Linear Algebra

Lecture 16

Inner product spaces,
orthogonality,

$\alpha_0 1 + \alpha_1 x + x^2 + \cdots + \alpha_n x^n$ $\qquad 2^n$

$\boxed{\cancel{x + \cdots + x^n}}$

$\boxed{1 + x + \cdots + x^n}$

$x + \cdots + x^{n+1}$

$$\left\{ \frac{1}{x-\alpha} : \alpha \in \mathbb{C} \right\}$$

$$\frac{\beta_1}{x-\alpha_1} + \frac{\beta_2}{x-\alpha_2} + \cdots + \frac{\beta_n}{x-\alpha_n} = 0$$

$$\beta_1 + \beta_2 \cancel{\frac{x-\alpha_1}{x-\alpha_2}} + \cdots + \beta_n \cancel{\frac{x-\alpha_1}{x-\alpha_n}} = 0$$

Definition: Inner product.

An inner product or dot product is on a vector space V (over $\mathbb{F}$) is a map $\langle \, , \rangle : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties:

For $x, y, z \in V$ and $\alpha \in \mathbb{R}$

i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

ii) $\langle x, y \rangle = \langle y, x \rangle$

iii) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$
$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

iv) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

$(V, \langle , \rangle)$ is called inner product space.

Example: Let $V = \mathbb{R}^n$, Define
$\langle x, y \rangle = x^T y$ for every $x, y \in \mathbb{R}^n$

Example: Let $A \in \mathbb{R}^{n \times n}$

Define $\langle \, , \, \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ as

$$\langle x, y \rangle = x^T A y$$

$$x^T A y \overset{??}{=} y^T A x$$

$$\| \qquad\qquad \|$$

$$\langle x, y \rangle \overset{??}{=} \langle y, x \rangle$$

$$\langle x, y+z \rangle = x^T A (y+z)$$

$$= x^T A y + x^T A z$$

$$= \langle x, y \rangle + \langle x, z \rangle$$

$\mathbb{R}^{2 \times 2}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad$ $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad$ $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\qquad$ $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle x, x \rangle$

$$= x^T A x = (x_1, x_2)\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 2x_1^2 + 2x_1x_2 + x_2^2$$

$\begin{pmatrix} + \\ + \end{pmatrix}, \begin{pmatrix} + \\ - \end{pmatrix}, \begin{pmatrix} - \\ + \end{pmatrix}, \begin{pmatrix} - \\ - \end{pmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = \begin{array}{c} \begin{matrix} f_{xx} & f_{xy} \end{matrix} \\ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \\ \begin{matrix} f_{xy} & f_{yy} \end{matrix} \end{array}$$

$$\boxed{\begin{array}{l} a > 0 \\ ac - b^2 > 0 \end{array}}$$

<u>Example:</u> Let $V = \mathbb{C}$ be a vector space over $\mathbb{R}$.

$$\langle z, w \rangle = Re(z\bar{w})$$

<u>Example:</u> Let $V = C[0,1]$, (the set of all continuous functions on $[0,1]$), over $\mathbb{R}$ be the given vector space.

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt \quad ; \forall f, g \in C[0,1]$$

$$\langle f, f \rangle = \int_0^1 [f(t)]^2\,dt \geq 0 \quad \checkmark$$

$$\langle f, f \rangle = 0 \iff f = 0$$

Let $f^2(t) = g(t)$ $\qquad \int_0^1 g(t)\,dt = 0$

$$\int_0^1 g(t)\,dt \geq \int_{t_0-\delta}^{t_0+\delta} g(t)\,dt \geq \int_{t_0-\delta}^{t_0+\delta} \frac{\varepsilon}{2}\,dt$$

$$= \varepsilon\delta$$

Example:

Let $V = M_{n\times n}(\mathbb{R})$ over $\mathbb{R}$.

$$\langle A, B\rangle = \operatorname{trace}(AB^T)$$

Exercise: Let V be an inner product space.

$$\langle ax+by, cv+dw\rangle = ac\langle x, v\rangle$$
$$+ ad\langle x,w\rangle + bc\langle y,v\rangle + bd\langle y,w\rangle$$

In particular,

$$\langle x+y, x-y\rangle = \langle x,x\rangle - \langle y,y\rangle$$
$$\langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$$

Definition (Norm)

Let V be an inner product space. The length or norm of a vector $v \in V$ is denoted as $\|v\|$ and is defined as the positive square root of $\langle v, v \rangle$.

$$\|v\| = \langle v, v \rangle^{1/2} = \sqrt{\langle v, v \rangle}$$

Lemma: Let V be an inner product space. The norm function $\|\cdot\| : V \to \mathbb{R}$, has following properties.

(1) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$

(2) $\|ax\| = |a| \, \|x\| \quad \forall \, x \in V, a \in \mathbb{R}$,

Furthermore, given a non-zero vector $v \in V$, there is a vector $u \in V$ such that $\|u\| = 1$ and $v = \|v\| u$.
u is called as a unit vector in the direction of v.

Cauchy-Schwarz Inequality.

Let V be an inner product space.

If $x, y \in V$, then

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|$$

Further, the equality holds if and only if one vector is a multiple of another vector.

Proof: Fix $x, y \in V$ and let $y \neq 0$.

$$f(t) = \langle x+ty, x+ty\rangle$$

$$= \langle x, x\rangle + 2t\langle x, y\rangle + t^2\langle y, y\rangle$$

$$f'(t) = 2\langle x, y\rangle + 2t\langle y, y\rangle$$

$$f'(t) = 0 \Rightarrow t = -\frac{\langle x, y\rangle}{\langle y, y\rangle} = t_0$$

is a point of extremum.

$$\Rightarrow \quad 0 \leq f(t_0) \leq f(t)$$

$$\Rightarrow \quad \langle x, x \rangle - 2\frac{(\langle x,y \rangle)^2}{\langle y,y \rangle} + \frac{(\langle x,y \rangle)^2}{\langle y,y \rangle} \geq 0$$

$$\Rightarrow \quad \langle x,x \rangle \langle y,y \rangle \geq (\langle x,y \rangle)^2$$

$$\Rightarrow \quad |\langle x,y \rangle| \leq \langle x,x \rangle^{1/2} \langle y,y \rangle^{1/2}$$

$$\Rightarrow \quad |\langle x,y \rangle| \leq \|x\| \, \|y\| \qquad \blacksquare$$

Corollary

$$\frac{|\langle x,y \rangle|}{\|x\| \, \|y\|} \leq 1$$

$$-1 \leq \frac{\langle x,y \rangle}{\|x\| \, \|y\|} \leq 1$$

$$\theta = \cos^{-1}\left(\frac{\langle x,y \rangle}{\|x\| \, \|y\|}\right)$$: angle between x & y

Particular case of C-S inequality.

Let $V = \mathbb{R}^n$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\left| \sum_{i=1}^{n} x_i y_i \right| \leq \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} y_i^2 \right)^{1/2}$$

$$\Rightarrow \left(\sum_{i=1}^{n} x_i y_i \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right)$$

<u>Corollarly</u>: Let V be an inner product space. Then the norm function $\|\cdot\| : V \longrightarrow \mathbb{R}$ has the following properties.

i) $\|x\| = 0 \iff x = 0 \quad \forall x \in V$

ii) $\|ax\| = |a| \, \|x\| \quad \forall x \in V, \; a \in \mathbb{R}$

iii) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$

Triangle inequality

Proof:

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$$
$$\underset{CS}{\leq} \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2$$

Rmk: Norm is a "metric/distance" on V.