Test - 12/4/21

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(*) $u_t + c u_x = 0$, FTCS $\rightarrow$ unstable ×

upwind scheme

if $c > 0$ FTBS, ie, $\dfrac{u_j^{n+1} - u_j^n}{\delta t} + c\dfrac{u_j^n - u_{j-1}^n}{\delta x} = 0$

if $c < 0$ FTFS, ie, $\dfrac{u_j^{n+1} - u_j^n}{\delta t} + c\dfrac{u_{j+1}^n - u_j^n}{\delta x} = 0$

upwind implies one sided differencing for the space derivative, i.e, u_x.

$c > 0$ $\qquad$ $c < 0$

$j-1 \quad j \quad j+1$

This scheme is stable for $\nu = |c|\dfrac{\delta t}{\delta x} \leq 1$

$\nu \rightarrow$ CFL number

Courant Fedrick Lewis no.

HT

Q

lab / compare with ex

$u_t + u_x = 0$, $u(x,0) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$, $u(0,t) = 0$

$\delta x = 1/4$, $\nu = 1/2$.

$c = 1$

$\dfrac{\delta t}{\delta x} = \dfrac{1}{2}$ $\qquad dt = \frac{1}{2} \times \delta x = \dfrac{1}{8}$

FTBS $\qquad u_j^{n+1} - u_j^n = -\nu\,(u_j^n - u_{j-1}^n)$

0 $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4

0 $\quad$ 0.25 $\quad$ 0.5 $\quad$ 0.75 $\quad$ 1

$u_j^{n+1} = (1-\nu)\,u_j^n + \nu\, u_{j-1}^n$, $n \geq 0$

$j = 1, 2, \ldots$

$u(0,0) = 0 = u_0$

$u_1^0 = \dfrac{1}{4^2}$, $u_2^0 = \left(\dfrac{2}{4}\right)^2$ $\quad u_3^0 = \left(\dfrac{3}{4}\right)^2$, $u_4^0 = 1$

$u_0^1 = 0$

$u_1^1 = \left(\dfrac{1}{2}\right) u_1^0 + \dfrac{1}{2} u_0^0 = \dfrac{1}{32}$

$u_2^1 = \dfrac{(u_2^0 + u_1^0)}{2} = \dfrac{5}{32}$

Solve ~~for n=0~~ $\quad$ 0.0156 $\quad$ 0.0937 $\quad$ 0.281 $\quad$ 0.5937

Q compare with exact

Lab HT

Q

$\dfrac{\partial T}{\partial t} + u\dfrac{\partial T}{\partial x} = 0$, $\quad T(x,0) = \begin{cases} 200x, & 0 < x < 0.5 \\ 200(1-x), & 0.5 < x < 1 \end{cases}$

$u = 0.1$, $\quad \nu = \dfrac{u\,\delta t}{\delta x} = 0.1$, $\delta x = 0.05$

Ans

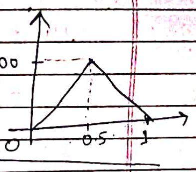
$c > 0$, FTBS

0 $\qquad$ 0.5

$u_j^{n+1} = (1-\nu)\,u_j^n + \nu\, u_{j-1}^n$

$u_j^{n+1} = 0.9\, u_j^n + 0.1\, u_{j-1}^n$

$= \dfrac{(9u_j^n + u_{j-1}^n)}{10}$



FTBS $\rightarrow O(\delta t, \delta x)$

<u>Modified Equation :</u>

$\dfrac{u_j^{n+1} - u_j^n}{\delta t} + c\,\dfrac{u_j^n - u_{j-1}^n}{\delta x} = 0$

Expand by Taylor Series:-

$\dfrac{1}{\delta t}\left[\left\{ u_j^n + \delta t\, u_t\Big|_j^n + \dfrac{\delta t^2}{2} u_{tt}\Big|_j^n + \dfrac{\delta t^3}{3!} u_{ttt}\Big|_j^n + \ldots \right\} - u_j^n \right]$

$+ \dfrac{c}{\delta x}\left[u_j^n - \left\{ u_j^n - \delta x\, u_x\Big|_j^n + \dfrac{\delta x^2}{2} u_{xx}\Big|_j - \dfrac{\delta x^3}{3!} u_{xxx}\Big|_j + \ldots \right\}\right] = 0.$

at (x_j, t_n) we get

$u_t + c u_x = -\dfrac{\delta t}{2} u_{tt} + \dfrac{c\,\delta x}{2} u_{xx} - \dfrac{\delta t^2}{3!} u_{ttt} - c\dfrac{\delta x^2}{3!} u_{xxx} + \ldots$ (*)

T.E $\sim O(\delta t, \delta x)$ consistent

Replace the time derivative by space derivative in R.HS.

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Differentiate (*) w.r.t. t

$$u_{tt} + c u_{xt} = -\frac{\delta t}{2} u_{ttt} + c\frac{\delta x}{2} u_{xxt} - \frac{\delta t^3}{6} u_{tttt} - \frac{c\,\delta x^3}{6} u_{xxxt} + \cdots \quad (i)$$

Diff (*) w.r.t. x

$$u_{tx} + c u_{xx} = -\frac{\delta t}{2} u_{ttx} + c\frac{\delta x}{2} u_{xxx} - \frac{\delta t^3}{6} u_{tttx} - c\frac{\delta x^3}{6} u_{xxxx} + \cdots \quad (ii)$$

(i) − c(ii)

$$u_{tt} - c^2 u_{xx} = \delta t\left[-\frac{u_{ttt}}{2} + \frac{c\,u_{ttx}}{2}\right] + O(\delta t^2) + \delta x\left[\frac{c\,u_{xxt}}{2} - \frac{c^2 u_{xxx}}{2}\right] + O(\delta x^2) \quad -(a)$$

Similarly, $u_{ttt} = -c^3 u_{xxx} + O(\delta t, \delta x) \quad -(b)$

Substitute (a), (b) in (*)

$$u_t + c u_x = -\frac{c^2 \delta t}{2} u_{xx} + c\frac{\delta x}{2} u_{xx} + \frac{\delta t^2}{6} c^3 u_{xxx} - c\frac{\delta x^2}{6} u_{xxx} + \cdots$$

i.e. $u_t + c u_x = c\frac{\delta x}{2}[1-\nu]u_{xx} + O(\delta x^2, \delta t^2)$, $\nu = c\,\delta t/\delta x$

If u is the solution of PDE. Then l.h.s. $u_t + c u_x = 0$

This eq^n is called the modified eq^n i.e. the PDE which the difference scheme (FTBC) is actually solving).

The least order term is $\frac{c}{2}(1-\nu)u_{xx}$, if $\nu \neq 1$.

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which is called the dissipation error.

$\frac{c}{2}(1-\nu)$ is the numerical diffusion.

Note

Presence of this 2nd order spatial derivatives tends to spread out the sharp discontinuity in u-profile.

Dissipative scheme whose least order term in T.E involves even-order derivative w.r.t. x

Dispersive scheme: T.E involves odd-order derivatives w.r.t. x

Lax Scheme: FTCS can be made stable by replacing $u_j^n = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} = 0$$

H.T

The modified eq^n (H.T)

$$u_t + c u_x = \frac{c}{2}\delta x\left(\frac{1}{\nu} - \nu\right)u_{xx} + c\frac{(\delta x)^3}{3}(1-\nu^2)u_{xxx} + \cdots$$

T.E $\to 0$ as $\delta x, \delta t \to 0$

consistent

Stable $|\nu| = |c|\,\delta t/\delta x \leq 1 \to$ H.T

T.E. $O(\delta t, \frac{\delta x^2}{\delta t})$

if $\nu \neq 1$, it is a dissipative scheme.

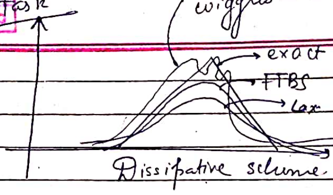
The Lax scheme involves large dissipation

Thus, Lax scheme

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\nu}{2}(u_{j+1}^n - u_{j-1}^n), \quad n \geq 0$$

Lab Task

Plot & check



Dissipative scheme.

Lax-Wendroff scheme:-

This scheme can be derived from the Taylor series expansion $u_i^{n+1} = u_j^n + \delta t\, u_t\big|_j^n + \frac{\delta t^2}{2} u_{tt}\big|_j^n + \frac{\delta t^3}{3!} u_{ttt}\big|_j^n$

Replace u_t, u_{tt} etc by space derivative by using PDE

i.e. u_x, u_{xx}, -

$u_t = -c u_x$

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial t} \equiv -c\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial t} u_t = -\frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial x}\right) = c\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)$$

$$= c^2 u_{xx}$$

$$u_{tt} = c^2 u_{xx}$$

or, $u_{tt} = -c u_{xt}$

$u_{xt} = -c u_{xx} \times (-c)$

$$\boxed{u_{tt} = c^2 u_{xx}}$$

So, $u_j^{n+1} = u_j^n - \delta t \cdot c\, u_x\big|_j^n + \frac{\delta t^2}{2!} c^2 u_{xx}\big|_j^n + O(\delta t^3)$

discretize space der. by Central diff

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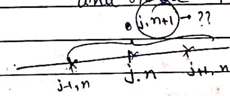
$$u_j^{n+1} = u_j^n - \frac{c}{2}\frac{\delta t}{\delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\delta t^2 c^2}{2\delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) + O(\delta t^3)$$

the Lax-Wendroff scheme is

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\nu^2}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$$

which is an explicit scheme and T.E $O(\delta t^2, \delta x^2)$

H.T → and stable if $|\nu| \le 1$



Modified Equation (H.T)

$$u_t + c u_x = -c\frac{(\delta x)^2}{6}(1-\nu)u_{xxx}$$

$$-c\frac{(\delta x)^3}{8}\nu(1-\nu^2)u_{xxxx} + \cdots$$

$T.E = O(\delta x^2, \delta t^2)$. → consistent

lowest-order term involves third order derivatives, which produces a dispersion error. This dispersion error is associated with wiggles formation, near a sharpness in the scheme

HT Lab Solve all the two 1st order PDEs by all the methods, : FTBS, Lax, Lax-Wendroff & Plot u vs x at $t = t^*$ (particular) along with the exact

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§ Two-step Lax-Wendroff scheme:-

Step I: Apply Lax method at the mid point $(j+\frac{1}{2})$ with half time step $(n+\frac{1}{2})$. $t_n \to t_{n+1/2}$

$$\frac{u_{j+1/2}^{n+1/2} - \frac{1}{2}\left(u_{j+1}^{n} + u_j^{n}\right)}{\delta t/2} + c\frac{\left(u_{j+1}^{n} - u_j^{n}\right)}{2\,\delta x/2} = 0$$

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}\left(u_{j+1}^{n} + u_j^{n}\right) - \frac{\nu}{2}\left(u_{j+1}^{n} - u_j^{n}\right) \rightarrow \text{Predictor}$$

Step II: $t_{n+1/2} \to t_{n+1}$: central difference at $(j, n+\frac{1}{2})$ with half space & time steps.

$$\frac{u_j^{n+1} - u_j^{n}}{2\,\delta t/2} + c\,\frac{u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}}{2\,\delta x/2} = 0$$

$$u_j^{n+1} = u_j^{n} - \nu\left(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}\right) \rightarrow \text{Corrector step}$$

Step II (~~Predictor~~ Corrector Step) Central difference approx. at $(n+\frac{1}{2})$ time step at the mid point $(j+\frac{1}{2})$.
Replace $u_{j+1/2}^{n+1/2}$, $u_{j-1/2}^{n+1/2}$ from Step I. to Step II,

we get the single step Lax-Wendroff scheme.
T.E $\sim O(\delta t^2, \delta x^2)$. Stable for $|\nu| \le 1$.

Q. $u_t + u_x = 0$, $u(x,0) = \exp(-20(x-2)^2) + \exp(-(x-5)^2)$

HW & Lab Task

solve upto $t = 17$.
$0 \le x \le 25$, $\delta x = 0.05$
$\nu = 0.8$

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Q. $u_t + 0.1\,u_x = 0$

$$u(x,0) = \begin{cases} 20x & 0 \le x \le 0.05 \\ 20(0.1 - x), & 0.05 \le x \le 0.1 \\ 0 & 0.1 < x \le 1. \end{cases}$$

$u(0,t) = 0$, $\nu = 0.8$, $\delta x = 1/4$,
Solve by all the methods.

A. $u_t + u\,u_x = 0$

HW T

$$u_t + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

$u_t = -F_x$
$u_{tt} = ??$

Test on 12/4

From 8:20 - 9:50 AM

Excluding linear m

(Q) $u_t + 0.1\, u_x = 0$

$$u(x,0) = \begin{cases} 20x & 0 \le x \le 0.05 \\ 20(0.1 - x), & 0.05 \le x \le 0.1 \\ 0 & 0.1 < x \le 1 \end{cases}$$

$u(0,t) = 0$, $\nu = 0.8$, $\delta x = 1/4$,

Solve by all the methods.

(A) $u_t + u u_x = 0$

$$u_t + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0$$

$u_t = -F_x$

$u_{tt} = ??$

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$u_t + c u_x = 0$

Mac Cormack Scheme:

$$u_j^{n+1} = u_j^n + u_t\big|_j^n \delta t + u_{tt}\big|_j^n \frac{\delta t^2}{2} + O(\delta t^3)$$

instead of replacing u_{tt} by using the PDE, we consider a 1^{st} order expansion $u_{tt}\big|_j^n$ as

$$u_{tt}\big|_j^n = \frac{\left(u_t\big|_j^{n+1} - u_t\big|_j^n\right)}{\delta t} \quad \text{--- (*)}$$

$$u_j^{n+1} = u_j^n + u_t\big|_j^n \delta t + \frac{\delta t}{2}\left[u_t\big|_j^{n+1} - u_t\big|_j^n\right] + O(\delta t^3)$$

$$= u_j^n + \frac{\delta t}{2}\left[u_t\big|_j^{n+1} + u_t\big|_j^n\right] + O(\delta t^3).$$

Since $u_t = -c u_x$ from the PDE

$$u_j^{n+1} = u_j^n - \frac{c\,\delta t}{2}\left[u_x\big|_j^n + u_x\big|_j^{n+1}\right] + O(\delta t^3)$$

If we replace u_x by the central diff. app. then it will be an implicit in $O(\delta t^2, \delta x^2)$ scheme which is difficult to handle due to the implicit nature. Mac-Cormac proposed a Prediction step to obtain u_j^{n+1} by FTFS, i.e. $\bar{u}_j^{n+1} = u_j^n - \nu(u_{j+1}^n - u_j^n)$

$$\nu = \frac{c\,\delta t}{\delta x}$$

Use $\bar{u}_j^{n+1}$. evaluate $u_x\big|_j^{n+1}$ by 1st-order backward scheme.

$$u_j^{n+1} = u_j^n - \frac{c\,\delta t}{2}\left[\frac{u_{j+1}^n - u_j^n}{\delta x} + \frac{\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}}{\delta x}\right]$$

[corrector step]

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}\left[u_{j+1}^n - u_j^n\right] - \frac{\nu}{2}\left[\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}\right]$$

Since $\left[\bar{u}_j^{n+1} = u_j^n - \nu(u_{j+1}^n - u_j^n)\right]$ using Predictor step.

$$u_j^{n+1} = u_j^n + \frac{1}{2}\left(\bar{u}_j^{n+1} - u_j^n\right) - \frac{\nu}{2}\left(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}\right)$$

$$u_j^{n+1} = \frac{1}{2}\left(u_j^n + \bar{u}_j^{n+1}\right) - \frac{\nu}{2}\left(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}\right)$$

↳ which is the corrector step.

The 2-Step Mac Cormack method is:

Predictor: $\bar{u}_j^{n+1} = u_j^n - \frac{c\,\delta t}{\delta x}\left[u_{j+1}^n - u_j^n\right]$ — (I)

Corrector: $u_j^{n+1} = \frac{1}{2}\left[u_j^n + \bar{u}_j^{n+1}\right] - \frac{c\,\delta t}{2\delta x}\left[\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}\right]$ (II)

At every grid at time level t_n, i.e. (x_j, t_n) evaluate $\bar{u}_j^{n+1}$ by (I), then obtain u_j^{n+1} by (II)

For a linear PDE, this scheme is identical to the Lax-Wendroff single step method. T.E. $O(\delta t^2, \delta x^2)$

Stable $|\nu| \le 1$ (H.T)

Prove

Solve Previous problems by this scheme as well.

Q. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$

$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0$, $F = F(u)$

$$\frac{\partial F(u)}{\partial x} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x} = A\frac{\partial u}{\partial x}$$

$A = \frac{\partial F}{\partial u}$ → Jacobian

$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$, $A = \frac{\partial F}{\partial u}$ If u is a vector, $A = \frac{\partial F}{\partial u}$ → matrix

$\frac{\partial u}{\partial t} = -A\frac{\partial u}{\partial x}$ referred as Jacobian

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 F}{\partial t\,\partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial t}\right)$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial t}\right)$$

$$= -\frac{\partial}{\partial x}\left(-A\cdot\frac{\partial F}{\partial x}\right) \Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right)$$

$$u_j^{n+1} = u_j^n + \delta t\frac{\partial u}{\partial t}\Big|_j^n + \frac{\delta t^2}{2}\frac{\partial^2 u}{\partial t^2}\Big|_j^n + O(\delta t^3)$$

$$= u_j^n - \delta t\frac{\partial F}{\partial x}\Big|_j^n + \frac{\delta t^2}{2}\frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right)\Big|_j^n + O(\delta t^3)$$

Space derivatives are discretized by central differences

$$\frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right)\Big|_j^n = \frac{1}{\delta x}\left(A\frac{\partial F}{\partial x}\Big|_{j+1/2}^n - A\frac{\partial F}{\partial x}\Big|_{j-1/2}^n\right)$$

$$= \frac{1}{\delta x^2}\left[A\big|_{j+1/2}^n\left(F_{j+1}^n - F_j^n\right) - A\big|_{j-1/2}^n\left(F_j^n - F_{j-1}^n\right)\right]$$

$$A\big|_{j+1/2}^n = A\left(\frac{u_j^n + u_{j+1}^n}{2}\right)$$

$$A\big|_{j-1/2}^n = A\left(\frac{u_j^n + u_{j-1}^n}{2}\right)$$

Single Step Lax-Wendroff scheme

$$U_j^{n+1} = U_j^n - \frac{\delta t}{\delta x}\cdot\frac{F_{j+1}^n - F_{j-1}^n}{2} + \frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2\left[A^n_{j+1/2}\left(F^n_{j+1} - F_j^n\right) - A^n_{j-1/2}\left(F_j^n - F_{j-1}^n\right)\right]$$

$$T\text{-}E = + O(\delta t^2, \delta x^2)$$

which involves the evaluation of the Jacobian $A^n_{j+1/2}$ and $A^n_{j-1/2}$ which become complicated when u is a vector

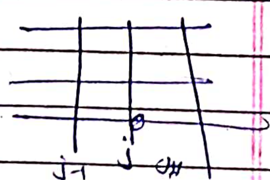
Two-step Lax-Wendroff scheme, proposed by <Richtmyer> as

Predictor: $\bar{U}_j = \frac{1}{2}\left(U_j^n + U_{j+1}^n\right) - \frac{\delta t}{2\delta x}\left(F_{j+1}^n - F_j^n\right)$

Corrector: $U_j^{n+1} = U_j^n - \frac{\delta t}{\delta x}\left(F(\bar{U}_j) - F(\bar{U}_{j-1})\right)$

Predictor step is computed at the centre of each cell

H.W. Check the two steps is 1



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Here we have,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = \begin{cases} \sqrt{x}, & 0 \le x \le 1 \\ 0 : & \text{otherw} \end{cases}$$

$\delta x = 0.2, \qquad u(0,t) = 0.$

$\nu = \delta t/\delta x = 0.5$

Now, in order to solve this eqⁿ we have to convert it into this form

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2\right) = 0$$

Now, let us consider $F(u) = \frac{u^2}{2}$

So the equation is reduced to.

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

Now, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}$ $\left(\text{let } \frac{\partial F}{\partial u} = A\right)$

$$= A\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} = -A\frac{\partial u}{\partial x} = -\frac{\partial F}{\partial x}$$

diff wrt time

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-\frac{\partial F}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial t}\right)$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}\right)$$

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Now, $\frac{\partial u}{\partial t} = -A\frac{\partial u}{\partial x} = -\frac{\partial F}{\partial x}$ & $A = \frac{\partial F}{\partial u}$

replacing $\frac{\partial u}{\partial t}$ with $-\frac{\partial F}{\partial x}$ & $\frac{\partial F}{\partial u} \to A$

$$= -\frac{\partial}{\partial x}\left(-A\cdot\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(A\cdot\frac{\partial F}{\partial x}\right).$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right).$$

~~Now, using Backward Time eq.~~

Now, using Taylor Series exp. on u_j^{n+1}

$$u_j^{n+1} = u_j^n + \delta t\left.\frac{\partial u}{\partial t}\right|_j^n + \frac{(\delta t)^2}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_j^n + O(\delta t^3)$$

$$U_j^{n+1} = U_j^n - \delta t\left.\frac{\partial F}{\partial x}\right|_j^n + \frac{(\delta t)^2}{2}\left.\frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right)\right|_j^n + O(\delta t^3)$$

Discretizing the space derivatives as

$$\left.\frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right)\right|_j^n = \frac{1}{\delta x}\left(\left.A\frac{\partial F}{\partial x}\right|_{j+1/2}^n - \left.A\frac{\partial F}{\partial x}\right|_{j-1/2}^n\right)$$

↳ using central diff at (j)

$$= \frac{1}{(\delta x)^2}\left(A\Big|_{j+1/2}^n\left(F_{j+1}^n - F_j^n\right) - A\Big|_{j-1/2}^n\left(F_j^n - F_{j-1}^n\right)\right),$$

$$\& \quad A\Big|_{j+1/2}^n = A_j^n\left(\frac{U_j^n + U_{j+1}^n}{2}\right)$$

$$A\Big|_{j-1/2}^n = A_j^n\left(\frac{U_j^n + U_{j-1}^n}{2}\right)$$

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Putting it into the equation, we get,

$$U_j^{n+1} = U_j^n - \nu \left(\frac{F_{j+1}^n - F_{j-1}^n}{2}\right) + \frac{\nu^2}{2}\left[A_{j+1/2}^n\left(F_{j+1}^n - F_j^n\right) - A_{j-1/2}^n\left(F_j^n - F_{j-1}^n\right)\right]$$

this can be broken down into two steps:

Step-I : $$\bar{U}_j = \left(\frac{U_j^n + U_{j+1}^n}{2}\right) - \frac{\nu}{2}\left(F_{j+1}^n - F_j^n\right) \quad (\text{predictor})$$

Step II : $$U_j^{n+1} = U_j^n - \nu\left(F(\bar{U}_j) - F(\bar{U}_{j-1})\right) \quad (\text{corrector})$$

$\hookrightarrow$ order is $O(\delta t^2, \delta x^2)$

& this is an explicit scheme