Integrality of the Multinomial Coefficients

Vishal Johnson

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Introduction

Consider a set of 10 balls; 3 of them are coloured blue, 4 of them green, 2 of them red and 1 of them orange. What are the number of ways of arranging these objects? It is assumed that the reader is familiar with the idea of permutations and combinations and is thus privy to the fact that the answer to this question is,

$$\binom{10}{3,4,2,2} = \frac{10!}{3!4!2!1!} = 12600.$$

One way to reason this out is to observe that the 10 balls can be arranged in 10! ways and for each of these arrangements any 3! arrangements of the blue balls, 4! arrangements of the green balls, 2! arrangements of the red balls and 1! arrangements of the orange balls are equivalent (indistinguishable). One must thus divide by each of the indistinguishable arrangements to get the total number of distinguishable arrangements.

Another related but quite distinct problem is finding out the coefficients in the expansion of a multinomial. Consider the multinomial, $(a+b+c+d)^{10}$. What is the coefficient of the term $a^3b^4c^2d^1$ in the expansion of this multinomial? Consider an expansion of the multinomial,

$$(a+b+c+d)^{10} = (a+b+c+d)_1(a+b+c+d)_2..(a+b+c+d)_{10}.$$

The numbering on the product terms is artificially introduced to keep track of number of terms. For each choice of 3 terms out of the 10 for 'a', 4 terms out of 10 for 'b', 2 terms for 'c' and 1 for 'd' there is a term in the product formed with exactly the required $a^3b^4c^2d^1$. Also, for each such choice of the coefficient, there is a unique arrangement of 10 balls into 10 numbered boxes with 'a' corresponding to blue balls, 'b' to green, 'c' to red and 'd' to orange. Additionally, for each arrangement of 10 balls discussed above, one can form a selection corresponding to $a^3b^4c^2d^1$. This is done by letting the positions of the blue balls indicate the terms in the product that lead to 'a', positions of the green balls indicate the terms that lead to 'b', red to 'c' and orange to 'd'. There is, thus, a bijection from arrangements of coloured balls to selections of positions for letters. This indicates that the coefficient of $a^3b^4c^2d^1$ is just, $\binom{10}{3,4,2,2}$.

Integrality

There is no justification provided for the number of arrangements actually being an integer; except of course that one cannot have a fractional number of arrangements. This article aims to give such a justification in an independent manner, using the ideas of the multinomial coefficients. The first step is to define the multinomial coefficient axiomatically, independent of any correlation with arrangements and selections; considering their equivalence as a separate fact of the matter.

Definition. The multinomial coefficient $\binom{n}{n_1,n_2,..n_k}$ is defined as,

$$\binom{n}{n_1, n_2, ..n_k} = \frac{n!}{n_1!n_2!..n_k!}.$$

Of course, the constraint that $n = n_1 + n_2 + ..n_k$ applies.

The independent justification is motivated from the pascal triangle. This article is essentially a generalisation of the idea of the pascal triangle into higher dimensions.

Motivation from the Pascal Triangle. Notation.

The pascal triangle is a tool that helps keep track of the multinomial coefficients for the case of a binomial. One can obtain the binomial coefficients for $(a+b)^n$ using the binomial coefficients for $(a+b)^{n-1}$ and thus, this usage lends itself to analysis using mathematical induction and convenient recursive notation. The binomial coefficients are commonly written as, $\binom{n}{r}$. But to be consistent with notation in this paper, they would be indicated with the slightly redundant $\binom{n}{r,s}$, where $s = n - r$.

In order to facilitate the idea, a geometric analogy is provided. Imagine an "rs-plane", consisting of the integral points, this shall be denoted as $\mathbb{Z}^2$ as the integral points simply consist of pairs of integers. All points with $r + s = n$ lie on the line that passes through $(0, n)$ and $(n, 0)$. Consider a function,

$$\mathcal{F} : \mathbb{Z}^2 \to \mathbb{Q}$$
$$(r, s) \to \binom{r+s}{r, s}$$

It is the task of this section to prove that the range of $\mathcal{F}$ only consists of integers.

Selecting r objects out of n can be done in two ways,

1. Selecting all r of the objects from $n - 1$ of the n objects
2. Selecting $r - 1$ of the objects from $n - 1$ of the n objects and also choosing the left out object

As these two cases are distinct, the total number of ways of choosing r out of n objects is the sum of the two different ways of choosing them. And thus, it must be that the formula for the binomial coefficients reflect this fact.

Theorem 1.
$$\binom{n}{r, s} = \binom{n-1}{r, s-1} + \binom{n-1}{r-1, s} \times 1$$
Here, $s = n - r$ as required by the constraint.

Remark. An important point to be noted is the fact that the terms in the formula for the multinomial coefficient should all be non-negative. If for example s in the above formula was 0, the multinomial coefficient would not make any sense. In view of brevit of notation, it shall be assumed (in the scope of this paper) that any formula for a multinomial coefficient with negative terms shall be treated as 0. This leads to the following definition.

Definition. If any of $n, n_1, n_2, ..n_k < 0$,
$$\binom{n}{n_1, n_2, ..n_k = 0}$$

The theorem shall be proved in the more general setting of the multinomial coefficients. The reader is to observe that the points $(r, s-1)$ and $(r-1, s)$ are the neighbours of the point (r, s) that lie closer to the origin. This geometrical idea would also be used for the higher dimensional case.

The General Multinomial Coefficients

In order to deal with the general multinomial coefficients it would be instructive to construct another function similar to the previous section $\mathcal{F}$. Consider a k-nomial with formal variables a_1 through a_k. Then the nth power of this k-nomial is given by,

$$\left(\sum_{k=1,2,..n} a_k\right)^n = (a_1 + a_2 + ..a_k)_1(a_1 + a_2 + ..a_k)_2..(a_1 + a_2 + ..a_k)_n,$$

and the coefficient of the term $a_1^{n_1} a_2^{n_2}..a_k^{n_k}$ is sought. Of course it is required for consistency that $n_1 + n_2 + ..n_k = n$.

We shall use the geometric tool of a k-dimensional integral lattice in order to facilitate the discussion. The k dimensions correspond to the k terms a_i, and the points in the lattice correspond to the multinomial coefficients. Define a function between the points on the lattice and the multinomial coefficients.

$$\mathcal{G} : \mathbb{Z}^k \to \mathbb{Q}$$
$$(n_1, n_2, ..n_k) \to \binom{n_1 + n_2 + ..n_k}{n_1, n_2, ..nk}$$

Partitioning the n spots into n_1, n_2,.. n_k of a_1, a_2,.. a_k can be done in n different ways,

1. Selecting $n_1 - 1$, n_2,.. n_k objects out of $n-1$ from a_1, a_2,.. a_k having chosen 1 from a_1 already
2. Select n_1, $n_2 - 1$,.. n_k objects out of $n-1$ from a_1, a_2,.. a_k having chosen 1 from a_2 already...
3. Select n_1, n_2,.. $n_k - 1$ objects out of $n-1$ from a_1, a_2,.. a_k having chosen 1 from a_k already

Theorem 2.

$$\binom{n}{n_1, n_2, ..n_k} = \binom{n-1}{n_1 - 1, n_2, ..n_k} \times 1 + \binom{n-1}{n_1, n_2 - 1, ..n_k} \times 1 + \\ ..\binom{n-1}{n_1, n_2 - 1, ..n_k - 1} \times 1$$

Here, $n = n_1 + n_2 + ..n_k$ *as required by the constraint.*

Proof.

$$\binom{n-1}{n_1 - 1, n_2, ..n_k} \times 1 + \binom{n-1}{n_1, n_2 - 1, ..n_k} \times 1 + ..\binom{n-1}{n_1, n_2 - 1, ..n_k - 1} \times 1$$

$$= \frac{(n-1)!}{(n_1 - 1)!n_2!..n_k!} + \frac{(n-1)!}{n_1!(n_2 - 1)!..n_k!} + ..\frac{(n-1)!}{n_1!n_2!..(n_k - 1)!}$$

$$= \frac{(n-1)! \times n_1}{(n_1 - 1)!n_2!..n_k! \times n_1} + \frac{(n-1)! \times n_2}{n_1!(n_2 - 1)!..n_k! \times n_2} + ..\frac{(n-1)! \times n_k}{n_1!n_2!..(n_k - 1)! \times n_k}$$

$$= \frac{(n-1)!}{n_1!n_2!..n_k!} \times (n_1 + n_2 + ..n_k)$$

$$= \frac{n!}{n_1!n_2!..n_k!}$$

$$= \binom{n}{n_1, n_2, ..n_k}$$

□

Remark. This also proves theorem 1.

Armed with this result, the integrality of the multinomial coefficients can easily be shown.

Integrality

Lemma 3. *Any point lying on a non-negative part of any axis takes a value 1. That is, any point such as* $(0, ..n, ..0)$ *gives a binomial coefficient of 1.*

Proof.

$$\mathcal{G}(0, ..n_i = n, ..0) = \frac{n!}{0!..n!..0!} = 1$$

□

Lemma 4. *Any point lying outside the positive k-drant takes a value of 0. That is, any point for which at least one of* $n_1, n_2..n_k < 0$ *takes on a binomial coefficient of 0.*

Proof. See the important remark above. □

Theorem 5. *All multinomial coefficients take on integral values.*

Proof. The cases corresponding to negative[1] regions and points on the axes have been dealt with in the above two lemmas.

The rest of the proof shall proceed by induction on n, the order of the multinomial.

Base Case For $n = 0$ the case holds because of lemma 1 above. The only point is $(0, 0, ..0)$.

Induction Assume that the hypothesis holds for all multinomial coefficients for multinomials upto order $n - 1$. Then any binomial coefficient $\binom{n}{n_[1, n_2, ..n_k}$ can be written as,

$$\binom{n}{n_1, n_2, ..n_k} = \binom{n-1}{n_1 - 1, n_2, ..n_k} + \binom{n-1}{n_1, n_2 - 1, ..n_k} + \\ ..\binom{n-1}{n_1, n_2 - 1, ..n_k - 1}.$$

Using the fact that all the RHS terms are integers (the inductive hypothesis for $n - 1$) it is seen that the theorem holds for all n.

□

References

[1] Mathematics for Computer Science, 2018, Eric Lehman, F Tom Leighton, Albert R Meyer.
`https://courses.csail.mit.edu/6.042/spring18/mcs.pdf`

[1]Negative regions contain point for which at least one of the n_i are negative.