Chapter 2

Linear Transformations and Matrices

2.1 Linear Transformations, Null Spaces, and Ranges

1. (a) Yes. That's the definition.
   (b) No. Consider a map f from $\mathbb{C}$ over $\mathbb{C}$ to $\mathbb{C}$ over $\mathbb{C}$ by letting $f(x+iy) = x$. Then we have $f(x_1 + iy_1 + x_2 + iy_2) = x_1 + x_2$ but $f(iy) = 0 \neq= if(y) = iy$.
   (c) No. This is right when T is a linear trasformation but not right in general. For example,

   $$\begin{aligned} T :& \mathbb{R} \to \mathbb{R} \\ & x \mapsto x + 1 \end{aligned}$$

   It's one-to-one but that $T(x) = 0$ means $x = -1$. For the counterexample of converse statement, consider $f(x) = |x|$.
   (d) Yes. We have $T(0_V) = T(0x) = 0T(0_V)_W = 0_W$, for arbitrary $x \in V$.
   (e) No. It is $\dim(V)$. For example, the transformation mapping the real line to $\{0\}$ will be.
   (f) No. We can map a vector to zero.
   (g) Yes. This is the Corollory after Theorem 2.6.
   (h) No. If $x_2 = 2x_1$, then $T(x_2)$ must be $2T(x_1) = 2y_1$.
2. It's a linear transformation since we have

$$T((a_1, a_2, a_3)+(b_1, b_2, b_3)) = T(a_1+b_1, a_2+b_2, a_3+b_3) = (a_1+b_1-a_2-b_2, 2a_3+2b_3)$$
$$= (a_1 - a_2, 2a_3) + (b_1 - b_2, 2b_3) = T(a_1, a_2, a_3) + T(b_1, b_2, b_3)$$

and

$$T(ca_1, ca_2, ca_3) = (c(a_1 - a_2), 2ca_3) = cT(a_1, a_2, a_3).$$

$N(T) = \{(a_1, a_1, 0)\}$ with basis $\{(1,1,0)\}$; $R(T) = \mathbb{R}^2$ with basis $\{(1,0),(0,1)\}$. Hence T is not one-to-one but onto.

3. Similarly check this is a linear transformation. $N(T) = \{0\}$ with basis $\varnothing$; $R(T) = \{a_1(1,0,2) + a_2(1,0,-1)\}$ with basis $\{(1,0,2),(1,0,-1)\}$. Hence T is one-to-one but not onto.

4. It's a linear transformation. And $N(T) = \{\begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\}$ with basis

$$\{\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

; $R(T) = \{\begin{pmatrix} s & t \\ 0 & 0 \end{pmatrix}\}$ with basis $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}$.

5. It's a linear transformation. And $N(T) = \{0\}$ with basis $\varnothing$; $R(T) = \{ax^3 + b(x^2+1) + cx\}$ with basis $\{x^3, x^2+1, x\}$. Hence T is one-to-one but not onto.

6. $N(T)$ is the set of all matrix with trace zero. Hence its basis is $\{E_{ij}\}_{i\neq j} \cup \{E_{ii} - E_{nn}\}_{i=1,2,\dots,n-1}$. $R(T) = \mathbb{F}$ with basis 1.

7. For property 1, we have $T(0) = T(0x) = 0T(x) = 0$,where x is an arbitrary element in V. For property 2, if T is linear, then $T(cx+y) = T(cx)+T(y) = cT(x)+T(y)$; if $T(cx+y) = cT(x)+T(y)$, then we may take $c = 1$ or $y = 0$ and conclude that T is linear. For property 3, just take $c = -1$ in property 3. For property 4, if T is linear, then

$$T(\sum_{i=1}^{n} a_i x_i) = T(a_1 x_1) + T(\sum_{i=1}^{n-1} a_i x_i) = \cdots = \sum_{i=1}^{n} T(a_i x_i) = \sum_{i=1}^{n} a_i T(x_i);$$

if the equation holds, just take $n = 2$ and $a_1 = 1$.

8. Just check the two condition of linear transformation.

9. (a) $T(0,0) \neq (0,0)$.
   (b) $T(2(0,1)) = (0,4) \neq 2T(0,1) = (0,2)$.
   (c) $T(2\frac{\pi}{2}, 0) = (0,0) \neq 2T(\frac{\pi}{2}, 0) = (2,0)$.
   (d) $T((1,0)+(-1,0)) = (0,0) \neq T(1,0) + T(0,1) = (2,0)$.
   (e) $T(0,0) \neq (0,0)$.

10. We may take $U(a,b) = a(1,4) + b(1,1)$. By Theorem 2.6, the mapping must be $T = U$. Hence we have $T(2,3) = (5,11)$ and T is one-to-one.

11. This is the result of Theorem 2.6 since $\{(1,1),(2,3)\}$ is a basis. And $T(8,11) = T(2(1,1)+3(2,3)) = 2T(1,1)+3T(2,3) = (5,-3,16)$.

12. No. We must have $T(-2,0,-6) = -2T(1,0,3) = (2,2) \neq (2,1)$.

13. Let $\sum_{i=0}^{k} a_i v_i = 0$. Then we have $T(\sum_{i=0}^{k} a_i v_i) = \sum_{i=0}^{k} a_i T(v_i) = 0$ and this implies $a_i = 0$ for all i.

14. (a) The sufficiency is due to that if $T(x) = 0$, $\{x\}$ can not be independent and hence $x = 0$. For the necessity, we may assume $\sum a_i T(v_i) = 0$. Thus we have $T(\sum a_i v_i) = 0$. But since T is one-to-one we have $\sum a_i v_i = 0$ and hence $a_i = 0$ for all proper i.

    (b) The sufficiency has been proven in Exercise 2.1.13. But note that S may be an infinite set. And the necessity has been proven in the previous exercise.

    (c) Since T is one-to-one, we have $T(\beta)$ is linear independent by the previous exercise. And since T is onto, we have $R(T) = W$ and hence $\text{span}(T(\beta)) = R(T) = W$.

15. We actually have $T(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1}$. Hence by detailed check we know it's one-to-one. But it's not onto since no function have integral$= 1$.

16. Similar to the previous exercise we have $T(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} i a_i x^{i-1}$. It's onto since $T(\sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1}) = \sum_{i=0}^{n} a_i x^i$. But it's not one-to-one since $T(1) = T(2) = 0$.

17. (a) Because $\text{rank}(T) \leq \dim(V) < \dim(W)$ by Dimension Theorem, we have $R(T) \subsetneq W$.

    (b) Because $\text{nullity}(T) = \dim(V) - \text{rank}(T) \geq \dim(V) - \dim(W) > 0$ by Dimension Theorem, we have $N(T) \neq \{0\}$.

18. Let $T(x,y) = (y,0)$. Then we have $N(T) = R(T) = \{(x,0) : x \in \mathbb{R}\}$.

19. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T(x,y) = (y,x)$ and U is the identity map from $\mathbb{R}^2 \to \mathbb{R}^2$. Then we have $N(T) = N(U) = \{0\}$ and $R(T) = R(U) = \mathbb{R}^2$.

20. To prove $A = T(V_1)$ is a subspace we can check first $T(0) = 0 \in A$. For $y_1, y_2 \in A$, we have for some $x_1, x_2 \in V_1$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$. Hence we have $T(x_1 + x_2) = y_1 + y_2$ and $T(cx_1) = xy_1$. This means both $y_1 + y_2$ and cy_1 are elements of A.

    To prove that $B = \{x \in V : T(x) \in W_1\}$ is a subspace we can check $T(0) = 0 \in W_1$ and hence $0 \in B$. For $x_1, x_2 \in B$, we have $T(x_1), T(x_2) \in W_1$. Hence we have $T(x_1 + x_2) = T(x_1), T(x_2) \in W_1$ and $T(cx_1) = cT(x_1) \in W_1$. This means both $x_1 + x_2$ and cx_1 are elements of B.

21. (a) To prove T is linear we can check

$$T(\sigma_1 + \sigma_2)(n) = \sigma_1(n+1) + \sigma_2(n+1) = T(\sigma_1)(n) + T(\sigma_2)(n)$$

and

$$T(c\sigma)(n) = c\sigma(n+1) = cT(\sigma)(n).$$

And it's similar to prove that U is linear.

(b) It's onto since for any σ in V. We may define another sequence τ such that $\tau(0) = 0$ and $\tau(n+1) = \sigma(n)$ for all $n \geq 1$. Then we have $T(\tau) = \sigma$. And it's not one-to-one since we can define a new σ_0 with $\sigma_0(0) = 1$ and $\sigma_0(n) = 0$ for all $n \geq 2$. Thus we have $\sigma_0 \neq 0$ but $T(\sigma_0) = 0$.

(c) If $T(\sigma)(n) = \sigma(n-1) = 0$ for all $n \geq 2$, we have $\sigma(n) = 0$ for all $n \geq 1$. And let σ_0 be the same sequence in the the previous exercise. We cannot find any sequence who maps to it.

22. Let $T(1,0,0) = a$, $T(0,1,0) = b$, and $T(0,0,1) = c$. Then we have

$$T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = ax + by + cz.$$

On the other hand, we have $T(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ if T is a mapping from $\mathbb{F}^n$ to $\mathbb{F}$. To prove this, just set $T(e_i) = a_i$, where $\{e_i\}$ is the standard of $\mathbb{F}^n$.

For the case that $T : \mathbb{F}^n \to \mathbb{F}$, actually we have

$$T(x_1, x_2, \ldots, x_n) = (\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=2}^{n} a_{2j}x_j, \ldots, \sum_{j=m}^{n} a_{mj}x_j)$$

. To prove this, we may set $T(e_j) = (a_{1j}, a_{2j}, \ldots, a_{mj})$.

23. With the help of the previous exercise, we have

$$N(T) = \{(x,y,z) : ax + by + cz = 0\}.$$

Hence it's a plane.

24. (a) It will be $T(a,b) = (0,b)$, since $(a,b) = (0,b) + (a,0)$.

(b) It will be $T(a,b) = (0,b-a)$,since $(a,b) = (0,b-a) + (a,a)$.

25. (a) Let W_1 be the xy-plane and W_2 be the z-axis. And $(a,b,c) = (a,b,0) + (0,0,c)$ would be the unique representation of $W_1 \oplus W_2$.

(b) Since $(a,b,c) = (0,0,c) + (a,b,0)$, we have $T(a,b,c) = (0,0,c)$.

(c) Since $(a,b,c) = (a-c,b,0) + (c,0,c)$, we have $T(a,b,c) = (a-c,b,0)$.

26. (a) Since $V = W_1 \oplus W_2$, every vector x have an unique representation $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$. So, now we have

$$T(x + cy) = T(x_1 + x_2 + cy_1 + cy_2)$$

$$= T((x_1 + cy_1) + (x_2 + cy_2)) = x_1 + cy_1 = T(x) + cT(y).$$

And hence it's linear.

On the other hand, we have $x = x + 0$ and hence $T(x) = x$ if $x \in W_1$. And if $x \notin W_1$, this means $x = x_1 + x_2$ with $x_2 \neq 0$ and hence we have $T(x) = x_1 \neq x_1 + x_2$.

(b) If $x_1 \in W_1$ then we have $T(x_1 + 0) = x_1 \in R(T)$; and we also have $R(T) \subset W_1$. If $x_2 \in W_2$ then we have $T(x_2) = T(0 + x_2) = 0$ and hence $x_2 \in N(T)$; and if $x \in N(T)$, we have $x = T(x) + x = 0 + x$ and hence $x \in W_2$.

(c) It would be $T(x) = x$ by (a).

(d) It would be $T(x) = 0$.

27. (a) Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for W and we can extend it to a basis $\beta = \{v_1, v_2, \ldots, v_n\}$ of V. Then we may set W' =span($\{v_{k+1}, v_{k+2}, \ldots, v_n\}$). Thus we have $V = W \oplus W'$ and we can define T be the projection on W along W'.

(b) The two projection in Exercise 2.1.24 would be the example.

28. We have $T(0) = 0 \in \{0\}$, $T(x) \in R(T)$, $T(x) = 0 \in N(T)$ if $x \in N(T)$ and hence they are T-invariant.

29. For $x, y \in W$, we have $x + cy \in W$ since it's a subspace and $T(x), T(y) \in W$ since it's T-invariant and finally $T(x + cy) = T(x) + cT(y)$.

30. Since $T(x) \in W$ for all x, we have W is T-invariant. And that $T_W = I_W$ is due to Exercise 2.1.26(a).

31. (a) If $x \in W$, we have $T(x) \in R(T)$ and $T(x) \in W$ since W is T-invariant. But by the definition of direct sum, we have $T(x) \in R(T) \cap W = \{0\}$ and hence $T(x) = 0$.

(b) By Dimension Theorem we have dim($N(T)$) =dim(V)−dim($R(T)$). And since $V = R(T) \oplus W$, we have dim$W$ =dim(V)−dim($R(T)$). In addition with $W \subset N(T)$ we can say that $W = N(T)$.

(c) Take T be the mapping in Exercise 2.1.21 and $W = \{0\}$. Thus $W \neq N(T) = \{(a_1, 0, 0, \ldots)\}$.

32. We have $N(T_W) \subset W$ since T_W is a mapping from W to W. For $x \in W$ and $x \in N(T_W)$, we have $T_W(x) = T(x) = 0$ and hence $x \in N(T)$. For the converse, if $x \in N(T) \cap W$, we have $x \in W$ and hence $T_W(x) = T(x) = 0$. So we've proven the first statement. For the second statement, we have

$$R(T_W) = \{y \in W : T_W(x) = y, x \in W\} = \{T_W(x) : x \in W\} = \{T(x) : x \in W\}.$$

33. It's natural that $R(T) \supset$span($\{T(v) : v \in \beta\}$) since all $T(v)$ is in $R(T)$. And for any $y \in R(T)$ we have $y = T(x)$ for some x. But every x is linear

combination of finite many vectors in basis. That is, $x = \sum_{i=1}^{k} a_i v_i$ for some $v_i \in \beta$. So we have

$$y = T(\sum_{i=1}^{k} a_i v_i) = \sum_{i=1}^{k} a_i T(v_i)$$

is an element in $\operatorname{span}(\{T(v) : v \in \beta\})$.

34. Let T be one linear transformation satisfying all conditions. We have that for any $x \in V$ we can write $x = \sum_{i=1}^{k} a_i v_i$ for some $v_i \in \beta$. So by the definition of linear transformation we have $T(x)$ must be

$$T(\sum_{i=1}^{k} a_i v_i) = \sum_{i=1}^{k} a_i T(v_i) = \sum_{i=1}^{k} a_i f(v_i)$$

,a fixed vector in W. So it would be the unique linear transformation.

35. (a) With the hypothesis $V = R(T) + N(T)$, it's sufficient to say that $R(T) \cap N(T) = \{0\}$. But this is easy since

$$\begin{aligned}\dim(R(T) \cap N(T)) &= \dim(R(T)) + \dim(N(T)) - \dim(R(T) + N(T)) \\ &= \dim(R(T)) + \dim(N(T)) - \dim(V) = 0\end{aligned}$$

by Dimension Theorem.

(b) Similarly we have

$$\begin{aligned}\dim(R(T) + N(T)) &= \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T)) \\ &= \dim(R(T)) + \dim(N(T)) - \dim(\{0\}) = \dim(V)\end{aligned}$$

by Dimension Theorem. So we have $V = R(T) + N(T)$.

36. (a) In this case we have $R(T) = V$ and $N(T) = \{(a_1, 0, 0, \ldots)\}$. So naturally we have $V = R(T) + N(T)$. But V is a direct sum of them since $R(T) \cap N(T) = N(T) \neq \{0\}$.

(b) Take $T_1 = U$ in the Exercise 2.1.21. Thus we have $R(T_1) = \{(0, a_1, a_2, \ldots)\}$ and $N(T_1) = \{(0, 0, \ldots)\}$. So we have $R(T_1) \cap N(T_1) = \{0\}$ but $R(T_1) + N(T_1) = R(T_1) \neq V$.

37. Let $c = \frac{a}{b} \in \mathbb{Q}$. We have that

$$T(x) = T(\underbrace{\frac{1}{b}x + \frac{1}{b}x + \cdots + \frac{1}{b}x}_{\text{b times}}) = bT(\frac{1}{b}x)$$

and hence $T(\frac{1}{b}x) = \frac{1}{b}T(x)$. So finally we have

$$\begin{aligned}T(cx) = T(\frac{a}{b}x) &= T(\underbrace{\frac{1}{b}x + \frac{1}{b}x + \cdots + \frac{1}{b}x}_{\text{a times}}) \\ &= aT(\frac{1}{b}x) = \frac{a}{b}T(x) = cT(x).\end{aligned}$$

38. It's additive since

$$T((x_1+iy_1)+(x_2+iy_2)) = (x_1+x_2)-i(y_1+y_2) = T(x_1+iy_1)+T(x_2+iy_2).$$

But it's not linear since $T(i) = -i \neq iT(1) = 0$.

39. It has been proven in the Hint.

40. (a) It's linear since

$$\eta(u+v) = (u+v)+W = (u+W)+(v+W) = \eta(u)+\eta(v)$$

and

$$\eta(cv) = cv+W = c(v+W) = c\eta(v)$$

by the definition in Exercise 1.3.31. And for all element $v+W$ in V/W we have $\eta(v) = v+W$ and hence it's onto. Finally if $\eta(v) = v+W = 0+W$ we have $v-0 = v \in W$. Hence $N(\eta) = W$.

(b) Since it's onto we have $R(T) = V/W$. And we also have $N(\eta) = W$. So by Dimension Theorem we have $\dim(V)$ =$\dim(V/W)$+$\dim(W)$.

(c) They are almost the same but the proof in Exercise 1.6.35 is a special case of proof in Dimension Theorem.

2.2 The Matrix Representation of a Linear Transformation

1. (a) Yes. This is result of Theorem 2.7.

(b) Yes. This is result of Theorem 2.6.

(c) No. It's a $n \times m$ matrix.

(d) Yes. This is Theorem 2.8.

(e) Yes. This is Theorem 2.7.

(f) No. A transformaion of $\mathcal{L}(V,W)$ can not map element in W in general.

2. (a) We have $T(1,0) = (2,3,1) = 2(1,0,0)+3(0,1,0)+1(0,0,1)$ and $T(0,1) = (-1,4,0) = -1(1,0,0)+4(0,1,0)+0(0,0,1)$. Hence we get

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}.$$

(b)

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

(c)

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 1 & -3 \end{pmatrix}.$$

(d)

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{pmatrix}.$$

(e)

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

(f)

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.$$

(g)

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

3. Since

$$\begin{aligned} T(1,0) &= (1,1,2) = -\frac{1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3) \\ T(0,1) &= (-1,0,1) = -1(1,1,0) + 1(0,1,1) + 0(2,2,3) \\ T(1,2) &= (-1,1,4) = -\frac{7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3) \\ T(2,3) &= (-1,2,7) = -\frac{11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3) \end{aligned}$$

we have

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$$

and

$$[T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}.$$

4. Since

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 + 0x + 0x^2$$
$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + 0x + 1x^2$$
$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 + 0x + 0x^2$$
$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 + 2x + 0x^2$$

we have

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

5. (a)

$$[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b)

$$[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(c)

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}.$$

(d)

$$[A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}.$$

(e)

$$[f(x)]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}.$$

(f)

$$[a]_\gamma = \begin{pmatrix} a \end{pmatrix}.$$

6. It would be a vector space since all the condition would be true since they are true in V and W. So just check it.

7. If we have $([T]_\beta^\gamma)_{ij} = A_{ij}$, this means

$$T(v_j) = \sum_{i=1}^{m} A_{ij} w_i.$$

And hence we have

$$aT(v_j) = \sum_{i=1}^{m} aA_{ij} w_i.$$

and thus $(a[T]_\beta^\gamma)_{ij} = aA_{ij}$.

8. If $\beta = \{v_1, v_2, \ldots, v_n\}$ and $x = \sum_{i=1}^n a_i v_i$ and $y = \sum_{i=1}^n b_i v_i$, then

$$T(x+cy) = T(\sum_{i=1}^{n} a_i v_i + c\sum_{i=1}^{n} b_i v_i) = (a_1+cb_1, a_2+cb_2, \ldots, a_n+cb_n) = T(x)+cT(y).$$

9. It would be linear since for $c \in \mathbb{R}$ we have

$$T((x_1+iy_1)+c(x_2+iy_2)) = (x_1+cx_2)+i(x_2+cy_2) = T(x_1+iy_1)+cT(x_2+iy_2).$$

And the matrix would be

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

10. It would be

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

11. Take $\{v_1, v_2, \ldots, v_k\}$ be a basis of W and extend it to be $\beta = \{v_1, v_2, \ldots, v_n\}$, the basis of V. Since W is T-invariant, we have $T(v_j) = \sum_{i=1}^k a_{ij} v_i$ if $j = 1, 2, \ldots, k$. This means $([T]_\beta)_{ij} = 0$ if $j = 1, 2, \ldots, k$ and $i = k+1, k+2, \ldots, n$.

12. Let $\{v_1, v_2, \ldots, v_k\}$ and $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ be the basis of W and W' respectly. By Exercise 1.6.33(b), we have $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis of V. And thus we have

$$[T]_\beta = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}$$

is a diagonal matrix.

13. Suppose, by contradiction, that $cT = U$ for some c. Since T is not zero mapping, there is some $x \in V$ and some nonzero vector $y \in W$ such that $T(x) = y \neq 0$. But thus we have $y = \frac{1}{c}cy = \frac{1}{c}U(x) = U(\frac{1}{c}x) \in R(U)$. This means $y \in R(T) \cap R(U)$, a contradiction.

14. It can be checked that differentiation is a linear operator. That is, T_i is an element of $\mathcal{L}(V)$ for all i. Now fix some n, and assume $\sum_{i=1}^n a_i T_i = 0$. We have $T_i(x^n) = \frac{n!}{(n-i)!}x^{n-i}$ and thus $\{T_i(x^n)\}_{i=1,2,\ldots,n}$ would be an independent set. Since $\sum_{i=1}^n a_i T_i(x^n) = 0$, we have $a_i = 0$ for all i.

15. (a) We have zero map is an element in S^0. And for $T, U \in S^0$, we have $(T + cU)(x) = T(x) + cU(x) = 0$ if $x \in S$.
    (b) Let T be an element of S_2^0. We have $T(x) = 0$ if $x \in S_1 \subset S_2$ and hence T is an element of S_1^0.
    (c) Since $V_1 + V_2$ contains both V_1 and V_2, we have $(V_1 + V_2)^0 \subset V_1^0 \cap V_2^0$ by the previous exercise. To prove the converse direction, we may assume that $T \in V_1^0 \cap V_2^0$. Thus we have $T(x) = 0$ if $x \in V_1$ or $x \in V_2$. For $z = u + v \in V_1 + V_2$ with $u \in V_1$ and $v \in V_2$, we have $T(z) = T(u) + T(v) = 0 + 0 = 0$. So T is an element of $(V_1 + V_2)^0$ and hence we have $(V_1 + V_2)^0 \supset V_1^0 \cap V_2^0$.

16. As the process in the proof of Dimension Theorem, we may pick the same basis $\beta = \{v_1, v_2, \ldots v_n\}$ for V. Write $u_{k+1} = T(v_{k+1}$. It has been proven that $\{u_{k+1}, u_{k+2}, \ldots, u_n\}$ is a basis for $R(T)$. Since $\dim(V)$ =$\dim(W)$, we can extend to a basis $\gamma = \{u_1, u_2, \ldots, u_n\}$ for W. Thus we have

$$[T]_\beta^\gamma = \begin{pmatrix} O & O \\ O & I_{n-k} \end{pmatrix}$$

is a diagonal matrix.

2.3 Composition of Linear Transformations and Matrix Multiplication

1. (a) It should be $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$.
   (b) Yes. That's Theorem 2.14.
   (c) No. In general β is not a basis for V.
   (d) Yes. That's Theorem 2.12.
   (e) No. It will be true when $\beta = \alpha$.
   (f) No. We have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I$.
   (g) No. T is a transformation from V to W but L_A can only be a transformation from $\mathbb{F}^m$ to $\mathbb{F}^n$.
   (h) No. We have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = I$.
   (i) Yes. That's Theorem 2.15.
   (j) Yes. Since $\delta_{ij} = 1$ only when $i = j$, we have $A_{ij} = \delta_i j$.

2. (a)

$$A(2B+3C) = \begin{pmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{pmatrix}.$$

$$(AB)D = A(BD) = \begin{pmatrix} 29 \\ -26 \end{pmatrix}.$$

(b)

$$A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}.$$

$$A^tB = \begin{pmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{pmatrix}.$$

$$BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}.$$

$$CB = \begin{pmatrix} 27 & 7 & 9 \end{pmatrix}.$$

$$CA = \begin{pmatrix} 20 & 26 \end{pmatrix}.$$

3. (a) We can calculate that $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ and $[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$ and finally

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}.$$

(b) We can calculate $[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ and

$$[U(h(x))]_\beta = [U]_\beta^\gamma[h(x)]_\beta \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}.$$

4. (a) $[T(A)]_\alpha = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix}.$

(b) $[T(f(x))]_\alpha = \begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}.$

(c) $[T(A)]_\gamma = \begin{pmatrix} 5 \end{pmatrix}.$

(d) $[T(f(x))]_\gamma = \begin{pmatrix} 12 \end{pmatrix}$.

5. **(b)** We have

$$\begin{aligned}(a(AB))_{ij} &= a\sum_{k=1}^{n} A_{ik}B_{kj} \\ &\sum_{k=1}^{n} aA_{ik}B_{kj} = ((aA)B)_{ij} \\ &\sum_{k=1}^{n} A_{ik}aB_{kj} = (A(aB))_{ij}\end{aligned}$$

(d) We have $[I(v_i)]_\alpha = e_i$ where v_i is the i-th vector of β.

Corollary. We have by Theorem 2.12

$$A(\sum_{i=1}^{k} a_iB_i) = \sum_{i=1}^{k} A(a_iB_i) = \sum_{i=1}^{k} a_iAB_i$$

and

$$(\sum_{i=1}^{k} a_iC_i)A = \sum_{i=1}^{k} (a_iC_i)A = \sum_{i=1}^{k} a_iC_iA.$$

6. We have $(Be_j)_i = \sum_{k=1}^{n} B_{ik}(e_j)_i = B_{ij}$, since $(e_j)_i = 1$ only when $i = j$ and it's 0 otherwise.

7. **(c)** Just check that for all vector $v \in \mathbb{F}^n$ we have $L_{A+B}(v) = (A+B)v = Av + Bv = L_A(v) + L_B(v)$ and $L_{aA}(v) = aA(v) = aL_A(v)$.

   (f) For all vector $v \in \mathbb{F}^n$ we have $L_{I_n}(v) = I_n(v) = v$.

8. In general we may set $T_1, T_2 \in \mathcal{L}(X,Y)$ and $U_1, U_2 \in \mathcal{L}(W,X)$, and $S \in (V,W)$, and thus we have the following statements.

   (a) $T_1(U_1 + U_2) = TU_1 + TU_2$ and $(U_1 + U_2)T = U_1T + U_2T$.

   (b) $T_1(U_1S) = (T_1U_1)S$.

   (c) $TI_X = I_YT = T$.

   (d) $a(T_1U_1) = (aT_1)U_1 = T_1(aU_1)$ for all scalars a.

   To prove this, just map arbitrary vector in domain by linear transformations and check whether the vectors producted by different transformations meet.

9. Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $U = L_A$ and $T = L_B$.

10. If A is a diagonal matrix then $A_{ij} \neq 0$ only when $i = j$. Hence we have $A_{ij} = \delta_{ij}A_{ij}$. Conversely, if A is not diagonal, we can find $A_{ij} \neq 0$ for some i, j and $i \neq j$. Thus we have $\delta_{ij}A_{ij} = 0 \neq A_{ij}$.

11. If $T^2 = T$ we may pick $y \in R(T)$ and thus we have $y = T(x)$ for some x and $T(y) = T(T(x)) = T^2(x) = 0$. Hence we conclude that $y \in N(T)$. Conversely if we have $R(T) \subset N(T)$, we have $T^2(x) = T(T(x)) = 0$ since $T(x)$ is an element in $R(T)$ and hence in $N(T)$.

12. (a) If UT is injective, we have that $UT(x) = 0$ implies $x = 0$. Thus we have that if $T(x) = 0$ we also have $UT(x) = 0$ and hence $x = 0$. So T is injective. But U may not be injective. For example, pick $U(x, y, z) = (x, y)$, a mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$, and $T(x, y) = (x, y, 0)$, a mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$.

    (b) If UT is surjective, we have that for all $z \in Z$ there is a vector $x \in V$ such that $UT(x) = z$. Thus we have that if for all $z \in Z$ we have $z = U(T(x))$ and hence U is surjective. But T may not surjective. The example in the previous question could also be the example here.

    (c) For all $z \in Z$, we can find $z = U(y)$ for some $y \in W$ since U is surjective and then find $y = T(x)$ for some $x \in V$ since T is surjective. Thus we have $z = UT(x)$ for some x and hence UT is surjective. On the other hand, if $UT(x) = 0$, this means $T(x) = 0$ since U is injective and $x = 0$ since T is injective.

13. It's natural that we have tr(A) =tr(A^t) since $A_{ii} = A^t_{ii}$ for all i. On the other hand we have

$$\begin{aligned}\operatorname{tr}(AB) &= \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n}A_{ik}B_{ki}\\ &= \sum_{k=1}^{n}\sum_{ik=1}^{n}B_{ki}A_{ik} = \sum_{k=1}^{n}(BA)_{kk}\\ &= \operatorname{tr}(BA).\end{aligned}$$

14. (a) We can write

$$Bz = \begin{pmatrix} \sum_{j=1}^{p} a_j B_{1j} \\ \sum_{j=2}^{p} a_j B_{2j} \\ \vdots \\ \sum_{j=1}^{p} a_j B_{nj} \end{pmatrix}$$

$$= \sum_{j=1}^{p} a_j \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix} = \sum_{j=1}^{p} a_j v_j.$$

    (b) This is the result of Theorem 2.13(b) and the previous exercise.

    (c) This is instant result of the fact that $wA = A^t w^t$.

    (d) This is also because $AB = B^t A^t$.

15. Let v_t be the t-th column vector of A and we have $v_j = \sum_{t \neq j} a_t v_t$. Thus we have $Mv_j = \sum_{t \neq j} a_t M v_t$. And hence we get the desired result since Mv_t is the column vector of MA.

16. (a) Since we know $R(T)$ is a T-invariant space, we can view T as a mapping from $R(T)$ to $R(T)$ and call this restricted mapping $T|_{R(T)}$. So now we have that

$$\dim R(T) = \mathrm{rank}(T) = \mathrm{rank}(T^2) = \dim(T(T(V))$$

$$= \dim(T(R(T)) = \mathrm{rank}(T|_{R(T)}).$$

And so the mapping $T|_{R(T)}$ is surjective and hence injective with the help of the fact $R(T)$ is finite dimensional. This also means $N(T|_{R(T)} = R(T) \cap N(T) = 0$. This complete the proof of the first statement. For the other, it's sufficient to say that $R(T)+N(T) = V$. But this is instant conclusion of the fact that $R(T) + N(T) \subset V$ and that

$$\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T))$$

$$= \dim(R(T)) + \dim(N(T)) = \dim(V).$$

(b) In general we have $\mathrm{rank}(T^{s+1}) \leq \mathrm{rank}(T^s)$ since the fact $T^{s+1}(V) = T^s(R(T)) \subset T^s(V)$. But the integer $\mathrm{rank}(T^s)$ can only range from 0 to $\dim(V)$. So there must be some integer k such that

$$\mathrm{rank}(T^k) = \mathrm{rank}(T^{k+1}).$$

And this means $T^{k+1}(V) = T^k(V)$ and hence $T^s(V) = T^k(V)$ for all $s \geq k$. Since $2k \geq k$, we can conclude that $\mathrm{rank}(T^k) = \mathrm{rank}(T^{2k})$ and hence we have $V = R(T^k) \oplus N(T^k)$ by the previous exercise.

17. If $T = T^2$, then we have $V = \{y : T(y) = y\} + N(T)$ since $x = T(x) + (x - T(x))$ and we have $T(T(x)) = T(x)$ and $T(x - T(x)) = T(x) - T(x) = 0$. On the other hand, we have if $x \in \{y : T(y) = y\} \cap N(T)$ then we also have $x = T(x) = 0$. So by arguments above we have $V = \{y : T(y) = y\} \oplus N(T)$. Finally we have that T must be the projection on W_1 along W_2 for some W_1 and W_2 such that $W_1 \oplus W_2 = V$.

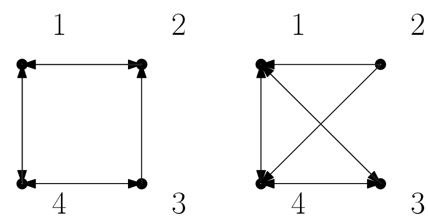
18. Let A, B, and C be $m \times n$, $n \times p$, and $p \times q$ matrices respectly. Next we want to claim that $(AB)C = A(BC)$ since

$$\begin{aligned}((AB)C)_{ij} &= \sum_{k=1}^{p} (AB)_{ik} C_{kj} = \sum_{k=1}^{p} (\sum_{l=1}^{n} A_{il} B_{lk}) C_{kj} \\ &= \sum_{k=1}^{p} \sum_{l=1}^{n} A_{il} B_{lk} C_{kj} = \sum_{l=1}^{n} \sum_{k=1}^{p} A_{il} B_{lk} C_{kj} \\ &= \sum_{l=1}^{n} A_{il} (\sum_{k=1}^{p} B_{lk} C_{kj}) = \sum_{l=1}^{n} A_{il} (BC)_{lj} = (A(BC))_{ij}.\end{aligned}$$

For the following questions, I would like to prove them in the languague of Graph Theory. So there are some definitions and details in Appendices.

19. Let $G = \mathcal{G}(B)$ be the graph associated to the symmetric matric B. And $(B^3)_{ii}$ is the number of walk of length 3 from i to i. If i is in some clique, then there must be a walk of length 3 from i back to i since a clique must have number of vertex greater than 3. Conversely, if $(B^3)_{ii}$ is greater than zero, this means there is at least one walke of length 3 from i to i, say $i \to j \to k \to i$. Note that i, j, and k should be different vertices since length is 3 and there is no loop. So i, j, and k must be a triangle, this means three vertices adjacent to each others. So i is contained in $\{i, j, k\}$ and so contained in some clique.

20. We can draw the associated digraph and find the cliques as follow:



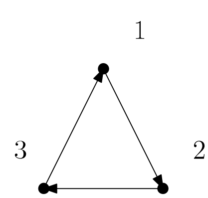
(a) (b)

    (a) There is no clique.
    (b) The only clique would be the set $\{1, 3, 4\}$.

21. A vertex v in a tournament is called a **king** if v can reach all other vertices within two steps. That is, for all vertex u other than v, we have either $v \to u$ or $v \to w \to u$ for some w. So $(A + A^2)_{ij} > 0$ is equivalent to that i can reach j within two steps. And the statement of this question also means that every tournament exists a king.

    To prove this statement, we can begin by arbitrary vertex v_1. If v_1 is a king, then we've done. If v_1 is not a king, this means v_1 can not reach some vertex, say v_2, within two steps. Now we have that $d^+(v_2) > d^+(v_1)$ since we have $v_2 \to v_1$ and that if $v_1 \to w$ for some w then we have $v_2 \to w$ otherwise we'll have that $v_1 \to w \to v_2$. Continuing this process and we can find $d^+(v_1) < d^+(v_2) < \cdots$ and terminate at some vertex v_k since there are only finite vertces. And so v_k would be a king.

22. We have $G = \mathcal{G}(A)$ is a tournament drawn below. And every vertex in this tournament could be a king.



23. The number of nonzero entries would be the number of the edges in a tournament. So it would be $n(n-1)/2$.

2.4 Invertibility and Isomorphisms

1. (a) No. It should be $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\beta^\alpha$.
   (b) Yes. See Appendix B.
   (c) No. L_A can only map $\mathbb{F}^n$ to $\mathbb{F}^m$.
   (d) No. It isomorphic to $\mathbb{F}^5$.
   (e) Yes. This is because $P_n(\mathbb{F}) \cong \mathbb{F}^n$.
   (f) No. We have that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = I$ but A and B are not invertible since they are not square.
   (g) Yes. Since we have both A and $(A^{-1})^{-1}$ are the inverse of A^{-1}, by the uniqueness of inverse we can conclude that they are the same.
   (h) Yes. We have that $L_{A^{-1}}$ would be the inverse of L_A.
   (i) Yes. This is the definition.

2. (a) No. They have different dimension 2 and 3.
   (b) No. They have different dimension 2 and 3.
   (c) Yes. $T^{-1}(a_1, a_2, a_3) = (-\frac{4}{3}a_2 + \frac{1}{3}a_3, a_2, -\frac{1}{2}a_1 - 2a_2 + \frac{1}{2}a_3)$.
   (d) No. They have different dimension 4 and 3.
   (e) No. They have different dimension 4 and 3.
   (f) Yes. $T^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & a-b \\ c & d-c \end{pmatrix}$.

3. (a) No. They have different dimension 3 and 4.
   (b) Yes. They have the same dimension 4.
   (c) Yes. They have the same dimension 4.
   (d) No. They have different dimension 3 and 4.

4. This is because that $(B^{-1}A^{-1})(AB) = (AB)(B^{-1}A^{-1}) = I$.

5. This is because that $(A^{-1})^t A^t = (AA^{-1})^t = I$ and $A^t(A^{-1})^t = (A^{-1}A)^t = I$.

6. If A is invertible, then A^{-1} exists. So we have $B = A^{-1}AB = A^{-1}O = O/$

7. (a) With the result of the previous exercise, if A is invertible we have that $A = O$. But O is not invertible. So this is a contradiction.
   (b) No. If A is invertible then $B = O$ by the previous exercise.

8. For Corollary 1 we may just pick $W = V$ and $\alpha = \beta$. For Corollary 2 we may just pick $V = \mathbb{F}^n$ and use Corollary 1.

9. If AB is invertible then L_{AB} is invertible. So $L_A L_B = L_{AB}$ is surjective and injective. And thus L_A is injective and L_B surjective by Exercise 2.3.12. But since their domain and codomain has the same dimension, actually they are both invertible, so are A and B.

10. (a) Since $AB = I_n$ is invertible, we have A and B is invertible by the previous exercise.

    (b) We have that $AB = I_n$ and A is invertible. So we can conclude that

    $$A^{-1} = A^{-1}I_n = A^{-1}AB = B.$$

    item Let T is a mapping from V to W and U is a mapping from W to V with dim$W$ =dim$V$. If TU be the identity mapping, then both T and U are invertible. Furthermore $T^{-1} = U$.

    To prove this we may pick bases α of V and β of W and set $A = [T]_\alpha^\beta$ and $B = [U]_\beta^\alpha$. Now apply the above arguments we have that A and B is invertible, so are T and U by Theorem 2.18.

11. If $T(f) = 0$ then we have that $f(1) = f(2) = f(3) = f(4) = 0$, then we have that f is zero function since it has degree at most 3 and it's impossible to have four zeroes if f is nonzero.

12. We can check ϕ_β is linear first. For $x = \sum_{i=1}^n a_i v_i$ and $y = \sum_{i=1}^n b_i v_i$, where

    $$\beta = \{v_1, v_2, \dots, v_n\}$$

    we have that

    $$\phi_\beta(x + cy) = \begin{pmatrix} a_1 + cb_1 \\ a_2 + cb_2 \\ \vdots \\ a_n + cb_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + c \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \phi_\beta(x) + c\phi_\beta(y).$$

    And we can check whether it is injective and surjective. If $\phi_\beta(x) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ then this means $x = \sum_{i=1}^n 0v_i = 0$. And for every $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + c$ in $\mathbb{F}^n$, we have that $x = \sum_{i=1}^n a_i v_i$ will be associated to it.

13. First we have that V is isomorphic to V by identity mapping. If V is isomorphic to W by mapping T, then T^{-1} exist by the definition of isomorphic and W is isomorphic to V by T^{-1}. If V is isomorphic to W by mapping T and W is isomorphic to X by mapping U, then V is isomorphic to X by mapping UT.

14. Let

$$\beta = \{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$$

be the basis of V. Then we have that ϕ_β in Theorem 2.21 would be the isomorphism.

15. We have that T is isomorphism if and only if that T is injective and surjective. And we also have that the later statement is equivalent to $T(\beta)$ is a baiss for W by Exercise 2.1.14(c).

16. We can check that Φ is linear since

$$\Phi(A + cD) = B^{-1}(A + cD)B = B^{-1}(AB + cDB)$$

$$= B^{-1}AB + cB^{-1}DB = \Phi(A) + c\Phi(D).$$

And it's injective since if $\Phi(A) = B^{-1}AB = O$ then we have $A = BOB^{-1} = O$. It's also be surjective since for each D we have that $\Phi(BDB^{-1}) = D$.

17. (a) If $y_1, y_2 \in T(V_0)$ and $y_1 = T(x_1)$, $y_2 = T(x_2)$, we have that $y_1 + y_2 = T(x_1 + x_2) \in T(V_0)$ and $cy_1 = T(cx_1) = T(V_0)$. Finally since V_0 is a subspace and so $0 = T(0) \in T(V_0)$, $T(V_0)$ is a subspace of W.

    (b) We can consider a mapping T' from V_0 to $T(V_0)$ by $T'(x) = T(x)$ for all $x \in V_0$. It's natural that T' is surjective. And it's also injective since T is injective. So by Dimension Theorem we have that

$$\dim(V_0) = \dim(N(T')) + \dim(R(T')) = \dim(T(V_0)).$$

18. With the same notation we have that

$$L_A\phi_\beta(p(x)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

and

$$\phi_\gamma T(p(x)) = \phi_\gamma(1 + 4x + 3x^2) = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

So they are the same.

19. (a) It would be

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) We may check that

$$L_A\phi_\beta\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\\ 3\\ 4\end{pmatrix} = \begin{pmatrix}1\\ 3\\ 2\\ 4\end{pmatrix}$$

and

$$\phi_\beta T\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \phi_\beta\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix} = \begin{pmatrix}1\\ 3\\ 2\\ 4\end{pmatrix}.$$

So they are the same.

20. With the notation in Figure 2.2 we can prove first that $\phi_\gamma(R(T)) = L_A(\mathbb{F}^n)$. Since ϕ_β is surjective we have that

$$L_A(\mathbb{F}^n) = L_A\phi_\beta(V) = \phi_\gamma T(V) = \phi_\gamma(R(T)).$$

Since $R(T)$ is a subspace of W and ϕ_γ is an isomorphism, we have that rank(T) =rank(L_A) by Exercise 2.4.17.

On the other hand, we may prove that $\phi_\beta(N(T)) = N(L_A)$. If $y \in \phi_\beta(N(T))$, then we have that $y = \phi_\beta(x)$ for some $x \in N(T)$ and hence

$$L_A(y) = L_A(\phi_\beta(x)) = \phi_\gamma T(x) = \phi_\gamma(0) = 0.$$

Conversely, if $y \in N(L_A)$, then we have that $L_A(y) = 0$. Since ϕ_β is surjective, we have $y = \phi_\beta(x)$ for some $x \in V$. But we also have that

$$\phi_\gamma(T(x)) = L_A(\phi_\beta(x)) = L_A(y) = 0$$

and $T(x) = 0$ since ϕ_γ is injective. So similarly by Exercise 2.4.17 we can conclude that nullity(T) =nullity(L_A).

21. First we prove the independence of $\{T_{ij}\}$. Suppose that $\sum_{i,j} a_{ij}T_{ij} = 0$. We have that

$$(\sum_{i,j} a_{ij}T_{ij})(v_k) = \sum_i a_{ij}T_{ik}(v_k) = \sum_i a_{ik}w_i = 0.$$

This means $a_{ik} = 0$ for all proper i since $\{w_i\}$ is a basis. And since k is arbitrary we have that $a_{ik} = 0$ for all i and k.

Second we prove that $[T_{ij}]_\beta^\gamma = M^{ij}$. But this is the instant result of

$$T_{ij}(v_j) = w_j$$

and

$$T_{ij}(v_k) = 0$$

for $k \neq j$. Finally we can observe that $\Phi(\beta) = \gamma$ is a basis for $M_{m\times n}(\mathbb{F})$ and so Φ is a isomorphism by Exercise 2.4.15.

22. It's linear since

$$T(f+cg) = ((f+cg)(c_0), (f+cg)(c_1), \dots (f+cg)(c_n))$$

$$= (f(c_0)+cg(c_0), f(c_1)+cg(c_1), \dots f(c_n)+cg(c_n)) = T(f)+cT(g).$$

Since $T(f)=0$ means f has $n+1$ zeroes, we know that f must be zero function(This fact can be proven by Lagrange polynomial basis for $P_n(\mathbb{F})$.). So T is injective and it will also be surjective since domain and codomain have same finite dimension.

23. The transformation is linear since

$$T(\sigma+c\tau) = \sum_{i=0}^{m}(\sigma+c\tau)(i)x^i$$

$$= \sum_{i=0}^{m}\sigma(i)x^i + c\tau(i)x^i = T(\sigma)+cT(\tau),$$

where m is a integer large enough such that $\sigma(k)=\tau(k)=0$ for all $k>m$. It would be injective by following argument. Since $T(\sigma)=\sum_{i=0}^{n}\sigma(i)x^i=0$ means $\sigma(i)=0$ for all integer $i\le n$, with the help of the choice of n we can conclude that $\sigma=0$. On the other hand, it would also be surjective since for all polynomial $\sum_{i=0}^{n}a_ix^i$ we may let $\sigma(i)=a_i$ and thus T will map σ to the polynomial.

24. (a) If $v+N(T)=v'+N(T)$, we have that $v-v'\in N(T)$ and thus $T(v)-T(v')=T(v-v')=0$.

    (b) We have that

$$\bar{T}((v+N(T))+c(u+N(T))) = \bar{T}((v+cu)+N(T))$$

$$= T(v+cu) = T(v)+cT(u).$$

    (c) Since T is surjective, for all $y\in Z$ we have $y=T(x)$ for some x and hence $y=\bar{T}(x+N(T))$. This means $\bar{T}$ is also surjective. On the other hand, if $\bar{T}(x+N(T))=T(x)=0$ then we have that $x\in N(T)$ and hence $x+N(T)=0+N(T)$. So $\bar{T}$ is injective. With these argument $\bar{T}$ is an isomorphism.

    (d) For arbitrary $x\in V$, we have

$$\bar{T}_\eta(x) = \bar{T}(x+N(T)) = T(x).$$

25. The transformation Ψ would be linear since

$$\Psi(f+cg) = \sum_{(f+cg)(s)\neq 0}(f+cg)(s)s = \sum_{(f+cg)(s)\neq 0} f(s)s+cg(s)s$$

$$= \sum_{(f \text{ or } cg)(s)\neq 0} f(s)s+cg(s)s = \sum_{(f \text{ or } cg)(s)\neq 0} f(s)+c\sum_{(f \text{ or } cg)(s)\neq 0} g(s)s$$

$$= \Psi(f) + c\Psi(g).$$

It will be injective by following arguments. If $\Psi(f) = \sum_{f(s)\neq 0} f(s)s = 0$ then we have that $f(s) = 0$ on those s such that $f(s) \neq 0$ since $\{s : f(s) \neq 0\}$ is finite subset of basis. But this can only be possible when $f = 0$. On the other hand, we have for all element $x \in V$ we can write $x = \sum_i a_i s_i$ for some finite subset $\{s_i\}$ of S. Thus we may pick a function f sucht that $f(s_i) = a_i$ for all i and vanish outside. Thus Ψ will map f to x. So Ψ is surjective. And thus it's an isomorphism.

2.5 The Change of Coordinate Matrix

1. (a) No. It should be $[x'_j]_\beta$.
   (b) Yes. This is Theorem 2.22.
   (c) Yes. This is Theorem 2.23.
   (d) No. It should be $B = Q^{-1}AQ$.
   (e) Yes. This is the instant result of the definition of similar and Theorem 2.23.

2. For these problem, just calculate $[I]_{\beta'}^{\beta}$.
   (a) $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$.
   (b) $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.
   (c) $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$.
   (d) $\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$.

3. (a) $\begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix}$.
   (b) $\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$.
   (c) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix}$.
   (d) $\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$.

(e) $\begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}$.

(f) $\begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}$.

4. We have that
$$[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$
and
$$\begin{aligned} [T]_{\beta'} &= [I]_\beta^{\beta'}[[T]_\beta[I]_{\beta'}^\beta \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 13 \\ -5 & 9 \end{pmatrix}. \end{aligned}$$

5. We have that
$$[T]_\beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
and
$$\begin{aligned} [T]_{\beta'} &= [I]_\beta^{\beta'}[[T]_\beta[I]_{\beta'}^\beta \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \end{aligned}$$

6. Let α be the standard basis (of $\mathbb{F}^2$ or $\mathbb{F}^3$). We have that $A = [L_A]_\alpha$ and hence $[L_A]_\beta = [I]_\alpha^\beta[L_A]_\alpha[I]_\beta^\alpha$. So now we can calculate $[L_A]_\beta$ and $Q = [I]_\beta^\alpha$ and $Q^{-1} = [I]_\alpha^\beta$.

(a) $[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

(b) $[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(c) $[L_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

(d) $[L_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$.

7. We may let β be the standard basis and $\alpha = \{(1,m),(-m,1)\}$ be another basis for $\mathbb{R}^2$.

   (a) We have that $[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q^{-1} = [I]_\alpha^\beta = \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$. We also can calculate that $Q = [I]_\beta^\alpha = \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ -\frac{m}{m^2+1} & \frac{1}{m^2+1} \end{pmatrix}$. So finally we get

   $$[T]_\beta = Q^{-1}[T]_\alpha Q = \begin{pmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{pmatrix}.$$

   That is, $T(x,y) = (\frac{x+2ym-xm^2}{m^2+1}, \frac{-y+2xm+ym^2}{m^2+1})$.

   (b) Similarly we have that $[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. And with the same Q and Q^{-1} we get

   $$[T]_\beta = Q^{-1}[T]_\alpha Q = \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{m}{m^2+1} & \frac{m^2}{m^2+1} \end{pmatrix}.$$

   That is, $T(x,y) = (\frac{x+ym}{m^2+1}, \frac{xm+ym^2}{m^2+1})$.

8. This is similar to the proof of Theorem 2.23 since

   $$[T]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'}[T]_\beta^\gamma[I_V]_{\beta'}^\beta = P^{-1}[T]_\beta^\gamma Q.$$

9. We may denote that A is similar to B by $A \sim B$. First we have $A = I^{-1}AI$ and hence $A \sim A$. Second, if $A \sim B$ we have $A = Q^{-1}BQ$ and $B = (Q^{-1})^{-1}AQ^{-1}$ and hence $B \sim A$. Finally, if $A \sim B$ and $B \sim C$ then we have $A = P^{-1}BP$ and $B = Q^{-1}CQ$. And this means $A = (QP)^{-1}C(QP)$ and hence $A \sim C$. So it's a equivalence relation.

10. If A and B are similar, we have $A = Q^{-1}BQ$ for some invertible matrix Q. So we have

    $$\operatorname{tr}(A) = \operatorname{tr}(Q^{-1}BQ) = \operatorname{tr}(QQ^{-1}B) = \operatorname{tr}(B)$$

    by Exercise 2.3.13.

11. (a) This is because

    $$RQ = [I]_\beta^\gamma[I]_\alpha^\beta = [I]_\alpha^\gamma.$$

    (b) This is because

    $$Q^{-1} = ([I]_\alpha^\beta)^{-1} = [I^{-1}]_\beta^\alpha = [I]_\beta^\alpha.$$

12. This is the instant result that $A = [L_A]_\beta$ and Q defined in the Corollary is actually $[I]_\gamma^\beta$.

13. Since Q is invertible, we have that L_Q is invertible. We try to check β' is an independent set and hence a basis since V has dimension n. Suppose that $\sum_{j=1}^{n} a_j x'_j = 0$. And it means that

$$\sum_{j=1}^{n} a_j \sum_{i=1}^{n} Q_{ij} x_i = \sum_{i=1}^{n} (\sum_{j=1}^{n} a_j Q_{ij}) x_i = 0.$$

Since β is a basis, we have that $\sum_{j=1}^{n} a_j Q_{ij} = 0$ for all i. Actually this is a system of linear equations and can be written as

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix} = vQ = 0,$$

where $v = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$. But since Q is invertible and so Q^{-1} exist, we can deduce that $v = vQQ^{-1} = 0Q^{-1} = 0$. So we know that β is a basis. And it's easy to see that $Q = [I]_{\beta}^{\beta}$ is the change of coordinate matrix changing β'-coordinates into β-coordinates.

14. Let $V = \mathbb{F}^n$, $W = \mathbb{F}^m$, $T = L_A$, β, and γ be the notation defined in the Hint. Let β' and γ' be the set of column vectors of Q and P respectly. By Exercise 2.5.13 we have that β' and γ' are bases and $Q = [I]_{\beta'}^{\beta}$, $P = [I]_{\gamma'}^{\gamma}$. Since we have that $[T]_{\beta'}^{\gamma'} = [I]_{\gamma}^{\gamma'} [T]_{\beta}^{\gamma} [I]_{\beta'}^{\beta}$, we have that $B = P^{-1}AQ$.

2.6 Dual Spaces

1. (a) No. Every linear functional is a linear transformation.
   (b) Yes. It's domain and codomain has dimension 1.
   (c) Yes. They have the same dimension.
   (d) Yes. It's isomorphic to the dual space of its dual space. But if the "is" here in this question means "equal", then it may not be true since dual space must has that its codomain should be $\mathbb{F}$.
   (e) No. For an easy example we may let T be the linear transformation such that $T(x_i) = 2f_i$, where $\beta\{x_1, x_2, \dots, x_n\}$ is the basis for V and $\beta^*\{f_1, f_2, \dots, f_n\}$ is the corresponding dual basis for V^*.
   (f) Yes.
   (g) Yes. They have the same dimension.
   (h) No. Codomain of a linear functional should be the field.

2. In these question we should check whether it's linear and whether its domain and codomain are V and F respectly.

(a) Yes. We may check that

$$f(p(x) + cq(x)) = 2p'(0) + 2cp'(0) + p''(1) + cq''(1)$$

$$2p'(0) + p''(1) + c(2p'(0) + q''(1)) = f(p(x)) + cf(q(x)).$$

(b) No. It's codomain should be the field.

(c) Yes. We may check that

$$\operatorname{tr}(A + cB) = \sum_{i=1}^{n} (A + cB)_{ii}$$

$$= \sum_{i=1}^{n} A_{ii} + cB_{ii} = \operatorname{tr}(A) + c\operatorname{tr}(B).$$

(d) No. It's not linear.

(e) Yes. We may check that

$$f(p(x) + cq(x)) = \int_0^1 (p(t) + cq(t))dt$$

$$= \int_0^1 p(t)dt + c\int_0^1 q(t)dt = f(p(x)) + cf(q(x)).$$

(f) Yes. We may check that

$$f(A + cB) = (A + cB)_{11} = A_{11} + cB_{11} = f(A) + cf(B).$$

3. (a) We may find out that for all vector $(x, y, z) \in \mathbb{R}^3$ we can express it as

$$(x, y, z) = (x - \frac{y}{2})(1, 0, 1) + \frac{y}{2}(1, 2, 1) + (z - x)(0, 0, 1).$$

So we can write

$$\begin{cases} f_1(x, y, z) = x - \frac{y}{2}; \\ f_2(x, y, z) = \frac{y}{2}; \\ f_3(x, y, z) = z - x. \end{cases}$$

(b) This is much easier and we have that

$$\begin{cases} f_1(a_0 + a_1x + a_2x^2) = a_0; \\ f_2(a_0 + a_1x + a_2x^2) = a_1; \\ f_3(a_0 + a_1x + a_2x^2) = a_2. \end{cases}$$

4. We may returen the representation such that

$$(x, y, z) = (x-2y)(\frac{2}{5}, -\frac{3}{10}, -\frac{1}{10}) + (x+y+z)(\frac{3}{5}, \frac{3}{10}, \frac{1}{10}) + (y-3z)(\frac{1}{5}, \frac{1}{10}, -\frac{3}{10}).$$

We may check that the set $\{(\frac{2}{5}, -\frac{3}{10}, -\frac{1}{10}), (\frac{3}{5}, \frac{3}{10}, \frac{1}{10}), (\frac{1}{5}, \frac{1}{10}, -\frac{3}{10})\}$ is a basis and hence the desired set. By Theorem 2.24 $\{f_1, f_2, f_3\}$ is a basis for V^*.

5. Assume that $p(t) = a+bx$. We have $\int_0^1 (a+bt)dt = a+\frac{b}{2}$ and $\int_0^2 (a+bt)dt = 2a+2b$. So we may returen the representation such that

$$a + bx = (a + \frac{b}{2})(2 - 2x) + (2a + 2b)(-\frac{1}{2} + x).$$

We may check that the set $\{2-2x, -\frac{1}{2}+x\}$ is a basis and hence the desired set. By Theorem 2.24 $\{f_1, f_2, f_3\}$ is a basis for V^*.

6. (a) Calculate directly that

$$T^t(f)(x,y) = fT(x,y) = f(3x+2y, x) = 7x + 4y.$$

(b) Since $\beta = \{(1,0),(0,1)\}$ and $(x,y) = x(1,0) + y(0,1)$, we have that $f_1(x,y) = x$ and $f_2(x,y) = y$. So we can find out that

$$T^t(f_1)(x,y) = f_1T(x,y) = f_1(3x+2y, x) = 3x+2y = 3f_1(x,y)+2f_2(x,y);$$

$$T^t(f_2)(x,y) = f_2T(x,y) = f_2(3x+2y, x) = x = 1f_1(x,y) + 0f_2(x,y).$$

And we have the matrix $[T^t]_{\beta^*} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$.

(c) Since $T(x,y) = (3x+2y, x)$, we can calculate that

$$[T]_\beta = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$$

and

$$([T]_\beta)^t = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}.$$

So we have that $[T^t]_{\beta^*} = ([T]_\beta)^t$.

7. (a) Calculate directly that

$$T^t(f)(a+bx) = fT(a+bx) = f(-a-2b, a+b) = -3a - 4b.$$

(b) Since $\beta = \{1, x\}$ and $a + bx = a \times 1 + b \times x$, we have that $f_1(a+bx) = a$ and $f_2(a+bx) = b$. And since $\gamma = \{(1,0),(0,1)\}$ and $(a,b) = a(1,0) + b(0,1)$, we have that $g_1(a,b) = a$ and $g_2(a,b) = b$. So we can find out that

$$T^t(g_1)(a+bx) = g_1T(a+bx) = g_1(-a-2b, a+b) = -a - 2b$$

$$= -1 \times g_1(a,b) + (-2) \times g_2(a,b);$$

$$T^t(g_2)(a+bx) = g_2T(a+bx) = g_2(-a-2b, a+b) = a + b$$

$$= 1 \times g_1(a,b) + 1 \times g_2(a,b).$$

And we have the matrix $[T^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$.

(c) Since $T(a+bx) = (-a-2b, a+b)$, we can calculate that

$$[T]_\beta^\gamma = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

and

$$([T]_\beta^\gamma)^t = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}.$$

So we have that $[T^t]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$.

8. Every plane could be written in the form $P = \{(x,y,z) : ax+by+cz = 0\}$ for some scalar a, b and c. Consider a transformation $T(x,y,z) = ax+by+cz$. It can be shown that T is an element in $(\mathbb{R}^3)^*$ and $P = N(T)$. For the case in $\mathbb{R}^2$, actually every line has the form $L = \{(x,y) : ax+by = 0\}$ and hence is the null space of a vector in $(\mathbb{R}^2)^*$.

9. If T is linear, we can set f_i be g_iT as the Hint. Since it's conposition of two linear function, it's linear. So we have

$$T(x) = (g_1(T(x)), g_2(T(x)), \dots, g_m(T(x)))$$
$$= (f_1(x), f_2(x), \dots, f_m(x)).$$

For the converse, let $\{e_i\}_{i=1,2,\dots,m}$ be the standard basis of $\mathbb{F}^m$. So if we have that $T(x) = \sum_{i=1}^m f_i(x)e_i$ with f_i linear, we can define $T_i(x) = f_i(x)e_i$ and it would be a linear transformation in $\mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$. Thus we know T is linear since T is summation of all T_i.

10. (a) Since we can check that $f_i(p(x)+cq(x)) = p(c_i)+cq(c_i) = f_i(p(x)) + cf_i(q(x))$, f_i is linear and hence in V^*. And we know that $\dim(V^*) = \dim(V) = \dim(P_n(\mathbb{F})) = n+1$. So now it's enough to show that $\{f_0, f_1, \dots, f_n\}$ is independent. So assume that $\sum_{i=1}^n a_if_i = 0$ for some a_i. We may define polynomials $p_i(x) = \prod_{j\neq i}(x-c_j)$ such that we know $p_i(c_i) \neq 0$ but $p_i(c_j) = 0$ for all $j \neq i$. So now we have that

$$\sum_{i=1}^n a_if_i(p_1) = a_1f_1(p_1) = 0$$

implies $a_1 = 0$. Similarly we have $a_i = 0$ for all proper i.

(b) By the Corollary after Theorem 2.26 we have an ordered basis

$$\beta = \{p_0, p_1, \dots, p_n\}$$

for V such that $\{f_1, f_2, \dots, f_n\}$ defined in the previous exercise is its dual basis. So we know that $p_i(c_j) = \delta_{ij}$. Since β is a basis, every polynomial in V is linear combination of β. If a polynomial q has the property that $q(c_j) = \delta_{0j}$, we can assume that $q = \sum_{i=0}^{n} a_i p_i$. Then we have

$$1 = q(c_0) = \sum_{i=0}^{n} a_i p_i(c_0) = a_1$$

and

$$0 = q(c_j) = \sum_{i=0}^{n} a_i p_i(c_j) = a_j$$

for all j other than 1. So actually we know $q = p_0$. This means p_0 is unique. And similarly we know all p_i is unique. Since the Lagrange polynomials,say $\{r_i\}_{i=1,2,\ldots n}$, defined in Section 1.6 satisfy the property $r_i(c_j) = \delta_{ij}$, by uniqueness we have $r_i = p_i$ for all i.

(c) Let $\beta = \{p_0, p_1, \ldots, p_n\}$ be those polynomials defined above. We may check that

$$q(x) = \sum_{i=0}^{n} a_i p_i(x)$$

has the property $q(c_i) = a_i$ for all i, since we know that $p_i(c_j) = \delta_{ij}$. Next if $r(x) \in V$ also has the property, we may assume that

$$r(x) = \sum_{i=0}^{n} b_i p_i(x)$$

since β is a basis for V. Similarly we have that

$$a_i = r(c_i) = \sum_{i=0}^{n} b_i p_i(c_i) = b_i.$$

So we know $r = q$ and q is unique.

(d) This is the instant result of 2.6.10(a) and 2.6.10(b) by setting $a_i = p(c_i)$.

(e) Since there are only finite term in that summation, we have that the order of integration and summation can be changed. So we know

$$\int_a^b p(t)dt = \int_a^b (\sum_{i=0}^{n} p(c_i)p_i(t))dt$$

$$= \sum_{i=0}^{n} \int_a^b p(c_i)p_i(t)dt.$$

11. It will be more clearer that we confirm that the domain and codomain of both $\psi_2 T$ and $T^{tt}\psi_2$ are V and W^{**} respectly first. So for all $x \in V$ we have

$$\psi_2 T(x) = \psi(T(x)) = \hat{T(x)} \in W^{**}$$

and

$$T^{tt}\psi_1(x) = T^{tt}(\hat{x})$$

$$= (T^t)^t(\hat{x}) = \hat{x}T^t \in W^{**}.$$

But to determine whether two elements f and g in W^{**} are the same is to check whether the value of $f(h)$ and $g(h)$ are the same for all $h \in W^*$. So let h be an element in $W*$. Let's check that

$$T(\hat{x})(h) = h(T(x))$$

and

$$\hat{x}T^t(h) = \hat{x}(hT) = h(T(x)).$$

So we know they are the same.

12. Let $\beta = \{x_1, x_2, \ldots, x_n\}$ be a basis for V. Then we know the functional $\hat{x_i} \in V^{**}$ means $\hat{x_i}(f) = f(x_i)$ for all funtional f in V^*. On the other hand, we have the dual basis $\beta^* = \{f_1, f_2, \ldots, f_n\}$ is defined by $f_i(x_j) = \delta_{ij}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$ such that f_i is lineaer. And we can further ferret what elements are in β^{**}. By definition of β^{**} we know $\beta^{**} = \{F_1, F_2, \ldots, F_n\}$ and $F_i(f_j) = \delta_{ij}$ and F_i is linear. So we may check that whether $F_i = \hat{x_i}$ by

$$\hat{x_i}(f_j) = f_j(x_i) = \delta_{ij} = F_i(f_j).$$

Since they are all linear functional and the value of them meets at basis β, they are actually equal by the Corollary after Theorem 2.6.

13. (a) We can check that $f + g$ and cf are elements in S^0 if f and g are elements in S^0 since $(f + g)(x) = f(x) + g(x) = 0$ and $(cf)(x) = cf(x) = 0$. And the zero function is an element in S^0.

(b) Let $\{v_1, v_2, \ldots, v_k\}$ be the basis of W. Since $x \notin W$ we know that $\{v_1, v_2, \ldots, v_{k+1} = x\}$ is an independent set and hence we can extend it to a basis $\{v_1, v_2, \ldots, v_n\}$ for V. So we can define a linear transformation T such that $f(v_i) = \delta_{i(k+1)}$. And thus f is the desired functional.

(c) Let W be the subspace span(S). We first prove that $W^0 = S^0$. Since every function who is zero at W must be a function who is zero at S. we know $W^0 \subset S^0$. On the other hand, if a linear function has the property that $f(x) = 0$ for all $x \in S$, we can deduce that $f(y) = 0$ for all $y \in W$ =span(S). Hence we know that $W^0 \supset S^0$ and $W^0 = S^0$. Since $(W^0)^0 = (S^0)^0$ and span$(\psi(S)) = \psi(W)$ by the fact ψ is an isomorphism, we can just prove that $(W^0)^0 = \psi(W)$.

Next, by Theorem 2.26 we may assume every element in $(W^0)^0 \subset V^{**}$ has the form $\hat{x}$ for some x. Let $\hat{x}$ is an element in $(W^0)^0$. We have that $\hat{x}(f) = f(x) = 0$ if $f \in W^0$. Now if x is not an element in W, by the previous exercise there exist some functional $f \in W^0$ such that $f(x) \neq 0$. But this is a contradiction. So we know that $\hat{x}$ is an element in $\psi(W)$ and $(W^0)^0 \subset \psi(W)$.

For the converse, we may assume that $\hat{x}$ is an element in $\psi(W)$. Thus for all $f \in W^0$ we have that $\hat{x}(f) = f(x) = 0$ since x is an element in W. So we know that $(W^0)^0 \supset \psi(W)$ and get the desired conclusion.

(d) It's natural that if $W_1 = W_2$ then we have $W_1^0 = W_2^0$. For the converse, if $W_1^0 = W_2^0$ then we have

$$\psi(W_1) = (W_1^0)^0 = (W_2^0)^0 = \psi(W_2)$$

and hence

$$W_1 = \psi^{-1}\psi(W_1) = \psi^{-1}\psi(W_2) = W_2$$

by the fact that ψ is an isomorphism.

(e) If f is an element in $(W_1 + W_2)^0$, we have that $f(w_1 + w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. So we know that $f(w_1 + 0) = 0$ and $f(0 + w_2) = 0$ for all proper w_1 and w_2. This means f is an element in $W_1^0 \cap W_2^0$. For the converse, if f is an element in $W_1^0 \cap W_2^0$, we have that $f(w_1 + w_2) = f(w_1) + f(w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. Hence we have that f is an element in $(W_1 + W_2)^0$.

14. We use the notation in the Hint. To prove that $\alpha = \{f_{k+1}, f_{k+2}, \dots, f_n\}$ is a basis for W^0, we should only need to prove that $\mathrm{span}(\alpha) = W^0$ since by $\alpha \subset \beta^*$ we already know that α is an independent set. Since $W^0 \subset V^*$, every element $f \in W^0$ we could write $f = \sum_{i=1}^n a_i f_i$. Next since for $1 \le i \le k$ x_i is an element in W, we know that

$$0 = f(x_i) = \sum_{i=1}^n a_i f_i(x_i) = a_i.$$

So actually we have $f = \sum_{i=k+1}^n a_i f_i$ is an element in $\mathrm{span}(\alpha)$. And finally we get the conclusion by

$$\dim(W) + \dim(W_0) = k + (n - k) = n = \dim(V).$$

15. If $T^t(f) = fT = 0$, this means $f(y) = 0$ for all $y \in R(T)$ and hence $f \in (R(T))^0$. If $f \in (R(T))^0$, this means $f(y) = 0$ for all $y \in R(T)$ and hence $T^t(f)(x) = f(T(x)) = 0$ for all x. This means f is an element in $N(T^t)$.

16. We have that

$$\mathrm{rank}(L_A) = \dim(R(L_A)) = m - \dim(R(L_A)^0) = m - \dim(N((L_A)^t))$$

$$= \dim((\mathbb{F}^m)^*) - \dim(N((L_A)^t)) = \dim(R((L_A)^t)).$$

Next, let α, β be the standard basis for $\mathbb{F}^n$ and $\mathbb{F}^m$. Let α^*, β^* be their dual basis. So we have that $[L_A)^t]_{\beta^*}^{\alpha^*} = ([L_A]_\alpha^\beta)^t = A^t$ by Theorem 2.25. Let ϕ_{β^*} be the isomorphism defined in Theorem 2.21. We get

$$\dim(R((L_A)^t)) = \dim(\phi_{\beta^*}(R((L_A)^t))) = \dim(R(L_{A^t})) = \mathrm{rank}(L_{A^t}).$$

17. If W is T-invariant, we have that $T(W) \subset W$. Let f be a functional in W^0. We can check $T^t(f) = fT$ is an element in W^0 since $T(w) \in W$ by the fact that T-invariant and thus $f(T(w)) = 0$.

    For the converse, if W^0 is T^t-invariant, we know $T^t(W^0) \subset W^0$. Fix one w in W, if $T(w)$ is not an element in W, by Exercise 2.6.13(b) there exist a functional $f \in W^0$ such that $f(T(w)) \neq 0$. But this means $T^t(f)(w) = fT(w) \neq 0$ and hence $T^t(f) \notin W^0$. This is a contradiction. So we know that $T(w)$ is an element in W for all w in W.

18. First check that Φ is a linear transformation by

    $$\Phi(f + cg)(s) = (f + cg)_S(s) = f_S(s) + cg_S(s) = (\Phi(f) + c\Phi(g))(s).$$

    Second we know Φ is injective and surjective by Exercise 2.1.34.

19. Let S' is a basis for W and we can extend it to be a basis S for V. Since W is a proper subspace of V, we have at least one element $t \in S$ sucht that $t \notin W$. And we can define a function g in $\mathcal{F}(S, \mathbb{F})$ by $g(t) = 1$ and $g(s) = 0$ for all $s \in S$. By the previous exercise we know there is one unique linear functional $f \in V^*$ such that $f_S = g$. Finally since $f(s) = 0$ for all $s \in S'$ we have $f(s) = 0$ for all $s \in W$ but $f(t) = 1$. So f is the desired functional.

20. (a) Assume that T is surjective. We may check whether $N(T^t) = \{0\}$ or not. If $T^t(f) = fT = 0$, we have that $f(y) = f(T(x)) = 0$ for all $y \in W$ since there exist some $x \in V$ such that $T(x) = y$. For the converse, assume that T^t is injective. Suppose, by contradiction, $R(T) \neq W$. By the previous exercise we can construct a nonzero linear functional $f(y) \in W^*$ such that $f(y) = 0$ for all $y \in R(T)$. Let f_0 be the zero functional in W^*. But now we have that $T^t(f)(x) = f(T(x)) = 0 = T^t(g)(x)$, a contradiction. So T must be surjective.

    (b) Assume that T^t is surjective. Suppose, by contradiction, $T(x) = 0$ for some nonzero $x \in V$. We can construct a nonzero linear functional $g \in V^*$ such that $g(x) \neq 0$. Since T^t is surjective, we get some functional $f \in W^*$ such that $T^t(f) = g$. But this means

    $$0 = f(T(x)) = T^t(f)(x) = g(x) \neq 0,$$

    a contradiction.

    For the converse, assume that T is injective and let S is a basis for V. Since T is injective, we have $T(S)$ is an independent set in W. So we can extend it to be a basis S' for W. Thus for every linear functional $g \in V^*$ we can construct a functional $f \in W^*$ such that $T^t(f) = g$ by the argument below. First we can construct a function $h \in \mathcal{F}(S, \mathbb{F})$ by $h(T(s)) = g(s)$ for $s \in S$ and $h(t) = 0$ for all $t \in S' \backslash T(S)$. By Exercise 2.6.18 there is a lineaer functional $f \in W^*$ such that $f_{S'} = h$. So now we have for all $s \in S$

    $$g(s) = h(T(s)) = f(T(s)) = T^t(f)(s).$$

    By Exercise 2.1.34 we have $g = T^t(f)$ and get the desired conclusion.

2.7 Homogeneous Linear Differential Equations with Constant Coeficients

1. (a) Yes. It comes from Theorem 2.32.
   (b) Yes. It comes from Theorem 2.28.
   (c) No. The equation $y = 0$ has the auxiliary polynomial $p(t) = 1$. But $y = 1$ is not a solution.
   (d) No. The function $y = e^t + e^{-t}$ is a solution to the linear differential equation $y'' - y = 0$.
   (e) Yes. The differential operator is linear.
   (f) No. The differential equation $y'' - 2y' + y = 0$ has a solution space of dimension two. So $\{e^t\}$ could not be a basis.
   (g) Yes. Just pick the differential equation $p(D)(y) = 0$.

2. (a) No. Let W be a finite-dimensional subspace generated by the function $y = t$. Thus y is a solution to the trivial equation $0y = 0$. But the solution space is C^∞ but not W. Since $y^{(k)} = 0$ for $k \geq 2$ and it is impossible that
   $$ay' + by = a + bt = 0$$
   for nonzero a, W cannot be the solution space of a homogeneous linear differential equation with constant coefficients.
   (b) No. By the previous argument, the solution subspace containing $y = t$ must be C^∞.
   (c) Yes. If x is a solution to the homogeneous linear differential equation with constant coefficients whose is auxiliary polynomial $p(t)$, then we can compute that $p(D)(x') = D(p(D)(x)) = 0$.
   (d) Yes. Compute that
   $$p(D)q(D)(x + y) = q(D)p(D)(x) + p(D)q(D)(x) = 0.$$
   (e) No. For example, e^t is a solution for $y' - y = 0$ and e^{-t} is a solution for $y' + y = 0$, but $1 = e^t e^{-t}$ is not a solution for $y'' - y = 0$.

3. Use Theorem 2.34.
   (a) The basis is $\{e^{-t}, te^{-t}\}$.
   (b) The basis is $\{1, e^t, e^{-t}\}$.
   (c) The basis is $\{e^t, te^t, e^{-t}, te^{-t}\}$.
   (d) The basis is $\{e^{-t}, te^{-t}\}$.
   (e) The basis is $\{e^{-t}, e^{\alpha t}, e^{\overline{\alpha} t}\}$, where α is the complex value $1 + 2i$.

4. Use Theorem 2.34.

(a) The basis is $\{e^{\alpha t}, e^{\beta t}\}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

(b) The basis is $\{e^t, te^t, t^2e^t\}$.

(c) The basis is $\{1, e^{-2t}, e^{-4t}\}$.

5. If f and g are elements in C^∞, then we know that the k-th derivative of $f+g$ exists for all integer k since

$$(f+g)^{(k)} = f^{(k)} + g^{(k)}.$$

So $f+g$ is also an element in C^∞. Similarly, for any scalar c, the k-th derivative of cf exists for all integer k since $(cf)^{(k)} = cf^{(k)}$. Finally, the function $f = 0$ is an element in C^∞ naturally.

6. (a) Use the fact

$$D(f+cg) = D(f) + cD(g)$$

for functions $f, g \in C^\infty$ and scalar c. This fact is a easy property given in the Calculus course.

(b) If $p(t)$ is a polynomial, then the differential operator $p(D)$ is linear by Theorem E.3.

7. Let W and V be the two subspaces generated by the two sets $\{x, y\}$ and $\{\frac{1}{2}(x+y), \frac{1}{2i}(x-y)\}$ separately. We know that $W \supset V$ since $\frac{1}{2}(x+y)$ and $\frac{1}{2i}(x-y)$ are elements in W. And it is also true that $W \subset V$ since

$$x = \frac{1}{2}(x+y) + \frac{i}{2i}(x-y)$$

and

$$y = \frac{1}{2}(x+y) - \frac{i}{2i}(x-y)$$

are elements in V.

8. Compute that $e^{(a\pm ib)t} = e^{at}e^{ibt} = (\cos bt + i\sin bt)e^{at}$. By Theorem 2.34 and the previous exercise we get the result.

9. Since those U_i are pairwise commutative, we may just assume that $i = n$. Hence if $U_n(x) = 0$ for some $x \in V$, then

$$U_1U_2 \cdots U_n(x) = U_1U_2 \cdots U_{n-1}(0) = 0.$$

10. Use induction on the number n of distinct scalar c_i's. When $n = 1$, the set $\{e^{c_1t}\}$ is independent since e^{c_1t} is not identically zero. Suppose now the set $\{e^{c_1t}, e^{c_2t}, \ldots, e^{c_nt}\}$ is independent for all $n < k$ and for distinct c_i's. Assume that

$$\sum_{i=1}^{k} b_i e^{c_it} = 0.$$

Since any differential operator is linear, we have

$$0 = (D - c_k I)(\sum_{i=1}^{k} b_i e^{c_k t}) = \sum_{i=1}^{k-1} (c_i - c_k) b_i e^{c_i t}.$$

This means that $(c_i - c_k)b_i = 0$ and so $b_i = 0$ for all $i < k$ by the fact that c_i's are all distinct. Finally b_k is also zero since

$$b_k e^{c_k t} = 0.$$

11. Denote the given set in Theorem 2.34 to be S. All the element in the set S is a solution by the proof of the Lemma before Theorem 2.34. Next, we prove that S is linearly independent by induction on the number k of distinct zeroes. For the case $k = 1$, it has been proven by the Lemma before Theorem 2.34. Suppose now the set S is linearly independent for the case $k < m$. Assume that

$$\sum_{i=1}^{m} \sum_{j=0}^{n_i - 1} b_{i,j} t^j e^{c_i t} = 0$$

for some coefficient $b_{i,j}$. Observe that

$$(D - c_m I)(t^j e^{c_i t}) = j t^{j-1} e^{c_i t} + (c_i - c_m) t^j e^{c_i t}.$$

Since any differential operator is linear, we have

$$(D - c_m I)^{n_m} (\sum_{i=1}^{m} \sum_{j=0}^{n_i - 1} b_{i,j} t^j e^{c_i t}) = 0.$$

Since all terms fo $i = m$ are vanished by the differential operator, we may apply the induction hypothesis and know the coefficients for all terms in the left and side is zero. Observer that the coefficient of the term $t^{n_i - 1} e^{c_i t}$ is $(c_i - c_m)^{n_m} b_{i,n_i - 1}$. This means $(c_i - c_m)^{n_m} b_{i,n_i - 1} = 0$ and so $b_{i,n_i - 1} = 0$ for all $i < m$. Thus we know that the coefficient of the term $t^{n_i - 2} e^{c_i t}$ is $(c_i - c_m)^{n_m} b_{i,n_i - 2}$. Hence $b_{i,n_i - 2} = 0$ for all $i < m$. Doing this inductively, we get $b_{i,j} = 0$ for all $i < m$. Finally, the equality

$$\sum_{j=0}^{n_m - 1} b_{m,j} t^j e^{c_m t} = 0$$

implies $b_{m,j} = 0$ for all j by the Lemma before Theorem 2.34. Thus we complete the proof.

12. The second equality is the definition of range. To prove the first equality, we observe that $R(g(D_V)) \subset N(h(D))$ since

$$h(D)(g(D)(V)) = p(D)(V) = \{0\}.$$

Next observe that

$$N(g(D_V)) = N(g(D))$$

since $N(g(D))$ is a subspace in V. By Theorem 2.32, the dimension of $N(g(D_V)) = N(g(D))$ is the degree of g. So the dimension of $R(g(D_V))$ is the degree of $h(t)$ minus the degree of $g(t)$, that is the degree of $h(t)$. So $N(h(D))$ and $R(g(D_V))$ have the same dimension. Hence they are the same.

13. (a) The equation could be rewriten as $p(D)(y) = x$, where $p(t)$ is the auxiliary polynomial of the equation. Since D is surjective by the Lemma 1 after Theorem 2.32, the differential operator $p(D)$ is also surjective. Hence we may find some solution y_0 such that $p(D)(y_0) = x$.

    (b) Use the same notation in the previous question. We already know that $p(D)(z) = x$. If w is also a solution such that $p(D)(w) = x$, then we have

    $$p(D)(w - z) = p(D)(w) - p(D)(z) = x - x = 0.$$

    So all the solution must be of the form $z+y$ for some y in the solution space V for the homogeneous linear equation.

14. We use induction on the order n of the equation. Let $p(t)$ be the auxiliary polynomial of the equation. If now $p(t) = t - c$ for some coeficient c, then the solution is Ce^{ct} for some constant C by Theorem 2.34. So if $Ce^{ct_0} = 0$ for some $t_0 \in \mathbb{R}$, then we know that $C = 0$ and the solution is the zero function. Suppose the statement is true for $n < k$. Now assume the degree of $p(t)$ is k. Let x be a solution and t_0 is a real number. For an arbitrary scalar c, we factor $p(t) = q(t)(t - c)$ for a polynomial $q(t)$ of degree $k - 1$ and set $z = q(D)(x)$. We have $(D - cI)(z) = 0$ since x is a solution and $z(t_0) = 0$ since $x^{(i)}(t_0) = 0$ for all $0 \le i \le n-1$. Again, z must be of the form Ce^{ct}. And so $Ce^{ct_0} = 0$ implies $C = 0$. Thus z is the zero function. Now we have $q(D)(x) = z = 0$. By induction hypothesis, we get the conclusion that x is identically zero. This complete the proof.

15. (a) The mapping Φ is linear since the differential operator D is linear. If $\Phi(x) = 0$, then x is the zero function by the previouse exercise. Hence Φ is injective. And the solution space is an n-dimensional space by Theorem 2.32. So The mapping is an isomorphism.

    (b) This comes from the fact the transformation Φ defined in the previous question is an isomorphism.

16. (a) Use Theorem 2.34. The auxiliary polynomial is $t^2 + \frac{g}{l}$. Hence the basis of the solution space is

    $$\{e^{it\sqrt{\frac{g}{l}}}, e^{-it\sqrt{\frac{g}{l}}}\}$$

or

$$\{\cos t\sqrt{\frac{g}{l}}, \sin t\sqrt{\frac{g}{l}}\}$$

by Exercise 2.7.8. So the solution should be of the form

$$\theta(t) = C_1 \cos t\sqrt{\frac{g}{l}} + C_2 \sin t\sqrt{\frac{g}{l}}$$

for some constants C_1 and C_2.

(b) Assume that

$$\theta(t) = C_1 \cos t\sqrt{\frac{g}{l}} + C_2 \sin t\sqrt{\frac{g}{l}}$$

for some constants C_1 and C_2 by the previous argument. Consider the two initial conditions

$$\theta(0) = C_1\sqrt{\frac{g}{l}} = \theta_0$$

and

$$\theta'(0) = C_2\sqrt{\frac{g}{j}} = 0.$$

Thus we get

$$C_1 = \theta_0\sqrt{\frac{l}{g}}$$

and

$$C_2 = 0.$$

So we get the unique solution

$$\theta(t) = \theta_0\sqrt{\frac{l}{g}} \cos t\sqrt{\frac{g}{l}}.$$

(c) The period of $\cos t\sqrt{\frac{g}{l}}$ is $2\pi\sqrt{\frac{l}{g}}$. Since the solution is unique by the previous argument, the pendulum also has the same period.

17. The auxiliary polynomial is $t^2 + \frac{k}{m}$. So the general solution is

$$y(t) = C_1 \cos t\sqrt{\frac{k}{m}} + C_2 \sin t\sqrt{\frac{k}{m}}$$

for some constants C_1 and C_2 by Exercise 2.7.8.

18. (a) The auxiliary polynomial is $mt^2 + rt + k$. The polynomial has two zeroes

$$\alpha = \frac{-r + \sqrt{r^2 - 4mk}}{2m}$$

and

$$\beta = \frac{-r - \sqrt{r^2 - 4mk}}{2m}.$$

So the general solution to the equation is

$$y(t) = C_1 e^{\alpha t} + C_2 e^{\beta t}.$$

(b) By the previous argument assume the solution is

$$y(t) = C_1 e^{\alpha t} + C_2 e^{\beta t}.$$

Consider the two initial conditions

$$y(0) = C_1 + C_2 = 0$$

and

$$y'(0) = \alpha C_1 + \beta C_2 = v_0.$$

Solve that

$$C_1 = (\alpha - \beta)^{-1} v_0$$

and

$$C_2 = (\beta - \alpha)^{-1} v_0.$$

(c) The limit tends to zero since the real parts of α and β is both $-\frac{r}{2m}$, a negative value, by assuming the $r^2 - 4mk \le 0$. Even if $r^2 - 4mk > 0$, we still know that α and β are negative real number.

19. Since $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a subset of $\mathcal{F}(\mathbb{C}, \mathbb{C})$, so if the solution which is useful in describing physical motion, then it will still be a real-valued function.

20. (a) Assume the differential equation has monic auxiliary polynomial $p(t)$ of degree n. Thus we know that $p(D)(x) = 0$ if x is a solution. This means that $x^{(k)}$ exists for all integer $k \le n$. We may write $p(t)$ as $t^n + q(t)$, where $q(t) = p(t) - t^n$ is a polynomial of degree less than n. Thus we have

$$x^{(n)} = -q(D)(x)$$

is differentiable since $x^{(n)}$ is a linear combination of lower order terms $x^{(k)}$ with $k \le n - 1$. Doing this inductively, we know actualy x is an element in C^∞.

(b) For complex number c and d, we may write $c = c_1 + ic_2$ and $d = d_1 + id_2$ for some real numbers c_1, c_2, d_1, and d_2. Thus we have

$$e^{c+d} = e^{(c_1+d_1)+i(c_2+d_2)} = e^{c_1} e^{d_1} (\cos(c_2 + d_2) + i \sin(c_2 + d_2))$$

and

$$e^c e^d = e_1^c e_1^d (\cos c_2 + i \sin c_2)(\cos d_2 + i \sin d_2)$$

$$= e_1^c e_1^d [(\cos c_2 \cos d_2 - \sin c_2 \sin d_2) + i(\sin c_2 \cos d_2 + \cos c_2 \sin d_2)]$$

$$= e^{c_1}e^{d_1}(\cos(c_2+d_2)+i\sin(c_2+d_2)).$$

This means $e^{c+d}=e^ce^d$ even if c and d are complex number.[1] For the second equality, we have

$$1=e^0=e^{c-c}=e^ce^{-c}.$$

So we get

$$e^{-c}=\frac{1}{e^c}.$$

(c) Let V be the set of all solution to the homogeneous linear differential equation with constant coefficient with auxiliary polynomial $p(t)$. Since each solution is an element in C^∞, we know that $V \supset N(p(D))$, where $N(p(D))$ is the null space of $p(D)$, since $p(D)(x)=0$ means that x is a solution. Conversely, if x is a solution, then we have $p(D)(x)=0$ and so $x \in N(p(D))$.

(d) Let $c=c_1+ic_2$ for some real numbers c_1 and c_2. Directly compute that

$$(e^{ct})'=(e^{c_1t+ic_2t})'=(e^{c_1t}(\cos c_2t+i\sin c_2t))'$$

$$c_1e^{c_1t}(\cos c_2t+i\sin c_2t))+ic_2e^{c_1t}(\cos c_2t+i\sin c_2t)$$

$$(c_1+ic_2)e^{c_1t}(\cos c_2t+i\sin c_2t)=ce^{ct}.$$

(e) Assume that $x=x_1+ix_2$ and $y=y_1+iy_2$ for some x_1, x_2, y_1, and y_2 in $\mathcal{F}(\mathbb{R},\mathbb{R})$. Compute that

$$(xy)'=(x_1y_1-x_2y_2)'+i(x_1y_2+x_2y_1)'$$

$$=(x_1'y_1+x_1y_1'-x_2'y_2-x_2y_2')+i(x_1'y_2+x_1y_2'+x_2'y_1+x_2y_1')$$

$$=(x_1'+ix_2')(y_1+iy_2)+(x_1+ix_2)(y_1'+iy_2')=x'y+xy'.$$

(f) Assume that $x=x_1+ix_2$ for some x_1 and x_2 in $\mathcal{F}(\mathbb{R},\mathbb{R})$. If

$$x'=x_1'+ix_2'=0,$$

then $x_1'=0$ and $x_2'=0$ since x_1' and x_2' are real-valued functions. Hence x_1 and x_2 are constant in $\mathbb{R}$. Hence x is a constant in $\mathbb{C}$.

[1]The textbook has a typo that $e^{c+d}=c^ce^d$.