SOLUTION USING ITERATIVE METHODS: $(Ax=b)$ $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^{n\times 1}$, $b \in \mathbb{R}^{n\times 1}$

Def: Diagonally dominant matrix

A matrix $A \in \mathbb{R}^{n\times n}$ is called diagonally dominant by rows

if $$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad \text{with } i=1,2,\dots n.$$

While it is called diagonally dominant by columns if

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ji}| \quad \text{with } i=1,\dots n.$$

If the inequalities above hold in a strict sense, A is called strictly diagonally dominant (by rows or by columns, respectively).

IDEA FOR ITERATIVE METHODS:

Consider a system of linear equations

$$Ax = b. \quad -(1)$$

A method for solving the linear system (1) is called iterative if it is a numerical method computing a sequence of approximate solutions $x^{(k)}$ that converges to the exact solution $\underline{x}$ as the number of iterations k goes to $+\infty$.

Idea of iterative schemes is based on the splitting

$$A = P - N$$

where P is a non-singular matrix.

Consider $Ax = b \Rightarrow Px = Nx + b$

Take the iteration $Px^{(k+1)} = Nx^{(k)} + b$ with a suitable initial guess $x^{(0)}$

$$\Rightarrow x^{(k+1)} = Gx^{(k)} + Hb$$

where $G = P^{-1}N$ is called iteration matrix and $H = P^{-1}$

DEF: An iterative method is said to converge if for any choice of initial vector $x^{(0)} \in \mathbb{R}^n$ the sequence of approximate solutions $x^{(k)}$ converges to the exact solution x.

DEF.: We call the vector $r_K = b - Ax^{(K)}$ (respectively $e_K = x^{(K)} - x$) residual (respectively error) at the Kth iteration.

Remark: In general, we have no knowledge of e_K because x is unknown. However it is easy to compute the residual r_K, so convergence is dedicated on the residual in practice.

Jacobi Iteration Method:

Consider splitting the matrix A into its

- lower triangular part L.
- Diagonal part D
- Upper triangular part U

$$A = L + D + U \qquad \begin{bmatrix} d_{11} & & U \\ & d_{22} & \\ L & & \ddots \; d_{nn} \end{bmatrix}$$

Assume that D^{-1} exist then $Ax=b$ equals to:

$$Dx = -(L+U)x + b$$

$$\Leftrightarrow \quad x = -D^{-1}(L+U)x + D^{-1}b$$

The iterative method, known as Jacobi iteration method, becomes:

$$\boxed{x^{(k+1)} = -D^{-1}(L+U)\,x^{(k)} + D^{-1}b}$$

In component form:

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij}\, x_j^{(k)}\right)}$$

Gauss-Seidel method: To improve Jacobi method we can use the newly computed components $x_j^{(k+1)}$ $(j<i)$ to calculate $x_i^{(k+1)}$. The algorithm in matrix form takes the following form:

$$x^{(k+1)} = D^{-1}\left(b - Lx^{(k+1)} - Ux^{(k)}\right)$$

$$\Rightarrow \quad (D+L)\,x^{(k+1)} = \left(b - Ux^{(k)}\right)$$

$$\Rightarrow \quad \boxed{x^{(k+1)} = -(L+D)^{-1}Ux^{(k)} + (L+D)^{-1}b.}$$

Gauss-Seidel method in component form:

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right)}$$

Theorem: If A is strictly diagonally dominant by rows then the Jacobi and Gauss-Seidel methods converge for any initial guess. (sufficient condition)

Theorem: (Necessary & sufficient condition)

The Gauss-Seidel and Jacobi iterations converge for every initial guess if and only if all the eigenvalues of the iteration matrix G have absolute value less than 1.

Assignment: Given the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

Decide whether (a) Jacobi and (b) Gauss-Seidel methods converge to the solution of $Ax = b$.

Example: Consider the following system of equations

$$5x + y + 2z = 13$$
$$x + 3y + z = 12$$
$$-x + 2y + 4z = 8$$

Does the Gauss-Seidel method for solving the above system converges for any initial guess? Justify your answer. Compute the solution $x^{(1)}$ after one Gauss-Seidel and Jacobi method. Take $\underline{x}^{(0)} = [1, 1, 1]^T$.

Solution: Both methods will converge for any initial guess because the coeff. matrix is strictly diagonally dominant.

Gauss-Seidel method:

$$x^{(1)} = \frac{1}{5}\left[13 - y^{(0)} - 2z^{(0)}\right]$$

$$= \frac{1}{5} * 10 = 2.$$

$$y^{(1)} = \frac{1}{3}\left[12 - x^{(1)} - z^{(0)}\right] = \frac{1}{3}\left[12 - 2 - 1\right] = 3$$

$$z^{(1)} = \frac{1}{4}\left[8 + x^{(1)} - 2y^{(1)}\right] = 1.$$

Jacobi-iteration method:

$$x^{(1)} = \frac{1}{5}\left[13 - y^{(0)} - 2z^{(0)}\right] = 2$$

$$y^{(1)} = \frac{1}{3}\left[12 - x^{(0)} - z^{(0)}\right] = \frac{1}{3}\left[12 - 1 - 1\right] = \frac{10}{3}.$$

$$z^{(1)} = \frac{1}{4}\left[8 + x^{(0)} - 2y^{(0)}\right] = \frac{1}{4}(8 + 1 - 2) = \frac{7}{4}.$$

(e): Solve the system of equations:

$$4x_1 + 2x_2 + x_3 = 4$$
$$x_1 + 3x_2 + x_3 = 4$$
$$3x_1 + 2x_2 + 6x_3 = 7$$

Using Gauss-Jacobi and Gauss-Seidel method. Perform three iterations using the initial guess $x^{(0)} = [0.1, 0.8, 0.5]^T$.

Sol: Gauss-Jacobi:

$$x_1^{(k+1)} = \frac{1}{4}\left[4 - 2x_2^{(k)} - x_3^{(k)}\right]$$
$$x_2^{(k+1)} = \frac{1}{3}\left[4 - x_1^{(k)} - x_3^{(k)}\right] \qquad x^{(0)} = [0.1, 0.8, 0.5]^T.$$
$$x_3^{(k+1)} = \frac{1}{6}\left[7 - 3x_1^{(k)} - 2x_2^{(k)}\right]$$

$$x_1^{(1)} = \frac{1}{4}[4 - 2*0.8 - 0.5] = 0.475$$
$$x_2^{(1)} = \frac{1}{3}[4 - 0.1 - 0.5] = 1.1333$$
$$x_3^{(1)} = \frac{1}{6}[7 - 3*0.1 - 2*0.8] = 0.85$$

$x_1^{(2)} = 0.2209$ $\quad x_2^{(2)} = 0.8917$ $\quad x_3^{(2)} = 0.5514$

$x_1^{(3)} = 0.4163$ $\quad x_2^{(3)} = 1.0759$ $\quad x_3^{(3)} = 0.7590.$

Gauss-Seidel:

$$x_1^{(k+1)} = \frac{1}{4}\left[4 - 2x_2^{(k)} - x_3^{(k)}\right]$$
$$x_2^{(k+1)} = \frac{1}{3}\left[4 - x_1^{(k+1)} - x_3^{(k)}\right]$$
$$x_3^{(k+1)} = \frac{1}{6}\left[7 - 3x_1^{(k+1)} - 2x_2^{(k+1)}\right]$$

$x_1^{(1)} = 0.475$ $\quad x_2^{(1)} = 1.0083$ $\quad x_3^{(1)} = 0.5931$

$x_1^{(2)} = 0.3476$ $\quad x_2^{(2)} = 1.0198$ $\quad x_3^{(2)} = 0.6529$

$x_1^{(3)} = 0.3269$ $\quad x_2^{(3)} = 1.0069$ $\quad x_3^{(3)} = 0.6677.$

Assignment: Solve above two example with the matrix form of Jacobi and Gauss-Seidel method.

Using matrix notation:

Iteration matrix G for Jacobi method:

$$G = -D^{-1}(L+U)$$

$$= -\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1/5 & \frac{2}{5} \\ 1/3 & 0 & 1/3 \\ -1/4 & 1/2 & 0 \end{bmatrix}$$

Iteration method: $x^{(k+1)} = Gx^{(k)} + D^{-1}b$

$$\underline{x}^{(k+1)} = -\begin{bmatrix} 0 & 1/5 & 2/5 \\ 1/3 & 0 & 1/3 \\ -1/4 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(k)} + \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}\begin{bmatrix} 13 \\ 12 \\ 8 \end{bmatrix}$$

$$\underline{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\underline{x}^{(1)} = -\begin{bmatrix} 0 & 1/5 & 2/5 \\ 1/3 & 0 & 1/3 \\ -1/4 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 13/5 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 \\ -2/3 \\ -1/4 \end{bmatrix} + \begin{bmatrix} 13/5 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 10/3 \\ 7/4 \end{bmatrix}.$$

Similarly for the Gauss-Seidel iteration method solution can be obtained in vector form.

A comparative Study:

$$5x + y + 2z = 13; \qquad x + 3y + z = 12; \qquad -x + 2y + 4z = 8$$

System of Linear Equation Ax = b

Iterative Method $x^{(k+1)} = Gx^{(k)} + \text{Hb}$

Exact Solution: [1.6364 3.1818 0.8182]

	Initial Guess = [1 1 1]		Initial Guess = [0 0 0]	
I	**Jacobi**	**Gauss Seidel**	**Jacobi**	**Gauss Seidel**
1	2.0000 3.3333 1.7500	2.0000 3.0000 1.0000	2.6000 4.0000 2.0000	2.6000 3.1333 1.0833
2	1.2333 2.7500 0.8333	1.6000 3.1333 0.8333	1.0000 2.4667 0.6500	1.5400 3.1256 0.8222
3	1.7167 3.3111 0.9333	1.6400 3.1756 0.8222	1.8467 3.4500 1.0167	1.6460 3.1773 0.8229
4	1.5644 3.1167 0.7736	1.6360 3.1806 0.8187	1.5033 3.0456 0.7367	1.6354 3.1806 0.8186
5	1.6672 3.2206 0.8328	1.6364 3.1816 0.8183	1.6962 3.2533 0.8531	1.6365 3.1817 0.8183
6	1.6228 3.1667 0.8065	**1.6364 3.1818 0.8182**	1.6081 3.1502 0.7974	**1.6364 3.1818 0.8182**
7	1.6441 3.1903 0.8224	1.6364 3.1818 0.8182	1.6510 3.1982 0.8269	1.6364 3.1818 0.8182
8	1.6330 3.1779 0.8159		1.6296 3.1740 0.8137	
9	1.6381 3.1837 0.8193		1.6397 3.1856 0.8204	
10	1.6355 3.1809 0.8177		1.6347 3.1800 0.8171	
11	1.6368 3.1823 0.8184		1.6372 3.1827 0.8187	
12	1.6362 3.1816 0.8181		1.6360 3.1814 0.8179	
13	1.6365 3.1819 0.8182		1.6366 3.1820 0.8183	
14	1.6363 3.1818 0.8182		1.6363 3.1817 0.8181	
15	**1.6364 3.1818 0.8182**		1.6364 3.1819 0.8182	
16	1.6364 3.1818 0.8182		1.6363 3.1818 0.8182	
17			**1.6364 3.1818 0.8182**	
18			1.6364 3.1818 0.8182	

Remark:

1. Gauss Seidel as by construction seems to be faster than Jacobi method. However this is not true in general. There are examples where Jacobi converges faster than Gauss Seidel.

2. If we change the order of the equations the iterative methods may not be convergent. See the diverging behavior of the iterative solution below:

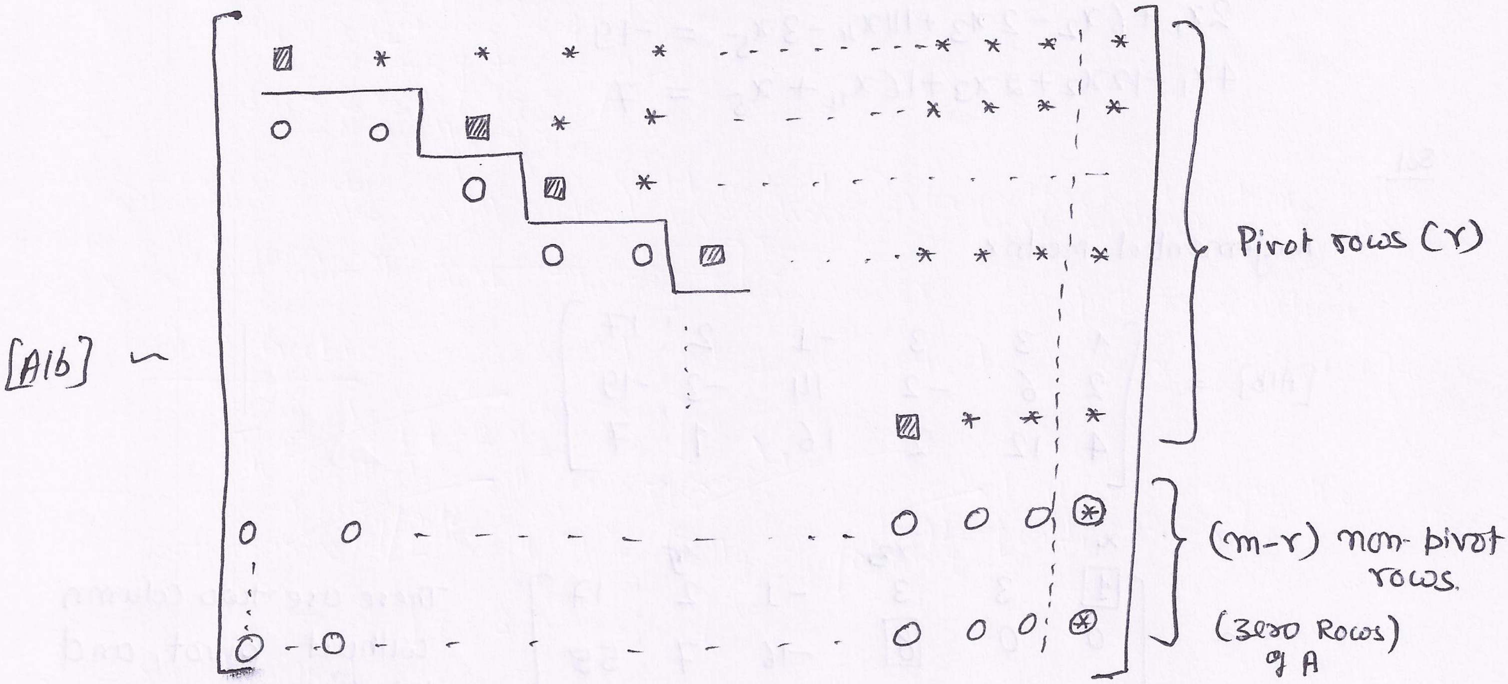
$$x + 3y + z = 12; \qquad -x + 2y + 4z = 8; \qquad 5x + y + 2z = 13$$

Iterative Method $x^{(k+1)} = Gx^{(k)} + \text{Hb}$

Exact Solution: [1.6364 3.1818 0.8182]

	Initial Guess = [1 1 1]	
I	**Jacobi**	**Gauss Seidel**
1	8.0000 2.5000 3.5000	8.0000 6.0000 -16.5000
2	1.0000 1.0000 -14.7500	10.5000 42.2500 -40.8750
3	23.7500 34.0000 3.5000	-73.8750 48.8125 166.7813
4	-93.5000 8.8750 -69.8750	-301.2188 -480.1719 999.6328
:	:	:
10	1.0e+004 * [-2.5925 -0.6066 -2.9968]	1.0e+006 * [0.5066 -1.4169 -0.5580]

- Gauss-Elimination: $Ax = b$.



▨ - pivot element $\neq 0$

$*$ ⊛ other elements (can be zero)

Case I: If ⊛ $\neq 0$

No solution

Case II: If ⊛ $= 0$ & $r = n$

Unique solution

Case III: If ⊛ $= 0$ & $r < n$

Infinitely many solutions.

free variables $(n-r)$.

- Iterative methods: (Gauss-Jacobi & Gauss-Seidel method)
  - Convergence (necessary, necessary & sufficient condition)
  - Working steps.
  -

Ex: Solve the system of equations

$$x_1 + 3x_2 + 3x_3 - x_4 + 2x_5 = 17$$
$$2x_1 + 6x_2 - 2x_3 + 14x_4 - 3x_5 = -19$$
$$4x_1 + 12x_2 + 2x_3 + 16x_4 + x_5 = 7$$

Sol:

Augmented matrix

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 3 & 3 & -1 & 2 & 17 \\ 2 & 6 & -2 & 14 & -3 & -19 \\ 4 & 12 & 2 & 16 & 1 & 7 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{cccccc} x_1 & & x_3 & & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{1} & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & \boxed{8} & -16 & 7 & 53 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \end{array}\right] \\ \begin{array}{cccccc} & x_2 & & x_4 & & \end{array} \end{array}$$

There are two column without pivot, and this indicate presence of free variables (x_2 & x_4).

Back substitution:

$$\boxed{x_5 = 3}$$

$$\boxed{x_4 = \alpha_1}$$

$$x_3 = \frac{1}{8}\left[53 - 21 + 16\alpha_1\right]$$
$$= \frac{1}{8}\left[32 + 16\alpha_1\right]$$
$$\boxed{x_3 = 4 + 2\alpha_1}$$

$$\boxed{x_2 = \alpha_2}$$

$$x_1 = 17 - 3\alpha_2 - 3(4 + 2\alpha_1) + \alpha_1 - 3\times 2$$
$$\boxed{x_1 = -1 - 5\alpha_1 - 3\alpha_2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \\ 3 \end{bmatrix} + \alpha_1 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} ; \quad \alpha_1 \alpha_2 \in \mathbb{R}.$$