

1 Two-Body Problem

To discuss the solution presented in [?], we need to formulate the two-body problem in terms of Delaunay elements. Thus, this section is dedicated to canonical transformations and the Hamilton-Jacobi Equation. Starting from a Hamiltonian formulation in spherical polar coordinates, we will first use Hamilton-Jacobi equations to derive canonical coordinates corresponding to zero Hamiltonian and then use a canonical transformation to derive a Hamiltonian formulation in terms of Delaunay Elements.

The discussion in this section is roughly based on sections 6 and 7 of [?].

1.1 Hamiltonian Formulation

Consider the problem of a point particle with mass m in a gravitational potential V , where the kinetic and potential energy can be expressed via the equations

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \cos^2 \theta \dot{\phi}^2) \quad (1)$$

$$V = -\frac{\mu m}{r} \quad (2)$$

in terms of spherical polar coordinates (r, θ, ϕ) and canonical momentums

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \quad (3)$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad (4)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mr^2 \cos^2 \theta \dot{\phi}. \quad (5)$$

Thus, the Hamiltonian (using the notation in [?]) can be expressed

$$F(r, \theta, \phi, p_r, p_\theta, p_\phi) = T + V = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \cos^2 \theta} \right) - \frac{\mu m}{r}. \quad (6)$$

In this section, we will use canonical transformations and the Hamilton-Jacobi equation to derive the Hamiltonian in terms of canonical coordinates and momentums called Delaunay Elements.

1.1.1 Canonical Transformations

Suppose F^* is a Hamiltonian obtained from an another Hamiltonian F with a **generating function** (called **determining function** in [?]) S . Then,

$$F^*(q', p') = F(q, p) + \frac{\partial S}{\partial t} \quad (7)$$

In this document, we only use generating functions of the type $S = S(q, p')$, which satisfy

$$p_i = \frac{\partial S}{\partial q_i}, \quad q'_i = \frac{\partial S}{\partial p'_i} \quad (8)$$

1.1.2 Hamilton-Jacobi Equation

With Hamilton-Jacobi equation, we can derive canonical coordinates, which do not appear explicitly in the transformed Hamiltonian. Then, each canonical coordinate and momentum is independent of time. That is, for $F = F(r, \theta, \phi, p_r, p_\theta, p_\phi)$ we wish find a generating function S such that

$$F\left(r, \theta, \phi, \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}, \frac{\partial S}{\partial \phi}\right) + \frac{\partial S}{\partial t} = 0. \quad (9)$$

Substitution to (6), yields

$$\frac{1}{2m} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \cos^2 \theta} \left(\frac{\partial S}{\partial \phi} \right)^2 \right] + \frac{\partial S}{\partial t} - \frac{\mu m}{r} = 0. \quad (10)$$

We attempt to solve (10) via separation of variables

$$S(r, \theta, \phi, t) = S_r(r) + S_\theta(\theta) + S_\phi(\phi) + S_t(t). \quad (11)$$

This yields

$$\left(\frac{dS_r}{dr} \right)^2 + \frac{1}{r^2} \left(\frac{dS_\theta}{d\theta} \right)^2 + \frac{1}{r^2 \cos^2 \theta} \left(\frac{dS_\phi}{d\phi} \right)^2 = 2m \left(-\frac{dS_t}{dt} + \frac{\mu m}{r} \right). \quad (12)$$

The dependence w.r.t. variables ϕ and t is limited to $dS_\phi/d\phi$ and dS_t/dt . Thus, there exists α_2 and α_1 such that

$$\frac{dS_\phi}{d\phi} = -\alpha_2, \quad \frac{dS_t}{dt} = -\alpha_1. \quad (13)$$

The equation (12) can be now written

$$\left(\frac{dS_r}{dr} \right)^2 + \underbrace{\frac{1}{r^2} \left(\frac{dS_\theta}{d\theta} \right)^2 + \frac{\alpha_2^2}{r^2 \cos^2 \theta}}_{\alpha_1/r^2} = 2m \left(\alpha_1 + \frac{\mu m}{r} \right) \quad (14)$$

and denoting the middle two terms with α_1/r^2 we obtain

$$\frac{dS_r}{dr} = \sqrt{2m \left(\alpha_1 + \frac{\mu m}{r} \right) - \frac{\alpha_1}{r^2}}. \quad (15)$$

Similarly for S_θ

$$\left(\frac{dS_\theta}{d\theta} \right)^2 = 2mr^2 \left(\alpha_1 + \frac{\mu m}{r} \right) - r^2 \left(\frac{dS_r}{dr} \right)^2 - \frac{\alpha_2^2}{\cos^2 \theta} = \alpha_3^2 + \frac{\alpha_2^2}{\cos^2 \theta}. \quad (16)$$

or

$$\frac{dS_\theta}{d\theta} = \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2 \theta}} \quad (17)$$

The generating function satisfying (9) can be thus written

$$S = -\alpha_1 t + \alpha_2 \phi + \int d\theta \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2 \theta}} + \int dr \sqrt{2m \left(\alpha_1 + \frac{\mu m}{r} \right) - \frac{\alpha_1^2}{r^2}} \quad (18)$$

We will select the canonical momentums

$$p'_1 := \alpha_1 = mh \quad (19)$$

$$p'_2 := \alpha_2 = m\sqrt{a\mu(1-e^2)} \cos I \quad (20)$$

$$p'_3 := \alpha_3 = m\sqrt{a\mu(1-e^2)} \quad (21)$$

To derive (19), we can compute

$$\alpha_1 = -\frac{\partial S}{\partial t} = F = T + V = mh, \quad (22)$$

where $h = v^2/2 - \mu/r$ is the **energy integral**.

Derivation of (20) is somewhat more complicated. Note that in spherical polar coordinates

$$\begin{aligned} x &= r \cos \theta \cos \phi \\ y &= r \cos \theta \sin \phi \\ z &= r \sin \theta \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= \dot{r} \cos \theta \cos \phi - r\dot{\theta} \sin \theta \cos \phi - r\dot{\phi} \cos \theta \sin \phi \\ \dot{y} &= \dot{r} \cos \theta \sin \phi - r\dot{\theta} \sin \theta \sin \phi + r\dot{\phi} \cos \theta \cos \phi \\ \dot{z} &= \dot{r} \sin \theta + r\dot{\theta} \cos \theta \end{aligned} \quad (23)$$

Now, we can note that the z-component of the angular momentum vector can be expanded in spherical polar coordinates

$$\begin{aligned} k_z = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \hat{\mathbf{z}} &= \begin{vmatrix} 0 & 0 & 1 \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} \\ &= x\dot{y} - y\dot{x} \\ &= r \cos \theta \cos \phi (\dot{r} \cos \theta \sin \phi - r\dot{\theta} \sin \theta \sin \phi + r\dot{\phi} \cos \theta \cos \phi) \\ &\quad - r \cos \theta \sin \phi (\dot{r} \cos \theta \cos \phi - r\dot{\theta} \sin \theta \cos \phi - r\dot{\phi} \cos \theta \sin \phi) \\ &= r\dot{r} \cos^2 \theta (\cos \phi \sin \phi - \cos \phi \sin \phi) \\ &\quad + r^2 \dot{\theta} \cos \phi \sin \phi (\sin \theta \cos \theta - \sin \theta \cos \theta) \\ &\quad + r^2 \dot{\phi} \cos^2 \theta (\cos^2 \phi + \sin^2 \phi) \\ &= r^2 \cos^2 \theta \dot{\phi}. \end{aligned} \quad (24)$$

On the other hand, for elliptic orbits

$$a = \frac{k^2}{\mu(1-e^2)}, \quad (25)$$

Thus,

$$\alpha_2 = \frac{\partial S}{\partial \phi} = p_\phi = mr^2 \cos^2 \theta \dot{\phi} = mk_z = m\sqrt{a\mu(1-e^2)} \cos I \quad (26)$$

To derive (21), we can compute

$$\begin{aligned} \alpha_3 &= \sqrt{\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\alpha_2^2}{\cos^2 \theta}} \\ &= \sqrt{p_\theta^2 + \frac{p\phi^2}{\cos^2 \theta}} \\ &= mr^2 \sqrt{\dot{\theta}^2 + \cos^2 \theta \dot{\phi}^2} = mr^2 \dot{f} = mk = m\sqrt{a\mu(1-e^2)}, \end{aligned} \quad (27)$$

where $\dot{f} = \dot{\theta}^2 + \cos^2 \theta \dot{\phi}^2$ is the time derivative of the natural anomaly since the sum is the square of total scalar angular speed.

The new canonical coordinates can be computed

$$\begin{aligned} q'_1 = \frac{\partial S}{\partial \alpha_1} &= -t + \frac{\partial}{\partial \alpha_1} \int dr \sqrt{2m \left(\alpha_1 + \frac{m\mu}{r} \right) - \frac{\alpha_3^2}{r^2}} \\ &= -t + \int m \left[2m \left(\alpha_1 + \frac{m\mu}{r} \right) - \frac{\alpha_3^2}{r^2} \right]^{-1/2} dr \\ &= -t + I_1 \end{aligned} \quad (28)$$

$$\begin{aligned} q'_2 = \frac{\partial S}{\partial \alpha_2} &= \phi + \frac{\partial}{\partial \alpha_2} \int d\theta \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2 \theta}} \\ &= \phi - \int \frac{\alpha_2}{\cos^2 \theta} \left(\alpha_3^2 - \frac{\alpha_2^2}{\cos^2 \theta} \right)^{-1/2} d\theta \\ &= \phi - I_2 \alpha_2 \end{aligned} \quad (29)$$

$$\begin{aligned} q'_3 = \frac{\partial S}{\partial \alpha_3} &= \frac{\partial}{\partial \alpha_3} \int \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2 \theta}} d\theta + \frac{\partial}{\partial \alpha_3} \int \sqrt{2m \left(\alpha_1 + \frac{\mu m}{r} \right) - \frac{\alpha_3^2}{r^2}} dr \\ &= \alpha_3 \int \left(\alpha_3^2 - \frac{\alpha_2^2}{\cos^2 \theta} \right)^{-1/2} d\theta - \alpha_3 \int \frac{1}{r} \left[2m \left(\alpha_1 + \frac{\mu m}{r} \right) - \frac{\alpha_3^2}{r^2} \right]^{-1/2} dr \\ &= \alpha_3 I_4 - \frac{\alpha_3}{m} I_3. \end{aligned} \quad (30)$$

The evaluation of the integrals I_1, I_2, I_3, I_4 is non-trivial and will be performed next.

For I_1 , substituting (19), (21) and $mh = -\mu m/2a$, we obtain

$$\begin{aligned}
I_1 &= m \int \left[2m \left(mh + \frac{m\mu}{r} \right) - \frac{m^2}{r^2} (a\mu(1-e^2)) \right]^{-1/2} dr \\
&= \int m dr \left[2m \left(-\frac{\mu m}{2a} + \frac{\mu m}{r} - \frac{m^2 a\mu(1-e^2)}{r^2} \right) \right]^{-1/2} \\
&= \frac{1}{\sqrt{\mu}} \int \frac{r dr}{\sqrt{-r^2/a + 2r - a(1-e^2)}}
\end{aligned} \tag{31}$$

Substitute $r = a(1 - e \cos E)$, which leads to $dr = ae \sin E dE$ and

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{\mu}} \int \frac{a(1 - e \cos E) ae \sin E dE}{\sqrt{-a(1 - e \cos E)^2 + 2a(1 - e \cos E) - a(1 - e^2)}} \\
&= \frac{a^{3/2}}{\sqrt{\mu}} \int (1 - e \cos E) dE \\
&= \frac{a^{3/2}}{\sqrt{\mu}} (E - e \sin E)
\end{aligned} \tag{32}$$

Application of Kepler's equation and Kepler's third law $n = \sqrt{\mu/a^3}$, we obtain the additive inverse for the time of perihelion

$$q'_1 = -t + \frac{M}{n} = -\tau. \tag{33}$$

For I_2 , it follows from (20), (21) that $\alpha_2/\alpha_3 = \cos I$

$$\alpha_2 I_2 = \int \frac{\alpha_2}{\cos^2 \theta \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2 \theta}}} = \frac{\alpha_2}{\alpha_3} \int \frac{d\theta}{\cos^2 \theta \sqrt{1 - \frac{(\alpha_2/\alpha_3)^2}{\cos^2 \theta}}} = \int \frac{\cos I d\theta}{\cos^2 \theta \sqrt{1 - \frac{\cos^2 I}{\cos^2 \theta}}} \tag{34}$$

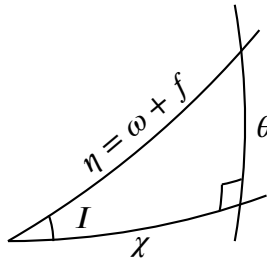


Figure 1: The relationship between the angles η, θ and χ .

Via application of Napier's rule to the angles in Figure 1, we obtain

$$\sin \chi = \cot(\pi/2 - \theta) \cot I = \tan \theta \cot I \tag{35}$$

and

$$\frac{d\theta}{\cos^2 \theta} = \tan i \cos \chi d\chi. \quad (36)$$

Thus,

$$\alpha_2 I_2 = \int \frac{\cos I \tan I \cos \chi d\chi}{\sqrt{1 - \cos^2 I (1 + \tan^2 I \sin^2 \chi)}} = \int \frac{\sin I \cos \chi d\chi}{\sqrt{\sin^2 I (1 - \sin^2 \chi)}} = \int d\chi = \chi. \quad (37)$$

and

$$q'_2 = \phi - \alpha_2 I_2 = \phi - \chi = \Omega. \quad (38)$$

For I_3 ,

$$\begin{aligned} I_3 &= m \int \frac{1}{r^2} \left[2m \left(-\frac{\mu m}{2a} + \frac{\mu m}{r} \right) - m^2 \frac{a\mu(1 - e^2)}{r^2} \right]^{-1/2} dr \\ &= \frac{1}{\sqrt{\mu}} \int \frac{dr}{r \sqrt{-r^2/a + 2r - a(1 - e^2)}} \end{aligned} \quad (39)$$

Substitute $r = a(1 - e \cos E)$, $dr = ae \sin E dE$, to obtain

$$\begin{aligned} I_3 &= \frac{1}{\sqrt{\mu}} \int \frac{ae \sin E dE}{a(1 - e \cos E) \sqrt{-a(1 - 2e \cos E + e^2 \cos^2 E) + 2a(1 - e \cos E) - a(1 - e^2)}} \\ &= \frac{1}{\sqrt{\mu}} \int \frac{e \sin E}{(1 - e \cos E) \sqrt{a(e^2 \sin^2 E)}} \\ &= \frac{1}{\sqrt{a\mu}} \int \frac{\sin E dE}{\sin E (1 - e \cos E)} \\ &= \frac{1}{\sqrt{a\mu(1 - e^2)}} \int \frac{\sqrt{1 - e^2} dE}{1 - e \cos E} \end{aligned} \quad (40)$$

Substituting $\sin f = \sqrt{1 - e^2} \sin E / (1 - e \cos E)$ leads to

$$\cos f df = \sqrt{1 - e^2} \frac{\cos f}{1 - e \cos E} dE \quad (41)$$

Dividing by $\cos f$ and substituting back, we obtain

$$I_3 = \frac{f}{\sqrt{a\mu(1 - e^2)}}. \quad (42)$$

For I_4 , substituting (20), (21)

$$I_4 = \int \left(\alpha_3^2 - \frac{\alpha_2^2}{\cos^2 \theta} \right)^{-1/2} = \frac{1}{m \sqrt{a\mu(1 - e^2)}} \int \left(1 - \frac{\cos^2 I}{\cos^2 \theta} \right)^{-1/2} d\theta \quad (43)$$

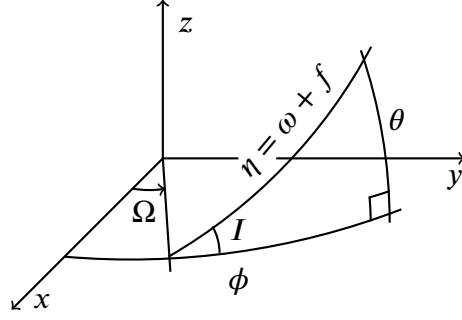


Figure 2: The relationship between the spherical polar coordinates and the Keplerian elements.

Multiplying with α_3 leads

$$\alpha_3 I_4 = \int \frac{\cos \theta d\theta}{\sqrt{\cos^2 \theta - \cos^2 I}} \quad (44)$$

From Figure 2, we obtain

$$\frac{\sin \theta}{\sin I} = \frac{\sin \eta}{\sin \pi/2} \quad (45)$$

which leads to

$$\cos^2 \theta = 1 - \sin^2 I \sin^2 \eta \quad (46)$$

$$\cos \theta d\theta = \sin i \cos \eta d\eta. \quad (47)$$

$$\alpha_3 I_4 = \int \frac{\sin I \cos \eta d\eta}{\sqrt{1 - \sin^2 I \sin^2 \eta - \cos^2 I}} = \int \frac{\sin I \cos \eta d\eta}{\sqrt{\sin^2 I \cos^2 \eta}} = \int d\eta = \eta. \quad (48)$$

Thus,

$$q'_3 = \alpha_3 I_4 - \frac{\alpha_3}{m} I_3 = \eta - \frac{m\sqrt{a\mu(1-e^2)}}{m} \frac{f}{\sqrt{a\mu(1-e^2)}} = \eta - f = \omega. \quad (49)$$

We now have the new canonical coordinates and momentums

$$q'_1 = -\tau, \quad (50)$$

$$q'_2 = \Omega, \quad (51)$$

$$q'_3 = \omega, \quad (52)$$

$$p'_1 = mh, \quad (53)$$

$$p'_2 = m\sqrt{a\mu(1-e^2)} \cos I, \quad (54)$$

$$p'_3 = m\sqrt{a\mu(1-e^2)}, \quad (55)$$

as well as the new Hamiltonian

$$H' = 0. \quad (56)$$

1.1.3 Delaunay Elements

2 Spherical Harmonics

2.1 Spheroid Potential

The gravitational potential of an arbitrary distribution of mass can be expanded in terms of spherical harmonics

$$\begin{aligned} V(\mathbf{r}) &= -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \\ &= -\frac{G}{r} \left(1 - \sum_{n=1}^{\infty} \left(\frac{R}{r} \right)^n J_n P_n(\cos \theta) - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{R}{r} \right)^n P_n^m(\cos \theta) (c_{nm} \cos m\phi + s_{nm} \sin m\phi) \right) \end{aligned} \quad (57)$$

where θ is the polar angle, G is the gravitational constant, and R is an arbitrary scaling factor. The Zonal and Tesseral harmonics can be written

$$J_n = -\frac{1}{MR^n} \int P_n(\cos \theta') r'^n \rho(\mathbf{r}') d^3\mathbf{r}', \quad (58)$$

$$c_{nm} = -2 \frac{(n-m)!}{(n+m)!} \frac{1}{MR^n} \int P_n^m(\cos \theta') \cos m\phi' r'^n \rho(\mathbf{r}') d^3\mathbf{r}' \quad (59)$$

$$s_{nm} = -2 \frac{(n-m)!}{(n+m)!} \frac{1}{MR^n} \int P_n^m(\cos \theta') \sin m\phi' r'^n \rho(\mathbf{r}') d^3\mathbf{r}', \quad (60)$$

where P_n is the Legendre polynomial of degree n and P_n^m is the Associated Legendre Polynomial of degree n and order m . If the body is axially symmetric w.r.t. z axis, the tesseral harmonics c_{nm} and s_{nm} disappear. If the body is symmetric w.r.t. the equatorial $z = 0$ plane $P_n(-x) = (-1)^n P_n(x)$ and J_n will disappear for odd-valued n .

Since the body of interest is a spheroid, the tesseral harmonics disappear, and we have an expansion in terms of only zonal harmonics

$$V(r, \theta) = -\frac{\mu}{r} \left(1 - \sum_{n=2}^{\infty} \left(\frac{R}{r} \right)^n J_n P_n(\cos \theta) \right). \quad (61)$$

For SGP4, the zonal harmonics are obtained from WGS72[TODO] with the values

$$\begin{aligned} J_2 &= 1082.616 \cdot 10^{-6} \\ J_3 &= -2.53881 \cdot 10^{-6} \\ J_4 &= -1.65597 \cdot 10^{-6}. \end{aligned}$$

The first four Legendre polynomials can be written

$$P_1(\cos \theta) = \cos \theta \quad (62)$$

$$P_2(\cos \theta) = \frac{1}{2} (3 \cos^2 \theta - 1) \quad (63)$$

$$P_3(\cos \theta) = \frac{1}{2} (5 \cos^3 \theta - 3 \cos \theta) \quad (64)$$

$$P_4(\cos \theta) = \frac{1}{8} (35 \cos^4 \theta - 30 \cos^2 \theta + 3) \quad (65)$$

$$P_5(\cos \theta) = \frac{1}{8} (63 \cos^5 \theta - 70 \cos^3 \theta + 15 \cos \theta) \quad (66)$$

Substituting (63) – (65) back to (61), we obtain the potential

$$\begin{aligned} V(r, \theta) = & -\frac{\mu}{r} \\ & + \frac{\mu J_2 R^2}{2r^3} (3 \cos^2 \theta - 1) \\ & + \frac{\mu J_3 R^3}{2r^4} (5 \cos^3 \theta - 3 \cos \theta) \\ & + \frac{\mu J_4 R^4}{8r^5} (35 \cos^4 \theta - 30 \cos^2 \theta + 3) \\ & + \frac{\mu J_5 R^5}{8r^6} (63 \cos^5 \theta - 70 \cos^3 \theta + 15 \cos \theta) \end{aligned} \quad (67)$$

We use $R = a_E$ to denote the equatorial radius of Earth and define

$$k_2 = \frac{J_2 a_E^2}{2}, \quad k_4 = -\frac{3J_4 a_E^4}{8}, \quad A_{3,0} = -J_3 a_E^3, \quad A_{5,0} = -J_5 a_E^5. \quad (68)$$

Thereafter, using β to denote the geocentric latitude, we obtain $\cos \theta = \sin \beta$ and

$$\begin{aligned} V(r, \theta) = & -\frac{\mu}{r} \\ & - \frac{\mu k_2}{r^3} (1 - 3 \sin^2 \beta) \\ & - \frac{\mu k_4}{r^5} \left(1 - 10 \sin^2 \beta + \frac{35}{3} \sin^4 \beta \right) \\ & - \frac{\mu A_{3,0}}{r^4} \left(-\frac{3}{2} \sin \beta + \frac{5}{2} \sin^3 \beta \right) \\ & - \frac{\mu A_{5,0}}{r^5} \left(\frac{15}{8} \sin \beta - \frac{35}{4} \sin^3 \beta + \frac{63}{8} \sin^5 \beta \right). \end{aligned} \quad (69)$$

Note that Brouwer calls the gravitational potential the force function and uses different sign [?]. That is, $U = -V$.

2.2 Disturbing Function

The first term in (69) corresponds to the unperturbed potential of a two-body problem, where the Keplerian elements are constants. The additional terms constitute a perturbing potential that leads to perturbations in the Keplerian elements. That is, instead of the Hamiltonian (??), we have

$$F = \frac{\mu}{2L^2} + R, \quad (70)$$

where $R = R_2 + R_3 + R_4 + R_5$ is the **disturbing function** and R_k denote the part of R corresponding to k :th degree in (69). In this section, we will derive R in terms of Delaunay variables.

In perifocal frame, where the non-disturbed orbit is on the $z = 0$ plane and positive x axis corresponds to the rising node, we can write the position vector as

$$r [\cos(g + f), \sin(g + f), 0]^T. \quad (71)$$

The position vector in an equatorial inertial frame with x axis corresponding to the rising node can be obtained via application of an anti-clockwise rotation w.r.t. x coordinate

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \begin{bmatrix} \cos(g + f) \\ \sin(g + f) \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(g + f) \\ \cos I \sin(g + f) \\ \sin I \sin(g + f) \end{bmatrix} \quad (72)$$

The z -coordinate can be related to the geocentric latitude β in (69) with the equation

$$\sin \beta = \sin I \sin(g + f). \quad (73)$$

Then, by using the relations

$$\begin{aligned} \sin^2(g + f) &= (1 - \cos(2g + 2f))/2, \\ \sin(g + f) \cos(2g + 2f) &= \frac{1}{2} \sin(3g + 3f) - \frac{1}{2} \sin(g + f), \\ \cos 2\theta &= 2 \cos^2 \theta - 1, \\ \sin(g + f) \cos(4g + 4f) &= \frac{1}{2} \sin(5g + 5f) - \frac{1}{2} \sin(3g + 3f), \end{aligned}$$

we obtain

$$\begin{aligned} \sin^2 \beta &= \frac{1}{2} \sin^2 I (1 - \cos(2g + 2f)), \\ \sin^3 \beta &= \sin^3 I \left(\frac{3}{4} \sin(g + f) - \frac{1}{4} \sin(3g + 3f) \right), \\ \sin^4 \beta &= \frac{1}{4} \sin^4 I (1 - 2 \cos(2g + 2f) + \cos^2(2g + 2f)) \\ &= \frac{1}{8} \sin^4 I (3 - 4 \cos(2g + 2f) + \cos(4g + 4f)) \\ \sin^5 \beta &= \frac{1}{16} [6 \sin(g + f) - 4 \sin(3g + 3f) + 4 \sin(g + f) + \sin(5g + 5f) - \sin(3g + 3f)] \\ &= \sin^5 I \left[\frac{5}{8} \sin(g + f) - \frac{5}{16} \sin(3g + 3f) + \frac{1}{16} \sin(5g + 5f) \right]. \end{aligned} \quad (74)$$

2.2.1 The J_2 Harmonic

We can expand

$$1 - 3 \sin^2 \beta = 1 - \frac{3}{2} \sin^2 I (1 - \cos(2g + 2f)) \quad (75)$$

$$= 1 - \frac{3}{2} (1 - \cos(2g + 2f)) + \frac{3}{2} \cos^2 I (1 - \cos(2g + 2f)) \quad (76)$$

$$= \left(-\frac{1}{2} + \frac{3}{2} \cos^2 I\right) + \left(\frac{3}{2} - \frac{3}{2} \cos^2 I\right) \cos(2g + 2f) \quad (77)$$

to obtain the disturbing function R_2 corresponding to the J_2 harmonic

$$\begin{aligned} R_2 &:= \frac{\mu k_2}{r^3} (1 - 3 \sin^2 \beta) \quad (78) \\ &= \frac{\mu k_2}{a^3} \left[\left(-\frac{1}{2} + \frac{3}{2} \cos^2 I\right) \frac{a^3}{r^3} + \left(\frac{3}{2} - \frac{3}{2} \cos^2 I\right) \cos(2g + 2f) \frac{a^3}{r^3} \right] \\ &= \frac{\mu k_2}{a^3} \left[\left(-\frac{1}{2} + \frac{3}{2} \frac{H^2}{G^2}\right) \frac{a^3}{r^3} + \left(\frac{3}{2} - \frac{3}{2} \frac{H^2}{G^2}\right) \cos(2g + 2f) \frac{a^3}{r^3} \right]. \end{aligned}$$

2.2.2 The J_4 Harmonic

We can expand

$$\begin{aligned} &1 - 10 \sin^2 \beta + \frac{35}{3} \sin^4 \beta \quad (79) \\ &= 1 - 5 \sin^2 I (1 - \cos(2g + 2f)) + \frac{35}{24} \sin^4 I (3 - 4 \cos(2g + 2f) + \cos(4g + 4f)) \\ &= \left[1 - 5(1 - \cos^2 I) + \frac{35}{8}(1 - \cos^2 I)^2\right] + \left[5(1 - \cos^2 I) - \frac{35}{6}(1 - \cos^2 I)^2\right] \cos(2g + 2f) \\ &+ \frac{35}{24}(1 - \cos^2 I)^2 \cos(4g + 4f) \\ &= \left(\frac{3}{8} - \frac{15}{4} \cos^2 I + \frac{35}{8} \cos^4 I\right) + \left(-\frac{5}{6} + \frac{20}{3} \cos^2 I - \frac{35}{6} \cos^4 I\right) \cos(2g + 2f) \\ &+ \left(\frac{35}{24} - \frac{35}{12} \cos^2 I + \frac{35}{24} \cos^4 I\right) \cos(4g + 4f). \end{aligned}$$

to obtain the disturbing function R_4 corresponding to the J_4 harmonic

$$\begin{aligned}
R_4 &:= \frac{\mu k_4}{r^5} \left[\left(\frac{3}{8} - \frac{15}{4} \cos^2 I + \frac{35}{8} \cos^4 I \right) \right. \\
&\quad + \left(-\frac{5}{6} + \frac{20}{3} \cos^2 I - \frac{35}{6} \cos^4 I \right) \cos(2g + 2f) \\
&\quad + \left. \left(\frac{35}{24} - \frac{35}{12} \cos^2 I + \frac{35}{24} \cos^4 I \right) \cos(4g + 4f) \right] \\
&= \frac{\mu^6 k_4}{L^{10}} \left[\left(\frac{3}{8} - \frac{15}{4} \frac{H^2}{G^2} + \frac{35}{8} \frac{H^4}{G^4} \right) \frac{a^5}{r^5} \right. \\
&\quad + \left(-\frac{5}{6} + \frac{20}{3} \frac{H^2}{G^2} - \frac{35}{6} \frac{H^4}{G^4} \right) \frac{a^5}{r^5} \cos(2g + 2f) \\
&\quad + \left. \left(\frac{35}{24} - \frac{35}{12} \frac{H^2}{G^2} + \frac{35}{24} \frac{H^4}{G^4} \right) \frac{a^5}{r^5} \cos(4g + 4f) \right].
\end{aligned} \tag{80}$$

2.2.3 The J_3 Harmonic

We can expand

$$\begin{aligned}
-\frac{3}{2} \sin \beta + \frac{5}{2} \sin^3 \beta &= -\frac{3}{2} \sin I \sin(g + f) + \frac{5}{2} \sin^3 I \left(\frac{3}{4} \sin(g + f) - \frac{1}{4} \sin(3g + 3f) \right) \\
&= \left(-\frac{3}{2} \sin I + \frac{15}{8} \sin^3 I \right) \sin(g + f) - \frac{5}{8} \sin^3 I \sin(3g + 3f).
\end{aligned} \tag{81}$$

to obtain the disturbing function R_3 corresponding to the J_3 harmonic

$$\begin{aligned}
R_3 &:= \frac{\mu A_{3,0}}{r^4} \left[\left(-\frac{3}{2} \sin I + \frac{15}{8} \sin^3 I \right) \sin(g + f) - \frac{5}{8} \sin^3 I \sin(3g + 3f) \right] \\
&= \frac{\mu A_{3,0}}{r^4} \sin I \left[\left(-\frac{3}{2} + \frac{15}{8} \frac{H^2}{G^2} \right) \sin(g + f) - \frac{5}{8} \frac{H^2}{G^2} \sin(3g + 3f) \right].
\end{aligned} \tag{82}$$

2.2.4 The J_5 Harmonic

We can expand

$$\begin{aligned}
\frac{15}{8} \sin \beta &= \sin I \left[\frac{15}{8} \sin(g + f) \right] \\
-\frac{35}{4} \sin^3 \beta &= \sin^3 I \left[-\frac{105}{16} \sin(g + f) + \frac{35}{16} \sin(3g + 3f) \right] \\
\frac{63}{8} \sin^5 \beta &= \sin^5 I \left[\frac{315}{64} \sin(g + f) - \frac{315}{128} \sin(3g + 3f) + \frac{63}{128} \sin(5g + 5f) \right]
\end{aligned} \tag{83}$$

to obtain the disturbing function R_5 corresponding to the J_5 harmonic

$$\begin{aligned}
R_5 &:= \frac{\mu A_{5,0}}{r^6} \left[\left(\frac{15}{8} \sin I - \frac{105}{16} \sin^3 I + \frac{315}{64} \sin^5 I \right) \sin(g + f) \right. \\
&\quad + \left(\frac{35}{15} \sin^3 I - \frac{315}{128} \sin^5 I \right) \sin(3g + 3f) \\
&\quad + \left. \left(\frac{63}{128} \sin^5 I \right) \sin(5g + 5f) \right]
\end{aligned} \tag{84}$$

2.3 Hamiltonian Formulation

3 Solution of the Problem

3.1 Von Zeipel Method and Canonical Transformations

3.2 Explicit Solution for J_2 Harmonic

3.2.1 Short-Period Terms

3.2.2 Long-Period Terms

3.2.3 Secular Terms

3.3 Elements and Orbit State Vectors

4 SGP4 Solution

4.1 Approximation

4.2 Coordinate Systems and Frames

4.2.1 Mean-of-Date (MoD)

4.2.2 True-of-Date (ToD)

4.2.3 True-Equator, Mean-Equinox (TEME)

4.2.4 Pseudo Earth-Fixed (PEF)

4.3 Atmospheric Drag

4.4 Two-Line Elements

4.5 Kozai Mean Elements

References