Date
22/08/2017

Lecture 11

proof:- In order to prove the theorem, we in fact show that

$$F(s)\,G(s) = \mathcal{L}\{f(t) * g(t)\}$$

by direct integration in the R.H.S.

Now,

$$\mathcal{L}\{f(t) * g(t)\} = \int_0^\infty e^{-st}\{f(t) * g(t)\}\,dt$$

(by def^n)

$$= \int_{t=0}^{\infty} e^{-st} \int_{\tau=0}^{t} f(\tau)\,g(t-\tau)\,d\tau\,dt$$

The domain of this repeated integral takes the form of a wedge in the t, τ plane.

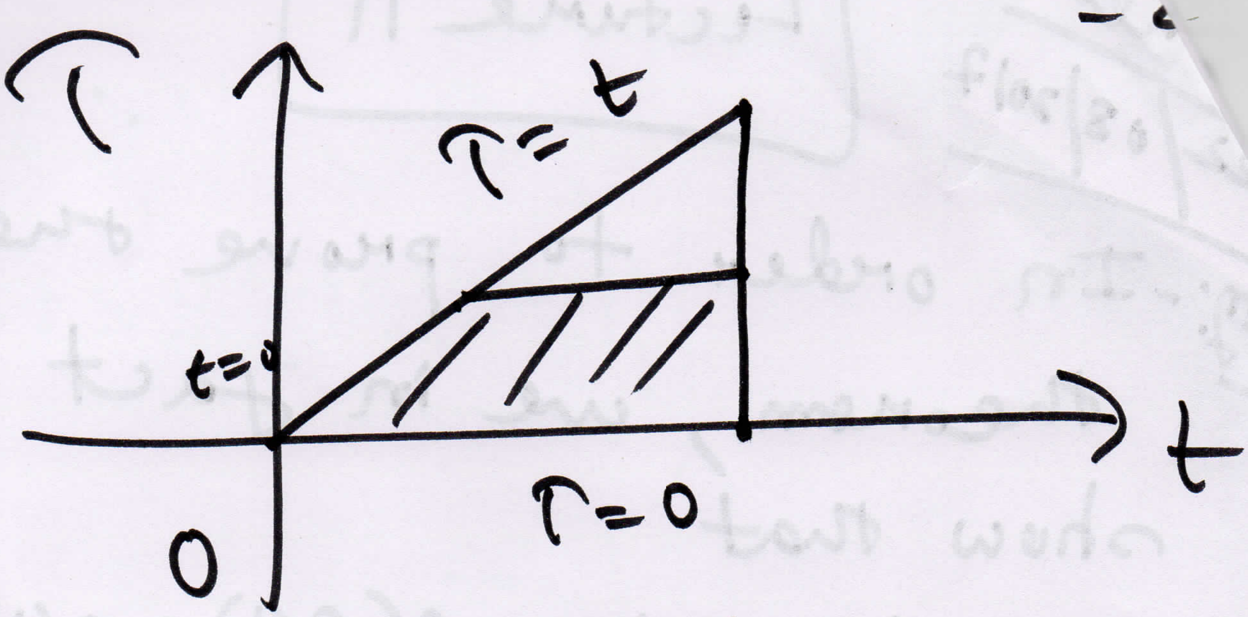


Fig 1

we rewrite this double integra~~te~~ to facilitate changing the order of integration.

$$\mathcal{L}\{f(t)*g(t)\} = \int_0^\infty \int_0^t e^{-st} f(\tau)\, g(t-\tau)\, d\tau\, dt$$

$$= \int_{\tau=0}^\infty \left[\int_{t=\tau}^\infty e^{-st} f(\tau)\, g(t-\tau)\, dt\right] d\tau$$

(how??)

$$\int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right]d\tau$$

Using the change of variable

$$u = t - \tau$$

in the inner integral where τ is constant, so that it becomes

$$|J| = \frac{\partial(t,\tau)}{\partial(u,\tau)} = 1.$$

$$\int_{t=\tau}^{\infty} e^{-st} g(t-\tau)\,dt$$

$u = t - \tau \Rightarrow t = u + \tau$

$du = dt$

Limits of

When $t = \tau$, $u = 0$

$t = \infty$, $u = \infty$

$$= \int_{u=0}^{\infty} e^{-s(u+\tau)} g(u)\,du.$$

$$= e^{-s\tau}\int_0^\infty e^{-su} g(u)\,du$$

$$= \mathcal{L}\{g(t)\}$$

$$= e^{-s\tau} G(s)$$

$\mathcal{L}\{f*g\}$

$$= \int_0^\infty f(\tau) \cdot e^{-s\tau} \underline{G(s)}\, d\tau$$

$$= G(s) \underbrace{\int_0^\infty e^{-s\tau} f(\tau)\, d\tau}_{= \mathcal{L}\{f(t)\}.}$$

$$= G(s)\, F(s)$$

$$\Rightarrow \mathcal{L}\{f * \underline{g}\} = G(s) \cdot F(s)$$

$$\Rightarrow f*g = \mathcal{L}^{-1}\{F(s)\, \underline{G(s)}\}$$

$\rightarrow$ <u>Borel's Theorem</u>

Convolution — Faltung

$$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$$

$$\mathcal{L}\{fg\} \neq \mathcal{L}\{f\}\,\mathcal{L}\{g\}.$$

$f * g$

in general

$$\mathcal{L}\{f*g\} = \mathcal{L}(f)\,\mathcal{L}\{g\}.$$

eg; $f = e^t$, $g = 1$.

$$\mathcal{L}\{fg\} = \mathcal{L}\{e^t\} = \frac{1}{s-1}$$

$$\mathcal{L}(f) = \mathcal{L}(e^t) = \frac{1}{s-1}$$

$$\mathcal{L}(g) = 1/s.$$

$$\mathcal{L}(f)\cdot\mathcal{L}(g) = \frac{1}{(s^2-s)}$$

$\neq \mathcal{L}(fg)$

H.W.

Note:- The convolution $f*g$ has the properties.

(i). $f*g = g*f$

(Commutative law / Symmetry)

(ii) $f*(g_1+g_2) = f*g_1 + f*g_2$

(Distributive law)

(iii) $(f*g)*v = f*(g*v)$

(Associative law)

(iv) $f*0 = 0*f = 0$

just as for numbers.

ut $f * 1 \neq f$ in general

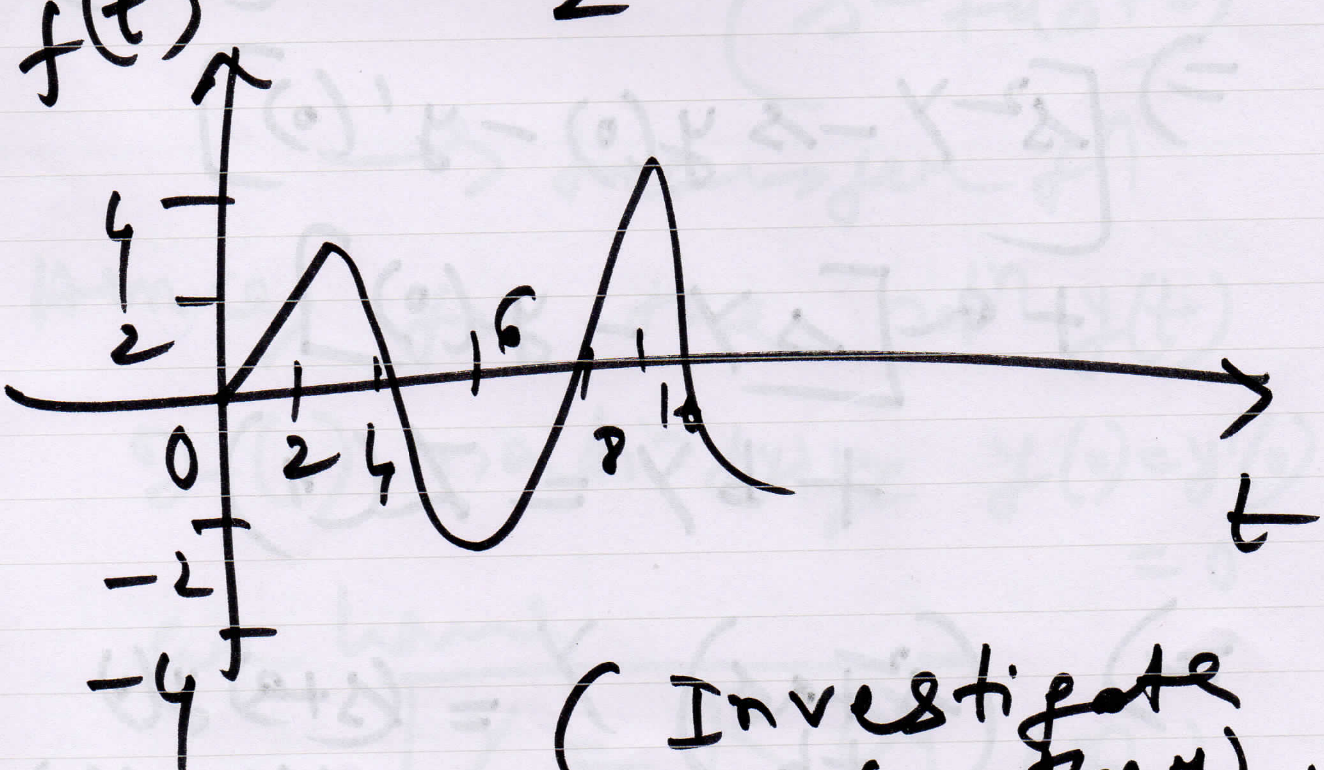
1) eg, $t * 1 = \int_0^t \tau \, d\tau$

$= t^2/2 \neq t$.

2) $(f * f)(t) \geq 0$ may not hold.

eg, $\sin t * \sin t$

$= -\frac{1}{2} t \cos t + \frac{1}{2} \sin t$.



(Investigate further).

Differential eq^ns

We know that the di. eq^n

$$y'' + ay' + by = r(t)$$

$\rightarrow$ (1) $r(t) \neq 0$

$$\Rightarrow \mathcal{L}\{y''\} + a\,\mathcal{L}\{y'\} + b\,\mathcal{L}\{y\} = \mathcal{L}\{r(t)\}$$

$$\Rightarrow \left[s^2 Y - s\,y(0) - y'(0)\right] + a\left[sY - y(0)\right] + bY = \mathcal{L}(r)$$

$$\Rightarrow (s^2 + as + b)\,Y = (s+a)\,y(0) + y'(0) + \mathcal{L}(r)$$

the sol^n of the subsidiary eq^n is

$$Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)\,Q(s) \rightarrow (2)$$

with $R(s) = \mathcal{L}\{r\}$

& $Q(s) = \dfrac{1}{(s^2+as+b)}$

$\rightarrow$ transfer f^n

~~Hen~~ce, for the sol^n $y(t)$ of (1) satisfying $y(0)=y'(0)=0$

we have

$$\boxed{Y = RQ} \quad (\text{fr } (2))$$

g obtain from the convolution theorem

the integral representation

$$y(t) = \int_0^t g(t-\tau)\, r(\tau)\, d\tau$$

$\longrightarrow (3)$

$$g(t) = \mathcal{L}^{-1}\{Q(s)\}$$

Ex, Response of a damped system to a single square wave.

$$y'' + 3y' + 2y = r(t)$$

$$r(t) = 1, \text{ if } 1 < t < 2$$

$$= 0 \text{ otherwise}$$

$$y(0) = y'(0) = 0.$$

we have

$$Q(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

hence $\quad q(t) = \mathcal{L}^{-1}\{Q(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$q(t) = e^{-t} - e^{-2t}.$$

We solve it by convolution technique

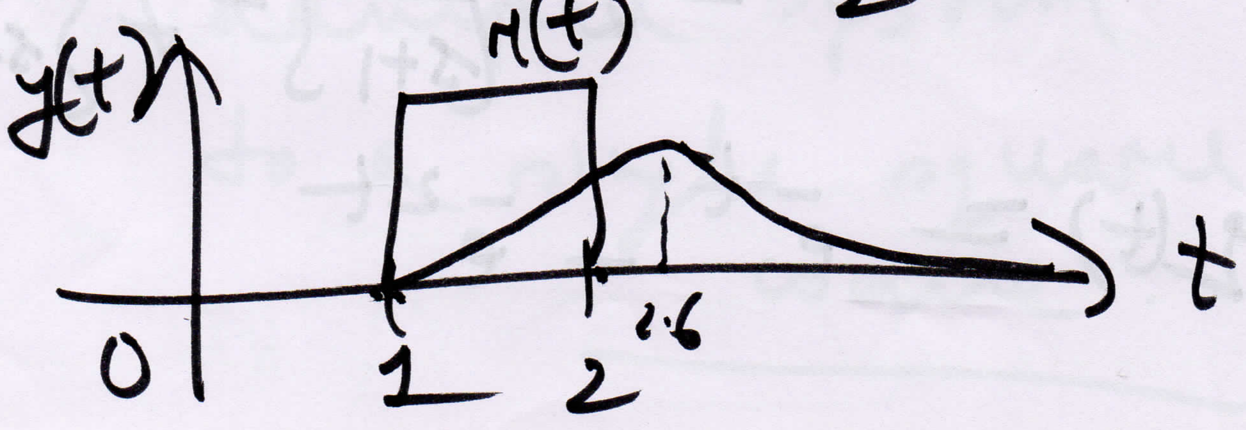
$$y(t) = \int g(t-\tau) \cdot n(\tau)\, d\tau$$

$$= \int g(t-\tau) \cdot 1\, d\tau$$

$$= \int g(t-\tau)\, d\tau$$

$$= \int \left[e^{-(t-\tau)} - e^{-2(t-\tau)} \right] d\tau$$

$$= e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)}$$



$$\Rightarrow y(t) = \begin{cases} 0, & t < 1 \\ \int_1^t \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau, & 1 < t < 2 \\ \int_1^2 \left[e^{-(t-\tau)} - e^{-(t-\tau)}\right] d\tau, & t > 2 \end{cases}$$

= same answer (check!)

~~Fredholm integral eqn.~~

Integral eqns

$$f(t) = g(t) + \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

Volterra integral eqn 2nd kind. where $g(t), h(t)$ are known

=x=