Transcendence of Newton's Laws

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Newton's laws are not the most accurate description of nature. Once things get very fast or very small the laws start to break down and more accurate theories are required to explain the goings on in these domains. However, in the domain of dogs, cats and cars newton's laws work amazingly well. So much so that many an engineer can get by without studying much more than a newtonian description of nature. One explanation is the simplicity and intuitiveness of newton's laws. This article explores another possibility.

1 Constant Mass Constant Boundary

Consider a system of particles indexed using an index set, $\mathfrak{I} = \{i\}$.

The laws are studied in an inertial reference frame. The position of the i'th particle is indicated as $\overrightarrow{\mathbf{r}}_i$ and its mass as m_i. Its velocity, momentum and acceleration are given by expressions,

$$\begin{aligned}
\overrightarrow{\mathbf{v}}_i &= \dot{\overrightarrow{\mathbf{r}}}_i \\
\overrightarrow{\mathbf{p}}_i &= m_i \dot{\overrightarrow{\mathbf{r}}}_i, \text{ and,} \\
\overrightarrow{\mathbf{a}}_i &= \ddot{\overrightarrow{\mathbf{r}}}_i.
\end{aligned}$$

In this section the masses are assumed to be constant. In this case newton's laws are stated as:

1. There exist inertial reference frames in which an isolated object would continue to remain at uniform motion unless acted upon by an external force[1].

$$\overrightarrow{\mathbf{F}}_i = 0 \implies \frac{\mathrm{d}\dot{\overrightarrow{\mathbf{r}}}_i}{\mathrm{d}t} = 0$$

2. The external force acting on a particle equals the rate of change of its momentum.

$$\overrightarrow{\mathbf{F}}_i = \frac{\mathrm{d}m_i \dot{\overrightarrow{\mathbf{r}}}_i}{\mathrm{d}t}$$

[1] For the remainder of the article an inertial reference frame would be used unless stated otherwise.

3. Forces always act in pairs such that they are equal and opposite and act on the two particles causing the forces.

$$\overrightarrow{\mathbf{F}}_{i\leftarrow j} = -\overrightarrow{\mathbf{F}}_{j\leftarrow i}$$

There needs to be some clarification about the forces and their description. It is assumed that the force on particle i is given by,

$$\overrightarrow{\mathbf{F}}_i = \sum_{j\in\mathfrak{I}} \overrightarrow{\mathbf{F}}_{i\leftarrow j}, \tag{1}$$

where the $\overrightarrow{\mathbf{F}}_{i\leftarrow j}$ are the atomic forces. This is equivalent to saying that the only kinds of forces are those that act between pairs of particles or that there are no fields.

If the third law is to be valid in case the indices are the same it must be that,

$$\overrightarrow{\mathbf{F}}_{i\leftarrow i} = \overrightarrow{\mathbf{0}}.$$

Consider now a partition of the index set,

$$\mathfrak{I}^{\diamond} = \{I\}, \text{ where } I \subset \mathfrak{I}, \text{ such that,}$$
$$\bigcup_{I\in\mathfrak{I}^{\diamond}} = \mathfrak{I} \text{ and } I\cap J = \delta_{IJ}{}^{2} \text{ when } I, J \in \mathfrak{I}^{\diamond}.$$

In this section it shall be assumed that the sets I do not change with time. The centre of mass can be viewed as a function from the sets I into the space of the inertial reference frame. It is defined as:

$$\overrightarrow{\mathbf{r}}_I = \frac{\sum_{i\in I} m_i \overrightarrow{\mathbf{r}}_i}{m_I}, \text{ where, } m_I = \sum_{i\in I} m_i$$
$$\implies m_I \overrightarrow{\mathbf{r}}_I = \sum_{i\in I} m_i \overrightarrow{\mathbf{r}}_i.$$

The force acting on I is the sum of the forces on each particle in I due to all the particles outside of I.

$$\begin{aligned} \overrightarrow{\mathbf{F}}_I = \sum_{i\in I} \overrightarrow{\mathbf{F}}_{i\leftarrow I^c}, \text{ where, } \overrightarrow{\mathbf{F}}_{i\leftarrow I^c} = \sum_{j\notin I} \overrightarrow{\mathbf{F}}_{i\leftarrow j} \\ \implies \overrightarrow{\mathbf{F}}_I = \sum_{i\in I, j\notin I} \overrightarrow{\mathbf{F}}_{i\leftarrow j}. \end{aligned} \tag{2}$$

However,

$$\begin{aligned} \overrightarrow{\mathbf{F}}_{i\leftarrow j} &= -\overrightarrow{\mathbf{F}}_{j\leftarrow i} \\ \implies \sum_{i,j\in I} \overrightarrow{\mathbf{F}}_{i\leftarrow j} &= \overrightarrow{\mathbf{0}}, \end{aligned} \tag{3}$$

[2] δ is the dirac-delta function.

the forces cancel in pairs so that the internal forces all add to $\overrightarrow{\mathbf{0}}$. As a consequence the formula for the total force acting on system I is simplified,

$$\begin{aligned}\overrightarrow{\mathbf{F}}_I &= \sum_{i\in I, j\notin I} \overrightarrow{\mathbf{F}}_{i\leftarrow j} \\ &= \sum_{i\in I, j\notin I} \overrightarrow{\mathbf{F}}_{i\leftarrow j} + \sum_{i,j\in I} \overrightarrow{\mathbf{F}}_{i\leftarrow j} \\ &= \sum_{i\in I, j\in \mathfrak{I}} \overrightarrow{\mathbf{F}}_{i\leftarrow j} \\ &= \sum_{i\in I} \overrightarrow{\mathbf{F}}_i .\end{aligned} \tag{4}$$

Assume that newton's laws hold for all the particles in the set $\mathfrak{I}$, then[3],

$$\overrightarrow{\mathbf{F}}_{I\leftarrow J} = \sum_{i\in I, j\in J} \overrightarrow{\mathbf{F}}_{i\leftarrow j} = \sum_{j\in J, i\in I} -\overrightarrow{\mathbf{F}}_{j\leftarrow i} = -\overrightarrow{\mathbf{F}}_{J\leftarrow I}, \tag{5}$$

and it is seen that the newton's third law is true for the sets of particles I. A crucial point to note is the exchange of the indices of summation in equation 5. This is valid because the indices of I and J are independent of one another.

In the following derivation it is to be noted that the time derivative and sum can be exchanged because of the fact that the boundaries are fixed, meaning the indices in the sets I are fixed, and because the masses of the particles are fixed.

$$\begin{aligned} m_I \overrightarrow{\mathbf{r}}_I &= \sum_{i\in I} m_i \overrightarrow{\mathbf{r}}_i \\ \implies m_I \dot{\overrightarrow{\mathbf{r}}}_I &= \frac{\mathrm{d} m_I \overrightarrow{\mathbf{r}}_I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i\in I} m_i \overrightarrow{\mathbf{r}}_i = \sum_{i\in I} m_i \dot{\overrightarrow{\mathbf{r}}}_i \end{aligned} \tag{6}$$

The second law is immediately obtained combining equations 4 and 6,

$$\overrightarrow{\mathbf{F}}_I = \sum_{i\in I} \overrightarrow{\mathbf{F}}_i = \sum_{i\in I} \frac{\mathrm{d} m_i \dot{\overrightarrow{\mathbf{r}}}_i}{\mathrm{d}t} = \frac{\mathrm{d} m_I \dot{\overrightarrow{\mathbf{r}}}_I}{\mathrm{d}t}, \tag{7}$$

and again a crucial point is the ability to exchange the time derivatives and the summations.

Finally, for the first law, it is assumed that the measurements are being done in an inertial frame.

[3]The force on I due to J is the sum of the forces on all the particles on I due to the particles in J.

$$\begin{aligned} \overrightarrow{\mathbf{F}}_I &= 0 \\ \implies \frac{\mathrm{d}m_I \dot{\overrightarrow{\mathbf{r}}}_I}{\mathrm{d}t} &= 0 \\ \implies \frac{\mathrm{d}\dot{\overrightarrow{\mathbf{r}}}_I}{\mathrm{d}t} &= 0 \end{aligned} \tag{8}$$

It is thus seen that newton's laws are obeyed by systems of newtonian particles. A critical requirement for all of these derivations are for the masses to be constant and also for the boundaries to be fixed. If these were not true the derivatives, dots and summation would not commute and the results above would not be obtained. Section 3 later explores some generalisations.

2 Error Propagation

Section 1 shows that if newton's laws are obeyed by a closed set of particles in the absence of fields then they would be obeyed by groups of such particles grouped in an arbitrary manner. This would include rigid bodies and fluids with a fixed boundary but not those with variable boundaries, the variable boundary case is explored in section 3. Even then, electromagnetism would not fit in this context because the entire paper ignores fields.

This section explores the case in which the laws are obeyed only approximately. The laws would then read:

1. $$\overrightarrow{\mathbf{F}}_i = 0 \implies \frac{\mathrm{d}\dot{\overrightarrow{\mathbf{r}}}_i}{\mathrm{d}t} = \mathcal{E}_i$$

2. $$\overrightarrow{\mathbf{F}}_i = \frac{\mathrm{d}m_i \dot{\overrightarrow{\mathbf{r}}}_i}{\mathrm{d}t} + \mathcal{E}_i$$

3. $$\overrightarrow{\mathbf{F}}_{i\leftarrow j} = -\overrightarrow{\mathbf{F}}_{j\leftarrow i} + \mathcal{E}_i$$

It is possible that the errors[4] depend on the index of the particle, on the positions of other particles and on time. These errors may give rise to very rich statistical phenomena such as ferromagnetism and perhaps even life. But this paper shall only explore the case where the errors are random and independent of one other, the positions of any particles and time. There are many situations in which such errors might obtain: the random errors that cause imprecision in measurement, brownian motion etc. and the causes of the errors might be

[4] In this sense the error is not really a "small" deviation from the actual value but could be quite large. It is better to think of the error as the difference between the actual behaviour and newtonian behaviour.

classical or quantum in nature. However, any such details are ignored and it is assumed that the errors are uniformly distributed with mean 0 and variance σ^2.

The errors are all represented by the random variables $\mathcal{E}_i$[5].

$$\begin{aligned} \mathbb{E}(\mathcal{E}_i) &= 0 \\ \mathbb{var}(\mathcal{E}_i) &= \sigma^2 \\ \mathbb{cov}(\mathcal{E}_i, \mathcal{E}_j) &= \delta_{ij} \end{aligned} \tag{9}$$

As a result when the laws are analysed for collections of particles,

1. $$\overrightarrow{\mathbf{F}}_I = 0 \implies \frac{\mathrm{d}\dot{\overrightarrow{\mathbf{r}}}_i}{\mathrm{d}t} = \mathcal{E}_I$$

2. $$\overrightarrow{\mathbf{F}}_I = \frac{\mathrm{d}m_I \dot{\overrightarrow{\mathbf{r}}}_I}{\mathrm{d}t} + \mathcal{E}_I$$

3. $$\overrightarrow{\mathbf{F}}_{I \leftarrow J} = -\overrightarrow{\mathbf{F}}_{J \leftarrow I} + \mathcal{E}_I$$

obtains.

Here,

$$\begin{aligned} \mathcal{E}_I &= |I| \times \mathcal{E}_i \\ \mathbb{E}(\mathcal{E}_I) &= 0 \\ \mathbb{var}(\mathcal{E}_I) &= \frac{\sigma^2}{|I|} \end{aligned} \tag{10}$$

as the error variance of a sum of independent random variables tends to have a variance that is $\frac{1}{number}$ as large[1].

Appealing to the reduction of errors due to the effect of adding many random terms reveals the reduction in error for each stage of abstracting the particles into systems of particles. It follows that that at each stage of abstraction, the laws of newton are obeyed more closely and thus there is an amplification of accuracy of the laws. The purpose of this paper was to show exactly this. This is the alternate viewpoint referenced at the beginning. One advantage of newton's laws might be the idea that abstractions of particles into systems reduce random errors that might be present in newton's laws. And thus, newton's laws are obeyed more and more closely as the "sizes"[6] of the systems increase.

[5] To be pedantic, one must distinguish between the error term in the first, second and third law. However, these details would also be ignored. All that needs to be remembered is that the error returns a random value whenever it is measured.

[6] Size, here, characterised by the number of particles.

3 Variable Mass Variable Boundaries

This section explores some generalisations, such as changing mass and changing boundaries, of the systems that are defined.

As shall be shown, it is not appropriate to use the centre of mass defined in equation 6, instead it is beneficial to redefine the centre of mass as[7]:

$$m'_I \dot{\vec{\mathbf{r}}}'_I = \sum_{i \in I} m_i \dot{\vec{\mathbf{r}}}_i. \tag{11}$$

It is assumed that the boundaries change with time and hence one must take this into account while doing the calculations. The third law can directly be obtained.

$$\vec{\mathbf{F}}_{I(t) \leftarrow J(t)} = \sum_{i \in I(t), j \in J(t)} \vec{\mathbf{F}}_{i \leftarrow j} = \sum_{j \in J(t), i \in I(t)} -\vec{\mathbf{F}}_{j \leftarrow i} = -\vec{\mathbf{F}}_{J(t) \leftarrow I(t)}, \tag{12}$$

and once again it is to be noted that the exchange of summations is valid because $I(t)$ and $J(t)$ are independent of each other at any instant of time.

As the boundaries change care must be taken to make sure that the various commutations are valid. The summations cannot be exchanged with the time derivative as the sets are dynamic. As a result, the second law would not be as simple to derive. Let it be assume that the second law is to be held valid for the system.

$$\begin{aligned}
\implies \vec{\mathbf{F}}'_I &= \frac{\mathrm{d}m'_I \dot{\vec{\mathbf{r}}}'_I}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i \in I(t)} m_i \dot{\vec{\mathbf{r}}}_i \\
&= \lim_{\Delta t \to 0} \frac{\sum_{i \in I(t+\Delta t)} m_i(t+\Delta t) \dot{\vec{\mathbf{r}}}_i(t+\Delta t) - \sum_{i \in I(t)} m_i(t) \dot{\vec{\mathbf{r}}}_i(t)}{\Delta t},
\end{aligned}$$

so far, it is only the definition of differentiation that has been used. By now splitting the sum into two parts, the derivative can be simplified,

$$\begin{aligned}
= \lim_{\Delta t \to 0} &\frac{\sum_{i \in I(t+\Delta t)} m_i(t+\Delta t) \dot{\vec{\mathbf{r}}}_i(t+\Delta t) - \sum_{i \in I(t+\Delta t)} m_i(t) \dot{\vec{\mathbf{r}}}_i(t)}{\Delta t} \\
&+ \lim_{\Delta t \to 0} \frac{\sum_{i \in I(t+\Delta t)} m_i(t) \dot{\vec{\mathbf{r}}}_i(t) - \sum_{i \in I(t)} m_i(t) \dot{\vec{\mathbf{r}}}_i(t)}{\Delta t},
\end{aligned}$$

and now it is seen that the first limit has to do with changing masses and velocities[8] whereas the second equation has to do with changing boundaries. It

[7]A prime is used to differentiate it from the previous definition of the centre of mass.

[8]It is hoped that this justifies the renewal of the definition of the centre of mass. With the original centre of mass the equations would get much more complicated.

is also to be observed that the two summations in the first limit can be combined and also that the Δt can be taken into the summation as per the law of limits. The second limit is also changed by introducing the idea of changing I. This makes sense when the summation to obtain a centre of mass is viewed as a function from the set of indices to the space of the vectors, and the operation is simply viewed as a consequence of the chain rule. Making these changes yields,

$$\begin{aligned}
&= \lim_{\Delta t \to 0} \sum_{i \in I(t+\Delta t)} \frac{m_i(t+\Delta t)\dot{\vec{\mathbf{r}}}_i(t+\Delta t) - m_i(t)\dot{\vec{\mathbf{r}}}_i(t)}{\Delta t} \\
&\qquad + \lim_{\Delta t \to 0} \frac{\sum_{i \in I(t+\Delta t)} m_i(t)\dot{\vec{\mathbf{r}}}_i(t) - \sum_{i \in I(t)} m_i(t)\dot{\vec{\mathbf{r}}}_i(t)}{\Delta I} \frac{\Delta I}{\Delta t}, \\
&= \sum_{i \in I(t)} \frac{\mathrm{d}m_i \dot{\vec{\mathbf{r}}}_i}{\mathrm{d}t} + \frac{\partial m_I' \dot{\vec{\mathbf{r}}}_I'}{\partial I} \frac{\mathrm{d}I}{\mathrm{d}t} \\
&= \sum_{i \in I(t)} F_i' + \frac{\partial m_I' \dot{\vec{\mathbf{r}}}_I'}{\partial I} \frac{\mathrm{d}I}{\mathrm{d}t}.
\end{aligned} \tag{13}$$

And thus, it is seen that that the force on a system contains two parts. The part due to the forces that act on each of the particles and due to the change in I[9]. However, the keen reader would notice that the definition of the forces in the third law and second law are of a different form and are not compatible with each other[10].

The change that needs to be made is to redefine equation 2 so that it reads,

$$\vec{\mathbf{F}}_I' = \sum_{i \in I, j \notin J} \vec{\mathbf{F}}_{i \leftarrow j} + \frac{\partial m_I' \dot{\vec{\mathbf{r}}}_I'}{\partial I} \frac{\mathrm{d}I}{\mathrm{d}t}. \tag{14}$$

As the definition of the atomic forces, $\vec{\mathbf{F}}_{i \leftarrow j}$, is not changed it is readily seen that equation 3 follows. The only change that needs to be done is to update equation 4 and the reader is implored to verify this for themselves. It is to be noted that the only changes relating the third law and second law is the updated equation 14. Thus, the applicability of the second and third laws follow independent of each other and only when they are to be related is there a need to introduce something like equation 14.

On the other hand the first law cannot be recovered at all from these considerations. It is written as,

$$\begin{aligned}
&\vec{\mathbf{F}}_I = 0 \\
\implies &\frac{\mathrm{d}m_I \dot{\vec{\mathbf{r}}}_I}{\mathrm{d}t} = 0,
\end{aligned} \tag{15}$$

[9] It is to be noted that the $\frac{\partial m_I \dot{\vec{\mathbf{r}}}_I}{\partial I}$ term denotes the operation of dividing the change in the summation due to an infinitesimal change in the index set by the infinitesimal change in the index set.

[10] That is the reason the force is also primed in this section.

and thus, it is only momentum as a whole that is conserved and the velocity could change with time even if the force is zero[11]. Additionally, even this is true only when the total force is zero, including the term due to changing boundaries[12].

4 Discussion

It is thus seen that newton's laws are transcendent in the sense that systems composed of particles that obey newton's laws also obey newton's laws. More so, if the particles obey the laws only approximately, the systems made of them obey the laws to a better approximation. If one views the deviation in newton's laws observed in section 3 as an error term, then section 2 tells us that systems composed of such systems would obey newton's laws. Indeed if the boundaries change in a random and unpredictable manner, then this approximation is well justified. Everyday objects fall into this category. A cell would obey newton's laws only approximately, tissue would obey them to a better approximation, humans as a whole would obey newton's laws quite very well and planets and stars would obey them even better. It is hoped that the reader is convinced that this could be a plausible reason for the wide applicability of newton's laws, it's transcendence.

5 Acknowldgement

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References

[1] Feller W. *An introduction to probability theory and its applications*, volume Volume 1. Wiley, 3 edition, 1968.

[2] Kolenkow R. Kleppner D. *An Introduction to Mechanics.* CUP, 2ed. edition, 2013.

[3] David Morin. *Introduction to classical mechanics: with problems and solutions.* Cambridge University Press, 1 edition, 2008.

[11] As the definition of an inertial system and the first law are so intricately tied, this might seem to be a huge impediment to the definition of an inertial system. However, this is far from true. One can easily define an inertial frame to be one for which a system of particles of fixed mass and unchanging boundary obeys the first law 15 and thus 8. For more details please check [2] or [3].

[12] This can also be seen as the law of conservation of momentum.