1 Cartesian Coordinates from Orbit Parameters

Cartesian coordinates of a Keplerian Orbit in an inertial frame can be expressed as

$$\begin{aligned}
\boldsymbol{r} &= R_z(\Omega)R_x(i)R_z(\omega)\begin{bmatrix}\xi\\ \eta\\ 0\end{bmatrix} \qquad (1)\\
&= \begin{bmatrix}\cos\Omega & -\sin\Omega & 0\\ \sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos i & -\sin i\\ 0 & \sin i & \cos i\end{bmatrix}\begin{bmatrix}\cos\omega & -\sin\omega & 0\\ \sin\omega & \cos\omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a\left(\cos E - e\right)\\ b\sin E\\ 0\end{bmatrix}\\
&= \begin{bmatrix}\cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i\\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i\\ \sin i\sin\omega & \sin i\cos\omega & \cos i\end{bmatrix}\begin{bmatrix}a\left(\cos E - e\right)\\ b\sin E\\ 0\end{bmatrix}
\end{aligned}$$

where

- Ω is the *longitude of ascending node*.
- i is the *inclination*.
- ω is the *argument of periapsis*.
- e is the *eccentricity*.
- a and b are the *semi-major axis* and the *semi-minor axis*, respectively.

The *eccentric anomaly* E is related to the *mean anomaly* M by the *Kepler Equation*

$$M = E - e\sin E. \qquad (2)$$

The Cartesian coordinates ξ and η on the orbital plane are obtained with

$$\begin{aligned}
\xi &= a\left(\cos E - e\right) \qquad (3)\\
\eta &= b\sin E. \qquad (4)
\end{aligned}$$

Since the orbital parameters for a Keplerian orbit are constant by definition, the velocity vector is obtained from time derivative of the orbital plane coordinates

$$\begin{aligned}
\dot{\xi} &= -a\dot{E}\sin E \qquad (5)\\
\dot{\eta} &= b\dot{E}\cos E. \qquad (6)
\end{aligned}$$

If we denote the *standard gravitational parameter* $\mu := GM$ defined in terms of the gravitational constant G and the mass of the central body M, we obtain

$$\dot{E} = \frac{\sqrt{\mu}\,a^{-3/2}}{1 - e\cos E}. \qquad (7)$$

2 Orbit Parameters from Cartesian Position and Velocity

Suppose $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are available. First compute the *angular momentum per unit mass*

$$\boldsymbol{k} \;=\; \boldsymbol{r} \times \boldsymbol{v}. \tag{8}$$

2.1 Eccentricity Vector

From elementary discussion of the two-body problem, it follows that the *eccentricity vector* is computed via:

$$\boldsymbol{e} \;=\; \frac{\boldsymbol{v} \times \boldsymbol{k}}{\mu} - \frac{\boldsymbol{r}}{||\boldsymbol{r}||}. \tag{9}$$

2.2 Inclination

Inclination is computed simply as the angle between $\hat{\mathbf{z}}$ and $\boldsymbol{c}$:

$$i \;=\; \arccos\left(c_z/c\right). \tag{10}$$

2.3 Semi-Major Axis

For an elliptical orbit, the distance from the central body is expressed by the formula for conic section

$$r = \frac{k^2/\mu}{1 + e\cos f}, \tag{11}$$

where f is the *natural anomaly* and $0 < e < 1$. The semi-major axis is obtained as half of the sum of maximum and minimum distances

$$a \;=\; \frac{k^2}{2\mu}\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{k^2}{2\mu}\left(\frac{1-e+1+e}{1-e^2}\right) = \frac{k^2}{2\mu(1-e^2)}. \tag{12}$$

In addition, it can be shown that $k^2 = \mu^2(e^2-1)/2h$ (See below). Using this, we obtain

$$a \;=\; \frac{1}{2\mu(1-e^2)}\frac{\mu^2(e^2-1)}{2h} = \frac{\mu(e^2-1)}{2h|1-e^2|}. \tag{13}$$

For an ellipse $e^2 - 1 < 1$ and

$$\boxed{a = -\frac{\mu}{2h}.} \tag{14}$$

To obtain the required relation, let's compute time derivative of kinetic energy per unit mass

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) = \frac{d}{dt}\left(\frac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}\right) = \dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}} = -\mu\dot{\boldsymbol{r}}\cdot\frac{\boldsymbol{r}}{r^3} = -\mu\frac{\dot{r}}{r^2} = -\frac{d}{dt}\frac{\mu}{r}, \tag{15}$$

or rearranging

$$\frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right) = 0. \tag{16}$$

This implies, that there exists a constant called the *energy integral.*

$$h := \frac{1}{2}v^2 - \frac{\mu}{r}. \tag{17}$$

The semi-major axis is computed via

$$a \quad = \quad \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1}. \tag{18}$$

Rearranging (9) yields

$$\mu\frac{\boldsymbol{r}}{r} + \mu\boldsymbol{e} \quad = \quad \dot{\boldsymbol{r}} \times \boldsymbol{k}. \tag{19}$$

Dot product of both sides with themselves, yields

$$\mu^2\left(1 + 2\frac{\boldsymbol{r}\cdot\boldsymbol{e}}{r} + e^2\right) \quad = \quad v^2k^2 \tag{20}$$

Rearranging (17), yields

$$v^2 = 2h + 2\mu/r. \tag{21}$$

Now computation of the dot product of (9) with $\boldsymbol{r}$ and using (8), yields

$$\boldsymbol{r}\cdot\boldsymbol{e} \quad = \quad -\frac{1}{\mu}\left(\boldsymbol{r}\cdot\boldsymbol{k}\times\dot{\boldsymbol{r}} + \mu\frac{\boldsymbol{r}\cdot\boldsymbol{r}}{r}\right) = -\frac{1}{\mu}\left(\boldsymbol{k}\cdot\dot{\boldsymbol{r}}\times\boldsymbol{r} + \mu r\right) = k^2/\mu - r. \tag{22}$$

Substitution of (21) and (22) to (20), yields

$$\mu^2\left(1 + 2\frac{k^2}{\mu r} - 2 + e^2\right) \quad = \quad 2hk^2 + 2\frac{\mu k^2}{r} \tag{23}$$

and by canceling the common terms and dividing by $2h$, we obtain

$$k^2 \quad = \quad \frac{\mu^2(e^2-1)}{2h}. \tag{24}$$

2.4 Longitude of Ascending Node

Ascending and descending nodes both occur at the intersections of the orbital plane with the $z = 0$ plane. The vector $\boldsymbol{k}$ is the constant normal of the orbital plane so any intersection must occur in the line parallel to the cross product

$$\boldsymbol{k}\times\hat{\mathbf{z}} \quad = \quad \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ c_x & c_y & c_z \\ 0 & 0 & 1 \end{vmatrix} = \begin{bmatrix} c_y \\ -c_x \\ 0 \end{bmatrix}. \tag{25}$$

This is consistent with the following two formulas

$$\begin{aligned} \Omega &= \operatorname{atan2}(-c_x, c_y) && (26) \\ \Omega &= \operatorname{atan2}(c_x, -c_y), && (27) \end{aligned}$$

which correspond to the ascending and descending node. At ascending node, $z = 0$ and $v_z > 0$. Computation of (8), yields

$$\boldsymbol{c} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & y & 0 \\ v_x & v_y & v_z \end{vmatrix} = \begin{bmatrix} v_z y \\ -v_z x \\ x v_y - y v_x \end{bmatrix}. \tag{28}$$

Since the computation is independent of the eccentricity of the ellipse or a circle, let's assume that $e = 0$. Then

$$\begin{aligned} c_x &= v_z r \sin\Omega = |v_z r| \sin\Omega && (29) \\ c_y &= -v_z r \cos\Omega = -|v_z r| \cos\Omega && (30) \end{aligned}$$

Substituting these to (26) and (27), yields

$$\begin{aligned} \Omega &= \operatorname{atan2}(-|v_z r| \sin\Omega, -|v_z r| \cos\Omega) && (31) \\ \Omega &= \operatorname{atan2}(|v_z r| \sin\Omega, |v_z r| \cos\Omega). && (32) \end{aligned}$$

(31) is a contradiction. Thus,

$$\boxed{\Omega = \operatorname{atan2}(c_x, -c_y)}. \tag{33}$$

2.5 Argument of Periapsis

At periapsis, the orbital plane coordinates satisfy

$$\begin{aligned} \xi &= a(\cos E - e) = a(1-e) && (34) \\ \eta &= b \sin E = 0. && (35) \end{aligned}$$

Transformation of this according to (1) must be equal to the eccentricity vector. That is,

$$R_z(\Omega) R_x(i) R_z(\omega) \begin{bmatrix} a(1-e) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{36}$$

Expanding this equation according to (1), yields

$$a(1-e) \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\cos i \sin\omega \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega \\ \sin i \sin\omega \end{bmatrix} = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{37}$$

Since we are only interested in angles, we can forget the term $a(1-e)$. First if, $i=0$, we obtain

$$\begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega \\ 0 \end{bmatrix} = \begin{bmatrix} e_x \\ e_y \\ 0 \end{bmatrix} \tag{38}$$

or with application of trigonometric identities:

$$\cos(\Omega+\omega) = e_x \tag{39}$$

$$\sin(\Omega+\omega) = e_y. \tag{40}$$

That is,

$$\boxed{\omega = \operatorname{atan2}(e_y, e_x) - \Omega.} \tag{41}$$

If $\sin i \neq 0$, we obtain

$$\frac{e_z}{\sin i} = \sin\omega. \tag{42}$$

Also, depending on the value of Ω, we obtain the relations.

$$\frac{1}{\cos\Omega}\left(e_x + \frac{\sin\Omega\cos i}{\sin i}e_z\right) = \cos\omega \tag{43}$$

$$\frac{1}{\sin\Omega}\left(e_y - \frac{\cos\Omega\cos i}{\sin i}e_z\right) = \cos\omega \tag{44}$$

and

$$\boxed{\omega = \begin{cases} \operatorname{atan2}\left(\dfrac{e_z}{\sin i}, \dfrac{1}{\cos\Omega}\left[e_x + \dfrac{\sin\Omega\cos i}{\sin i}e_z\right]\right), & \cos\Omega \neq 0 \\ \operatorname{atan2}\left(\dfrac{e_z}{\sin i}, \dfrac{1}{\sin\Omega}\left[e_y - \dfrac{\cos\Omega\cos i}{\sin i}e_z\right]\right), & \sin\Omega \neq 0 \end{cases}} \tag{45}$$

2.6 Eccentric Anomaly

From (1), application of inverse rotations leads to an expression for the Cartesian coordinates on the orbital plane:

$$\begin{bmatrix} \xi \\ \eta \\ 0 \end{bmatrix} = \begin{bmatrix} a(\cos E - e) \\ b\sin E \\ 0 \end{bmatrix} = R_z(-\omega)R_x(-i)R_z(-\Omega)\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{46}$$

This allows us to solve

$$\cos E = \xi/a + e \tag{47}$$

$$\sin E = \eta/b. \tag{48}$$

and

$$\boxed{E = \operatorname{atan2}(\eta/b, \xi/a + e)} \tag{49}$$

The *mean anomaly* is then computed from the *Kepler Equation*

$$\boxed{M = E - e\sin E.} \tag{50}$$