Contents

1 Two-Body Problem

To discuss the solution presented in [1], we need to formulate the two-body problem in terms of Delaunay elements. Thus, this section is dedicated to canonical transformations and the Hamilton-Jacobi Equation. Starting from a Hamiltonian formulation in spherical polar coordinates (section 1.1), we will first use Hamilton-Jacobi equations to derive canonical coordinates corresponding to zero Hamiltonian (section 1.3) and then use a canonical transformation to derive a Hamiltonian formulation in terms of Delaunay Elements (section 1.4).

The discussion in this section is roughly based on sections 6 and 7 of [2].

1.1 Hamiltonian Formulation in Spherical Coordinates

Consider the problem of a point particle with mass m in a gravitational potential V, where the kinetic and potential energy can be expressed via the equations

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\dot{\phi}) \tag{1}$$

$$V = -\frac{\mu m}{r} \tag{2}$$

in terms of spherical polar coordinates (r, θ, ϕ) and canonical momenta

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \tag{3}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{4}$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mr^2\cos^2\theta\dot{\phi}. \tag{5}$$

Thus, the Hamiltonian (using the notation in [1]) can be expressed

$$F(r, \theta, \phi, p_r, p_\theta, p_\phi) = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\cos^2\theta}\right) - \frac{\mu m}{r}. \tag{6}$$

In this section, we will use canonical transformations and the Hamilton-Jacobi equation to derive the Hamiltonian in terms of canonical coordinates and momenta called Delaunay Elements.

1.2 Canonical Transformations

Suppose F^* is a Hamiltonian obtained from an another Hamiltonian F with a **generating function** (called **determining function** in [1]) S. Then,

$$F^*(\boldsymbol{q}', \boldsymbol{p}') = F(\boldsymbol{q}, \boldsymbol{p}) + \frac{\partial S}{\partial t} \tag{7}$$

In this document, we only use generating functions of the type $S = S(\boldsymbol{q}, \boldsymbol{p}')$, which satisfy

$$p_i = \frac{\partial S}{\partial q_i}, \quad q_i' = \frac{\partial S}{\partial p_i'} \tag{8}$$

1.3 Hamilton-Jacobi Equation

With Hamilton-Jacobi equation, we can derive canonical coordinates, which do not appear explicitly in the transformed Hamiltonian. Then, each canonical coordinate and momentum is independent of time. That is, for $F = F(r, \theta, \phi, p_r, p_\theta, p_\phi)$ we wish find a generating function S such that

$$F\left(r, \theta, \phi, \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}, \frac{\partial S}{\partial \phi}\right) + \frac{\partial S}{\partial t} = 0. \tag{9}$$

Substitution to (6), yields

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\cos^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right] + \frac{\partial S}{\partial t} - \frac{\mu m}{r} = 0. \tag{10}$$

1.3.1 Separation of Variables

We attempt to solve (10) via separation of variables

$$S(r, \theta, \phi, t) = S_r(r) + S_\theta(\theta) + S_\phi(\phi) + S_t(t). \tag{11}$$

This yields

$$\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{1}{r^2\cos^2\theta}\left(\frac{dS_\phi}{d\phi}\right)^2 = 2m\left(-\frac{dS_t}{dt} + \frac{\mu m}{r}\right). \tag{12}$$

The dependence w.r.t. variables ϕ and t is limited to $dS_\phi/d\phi$ and dS_t/dt. Thus, there exists α_2 and α_1 such that

$$\frac{dS_\phi}{d\phi} = -\alpha_2, \quad \frac{dS_t}{dt} = -\alpha_1. \tag{13}$$

The equation (12) can be now written

$$\left(\frac{dS_r}{dr}\right)^2 + \underbrace{\frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{\alpha_2^2}{r^2\cos^2\theta}}_{\alpha_1/r^2} = 2m\left(\alpha_1 + \frac{\mu m}{r}\right) \tag{14}$$

and denoting the middle two terms with α_1/r^2 we obtain

$$\frac{dS_r}{dr} = \sqrt{2m\left(\alpha_1 + \frac{\mu m}{r}\right) - \frac{\alpha_1}{r^2}}. \tag{15}$$

Similarly for S_θ

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = 2mr^2\left(\alpha_1 + \frac{\mu m}{r}\right) - r^2\left(\frac{dS_r}{dr}\right)^2 - \frac{\alpha_2^2}{\cos^2\theta} = \alpha_3^2 + \frac{\alpha_2^2}{\cos^2\theta}. \tag{16}$$

or

$$\frac{dS_\theta}{d\theta} = \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}} \tag{17}$$

The generating function satisfying (9) can be thus written

$$S = -\alpha_1 t + \alpha_2\phi + \int d\theta\sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}} + \int dr\sqrt{2m\left(\alpha_1 + \frac{\mu m}{r}\right) - \frac{\alpha_1}{r^2}} \tag{18}$$

1.3.2 Canonical Momenta

We will select the canonical momenta

$$p_1' := \alpha_1 = mh \tag{19}$$
$$p_2' := \alpha_2 = m\sqrt{a\mu(1-e^2)}\cos I \tag{20}$$
$$p_3' := \alpha_3 = m\sqrt{a\mu(1-e^2)} \tag{21}$$

To derive (19), we can compute

$$\alpha_1 = -\frac{\partial S}{\partial t} = F = T + V = mh, \tag{22}$$

where $h = v^2/2 - \mu/r$ is the **energy integral**.

Derivation of (20) is somewhat more complicated. Note that in spherical polar coordinates

$$\begin{aligned} x &= r\cos\theta\cos\phi \\ y &= r\cos\theta\sin\phi \\ z &= r\sin\theta \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta\cos\phi - r\dot{\theta}\sin\theta\cos\phi - r\dot{\phi}\cos\theta\sin\phi \\ \dot{y} &= \dot{r}\cos\theta\sin\phi - r\dot{\theta}\sin\theta\sin\phi + r\dot{\phi}\cos\theta\cos\phi \\ \dot{z} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned} \tag{23}$$

Now, we can note that that the z-component of the angular momentum vector can be expanded in spherical polar coordinates

$$\begin{aligned} k_z = (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \cdot \hat{z} &= \begin{vmatrix} 0 & 0 & 1 \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} \\ &= x\dot{y} - y\dot{x} \\ &= r\cos\theta\cos\phi(\dot{r}\cos\theta\sin\phi - r\dot{\theta}\sin\theta\sin\phi + r\dot{\phi}\cos\theta\cos\phi) \\ &- r\cos\theta\sin\phi(\dot{r}\cos\theta\cos\phi - r\dot{\theta}\sin\theta\cos\phi - r\dot{\phi}\cos\theta\sin\phi) \\ &= r\dot{r}\cos^2\theta(\cos\phi\sin\phi - \cos\phi\sin\phi) \\ &+ r^2\dot{\theta}\cos\phi\sin\phi(\sin\theta\cos\theta - \sin\theta\cos\theta) \\ &+ r^2\dot{\phi}\cos^2\theta(\cos^2\phi + \sin^2\phi) \\ &= r^2\cos^2\theta\dot{\phi}. \end{aligned} \tag{24}$$

On the other hand, for elliptic orbits

$$a = \frac{k^2}{\mu(1-e^2)}, \tag{25}$$

Thus,

$$\alpha_2 = \frac{\partial S}{\partial \phi} = p_\phi = mr^2\cos^2\theta\dot{\phi} = mk_z = m\sqrt{a\mu(1-e^2)}\cos I \tag{26}$$

To derive (21), we can compute

$$\begin{aligned} \alpha_3 &= \sqrt{\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\alpha_2^2}{\cos^2\theta}} \\ &= \sqrt{p_\theta^2 + \frac{p\phi^2}{\cos^2\theta}} \\ &= mr^2\sqrt{\dot{\theta}^2 + \cos^2\theta\dot{\phi}^2} = mr^2\dot{f} = mk = m\sqrt{a\mu(1-e^2)}, \end{aligned} \tag{27}$$

where $\dot{f} = \dot{\theta}^2 + \cos^2\theta\dot{\phi}^2$ is the time derivative of the natural anomaly since the sum is the square of total scalar angular speed.

1.3.3 Canonical Coordinates

The new canonical coordinates can be computed

$$
\begin{aligned}
q_1' = \frac{\partial S}{\partial \alpha_1} &= -t + \frac{\partial}{\partial \alpha_1} \int dr \sqrt{2m\left(\alpha_1 + \frac{m\mu}{r}\right) - \frac{\alpha_3^2}{r^2}} \\
&= -t + \int m \left[2m\left(\alpha_1 + \frac{m\mu}{r}\right) - \frac{\alpha_3^2}{r^2}\right]^{-1/2} dr \\
&= -t + I_1
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
q_2' = \frac{\partial S}{\partial \alpha_2} &= \phi + \frac{\partial}{\partial \alpha_2} \int d\theta \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}} \\
&= \phi - \int \frac{\alpha_2}{\cos^2\theta}\left(\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}\right)^{-1/2} d\theta \\
&= \phi - I_2\alpha_2
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
q_3' = \frac{\partial S}{\partial \alpha_3} &= \frac{\partial}{\partial \alpha_3} \int \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}} d\theta + \frac{\partial}{\partial \alpha_3} \int \sqrt{2m\left(\alpha_1 + \frac{\mu m}{r}\right) - \frac{\alpha_3^2}{r^2}} dr \\
&= \alpha_3 \int \left(\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}\right)^{-1/2} d\theta - \alpha_3 \int \frac{1}{r}\left[2m\left(\alpha_1 + \frac{\mu m}{r}\right) - \frac{\alpha_3^2}{r^2}\right]^{-1/2} dr \\
&= \alpha_3 I_4 - \frac{\alpha_3}{m} I_3.
\end{aligned} \tag{30}
$$

The evaluation of the integrals I_1, I_2, I_3, I_4 is non-trivial and will be performed next.

For I_1, substituting (19), (21) and $mh = -\mu m/2a$, we obtain

$$
\begin{aligned}
I_1 &= m \int \left[2m\left(mh + \frac{m\mu}{r}\right) - \frac{m^2}{r^2}(a\mu(1-e^2))\right]^{-1/2} dr \\
&= \int m\, dr \left[2m\left(-\frac{\mu m}{2a} + \frac{\mu m}{r} - \frac{m^2 a\mu(1-e^2)}{r^2}\right)\right]^{-1/2} \\
&= \frac{1}{\sqrt{\mu}} \int \frac{r\, dr}{\sqrt{-r^2/a + 2r - a(1-e^2)}}
\end{aligned} \tag{31}
$$

Substitute $r = a(1 - e\cos E)$, which leads to $dr = ae\sin E\, dE$ and

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{\mu}} \int \frac{a(1-e\cos E)\, ae\sin E\, dE}{\sqrt{-a(1-e\cos E)^2 + 2a(1-e\cos E) - a(1-e^2)}} \\
&= \frac{a^{3/2}}{\sqrt{\mu}} \int (1 - e\cos E)\, dE \\
&= \frac{a^{3/2}}{\sqrt{\mu}} (E - e\sin E)
\end{aligned} \tag{32}
$$

Application of Kepler's equation and Kepler's third law $n = \sqrt{\mu/a^3}$, we obtain the additive inverse for the time of perihelion

$$q_1' = -t + \frac{M}{n} = -\tau. \tag{33}$$

For I_2, it follows from (20), (21) that $\alpha_2/\alpha_3 = \cos I$

$$\alpha_2 I_2 = \int \frac{\alpha_2}{\cos^2\theta\sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}}} = \frac{\alpha_2}{\alpha_3}\int \frac{d\theta}{\cos^2\theta\sqrt{1 - \frac{(\alpha_2/\alpha_3)^2}{\cos^2\theta}}} = \int \frac{\cos I d\theta}{\cos^2\theta\sqrt{1 - \frac{cos^2 I}{\cos^2\theta}}} \tag{34}$$

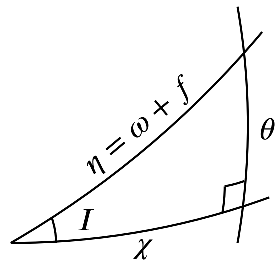

Figure 1: The relationship between the angles η, θ and χ.

Via application of Napier's rule to the angles in Figure 1, we obtain

$$\sin\chi = \cot(\pi/2 - \theta)\cot I = \tan\theta\cot I \tag{35}$$

and

$$\frac{d\theta}{\cos^2\theta} = \tan i \cos\chi d\chi. \tag{36}$$

Thus,

$$\alpha_2 I_2 = \int \frac{\cos I \tan I \cos\chi d\chi}{\sqrt{1 - cos^2 I(1 + \tan^2 I \sin^2\chi)}} = \int \frac{\sin I \cos\chi d\chi}{\sqrt{\sin^2 I(1 - \sin^2\chi)}} = \int d\chi = \chi. \tag{37}$$

and

$$q_2' \;=\; \phi - \alpha_2 I_2 = \phi - \chi = \Omega. \tag{38}$$

For I_3,

$$\begin{aligned} I_3 &= m\int \frac{1}{r^2}\left[2m\left(-\frac{\mu m}{2a} + \frac{\mu m}{r}\right) - m^2\frac{a\mu(1-e^2)}{r^2}\right]^{-1/2} dr \\ &= \frac{1}{\sqrt{\mu}}\int \frac{dr}{r\sqrt{-r^2/a + 2r - a(1-e^2)}} \end{aligned} \tag{39}$$

Substitute $r = a(1 - e\cos E)$, $dr = ae\sin E dE$, to obtain

$$\begin{aligned}
I_3 &= \frac{1}{\sqrt{\mu}}\int \frac{ae\sin E dE}{a(1-e\cos E)\sqrt{-a(1-2e\cos E+e^2\cos^2 E)+2a(1-e\cos E)-a(1-e^2)}} \\
&= \frac{1}{\sqrt{\mu}}\int \frac{e\sin E}{(1-e\cos E)\sqrt{a(e^2\sin^2 E)}} \qquad (40)\\
&= \frac{1}{\sqrt{a\mu}}\int \frac{\sin E dE}{\sin E(1-e\cos E)} \\
&= \frac{1}{\sqrt{a\mu(1-e^2)}}\int \frac{\sqrt{1-e^2}dE}{1-e\cos E}
\end{aligned}$$

Substituting $\sin f = \sqrt{1-e^2}\sin E/(1-e\cos E)$ leads to

$$\cos f df = \sqrt{1-e^2}\frac{\cos f}{1-e\cos E}dE \qquad (41)$$

Dividing by $\cos f$ and substituting back, we obtain

$$I_3 = \frac{f}{\sqrt{a\mu(1-e^2)}}. \qquad (42)$$

For I_4, substituting (20), (21)

$$I_4 = \int\left(\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}\right)^{-1/2} = \frac{1}{m\sqrt{a\mu(1-e^2)}}\int\left(1-\frac{\cos^2 I}{\cos^2\theta}\right)^{-1/2} d\theta \qquad (43)$$

Multiplying with α_3 leads

$$\alpha_3 I_4 = \int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta - \cos^2 I}} \qquad (44)$$

From Figure 2, we obtain

$$\frac{\sin\theta}{\sin I} = \frac{\sin\eta}{\sin\pi/2} \qquad (45)$$

which leads to

$$\begin{aligned}
\cos^2\theta &= 1 - \sin^2 I\sin^2\eta \qquad (46)\\
\cos\theta d\theta &= \sin i\cos\eta d\eta. \qquad (47)
\end{aligned}$$

$$\alpha_3 I_4 = \int \frac{\sin I\cos\eta d\eta}{\sqrt{1-\sin^2 I\sin^2\eta - \cos^2 I}} = \int \frac{\sin I\cos\eta d\eta}{\sqrt{\sin^2 I\cos^2\eta}} = \int d\eta = \eta. \qquad (48)$$

Thus,

$$q_3' = \alpha_3 I_4 - \frac{\alpha_3}{m}I_3 = \eta - \frac{m\sqrt{a\mu(1-e^2)}}{m}\frac{f}{\sqrt{a\mu(1-e^2)}} = \eta - f = \omega. \qquad (49)$$

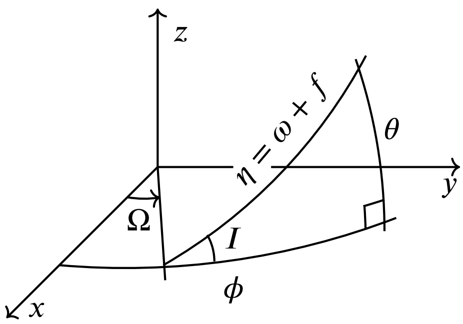

Figure 2: The relationship between the spherical polar coordinates and the Keplerian elements.

1.3.4 Hamiltonian Formulation with Zero Hamiltonian

We now have the new canonical coordinates and momenta

$$
\begin{aligned}
q_1' &= -\tau, && (50)\\
q_2' &= \Omega, && (51)\\
q_3' &= \omega, && (52)\\
p_1' &= mh, && (53)\\
p_2' &= m\sqrt{a\mu(1-e^2)}\cos I, && (54)\\
p_3' &= m\sqrt{a\mu(1-e^2)}, && (55)
\end{aligned}
$$

as well as the new Hamiltonian

$$F' = 0. \tag{56}$$

The mass is typically left out leading to variables

$$
\begin{aligned}
q_1' &= -\tau, && (57)\\
q_2' &= \Omega, && (58)\\
q_3' &= \omega, && (59)\\
p_1' &= h, && (60)\\
p_2' &= \sqrt{a\mu(1-e^2)}\cos I, && (61)\\
p_3' &= \sqrt{a\mu(1-e^2)}, && (62)
\end{aligned}
$$

Since none of the variables are present in the Hamiltonian $F' = 0$, all of the variables are constant.

1.4 Delaunay Elements

To obtain the Delaunay Elements used in [1] from (56) − (62), we seek the following variables

$$
\begin{aligned}
h &= \Omega && (63)\\
g &= \omega && (64)\\
l &= n(t+q_1) && (65)\\
H &= \sqrt{a\mu(1-e^2)} && (66)\\
G &= \sqrt{a\mu(1-e^2)}\cos I && (67)\\
L &= ? && (68)
\end{aligned}
$$

These are found with the generating function

$$
\begin{aligned}
S &= \left(nL - \frac{3\mu}{2a}\right)(t+q_1') + q_2'H + q_3'G && (69)\\
&= \left(nL - \frac{3\mu}{2a}\right)(t-\tau) + \Omega H + \omega G
\end{aligned}
$$

Now

$$
p_1 = \frac{\partial S}{\partial q_1} = -\frac{\partial S}{\partial \tau} = -\frac{3\mu}{2a} + nL = h \tag{70}
$$

from which by using $h = -\mu/2a$ (for ellipses) and Kepler's third law $n = \sqrt{\mu/a^3}$, we can solve

$$
L = \frac{1}{n}\left(h + \frac{3\mu}{2a}\right) = \frac{1}{n}\left(-\frac{\mu}{2a} + \frac{3\mu}{2a}\right) = \frac{\mu}{an} = \sqrt{a\mu} \tag{71}
$$

The new Hamiltonian

$$
F = 0 + \frac{\partial S}{\partial t} = nL - \frac{3\mu}{2a} = \sqrt{\mu/a^3}\sqrt{a\mu} - \frac{3\mu}{2a} = -\frac{\mu}{2a} = -\frac{\mu^2}{2L^2}. \tag{72}
$$

Thus, we have the Hamiltonian formulation in terms of Delaunay variables

$$
\boxed{
\begin{array}{lll}
l = M, & L = \sqrt{a\mu}, & \\
g = \omega, & G = \sqrt{a\mu(1-e^2)}, & F = -\dfrac{\mu}{2L^2} \\
h = \Omega, & H = \sqrt{a\mu(1-e^2)}\cos I, &
\end{array}
} \tag{73}
$$

1.5 Useful Relations

In this section, we derive several relations used by Brouwer in [1] involving the Delaunay variables and Keplerian orbits.

We assume that the following equations for true anomaly are known

$$
\begin{aligned}
\cos f &= \frac{\cos E - e}{1 - e\cos E}, && (74)\\
\sin f &= \sqrt{1-e^2}\frac{\sin E}{1 - e\cos E}. && (75)
\end{aligned}
$$

1.5.1 $\partial E/\partial e$

Consider Kepler's equation

$$M = E - e\sin E \tag{76}$$

as an equation relating the pairs (e', M) and (e, E). Then, partial derivative of M w.r.t. e' must disappear. That is,

$$\begin{aligned}\frac{\partial M}{\partial e'} &= \frac{\partial M}{\partial e}\frac{\partial e}{\partial e'} + \frac{\partial M}{\partial E}\frac{\partial E}{\partial e'} \\ &= \frac{\partial}{\partial e}(E - e\sin E) + \frac{\partial E}{\partial e'}\frac{\partial}{\partial E}(E - e\sin E) \\ &= -\sin E + \frac{\partial E}{\partial e'}(1 - e\cos E) \\ &= 0.\end{aligned} \tag{77}$$

Thus,

$$\frac{\partial E}{\partial e} = \frac{\sin E}{1 - e\cos E}. \tag{78}$$

1.5.2 $\partial/\partial e(a/r)$

We wish to show that

$$\frac{\partial}{\partial e}\frac{a}{r} = \frac{a^2}{r^2}\cos f \tag{79}$$

To derive this, we can simply compute

$$\begin{aligned}\frac{\partial}{\partial e'}\frac{a}{r} &= \frac{\partial}{\partial e}\frac{a}{r} + \frac{\sin E}{1 - e\cos E}\frac{\partial}{\partial E}\frac{a}{r} \\ &= \frac{\cos E}{(1 - e\cos E)^2} + \frac{\sin E}{1 - e\cos E}\frac{-e\sin E}{(1 - e\cos E)^2} \\ &= \frac{cos E - e\cos^2 E - e\sin^2 E}{(1 - e\cos E)^3} \\ &= \frac{a^2}{r^2}\frac{\cos E - e}{1 - e\cos E} \\ &= \frac{a^2}{r^2}\cos f \quad \square\end{aligned} \tag{80}$$

where we have used

$$\frac{\partial}{\partial e}\frac{a}{r} = \frac{\partial}{\partial e}\frac{1}{1 - e\cos E} = \frac{\cos E}{(1 - e\cos E)^2} \tag{81}$$

$$\frac{\partial}{\partial E}\frac{a}{r} = \frac{\partial}{\partial E}\frac{1}{1 - e\cos E} = \frac{-e\sin E}{(1 - e\cos E)^2} \tag{82}$$

1.5.3 $\partial l / \partial f$

We wish to show that

$$\frac{\partial l}{\partial f} = \frac{L}{G}\frac{r^2}{a^2}. \tag{83}$$

From (74), we obtain

$$f = \operatorname{acos}\left(\frac{\cos E - e}{1 - e\cos E}\right) \tag{84}$$

Now by application of the chain rule

$$\frac{df}{dE} = \left(\frac{d}{dE}\frac{\cos E - e}{1 - e\cos E}\right)\left[-\left(1 - \left[\frac{\cos E - e}{1 - e\cos E}\right]^2\right)^{-1/2}\right]. \tag{85}$$

For the first part

$$\begin{aligned} \frac{d}{dE}\frac{\cos E - e}{1 - e\cos E} &= \frac{(1 - e\cos E)(-\sin E) - (\cos E - e)(e\sin E)}{(1 - e\cos E)^2} \\ &= \frac{-\sin E + e\sin E\cos E - e\sin E\cos E + e^2\sin E}{(1 - e\cos E)^2} \\ &= -\frac{\sin E(1 - e^2)}{(1 - e\cos E)^2}. \end{aligned} \tag{86}$$

For the latter part

$$\begin{aligned} \frac{-1}{\sqrt{1 - [\cdot]^2}} &= -\left(\frac{1 - 2e\cos E + e^2\cos^2 E - \cos^2 E + 2e\cos E - e^2}{(1 - e\cos E)^2}\right)^{-1/2} \\ &= -\left(\frac{(1 - e^2)(1 - \cos^2 E)}{(1 - e\cos E)^2}\right)^{-1/2} \\ &= -\frac{1 - e\cos E}{\sin E\sqrt{1 - e^2}}. \end{aligned} \tag{87}$$

Thus

$$\frac{df}{dE} = \frac{\sin E(1 - e^2)}{(1 - e\cos E)^2}\cdot\frac{1 - e\cos E}{\sin E\sqrt{1 - e^2}} = \frac{\sqrt{1 - e^2}}{1 - e\cos E}. \tag{88}$$

Also from the Kepler equation $M = -E - e\sin E$, we obtain

$$\frac{dM}{dE} = 1 - e\cos E. \tag{89}$$

Thus,

$$\frac{dM}{df} = \frac{dM}{dE}\frac{dE}{df} = \frac{(1 - e\cos E)^2}{\sqrt{1 - e^2}} \tag{90}$$

From (74), we can also solve

$$\cos E = \frac{\cos f + e}{1 + e\cos f} \tag{91}$$

and

$$1 - e\cos E = \frac{1 + e\cos f - e\cos f - e^2}{1 + e\cos f} = \frac{1 - e^2}{1 + e\cos f}. \tag{92}$$

Substituting this back to (90), yields

$$\frac{dM}{df} \;=\; \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2} \tag{93}$$

For elliptic orbits,

$$\frac{r^2}{a^2} \;=\; \left(\frac{1-e^2}{1+e\cos f}\right)^2 = \sqrt{1-e^2}\frac{dM}{df} \tag{94}$$

Thus, we finally have

$$\frac{dM}{df} \;=\; \frac{1}{\sqrt{1-e^2}}\frac{r^2}{a^2} = \frac{L}{G}\frac{r^2}{a^2}. \quad \square \tag{95}$$

1.5.4 $\partial f/\partial e$

We wish to show that

$$\frac{\partial f}{\partial e} = \left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f \tag{96}$$

From (74), we obtain

$$\frac{df}{de} \;=\; \left(\frac{d}{de}\frac{\cos E - e}{1 - e\cos E}\right)\frac{-1}{\sqrt{1-(\cdot)^2}} \tag{97}$$

The latter part is obtained from (87). For the first part

$$\begin{aligned}
\frac{d}{de}\frac{\cos E - e}{1 - e\cos E} &= \frac{(1-e\cos E)(\partial E/\partial e(-\sin E) - 1) + (e - \cos E)(\partial E/\partial e\, e\sin E - \cos E)}{(1-e\cos E)^2} \\
&= \frac{\cos^2 E - 1}{(1-e\cos E)^2} + \frac{\partial E}{\partial e}\frac{\sin E(e^2-1)}{(1-e\cos E)^2} \\
&= \frac{\cos^2 E - 1}{(1-e\cos E)^2} + \frac{\sin^2 E(e^2-1)}{(1-e\cos E)^3},
\end{aligned} \tag{98}$$

where we have used (78). Using (87), we can now compute

$$\begin{aligned}
\frac{df}{de} &= -\frac{1-e\cos E}{\sin E\sqrt{1-e^2}}\left(\frac{\cos^2 E-1}{(1-e\cos E)^2}+\frac{\sin^2 E(e^2-1)}{(1-e\cos E)^3},\right) \\
&= \frac{1}{\sqrt{1-e^2}}\frac{\sin E}{1-e\cos E}+\sqrt{1-e^2}\frac{\sin E}{(1-e\cos E)^2} \\
&= \frac{\sin f}{1-e^2}+\frac{\sin f}{1-e\cos E} \\
&= \left(\frac{1}{1-e^2}+\frac{a}{r}\right)\sin f \\
&= \left(\frac{L^2}{G^2}+\frac{a}{r}\right)\sin f, \quad \square
\end{aligned} \tag{99}$$

1.5.5 $\int \sigma_1 dl$

We wish to show that

$$\int \sigma_1\, dl \;=\; \frac{L^3}{G^3}(f-l+e\sin f) \quad : \quad \sigma_1=\frac{a^3}{r^3}-\frac{L^3}{G^3} \tag{100}$$

First note that using (83),

$$\frac{\partial}{\partial l}(f+e\sin f)=\frac{\partial f}{\partial l}+e\frac{\partial f}{\partial l}\cos f=(1+e\cos f)\frac{\partial f}{\partial l}=(1+e\cos f)\frac{a^2}{r^2}\sqrt{1-e^2}. \tag{101}$$

Substituting

$$\frac{a}{r}=\frac{1+e\cos f}{1-e^2}, \tag{102}$$

we obtain

$$\frac{\partial}{\partial l}(f+e\sin f)=\frac{a^3}{r^3}(1-e^2)^{3/2} \tag{103}$$

and

$$\frac{\partial}{\partial l}\frac{f+e\sin f}{(1-e^2)^{3/2}}=\frac{a^3}{r^3} \tag{104}$$

Thus,

$$\int\left(\frac{a^3}{r^3}-\frac{L^3}{G^3}\right)dl=\frac{f+e\sin f}{(1-e^2)^{3/2}}-\frac{L^3}{G^3}l=\frac{L^3}{G^3}(f-l+e\sin f). \quad \square \tag{105}$$

1.5.6 $\int \sigma_2 dl$

We wish to show that

$$\int \sigma_2\, dl \;=\; \frac{1}{(1-e^2)^{3/2}}\left[\frac{1}{2}\sin(2g+2f)+\frac{e}{2}\sin(2g+f)+\frac{e}{6}\sin(2g+3f)\right] \tag{106}$$

where

$$\sigma_2 = \frac{a^3}{r^3}\cos(2g+2f). \tag{107}$$

Differentiating the square brackets part of (106), we obtain

$$\frac{\partial}{\partial dl}[\cdot] \;=\; \frac{\partial f}{\partial l}\left[\cos(2g+2f)+\frac{e}{2}\cos(2g+f)+\frac{e}{2}\cos(2g+3f)\right] \tag{108}$$

With trigonometric summation formulas, we can compute

$$\cos(2g+f) \;=\; \cos(2g+2f-f) = \cos(2g+2f)\cos f + \sin(2g+2f)\sin f \tag{109}$$
$$\cos(2g+3f) \;=\; \cos(2g+2f+f) = \cos(2g+2f)\cos f - \sin(2g+2f)\sin f. \tag{110}$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial dl}[\cdot] &= \frac{\partial f}{\partial l}\left[\cos(2g+2f)+e\cos(2g+2f)\cos f\right] \qquad (111)\\ &= \frac{\partial f}{\partial l}\left[\cos(2g+2f)(1+e\cos f)\right]\\ &= \sqrt{1-e^2}\frac{a^2}{r^2}(1+e\cos f)\cos(2g+2f).\\ &= \frac{a^3}{r^3}(1-e^2)^{3/2}\cos(2g+2f). \quad \square\end{aligned}$$

1.5.7 $(1+e\cos f)\partial f/\partial e$

We wish to show that

$$(1+e\cos f)\frac{\partial f}{\partial e} \;=\; \left(\frac{a^2}{r^2}\frac{G^2}{L^2}+\frac{a}{r}\right)\sin f \tag{112}$$

To start, note that

$$\begin{aligned}(1+e\cos f)\frac{\partial f}{\partial e} &= (1+e\cos f)\left(\frac{a}{r}+\frac{L^2}{G^2}\right)\sin f \qquad (113)\\ &= \left(\frac{a}{r}+\frac{L^2}{G^2}\right)\sin f + \left(\frac{a}{r}+\frac{L^2}{G^2}\right)e\sin f\cos f\\ &= \frac{a}{r}(1+e\cos f)\sin f + \frac{L^2}{G^2}(1+e\cos f)\sin f\\ &= \left[\frac{a}{r}+\frac{a}{r}e\cos f+\frac{L^2}{G^2}(1+e\cos f)\right]\sin f\\ &= \left[\frac{a}{r}+\frac{a}{r}(1+e\cos f)\right]\sin f\\ &= \left[\frac{a}{r}+\frac{a^2}{r^2}(1-e^2)\right]\sin f. \quad \square\end{aligned}$$

2 Spherical Harmonics and the Perturbed Problem

2.1 Spheroid Potential

The gravitational potential of an arbitrary distribution of mass can be expanded in terms of spherical harmonics

$$
\begin{aligned}
V(\boldsymbol{r}) &= -G\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3\boldsymbol{r}^3 \qquad (114)\\
&= -\frac{G}{r}\left(1-\sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n J_n P_n(\cos\theta)-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{R}{r}\right)^n P_n^m(\cos\theta)\left(c_{nm}\cos m\phi+s_{nm}\sin n\phi\right)\right)
\end{aligned}
$$

where θ is the polar angle, G is the gravitational constant, and R is an arbitrary scaling factor. The Zonal and Tesseral harmonics can be written

$$J_n = -\frac{1}{MR^n}\int P_n(\cos\theta')r'^n\rho(\boldsymbol{r}')\, d^3\boldsymbol{r}', \qquad (115)$$

$$c_{nm} = -2\frac{(n-m)!}{(n+m)!}\frac{1}{MR^n}\int P_n^m(\cos\theta')\cos m\phi' r'^n\rho(\boldsymbol{r}')d^3\boldsymbol{r}' \qquad (116)$$

$$s_{nm} = -2\frac{(n-m)!}{(n+m)!}\frac{1}{MR^n}\int P_n^m(\cos\theta')\sin m\phi' r'^n\rho(\boldsymbol{r}')d^3\boldsymbol{r}, \qquad (117)$$

where P_n is the Legendre polynomial of degree n and P_n^m is the Associated Legendre Polynomial of degree n and order m. If the body is axially symmetric w.r.t. z axis, the tesseral harmonics c_{nm} and s_{nm} disappear. If the body is symmetric w.r.t. the equatorial $z=0$ plane $P_n(-x)=(-1)^nP_n(x)$ and J_n will disappear for odd-valued n.

Since the body of interest is a spheroid, the tesseral harmonics disappear, and we have an expansion in terms of only zonal harmonics

$$V(r,\theta) = -\frac{\mu}{r}\left(1-\sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^n J_nP_n(\cos\theta)\right). \qquad (118)$$

For SGP4, the zonal harmonics are obtained from WGS72[TODO] with the values

$$
\begin{aligned}
J_2 &= 1082.616\cdot 10^{-6}\\
J_3 &= -2.53881\cdot 10^{-6}\\
J_4 &= -1.65597\cdot 10^{-6}.
\end{aligned}
$$

The first four Legendre polynomials can be written

$$P_1(\cos\theta) = \cos\theta \qquad (119)$$

$$P_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta-1\right) \qquad (120)$$

$$P_3(\cos\theta) = \frac{1}{2}\left(5\cos^3\theta-3\cos\theta\right) \qquad (121)$$

$$P_4(\cos\theta) = \frac{1}{8}\left(35\cos^4\theta-30\cos^2\theta+3\right) \qquad (122)$$

$$P_5(\cos\theta) = \frac{1}{8}\left(63\cos^5\theta-70\cos^3\theta+15\cos\theta\right) \qquad (123)$$

Substituting (120) – (122) back to (118), we obtain the potential

$$
\begin{aligned}
V(r,\theta) &= -\frac{\mu}{r} \\
&+ \frac{\mu J_2 R^2}{2r^3}\left(3\cos^2\theta - 1\right) \\
&+ \frac{\mu J_3 R^3}{2r^4}\left(5\cos^3\theta - 3\cos\theta\right) \\
&+ \frac{\mu J_4 R^4}{8r^5}\left(35\cos^4\theta - 30\cos^2\theta + 3\right) \\
&+ \frac{\mu J_5 R^5}{8r^6}\left(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta\right)
\end{aligned} \tag{124}
$$

We use $R = a_E$ to denote the equatorial radius of Earth and define

$$
k_2 = \frac{J_2 a_E^2}{2}, \quad k_4 = -\frac{3J_4 a_E^4}{8}, \quad A_{3,0} = -J_3 a_E^3, \quad A_{5,0} = -J_5 a_E^5. \tag{125}
$$

Thereafter, using β to denote the geocentric latitude, we obtain $\cos\theta = \sin\beta$ and

$$
\begin{aligned}
V(r,\theta) &= -\frac{\mu}{r} \\
&- \frac{\mu k_2}{r^3}\left(1 - 3\sin^2\beta\right) \\
&- \frac{\mu k_4}{r^5}\left(1 - 10\sin^2\beta + \frac{35}{3}\sin^4\beta\right) \\
&- \frac{\mu A_{3,0}}{r^4}\left(-\frac{3}{2}\sin\beta + \frac{5}{2}\sin^3\beta\right) \\
&- \frac{\mu A_{5,0}}{r^5}\left(\frac{15}{8}\sin\beta - \frac{35}{4}\sin^3\beta + \frac{63}{8}\sin^5\beta\right).
\end{aligned} \tag{126}
$$

Note that Brouwer calls the gravitational potential the force function and uses different sign [1]. That is, $U = -V$.

2.2 Disturbing Function

The first term in (126) corresponds to the unperturbed potential of a two-body problem, where the Keplerian elements are constants. The additional terms constitute a perturbing potential that leads to perturbations in the Keplerian elements. That is, instead of the Hamiltonian (**??**), we have

$$
F = \frac{\mu}{2L^2} + R, \tag{127}
$$

where $R = R_2 + R_3 + R_4 + R_5$ is the **disturbing function** and R_k denote the part of R corresponding to k:th degree in (126). In this section, we will derive R in terms of Delaunay variables.

In perifocal frame, where the non-disturbed orbit is on the $z = 0$ plane and positive x axis corresponds to to the rising node, we can write the position vector as

$$
r\left[\cos(g+f), \quad \sin(g+f), \quad 0\right]^T. \tag{128}
$$

The position vector in an equatorial inertial frame with x axis corresponding to the rising node can be obtained via application of an anti-clockwise rotation w.r.t. x coordinate

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \begin{bmatrix} \cos(g+f) \\ \sin(g+f) \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(g+f) \\ \cos I \sin(g+f) \\ \sin I \sin(g+f) \end{bmatrix} \tag{129}$$

The z-coordinate can be related to the geocentric latitude β in (126) with the equation

$$\sin\beta = \sin I \sin(g+f). \tag{130}$$

Then, by using the relations

$$\begin{aligned} \sin^2(g+f) &= (1-\cos(2g+2f))/2, \\ \sin(g+f)\cos(2g+2f) &= \frac{1}{2}\sin(3g+3f) - \frac{1}{2}\sin(g+f), \\ \cos 2\theta &= 2\cos^2\theta - 1, \\ \sin(g+f)\cos(4g+4f) &= \frac{1}{2}\sin(5g+5f) - \frac{1}{2}\sin(3g+3f), \end{aligned}$$

we obtain

$$\begin{aligned} \sin^2\beta &= \frac{1}{2}\sin^2 I\,(1-\cos(2g+2f))\,, \qquad (131) \\ \sin^3\beta &= \sin^3 I\left(\frac{3}{4}\sin(g+f) - \frac{1}{4}\sin(3g+3f)\right), \\ \sin^4\beta &= \frac{1}{4}\sin^4 I\left(1 - 2\cos(2g+2f) + \cos^2(2g+2f)\right) \\ &= \frac{1}{8}\sin^4 I\,(3 - 4\cos(2g+2f) + \cos(4g+4f)) \\ \sin^5\beta &= \frac{1}{16}\left[6\sin(g+f) - 4\sin(3g+3f) + 4\sin(g+f) + \sin(5g+5f) - \sin(3g+3f)\right] \\ &= \sin^5 I\left[\frac{5}{8}\sin(g+f) - \frac{5}{16}\sin(3g+3f) + \frac{1}{16}\sin(5g+5f)\right]. \end{aligned}$$

2.2.1 The J_2 Harmonic

We can expand

$$\begin{aligned} 1 - 3\sin^2\beta &= 1 - \frac{3}{2}\sin^2 I\,(1-\cos(2g+2f)) \qquad (132) \\ &= 1 - \frac{3}{2}(1-\cos(2g+2f)) + \frac{3}{2}\cos^2 I\,(1-\cos(2g+2f)) \qquad (133) \\ &= \left(-\frac{1}{2} + \frac{3}{2}\cos^2 I\right) + \left(\frac{3}{2} - \frac{3}{2}\cos^2 I\right)\cos(2g+2f) \qquad (134) \end{aligned}$$

to obtain the disturbing function R_2 corresponding to the J_2 harmonic

$$\begin{aligned} R_2 \quad := \quad & \frac{\mu k_2}{r^3}\left(1-3\sin^2\beta\right) \qquad (135)\\ = \quad & \frac{\mu k_2}{a^3}\left[\left(-\frac{1}{2}+\frac{3}{2}\cos^2 I\right)\frac{a^3}{r^3}+\left(\frac{3}{2}-\frac{3}{2}\cos^2 I\right)\cos(2g+2f)\frac{a^3}{r^3}\right]\\ = \quad & \frac{\mu^4 k_2}{L^6}\left[\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3}+\left(\frac{3}{2}-\frac{3}{2}\frac{H^2}{G^2}\right)\cos(2g+2f)\frac{a^3}{r^3}\right]. \end{aligned}$$

2.2.2 The J_4 Harmonic

We can expand

$$\begin{aligned} & 1-10\sin^2\beta+\frac{35}{3}\sin^4\beta \qquad (136)\\ = \; & 1-5\sin^2 I\,(1-\cos(2g+2f))+\frac{35}{24}\sin^4 I\,(3-4\cos(2g+2f)+\cos(4g+4f))\\ = \; & \left[1-5(1-\cos^2 I)+\frac{35}{8}(1-\cos^2 I)^2\right]+\left[5(1-\cos^2 I)-\frac{35}{6}(1-\cos^2 I)^2\right]\cos(2g+2f)\\ + \; & \frac{35}{24}(1-\cos^2 I)^2\cos(4g+4f)\\ = \; & \left(\frac{3}{8}-\frac{15}{4}\cos^2 I+\frac{35}{8}\cos^4 I\right)+\left(-\frac{5}{6}+\frac{20}{3}\cos^2 I-\frac{35}{6}\cos^4 I\right)\cos(2g+2f)\\ + \; & \left(\frac{35}{24}-\frac{35}{12}\cos^2 I+\frac{35}{24}\cos^4 I\right)\cos(4g+4f). \end{aligned}$$

to obtain the disturbing function R_4 corresponding to the J_4 harmonic

$$\begin{aligned} R_4 \quad := \quad & \frac{\mu k_4}{r^5}\left[\left(\frac{3}{8}-\frac{15}{4}\cos^2 I+\frac{35}{8}\cos^4 I\right)\right. \qquad (137)\\ + \quad & \left(-\frac{5}{6}+\frac{20}{3}\cos^2 I-\frac{35}{6}\cos^4 I\right)\cos(2g+2f)\\ + \quad & \left.\left(\frac{35}{24}-\frac{35}{12}\cos^2 I+\frac{35}{24}\cos^4 I\right)\cos(4g+4f)\right]\\ = \quad & \frac{\mu^6 k_4}{L^{10}}\left[\left(\frac{3}{8}-\frac{15}{4}\frac{H^2}{G^2}+\frac{35}{8}\frac{H^4}{G^4}\right)\frac{a^5}{r^5}\right.\\ + \quad & \left(-\frac{5}{6}+\frac{20}{3}\frac{H^2}{G^2}-\frac{35}{6}\frac{H^4}{G^4}\right)\frac{a^5}{r^5}\cos(2g+2f)\\ + \quad & \left.\left(\frac{35}{24}-\frac{35}{12}\frac{H^2}{G^2}+\frac{35}{24}\frac{H^4}{G^4}\right)\frac{a^5}{r^5}\cos(4g+4f)\right]. \end{aligned}$$

2.2.3 The J_3 Harmonic

We can expand

$$\begin{aligned} -\frac{3}{2}\sin\beta+\frac{5}{2}\sin^3\beta \quad = \quad & -\frac{3}{2}\sin I\sin(g+f)+\frac{5}{2}\sin^3 I\left(\frac{3}{4}\sin(g+f)-\frac{1}{4}\sin(3g+3f)\right) (138)\\ = \quad & \left(-\frac{3}{2}\sin I+\frac{15}{8}\sin^3 I\right)\sin(g+f)-\frac{5}{8}\sin^3 I\sin(3g+3f). \end{aligned}$$

to obtain the disturbing function R_3 corresponding to the J_3 harmonic

$$\begin{aligned} R_3 &:= \frac{\mu A_{3,0}}{r^4}\left[\left(-\frac{3}{2}\sin I + \frac{15}{8}\sin^3 I\right)\sin(g+f) - \frac{5}{8}\sin^3 I \sin(3g+3f)\right] \\ &= \frac{\mu A_{3,0}}{r^4}\sin I\left[\left(-\frac{3}{2} + \frac{15}{8}\frac{H^2}{G^2}\right)\sin(g+f) - \frac{5}{8}\frac{H^2}{G^2}\sin(3g+3f)\right]. \end{aligned} \tag{139}$$

2.2.4 The J_5 Harmonic

We can expand

$$\begin{aligned} \frac{15}{8}\sin\beta &= \sin I\left[\frac{15}{8}\sin(g+f)\right] \\ -\frac{35}{4}\sin^3\beta &= \sin^3 I\left[-\frac{105}{16}\sin(g+f) + \frac{35}{16}\sin(3g+3f)\right] \\ \frac{63}{8}\sin^5\beta &= \sin^5 I\left[\frac{315}{64}\sin(g+f) - \frac{315}{128}\sin(3g+3f) + \frac{63}{128}\sin(5g+5f)\right] \end{aligned} \tag{140}$$

to obtain the disturbing function R_5 corresponding to the J_5 harmonic

$$\begin{aligned} R_5 := \frac{\mu A_{5,0}}{r^6}\Bigg[&\left(\frac{15}{8}\sin I - \frac{105}{16}\sin^3 I + \frac{315}{64}\sin^5 I\right)\sin(g+f) \\ + &\left(\frac{35}{15}\sin^3 I - \frac{315}{128}\sin^5 I\right)\sin(3g+3f) \\ + &\left(\frac{63}{128}\sin^5 I\right)\sin(5g+5f)\Bigg] \end{aligned} \tag{141}$$

2.3 Hamiltonian Formulation

3 Solution of the Problem

3.1 Von Zeipel Method and Canonical Transformations

The basic idea of the Von Zeipel method is to perform successive canonical transformations so that each transformation gets rid of the dependence of the Hamiltonian on a single canonical coordinate. This is continued until the final Hamiltonian is independent of all canonical coordinates. The canonical momenta in the resulting Hamiltonian are then constants and canonical coordinates are linear functions of time. To obtain a solution to the original problem, the solution is then transformed back to the original coordinates, where each canonical transformation introduces periodics (trigonometric functions) w.r.t variable reintroduced to the Hamiltonian.

$$\begin{tikzcd}
F(l,g,L,G,H) \arrow[d, bend right, "S"'] & \\
F^*(g',L',G',H) \arrow[u, bend right, "\text{short-period terms (l)}"'] \arrow[d, bend right, "S^*"'] & \\
F^{**}(L',G'',H) \arrow[u, bend right, "\text{long-period terms (g)}"'] & \text{secular terms (t)}
\end{tikzcd}$$

Figure 3: Canonical transformations in [1].

Three systems will be used as shown in Figure 3:

- The first system uses undotted variables (L, G, H, l, g, h) and the Hamiltonian $F = F(L, G, H, l, g)$ explicitly dependent on both l and g.
- The second system uses singly dotted variables (L', G', H', l', g', h') and the Hamiltonian $F^* = F^*(L', G', H', g')$ explicitly dependent on g.
- The third system uses double dotted variables $(L'', G'', H'', l'', g'', h'')$ and the Hamiltonian $F^{**} = F^{**}(L'', G'', H'')$ independent from all angle variables.

The generating functions $S = S(L', G', H', l, g, h)$ and $S^* = S^*(L'', G'', H'', l', g', h')$ are used for transformations $F \mapsto F^*$ and $F^* \mapsto F^{**}$, respectively.

3.1.1 Decomposition of the Hamiltonian

In order to derive each transformation, the Hamiltonians as well as the generating functions are expanded as Taylor series in powers of some parameter (e.g. k_2 in [1]) and terms of equal order are equated. For example, for the problem involving only the J_2 harmonic, we can divide F as follows:

$$F = F_0 + F_1 = \underbrace{-\frac{\mu^4}{2L^2}}_{F_0} + \underbrace{\frac{\mu^4 k_2}{L^6}\left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3} + \left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G^2}\right)\cos(2g + 2f)\frac{a^3}{r^3}\right]}_{F_1} \tag{142}$$

The first-order term F_1 is further decomposed into periodic and secular parts. Define

$$A := -\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}, \quad B := \frac{3}{2} - \frac{3}{2}\frac{H^2}{G^2}, \quad \sigma_1 := \frac{a^3}{r^3} - \frac{L^3}{G^3}, \quad \sigma_2 := \frac{a^3}{r^3}\cos(2g+2f). \tag{143}$$

Then,

$$\begin{aligned} F_1 &= \frac{k_2\mu^4}{L^6}\left(A\frac{a^3}{r^3} + B\frac{a^3}{r^3}\cos(2g+2f)\right) &(144)\\ &= \frac{k_2\mu^4}{L^6}\left[A\left(\sigma_1 + \frac{L^3}{G^3}\right) + B\sigma_2\right] \\ &= \underbrace{\frac{k_2\mu^4}{L^6}(A\sigma_1 + B\sigma 2)}_{F_{1p}} + \underbrace{\frac{k_2\mu^4}{G^3L^3}A}_{F_{1s}} =: F_{1p} + F_{1s}. \end{aligned}$$

The dependence on l is limited to the terms σ_1 and σ_2.

3.1.2 First Transformation of the Hamiltonian

We want the new Hamiltonian F^* to be equal so we require

$$F = F_0 + F_1 = F_0^* + F_1^* + F_2^* = F^*, \tag{145}$$

$$S = S_0 + S_1 + S_2, \tag{146}$$

where the Hamiltonians and the generating function have the following dependence on the variables

$$F = F(L, G, H, l, g, -) \tag{147}$$

$$F^* = F^*(L', G', H, -, g', -) \tag{148}$$

$$S = S(L', G', H', l, g, h) \tag{149}$$

We select

$$S_0 = L'l + G'g + H'h. \tag{150}$$

In F_1, the dependence on mean anomaly l is via true anomaly f. To derive the relationship between the original and transformed variables, we require

$$\begin{aligned} F_0(L) + F_1(L, G, H, l, g, -) &= F_0^* + F_1^*(L', G', H, -, g', -) &(151)\\ &+ \quad F_2^*(L', G', H, -, g', -). \end{aligned}$$

From (8), we can substitute

$$
\begin{aligned}
L &= \frac{\partial S}{\partial l} = \frac{\partial}{\partial l}(S_0 + S_1 + S_2) = L' + \frac{\partial S_1}{\partial l} + \frac{\partial S_2}{\partial l}, && (152)\\
G &= \frac{\partial S}{\partial g} = \frac{\partial}{\partial g}(S_0 + S_1 + S_2) = G' + \frac{\partial S_1}{\partial g} + \frac{\partial S_2}{\partial g}, && (153)\\
H &= \frac{\partial S}{\partial h} = \frac{\partial}{\partial h}(S_0 + S_1 + S_2) = H + \frac{\partial S_1}{\partial h} + \frac{\partial S_2}{\partial h}, && (154)\\
l' &= \frac{\partial S}{\partial L'} = \frac{\partial}{\partial L'}(S_0 + S_1 + S_2) = l + \frac{\partial S_1}{\partial L'} + \frac{\partial S_2}{\partial L'}, && (155)\\
g' &= \frac{\partial S}{\partial G'} = \frac{\partial}{\partial G'}(S_0 + S_1 + S_2) = g + \frac{\partial S_1}{\partial G'} + \frac{\partial S_2}{\partial G'}, && (156)\\
h' &= \frac{\partial S}{\partial H'} = \frac{\partial}{\partial H}(S_0 + S_1 + S_2) = h + \frac{\partial S_1}{\partial H'} + \frac{\partial S_2}{\partial H'}. && (157)
\end{aligned}
$$

to obtain

$$
\begin{aligned}
& F_0\left(L' + \frac{\partial S_1}{\partial l} + \frac{\partial S_2}{\partial l}\right) + F_1\left(L' + \frac{\partial S_1}{\partial l} + \frac{\partial S_2}{\partial l}, G' + \frac{\partial S_1}{\partial g} + \frac{\partial S_2}{\partial g}, H', l, g, -\right) && (158)\\
= \; & F_0^*\left(L'\right) + F_1^*\left(L', G', H', -, g + \frac{\partial S_1}{\partial G'} + \frac{\partial S_2}{\partial G'}, -\right) + F_2^*\left(L', G', H', -, g + \frac{\partial S_1}{\partial G'} + \frac{\partial S_2}{\partial G'}, -\right)
\end{aligned}
$$

We expand each as a Taylor series w.r.t. (L', G', H', l, g, h) to obtain

$$
\begin{aligned}
F_0(\cdot) &\approx F_0(L') + \frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l} + \frac{\partial S_2}{\partial l}\right)^2, && (159)\\
F_1(\cdot) &\approx F_1(L', G', H', l, g, -) + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_1}{\partial L'^2}\left(\frac{\partial S_1}{\partial l} + \frac{\partial S_2}{\partial l}\right)^2 && (160)\\
&+ \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} + \frac{\partial F_1}{\partial G'}\frac{\partial S_2}{\partial g} + \frac{1}{2}\frac{\partial^2 F_1}{\partial G'^2}\left(\frac{\partial S_1}{\partial g} + \frac{\partial S_2}{\partial g}\right)^2,\\
F_1^*(\cdot) &\approx F_1^*(L', G', H', -, g, -) + \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'} + \frac{\partial F_1^*}{\partial g}\frac{\partial S_2}{\partial G'} + \frac{1}{2}\frac{\partial^2 F_1^*}{\partial g^2}\left(\frac{\partial S_1}{\partial G'} + \frac{\partial S_2}{\partial G'}\right)^2, && (161)\\
F_2^*(\cdot) &\approx F_2^*(L', G', H', -, g, -) + \frac{\partial F_2^*}{\partial g}\frac{\partial S_1}{\partial G'} + \frac{\partial F_2^*}{\partial g}\frac{\partial S_2}{\partial G'} + \frac{1}{2}\frac{\partial^2 F_2^*}{\partial g^2}\left(\frac{\partial S_1}{\partial G'} + \frac{\partial S_2}{\partial G'}\right)^2. && (162)
\end{aligned}
$$

Here S_k and F_k will have order k w.r.t. k_2. We will only retain the terms that are at most of order 2. Thus,

$$
\begin{aligned}
F_0(\cdot) &\approx F_0 + \frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 && (163)\\
F_1(\cdot) &\approx F_1 + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} && (164)\\
F_1^*(\cdot) &\approx F_1^* + \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'} && (165)\\
F_2^*(\cdot) &\approx F_2^* && (166)
\end{aligned}
$$

and (158) can written

$$
\begin{aligned}
F_0(L') \quad &+ \quad \frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 \\
&+ \quad F_1(L', G', H', l, g) + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} \\
&= \quad F_0^*(L') + F_1^*(L', G', H', g) + \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'} + F_2^*(L', G', H', g).
\end{aligned} \tag{167}
$$

Equating terms of each order of k_2, we get the equations

$$F_0 = F_0^* \tag{168}$$

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + F_1 = F_1^* \tag{169}$$

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} = F_2^* + \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'}. \tag{170}$$

for orders 0, 1 and 2, respectively. Thus

$$F_0^*(L') = \frac{\mu}{2L'^2} \tag{171}$$

The first-degree part (169) can be written as

$$-\frac{\mu^2}{L'^3}\frac{\partial S_1}{\partial l} + \frac{\mu^4 k_2}{L'^3 G'^3}A + \frac{\mu^4 k_2}{L'^6}(A\sigma_1 + B\sigma_2) = F_1^*, \tag{172}$$

and is satisfied if

$$F_1^* = \frac{\mu^4 k_2}{L'^3 G'^3}A, \tag{173}$$

$$\frac{\partial S_1}{\partial l} = \frac{\mu^2 k_2}{L'^3}(A\sigma_1 + B\sigma_2) \tag{174}$$

hold. Thus, we can solve

$$
\begin{aligned}
S_1 &= \frac{\mu^2 k_2}{L'^3}\int (A\sigma_1 + B\sigma_2)\, dl \\
&= \frac{\mu^2 k_2}{L'^3}\left(A\int \sigma_1\, dl + B\int \sigma_2\, dl\right) \\
&= \frac{\mu^2 k_2}{G'^3}\left[A(f - l + e\sin f) + B\left(\frac{1}{2}\sin(2g+2f) + \frac{e}{2}\sin(2g+f) + \frac{e}{6}\sin(2g+3f)\right)\right].
\end{aligned} \tag{175}
$$

3.1.3 Short-Period Terms

The term S_2 is ignored so we finally have the complete generating function $S = S_0 + S_1$ and the short-period terms (periodic w.r.t. l) can be derived. The canonical momenta can be computed with (152) – (154). We will use the constant

$$\gamma_2 := \frac{\mu^2 k_2}{L'^4}. \tag{176}$$

The computation of L is simple using (174):

$$\begin{aligned} L &= L' + \frac{\partial S_1}{\partial l} \qquad (177)\\ &= L'\left[1 + \gamma_2(A\sigma_1 + B\sigma_2)\right] \\ &= L'\left\{1 + \gamma_2\left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^3}{r^3} - \frac{L'^2}{G'^3}\right) + \left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G'^2}\right)\frac{a^3}{r^3}\cos(2g+2f)\right]\right\}. \end{aligned}$$

Computation of G involves only the B part of (175):

$$\begin{aligned} G &= G' + \frac{\partial S_1}{\partial g} \qquad (178)\\ &= G'\left\{1 + \gamma_2\frac{L'^4}{G'^4}\left[\cos(2g+2f) + e\cos(2g+f) + \frac{e}{3}\cos(2g+3f)\right]\right\} \end{aligned}$$

Since S_1 is independent of h,

$$H = H'. \qquad (179)$$

The computation of the short-term periodics in the canonical coordinates is more complex. From (112), we obtain

$$\frac{\partial}{\partial e}(f - l + e\sin f) = (1 + e\cos f)\frac{\partial f}{\partial e} + \sin f = \left(\frac{a^2}{r^2}\frac{G'^2}{L'^2} + \frac{a}{r} + 1\right)\sin f. \qquad (180)$$

For the B part of (175), using (109) − (110) and (112), we obtain

$$\begin{aligned} \frac{\partial}{\partial e}(\cdot)_B &= \frac{\partial}{\partial e}\left(\frac{1}{2}\sin(2g+2f) + \frac{e}{2}\sin(2g+f) + \frac{e}{6}\sin(2g+3f)\right) \qquad (181)\\ &= \frac{\partial f}{\partial e}\left(\cos(2g+2f) + \frac{e}{2}\cos(2g+2f) + \frac{e}{2}\cos(2g+3f)\right) + \frac{1}{2}\sin(2g+f) + \frac{1}{6}\sin(2g+3f) \\ &= \cos(2g+2f)(1+e\cos f)\frac{\partial f}{\partial e} + \frac{1}{2}\sin(2g+f) + \frac{1}{6}\sin(2g+3f). \\ &= \left(\frac{a^2}{r^2}\frac{G'^2}{L'^2} + \frac{a}{r}\right)\cos(2g+2f)\sin f + \frac{1}{2}\sin(2g+f) + \frac{1}{6}\sin(2g+3f). \end{aligned}$$

Now application of standard trigonometric formulas yields

$$\begin{aligned} \cos(2g+2f)\sin f &= \sin(2g+3f) - \sin(2g+2f)\cos f \qquad (182)\\ &= -\sin(2g+f) + \sin(2g+2f)\cos f \\ &= \frac{1}{2}\sin(2g+3f) - \frac{1}{2}\sin(2g+f). \end{aligned}$$

Thus, substituting this back, yields

$$\begin{aligned} \frac{\partial}{\partial e}(\cdot)_B &= \frac{1}{2}\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2} + \frac{a}{r}\right)\sin(2g+3f) + \frac{1}{6}\sin(2g+3f) \qquad (183)\\ &- \frac{1}{2}\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2} + \frac{a}{r}\right)\sin(2g+f) + \frac{1}{2}\sin(2g+f). \\ &= \frac{1}{2}\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2} + \frac{a}{r} + \frac{1}{3}\right)\sin(2g+3f) + \frac{1}{2}\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2} - \frac{a}{r} + 1\right)\sin(2g+f). \end{aligned}$$

With (180) and (183), we can compute

$$\begin{aligned}
\frac{\partial S_1}{\partial e} &= \frac{\mu^2 k_2}{G'^3}\left\{A\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f\right. && (184)\\
&+ \left.\frac{B}{2}\left[\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)+\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)\right]\right\}.
\end{aligned}$$

To compute the partial derivatives w.r.t. L' and G', we first note that

$$e^2 = 1-\frac{G^2}{L^2} \tag{185}$$

and suppose that ψ is a variable of e. Then

$$\frac{\partial\psi}{\partial L} = \frac{d\psi}{de}\frac{\partial e}{\partial L} = \frac{d\psi}{de}\frac{1}{2}\frac{2G^2}{eL^3} = \frac{G^2}{L^3}\frac{1}{e}\frac{d\psi}{de}, \tag{186}$$

$$\frac{\partial\psi}{\partial G} = \frac{d\psi}{de}\frac{\partial e}{\partial G} = -\frac{d\psi}{de}\frac{1}{2}\frac{2G}{eL^2} = -\frac{G}{L^2}\frac{1}{e}\frac{d\psi}{de}. \tag{187}$$

Applying this to S_1, we obtain

$$\frac{\partial S_1}{\partial L'} = \frac{G'^2}{L'^3}\frac{1}{e}\frac{\partial S_1}{\partial e}. \tag{188}$$

Using (155) − (157), we can compute the short-period terms. The momentum L' does not appear explicitly in S_1 so the dependence is only via e. That is,

$$\begin{aligned}
l &= l' - \frac{G'^2}{L'^3}\frac{1}{e}\frac{\partial S_1}{\partial e} && (189)\\
&= l' - \frac{\gamma_2}{e}\frac{L'}{G'}\left\{\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f\right.\\
&+ \left.\left(\frac{3}{4}-\frac{3}{4}\frac{H^2}{G'^2}\right)\left[\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)+\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)\right]\right\}
\end{aligned}$$

Dependence of S_1 on G' is via direct appearance in expression (175) and via dependence (187) on e. That is,

$$g = g' - \frac{\partial S_1}{\partial G'} - \frac{G'}{L'^2}\frac{1}{e}\frac{\partial S_1}{\partial e}. \tag{190}$$

The direct part can be computed

$$\begin{aligned}
\frac{\partial S_1}{\partial G'} &= \frac{\mu^2 k_2}{G'^4}\left[\left(+\frac{3}{2}-\frac{9}{2}\frac{H^2}{G'^2}\right)(\cdot)+\left(-\frac{9}{2}+\frac{9}{2}\frac{H^2}{G'^2}\right)(\cdot)\right] && (191)\\
&+ \frac{\mu^2 k_2}{G'^4}\left[\left(\qquad -\frac{6}{2}\frac{H^2}{G'^2}\right)(\cdot)+\left(\qquad +\frac{6}{2}\frac{H^2}{G'^2}\right)(\cdot)\right]\\
&= -\gamma_2\frac{L'^4}{G'^4}\left[\left(-\frac{3}{2}+\frac{15}{2}\frac{H^2}{G'^2}\right)(\cdot)+\left(\frac{9}{2}-\frac{15}{2}\frac{H^2}{G'^2}\right)(\cdot)\right]
\end{aligned}$$

The indirect part can be computed

$$
\begin{aligned}
\frac{\partial S_1}{\partial e}\frac{\partial e}{\partial G'} &= -\frac{G'}{L'^2}\frac{1}{e}\frac{\partial S_1}{\partial e} \qquad (192)\\
&= -\frac{G'}{L'^2}\frac{1}{e}\frac{\mu^2 k_2}{G'^3}\left\{\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f\right.\\
&+ \left.\left(\frac{3}{4}-\frac{3}{4}\frac{H^2}{G'^2}\right)\left[\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)+\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)\right]\right\}.
\end{aligned}
$$

Thus, we obtain the lenghty expression

$$
\begin{aligned}
g &= g'+\gamma_2\frac{L'^4}{G'^4}\left[\left(-\frac{3}{2}+\frac{15}{2}\frac{H^2}{G'^2}\right)(f-l+e\sin f)\right. \qquad (193)\\
&+ \left.\left(\frac{9}{2}-\frac{15}{2}\frac{H^2}{G'^2}\right)\left(\frac{1}{2}\sin(2g+2f)+\frac{e}{2}\sin(2g+f)+\frac{e}{6}\sin(2g+3f)\right)\right]\\
&+ \frac{\gamma_2}{e}\frac{L'^2}{G'^2}\left\{\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f\right.\\
&+ \left.\left(\frac{3}{4}-\frac{3}{4}\frac{H^2}{G'^2}\right)\left[\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)+\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)\right]\right\}.
\end{aligned}
$$

Finally, we can compute

$$
\begin{aligned}
h &= h'-\frac{\partial S_1}{\partial H} \qquad (194)\\
&= h'-\frac{\mu k_2}{G'^2}\left\{3\frac{H}{G'^2}(f-l+e\sin f)-3\frac{H}{G'^2}\left[\frac{1}{2}\sin(2g+2f)+\frac{e}{2}\sin(2g+f)+\frac{e}{6}\sin(2g+3f)\right]\right\}\\
&= h'-3\gamma_2\frac{L'^4}{G'^4}\frac{H}{G'}\left[f-l+e\sin f-\frac{1}{2}\sin(2g+2f)-\frac{e}{2}\sin(2g+2)-\frac{e}{6}\sin(2g+3f)\right].
\end{aligned}
$$

3.1.4 Second Transformation of the Hamiltonian

Now the remaining part of the new Hamiltonian can be computed

$$
F_2^* = \frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l}+\frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2+\frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial l}+\frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g}-\frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'} \qquad (195)
$$

3.2 Explicit Solution for J_2 Harmonic

3.2.1 Short-Period Terms

3.2.2 Long-Period Terms

3.2.3 Secular Terms

3.3 Elements and Orbit State Vectors

4 SGP4 Solution

4.1 Approximation

4.2 Coordinate Systems and Frames

4.2.1 Mean-of-Date (MoD)

4.2.2 True-of-Date (ToD)

4.2.3 True-Equator, Mean-Equinox (TEME)

4.2.4 Pseudo Earth-Fixed (PEF)

4.3 Atmospheric Drag

4.4 Two-Line Elements

4.5 Kozai Mean Elements

References

[1] Brouwer - Solution of the Problem of Artificial Satellite Theory Without Drag, The Astronomical Journal, 64, No. 1274, 378-396, 1959 (`https://adsabs.harvard.edu/full/1959AJ.....64..378B`).

[2] Karttunen - Johdatus Taivaanmekaniikkaan, Ursan julkaisuja 1982, Tähtitieteellinen yhdistys Ursa, 2001.

[3] Vallado - Companion code for Fundamentals of Astrodynamics and Applications, 2013.

[4] Hoots, Roehrich - Spacetrack Report No. 3, 1980 (`https://celestrak.org/NORAD/documentation/sp`

[5] Lane, Hoots - General Perturbation Theories Derived from the 1965 Lane Drag Theory, Project Space Track, 1979 (`https://apps.dtic.mil/sti/citations/ADA081264`).

[6] Hoots, Schumacher, Glover - History of Analytical Orbit Modeling in the U.S. Space Surveillance System, Journal of Guidance, Control and Dynamics, Vol 27, No.2. 174-185 March-April 2004 (`https://arc.aiaa.org/doi/10.2514/1.9161`).

[7] Hujsak - A Restricted Four Body Solution for Resonating Satellites without Drag, Project Space Track, 1979 (`https://apps.dtic.mil/sti/citations/ADA081263`).

[8] Vallado, Crawford, Hujsak - Revisiting Spacetrack Report No.3, American Institute of Aeronautics and Astronautics, AIAA 2006-6753, 2006 (`https://celestrak.org/publications/AIAA/2006-6753/AIAA-2006-6753.pdf`).