ \implies &\lambda_\alpha \mathtt{V}_{\alpha\beta} = \mathtt{V}_{\alpha\beta}\lambda_\beta \end{aligned} \tag{B.10}$$

which means that if $\alpha \neq \beta$, the corresponding eigenvalues λ_α and λ_β are not equal implying that

$$\mathtt{V}_{\alpha\beta} = 0 \impliedby \lambda_\alpha \neq \lambda_\beta.$$

This implies that $\mathtt{V}$ has the same diagonal structure as $\mathtt{D}$,

$$\mathtt{V} = \begin{pmatrix} \ddots & & \\ & \mathtt{V}_{\alpha\alpha} & \\ & & \ddots \end{pmatrix} \tag{B.11}$$

with all non diagonal terms being 0.

In case the same analysis was repeated with $\Psi^\dagger\Psi$ one would obtain

$$\mathtt{UD}' = \mathtt{D}'\mathtt{U} \tag{B.12}$$

where

$$\begin{aligned} \mathtt{U} &= \mathtt{U}_1{\mathtt{U}_2}^\dagger \implies \mathtt{U}\mathtt{U}_2 = \mathtt{U}_1 \\ \mathtt{D}' &= \Sigma^\dagger\Sigma = \sum_\alpha \lambda'_\alpha \mathbb{P}'_\alpha. \end{aligned} \tag{B.13}$$

$\mathbb{P}'_\alpha$ are the same as $\mathbb{P}_\alpha$ except perhaps for the null eigenvalues $\lambda_\alpha = 0 = \lambda'_\alpha$. Then the result would be that

$$\mathtt{U} = \begin{pmatrix} \ddots & & \\ & \mathtt{U}_{\alpha\alpha} & \\ & & \ddots \end{pmatrix},$$

where $\mathtt{U}_{\alpha\alpha}$ and $\mathtt{V}_{\alpha\alpha}$ have the same dimensions except perhaps for the null subspaces corresponding to $\lambda_\alpha = 0 = \lambda'_\alpha$.

Now consider a unitary

$$\mathtt{V} = \begin{pmatrix} \ddots & & \\ & \mathtt{V}_{\alpha\alpha} & \\ & & \ddots \end{pmatrix},$$

with arbitrary unitaries $\mathtt{V}_{\alpha\alpha}$ in $\mathcal{H}_\alpha$. Then for any Schmidt decomposition $\Psi = \mathtt{V}_2\Sigma\mathtt{U}_2$ using $\mathtt{V}_2 = \mathtt{V}_1\mathtt{V}$ one gets,

$$\begin{aligned} \Psi &= \mathtt{V}_2\Sigma\mathtt{U}_2 \\ &= \mathtt{V}_1\mathtt{V}\Sigma\mathtt{U}_2 \\ &= \mathtt{V}_1 \begin{pmatrix} \ddots & & \\ & \mathtt{V}_{\alpha\alpha} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \ddots & & \\ & \Sigma_\alpha \mathbb{1}_{\mathcal{H}_\alpha} & \\ & & \ddots \\ & & \end{pmatrix} \mathtt{U}_2 \\ &= \mathtt{V}_1\Sigma\mathtt{U}\mathtt{U}_2 \\ &= \mathtt{V}_1\Sigma\mathtt{U}_1 \end{aligned}$$

where $\mathtt{U}$ is the same as $\mathtt{V}$ except for the 0 eigenspace, that is $\mathtt{V}_{\alpha\alpha} = \mathtt{U}_{\alpha\alpha}$ for all α expect perhaps when $\lambda_\alpha = 0 = \lambda'_\alpha$. The fact that $\mathtt{V}$ is diagonal like Σ is what helps them commute.

This means that all and only those $\mathtt{V}$ which are co-diagonal with Σ are the unitaries that are envariant (section 3.1). Which further means that within subspaces with the same absolute value of Schmidt coefficients any unitary in that subspace is a valid envariant transformation and therefore the most general envariant transformations are those which are the product of such transformations in the different subspaces.

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Declaration

I hereby declare that this thesis is my own work, and that I have not used any sources and aids other than those stated in the thesis.

München, 9 Aug 2022