Date
21/10/2019

Lecture 8

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Quiz

~~1^{st} November~~

~~(Friday)~~

~~Time 6 – 7 pm~~

~~Venue: Room No: N322,~~

~~Deptt. of Mathematics.~~

Syllabus :– Bessel's function

Hyper-geometric function.

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Note :– Hypergeometric function $F(a, b; c; x)$ can be put in the following different forms:

$$F(a,b;c;x) = (1-x)^{c-a-b} F(c-a, c-b; c; x)$$

$$= (1-x)^{-a} F\left(a, c-b; c; \frac{x}{x-1}\right)$$

$$= (1-x)^{-b} F\left(b, c-a; c; \frac{x}{x-1}\right)$$

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§/ Gauss's hyper-geometric equation or Gauss's eqn or, hyper-geometric equation

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta y = 0.$$

is called the hyper-geometric equation

Solution of the hyper-geometric equation

Let α, β and γ be constants, then

$$x(1-x)y'' + \{\gamma - (\alpha+\beta+1)x\}y' - \alpha\beta y = 0 \longrightarrow (1)$$

is known as the hyper-geometric eq^n

Let $\gamma \neq 0, -1, -2, \dots$ Then

a solⁿ of 1 is given by

$$ {}_2F_1(\alpha, \beta; \gamma; x) $$

$$ = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} \cdot x^2 $$

$$ + \cdots \longrightarrow 2 $$

Now, ${}_2F_1(\alpha, \beta; \gamma; x)$ is called

the hyper-geometric f^n.

If γ is not an integer

then the other soln of 1

& linearly independent of the soln ${}_2F_1(\alpha, \beta; \gamma; x)$ is

$$x^{1-\gamma}\ {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x).$$

Thus, if γ is not an integer then the general solⁿ of (1) is

$$y = a\ {}_2F_1(\alpha, \beta; \gamma; x) + b\ x^{1-\gamma}\ {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)$$

where a & b are arbitrary constants.

$\longrightarrow$ (3)

Note :- The hyper-geometric f^n $F(\alpha, \beta; \gamma; x)$ is defined only if

(i) α & β are real nos.

(ii) γ is any real no. such that

$$\gamma \neq 0, -1, -2, \dots$$

(iii) The variable x satisfies $|x| < 1$.

The general soln (3) of (1) exists if γ is not an integer.

Note :- If either α or β is a negative integer, then $F(\alpha, \beta; \gamma; x)$ reduces to a polynomial, because after finite no. of terms, the co-efficient of each term will be zero

e.g., consider the following :

$F(-2, b; c; x)$

$$= 1 + \frac{(-2)b}{1 \cdot c} x + \frac{(-2)(-1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} x^2 + 0$$

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which is a polynomial of degree 2.

$$\text{Slly, } F(a, -2; c; x) = 1 - \left(\frac{2a}{c}\right)x + \left\{\frac{a(a+1)}{c(c+1)}\right\}x^2$$

8/ <u>Symmetric Property of hyper-geometric f^n</u>

Hyper-geometric f^n does not change if the parameters α & β are interchanged, keeping γ fixed. Thus,

$$F(\alpha, \beta; \gamma; x) = F(\beta, \alpha; \gamma; x).$$

proof :- we have, by defn

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{x^n}{n!} \longrightarrow (1)$$

& $$F(\beta, \alpha; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\beta)_n (\alpha)_n}{(\gamma)_n} \cdot \frac{x^n}{n!} \longrightarrow (2)$$

$\therefore$ from (1) & (2), we have

$$F(\alpha, \beta; \gamma; x) = F(\beta, \alpha; \gamma; x).$$

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§ ~~Differentiation~~ Differentiation of hypergeometric f^n

Show that

$$\frac{d}{dx} F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; x) \rightarrow (*)$$

& deduce that

(i) $$\frac{d^n}{dx^n} F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F(\alpha+n, \beta+n, \gamma+n; x)$$

H-W (ii) $$\left[\frac{d^n}{dx^n} F(\alpha, \beta; \gamma; x)\right]_{x=0} = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n}.$$

Proof:- By defⁿ, we have

$$F(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \cdot \frac{x^n}{n!}$$

Differentiating both sides w.r.t 'x', we have

$$\frac{d}{dx}\left\{F(\alpha,\beta;\gamma;x)\right\} = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{\cancel{n}\, x^{n-1}}{(\cancel{n!})}$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n\, x^{n-1}}{(\gamma)_n\,(n-1)!}$$

[∵ the term with $n=0$ vanishes]

[Let $m = n-1$
or, $n = m+1$
so that when $n=1$, $m=0$
& $n=\infty$, $m=\infty$]

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1} (\beta)_{m+1}}{(\gamma)_{m+1}} \cdot \frac{x^m}{m!}$$

(taking m as the new variable of summation)

$$= \sum_{m=0}^{\infty} \frac{\alpha(\alpha+1)_m \beta(\beta+1)_m}{\gamma(\gamma+1)_m} \cdot \frac{x^m}{m!}$$

$$= \frac{\alpha\beta}{\gamma} \left[\sum_{m=0}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{(\gamma+1)_m} \frac{x^m}{m!} \right]$$

$$= \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; x)$$

To show

$$\frac{d^n}{dx^n}\left\{F(\alpha,\beta;\gamma;x)\right\} = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} F(\alpha+n, \beta+n; \gamma+n; x)$$

Since $\alpha = (\alpha)_1$, $\beta = (\beta)_1$, $\gamma = (\gamma)_1$,

~~we see that (1) is~~

(1) is true for $n = 1$.

Let us now assume that (1) is true for $n = m$ (say)

i.e, $$\frac{d^m}{dx^m}\left\{F(\alpha,\beta;\gamma;x)\right\} = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m} F(\alpha+m, \beta+m; \gamma+m; x)$$ $\rightarrow$ (3)

Diff. w.r.t x both sides of (3), we get

$$\frac{d^{m+1}}{dx^{m+1}} F(\alpha, \beta; \gamma; x)$$

$$= \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \frac{d}{dx} F(\alpha+m, \beta+m; \gamma+m; x)$$

$$= \frac{(\alpha)_m (\beta)_m (\alpha+m)(\beta+m)}{(\gamma)_m (\gamma+m)} F(\alpha+m+1, \beta+m+1; \gamma+m+1; x)$$

$$= \frac{(\alpha)_{m+1} (\beta)_{m+1}}{(\gamma)_{m+1}} F(\alpha+m+1, \beta+m+1; \gamma+m+1; x) \quad [\text{by } (*)]$$

$[\because (\alpha+m)(\alpha)_m = (\alpha)_{m+1}]$

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1st Integral Representation

$$F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt$$

or, $F(\alpha, \beta; \gamma; x)$

$$= \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt.$$

If $\gamma > \beta > 0$

Ex/ Prove that

(i) $e^x = {}_1F_1(\alpha; \alpha; x)$

Soln:- we know,

$${}_1F_1(\alpha; \beta; x) = 1 + \frac{\alpha}{\beta} \cdot \frac{x}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{x^2}{2!} + \cdots$$

Replacing β by α, we have,

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x.$$

(ii) $(1-x)^{-\alpha} = {}_2F_1(\alpha, \beta; \beta; x)$

(iii) $(1-x)^{-1} = F(1, 1; 1; x)$, $|x| < 1$

(iv) $(1+x)^n = F(-n, 1; 1; -x)$

(v) $\ln(1+x) = \ln_e(1+x)$

$= x\,{}_2F_1(1, 1; 2; -x).$

(vi) $\ln(1-x) = -x\,{}_2F_1(1, 1; 2; x)$

(vii) $\ln\left(\frac{1+x}{1-x}\right) = 2x F(1/2, 1; 3/2; x^2)$

(viii) $\sin^{-1} x = x F(1/2; 1/2; 3/2; x^2)$

(ix) $\tan^{-1} x = x F(1/2, 1, 3/2, -x^2)$

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xm

Q) Show that

$$\lim_{a \to \infty} {}_2F_1\left(1, a; 1, \frac{x}{a}\right) = e^x.$$

— x —