

VECTOR CALCULUS

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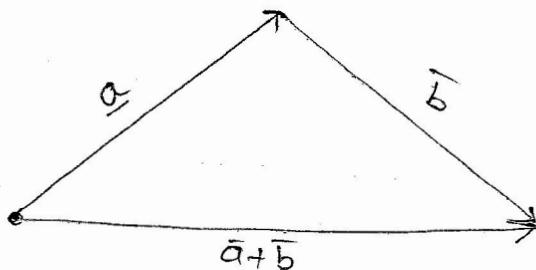
VECTORS and SCALARS:

- * A vector is a physical quantity which has both magnitude and direction.

Examples: velocity, force, electric field.

- * A scalar is a physical quantity which has magnitude only.

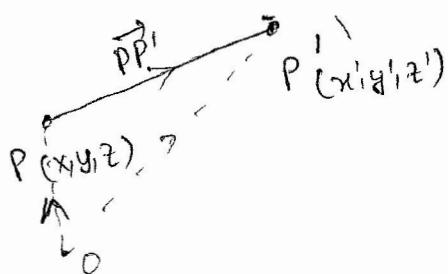
Examples: mass, temperature and pressure.

Addition of vectors: (Geometric definition)Addition of vectors: (Algebraic definition)

$$\underline{a} = (a_1, a_2, a_3) \quad \underline{b} = (b_1, b_2, b_3) \quad \text{OR} \quad \underline{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$\underline{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\underline{a} + \underline{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3). \quad \underline{a} + \underline{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$$

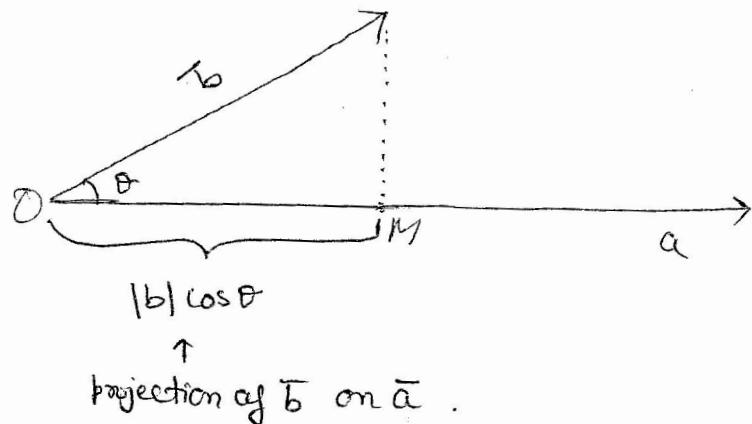
The vector joining two points:

$$\overline{PP'} = (x' - x, y' - y, z' - z).$$

$$\overline{OP} + \overline{PP'} = \overline{OP'} \Rightarrow \overline{PP'} = \overline{OP'} - \overline{OP}.$$

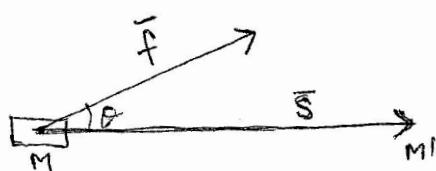
Dot product: (scalar product) \rightarrow scalar quantity.

$$\bar{a} \cdot \bar{b} = |\bar{a}| |\bar{b}| \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3.$$



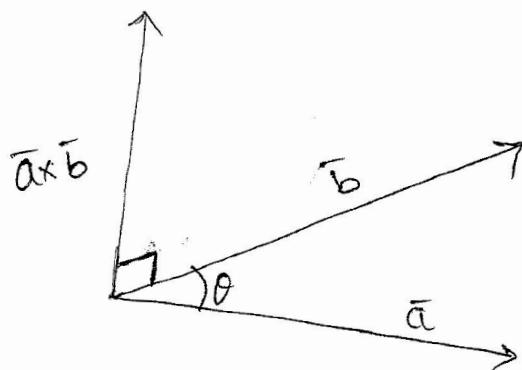
$$\boxed{\text{Projection of } \bar{b} \text{ on } \bar{a} = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|}}$$

Application: Evaluation of work done:



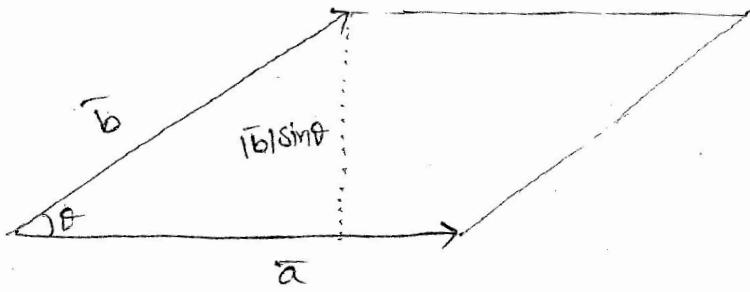
$$\boxed{W = \bar{f} \cdot \bar{s} = |\bar{f}| |\bar{s}| \cos \theta}$$

Cross product: (vector product) \rightarrow vector quantity



$$|\bar{a} \times \bar{b}| = |\bar{a}| |\bar{b}| \sin \theta$$

Direction: perpendicular to both \bar{a} and \bar{b} in a right handed sense.



Area of the parallelogram = $|\vec{a}| |\vec{b}| \sin\theta$.

$$= |\vec{a} \times \vec{b}|$$

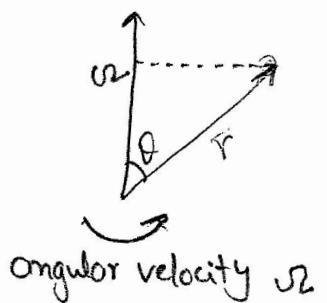
Evaluation of Cross product:

$$\vec{a} = (a_1, a_2, a_3) \quad \vec{b} = (b_1, b_2, b_3)$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i}(a_2 b_3 - b_2 a_3) + \vec{j}(a_3 b_1 - b_3 a_1) + \vec{k}(a_1 b_2 - b_1 a_2).$$

Application:

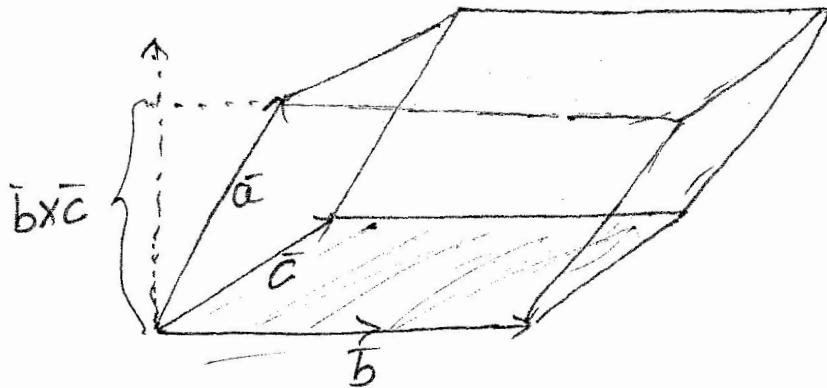


Velocity vector of the point r is

$$\vec{v} = v_r \times \vec{r}$$

Scalar triple product:

$$a \cdot b \times c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \cancel{(a \cdot b \cdot c)} = (a \cdot b \cdot c)$$



$$\text{height of the parallelepiped} = \frac{|a \cdot b \times c|}{|b \times c|}$$

$$\text{Area of the base} = |b \times c|$$

$$\text{Volume of the parallelepiped} = |a \cdot b \times c|$$

Volume of the tetrahedron made by the vectors a, b, c is

$$\frac{1}{6} |a \cdot b \times c|.$$

Scalar fields and vector fields:

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A scalar or vector quantity is said to be field if it is a function of position.

Examples: Scalar field: Temperature inside a room.

Vector field: Velocity of the air within a room.

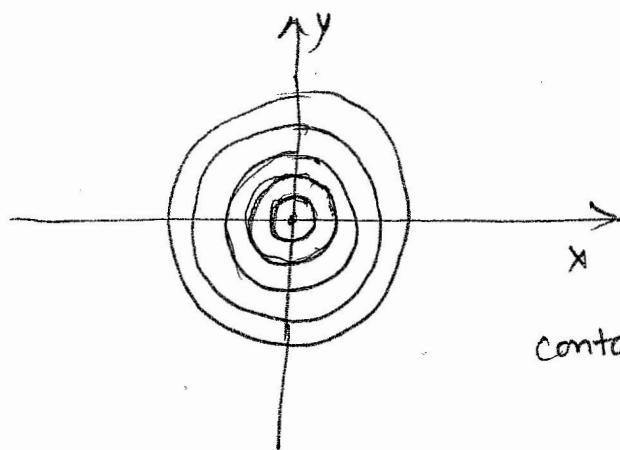
In general scalar fields depends ^{or vector} in three dimensional.

- difficult to visualise.

If, for example, a scalar field $T = T(x, y)$ depends on two coordinates then it can be visualised by sketching a contour plot i.e. the line $T(x, y) = \text{constant}$ is plotted for different values of the constant.

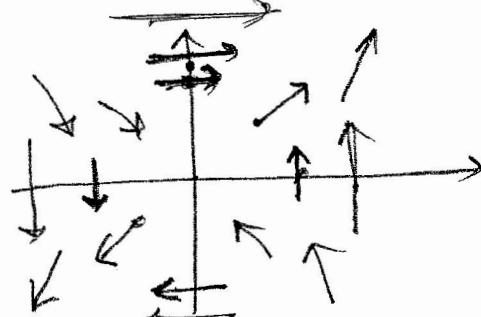
Consider $T(x, y) = x^2 + y^2$

contour lines $x^2 + y^2 = \text{constant}$.

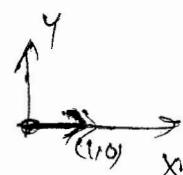


contours of the scalar field $T(x, y) = x^2 + y^2$

Consider vector field: $\underline{u}(x, y) = (y, x)$.



Vector field
 $u(x, y) = (y, x)$



Level Surfaces: Let $f(x, y, z)$ be a single valued continuous scalar function defined at every point in D .

Then the equation of surface

$$f(x, y, z) = c$$

is called a level surface.

Examples:

$$f(x, y, z) = z - \sqrt{x^2 + y^2}$$

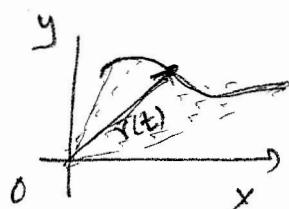
$$\Rightarrow z - \sqrt{x^2 + y^2} = c$$

$$\Rightarrow (z - c)^2 = (x^2 + y^2)$$

The level surfaces are cones.

Parametric representation of curves:

Curve in two dimension: $x = x(t)$ $y = y(t)$ $a \leq t \leq b$.

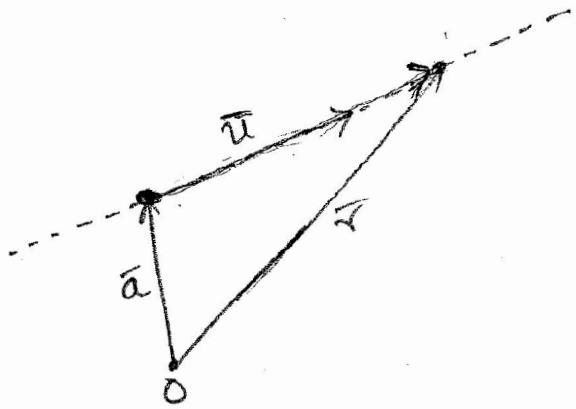


Position vector $\boxed{\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}}$

Similarly in 3D:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \quad a \leq t \leq b.$$

Equation of a straight line:



\bar{a} : position vector of a particular fixed point on the line

\bar{u} : vector pointing along the line.

equation of the straight line:

$$\bar{r} = \bar{a} + \lambda \bar{u}$$

$$= (a_1 + \lambda u_1) \hat{i} + (a_2 + \lambda u_2) \hat{j} + (a_3 + \lambda u_3) \hat{k}.$$

Equation of a circle in a plane in 3-dimensions:

Consider the circle as $x^2 + y^2 = a^2$, $z=d$.

$$\bar{r}(t) = a \cos \theta \hat{i} + a \sin \theta \hat{j} + d \hat{k}.$$

$$0 \leq \theta \leq 2\pi.$$

Parametric representation of surfaces: Let $f(x, y, z) = c$. be the equation of a surface.

(i) write $z = f(x, y)$ then.

or

$$\bar{r}(u, v) = u \hat{i} + v \hat{j} + f(u, v) \hat{k}.$$

$$(ii) \quad \bar{r}(u, v) = x(u, v) \hat{i} + y(u, v) \hat{j} + z(u, v) \hat{k}. \quad ; \quad (u, v) \in D.$$

Parametric representation of a sphere: $x^2 + y^2 + z^2 = a^2$. (5)

$$\gamma(u, v) = a \cos u \sin v \hat{i} + a \sin u \sin v \hat{j} + a \cos v \hat{k}$$
$$0 \leq u \leq 2\pi \quad 0 \leq v \leq \pi.$$

Limit and continuity of vector functions:

limit: $\lim_{t \rightarrow a} |\vec{v}(t) - \vec{d}| = 0$

continuity: $\vec{v}(t)$ is said to be continuous at $t = a$ if

- $\vec{v}(t)$ is defined in some neighbourhood of a
- $\lim_{t \rightarrow a} \vec{v}(t)$ exists and
- $\lim_{t \rightarrow a} \vec{v}(t) = \vec{v}(a)$

Differentiability: $\vec{v}(t)$ is said to be differentiable if

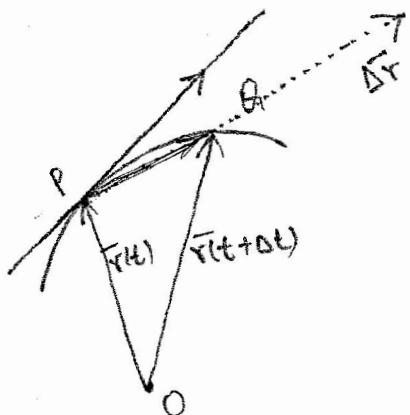
$$\lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} \text{ exists}$$

Let $\vec{v}(t) = \gamma(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be
the parametric representation of a curve C . Then

$$\frac{d\vec{r}}{dt} = \vec{v}'(t) = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}$$

Geometric representation of $\vec{r}(t)$: (tangent to a curve)

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Let the parametric equation of the curve be $\vec{r}(t)$.

Note that the direction of $\Delta \vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$ and

$\frac{\Delta \vec{r}}{\Delta t}$ is the same.

Then the limiting position vector of the vector $\frac{\Delta \vec{r}}{\Delta t}$, that is,

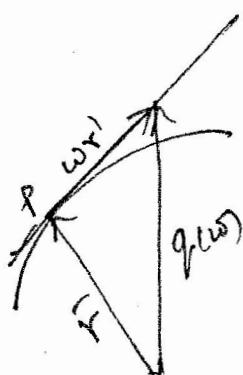
$\lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t}$ is the tangent to the curve at P.

$$\text{tangent vector} = \vec{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t}$$

$$\text{Unit tangent vector } \hat{u} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}.$$

Equation of tangent to C at P:

$$\bar{q}(\omega) = \vec{r} + \omega \vec{r}'$$



Rules of Differentiation:

$$(\bar{u} \cdot \bar{v})' = \bar{u} \cdot \bar{v}' + \bar{u}' \cdot \bar{v}$$

$$(\bar{u} \times \bar{v})' = \bar{u}' \times \bar{v} + \bar{u} \times \bar{v}'$$

$$(\bar{u} \cdot \bar{v} \cdot \bar{w})' = (\bar{u}' \cdot \bar{v} \cdot \bar{w}) + (\bar{u} \cdot \bar{v}' \cdot \bar{w}) + (\bar{u} \cdot \bar{v} \cdot \bar{w}')$$

Partial derivatives of a vector function:

Let $\bar{v} = v_1 \bar{i} + v_2 \bar{j} + v_3 \bar{k}$ & v_1, v_2, v_3 are differentiable functions of n variables t_1, t_2, \dots, t_n .

Then the partial derivative of \bar{v} with respect to t_j is given by.

$$\frac{\partial \bar{v}}{\partial t_j} = \frac{\partial v_1}{\partial t_j} \bar{i} + \frac{\partial v_2}{\partial t_j} \bar{j} + \frac{\partial v_3}{\partial t_j} \bar{k}.$$

Example: i) find $\bar{v}'(t)$ for $\bar{v}(t) = (\cos t + t^2)(\bar{i} + \bar{j} + 2\bar{k})$

$$\bar{v}'(t) = [(-\sin t + 2t)\bar{i} + (\cos t + t^2)\bar{j}]$$

$$+ [-\sin t + 2t](\bar{j} + 2\bar{k})$$

$$= (3t^2 - t \sin t + \cos t)\bar{i} + (2t - \sin t)(\bar{j} + 2\bar{k})$$

ii) Partial derivatives: $\bar{r}(t_1, t_2) = a \cos t_1 \bar{i} + a \sin t_1 \bar{j} + t_2 \bar{k}$.

$$\frac{\partial \bar{r}}{\partial t_1} = -a \sin t_1 \bar{i} + a \cos t_1 \bar{j}$$

$$\frac{\partial \bar{r}}{\partial t_2} = \bar{k}.$$

D.

Gradient and Directional Derivative

- * Gradient of a scalar function $f(x, y, z)$ is a vector given by:

$$\text{grad } f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

- * Nabla or Del operator:

$$\nabla \equiv \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \quad \text{or} \quad \nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

So, $\text{grad } f = \nabla f$

- If a surface is given by $f(x, y, z) = c$. Then $\nabla f(P)$ is the vector normal to the surface $f(x, y, z) = c$ at the point P .

Consider a smooth curve C on the surface passing through a point P on the surface. Let $\bar{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ be the position vector off P .

Since the curve lies on the surface we have

$$f(x(t), y(t), z(t)) = c$$

Then, $\frac{d}{dt} f(x(t), y(t), z(t)) = 0$

$$\Rightarrow \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = 0 \quad \text{by chain rule.}$$

$$\Rightarrow \left(\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \right) \cdot \left(\frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} + \frac{dz}{dt} \vec{k} \right) = 0$$

$$\Rightarrow \nabla f \cdot \bar{r}'(t) = 0$$

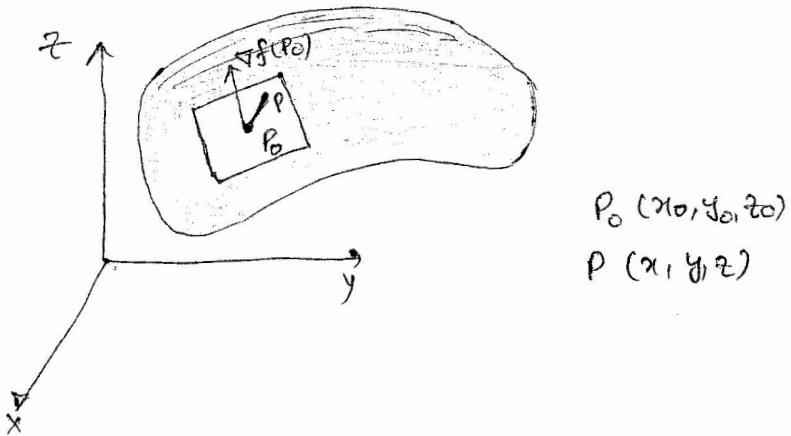
Note that $\bar{r}'(t)$ is a tangent vector at P and lies in the tangent plane of

$$\Rightarrow \nabla f(P) \text{ is the vector normal to the surface } f(x, y, z) = c \text{ at } P.$$

Unit normal vector to a surface $f(x, y, z) = c$:

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

Equation of the tangent plane:



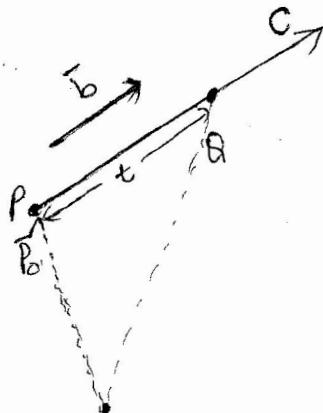
$$\overrightarrow{P_0P} \cdot \nabla f(P_0) = 0 \quad \text{since } \overrightarrow{P_0P} \text{ lies in the tangent plane.}$$

$$\Rightarrow [(x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}] \cdot \left[\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \right] = 0$$

$$\Rightarrow \boxed{(x-x_0)\frac{\partial f}{\partial x}(P_0) + (y-y_0)\frac{\partial f}{\partial y}(P_0) + (z-z_0)\frac{\partial f}{\partial z}(P_0) = 0}$$

Directional Derivative: Generalization of the notion of partial derivatives.

Partial derivative: direction is parallel to one of the coordinate axes.



$$D_b f = \frac{df}{dt} = \lim_{t \rightarrow 0} \frac{f(Q) - f(P)}{t}$$

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If the limit exists, it is called the directional derivative of f

at the point P in the direction of b . The position vector of the line C is: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \mathbf{P}_0 + t\mathbf{b}$.

Using chain rule:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \\ &= \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) \cdot \left(\frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \right) \\ &= \nabla f \cdot \frac{d\mathbf{r}}{dt} \\ &= \nabla f \cdot \mathbf{b} \end{aligned}$$

directional derivative $D_b f = \nabla f \cdot \mathbf{b}$

At any point P , the directional derivative of f represents the rate of change in f along \mathbf{b} at the point P .

Remark: $\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$

Note that component of ∇f in the \mathbf{i} direction = $\frac{\nabla f \cdot \mathbf{i}}{|\mathbf{i}|}$
 rate of change of f in x direction = $\frac{\partial f}{\partial x}$.

In general, the component of ∇f in any direction is the rate of change of f in that direction. This leads to the formula $\nabla f \cdot \mathbf{E}$: \mathbf{E} unit vector.

Maximum rate of change of a scalar field:

Note that

Rate of change of f in the direction of \vec{b}

$$= D_{\vec{b}} f = \nabla f \cdot \vec{b} = |\nabla f| |\vec{b}| \cos \theta \\ = |\nabla f| \cos \theta.$$

$$\Rightarrow -|\nabla f| \leq D_{\vec{b}} f \leq |\nabla f| \quad \text{since } -1 \leq \cos \theta \leq 1$$

\Rightarrow Rate of change is maximum when θ is 0, that is, in the direction of ∇f .

\Rightarrow Rate of change is minimum when θ is π , that is, in the opposite direction of ∇f .

\Rightarrow Gradient vector ∇f points in the direction in which f increases most rapidly and $-\nabla f$ points in the direction in which f decreases most rapidly.

Example: Find the unit normal to the surface $x^2 + y^2 - z = 0$ at the point $(1, 1, 2)$.

Sol: Define $f = x^2 + y^2 - z = 0 \Rightarrow \nabla f = (2x, 2y, -1)$

$$\nabla f(1, 1, 2) = (2, 2, -1)$$

Unit normal vector $= \frac{1}{\sqrt{7+4+1}} (2, 2, -1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$.
The other one is $-\vec{n} = (-2/3, -2/3, 1/3)$.

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Example: Find the directional derivative of the scalar field $f = 2x + y + z^2$ in the direction of the vector $(1, 1, 1)$ and evaluate this at the origin.

Sol: $\nabla f = (2, 1, 2z)$

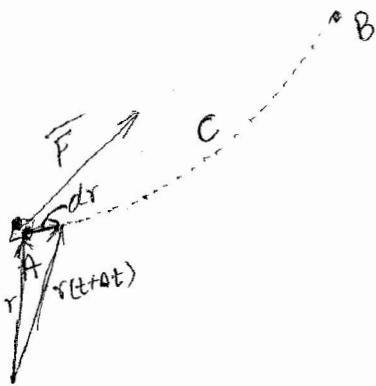
$$\begin{aligned} D_{(1,1,1)} f &= \nabla f \cdot (1, 1, 1) \frac{1}{\sqrt{3}} \\ &= (2, 1, 2z) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ &= \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2z}{\sqrt{3}}\right) \end{aligned}$$

At the origin: $Df = \sqrt{3}$. Ans.

Conservative vector field: A vector field \vec{V} is said to be conservative if the vector function can be written as the gradient of a scalar function f , that is,

$\vec{V} = \nabla f$. The function f is called a potential function or a potential of \vec{V} .

In such a vector field the work done in moving a particle from a point P to a point Q depends on the points P and Q and is independent of path along which the particle is displaced from P to Q .



Suppose $\bar{F} = \nabla f$

Total work done displacing the particle from A to B..

$$W = \int_C \bar{F} \cdot d\bar{r} = \int_C \nabla f \cdot d\bar{r}$$

Recall from the Directional derivative $df = \nabla f \cdot d\bar{r}$

$$\Rightarrow W = \int_C df = f|_a^b = f(b) - f(a).$$

$\Rightarrow \bar{F}$ is conservative.



Example: Show that the vector field $\bar{F} = (2xy, x, 2z)$ is conservative.

Sol: \bar{F} is conservative if it can be written as $\bar{F} = \nabla \varphi$.

$$\Rightarrow \frac{\partial \varphi}{\partial x} = 2xy, \quad \textcircled{1} \quad \frac{\partial \varphi}{\partial y} = x, \quad \textcircled{2} \quad \frac{\partial \varphi}{\partial z} = 2z, \quad \textcircled{3}$$

$$\textcircled{1} \Rightarrow \varphi = x^2 + xy + h(y, z).$$

$$\textcircled{2} \Rightarrow x + \frac{\partial h}{\partial y} = x \Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h \text{ is independent of } y. \\ h = h(z)$$

$$\textcircled{3} \quad 0 + \frac{dh}{dz} = 2z \Rightarrow h = z^2 + c$$

$$\Rightarrow \varphi = x^2 + xy + z^2 + c. \quad \text{Ans}$$

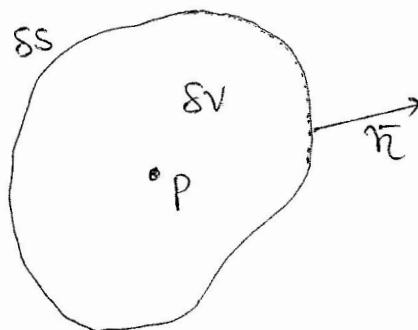
Divergence of a vector field:

The divergence of a vector field is defined as

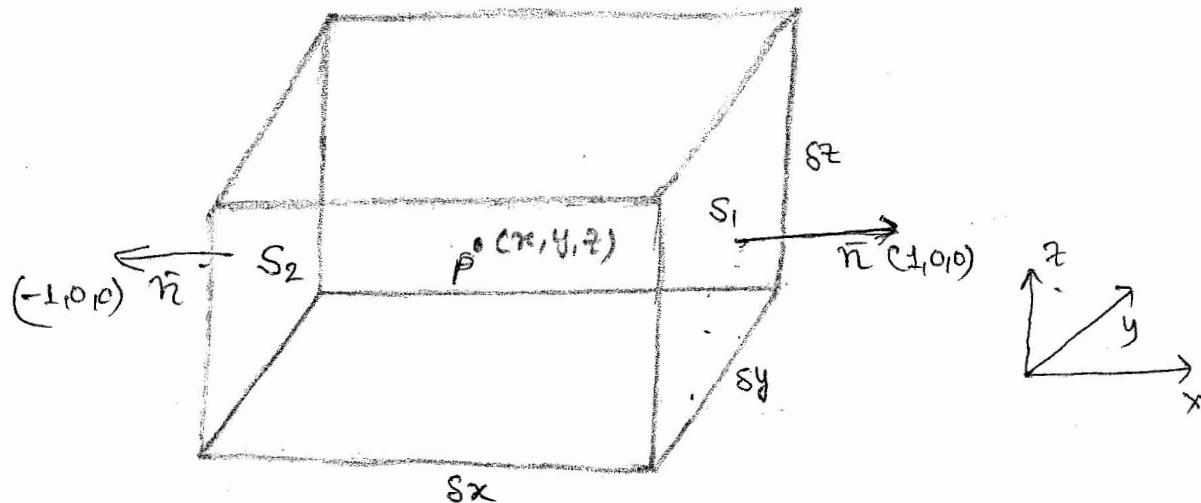
$$\operatorname{div} \vec{V} = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \iint_{\delta S} \vec{V} \cdot \vec{n} \, ds.$$

Flux of the vector field out
of a small closed
surface

where δV is a small volume enclosing P with surface δS and \vec{n} is the outward pointing normal to δS .



Divergence in terms of components:



$$\iint_{S_1} \vec{V} \cdot \vec{n} \, ds \approx u_1(x + \frac{\delta x}{2}, y, z) \, dy \, dz.$$

$$\iint_{S_2} \vec{V} \cdot \vec{n} \, ds \approx -u_1(x - \frac{\delta x}{2}, y, z) \, dy \, dz.$$

Adding the contribution from these two sides we get

$$\iint_{S_1+S_2} \bar{u} \cdot \bar{n} ds \approx \left(u_1(x + \frac{\delta x}{2}, y, z) - u_1(x - \frac{\delta x}{2}, y, z) \right) \delta y \delta z$$

$$\approx \frac{\partial u_1}{\partial x} \delta x \delta y \delta z$$

$$\approx \frac{\partial u_1}{\partial x} \delta v$$

Similarly from other sides.

$$\iint_{S_3+S_4} \bar{u} \cdot \bar{n} ds \approx \frac{\partial u_2}{\partial y} \delta v$$

$$\iint_{S_5+S_6} \bar{u} \cdot \bar{n} ds \approx \frac{\partial u_3}{\partial z} \delta v.$$

Therefore:

$$\boxed{\text{div } \bar{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}}$$

It can also be written as:

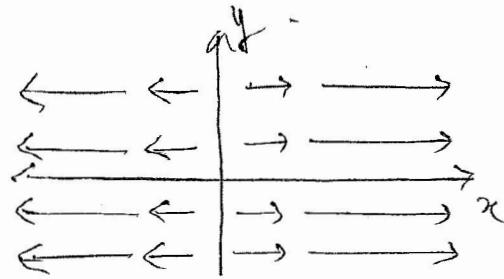
$$\boxed{\text{div } \bar{u} = \nabla \cdot \bar{u}}$$

Physical Interpretation :

Divergence can be interpreted as the rate of expansion or compression of the vector field.

(19)

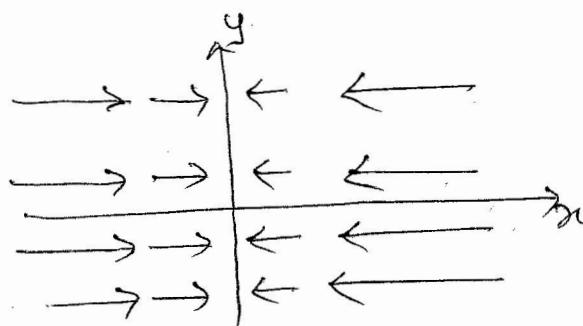
Examples: (1) $\vec{u} = (x, 0, 0)$



Tendency of the fluid is expansion.

$$\operatorname{div} \vec{u} = \nabla \cdot \vec{u} = 1. \text{ (positive)}$$

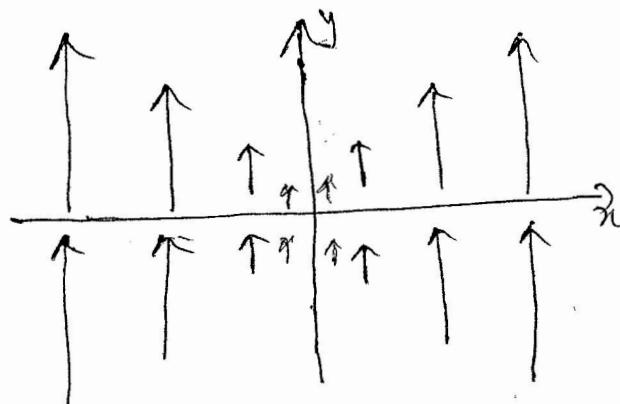
(2) $\vec{u} = (-x, 0, 0)$



Tendency of the fluid is compression.

$$\operatorname{div} \vec{u} = -1. \text{ (negative)}$$

(3) $\vec{u} = (0, x, 0)$



neither expanding
nor contracting.

$$\operatorname{div} \vec{u} = 0$$

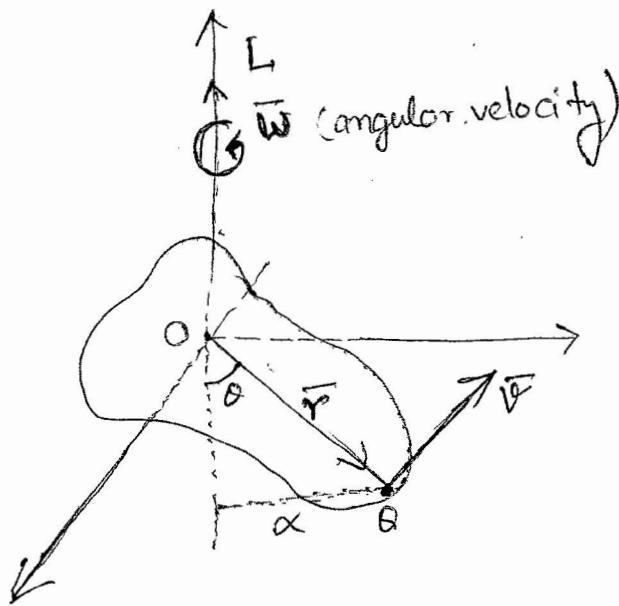
Note: A vector field \vec{V} for which $\nabla \cdot \vec{V} = 0$ everywhere is said to be solenoidal. The relation $\operatorname{div} \vec{v} = 0$ is also known as the condition of incompressibility.

* Curl of a vector field:

Curl of a differentiable vector function $\vec{V} = V_1 \vec{i} + V_2 \vec{j} + V_3 \vec{k}$ is given by

$$\begin{aligned}\text{Curl } \vec{V} &= \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} \\ &= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) \vec{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) \vec{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \vec{k}.\end{aligned}$$

Physical Interpretation:



$$\alpha = |\bar{\omega}| \sin \theta$$

$$|\bar{v}| = |\bar{\omega}| \alpha = |\bar{\omega}| |\bar{r}| \sin \theta = |\bar{\omega}| |\bar{r}| \sin \theta$$

$$\Rightarrow \bar{v} = \bar{\omega} \times \bar{r}$$

Also note that $\bar{\omega} = |\bar{\omega}| \bar{k}$ and $\bar{r} = x \bar{i} + y \bar{j} + z \bar{k}$

$$\text{Thus: } \bar{v} = \bar{\omega} \times \bar{r} = -|\bar{\omega}| y \bar{i} + |\bar{\omega}| x \bar{j}$$

$$\text{Also, } \text{Curl } \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -|\bar{\omega}| y & |\bar{\omega}| x & 0 \end{vmatrix} = 2|\bar{\omega}| \bar{k} = 2\bar{\omega}$$

Curl is directed along the axis of rotation with magnitude twice the angular speed.

Example: Given in the discussion of divergence.

(i) $\bar{u} = (x, 0, 0)$

$$\nabla \times \bar{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} = i(0) + j(0) + k(0) = 0.$$

No sense of rotation.

(ii) $\bar{u} = (-x, 0, 0)$ Again there is no sense of rotation.

(iii) $\bar{u} = (0, x, 0)$

$$\nabla \times \bar{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = i(0) + j(0) + k(1) = k$$

Since any point the velocity component in the y direction to the right of the particle is greater than that of the left, the rotation is about an axis in the z direction.

Note: A vector field \bar{u} for which $\nabla \times \bar{u} = 0$ everywhere is said to be irrotational.

Curl and Conservative vector field: Suppose \bar{u} is conservative.

that is, $\bar{u} = \nabla \psi = \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z} \right)$

$$\nabla \times \bar{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \end{vmatrix} = i \left[\frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial y} \right) \right] + j \left[\frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial z} \right) \right] + k \left[\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial x} \right) \right] = 0.$$

Any vector field that can be written as the gradient of a scalar field is irrotational.