# Half Average

IFT-Retreat Schneefernerhaus, Jan 2022

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# Rules of the Game

- The game is played amongst a group of $N$ players. Each player is required to provide a real number between 0 and 100 inclusive.
- The aim is for each player to guess closest to $\frac{1}{2}$ of the average of all the players,

$$\frac{1}{2}\langle X\rangle = \frac{1}{2}\Sigma_{i=1}^{N} X_i.$$

# A Round

- Please go to menti.com and enter the *code* or wait for the QR code in the next slide
- Please do not interact with others
- There is around a minute of time to answer

# A Round

Figure: QR code

- Please go to menti.com and enter 2608 3050
- Please do not interact with others
- There is around a minute of time to answer

# Depth of Reasoning [Level-0]

It is motivated intuitively.

- Assume game played amongst players who guess randomly.
- Call these "level-0" players and their average $A_0$. $A_0$ is quite likely around 50 if there is a large enough set of players.

# Depth of Reasoning [Level-1]

- Level-1 players acknowledge the existence of level-0 players but fail to recognise the existence of others of their own kind.
- As a result, the typical level-1 players simply guesses $\frac{1}{2}A_0$ and feels smug. Call such a player's guess $G_1$,

$$G_1 = \frac{1}{2}A_0.$$

# Depth of Reasoning [Level-2]

- Level-2 players acknowledge the existence of level-0 as well as level-1 players, but not themselves.
- However, they are now faced with the much harder task of guessing the distribution of level-0 and level-1 players.

# Depth of Reasoning [Level-2]

$$A_1 = G_1,$$
$$G_2 = \frac{1}{2}\Sigma_{i=0}^{1} p_i^{(2)} A_i,$$

where $A_1$ is the average of all level-1 players and $p_i^{(2)}$ is the distribution of level-0 and level-1 players assumed by level-2 players.

- There is a complication that arises in this level, it could be that the $p_i^{(2)}$ distribution is not a constant among level-2 players but it changes from player to player. However, it is assumed to be the same in this analysis.

# Depth of Reasoning [Level-3]

$$A_1 = G_1,$$
$$A_2 = G_2,$$
$$G_3 = \frac{1}{2}\Sigma_{i=0}^{2} p_i^{(3)} A_i,$$

- Note that the average of level-2 players, $A_2$, is assumed based on $\tilde{p}_i^{(2)}$ which may not be what the level-2 players assume, $p_i^{(2)}$, but rather what level-3 players assume that level-2 players assume.
- Again simplifying assumption is made that level-3 players completely understand level-2 players and so on.

# Depth of Reasoning [Level-n]

Generalising,

$$A_i = G_i(\forall i \in \{0, 1, ...n-1\}),$$
$$G_n = \frac{1}{2}\Sigma_{i=0}^{n-1} p_i^{(n)} A_i,$$

where level-n players only acknowledge the existence of players of level-0 to level-(n-1).

# Bounds

- Again, the argument is made intuitively in this section. It is given that $A_0 = 50$. What are the range of values of $G_1$?
- All level-1 players make the same guess,

$$G_1 = 25$$

# Bounds

- Now look at $G_2$. It depends on the assumed distribution $p_i^{(2)}$ but bounds can be placed on its range of values,

$$\begin{array}{ccccc} 12.5 & \leq & G_2 & \leq & 25 \\ (p_i^{(2)} = \delta_{i1}) & & & & (p_i^{(2)} = \delta_{i0}), \end{array}$$

- which implies for $G_3$,

$$\begin{array}{ccccc} 6.25 & \leq & G_3 & \leq & 25 \\ (p_i^{(3)} = \delta_{i2}) & & & & (p_i^{(3)} = \delta_{i0}), \end{array}$$

noting that the lower limit depends on the smallest value for $G_2$.

# Bounds

Once again continuing the induction intuitively,

$$\frac{50}{2^n} \leq G_n \leq \frac{50}{2}$$

# Concrete Model

This section attempts to generate a concrete model for each $p_i^{(n)}$ and analyse what happens. Consider that $p_i^{(n)}$ is a geometric series. That is,

$$
\begin{aligned}
p_i^{(n)} &\propto \alpha^i (\forall i \in \{0, 1, ...n-1\}) \\
\implies p_i^{(n)} &= \frac{\alpha^i}{S_n} \\
S_n &= \Sigma_{i=0}^{n-1} \alpha^i = \frac{1-\alpha^n}{1-\alpha}.
\end{aligned}
$$

# Concrete Model [Initial Conditions]

The initial conditions are:

$$
\begin{aligned}
G_1 &= 25 \\
G_0 \equiv A_0 &= 50
\end{aligned}
$$

and

$$
\begin{aligned}
G_2 &= \frac{1}{2}\frac{A_0 + \alpha A_1}{1+\alpha} \\
&= \frac{1}{2}\frac{A_0 + \alpha G_1}{S_2} \\
\Leftarrow G_1 &= A_1 = 25.
\end{aligned}
$$

# Concrete Model [Recursion]

A recursion relation for the guess can be derived,

$$G_n = \frac{1}{2}\Sigma_{i=0}^{n-1} p_i^{(n)} A_i$$

$$\begin{aligned}
\implies 2G_nS_n &= \Sigma_{i=0}^{n-1}\alpha^i A_i \\
&= \Sigma_{i=0}^{n-2}\alpha^i A_i + \alpha^{n-1}A_{n-1} \\
&= 2G_{n-1}S_{n-1} + \alpha^{n-1}A_{n-1} \\
&= (2S_{n-1} + \alpha^{n-1})G_{n-1} \\
\Leftarrow G_{n-1} &= A_{n-1}
\end{aligned}$$

which can be proved using induction.

# Concrete Model

The concrete recursion relation is,

$$\implies G_n = \frac{2S_{n-1} + \alpha^{n-1}}{2S_n} G_{n-1} = \frac{2 - \alpha^{n-1} - \alpha^n}{2 - 2\alpha^n} G_{n-1}$$

# Concrete Model [Numerical Analysis]

Observed results for 200 iterations of the recursions.

# Concrete Model [Numerical Analysis]

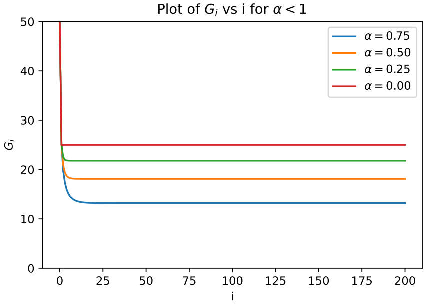



Figure: Observed graphs for $\alpha < 1$.

As hypothesised, the $G_\infty$ reaches a finite value when $\alpha < 1$. This makes sense as the higher level people exponentially decay and thus, the lower level players dominate the statistics.

# Concrete Model [Numerical Analysis]

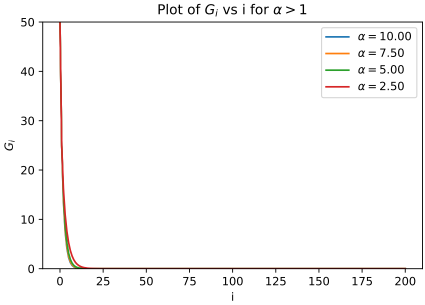



Figure: Obeserved graphs for $\alpha > 1$.

In this case, the higher level players dominate and so the average quickly decays to 0.

# Level-$(\infty + 1)$

- If $\alpha \geq 1$ it is seen that the optimal choice is 0 and there is no incentive to move away from that solution, it is a stable point. If the players are predominantly of a higher level the optimal choice is 0.
- The interesting case is when $G_\infty$ is non-zero. As seen through the previous numerical analysis this is, in fact, possible.

# Level-$(\infty + 1)$

Consider, for $\alpha < 1$,

$$\begin{aligned} G_\omega &= \frac{1}{2}\Sigma_{i=0}^{\infty} p_i^{(\infty)} A_i \\ \implies 2G_\omega S_\infty &= \Sigma_{i=0}^{\infty} \alpha^i A_i \\ &= \Sigma_{i=0}^{n-1} \alpha^i A_i + \Sigma_{i=n}^{\infty} \alpha^i A_i \\ &= 2G_n S_n + \alpha^n (\text{something finite}) \end{aligned}$$

using the recursion relation. Taking the limit of the right hand side as $n \to \infty$ and noticing that the finite terms are exponentially suppressed,

$$\begin{aligned} \implies 2G_\omega S_\infty &= \lim_{n\to\infty} 2G_n S_n \\ \implies G_\omega &= G_\infty, \end{aligned}$$

that is, that the expected guess given all the infinite terms is the same as the limit of the guess as the number of terms tend to infinity.

# Level-$(\infty + 1)$

There is a very important point to note however, the average of all the terms must be twice the guess to be consistent,

$$A_\infty = \Sigma_{i=0}^{\infty} p_i^{(\infty)} A_i = 2G_\omega = 2G_\infty .$$

# Level-$(\infty + 1)$

- For $\alpha \geq 1$ everybody is a level-$\infty$[1] player, however, for $\alpha < 1$ the number of level-$\infty$ players tends to 0 for any finite number of total players.
- Level-$(\infty + 1)$ players realise that as a consequence of the previous analysis there would be a finite set of people who are level-$\infty$. Say the ratio of level-$\infty$ people to finite level people is $\beta$. This means their guess is,

$$G_{\infty+1} = \frac{1}{2}\frac{A_\infty + \beta G_\infty}{1+\beta} = \frac{2+\beta}{2(1+\beta)} G_\infty \neq G_\infty,$$

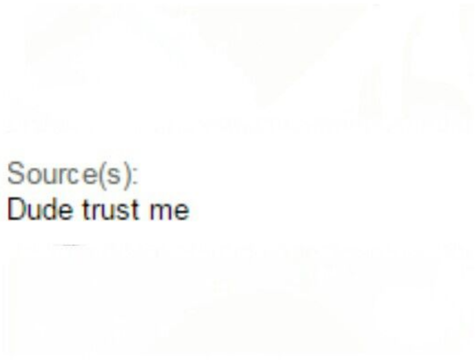
as $A_\infty = 2G_\infty$.

[1] Or rather a level-n player for $n \to \infty$

# Level-$(\infty + 1)$

Continuing as before leads to the recursion relation,

$$G_{\infty+n} = \frac{1}{1+\beta}\left(1 - \frac{S_{n-1}}{S_n}\right) G_\infty + \frac{2(1+\beta)S_{n-1} + \beta\alpha^{n-1}}{2(1+\beta)S_n} G_{\infty+n-1}$$



# Level-$(\infty + 1)$

To continue the analysis define[2],

$$T_n \equiv \Sigma_{i=0}^{n-1} \beta^i,$$

and,

$$q_i^{(n)} \equiv \frac{\beta^i}{T_n} (\forall i \in \{0, 1, ...n-1\}).$$

[2]Generalising this completely requires infinitely slicker notation. Literally.

# Level-($\infty^2$)

Then, having established all "smaller" sums $G_{k\infty}$ for $k < n$,

$$G_{n\infty+m} = \frac{1}{2}(\Sigma_{k=0}^{n-1} q_k^{(n+1)} 2G_{(k+1)\infty} + q_n^{(n+1)} \Sigma_{i=0}^{m-1} p_i^{(m)} A_{n\omega+i}),$$

and one can reach $G_{\omega\omega}$ and so on.

# Level-($\infty + 1$) [Numerical Analysis]

A similar numerical analysis as before reveals,

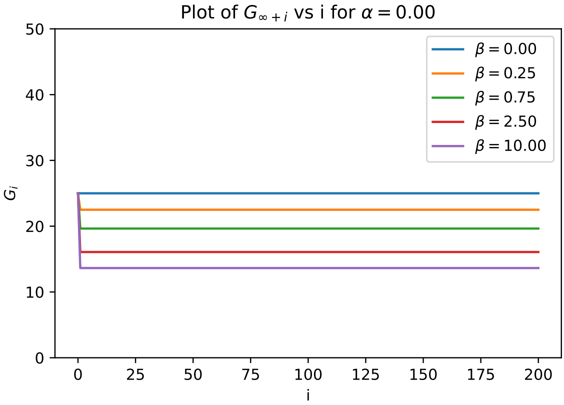



Figure: Range of $\beta$s for $\alpha = 0.00$.

# Level-$(\infty + 1)$ [Numerical Analysis]

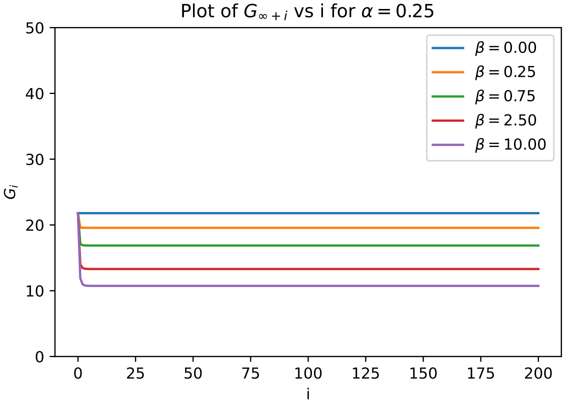



Figure: Range of $\beta$s for $\alpha = 0.25$.

# Level-$(\infty + 1)$ [Numerical Analysis]

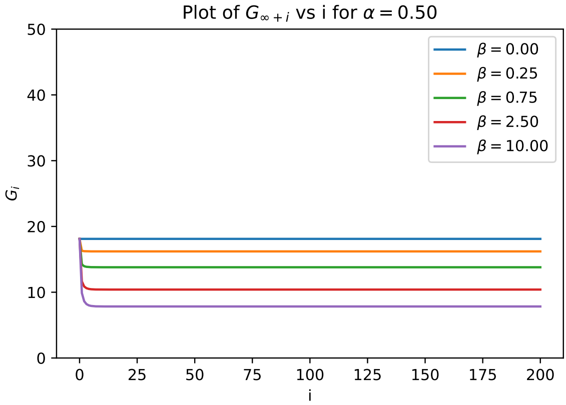



Figure: Range of $\beta$s for $\alpha = 0.50$.

# Level-($\infty + 1$) [Numerical Analysis]

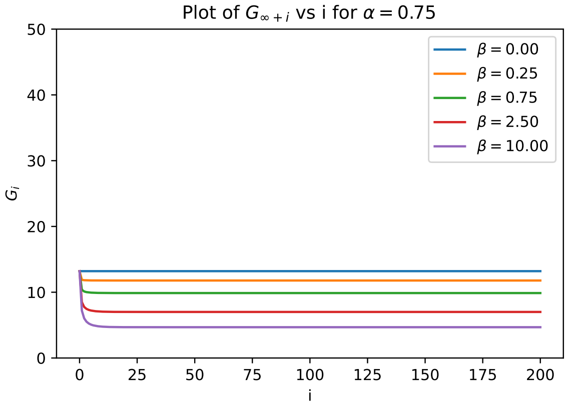



Figure: Range of $\beta$s for $\alpha = 0.75$.

# Level-($\infty + 1$) [Numerical Analysis]

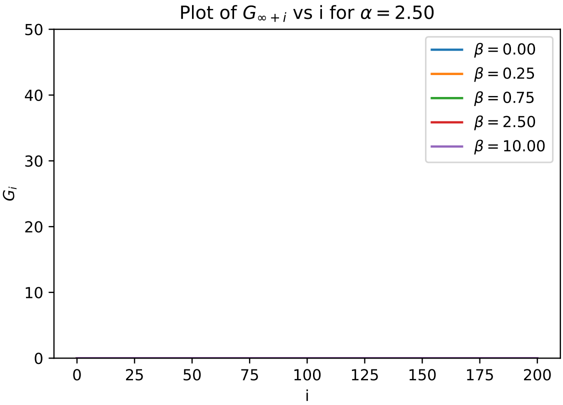



Figure: Range of $\beta$s for $\alpha = 2.50$.

Once 0 is reached, it stays there.

# Level-($\infty + 1$) [Numerical Analysis]

Now for some graphs of $n\infty$,

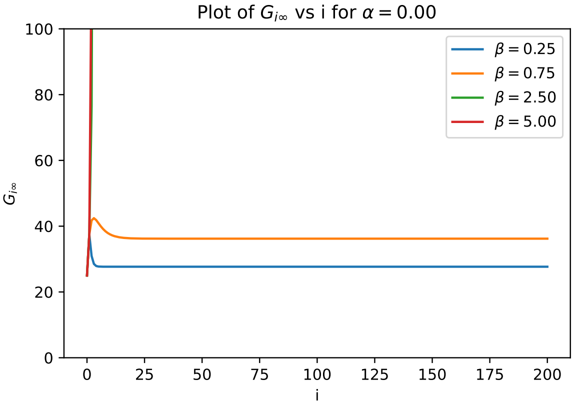



Figure: Range of $\beta$s for $\alpha = 0.00$.

# Level-$(\infty + 1)$ [Numerical Analysis]

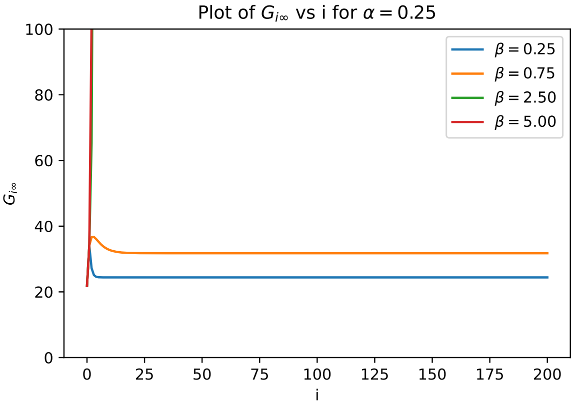



Figure: Range of $\beta$s for $\alpha = 0.25$.

# Level-($\infty + 1$) [Numerical Analysis]

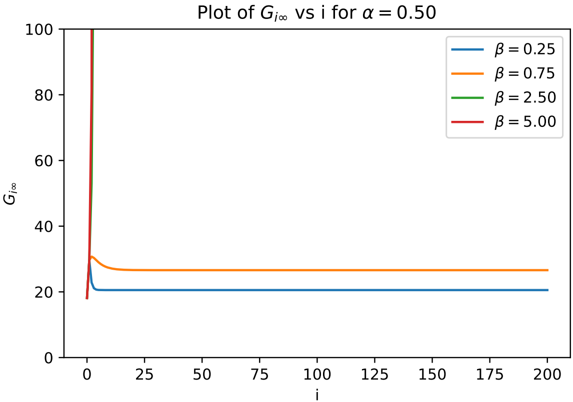



Figure: Range of $\beta$s for $\alpha = 0.50$.

# Level-($\infty + 1$) [Numerical Analysis]

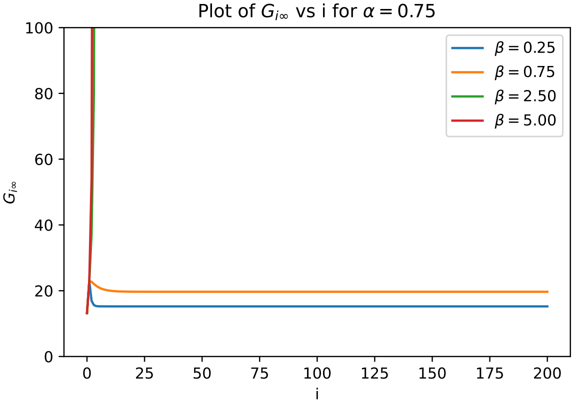



Figure: Range of $\beta$s for $\alpha = 0.75$.

# Level-($\infty + 1$) [Numerical Analysis]

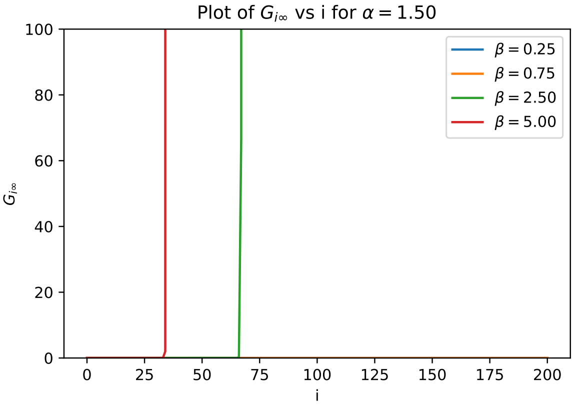



Figure: Range of $\beta$s for $\alpha = 1.50$.

# What Next?

- Perform analytical recursion calculations to make sure numerical results make sense.
- Improve notation so that ordinal arithmetic can be used.
- Check models other than geometric.
- Real world tests.

# Remarks

- There seems to be rich behaviour in expanding beyond finite depths of reasoning. Maybe indicates why human behaviour is so complicated.
- It lends some support to the frequentist interpretation of probability.

# References

Wikipedia, *Guess 2/3 of the Average*.

# Thanks

Thanks for listening!