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17/10/2017

Lecture 22

Fourier Transform

Fourier Integral
→ Fourier Transform

Integral transform

Def^n: An integral transform is a transformation that produces from given f^ns, new f^ns that depend on a different variable & appear in the form of an integral.

mainly applied in

- solving o.d.eq^ns, p.d.eq^ns
- integral eq^ns
- special f^ns

The Laplace transform is of one kind

(2) Fourier transform

(I) Fourier Cosine Transforms

For an even f^n $f(x)$, the Fourier integral is the Fourier cosine integral

(a) $$f(x) = \int_0^\infty A(\omega) \cos(\omega x)\, d\omega$$

where, $$A(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \cos(\omega v)\, dv$$ (b)

$\rightarrow$ (1)

we now get

$$\boxed{A(\omega) = \sqrt{\frac{2}{\pi}}\, \hat{f}_c(\omega)}, \text{ where}$$

c suggest "<u>cosine</u>"

Then from (1)(b), writing $v = x$, we have

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\,\hat{f}_c(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx$$

$$\Rightarrow \boxed{\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x)\,dx} \longrightarrow (2)$$

& from (1)(a),

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos(\omega x)\,d\omega} \longrightarrow (3)$$

In eq^n (2), we integrate w.r.t x
& in eq^n (3), we integrate w.r.t ω.

Formula (2) gives from $f(x)$, a new f^n $\hat{f}_c(\omega)$, called the Fourier cosine transform of $f(x)$. Formula (3) gives us back $f(x)$ from $\hat{f}_c(\omega)$ & we therefore call $f(x)$ the Inverse Fourier cosine transform of $\hat{f}_c(\omega)$.

The process of obtaining the transform $\hat{f}_c$ from a given f^n f is also called the Fourier cosine transform or, Fourier cosine transform method.

Fourier Sine Transforms

Similarly, for an <u>odd function</u> $f(x)$, the Fourier integral is the Fourier sine integral,

(a) $$f(x) = \int_0^\infty B(\omega) \sin(\omega x)\, d\omega,$$

where

(b) $$B(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \sin(\omega v)\, dv \longrightarrow (4)$$

We now set,

$$B(\omega) = \sqrt{\frac{2}{\pi}}\, \hat{f}_s(\omega),$$

where s suggests, "<u>sine</u>".

Then 4(b), writing $v = x$, we have

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\,\hat{f}_s(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx$$

$$\Rightarrow \boxed{\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x)\,dx} \longrightarrow (5)$$

This is called the <u>Fourier sine transform</u> of $f(x)$.

Slly, from (4)(a), we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin(\omega x)\,d\omega$$

$$\Rightarrow \boxed{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin(\omega x)\,d\omega} \longrightarrow (6)$$

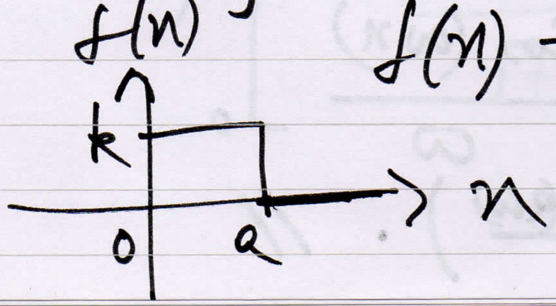
this is called the Inverse Fourier sine transform of $\hat{f}_s(\omega)$. The process of obtaining $\hat{f}_s(\omega)$ from $f(x)$ is also called the Fourier sine transform or, the Fourier sine transform method.

Other notations are :-

$$\mathcal{F}_c(f) = \hat{f}_c \,, \quad \mathcal{F}_s(f) = \hat{f}_s$$

& $\mathcal{F}_c^{-1}$ & $\mathcal{F}_s^{-1}$ are the inverses of $\mathcal{F}_c$ & $\mathcal{F}_s$ resly.

Ex1/ Find the Fourier cosine & sine transforms of the f^n



$$f(x) = \begin{cases} k, & \text{if } 0 < x < a \\ 0, & \text{if } x > a. \end{cases}$$

sol^n:- From def^n (2) & (5), we obtain by integration

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}}\, k \int_0^a \cos(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, k \cdot \left[\frac{\sin \omega x}{\omega}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\, k \left(\frac{\sin a\omega}{\omega}\right)$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, k \int_0^a \sin(\omega x)\, dx$$

$$= -\sqrt{\frac{2}{\pi}}\, k \left[\frac{\cos(\omega x)}{\omega}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\, k \left(\frac{1 - \cos(a\omega)}{\omega}\right). \;//$$

$f(x) = k, \quad 0 < x < \infty,$

$\hat{f}_c(\omega) = ?$

$\hat{f}_s(\omega) = ?$

<u>No</u> $\rightsquigarrow \int_0^\infty \cos x \, dx \rightarrow$ divergent.

or, $\int_0^\infty \sin x \, dx$ is divergent

<u>Ex2</u>/ Find $\mathcal{F}_c(e^{-x})$.

Sol^n:— By integration by parts, we obtain

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cdot \cos \omega x \, dx.$$

(how?) $$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+\omega^2} \left(-\cos \omega x + \omega \sin \omega x \right) \right]_{x=0}^{\infty}$$

$$= \frac{\sqrt{2/\pi}}{1+\omega^2} \, .$$

//

Note ! –

$\frac{2}{\pi}$

Q) What have we done in order to introduce the two integral transforms under consideration?)

we have changed the notations A & B to get a

"symmetric" ~~re~~distribution

of the constant $2/\pi$ in

their original formulas.

This redistribution is a standard convenience,

but it is not essential

L

One could do without it.)

what have we gained?

Q) what have we gained?

— We have got two transforms which have operational

properties that allow us

to convert differentiation

into algebraic operations

(just as Laplace transform)

— this is why it is

being applied to solving d.e.'s

Linearity, Transforms of Derivatives

If $f(x)$ is absolutely integrable on the positive x-axis & piece-wise continuous on every finite interval then the Fourier cosine & sine transforms of f exist.

Furthermore, for a function

$af(x) + bg(x)$

we have from eqn (2),

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x)\, dx$$

$$\therefore \hat{f}_c(af + bg) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[af(x) + bg(x) \right] \cos(\omega x)\, dx$$

$\therefore \hat{\mathcal{F}}_c(af+bg)$

$$= a\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(\omega x)\,dx$$

$$+ b\sqrt{\frac{2}{\pi}}\int_0^{\infty} g(x)\cos(\omega x)\,dx$$

$$= a\,\hat{\mathcal{F}}_c(f) + b\,\hat{\mathcal{F}}_c(g).$$

Ex

Slly, $\hat{\mathcal{F}}_s$ is also linear

ie, $\hat{\mathcal{F}}_s(af+bg)$

$$= a\,\hat{\mathcal{F}}_s(f) + b\,\hat{\mathcal{F}}_s(g).$$

$\longrightarrow$ (7)

$\not{x}$

1-1 / (Cosine & sine transforms of derivatives)

Let $f(x)$ be continuous & ~~is~~ absolutely integrable on the X-axis, let $f'(x)$ be piece-wise continuous on each finite interval, & let $f(x) \to 0$ as $x \to \infty$.

Then (a) $\mathcal{F}_c\{f'(x)\}$

$$= \omega \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0).$$

(b) $\mathcal{F}_s\{f'(x)\} = -\omega \mathcal{F}_c\{f(x)\}$

$\longrightarrow$ (8)

Proof:- (a) We apply integrati[on] by parts,

$$\mathcal{F}_c\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos(\omega x)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\{f(x)\cos(\omega x)\}_0^\infty + \omega\int_0^\infty f(x)\sin(\omega x)\,dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}f(0) + \omega\underbrace{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x)\,dx}$$

$$= -\sqrt{\frac{2}{\pi}}f(0) + \omega\mathcal{F}_s\{f(x)\}$$

$$\Rightarrow \boxed{\mathcal{F}_c\{f'(x)\} = \omega\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0).}$$

(2) $\mathcal{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin(\omega x)\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{f(x)\sin\omega x\right]_0^\infty_{(=0)} - \omega\int_0^\infty f(x)\cos(\omega x)\,dx\right]$$

$$= 0 - \omega\cdot\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x)\,dx$$

$$\therefore \boxed{\mathcal{F}_s\{f'(x)\} = -\omega\,\mathcal{F}_c\{f(x)\}}.$$

Formula (8a) with f' instead of f gives

$$\mathcal{F}_c\{f''(x)\} = \omega\,\mathcal{F}_s\{f'(x)\} - \sqrt{\frac{2}{\pi}}\,f'(0)$$

(Using 8(b))

$$\mathcal{F}_c\{f''(x)\} = \omega \mathcal{F}_s\{f'(x)\}$$

$$- \sqrt{\tfrac{2}{\pi}} f'(0)$$

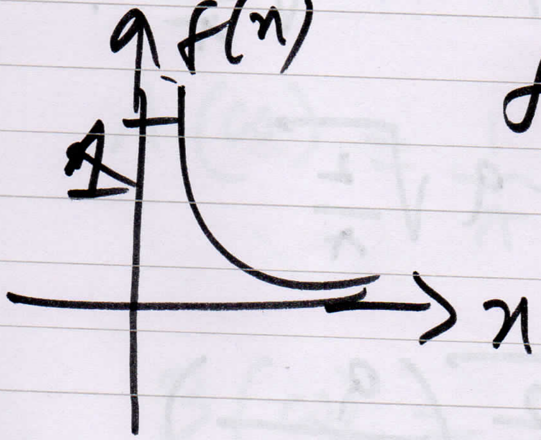
$$= -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\tfrac{2}{\pi}} f'(0).$$

$$\therefore \boxed{\mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\, f(x) - \sqrt{\tfrac{2}{\pi}} f'(0)} \rightarrow (9)$$

Slly, $\mathcal{F}_s\{f''(x)\} = -\omega \mathcal{F}_c\{f'(x)\}$

$$\Rightarrow \boxed{\mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \sqrt{\tfrac{2}{\pi}}\, \omega f(0)}$$

Ex1/ Find the Fourier cosine transform of



$$f(x) = e^{-ax}, \quad a > 0$$

Let $f(x) = e^{-ax}$

$$f'(x) = -a e^{-ax},$$

$f'(0) = -a$.

$$f''(x) = a^2 e^{-ax}.$$

$$\therefore f''(x) = a^2 e^{-ax} = a^2 f(x).$$

$$\Rightarrow a^2 f(x) = f''(x)$$

[Using q(a)]

$$\mathcal{F}_c\{a^2 f(x)\} = \mathcal{F}_c\{f''(x)\}$$

$$\Rightarrow a^2 \mathcal{F}_c\{f(x)\} = \mathcal{F}_c\{f''(x)\}$$

$$= -\omega^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}}\, f'(0).$$

$$\Rightarrow a^2 \tilde{F}_c(f) = -\omega^2 \tilde{F}_c(f) + a\cdot\sqrt{\frac{2}{\pi}}.$$

$$\Rightarrow (a^2+\omega^2)\tilde{F}_c(f) = a\sqrt{\frac{2}{\pi}}$$

$$\Rightarrow \tilde{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\omega^2}\right), \quad a>0$$

(3rd transform)

Fourier Transform

↓ obtained from

Complex form of Fourier Integral

The (real) Fourier integral is

$$f(x) = \int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] d\omega$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)\,dv.$$

Substituting A & B into the integral for f, we get

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\left[\cos(\omega v)\cos(\omega x) + \sin(\omega v)\sin(\omega x)\right] dv\,d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \underbrace{\left[\int_{-\infty}^{\infty} f(v)\cos(\omega x - \omega v)\,dv\right]}_{= F(\omega) \to \text{even in } \omega} d\omega \longrightarrow (1^*)$$

the integral of

$\therefore$ $F(\omega)$ from $\omega = 0$ to ∞

is $\frac{1}{2}$ times the integral

of $F(\omega)$ from $\omega = -\infty$ to ∞.

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\theta)\cos(\omega x - \omega\theta)\,d\theta\right] d\omega$$

$$\cancel{\sin(\omega x - \omega\theta)} \longrightarrow (1)$$

$$\cancel{G}(\omega)$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\underbrace{\int_{-\infty}^{\infty} f(\theta)\sin(\omega x - \omega\theta)\,d\theta}_{\substack{= G(\omega) \\ \Rightarrow \text{ odd f}^{n} \text{ of } \omega}}\right] d\omega$$ (why?)

$= 0$

We now take the integrand of (1) + i $(= \sqrt{-1})$ times the integrand of (2) & use the Euler formula

$$e^{ix} = \cos x + i \sin x \quad \longrightarrow (3)$$

Taking $(\omega x - \omega v)$ instead of x in (3) gives

$$f(v) \cos(\omega x - \omega v) + i f(v) \sin(\omega x - \omega v)$$
$$= f(v) e^{i(\omega x - \omega v)}$$

Hence we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{i\omega(x-v)}\, dv\, d\omega,$$

$$(i = \sqrt{-1})$$

$\longrightarrow (4)$

—x—