1 Two-Body Problem

To discuss the solution presented in [1], we need to formulate the two-body problem in terms of Delaunay elements. Thus, this section is dedicated to canonical transformations and the Hamilton-Jacobi Equation. Starting from a Hamiltonian formulation in spherical polar coordinates (section 1.1), we will first use Hamilton-Jacobi equations to derive canonical coordinates corresponding to zero Hamiltonian (section 1.3) and then use a canonical transformation to derive a Hamiltonian formulation in terms of Delaunay Elements (section 1.4).

The discussion in this section is roughly based on sections 6 and 7 of [2].

1.1 Hamiltonian Formulation in Spherical Coordinates

Consider the problem of a point particle with mass m in a gravitational potential V, where the kinetic and potential energy can be expressed via the equations

$$T = = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\dot{\phi}) \tag{1}$$

$$V = -\frac{\mu m}{r} \tag{2}$$

in terms of spherical polar coordinates (r, θ, ϕ) and canonical momentums

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \tag{3}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{4}$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mr^2\cos^2\theta\dot{\phi}. \tag{5}$$

Thus, the Hamiltonian (using the notation in [1]) can be expressed

$$F(r, \theta, \phi, p_r, p_\theta, p_\phi) = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\cos^2\theta}\right) - \frac{\mu m}{r}. \tag{6}$$

In this section, we will use canonical transformations and the Hamilton-Jacobi equation to derive the Hamiltonian in terms of canonical coordinates and momentums called Delaunay Elements.

1.2 Canonical Transformations

Suppose F^* is a Hamiltonian obtained from an another Hamiltonian F with a **generating function** (called **determining function** in [1]) S. Then,

$$F^*(\boldsymbol{q}', \boldsymbol{p}') = F(\boldsymbol{q}, \boldsymbol{p}) + \frac{\partial S}{\partial t} \tag{7}$$

In this document, we only use generating functions of the type $S = S(\boldsymbol{q}, \boldsymbol{p}')$, which satisfy

$$p_i = \frac{\partial S}{\partial q_i}, \quad q_i' = \frac{\partial S}{\partial p_i'} \tag{8}$$

1.3 Hamilton-Jacobi Equation

With Hamilton-Jacobi equation, we can derive canonical coordinates, which do not appear explicitly in the transformed Hamiltonian. Then, each canonical coordinate and momentum is independent of time. That is, for $F = F(r, \theta, \phi, p_r, p_\theta, p_\phi)$ we wish find a generating function S such that

$$F\left(r, \theta, \phi, \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}, \frac{\partial S}{\partial \phi}\right) + \frac{\partial S}{\partial t} = 0. \tag{9}$$

Substitution to (6), yields

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\cos^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right] + \frac{\partial S}{\partial t} - \frac{\mu m}{r} = 0. \tag{10}$$

1.3.1 Separation of Variables

We attempt to solve (10) via separation of variables

$$S(r, \theta, \phi, t) = S_r(r) + S_\theta(\theta) + S_\phi(\phi) + S_t(t). \tag{11}$$

This yields

$$\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{1}{r^2\cos^2\theta}\left(\frac{dS_\phi}{d\phi}\right)^2 = 2m\left(-\frac{dS_t}{dt} + \frac{\mu m}{r}\right). \tag{12}$$

The dependence w.r.t. variables ϕ and t is limited to $dS_\phi/d\phi$ and dS_t/dt. Thus, there exists α_2 and α_1 such that

$$\frac{dS_\phi}{d\phi} = -\alpha_2, \quad \frac{dS_t}{dt} = -\alpha_1. \tag{13}$$

The equation (12) can be now written

$$\left(\frac{dS_r}{dr}\right)^2 + \underbrace{\frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{\alpha_2^2}{r^2\cos^2\theta}}_{\alpha_1/r^2} = 2m\left(\alpha_1 + \frac{\mu m}{r}\right) \tag{14}$$

and denoting the middle two terms with α_1/r^2 we obtain

$$\frac{dS_r}{dr} = \sqrt{2m\left(\alpha_1 + \frac{\mu m}{r}\right) - \frac{\alpha_1}{r^2}}. \tag{15}$$

Similarly for S_θ

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = 2mr^2\left(\alpha_1 + \frac{\mu m}{r}\right) - r^2\left(\frac{dS_r}{dr}\right)^2 - \frac{\alpha_2^2}{\cos^2\theta} = \alpha_3^2 + \frac{\alpha_2^2}{\cos^2\theta}. \tag{16}$$

or

$$\frac{dS_\theta}{d\theta} = \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}} \tag{17}$$

The generating function satisfying (9) can be thus written

$$S = -\alpha_1 t + \alpha_2\phi + \int d\theta\sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}} + \int dr\sqrt{2m\left(\alpha_1 + \frac{\mu m}{r}\right) - \frac{\alpha_1}{r^2}} \tag{18}$$

1.3.2 Canonical Momentums

We will select the canonical momentums

$$\begin{aligned}
p_1' &:= \alpha_1 = mh &(19)\\
p_2' &:= \alpha_2 = m\sqrt{a\mu(1-e^2)}\cos I &(20)\\
p_3' &:= \alpha_3 = m\sqrt{a\mu(1-e^2)} &(21)
\end{aligned}$$

To derive (19), we can compute

$$\alpha_1 = -\frac{\partial S}{\partial t} = F = T + V = mh, \tag{22}$$

where $h = v^2/2 - \mu/r$ is the **energy integral**.

Derivation of (20) is somewhat more complicated. Note that in spherical polar coordinates

$$\begin{aligned}
x &= r\cos\theta\cos\phi\\
y &= r\cos\theta\sin\phi\\
z &= r\sin\theta
\end{aligned}$$

and

$$\begin{aligned}
\dot{x} &= \dot{r}\cos\theta\cos\phi - r\dot{\theta}\sin\theta\cos\phi - r\dot{\phi}\cos\theta\sin\phi &(23)\\
\dot{y} &= \dot{r}\cos\theta\sin\phi - r\dot{\theta}\sin\theta\sin\phi + r\dot{\phi}\cos\theta\cos\phi\\
\dot{z} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta
\end{aligned}$$

Now, we can note that that the z-component of the angular momentum vector can be expanded in spherical polar coordinates

$$\begin{aligned}
k_z = (\boldsymbol{r}\times\dot{\boldsymbol{r}})\cdot\hat{z} &= \begin{vmatrix} 0 & 0 & 1\\ x & y & 0\\ \dot{x} & \dot{y} & 0 \end{vmatrix} &(24)\\
&= x\dot{y} - y\dot{x}\\
&= r\cos\theta\cos\phi(\dot{r}\cos\theta\sin\phi - r\dot{\theta}\sin\theta\sin\phi + r\dot{\phi}\cos\theta\cos\phi)\\
&- r\cos\theta\sin\phi(\dot{r}\cos\theta\cos\phi - r\dot{\theta}\sin\theta\cos\phi - r\dot{\phi}\cos\theta\sin\phi)\\
&= r\dot{r}\cos^2\theta(\cos\phi\sin\phi - \cos\phi\sin\phi)\\
&+ r^2\dot{\theta}\cos\phi\sin\phi(\sin\theta\cos\theta - \sin\theta\cos\theta)\\
&+ r^2\dot{\phi}\cos^2\theta(\cos^2\phi + \sin^2\phi)\\
&= r^2\cos^2\theta\dot{\phi}.
\end{aligned}$$

On the other hand, for elliptic orbits

$$a = \frac{k^2}{\mu(1-e^2)}, \tag{25}$$

Thus,

$$\alpha_2 = \frac{\partial S}{\partial \phi} = p_\phi = mr^2\cos^2\theta\dot{\phi} = mk_z = m\sqrt{a\mu(1-e^2)}\cos I \tag{26}$$

To derive (21), we can compute

$$\begin{aligned} \alpha_3 &= \sqrt{\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\alpha_2^2}{\cos^2\theta}} \qquad (27) \\ &= \sqrt{p_\theta^2 + \frac{p\phi^2}{\cos^2\theta}} \\ &= mr^2\sqrt{\dot{\theta}^2 + \cos^2\theta\dot{\phi}^2} = mr^2\dot{f} = mk = m\sqrt{a\mu(1-e^2)}, \end{aligned}$$

where $\dot{f} = \dot{\theta}^2 + \cos^2\theta\dot{\phi}^2$ is the time derivative of the natural anomaly since the sum is the square of total scalar angular speed.

1.3.3 Canonical Coordinates

The new canonical coordinates can be computed

$$\begin{aligned} q_1' = \frac{\partial S}{\partial \alpha_1} &= -t + \frac{\partial}{\partial \alpha_1}\int dr\sqrt{2m\left(\alpha_1 + \frac{m\mu}{r}\right) - \frac{\alpha_3^2}{r^2}} \qquad (28) \\ &= -t + \int m\left[2m\left(\alpha_1 + \frac{m\mu}{r}\right) - \frac{\alpha_3^2}{r^2}\right]^{-1/2} dr \\ &= -t + I_1 \end{aligned}$$

$$\begin{aligned} q_2' = \frac{\partial S}{\partial \alpha_2} &= \phi + \frac{\partial}{\partial \alpha_2}\int d\theta\sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}} \qquad (29) \\ &= \phi - \int \frac{\alpha_2}{\cos^2\theta}\left(\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}\right)^{-1/2} d\theta \\ &= \phi - I_2\alpha_2 \end{aligned}$$

$$\begin{aligned} q_3' = \frac{\partial S}{\partial \alpha_3} &= \frac{\partial}{\partial \alpha_3}\int \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}}d\theta + \frac{\partial}{\partial \alpha_3}\int \sqrt{2m\left(\alpha_1 + \frac{\mu m}{r}\right) - \frac{\alpha_3^2}{r^2}}dr \qquad (30) \\ &= \alpha_3\int\left(\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}\right)^{-1/2} d\theta - \alpha_3\int \frac{1}{r}\left[2m\left(\alpha_1 + \frac{\mu m}{r}\right) - \frac{\alpha_3^2}{r^2}\right]^{-1/2} dr \\ &= \alpha_3 I_4 - \frac{\alpha_3}{m}I_3. \end{aligned}$$

The evaluation of the integrals I_1, I_2, I_3, I_4 is non-trivial and will be performed next.

For I_1, substituting (19), (21) and $mh = -\mu m/2a$, we obtain

$$\begin{aligned} I_1 &= m\int\left[2m\left(mh+\frac{m\mu}{r}\right)-\frac{m^2}{r^2}(a\mu(1-e^2))\right]^{-1/2} dr \\ &= \int m\,dr\left[2m\left(-\frac{\mu m}{2a}+\frac{\mu m}{r}-\frac{m^2 a\mu(1-e^2)}{r^2}\right)\right]^{-1/2} \\ &= \frac{1}{\sqrt{\mu}}\int\frac{r\,dr}{\sqrt{-r^2/a+2r-a(1-e^2)}} \end{aligned} \tag{31}$$

Substitute $r = a(1 - e\cos E)$, which leads to $dr = ae\sin E\,dE$ and

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{\mu}}\int\frac{a(1-e\cos E)\,ae\sin E\,dE}{\sqrt{-a(1-e\cos E)^2+2a(1-e\cos E)-a(1-e^2)}} \\ &= \frac{a^{3/2}}{\sqrt{\mu}}\int(1-e\cos E)\,dE \\ &= \frac{a^{3/2}}{\sqrt{\mu}}(E-e\sin E) \end{aligned} \tag{32}$$

Application of Kepler's equation and Kepler's third law $n = \sqrt{\mu/a^3}$, we obtain the additive inverse for the time of perihelion

$$q_1' = -t + \frac{M}{n} = -\tau. \tag{33}$$

For I_2, it follows from (20), (21) that $\alpha_2/\alpha_3 = \cos I$

$$\alpha_2 I_2 = \int\frac{\alpha_2}{\cos^2\theta\sqrt{\alpha_3^2-\frac{\alpha_2^2}{\cos^2\theta}}} = \frac{\alpha_2}{\alpha_3}\int\frac{d\theta}{\cos^2\theta\sqrt{1-\frac{(\alpha_2/\alpha_3)^2}{\cos^2\theta}}} = \int\frac{\cos I d\theta}{\cos^2\theta\sqrt{1-\frac{cos^2 I}{\cos^2\theta}}} \tag{34}$$

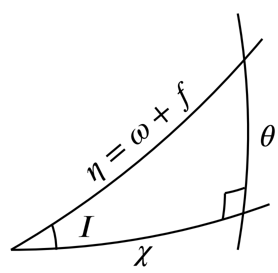

Figure 1: The relationship between the angles η, θ and χ.

Via application of Napier's rule to the angles in Figure 1, we obtain

$$\sin\chi = \cot(\pi/2-\theta)\cot I = \tan\theta\cot I \tag{35}$$

and

$$\frac{d\theta}{\cos^2\theta} = \tan i \cos\chi d\chi. \tag{36}$$

Thus,

$$\alpha_2 I_2 = \int \frac{\cos I \tan I \cos\chi d\chi}{\sqrt{1 - cos^2 I(1 + \tan^2 I \sin^2\chi)}} = \int \frac{\sin I \cos\chi d\chi}{\sqrt{\sin^2 I(1 - \sin^2\chi)}} = \int d\chi = \chi. \tag{37}$$

and

$$q_2' \quad = \quad \phi - \alpha_2 I_2 = \phi - \chi = \Omega. \tag{38}$$

For I_3,

$$\begin{aligned} I_3 \quad &= \quad m\int \frac{1}{r^2}\left[2m\left(-\frac{\mu m}{2a} + \frac{\mu m}{r}\right) - m^2\frac{a\mu(1-e^2)}{r^2}\right]^{-1/2} dr \\ &= \quad \frac{1}{\sqrt{\mu}}\int \frac{dr}{r\sqrt{-r^2/a + 2r - a(1-e^2)}} \end{aligned} \tag{39}$$

Substitute $r = a(1 - e\cos E)$, $dr = ae\sin E dE$, to obtain

$$\begin{aligned} I_3 \quad &= \quad \frac{1}{\sqrt{\mu}}\int \frac{ae\sin E dE}{a(1-e\cos E)\sqrt{-a(1-2e\cos E + e^2\cos^2 E) + 2a(1-e\cos E) - a(1-e^2)}} \\ &= \quad \frac{1}{\sqrt{\mu}}\int \frac{e\sin E}{(1-e\cos E)\sqrt{a(e^2\sin^2 E)}} \\ &= \quad \frac{1}{\sqrt{a\mu}}\int \frac{\sin E dE}{\sin E(1-e\cos E)} \\ &= \quad \frac{1}{\sqrt{a\mu(1-e^2)}}\int \frac{\sqrt{1-e^2}dE}{1-e\cos E} \end{aligned} \tag{40}$$

Substituting $\sin f = \sqrt{1-e^2}\sin E/(1-e\cos E)$ leads to

$$\cos f df = \sqrt{1-e^2}\frac{\cos f}{1-e\cos E}dE \tag{41}$$

Dividing by $\cos f$ and substituting back, we obtain

$$I_3 \quad = \quad \frac{f}{\sqrt{a\mu(1-e^2)}}. \tag{42}$$

For I_4, substituting (20), (21)

$$I_4 \quad = \quad \int\left(\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}\right)^{-1/2} = \frac{1}{m\sqrt{a\mu(1-e^2)}}\int\left(1 - \frac{\cos^2 I}{\cos^2\theta}\right)^{-1/2} d\theta \tag{43}$$

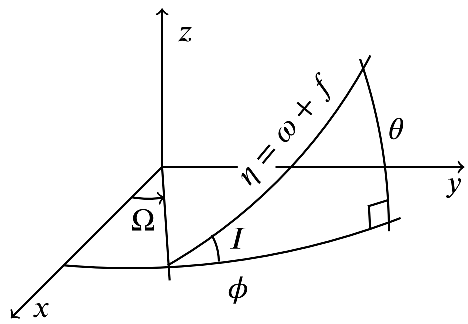

Figure 2: The relationship between the spherical polar coordinates and the Keplerian elements.

Multiplying with α_3 leads

$$\alpha_3 I_4 = \int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta - \cos^2 I}} \tag{44}$$

From Figure 2, we obtain

$$\frac{\sin\theta}{\sin I} = \frac{\sin\eta}{\sin\pi/2} \tag{45}$$

which leads to

$$\cos^2\theta = 1 - \sin^2 I \sin^2\eta \tag{46}$$

$$\cos\theta d\theta = \sin i \cos\eta d\eta. \tag{47}$$

$$\alpha_3 I_4 = \int \frac{\sin I \cos\eta d\eta}{\sqrt{1 - \sin^2 I \sin^2\eta - \cos^2 I}} = \int \frac{\sin I \cos\eta d\eta}{\sqrt{\sin^2 I \cos^2\eta}} = \int d\eta = \eta. \tag{48}$$

Thus,

$$q_3' = \alpha_3 I_4 - \frac{\alpha_3}{m} I_3 = \eta - \frac{m\sqrt{a\mu(1-e^2)}}{m} \frac{f}{\sqrt{a\mu(1-e^2)}} = \eta - f = \omega. \tag{49}$$

1.3.4 Hamiltonian Formulation with Zero Hamiltonian

We now have the new canonical coordinates and momentums

$$q_1' = -\tau, \tag{50}$$

$$q_2' = \Omega, \tag{51}$$

$$q_3' = \omega, \tag{52}$$

$$p_1' = mh, \tag{53}$$

$$p_2' = m\sqrt{a\mu(1-e^2)}\cos I, \tag{54}$$

$$p_3' = m\sqrt{a\mu(1-e^2)}, \tag{55}$$

as well as the new Hamiltonian

$$F' = 0. \tag{56}$$

The mass is typically left out leading to variables

$$\begin{aligned}
q'_1 &= -\tau, & (57)\\
q'_2 &= \Omega, & (58)\\
q'_3 &= \omega, & (59)\\
p'_1 &= h, & (60)\\
p'_2 &= \sqrt{a\mu(1-e^2)}\cos I, & (61)\\
p'_3 &= \sqrt{a\mu(1-e^2)}, & (62)
\end{aligned}$$

Since none of the variables are present in the Hamiltonian $F' = 0$, all of the variables are constant.

1.4 Delaunay Elements

To obtain the Delaunay Elements used in [1] from (56) − (62), we seek the following variables

$$\begin{aligned}
h &= \Omega & (63)\\
g &= \omega & (64)\\
l &= n(t + q_1) & (65)\\
H &= \sqrt{a\mu(1-e^2)} & (66)\\
G &= \sqrt{a\mu(1-e^2)}\cos I & (67)\\
L &= ? & (68)
\end{aligned}$$

These are found with the generating function

$$\begin{aligned}
S &= \left(nL - \frac{3\mu}{2a}\right)(t + q'_1) + q'_2 H + q'_3 G & (69)\\
&= \left(nL - \frac{3\mu}{2a}\right)(t - \tau) + \Omega H + \omega G
\end{aligned}$$

Now

$$p_1 = \frac{\partial S}{\partial q_1} = -\frac{\partial S}{\partial \tau} = -\frac{3\mu}{2a} + nL = h \tag{70}$$

from which by using $h = -\mu/2a$ (for ellipses) and Kepler's third law $n = \sqrt{\mu/a^3}$, we can solve

$$L = \frac{1}{n}\left(h + \frac{3\mu}{2a}\right) = \frac{1}{n}\left(-\frac{\mu}{2a} + \frac{3\mu}{2a}\right) = \frac{\mu}{an} = \sqrt{a\mu} \tag{71}$$

The new Hamiltonian

$$F = 0 + \frac{\partial S}{\partial t} = nL - \frac{3\mu}{2a} = \sqrt{\mu/a^3}\sqrt{a\mu} - \frac{3\mu}{2a} = -\frac{\mu}{2a} = -\frac{\mu^2}{2L^2}. \tag{72}$$

Thus, we have the Hamiltonian formulation in terms of Delaunay variables

$$\boxed{\begin{array}{lll} l = M, & L = \sqrt{a\mu}, & \\ g = \omega, & G = \sqrt{a\mu(1-e^2)}\cos I, & F = -\dfrac{\mu}{2L^2} \\ h = \Omega, & H = \sqrt{a\mu(1-e^2)}, & \end{array}} \tag{73}$$

1.5 Useful Relations

In this section, we derive several relations used by Brouwer in [1] involving the Delaunay variables and Keplerian orbits.

We assume that the following equations for true anomaly are known

$$\cos f = \frac{\cos E - e}{1 - e\cos E}, \tag{74}$$

$$\sin f = \sqrt{1-e^2}\frac{\sin E}{1 - e\cos E}. \tag{75}$$

1.5.1 $\partial E/\partial e$

Consider Kepler's equation

$$M = E - e\sin E \tag{76}$$

as an equation relating the pairs (e', M) and (e, E). Then, partial derivative of M w.r.t. e' must disappear. That is,

$$\begin{aligned} \frac{\partial M}{\partial e'} &= \frac{\partial M}{\partial e}\frac{\partial e}{\partial e'} + \frac{\partial M}{\partial E}\frac{\partial E}{\partial e'} \qquad (77) \\ &= \frac{\partial}{\partial e}(E - e\sin E) + \frac{\partial E}{\partial e'}\frac{\partial}{\partial E}(E - e\sin E) \\ &= -\sin E + \frac{\partial E}{\partial e'}(1 - e\cos E) \\ &= 0. \end{aligned}$$

Thus,

$$\frac{\partial E}{\partial e} = \frac{\sin E}{1 - e\cos E}. \tag{78}$$

1.5.2 $\partial/\partial e(a/r)$

We wish to show that

$$\frac{\partial}{\partial e}\frac{a}{r} = \frac{a^2}{r^2}\cos f \tag{79}$$

To derive this, we can simply compute

$$
\begin{aligned}
\frac{\partial}{\partial e'}\frac{a}{r} &= \frac{\partial}{\partial e}\frac{a}{r} + \frac{\sin E}{1 - e\cos E}\frac{\partial}{\partial E}\frac{a}{r} \\
&= \frac{\cos E}{(1 - e\cos E)^2} + \frac{\sin E}{1 - e\cos E}\frac{-e\sin E}{(1 - e\cos E)^2} \\
&= \frac{cos E - e\cos^2 E - e\sin^2 E}{(1 - e\cos E)^3} \\
&= \frac{a^2}{r^2}\frac{\cos E - e}{1 - e\cos E} \\
&= \frac{a^2}{r^2}\cos f
\end{aligned}
\tag{80}
$$

where we have used

$$
\frac{\partial}{\partial e}\frac{a}{r} = \frac{\partial}{\partial e}\frac{1}{1 - e\cos E} = \frac{\cos E}{(1 - e\cos E)^2} \tag{81}
$$

$$
\frac{\partial}{\partial E}\frac{a}{r} = \frac{\partial}{\partial E}\frac{1}{1 - e\cos E} = \frac{-e\sin E}{(1 - e\cos E)^2} \tag{82}
$$

1.5.3 $\partial l/\partial f$

We wish to show that

$$
\frac{\partial M}{\partial f} = \frac{L}{G}\frac{r^2}{a^2}. \tag{83}
$$

From (74), we obtain

$$
f = \mathrm{acos}\left(\frac{\cos E - e}{1 - e\cos E}\right) \tag{84}
$$

Now by application of the chain rule

$$
\frac{df}{dE} = \left(\frac{d}{dE}\frac{\cos E - e}{1 - e\cos E}\right)\left[-\left(1 - \left[\frac{\cos E - e}{1 - e\cos E}\right]^2\right)^{-1/2}\right]. \tag{85}
$$

For the first part

$$
\begin{aligned}
\frac{d}{dE}\frac{\cos E - e}{1 - e\cos E} &= \frac{(1 - e\cos E)(-\sin E) - (\cos E - e)(e\sin E)}{(1 - e\cos E)^2} \\
&= \frac{-\sin E + e\sin E\cos E - e\sin E\cos E + e^2\sin E}{(1 - e\cos E)^2} \\
&= -\frac{\sin E(1 - e^2)}{(1 - e\cos E)^2}.
\end{aligned}
\tag{86}
$$

For the latter part

$$\begin{aligned} \frac{-1}{\sqrt{1-[\cdot]^2}} &= -\left(\frac{1-2e\cos E+e^2\cos^2 E-\cos^2 E+2e\cos E-e^2}{(1-e\cos E)^2}\right)^{-1/2} \\ &= -\left(\frac{(1-e^2)(1-\cos^2 E)}{(1-e\cos E)^2}\right)^{-1/2} \\ &= -\frac{1-e\cos E}{\sin E\sqrt{1-e^2}}. \end{aligned} \tag{87}$$

Thus

$$\frac{df}{dE} = \frac{\sin E(1-e^2)}{(1-e\cos E)^2}\cdot\frac{1-e\cos E}{\sin E\sqrt{1-e^2}} = \frac{\sqrt{1-e^2}}{1-e\cos E}. \tag{88}$$

Also from the Kepler equation $M = -E - e\sin E$, we obtain

$$\frac{dM}{dE} = 1-e\cos E. \tag{89}$$

Thus,

$$\frac{dM}{df} = \frac{dM}{dE}\frac{dE}{df} = \frac{(1-e\cos E)^2}{\sqrt{1-e^2}} \tag{90}$$

From (74), we can also solve

$$\cos E = \frac{\cos f+e}{1+e\cos f} \tag{91}$$

and

$$1-e\cos E = \frac{1+e\cos f-e\cos f-e^2}{1+e\cos f} = \frac{1-e^2}{1+e\cos f}. \tag{92}$$

Substituting this back to (90), yields

$$\frac{dM}{df} = \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2} \tag{93}$$

For elliptic orbits,

$$\frac{r^2}{a^2} = \left(\frac{1-e^2}{1+e\cos f}\right)^2 = \sqrt{1-e^2}\frac{dM}{df} \tag{94}$$

Thus, we finally have

$$\frac{dM}{df} = \frac{1}{\sqrt{1-e^2}}\frac{r^2}{a^2} = \frac{L}{G}\frac{r^2}{a^2}. \quad \square \tag{95}$$

1.5.4 $\partial f/\partial e$

We wish to show that

$$\frac{\partial f}{\partial e} = \left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f \tag{96}$$

From (74), we obtain

$$\frac{df}{de} = \left(\frac{d}{de}\frac{\cos E - e}{1 - e\cos E}\right)\frac{-1}{\sqrt{1-(\cdot)^2}} \tag{97}$$

The latter part is obtained from (87). For the first part

$$\begin{aligned}
\frac{d}{de}\frac{\cos E - e}{1 - e\cos E} &= \frac{(1 - e\cos E)(\partial E/\partial e(-\sin E) - 1) + (e - \cos E)(\partial E/\partial e\, e\sin E - \cos E)}{(1 - e\cos E)^2} \\
&= \frac{\cos^2 E - 1}{(1 - e\cos E)^2} + \frac{\partial E}{\partial e}\frac{\sin E(e^2 - 1)}{(1 - e\cos E)^2} \\
&= \frac{\cos^2 E - 1}{(1 - e\cos E)^2} + \frac{\sin^2 E(e^2 - 1)}{(1 - e\cos E)^3},
\end{aligned} \tag{98}$$

where we have used (78). Using (87), we can now compute

$$\begin{aligned}
\frac{df}{de} &= -\frac{1 - e\cos E}{\sin E\sqrt{1 - e^2}}\left(\frac{\cos^2 E - 1}{(1 - e\cos E)^2} + \frac{\sin^2 E(e^2 - 1)}{(1 - e\cos E)^3},\right) \\
&= \frac{1}{\sqrt{1 - e^2}}\frac{\sin E}{1 - e\cos E} + \sqrt{1 - e^2}\frac{\sin E}{(1 - e\cos E)^2} \\
&= \frac{\sin f}{1 - e^2} + \frac{\sin f}{1 - e\cos E} \\
&= \left(\frac{1}{1 - e^2} + \frac{a}{r}\right)\sin f \\
&= \left(\frac{L^2}{G^2} + \frac{a}{r}\right)\sin f, \quad \square
\end{aligned} \tag{99}$$

where we have used (75).

2 Spherical Harmonics

2.1 Spheroid Potential

The gravitational potential of an arbitrary distribution of mass can be expanded in terms of spherical harmonics

$$\begin{aligned}
V(\boldsymbol{r}) &= -G\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d^3\boldsymbol{r}^3 \\
&= -\frac{G}{r}\left(1 - \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n J_n P_n(\cos\theta) - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{R}{r}\right)^n P_n^m(\cos\theta)\left(c_{nm}\cos m\phi + s_{nm}\sin n\phi\right)\right)
\end{aligned} \tag{100}$$

where θ is the polar angle, G is the gravitational constant, and R is an arbitrary scaling factor. The Zonal and Tesseral harmonics can be written

$$
\begin{aligned}
J_n &= -\frac{1}{MR^n}\int P_n(\cos\theta')r'^n\rho(\boldsymbol{r}')\,d^3\boldsymbol{r}', && (101)\\
c_{nm} &= -2\frac{(n-m)!}{(n+m)!}\frac{1}{MR^n}\int P_n^m(\cos\theta')\cos m\phi' r'^n\rho(\boldsymbol{r}')d^3\boldsymbol{r}' && (102)\\
s_{nm} &= -2\frac{(n-m)!}{(n+m)!}\frac{1}{MR^n}\int P_n^m(\cos\theta')\sin m\phi' r'^n\rho(\boldsymbol{r}')d^3\boldsymbol{r}, && (103)
\end{aligned}
$$

where P_n is the Legendre polynomial of degree n and P_n^m is the Associated Legendre Polynomial of degree n and order m. If the body is axially symmetric w.r.t. z axis, the tesseral harmonics c_{nm} and s_{nm} disappear. If the body is symmetric w.r.t. the equatorial $z = 0$ plane $P_n(-x) = (-1)^n P_n(x)$ and J_n will disappear for odd-valued n.

Since the body of interest is a spheroid, the tesseral harmonics disappear, and we have an expansion in terms of only zonal harmonics

$$
V(r,\theta) = -\frac{\mu}{r}\left(1-\sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^n J_n P_n(\cos\theta)\right). \qquad (104)
$$

For SGP4, the zonal harmonics are obtained from WGS72[TODO] with the values

$$
\begin{aligned}
J_2 &= 1082.616\cdot 10^{-6}\\
J_3 &= -2.53881\cdot 10^{-6}\\
J_4 &= -1.65597\cdot 10^{-6}.
\end{aligned}
$$

The first four Legendre polynomials can be written

$$
\begin{aligned}
P_1(\cos\theta) &= \cos\theta && (105)\\
P_2(\cos\theta) &= \frac{1}{2}\left(3\cos^2\theta-1\right) && (106)\\
P_3(\cos\theta) &= \frac{1}{2}\left(5\cos^3\theta-3\cos\theta\right) && (107)\\
P_4(\cos\theta) &= \frac{1}{8}\left(35\cos^4\theta-30\cos^2\theta+3\right) && (108)\\
P_5(\cos\theta) &= \frac{1}{8}\left(63\cos^5\theta-70\cos^3\theta+15\cos\theta\right) && (109)
\end{aligned}
$$

Substituting (106) – (108) back to (104), we obtain the potential

$$\begin{aligned} V(r,\theta) &= -\frac{\mu}{r} \\ &+ \frac{\mu J_2 R^2}{2r^3}\left(3\cos^2\theta - 1\right) \\ &+ \frac{\mu J_3 R^3}{2r^4}\left(5\cos^3\theta - 3\cos\theta\right) \\ &+ \frac{\mu J_4 R^4}{8r^5}\left(35\cos^4\theta - 30\cos^2\theta + 3\right) \\ &+ \frac{\mu J_5 R^5}{8r^6}\left(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta\right) \end{aligned} \tag{110}$$

We use $R = a_E$ to denote the equatorial radius of Earth and define

$$k_2 = \frac{J_2 a_E^2}{2}, \quad k_4 = -\frac{3J_4 a_E^4}{8}, \quad A_{3,0} = -J_3 a_E^3, \quad A_{5,0} = -J_5 a_E^5. \tag{111}$$

Thereafter, using β to denote the geocentric latitude, we obtain $\cos\theta = \sin\beta$ and

$$\begin{aligned} V(r,\theta) &= -\frac{\mu}{r} \\ &- \frac{\mu k_2}{r^3}\left(1 - 3\sin^2\beta\right) \\ &- \frac{\mu k_4}{r^5}\left(1 - 10\sin^2\beta + \frac{35}{3}\sin^4\beta\right) \\ &- \frac{\mu A_{3,0}}{r^4}\left(-\frac{3}{2}\sin\beta + \frac{5}{2}\sin^3\beta\right) \\ &- \frac{\mu A_{5,0}}{r^5}\left(\frac{15}{8}\sin\beta - \frac{35}{4}\sin^3\beta + \frac{63}{8}\sin^5\beta\right). \end{aligned} \tag{112}$$

Note that Brouwer calls the gravitational potential the force function and uses different sign [1]. That is, $U = -V$.

2.2 Disturbing Function

The first term in (112) corresponds to the unperturbed potential of a two-body problem, where the Keplerian elements are constants. The additional terms constitute a perturbing potential that leads to perturbations in the Keplerian elements. That is, instead of the Hamiltonian (**??**), we have

$$F = \frac{\mu}{2L^2} + R, \tag{113}$$

where $R = R_2 + R_3 + R_4 + R_5$ is the **disturbing function** and R_k denote the part of R corresponding to k:th degree in (112). In this section, we will derive R in terms of Delaunay variables.

In perifocal frame, where the non-disturbed orbit is on the $z = 0$ plane and positive x axis corresponds to to the rising node, we can write the position vector as

$$r\left[\cos(g+f), \quad \sin(g+f), \quad 0\right]^T. \tag{114}$$

The position vector in an equatorial inertial frame with x axis corresponding to the rising node can be obtained via application of an anti-clockwise rotation w.r.t. x coordinate

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \begin{bmatrix} \cos(g+f) \\ \sin(g+f) \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(g+f) \\ \cos I \sin(g+f) \\ \sin I \sin(g+f) \end{bmatrix} \tag{115}$$

The z-coordinate can be related to the geocentric latitude β in (112) with the equation

$$\sin\beta = \sin I \sin(g+f). \tag{116}$$

Then, by using the relations

$$\begin{aligned} \sin^2(g+f) &= (1-\cos(2g+2f))/2, \\ \sin(g+f)\cos(2g+2f) &= \frac{1}{2}\sin(3g+3f) - \frac{1}{2}\sin(g+f), \\ \cos 2\theta &= 2\cos^2\theta - 1, \\ \sin(g+f)\cos(4g+4f) &= \frac{1}{2}\sin(5g+5f) - \frac{1}{2}\sin(3g+3f), \end{aligned}$$

we obtain

$$\begin{aligned} \sin^2\beta &= \frac{1}{2}\sin^2 I\,(1-\cos(2g+2f))\,, \qquad (117) \\ \sin^3\beta &= \sin^3 I\left(\frac{3}{4}\sin(g+f) - \frac{1}{4}\sin(3g+3f)\right), \\ \sin^4\beta &= \frac{1}{4}\sin^4 I\left(1 - 2\cos(2g+2f) + \cos^2(2g+2f)\right) \\ &= \frac{1}{8}\sin^4 I\,(3 - 4\cos(2g+2f) + \cos(4g+4f)) \\ \sin^5\beta &= \frac{1}{16}\left[6\sin(g+f) - 4\sin(3g+3f) + 4\sin(g+f) + \sin(5g+5f) - \sin(3g+3f)\right] \\ &= \sin^5 I\left[\frac{5}{8}\sin(g+f) - \frac{5}{16}\sin(3g+3f) + \frac{1}{16}\sin(5g+5f)\right]. \end{aligned}$$

2.2.1 The J_2 Harmonic

We can expand

$$\begin{aligned} 1 - 3\sin^2\beta &= 1 - \frac{3}{2}\sin^2 I\,(1-\cos(2g+2f)) &(118) \\ &= 1 - \frac{3}{2}(1-\cos(2g+2f)) + \frac{3}{2}\cos^2 I\,(1-\cos(2g+2f)) &(119) \\ &= \left(-\frac{1}{2} + \frac{3}{2}\cos^2 I\right) + \left(\frac{3}{2} - \frac{3}{2}\cos^2 I\right)\cos(2g+2f) &(120) \end{aligned}$$

to obtain the disturbing function R_2 corresponding to the J_2 harmonic

$$\begin{aligned}
R_2 \quad &:= \quad \frac{\mu k_2}{r^3}\left(1-3\sin^2\beta\right) \qquad (121)\\
&= \quad \frac{\mu k_2}{a^3}\left[\left(-\frac{1}{2}+\frac{3}{2}\cos^2 I\right)\frac{a^3}{r^3}+\left(\frac{3}{2}-\frac{3}{2}\cos^2 I\right)\cos(2g+2f)\frac{a^3}{r^3}\right]\\
&= \quad \frac{\mu k_2}{a^3}\left[\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3}+\left(\frac{3}{2}-\frac{3}{2}\frac{H^2}{G^2}\right)\cos(2g+2f)\frac{a^3}{r^3}\right].
\end{aligned}$$

2.2.2 The J_4 Harmonic

We can expand

$$\begin{aligned}
& 1-10\sin^2\beta+\frac{35}{3}\sin^4\beta \qquad (122)\\
= \; & 1-5\sin^2 I\,(1-\cos(2g+2f))+\frac{35}{24}\sin^4 I\,(3-4\cos(2g+2f)+\cos(4g+4f))\\
= \; & \left[1-5(1-\cos^2 I)+\frac{35}{8}(1-\cos^2 I)^2\right]+\left[5(1-\cos^2 I)-\frac{35}{6}(1-\cos^2 I)^2\right]\cos(2g+2f)\\
+ \; & \frac{35}{24}(1-\cos^2 I)^2\cos(4g+4f)\\
= \; & \left(\frac{3}{8}-\frac{15}{4}\cos^2 I+\frac{35}{8}\cos^4 I\right)+\left(-\frac{5}{6}+\frac{20}{3}\cos^2 I-\frac{35}{6}\cos^4 I\right)\cos(2g+2f)\\
+ \; & \left(\frac{35}{24}-\frac{35}{12}\cos^2 I+\frac{35}{24}\cos^4 I\right)\cos(4g+4f).
\end{aligned}$$

to obtain the disturbing function R_4 corresponding to the J_4 harmonic

$$\begin{aligned}
R_4 \quad &:= \quad \frac{\mu k_4}{r^5}\left[\left(\frac{3}{8}-\frac{15}{4}\cos^2 I+\frac{35}{8}\cos^4 I\right)\right. \qquad (123)\\
&+ \quad \left(-\frac{5}{6}+\frac{20}{3}\cos^2 I-\frac{35}{6}\cos^4 I\right)\cos(2g+2f)\\
&+ \quad \left.\left(\frac{35}{24}-\frac{35}{12}\cos^2 I+\frac{35}{24}\cos^4 I\right)\cos(4g+4f)\right]\\
&= \quad \frac{\mu^6 k_4}{L^{10}}\left[\left(\frac{3}{8}-\frac{15}{4}\frac{H^2}{G^2}+\frac{35}{8}\frac{H^4}{G^4}\right)\frac{a^5}{r^5}\right.\\
&+ \quad \left(-\frac{5}{6}+\frac{20}{3}\frac{H^2}{G^2}-\frac{35}{6}\frac{H^4}{G^4}\right)\frac{a^5}{r^5}\cos(2g+2f)\\
&+ \quad \left.\left(\frac{35}{24}-\frac{35}{12}\frac{H^2}{G^2}+\frac{35}{24}\frac{H^4}{G^4}\right)\frac{a^5}{r^5}\cos(4g+4f)\right].
\end{aligned}$$

2.2.3 The J_3 Harmonic

We can expand

$$\begin{aligned}
-\frac{3}{2}\sin\beta+\frac{5}{2}\sin^3\beta \quad &= \quad -\frac{3}{2}\sin I\sin(g+f)+\frac{5}{2}\sin^3 I\left(\frac{3}{4}\sin(g+f)-\frac{1}{4}\sin(3g+3f)\right) \qquad (124)\\
&= \quad \left(-\frac{3}{2}\sin I+\frac{15}{8}\sin^3 I\right)\sin(g+f)-\frac{5}{8}\sin^3 I\sin(3g+3f).
\end{aligned}$$

to obtain the disturbing function R_3 corresponding to the J_3 harmonic

$$
\begin{aligned}
R_3 \quad &:= \quad \frac{\mu A_{3,0}}{r^4}\left[\left(-\frac{3}{2}\sin I + \frac{15}{8}\sin^3 I\right)\sin(g+f) - \frac{5}{8}\sin^3 I \sin(3g+3f)\right] \\
&= \quad \frac{\mu A_{3,0}}{r^4}\sin I\left[\left(-\frac{3}{2} + \frac{15}{8}\frac{H^2}{G^2}\right)\sin(g+f) - \frac{5}{8}\frac{H^2}{G^2}\sin(3g+3f)\right].
\end{aligned} \tag{125}
$$

2.2.4 The J_5 Harmonic

We can expand

$$
\begin{aligned}
\frac{15}{8}\sin\beta \quad &= \quad \sin I\left[\frac{15}{8}\sin(g+f)\right] \\
-\frac{35}{4}\sin^3\beta \quad &= \quad \sin^3 I\left[-\frac{105}{16}\sin(g+f) + \frac{35}{16}\sin(3g+3f)\right] \\
\frac{63}{8}\sin^5\beta \quad &= \quad \sin^5 I\left[\frac{315}{64}\sin(g+f) - \frac{315}{128}\sin(3g+3f) + \frac{63}{128}\sin(5g+5f)\right]
\end{aligned} \tag{126}
$$

to obtain the disturbing function R_5 corresponding to the J_5 harmonic

$$
\begin{aligned}
R_5 \quad &:= \quad \frac{\mu A_{5,0}}{r^6}\left[\left(\frac{15}{8}\sin I - \frac{105}{16}\sin^3 I + \frac{315}{64}\sin^5 I\right)\sin(g+f)\right. \\
&+ \qquad \left(\frac{35}{15}\sin^3 I - \frac{315}{128}\sin^5 I\right)\sin(3g+3f) \\
&+ \qquad \left.\left(\frac{63}{128}\sin^5 I\right)\sin(5g+5f)\right]
\end{aligned} \tag{127}
$$

2.3 Hamiltonian Formulation

3 Solution of the Problem

3.1 Von Zeipel Method and Canonical Transformations

3.2 Explicit Solution for J_2 Harmonic

3.2.1 Short-Period Terms

3.2.2 Long-Period Terms

3.2.3 Secular Terms

3.3 Elements and Orbit State Vectors

4 SGP4 Solution

4.1 Approximation

4.2 Coordinate Systems and Frames

4.2.1 Mean-of-Date (MoD)

4.2.2 True-of-Date (ToD)

4.2.3 True-Equator, Mean-Equinox (TEME)

4.2.4 Pseudo Earth-Fixed (PEF)

4.3 Atmospheric Drag

4.4 Two-Line Elements

4.5 Kozai Mean Elements

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