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Lecture - 33 & 34

Solution of Heat eqn (one dimensional)

We consider the heat flow in long thin bar of constant cross section and is perfectly insulated laterally so that the ~~flow~~ heat flows in the x-dirn only.

$u(x,t)$ denotes the temperature at a point x at any time t. So the heat eqn is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{---(A)} \qquad 0<x<l, \quad t>0$$

together with the condition

$$u(0,t) = u(l,t) = 0 \quad t>0 \quad \text{---(1)}$$

$$u(x,0) = f(x) \quad \text{---(2)}$$

Asb u remains finite as $t \to \infty$

Let the solution of eqn (A) be of the form

$$u(x,t) = F(x)\,T(t) = FT \text{ (say)}$$

where F ~~and~~ is a fn of x only and T is the fn of t only.

Substituting in (A)

$$F \cdot \frac{dT}{dt} = c^2 T \cdot \frac{d^2F}{dx^2}$$

$$\frac{1}{F} \cdot \frac{d^2F}{dx^2} = \frac{1}{c^2 T} \cdot \frac{dT}{dt}$$

Since LHS is a function of a independent variable x while RHS is a fn of indep. variable t. the two can't be equal to each other while both reduces to a const value.

$$\frac{1}{F}\cdot\frac{d^2F}{dx^2} = \frac{1}{c^2T}\cdot\frac{dT}{dt} = 0,\ k^2,\ -k^2$$

and the 3 cases are

(i) $\frac{d^2F}{dx^2} = 0$ & $\frac{dT}{dt} = 0$

(ii) $\frac{d^2F}{dx^2} - k^2F = 0$ & $\frac{dT}{dt} - c^2k^2T = 0$

(iii) $\frac{d^2F}{dx^2} + k^2F = 0$ & $\frac{dT}{dt} + c^2k^2T = 0$

The general solⁿ in these cases are

(a) $F = Ax + B$ $T = C$

(b) $F = Ae^{kx} + Be^{-kx}$ $T = C\cdot e^{k^2c^2t}$

(c) $F = A\cos kx + B\sin kx$ $T = C\cdot e^{-k^2c^2t}$

first case → from B.C. (1)

$u(0,t) = F(0)\cdot T(t) = 0$

$u(l,t) = F(l)\,T(t) = 0$

$F(0) = 0$ & $F(l) = 0$ because $T(t) \neq 0$

So first case doesn't give solⁿ as it gives $A = B = 0$ i.e. $F = 0$ and hence $u(x,t) = 0$

Second case →

This also doesn't give solⁿ as u is finite as $t \to \infty$.

3rd case →

$$F(0) = 0 \Rightarrow A = 0$$

$$F(l) = B \sin kl = 0$$

$B \neq 0 \quad \therefore \sin kl = 0 \Rightarrow k = \frac{n\pi}{l}$

$$k_n^2 = \left(\frac{n\pi}{l}\right)^2 \longrightarrow \text{eigen values}$$

$$F_n(x) = \sin \frac{n\pi x}{l} \longrightarrow \text{corresponding eigen function}$$

For some values of n,

$$u_n(x,t) = B_n \cdot \sin \frac{n\pi}{l} x \cdot e^{-\frac{n^2\pi^2c^2}{l^2} \cdot t}$$

is solution of (A) satisfying (1). Therefore we may take the solⁿ as

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x \cdot e^{-\frac{n^2\pi^2c^2}{l^2} \cdot t} \quad -(3)$$

From this and (2)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \cdot \sin \frac{n\pi}{l} \cdot x = f(x)$$

B_n can be chosen such that $u(x,0)$ becomes the Fourier sine series of $f(x)$ i.e.

$$B_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi}{l} x \, dx \quad (4)$$

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Thus the series (3) with the coefficient (4) is the sol^n of one dimen[sional] heat eq^n (A) with the conditions (1) & (2)

Ex Solve the IBVP $\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}$ $0 \le x < L$

B.C. $\theta(0,t) = 0$ $t \ge 0$

$\frac{\partial \theta}{\partial x}(L,t) = 0$ $t > 0$

I.C. $\theta(x,0) = \theta_0$ $0 \le x \le L$

Soln $\theta(x,t) = e^{-\alpha k^2 t}(A\cos kx + B\sin kx)$

First B.C. $\Rightarrow A = 0$

$\theta(x,t) = B \sin kx . e^{-\alpha k^2 t}$

$\frac{\partial \theta}{\partial x} = B.k. \cos kx . e^{-\alpha k^2 t}$

at $x = L$ $\Rightarrow 0$

$B.k. \cos kL . e^{-\alpha k^2 t} = 0$

$\cos kL = 0$

$kL = (n + 1/2)\pi$

$k_n = \frac{(2n+1)\pi}{2L}$ $n = 0, 1, 2, \ldots$

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Sol^n is of the form

$$\theta = B \cdot \exp\left[-\alpha\left(\frac{(2n+1)}{2L}\right)^2 \cdot \pi^2 t\right] \cdot \sin\left(\frac{2n+1}{2L}\right)\pi x$$

Using principle of superposition →

$$\theta(x,t) = \sum_{n=0}^{\infty} B_n \cdot \exp\left[-\alpha\left\{\frac{(2n+1)}{2L}\right\}^2 \cdot \pi^2 t\right] \cdot \sin\left(\frac{2n+1}{2L}\right)\pi x$$

Using I.C.

$$\theta(x,0) = \theta_0 = \sum_{n=0}^{\infty} B_n \cdot \sin\left(\frac{2n+1}{2L}\right)\pi x$$

with $B_n = \frac{2}{L}\int_0^L \theta_0 \cdot \sin\left(\frac{2n+1}{2L}\right)\pi x \, dx$

$$= \frac{2}{\cancel{L}} \cdot \frac{2\cancel{L}}{(2n+1)\pi} \cdot \left[-\cos\frac{(2n+1)\pi L}{2L} - \theta_0 1\right]$$

$$= \frac{4\theta_0}{(2n+1)\pi}\left(1 - \underbrace{\cos\frac{(2n+1)}{2\cancel{L}} \cdot \pi\cancel{L}}_{=0}\right)$$

$$B_n = \frac{4\theta_0}{(2n+1)\pi}$$

$$\theta(x,t) = \sum_{n=0}^{\infty} \frac{4\theta_0}{(2n+1)\pi} \cdot \exp\left[-\alpha\left(\frac{2n+1}{2L}\right)^2 \pi^2 t\right] \cdot \sin\frac{(2n+1)}{2L}\pi x$$

Ex.

Solve the one dimensional heat eq^n in $0 \le x \le \pi$ & $t > 0$

subject to (i) u remains finite as $t \to \infty$

(ii) $u(0,t) = 0$ & $u(\pi, t) = 0$

(iii) At $t = 0$ $u = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le \pi \end{cases}$

Case -(iii)

$$F = A \cos kx + B \sin kx \quad \& \quad T = c \cdot e^{-k^2 c^2 t}$$

using B.C.

$$A \cdot 1 + B \cdot 0 = 0 \Rightarrow A = 0$$

$$\cancel{A \cdot \cos \pi k} \quad B \cdot \sin k\pi = 0$$

$$\sin k\pi = 0$$

$$k = n \qquad n = 1, 2, 3 \ldots$$

eigen values.

$$u_n(x,t) = B_n \cdot \sin nx \cdot e^{-c^2 n^2 t}$$

$$B_n = \frac{2}{\pi} \int_0^{\pi/2} x \sin nx \, dx$$

$$\frac{2}{\pi} \left[\frac{x}{n} \cdot \cos nx \right]_{\pi/2}^{0} + \left(\frac{\sin nx}{n^2} \right)_0^{\pi/2}$$

$$\frac{2}{\pi} \left(0 - \frac{\pi}{2n} \cos \frac{n\pi}{2} \right) + \frac{\sin n\pi/2}{n^2}$$

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$$B_n = \frac{4\sin n\pi/2}{\pi n^2} - \frac{2\cos n\pi/2}{\cancel{n}}$$

$$B_n = \frac{4}{\pi}\int_{\pi/2}^{\pi} \pi \sin nx - x\sin nx$$

$$= 4\cdot\left(\frac{\cos nx}{n}\right)_{\pi}^{\pi/2} - \frac{4}{\pi}\left[\left(\frac{x}{n}\cos nx\right)_{\pi}^{\pi/2} + \left(\frac{\sin nx}{n^2}\right)_{\pi/2}^{\pi}\right]$$

$$= 4\,\frac{\cos n\pi/2}{n} - \cancel{\frac{4\cos n\pi}{n}} - \frac{4}{\pi}\left(\frac{\pi}{2n}\cdot\frac{\cos n\pi}{2} - \cancel{\frac{\pi}{n}\cos n\pi}\right)$$

$$- \cancel{\frac{\sin n\pi}{n^2}}^{\to 0} + \frac{4\sin n\pi/2}{\pi n^2}$$

$$= \frac{4\cos n\pi/2}{n} - \frac{2}{n}\cos\frac{n\pi}{2} - \frac{2\cos n\pi/2}{n}$$

$$B_n = \frac{4\sin\frac{n\pi}{2}}{\pi n^2}$$

Qu show that the solution of the eqn

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

satisfy (i) $u \to 0$ as $t \to \infty$
(ii) $u = 0$ for $x = 0$ & $x = a$ $\forall\, t > 0$
(iii) $u = x$ when $t = 0$ and $0 < x < a$ is

$$u(x,t) = \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{a}\cdot x \exp\left[-\left(\frac{n\pi}{a}\right)^2 t\right]$$

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Sol^n = $u(x,t) = (A\cos kx + B\sin kx) \cdot e^{-k^2 t}$

$u(0,t)_{=0} \Rightarrow A = 0$

$u(a,t) = 0 \Rightarrow \sin ka = 0$

$k = \frac{n\pi}{a}$

$u_n = B_n \cdot \sin \frac{n\pi}{a} x \cdot e^{-\frac{n^2\pi^2}{a^2} \cdot t}$

$$B_n = \frac{2}{a} \cdot \int_0^a x \sin \frac{n\pi}{a} \cdot x \, dx$$

$$= \frac{2}{a} \cdot \left[\frac{a}{n\pi} \cdot \left(x \cos \frac{n\pi x}{a} \right)_a^0 + \cancel{\frac{a^2}{n^2\pi^2} \left(\sin \frac{n\pi}{a} x \right)_0^a} \right]$$

$$= -\frac{2}{\cancel{a}} \cdot \left(\frac{\cancel{a}}{n\pi} \cdot a \cos n\pi \right)$$

$$= -\frac{2a(-1)^n}{n\pi}$$

$$B_n = \frac{2a}{n\pi} (-1)^{n+1}$$

$$u(x,t) = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \sin \frac{n\pi}{a} x \cdot e^{-\frac{n^2\pi^2}{a^2} t}$$

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Lecture - 35

Solution of laplace equation →

Let's consider the steady state T distribution in a rectangular sheet of metal. Hence the T is everywhere independent of time and therefore the T distribution within the plate is obtained by the differential eqn

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This is called laplace eqn.

Dirichlet problem for a rectangle →

PDE : $\nabla^2 u = 0$ — (A)

$0 \le x \le a$

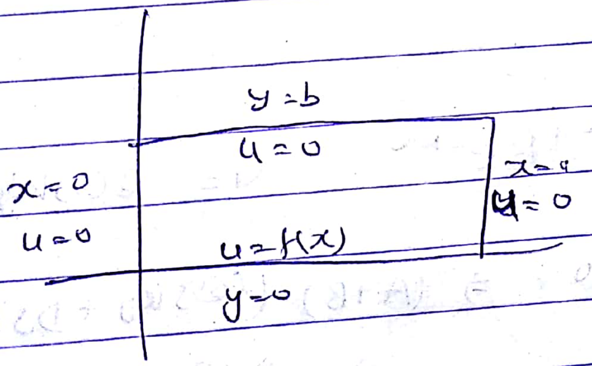
$0 \le y \le b$.

BCs → $u(0,y) = 0$

$u(a,y) = 0$

$u(x,b) = 0$

$u(x,0) = f(x)$



Soln of (A) is taken in the form

$u(x,y) = F(x) \cdot G(y) = FG$ (say) — (1)

where F is a fn of x alone and

G " " " y alone.

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$$G \cdot \frac{d^2F}{dx^2} + F \frac{d^2G}{dy^2} = 0$$

$$\frac{1}{F} \cdot \frac{d^2F}{dx^2} = -\frac{1}{G} \frac{d^2G}{dy^2}$$

$$\frac{1}{F} \cdot \frac{d^2F}{dx^2} = -\frac{1}{G} \cdot \frac{d^2G}{dx^2} = 0,\ k^2,\ -k^2$$

(i) $$\frac{d^2F}{dx^2} = \frac{d^2G}{dy^2} = 0$$

(ii) $$\frac{d^2F}{dx^2} - k^2F = 0 \quad \& \quad \frac{d^2G}{dy^2} + k^2G = 0$$

(iii) $$\frac{d^2F}{dx^2} + k^2F = 0 \quad \& \quad \frac{d^2G}{dy^2} - k^2G = 0$$

(i) $F = ax + b \qquad G = Cy + D$

$u(0,y) = u(a,y) = 0$

$\Rightarrow F = 0 \quad \Rightarrow u(x,y) = 0$

(ii) $F = Ae^{kx} + Be^{-kx} \qquad G = C\cos ky + D\sin ky$

$u(0,y) = 0 \quad \Rightarrow (A+B)(C\cos ky + D\sin ky) = 0$

$\Rightarrow A + B = 0$

$u(a,y) = 0 \quad \Rightarrow \underbrace{(A \cdot e^{ka} + B \cdot e^{-ka})}_{=0}(\qquad) = 0$

$\Rightarrow A = B = 0$

$\Rightarrow F = 0 \Rightarrow u(x,y) = 0$

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The only possible solution →

(iii) $F = A\cos kx + B\sin ky$ & $G = C\cdot e^{ky} + D\cdot e^{-ky}$

$u(x,y) = (A\cos kx + B\sin kx)(Ce^{ky} + D\cdot e^{-ky})$

using BC →

$u(0,y) = 0$

$A(\quad) = 0$

$\Rightarrow A = 0$

$u(a,y) = 0 \Rightarrow B\sin ka(\quad) = 0$

$B\sin ka = 0 \qquad (\because B \neq 0)$

$\sin ka = 0$

$ka = n\pi$

$k = \frac{n\pi}{a} \qquad n = 1, 2, 3 \ldots$

Eigen values are $k_n^2 = \left(\frac{n\pi}{a}\right)^2$

eigen functions $F_n(x) = \sin\frac{n\pi}{a}x$

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{a}x \left[C_n\cdot e^{\frac{n\pi}{a}y} + D_n\cdot e^{-\frac{n\pi}{a}y}\right]$$

using B.C. $u(x,b) = 0$

$$\sin\frac{n\pi}{a}x\left(C_n\cdot e^{\frac{n\pi b}{a}} + D_n\cdot e^{-\frac{n\pi b}{a}}\right) = 0$$

$$\Rightarrow C_n\cdot e^{\frac{n\pi b}{a}} + D_n\cdot e^{-\frac{n\pi b}{a}} = 0$$

$$D_n = -C_n\cdot e^{\frac{2n\pi b}{a}}$$

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{a}x \left[C_n e^{\frac{n\pi y}{a}} - C_n e^{-\frac{n\pi y}{a}} \cdot \frac{e^{\frac{n\pi b}{a}}}{e^{-\frac{n\pi b}{a}}} \right]$$

$$= \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi a}{x}\, 2C_n}{e^{-\frac{n\pi b}{a}}} \left[\frac{e^{\frac{n\pi(y-b)}{a}} - e^{-\frac{n\pi(y-b)}{a}}}{2} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2C_n}{e^{-\frac{n\pi b}{a}}} \cdot \sin\frac{n\pi x}{a} \sinh\frac{n\pi(y-b)}{a}$$

Let $\frac{2C_n}{e^{-\frac{n\pi b}{a}}} = A_n$

So the solution can be written as :-

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cdot \sin\frac{\pi n x}{a} \cdot \sinh\frac{n\pi(y-b)}{a}$$

Finally using the non Homogeneous BC
$u(x,0) = f(x)$

$$\sum_{n=1}^{\infty} A_n \cdot \sin\frac{n\pi x}{a} \cdot \sinh\left(\frac{-n\pi b}{a}\right) = f(x)$$

$$A_n \cdot \sin\left(\frac{-n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \cdot \sin\frac{n\pi}{a}x\,dx$$

So required solution for given Dirichlet problem is $u(x,y) = \sum_{n=1}^{\infty} A_n \cdot \sin\frac{n\pi x}{a} \cdot \sinh\frac{n\pi(y-b)}{a}$

where $A_n = \frac{1}{\sinh\left(\frac{-n\pi b}{a}\right)} \cdot \frac{2}{a} \int_0^a f(x) \cdot \sin\frac{n\pi x}{a}\,dx$

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Lecture - 36 & 37

Ex:

Solve $\nabla^2 u = 0$

B.C.s are $u(0,y) = 0$, $u(a,y) = 0$, $u(x,b) = 0$

$u(x,0) = Tx(x-a)$

Sol^n: $u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x \, \sinh\left(\frac{n\pi}{a}(y-b)\right)$

where $A_n \sinh\left(\frac{-n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right)dx$

$u(x,0) = Tx(x-a) = f(x)$

$$A_n \sinh\left(\frac{-n\pi b}{a}\right) = \frac{T \cdot 2}{a}\left[\int_0^a Tx \sin\left(\frac{n\pi x}{a}\right)dx\right.$$

$$\left. - \int_0^a Tx^2 \sin\left(\frac{n\pi x}{a}\right)dx\right]$$

$$\left[\left[-x\cos\left(\frac{n\pi x}{a}\right)\frac{a}{n\pi} + \frac{a^2}{n^2\pi^2}\left(\sin\frac{n\pi x}{a}\right)\right]_0^a\right.$$

$$- \left[\left(-x^2 \cdot \cos\left(\frac{n\pi x}{a}\right)\frac{a}{n\pi}\right)_0^a + \frac{a^2}{n^2\pi^2}\int_0^a 2x \cos\left(\frac{n\pi x}{a}\right)dx\right.$$

$$\left(-\frac{a^2 (-1)^n}{n\pi} + 0\right) - \left(\left(-\frac{a^3 (-1)^n}{n\pi} + \frac{a^2}{n^2\pi^2}\right)\right)$$

$$2\left[\left(x \sin\frac{n\pi x}{a} \cdot \frac{a}{n\pi}\right)_0^a - \frac{a^2}{n^2\pi^2}\left[-\cos\frac{n\pi x}{a}\right]_0^a\right]$$

$$= \frac{a^2(-1)^{n+1}}{n\pi} + \frac{a^3(-1)^n}{n\pi} + \frac{2a^2}{n^2\pi^2}\Big($$

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$$= \frac{4a^2T}{n^3\pi^3}\left((-1)^n - 1\right)$$

The interior Dirichlet problem for a circle:

To find the value of u at any point in the interior of circle r=a in terms of its values on the boundary r=a such that u is single valued and continous within and on the circular region satisfying the eqn →

$$\nabla^2 u = 0 \quad -(1) \quad 0 \le r \le a, \quad 0 \le \theta \le 2\pi$$

subject to $u(a,\theta) = f(\theta) \qquad 0 \le \theta \le 2\pi \quad -(2)$

The requirment of single valudness of u implies the periodicity condition i.e.

$$u(r, \theta + 2\pi) = u(r, \theta) \qquad 0 \le r \le a \quad -(3)$$

Eqn (1) in polar cordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} = 0 \quad -(4)$$

Let $u(r,\theta) = R(r)\,H(\theta)$ $-(5)$ be the soln of (4)

$$u(r,\theta) = R\cdot H$$

$$\frac{\partial^2 u}{\partial r^2} = H\cdot\frac{d^2R}{dr^2}$$

$$\frac{\partial u}{\partial r} = H\cdot\frac{dR}{dr} \quad \& \quad \frac{\partial^2 u}{\partial \theta^2} = R\cdot\frac{d^2H}{d\theta^2}$$

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$$H \frac{d^2R}{dr^2} + \frac{H}{r}\frac{dR}{dr} + \frac{R}{r^2}\frac{d^2H}{d\theta^2} = 0$$

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) = -\frac{1}{r^2H}\frac{d^2H}{d\theta^2}$$

$$\frac{r^2R'' + rR'}{R} = -\frac{H''}{H} = k \text{ (say)} \quad (6)$$

case - (i) Let $k>0$. then $k = \lambda^2$

$\rightarrow$ $r^2R'' + rR' - \lambda^2 R$

$r = e^z$

$R = C_1 e^{\lambda z} + C_2 e^{-\lambda z}$

$R = C_1 r^{\lambda} + C_2 r^{-\lambda}$

$\rightarrow$ Also $H'' + \lambda^2 H = 0$

$H = C_3 \cos(\lambda\theta) + C_4 \sin(\lambda\theta)$

Additional

$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

put $x = e^z$ $\quad z = \ln x$

$(\theta(\theta-1) + \theta - 1)\, y = 0$

$\theta = \frac{d}{dz}$

$m^2 - 1 = 0 \Rightarrow A.E.$

$y = C_1 e^{z} + C_2 e^{-z}$

$$u(r,\theta) = (C_1 r^{\lambda} + C_2 r^{-\lambda}) + C_3 \cos(\lambda\theta) + C_4 \sin(\lambda\theta) \quad (7)$$

case - (ii)

$r^2R'' + rR' + \lambda^2 R = 0$

$R = C_1 \cos(\lambda \ln r) + C_2 \sin(\lambda \ln r)$

$H'' - \lambda^2 H = 0 \Rightarrow H = C_3 e^{\lambda\theta} + C_4 e^{-\lambda\theta}$

$$u(r,\theta) = C_1 \cos(\lambda \ln r) + C_2 \sin(\lambda \ln r) + C_3 e^{\lambda\theta} + C_4 e^{-\lambda\theta} \quad (8)$$

Case - (iii) Let $k=0$

$\rightarrow \quad rR'' + R' = 0$

$r\cancel{R}' + x = 0 \qquad V = R'$

$$\frac{dV}{V} + \frac{dr}{r} = 0$$

$$\ln(Vr) = c$$

$$V = \frac{c_1}{r}$$

$$\frac{dR}{dr} = \frac{c_1}{r}$$

$$R = c_1 \ln r + c_2$$

$\Rightarrow \quad H'' = 0$

$$H = c_3\theta + c_4$$

$$u(r,\theta) = (c_1 \ln r + c_2)(c_3\theta + c_4) \quad \text{---} (9)$$

For the interior problem, $r=0$ is a point of the domain of definition of the problem and since $\ln r$ is not defined at $r=0$, so solution (8) & (9) are not acceptable. Thus the required soln is obtained from eqn (7). The periodicity condn (3) implies that ~~the periodicity~~

$$c_3\cos(\lambda\theta) + c_4\sin(\lambda\theta) = c_3\cos\lambda(\theta+2\pi) + c_4\sin\lambda(\theta+2\pi)$$

$$2\sin\lambda\pi \cdot \left(c_3\sin\lambda\theta + \lambda\pi - c_4\cos(\lambda\theta+\lambda\pi) \right)$$

implies $\sin\lambda\pi = 0$

$$\lambda\pi = n\pi$$

$$\lambda = n \qquad n = 0, 1, 2, 3, \ldots$$

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$$u(r,\theta) = \sum_{r=0}^{\infty} (C_n r^n + D_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta)$$

at $r=0$ $u(r,\theta)$ must be finite. So $D_n = 0 \ \forall n$

$$u(r,\theta) = \sum_{r=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

(with no loss of generality)

Which is a full range Fourier Series.

Now $u(a,\theta) = f(\theta)$ implies that

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta) \quad \text{--- (11)}$$

$$A_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta$$

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

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Thus we obtain the Solution →

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \frac{1}{\pi}\int_0^{2\pi} f(\phi) \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \times \{\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\}\,d\phi$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)\,d\phi\right] \quad -(12)$$

where the variable θ in (12) has been replaced by ϕ distinguish it from the current variable θ in eqn (11)

Let $C = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)$

and $S = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \sin n(\phi-\theta)$

$$C + iS = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cdot e^{i(\phi-\theta)n}$$

$$= \sum_{n=1}^{\infty}\left[\frac{r}{a}\cdot e^{i(\phi-\theta)}\right]^n$$

$$= \frac{\frac{r}{a}\cdot e^{i(\phi-\theta)}}{1 - \frac{r}{a}\cdot e^{i(\phi-\theta)}}$$

Equating the real parts both sides →

$$C = \frac{\frac{r}{a}\cos(\phi-\theta) - \frac{r^2}{a^2}}{1 - \frac{2r}{a}\cos(\phi-\theta) + \frac{r^2}{a^2}}$$

Thus the expression in the square brackets of (12) becomes

$$= \frac{a^2 - r^2}{2(a^2 - 2ar\cos(\phi - \theta) + r^2)}$$

The required sol^n takes the form

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{(a^2-r^2) f(\phi)}{(a^2 - 2ar\cos(\phi-\theta) + r^2)}\right) d\phi$$

This is known as poisson integral formula for a circle (interior dirichlet problem).

Lecture - 38 (class Test)
Lecture - 39 & 40

Apendix-1

Non - Homogeneous linear equation with constant coefficients →

$$(D - mD' - \alpha) z = 0$$

$$p - mq = \alpha z$$

Subsidiary eq^n →

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\alpha z}$$

$$mx + y = c_1$$

$$c_2 + \alpha x = \ln z$$

$$z = c_2 \cdot e^{\alpha x}$$

$$z \cdot e^{-\alpha x} = c_2$$

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$$z = e^{\alpha x} f(y+mx)$$

Simillarly it can be shown that integral of
$(D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2) \cdots (D - m_n D' - \alpha_n)z$

$$z = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x) + \cdots + e^{\alpha_n x} f_n(y+m_n x)$$

In case of Repeated factors:
$$(D - mD' - \alpha)^r z = 0$$

$$z = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx) + \cdots + x^{r-1} e^{\alpha x} f_r(y+mx)$$

Particular Integrals

Case -(i) When the R.H.S. of DE is of the form e^{ax+by} then

$$\frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a,b)} \cdot e^{ax+by}$$

provided $f(a,b) \neq 0$ i.e. put $D=a$ & $D'=b$

Case -(ii) When the R.H.S. of the differential eqn is of the form $\sin(ax+by)$ or $\cos(ax+by)$ then $\frac{1}{f(D, D')} \sin(ax+by)$ is obtained by

putting $D^2 = -a^2$, $DD' = -ab$ & $D'^2 = -b^2$
Provided denominator is not zero.

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Case -(iii) When the R.H.S is of the form $x^m y^n$ when m, n are positive integers.

then $\frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$

Ex. Solve

$$(D^2 - D'^2 + D - D')z = 0$$

$$(D + D' + 1)(D - D')z = 0$$

$$z = e^{-x} f_1(y - x) + f_2(y + x)$$

Ex. Solve $DD'(D - 2D' - 3)z = 0$

$$z = e^{0 \cdot x} f_1(y) + e^{0 \cdot x} f_2(0 - x)$$

$$+ e^{3x} f_3(y + 2x)$$

$$\therefore z = f_1(y) + f_2(-x) + e^{3x} f_3(y + 2x)$$

Ex. solve $r + 2s + t + 2p + 2q + z = 0$

$\cancel{D + D'}$

$$((D + D')^2 + 2(D + D') + 1)z = 0$$

$$(D + D' + 1)^2 z = 0$$

$$z = e^{-x} f_1(y - x) + x \cdot e^{-x} f_2(y - x)$$

Ex. solve $(r - s + p) = 1$

$$(D^2 - DD' + D)z = 1$$

$$D(D - D' + 1)z = 1$$

C.F. $\rightarrow$ $z = e^{0 \cdot x} f_1(y) + e^{-x} f_2(y + x)$

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P.I. $\frac{1}{D(D-D'+1)} \cdot 1$

$= \frac{1}{D}(1+D-D')^{-1} \cdot 1$

$= \frac{1}{D} \cdot 1$

$= x$

$z = f_1(y) + e^{-x} f_2(y+x) + x$

Ex. Solve $(D-D'-1)(D-D'-2)z = e^{2x-y} + x$

C.F. $= z = e^{x} f_1(y+x) + e^{2x} f_2(y+x)$

P.I. $= \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y} + \frac{1}{(D-D'-1)(D-D'-2)} x$

$= \frac{1}{(2+1-1)(2+1-2)} e^{2x-y} + \frac{1}{2(1-D+D')(1-\frac{D}{2}+\frac{D'}{2})} x$

$= \frac{1}{2} e^{2x-y} + \frac{1}{2} \frac{1}{(1-D+D')}(1-\frac{D}{2}) x$

$+ \frac{1}{2} \frac{1}{\cancel{(1-D+D')}} (1+D)(1+\frac{D}{2}) x$

$+ \frac{1}{2}(1 + \frac{3D}{2} + \cancel{\;}) x$

$+ \frac{x}{2} + \frac{3}{4}$

$= \frac{1}{2} e^{2x-y} + \frac{x}{2} + \cancel{\frac{3}{4} \cdot \frac{x}{2} + \frac{1}{8} \cdot \frac{x}{8}}$

$z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{2} e^{2x-y} + \frac{x}{2} + \cancel{\frac{3x}{8}} + \frac{3}{4}$

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Ex: Solve $(D^2 - DD' + D' - 1)z = \cos(x+2y)$

$$(D-1)(D+1) - D'(D-1)$$

$$(D-1)(D - D' + 1) z = 0$$

$$z = e^{x} f_1(y) + e^{-x} f_2(y+x)$$

P.I. $$\frac{1}{D^2 - DD' + D' - 1} \cos(x+2y)$$

$D'^2 = -4$

$a = 1 \quad b = 2 \quad D' = \pm 2i$

$$\cancel{Re\left[\frac{1}{-1+2+\;-1} e^{ix+2i2y}\right]}$$

$$= \frac{1}{D'} \cos(x+2y)$$

$$= \frac{D'}{D'^2} \cos(x+2y)$$

$$= \frac{D'}{-2^2} \cos(x+2y)$$

$$= \frac{-1}{2} \sin(x+2y)$$

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Apendix-2

Solve the initial value problem for the quasi-linear equation $zz_x + z_y = 1$ containing the initial data curve C:

$$x_0 = s, \quad y = s, \quad z_0 = \frac{s}{2}$$

Sol^n characteristic eq^n are

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 1$$

$$y = t + c_1 \qquad z = t + c_2$$

$$x = \frac{t^2}{2} + c_2 t + c_3$$

Given $x(s,0) = s$, $y(s,0) = s$, $z(s,0) = s/2$

$$y = t + s \quad \text{— (ii)}$$

$$z = t + s/2 \quad \text{— (iii)}$$

$$x = \frac{t^2}{2} + \frac{1}{2}st + s$$

$$x = \frac{t^2}{2} + \frac{1}{2}st + s \quad \text{— (i)}$$

$$(y = t + s)\, t/2$$

$$y t/2 = \frac{t^2}{2} + \frac{st}{2} \quad -[(i)]$$

$$y t/2 - x = -s \qquad \text{put in (ii)}$$

$$y = -y t/2 + x + t$$

$$y - x = \left(\frac{-y}{2} + 1\right) t \Rightarrow t = \frac{2(y-x)}{-y+2}, \quad s = y + \frac{2(y-x)}{y-2}$$

$$s = \frac{-(y^2-2x)}{y-2}$$

$$t = \frac{2(x-y)}{y-2} \qquad s = \frac{-(y^2-2x)}{y-2}$$

and put in (iii)

$$z = \frac{y-x}{1-y/2} + \frac{1}{2}\,\frac{x-y/2}{1-y/2}$$

$$z = \frac{4y-2x-y^2}{2(2-y)}$$

Ex. Solve the cauchy problem for

$$2z_x + yz_y = z$$

for the initial data curve

$$c \quad x_0 = s,\ y_0 = s^2,\ z_0 = s$$

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = y \qquad \frac{dz}{dt} = z$$

$$x = 2t + c_1 \qquad y = c_2 \cdot e^t \qquad z = c_3 \cdot e^t$$

$$x = 2t + s \qquad y = s^2 e^t \qquad z = s e^t$$

$$\frac{y}{z} = s$$

$$x = 2t + s$$

$$y = s^2 e^t$$

$$\ln y = \ln s^2 + \ln e^t$$

$$2\ln y = 4\ln s + 2t$$

$$\boxed{\text{Sol}^n = z^2 = y\exp\left(\frac{x+ \ln y}{2z}\right)}$$

$$(x - 2\ln y) = s - 4\ln s$$

$$s\ln e - 4\ln s$$

$$\ln \frac{e^s}{s^4} = x - 2\ln y$$

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Ex find the Soln of IVP for

$$z_x - 2z_y + z = 0$$

for initial value C: $x_0 = 0,\ y_0 = s,\ z_0 = -2s$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -2, \quad \frac{dz}{dt} = z$$

$x = t$ $\qquad s = \dfrac{y}{2e^{-x} + 8}$, $t = x$

$y = -2se^{-t} + 3s$

$z = -2s \cdot e^{-t}$

$$z = \frac{-2y}{-2e^{-x} + 3} e^{-x}$$

$$= \frac{-2y}{-2 + 8e^{x}}$$

$$\boxed{z = \frac{2y}{3e^{x} - 2}}$$

Ex. The ~~charac~~ characteristic eqn w.r.t the parameters t of a first order PDE are given by $\frac{dx}{dt} = x+2$, $\frac{dy}{dt} = 2y$, $\frac{dz}{dt} = 2z$

Find x, y, z in terms of t. Find a particular solution passing through the initial curve at t=0 given by $y = z^2$, $x = -1$

$$\frac{dx}{dt} = x+2 \qquad \frac{dy}{dt} = 2y \qquad \frac{dz}{dt} = 2t$$

$x_0 = -1$
$y = s^2$
$z = s$

$\ln(x+2) = t + \ln c_1 \qquad y = c_2 \cdot e^{2t} \qquad z = c_3 \cdot e^{2t}$

$x+2 = c_1 \cdot e^t$

$x = -2 + c_1 \cdot e^t$

$x = e^t - 2$ $\qquad e^t = x+2$

$y = s^2 e^{2t}$ $\qquad t = \ln(x+2)$

$z = s\, e^{2t}$ $\qquad y = s^2 \cdot (x+2)^2$

$$s = \frac{\sqrt{y}}{(x+2)}$$

$$z = \frac{\sqrt{y}}{\cancel{(x+2)}} \cdot \cancel{(x+2)}^2$$

$$z = \sqrt{y}\,(x+2)$$

Apendix - III

1. complete integral is given

$$z(x, y, a, b) = \frac{1}{2}\left(\frac{ax+y}{\sqrt{1+a^2}} + b\right)^2 - \frac{1}{2}$$

Sol^n $\quad z_a = 0$

$z_b = 0$

& $z = (\qquad)$

$$\frac{\partial z}{\partial a} = \left(\frac{ax+y}{\sqrt{1+a^2}} + b\right) \cdot \left(\qquad\qquad\right) = 0$$

$$\frac{\partial z}{\partial b} = \left(\frac{ax+y}{\sqrt{1+a^2}} + b\right) \cdot 1 = 0$$

$$\boxed{z = \frac{-1}{2}}$$

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2. $z = xy + a^2 - ay - ax$

$\frac{\partial z}{\partial a} = 2a - y - x = 0 \qquad x + y = 2a$

$\frac{\partial z}{\partial b} = 0 \Rightarrow 0$

$z = xy + a^2 - 2a^2$

$= xy - a^2$

$$\boxed{z = xy - \frac{1}{4}(x+y)^2}$$

3. $f(x, y, z, a, b) = (x-a)^2 + (y-b)^2 + z^2 - 1 = 0$

Find Singular Integral

$z = \pm 1$