Interaction Free Measurement

Vishal Johnson

October 2021

1 Introduction

Interaction free measurement is a procedure whereby the state of a system can be assessed without interacting with it. This may seem a bit surprising as measurement, by definition, seems to indicate that there must have been some sort of interaction. A simple example shows that interaction free measurement is, in fact, mundane.

Imagine there are two boxes and it is known that there is one ball contained in these boxes. If either box is opened and revealed to not contain the ball, it will have, without interaction, been found out that the other box contains a ball. The other box's state has been measured without interaction! Of help in this case is the information provided about the number of balls in boxes.

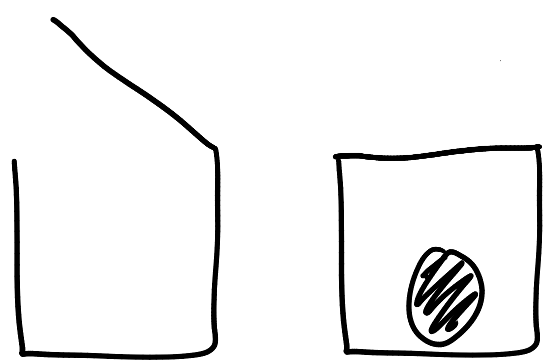


Figure 1: Given prior information, opening one box ascertains the contents of the other.

The surprising quantum mechanical extension is being able to probe the state of the system even without prior knowledge. This article first discusses the Elitzur-Vaidman Bomb Tester [1] and then explores the Quantum Zeno Effect following Vazirani [2].

2 Description of the Interferometer

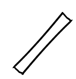
A crucial ingredient of this experiment (gedankenexperiment) is the interferometer. For the purpose of our discussion, one may use a working schematic model of the interferometer. What is to be ensured is that this agrees with experiment. An explanation in terms of electromagnetic waves can be found in reference [3].

This set up uses two kinds of mirrors, partially reflective and totally reflective. The photon states are: $|{\rightarrow}\rangle$, indicating that the photon is travelling to the right, and $|{\uparrow}\rangle$, likewise.

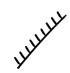
Equi Reflectance

The descriptions of the mirrors' actions are as follows[1],

$$\begin{aligned}
\text{partially reflective: } |{\rightarrow}\rangle &\rightarrow \frac{1}{\sqrt{2}}|{\rightarrow}\rangle + \frac{i}{\sqrt{2}}|{\uparrow}\rangle \\
|{\uparrow}\rangle &\rightarrow \frac{1}{\sqrt{2}}|{\uparrow}\rangle + \frac{i}{\sqrt{2}}|{\rightarrow}\rangle
\end{aligned}$$



$$\begin{aligned}
\text{totally reflective: } |{\rightarrow}\rangle &\rightarrow i\,|{\uparrow}\rangle \\
|{\uparrow}\rangle &\rightarrow i\,|{\rightarrow}\rangle
\end{aligned}$$



and the schematic diagram looks like (figure 2),

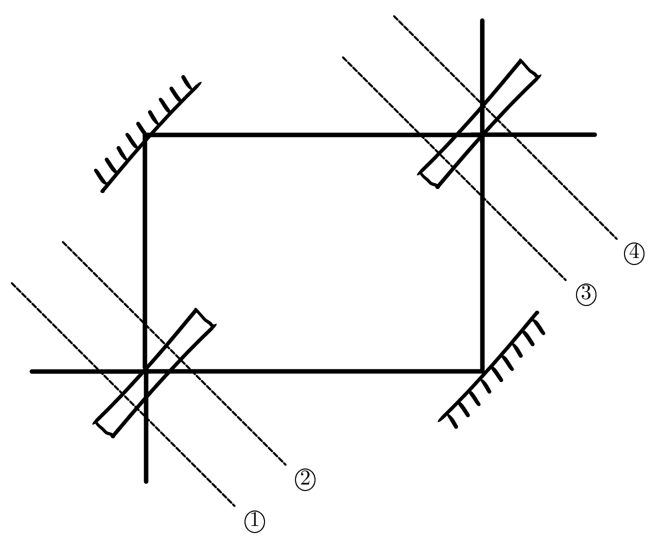


Figure 2: Mach-Zehnder Interferometer with 0.5 reflectance.

[1] Note that these operations are unitary.

Starting with a $|{\rightarrow}\rangle$ and tracing its path through the network, the following states are traversed,

$$
\begin{aligned}
\underbrace{|{\rightarrow}\rangle}_{1} &\rightarrow \underbrace{\frac{1}{\sqrt{2}}|{\rightarrow}\rangle + \frac{i}{\sqrt{2}}|{\uparrow}\rangle}_{2} \\
&\rightarrow \underbrace{\frac{i}{\sqrt{2}}|{\uparrow}\rangle - \frac{1}{\sqrt{2}}|{\rightarrow}\rangle}_{3} \\
&\rightarrow \frac{i}{\sqrt{2}}(\cancel{\frac{1}{\sqrt{2}}|{\uparrow}\rangle} + \frac{i}{\sqrt{2}}|{\rightarrow}\rangle) - \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}|{\rightarrow}\rangle + \cancel{\frac{i}{\sqrt{2}}|{\uparrow}\rangle}) \\
&= \underbrace{-|{\rightarrow}\rangle}_{4}!
\end{aligned}
$$

This means that starting with $|{\rightarrow}\rangle$ one can never end up with a $|{\uparrow}\rangle$.

$$\mathcal{P}(\uparrow) = 0$$

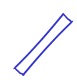
Again, a look at reference [3] confirms the physical correctness of this argument. It is instructive to explore arbitrary reflectance.

Arbitrary Reflectance

The descriptions of the additional mirrors' actions are as follows,

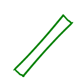
partially reflective (blue): $|{\rightarrow}\rangle \rightarrow p\,|{\rightarrow}\rangle + iq\,|{\uparrow}\rangle$

$|{\uparrow}\rangle \rightarrow p\,|{\uparrow}\rangle + iq\,|{\rightarrow}\rangle$



partially reflective (green): $|{\rightarrow}\rangle \rightarrow q\,|{\rightarrow}\rangle + ip\,|{\uparrow}\rangle$

$|{\uparrow}\rangle \rightarrow q\,|{\uparrow}\rangle + ip\,|{\rightarrow}\rangle$



$(p, q \in \mathbb{R},\ p^2 + q^2 = 1)$

and the schematic diagram looks like (figure 3),

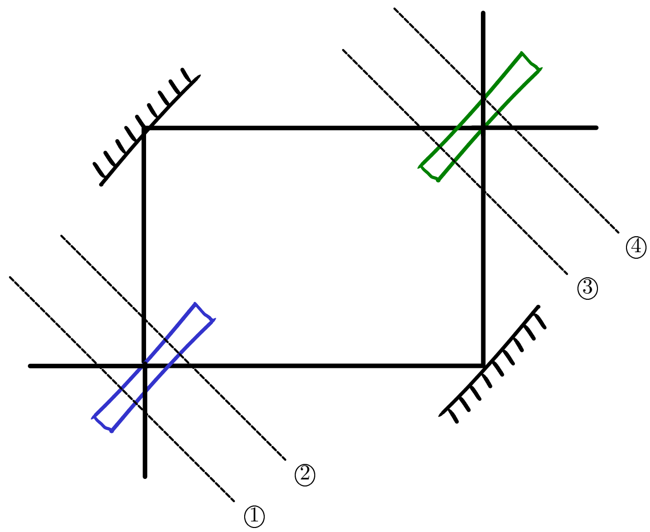


Figure 3: Mach-Zehnder Interferometer with arbitrary reflectance.

Starting with a $|\rightarrow\rangle$ and tracing its path through the network, the following set of states are traversed,

$$\begin{aligned}
\underbrace{|\rightarrow\rangle}_{\textcircled{1}} &\rightarrow \underbrace{p\,|\rightarrow\rangle + iq\,|\uparrow\rangle}_{\textcircled{2}} \\
&\rightarrow \underbrace{ip\,|\uparrow\rangle - q\,|\rightarrow\rangle}_{\textcircled{3}} \\
&\rightarrow ip(\cancel{q\,|\uparrow\rangle} + ip\,|\rightarrow\rangle) - q(q\,|\rightarrow\rangle + \cancel{ip\,|\uparrow\rangle}) \\
&= \underbrace{-\,|\rightarrow\rangle}_{\textcircled{4}}!
\end{aligned}$$

Once again, starting with $|\rightarrow\rangle$ one can never end up with a $|\uparrow\rangle$.

$$\mathcal{P}(\uparrow) = 0$$

What happens to vertical states?

Starting with a $|\uparrow\rangle$ and tracing its path through the network, the following set of states are traversed,

$$\begin{aligned}
|\uparrow\rangle &\rightarrow p\,|\uparrow\rangle + iq\,|\rightarrow\rangle \\
&\rightarrow ip\,|\rightarrow\rangle - q\,|\uparrow\rangle \\
&\rightarrow ip(\cancel{q\,|\rightarrow\rangle} + ip\,|\uparrow\rangle) - q(q\,|\uparrow\rangle + \cancel{ip\,|\rightarrow\rangle}) \\
&= -\,|\uparrow\rangle\,.
\end{aligned}$$

This means that an arbitrary state $\alpha\,|{\rightarrow}\rangle + \beta\,|{\uparrow}\rangle$ goes to $-(\alpha\,|{\rightarrow}\rangle + \beta\,|{\uparrow}\rangle)$[2].

3 Elitzur-Vaidman Bomb Tester

Consider a bomb that is triggered by a photon in its cavity. It is so sensitive that even a single photon would be able to trigger it. Once triggered, the bomb explodes and ceases to exist.

There are two classes of bombs, working ones that work as above and defective ones which simply allow photons to pass through their cavity without consequence. The aim is to find out if a bomb is defective or not. Of course, classically, as even one photon would trigger the bomb there is no way of finding out if a bomb is defective without risking it exploding. The Elitzur-Vaidman bomb tester can, with a certain, probability test the bomb without causing its explosion.

The trick is to place one such bomb in the lower leg of the interferometer previously described. Here is a schematic diagram (figure 4),

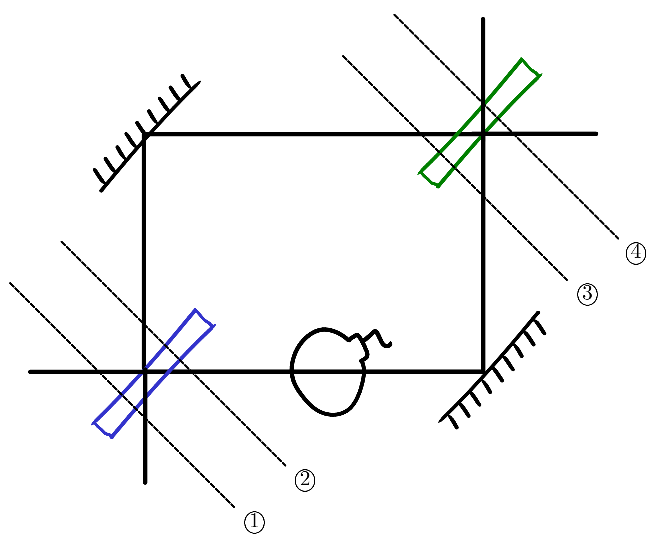


Figure 4: Elitzur-Vaidman bomb detector.

Tracing the states of the photon as it traverses the network, a defective bomb would allow the photon to pass through unaffected and it would behave as if there was nothing in the cavity (as in figure 3), there is no chance of obtaining a $|{\uparrow}\rangle$ starting from a $|{\rightarrow}\rangle$. On the other hand, if the bomb was working (as in

[2]Note that this shows that the action of the interferometer is unitary, $-\mathbb{1}$.

figure 4), the series of traversed states is,

$$\begin{aligned}\underbrace{|{\rightarrow}\rangle}_{①} &\rightarrow \underbrace{p\,|{\rightarrow}\rangle + iq\,|{\uparrow}\rangle}_{②}\\ &\rightarrow \underbrace{p\,|\star\rangle - q\,|{\rightarrow}\rangle}_{③}\\ &\rightarrow \underbrace{p\,|\star\rangle - q(q\,|{\rightarrow}\rangle + ip\,|{\uparrow}\rangle}_{④})\end{aligned}$$

with $|\star\rangle$ indicating that the photon has been absorbed by the bomb thus causing it to explode.

Notice now that there is a non-zero probability to obtain $|{\uparrow}\rangle$ as a final state! In this circumstance, the bomb will have been measured without interaction and without any prior information!

Maximising Yield

Given that the bomb is working, the probability to obtain $|{\uparrow}\rangle$ is,

$$\mathcal{P}(\uparrow|\text{working}) = p^2q^2$$

whereas,

$$\mathcal{P}(\uparrow|\text{defective}) = 0.$$

If this is applied to a large number of bombs there is a certain probability to cause a bomb to explode and a certain probability to deduce its operability without causing an explosion. Given that the bomb is working,

$$\begin{aligned}\mathcal{P}(\text{no explosion}|\text{successful test}) &= \frac{\mathcal{P}(\text{successful test} \wedge \text{no explosion})}{\mathcal{P}(\text{successful test})}\\ &= \frac{p^2q^2}{p^2q^2+p^2}\\ &= \frac{q^2}{1+q^2}.\end{aligned}$$

This probability can be maximised by taking the limit $q \rightarrow 1$[3],

$$\lim_{q\rightarrow 1}\frac{q^2}{1+q^2} = \frac{1}{2}.$$

4 Quantum Zeno Effect

Amazingly, even this limit can be surpassed. But new machinery must be introduced in order to achieve this. The discussion here follows that by Umesh Vazirani [2].

[3] In the equi reflectance case, $q^2 = \frac{1}{2}$ and the probability of successful detection sans explosion is $\frac{1}{3}$.

Along with the above Elitzur-Vaidman bomb tester another unitary rotation matrix is also required. Consider,

$$\mathrm{U}(\theta) = \begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix}$$

acting on the space,

$$span\left\{ |{\rightarrow}\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |{\uparrow}\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

This leads to the relations,

$$\begin{aligned} \mathrm{U}(\theta)\,|{\rightarrow}\rangle &= \cos(\theta)\,|{\rightarrow}\rangle + i\sin(\theta)\,|{\uparrow}\rangle \\ \mathrm{U}(\theta)\,|{\uparrow}\rangle &= \cos(\theta)\,|{\uparrow}\rangle + i\sin(\theta)\,|{\rightarrow}\rangle, \end{aligned}$$

which means,

$$\mathrm{U}(\theta)(\cos(k\theta)\,|{\rightarrow}\rangle + i\sin(k\theta)\,|{\uparrow}\rangle) = \cos((k+1)\theta)\,|{\rightarrow}\rangle + i\sin((k+1)\theta)\,|{\uparrow}\rangle.$$

The schematic diagram is as follows (figure 5),

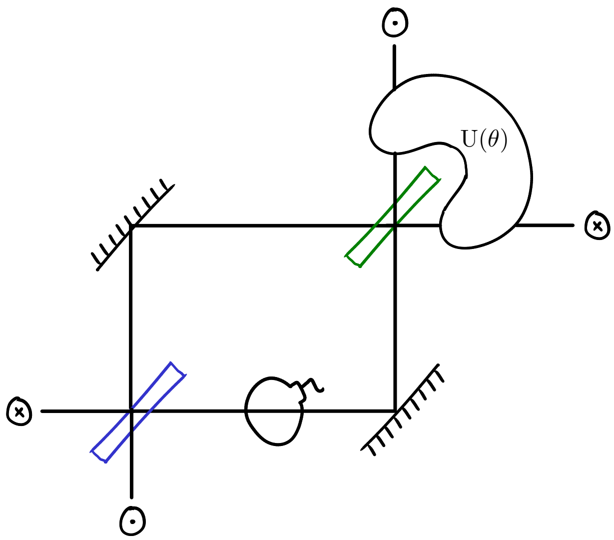


Figure 5: Circuit for nearly always measuring the state with nearly negligible probability of explosion.

Two cases are to be distinguished. In case the bomb is defective, the photon passes without any consequence,

$$\begin{aligned}
|{\rightarrow}\rangle &\rightarrow -|{\rightarrow}\rangle \\
&\rightarrow -(\cos(\theta)|{\rightarrow}\rangle + i\sin(\theta)|{\uparrow}\rangle) \\
&(\text{repeating N times ... }) \\
&\rightarrow (-1)^N(\cos(N\theta)|{\rightarrow}\rangle + i\sin(N\theta)|{\uparrow}\rangle),
\end{aligned}$$

and if $N\theta \approx \frac{\pi}{2}$, the final state is $\approx |{\uparrow}\rangle$.

On the other hand, if the bomb is working, there is a chance that the photon is absorbed by the bomb,

$$|{\rightarrow}\rangle \rightarrow p\,|{\star}\rangle - q(q\,|{\rightarrow}\rangle + ip\,|{\uparrow}\rangle).$$

The photon gets absorbed with probability p^2 and with a probability q^2 enters the rotation part of the circuit in the state $-(q\,|{\rightarrow}\rangle + ip\,|{\uparrow}\rangle)$[4]. Choosing $q = \cos(-\theta)$ and $p = \sin(-\theta)$ leads to,

$$\mathrm{U}(\theta)(-(q\,|{\rightarrow}\rangle + ip\,|{\uparrow}\rangle)) = -\mathrm{U}(\theta)(\cos(-\theta)\,|{\rightarrow}\rangle + i\sin(-\theta)\,|{\uparrow}\rangle) = -\,|{\rightarrow}\rangle\,.$$

This means that after N rounds the state is $|{\rightarrow}\rangle$ with probability[5],

$$(q^2)^N \approx \left(1 - \frac{\theta^2}{2}\right)^{2N} \approx 1 - \frac{\pi^2}{4N},$$

and gets absorbed with a probability $\approx \frac{\pi^2}{4N}$ and this can be made as small as possible by choosing θ (equivalently N) judiciously. And thus, the bomb can be detected almost certainly using this setup, without any prior information and with almost no chance of setting it off!

5 Conclusion

Not only can the state of a system be measured without interacting with it, but using quantum mechanics it can be done almost certainly with almost no chance of destroying the state!

There has been experimental verification of this effect (see reference [4]) but there is still some debate about details. The debate about how quantum states evolve and "collapse" continues.

A Appendix: Must the wavefunction collapse?

In case of the bomb tester, there is a possibility for the photon to be absorbed by the bomb and lost forever. This is indicated by the state $|{\star}\rangle$.

[4]Look at appendix A to see how this might be possible without wavefunction collapse.

[5]It is assumed that N is large, or that equivalently, θ is small.

However in section 4, it was necessary to consider the wavefuction to collapse in case it is intercepted by the bomb. The question is whether this can be analysed without considering wavefunction collapse. Attempting to trace the evolution of the photon state as it repeatedly traverses the circuit in figure 5,

$$\begin{aligned}
|{\rightarrow}\rangle &\rightarrow p\,|{\star}\rangle - q(q\,|{\rightarrow}\rangle + ip\,|{\uparrow}\rangle) \\
&\rightarrow p\,|{\star}\rangle - q\,|{\rightarrow}\rangle \\
&\rightarrow p\,|{\star}\rangle - q(p\,|{\star}\rangle - q\,|{\rightarrow}\rangle) \\
&\ldots \\
&\rightarrow p(1 + (-q) + (-q)^2 \ldots (-q)^{N-1})\,|{\star}\rangle + (-q)^N\,|{\rightarrow}\rangle\,.
\end{aligned}$$

The non-absorbed state $|{\rightarrow}\rangle$ with probability q^{2N} is reassuring, however, even a look at the state after two passes reveals that probability doesn't seem to be conserved in this case[6].

However, not all hope is lost. A very specific assumption was made that all the $|{\star}\rangle$ states are equivalent. Must the wavefunction still collapse if this requirement is relaxed?

Maybe not!

Assume now that the $|{\star}\rangle$ states at different times are not equivalent and label them using the iteration that the photon is currently in. Using the diagram in figure 6,

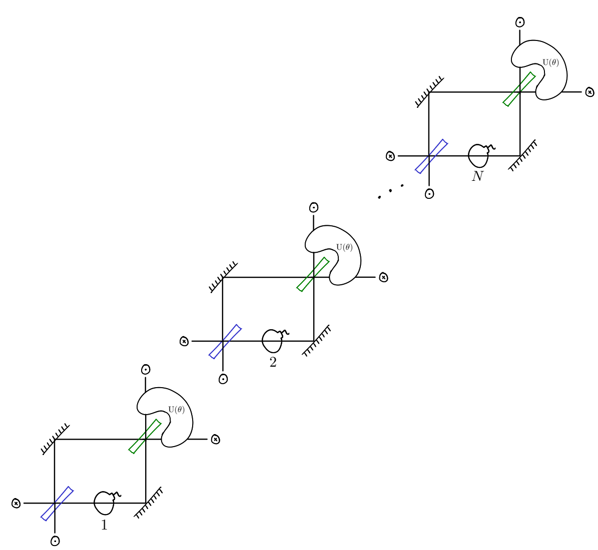


Figure 6: Circuit of figure 5 laid out in a convincing manner.

[6] $q^4 + p^2(1-q)^2$ need not equal 1.

the progress of the photon state can be traced,

$$\begin{aligned}
|{\rightarrow}\rangle &\rightarrow p\,|{\star_1}\rangle - q\,|{\rightarrow}\rangle \\
&\rightarrow p\,|{\star_1}\rangle - q(p\,|{\star_2}\rangle - q\,|{\rightarrow}\rangle) \\
&\dots \\
&\rightarrow p(|{\star_1}\rangle + (-q)\,|{\star_2}\rangle + (-q)^2\,|{\star_3}\rangle \dots (-q)^{N-1}\,|{\star_N}\rangle) + (-q)^N\,|{\rightarrow}\rangle
\end{aligned}$$

which ensures that probability is conserved!

Assuming that the bomb at different instances of time are different absorbing elements, one can reach a consistent explanation of the quantum zeno effect without wavefunction collapse!

References

[1] Avshalom C. Elitzur and Lev Vaidman, *Quantum Mechanical Interaction-Free Measurements*, 1993.

[2] Umesh Vazirani, *Applications of Quantum Search, Quantum Zeno Effect*, 2005.

[3] K.P. Zetie, S.F. Adams and R.M. Tocknell, *How does a Mach–Zehnder interferometer work?*, 2000.

[4] Paul G. Kwiat; H. Weinfurter; T. Herzog; A. Zeilinger; M. Kasevich, *"Experimental realization of "interaction-free" measurements"*, *Fundamental Problems in Quantum Theory. 755: 383–393*, 1994.