# INTERPOLATION:

Interpolation is the process of estimating intermediate values between precise data points or approximating complicated functions by simple polynomials:

Two main applications:

1) Constructing the function when it is not given explicitely and only he values of $f(x)$ or its certain derivatives are given at some points.

2) Replacing complicated function by an interpolating polynomials so that many operations such as determination of roots, differentiation and integration may be performed.

IDEA: Weirstrass approximation theorem:

Suppose that $f$ is defined and continuous on $[a,b]$. For each $\varepsilon>0$, there exists a polynomial $P(x)$ with the property that

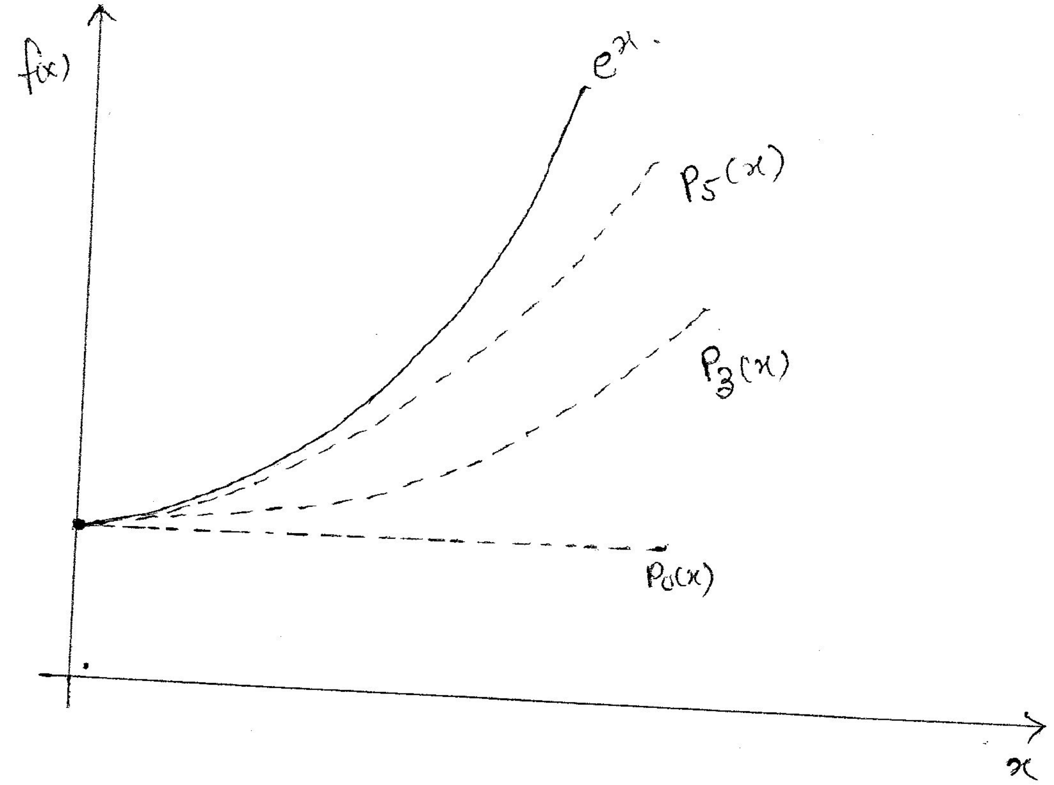
$$|f(x)-P(x)|<\varepsilon, \quad \text{for all } x \text{ in } [a,b].$$

Why not Taylor's Polynomials: Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point. Consider $f(x)=e^x$. Consider Taylor's polynomial around $x=0$.

$$P_0(x)=1 \qquad P_1(x)=1+x \qquad P_2(x)=1+x+\frac{x^2}{2}$$

$$P_3(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6} \qquad P_4(x)=1+x+\cdots+\frac{x^3}{6}+\frac{x^4}{24}$$

$$P_5(x)=1+x+\frac{x^2}{2}+\cdots+\frac{x^4}{24}+\frac{x^5}{120}.$$

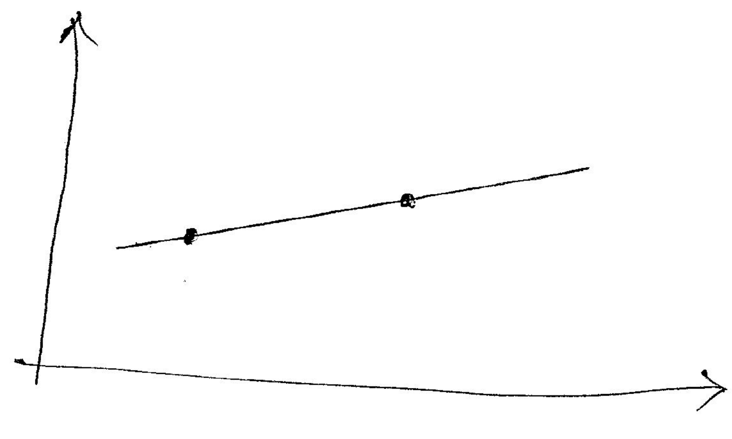



For ordinary computation purposes it is more efficient to use methods that include information at various points.

The most commonly used method for this purpose is polynomial interpolation

For $(n+1)$ data points there is one and only one polynomial of order $\leq n$ that passes through all the points.

For example there is only one ~~line~~ straight line (a first order polynomial) that passes through two points.



Although there is a unique nth order polynomial that fits (n+1) data points, there are a variety of mathematical formats in which this polynomial can be expressed. Example:

- Newton's divided difference formula ✓
- Lagrange interpolation formula ✓
- Newton's forward and backward interpolation formula ✓
- ....

We shall mainly discuss Newton's divided difference, Lagrange and Newtons forward and backward interpolation formulas.

## Fundamental approach for polynomial interpolation:

Consider a nth order polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

A straight forward method for computing the coefficients of this polynomial is based on the fact that (n+1) data points are required to determine (n+1) unknowns.

Consider a second order polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2.$$

Suppose there are three given data points $(x_0, f(x_0))$, $(x_1, f(x_1))$ $(x_2, f(x_2))$.

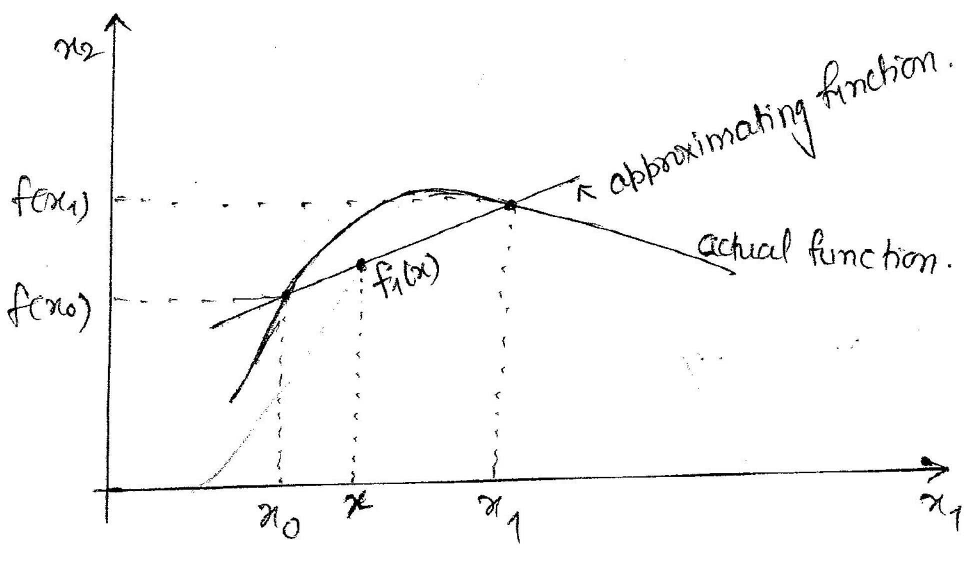
This polynomial must satisfy the data points

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 \\ f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 \\ f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 \end{aligned}\right\} \Rightarrow \text{get } a_i$$

In practice this is observed that this system of equations is ill-conditioned. Whether they are solved with an elimination method or with a more efficient algorithm, the resulting coeff. can be highly inaccurate in particular for large $n$. Therefore we have some mathematics formats (interpolating formula) in which such calculations can be avoided.

Newton's divided-difference interpolating polynomials:

1 Linear case: The simplest form of interpolation is to connect two data points with a straight line:



# INTERPOLATING POLYNOMIALS USING FINITE DIFFERENCES:

Finite difference operators:

Let the tabular point $x_0, x_1, \ldots x_n$ be equally spaced, that is,
$x_i = x_0 + ih, \quad i = 0, 1, \ldots n.$

- The shift operator: $E^1 f(x_i) = f(x_i + h)$
- The forward difference operator: $\Delta f(x_i) = f(x_i + h) - f(x_i)$
- The backward difference operator: $\nabla f(x_i) = f(x_i) - f(x_i - h)$
- The central difference operator: $\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$

It can be easily verified that

$$\Delta f_i = \nabla f_{i+1} = \delta f_{i+\frac{1}{2}}$$

$$\Delta = E - 1$$
$$\nabla = 1 - E^{-1}$$
$$\delta = E^{1/2} - E^{-1/2}$$

Higher order differences: forward:

$$\Delta^2 f(x_i) = \Delta[\Delta f(x_i)]$$
$$= \Delta[f_{i+1} - f_i] = f_{i+2} - f_{i+1} - f_{i+1} + f_i$$
$$= f_{i+2} - 2f_{i+1} + f_i$$

Similarly: $\Delta^3 f_i = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i$

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$$\Delta^n f(x_i) = (E-1)^n f(x_i) = \sum_{K=0}^{n} (-1)^K \frac{\lfloor n}{\lfloor K \lfloor n-K} f_{i+n-K}.$$

Backward: $\nabla^2 f(x_i) = \nabla[\nabla f(x_i)] = \nabla[f_i - f_{i-1}]$
$$= f_i - f_{i-1} - f_{i-1} + f_{i-2}$$
$$= f_i - 2f_{i-1} + f_{i-2}$$

$$\nabla^n f(x_i) = (1-E^{-1})^n f(x_i) = \sum_{K=0}^{n} (-1)^K \frac{\lfloor n}{\lfloor K \lfloor n-K} f_{i-K}.$$

Table 1: Forward difference table:

| $x$ | $f(x)$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ |
|---|---|---|---|---|
| $x_0$ | $\boxed{f_0}$ | | | |
| | | $\boxed{\Delta f_0}$ | | |
| $x_1$ | $f_1$ | | $\boxed{\Delta^2 f_0}$ | |
| | | $\Delta f_1$ | | $\boxed{\Delta^3 f_0}$ |
| $x_2$ | $f_2$ | | $\Delta^2 f_1$ | |
| | | $\Delta f_2$ | | |
| $x_3$ | $f_3$ | | | |

Table 2: Backward difference table:

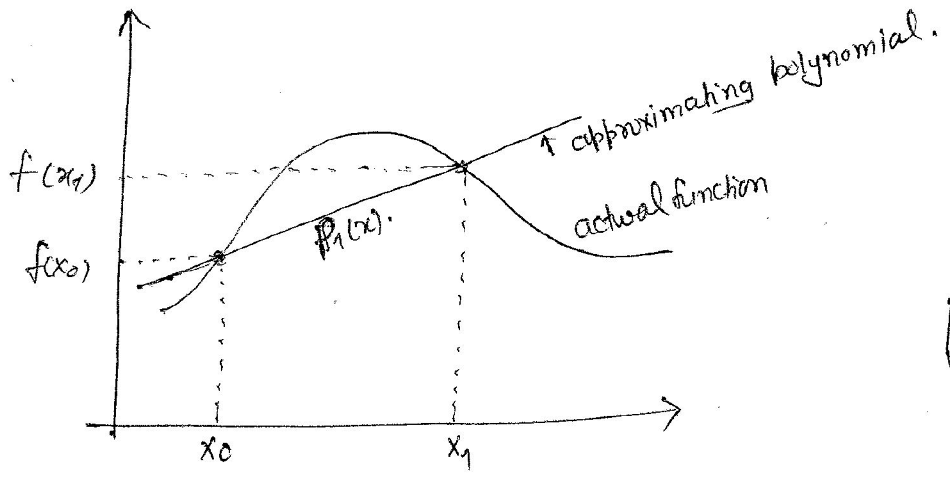
| $x$ | $f(x)$ | $\nabla f$ | $\nabla^2 f$ | $\nabla^3 f$ |
|---|---|---|---|---|
| $x_0$ | $f_0$ | | | |
| | | $\nabla f_1$ | | |
| $x_1$ | $f_1$ | | $\nabla^2 f_2$ | |
| | | $\nabla f_2$ | | $\boxed{\nabla^3 f_3}$. |
| $x_2$ | $f_2$ | | $\boxed{\nabla^2 f_3}$ | |
| $x_3$ | $\boxed{f_3}$ | $\boxed{\nabla f_3}$ | | |

Table 3: Central difference table:

| $x$ | $f(x)$ | $\delta f$ | $\delta^2 f$ | $\delta^3 f$ | $\delta^4 f$ |
|---|---|---|---|---|---|
| $x_0$ | $f_0$ | | | | |
| | | $\delta f_{1/2}$ | | | |
| $x_1$ | $f_1$ | | $\delta^2 f_1$ | | |
| | | $\delta f_{3/2}$ | | $\delta^3 f_{3/2}$ | |
| $x_2$ | $f_2$ | | $\delta^2 f_2$ | | $\delta^4 f_2$. |
| | | $\delta f_{5/2}$ | | $\delta^3 f_{5/2}$ | |
| $x_3$ | $f_3$ | | $\delta^2 f_3$ | | |
| | | $\delta f_{7/2}$ | | | |
| $x_4$ | $f_4$ | | | | |

# Newton's Forward Difference Interpolation

1. Linear Case: The simplest form of interpolation is to connect two data points with a straight line:




| | |
|---|---|
| $x_0$ | $x_1$ |
| $f(x_0)$ | $f(x_1)$ |

Let us consider the general equation of straight line:

$$P_1(x) = b_0 + b_1(x - x_0) \qquad y = a_0 + a_1 x$$

at the point $x = x_0$.

$$P_1(x_0) = \boxed{f(x_0) = b_0}$$

at the point $x = x_1$

$$P_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$

$$\Rightarrow \frac{f(x_1) - f(x_0)}{x_1 - x_0} = b_1$$

Let us consider the equidistant data point then.

$$b_1 = \frac{f(x_0+h) - f(x_0)}{h} = \frac{\Delta f_0}{h}$$

So. $$\boxed{P_1(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h}}$$

## II Quadratic Interpolation:



If three data points are available this can be accomplished with a second order polynomial. Let us consider a general second order polynomial

$$P_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$$

- at the point $x = x_0$.

$$\boxed{b_0 = f(x_0)}$$

- at the point $x = x_1$:

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} = b_1 \Rightarrow b_1 = \frac{\Delta f_0}{h}$$

(in case of equidistant data point)

- at the point $x = x_2$:

$$f(x_2) = f(x_0) + \frac{f(x_1)-f(x_0)}{h}(x_2-x_0) + b_2(x_2-x_0)(x_2-x_1)$$

$$\Rightarrow f(x_2) = f(x_0) + 2f(x_1) - 2f(x_0) + 2h^2 b_2$$

$$\Rightarrow \frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2} = b_2$$

$$\Rightarrow \frac{\Delta^2 f_0}{|2 \cdot h^2} = b_2.$$

Interpolating polynomial:

$$\boxed{P_2(x) = f(x_0) + (x-x_0)\frac{\Delta f_0}{|1 \cdot h} + (x-x_0)(x-x_1)\frac{\Delta^2 f_0}{|2 \cdot h^2}.}$$

We can now write the Newton's Forward Difference interpolation formula based on $(n+1)$ nodal points $x_0, x_1, \cdots x_n$ as

$$\boxed{\begin{aligned} P_n(x) &= f_0 + (x-x_0)\frac{\Delta f_0}{\lfloor 1\, h} + (x-x_0)(x-x_1)\frac{\Delta^2 f_0}{\lfloor 2 \cdot h^2} + \cdots \\ &\quad + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\frac{\Delta^n f_0}{\lfloor n \cdot h^n}. \end{aligned}} \quad \text{—} (*)$$

If we put $\frac{x-x_0}{h} = u$ then it takes the following form:

$$\boxed{\begin{aligned} P_n(x_0+hu) &= f_0 + u\,\Delta f_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 f_0 + \cdots \\ &\quad + \frac{u(u-1)\cdots(u-n+1)}{\lfloor n}\Delta^n f_0 . \end{aligned}} \quad \text{—} (**)$$

Alternative Derivation:

$$f(x) = f\left(x_0 + \frac{x-x_0}{h}h\right) = f(x_0+uh)$$

$$= E^u f(x_0)$$

$$= (1+\Delta)^u f(x_0)$$

$$= f_0 + u\,\Delta f_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 f_0 + \cdots + \frac{u(u-1)\cdots(u-n+1)}{\lfloor n}\Delta^n f_0$$

$$+ \cdots .$$

neglecting the difference $\Delta^{n+1} f_0$ and higher order differences we get $(**)$.

# Newton's Backward Difference Formula:

$$f(x) = f\left(x_n + \frac{x-x_n}{h}h\right) = f(x_n + hu)$$

$$= E^u f(x_n)$$

$$= (1-\nabla)^{-u} f(x_n)$$

$$= f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{\underline{|2}}\nabla^2 f(x_n) + \cdots +$$

$$\frac{u(u+1)\cdots(u+n-1)}{\underline{|n}}\nabla^n f(x_n) + \cdots$$

where $\frac{x-x_n}{h} = u$.

Neglecting the difference $\nabla^{n+1} f(x_n)$ and higher order differences we get interpolating polynomial

$$\boxed{P_n(x_n+hu) = f_n + u\nabla f_n + \frac{u(u+1)}{\underline{|2}}\nabla^2 f_n + \cdots + \frac{u(u+1)(u+2)\cdots(u+n-1)}{\underline{|n}}\nabla^n f_n.}$$

OR.

$$\boxed{P_n(x) = f_n + \frac{(x-x_n)}{\underline{|1}\cdot h}\nabla f_n + \frac{(x-x_n)(x-x_{n-1})}{\underline{|2}\cdot h^2}\nabla^2 f_n + \cdots + \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_1)}{\underline{|n}\; h^n}\nabla^n f_n.}$$

Example: Find the cubic polynomial which takes to following values

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | 1 | 2 | 1 | 10 |

Use Newton's Forward and Backward interpolation.

Solution: The difference table:

| $x$ | $f(x)$ | $\Delta f / \nabla f$ | $\Delta^2 f / \nabla^2 f$ | $\Delta^3 f / \nabla^3 f$ |
|---|---|---|---|---|
| 0 | 1 | | | |
| | | 1 | | |
| 1 | 2 | | $-2$ | |
| | | $-1$ | | 12 |
| 2 | 1 | | 10 | |
| | | 9 | | |
| 3 | 10 | | | |

Forward:

$$P_3(x) = f_0 + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{\lfloor 2\, h^2}\Delta^2 f_0 + \frac{(x-x_0)(x-x_1)(x-x_2)}{\lfloor 3 \cdot h^3}\Delta^3 f_0$$

$$= 1 + \frac{(x-0)}{1}\cdot 1 + \frac{(x-0)(x-1)}{2}(-2) + \frac{(x-0)(x-1)(x-2)}{6}\, 12$$

$$= 1 + x - x^2 + \cancel{x} + 2x^3 - 6x^2 + 4x$$

$$= 2x^3 - 7x^2 + 6x + 1. \quad \text{Ans.}$$

Backward:

$$P_3(x) = f_3 + \frac{(x-x_3)}{\lfloor 1\, h}\nabla f_3 + \frac{(x-x_3)(x-x_2)}{\lfloor 2\, h^2}\nabla^2 f_3 + \frac{(x-x_3)(x-x_2)(x-x_1)}{\lfloor 3 \cdot h^3}\nabla^3 f_3$$

$$= 10 + (x-3)\cdot 9 + \frac{(x-3)(x-2)}{2}\, 10 + \frac{(x-3)(x-2)(x-1)}{6}\cdot 12$$

$$= 10 + 9x - 27 + 5x^2 - 25x + 30 + 2(x^3 - 6x^2 + 11x - 6)$$

$$= 2x^3 - 7x^2 + 6x + 1. \quad \text{Ans.}$$

**Example:** Construct Newton's forward interpolation polynomial for the following table:

| $x$ | 4 | 6 | 8 | 10 |
|---|---|---|---|---|
| $y$ | 1 | 3 | 8 | 16. |

Hence evaluate $y$ for $x=5$.

**Solution:** Difference table.

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 4 | 1 | | | |
| | | 2 | | |
| 6 | 3 | | 3 | |
| | | 5 | | 0 |
| 8 | 8 | | 3 | |
| | | 8 | | |
| 10 | 16 | | | |

$$P(x) = 1 + \frac{(x-4)}{2} 2 + \frac{(x-4)(x-6)}{\lfloor 2 \cdot (2)^2} \cdot 3$$

$$= 1 + x - 4 + \frac{3}{8}(x^2 - 10x + 24)$$

$$= 6 - \frac{11}{4}x + \frac{3}{8}x^2$$

$$P(5) = 6 - \frac{11}{4} \times 5 + \frac{3}{8} \times 25$$

$$= 1.625$$ Ans.

**Example:** Estimate the values of $f(22)$ and $f(42)$ from the following available data:

| $x$ | 20 | 25 | 30 | 35 | 40 | 45 |
|---|---|---|---|---|---|---|
| $f(x)$ | 354 | 332 | 291 | 260 | 231 | 204 |

Difference table:

| $x$ | $f(x)$ | $\Delta f(x) / \nabla f(x)$ | $\Delta^2 f / \nabla^2 f$ | $\nabla^3 f / \nabla^3 f$ | $\nabla^4$ | $\nabla^5$ |
|---|---|---|---|---|---|---|
| 20 | 354 | | | | | |
| | | -22 | | | | |
| 25 | 332 | | -19 | | | |
| | | -41 | | 29 | | |
| 30 | 291 | | 10 | | -37 | |
| | | -31 | | -8 | | 45 |
| 35 | 260 | | 2 | | 8 | |
| | | -29 | | 0 | | |
| 40 | 231 | | 2 | | | |
| | | -27 | | | | |
| 45 | 204 | | | | | |

$f(22)$: $x = 22$. $x_0 = 20$ $h = 5$

$$u = \frac{2}{5}.$$

$$P(22) = f_0 + u\Delta f_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 f_0 + \cdots$$

$$= 354 + \frac{2}{5}\cdot(-22) + \frac{\frac{2}{5}\left(\frac{2}{5}-1\right)}{2}\cdot(-19) + \frac{\frac{2}{5}\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)}{6}\cdot(29)$$

$$+ \frac{\frac{2}{5}\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)\left(\frac{2}{5}-3\right)}{24}(-37) + \frac{\frac{2}{5}\left(\frac{2}{5}-1\right)\cdots\left(\frac{2}{5}-4\right)}{120}45$$

$$= 352.223.$$ Ans-

$f(42)$: $x = 42.$ $x_n = 45$ $u = \frac{x - x_n}{h} = \frac{42-45}{5} = -\frac{3}{5}.$

$$P(42) = f_n + u\nabla f_n + \frac{u(u+1)}{\lfloor 2}\nabla^2 f_n + \cdots + \frac{u(u+1)(u+2)(u+3)(u+4)}{\lfloor 5}\nabla^5 f_n$$

$$= 204 + \left(-\frac{3}{5}\right)\cdot(-27) + \frac{\left(-\frac{3}{5}\right)\left(-\frac{3}{5}+1\right)}{\lfloor 2}(2) + \frac{\left(-\frac{3}{5}\right)\left(-\frac{3}{5}+1\right)\left(-\frac{3}{5}+2\right)\left(-\frac{3}{5}+3\right)}{\lfloor 4}8$$

$$+ \frac{\left(-\frac{3}{5}\right)\cdots\left(-\frac{3}{5}+4\right)}{\lfloor 5}\cdot 45.$$

$$= 218.6630.$$ Ans.

Example:

Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree 4 in $x$:

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 1 | $-1$ | 1 | $-1$ | 1. |

Solution: Difference table:

| | $x$ | $y$ | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 f$ |
|---|---|---|---|---|---|---|
| $x_0$ | 1 | 1 | | | | |
| | | | $-2$ | | | |
| $x_1$ | 2 | $-1$ | | 4 | | |
| | | | 2 | | $-8$ | |
| $x_2$ | 3 | 1 | | $-4$ | | 16. |
| | | | $-2$ | | 8 | |
| $x_3$ | 4 | $-1$ | | 4 | | |
| | | | 2 | | | |
| $x_4$ | 5 | 1 | | | | |

Backward difference formula:

$$\varphi_4(x) = f_4 + \frac{(x-x_4)}{\lfloor 1 \cdot h}\nabla f_4 + \frac{(x-x_4)(x-x_3)}{\lfloor 2\, h^2}\nabla^2 f_4$$

$$+ \frac{(x-x_4)(x-x_3)(x-x_2)}{\lfloor 3 \cdot h^3}\nabla^3 f_4 + \frac{(x-x_4)(x-x_3)(x-x_2)(x-x_1)}{\lfloor 4 \cdot h^4}\nabla^4 f_4$$

$$= 1 + \frac{(x-5)}{1}(2) + \frac{(x-5)(x-4)}{\cancel{2}}\cdot \cancel{4}^{2} + \frac{(x-5)(x-4)(x-3)}{\cancel{6}\,3}\,\cancel{8}\,4$$

$$+ \frac{(x-5)(x-4)(x-3)(x-2)}{\cancel{24}\,3}\cdot\cancel{16}\,2$$

$$= \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31.$$

ERROR:

**Theorem:** Let $x_0, x_1, \cdots x_n$ be $(n+1)$ distinct nodes and let $x$ be a point belonging to the domain of a given function $f$. Assume that $f \in C^{n+1}(I_x)$, where $I_x$ is the smallest interval containing the nodes $x_0, x_1, \cdots x_n$ and $x$. Then the interpolation error at the point $x$ is given by

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{\lfloor(n+1)}(x-x_0)(x-x_1)\cdots(x-x_n).$$

where $\xi \in I_x$.

**Proof:** For simplicity let us assume $\omega_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

Note that the result is obviously true if $x$ coincides with any of the interpolation nodes.

Now Define for any $x \in I_x$ the function

$$G(t) = E_n(t) - \frac{\omega_{n+1}(t)\, E_n(x)}{\omega_{n+1}(x)} \qquad t \in I_x$$

Since $f \in C^{(n+1)}(I_x)$ and $\omega_{n+1}$ is a polynomial, then $G \in C^{(n+1)}(I_x)$ and it has $n+2$ distinct zeros in $I_x$ since

$$G(x_i) = E_n(x_i) - \frac{\omega_{n+1}(x_i)\, E_n(x)}{\omega_{n+1}(x)} = 0 \qquad i = 0, \cdots n.$$

$$G(x) = E_n(x) - \frac{\omega_{n+1}(x)\, E_n(x)}{\omega_{n+1}(x)} = 0.$$

Then Using mean value theorem: $G'$ has atleast $(n+1)$ distinct zeros. By recursion it follows that $G^{(j)}$ admits at least $n+2-j$ distinct zeros.

$\Rightarrow G^{(n+1)}$ has atleast one zero, which we denote by $\xi$.

We know $E_n^{(n+1)}(t) = f^{(n+1)}(t)$ (Since $E_n(t) = f(t) - P_n(t)$)

and $\omega_{n+1}^{(n+1)}(x) = \lfloor n+1$

We get



$$G^{(n+1)}(t) = f^{(n+1)}(t) - \frac{\lfloor(n+1)}{\omega_{n+1}(x)} E_n(x)$$

at $t=\xi$.

$$0 = f^{(n+1)}(\xi) - \frac{\lfloor(n+1)}{\omega_{n+1}(x)} E_n(x)$$

$$\Rightarrow \boxed{E_n(x) = \frac{f^{(n+1)}(\xi)}{\lfloor n+1} . \omega_{n+1}(x)}$$

Newton's Divided-Difference interpolating polynomials: (Works also for Non-equi-distant points)

Linear Case: A general equation of the straight line.

$$P_1(x) = b_0 + b_1(x - x_0)$$

at the point $x = x_0$:

$$P_1(x_0) = \boxed{f(x_0) = b_0}$$

at the point $x = x_1$

$$P_1(x_1) = f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

$$\Rightarrow b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_1, x_0] \quad \text{divided difference.}$$

So,

$$\boxed{P_1(x) = f(x_0) + f[x_1, x_0](x - x_0)}$$

Quadratic Interpolation: Let us assume we have three data point $(x_0, f(x_0))$ $(x_1, f(x_1))$ & $(x_2, f(x_2))$

In this case we can fit a polynomial of degree 2.

$$P_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

at $x = x_0$:

$$P_2(x) = \boxed{f(x_0) = b_0}$$

at $x = x_1$:

$$P_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$

$$\Rightarrow \underbrace{\frac{f(x_1) - f(x_0)}{x_1 - x_0}}_{f[x_1, x_0]} = b_1$$

at $x = x_2$:

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$\vdots$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= f[x_2; x_1, x_0].$$

So,

$$P_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General form of Newton's interpolating polynomials:

$$P_n(x) = f(x_0) + (x - x_0) f[x_1, x_0] + (x - x_0)(x - x_1) f[x_2, x_1, x_0] + \cdots$$

$$\cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f[x_n, x_{n-1}, \cdots x_0].$$

This is called Newton's divided difference interpolating polynomials.

Lagrange interpolating polynomials: (works also for non-equidistant points)

The Lagrange interpolating polynomial is again simply a reformulation of the Newton's polynomials that avoids the computation of divided differences.

Linear: Newton's divided difference formula:

$$P_1(x) = f(x_0) + f[x_1, x_0](x - x_0)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$= L_0(x) f(x_0) + L_1(x) f(x_1)$$

$$P_1(x) = \sum_{i=0}^{1} L_i(x) f(x_i)$$

$L_i(x)$ are called Lagrange coefficient, defined as: (Lagrange fundamental Polynomials).

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

Note that

$$L_i(x) = \begin{cases} 0 & x = x_0, x_1, \cdots x_n \\ 1 & x = x_i \end{cases}$$

In general, suppose $(n+1)$ data points $x_0, x_1, x_2 \cdots x_n$ are given then.

$$\boxed{P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)}$$

**Example:** Estimate, using Newton's divided difference formula, natural logarithm of 2 using linear interpolation. First perform the computation by interpolating between $\ln 1 = 0$ and $\ln 6 = 1.791759$. Then repeat the procedure but use a smaller interpolation $\ln 1$ to $\ln 4$ $(1.386294)$. Note that the true value of $\ln 2$ is $0.6931472$.

**Solution:** (I) $$f_1(2) = f(1) + \frac{f(6)-f(1)}{6-1}(2-1)$$

$$= 0 + \frac{1.791759}{5} = 0.3583518.$$

(II): $$f_1(2) = 0 + \frac{1.386294}{3} = 0.462098.$$

**Example:** Fit a second order polynomial to the given three points:

$x_0 = 1$ $\quad f(x_0) = 0$ $\quad \ln 1$

$x_1 = 4$ $\quad f(x_1) = 1.386294$ $\quad \ln 4$

$x_2 = 6$ $\quad f(x_2) = 1.791759$ $\quad \ln 6$

Use the polynomial to evaluate value at $x = 2$.

**Sol:** $$P_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

Divided difference table:

| $x$ | $f(x)$ | $f[x_1, x_0]$ | $f[x_2, x_1, x_0]$ |
|---|---|---|---|
| 1 | 0 | | |
| | | 0.462098 | |
| | | | −0.0518731 |
| 4 | 1.386294 | | |
| | | 0.2027325 | |
| 6 | 1.791759 | | |

$$P_2(x) = 0 + 0.462098(x-1) - 0.0518731(x-1)(x-4)$$

$$P_2(2) = 0.5658442$$

Example: Use Lagrange interpolating polynomial of the second order to evaluate $\ln 2$. Given data points are

| | | |
|---|---|---|
| $x_0 = 1$ | $f(x_0) = 0$ | $\ln 1$ |
| $x_1 = 4$ | $f(x_1) = 1.386294$ | $\ln 4$ |
| $x_2 = 6$ | $f(x_2) = 1.791760$ | $\ln 6$ |

Sol:

$$P_2(2) = \frac{(2-4)(2-6)}{(1-4)(1-6)} * 0 + \frac{(2-1)(2-6)}{(4-1)(4-6)} * 1.386294$$

$$+ \frac{(2-1)(2-4)}{(6-1)(6-4)} \, 1.791760$$

$$= 0.5658442.$$

Ans.

## Numerical Integration ( Numerical evaluation of integrals)

Complicated integrals: $\int_0^1 e^{-x^2}dx$ or $\int_0^{\pi} x^{\pi}\sin(\sqrt{x})dx$ etc.

### Newton's Cotes Integration formulas:

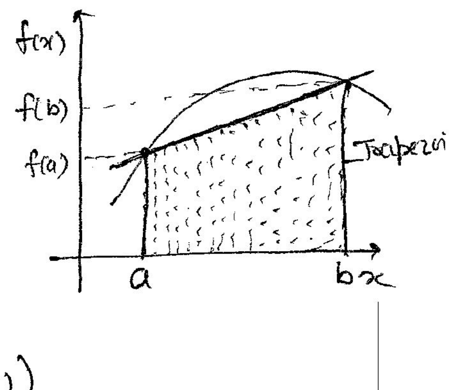
These formulas are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate

$$I = \int_a^b f(x)\,dx \simeq \int_a^b P_n(x)\,dx$$

where $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$.

### (1) The trapezoidal rule: (single application)

$$I = \int_a^b f(x)\,dx \simeq \int_a^b P_1(x)\,dx$$




$$= \int_a^b \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]dx$$

$$= f(a)(b-a) + \frac{f(b)-f(a)}{b-a}\,\frac{1}{2}(b-a)^2$$

$$= f(a)(b-a) + \frac{1}{2}(b-a)\cdot(f(b)-f(a))$$

$$\Rightarrow \boxed{\int_a^b f(x)\,dx \simeq (b-a)\frac{[f(a)+f(b)]}{2}}$$

Problem: Using trapezoidal rule integrate numerically the function

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a=0$ to $b=0.8$. Compare result with exact value of integral $1.640533$.

Solution: The function values

$$f(0) = 0.2$$

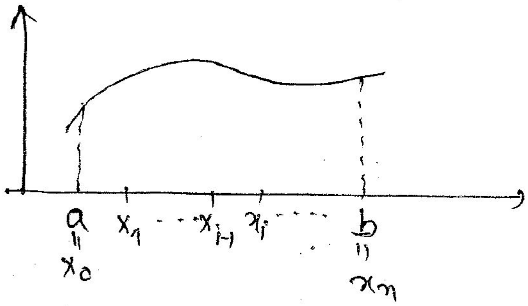
$$f(0.8) = 0.232$$

$$\int_0^{0.8} f(x)\,dx \approx \frac{0.2 + 0.232}{2} \cdot (0.8 - 0)$$

$$= 0.1728.$$

The multiple application of trapezoidal rule.

To improve accuracy of the trapezoidal rule we divide the integration interval from a to b into a number of segments and apply the method to each segment.




Consider there are $n+1$ equally spaced base points $x_0, x_1, \cdots x_n$.

Then. $h = \dfrac{b-a}{n}$

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$\approx h\,\frac{f(x_0)+f(x_1)}{2} + h\,\frac{f(x_1)+f(x_2)}{2} + \cdots + h\,\frac{f(x_{n-1})+f(x_n)}{2}$$

$$= \frac{h}{2}\left[f(x_0) + 2\{f(x_1)+f(x_2)+\cdots+f(x_{n-1})\} + f(x_n)\right]$$

$$\boxed{I \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]}$$

Problem: Use the two-segment trapezoidal rule to estimate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a=0$ to $b=0.8$.

Sol:

0     0.4     0.8

$$h = \frac{0.8-0}{2} = 0.4.$$

$$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$$

$$I = \int_0^{0.8} f(x)\,dx \simeq \frac{0.4}{2}\left[0.2 + 2(2.456) + 0.232\right]$$

$$= 1.0688. \qquad \text{Ans.}$$

ERROR BOUNDS for the trapezoidal rule:

1) Single application:

We know
$$f(x) - P_1(x) = (x-x_0)(x-x_1)\frac{f''(t)}{2} \qquad (1)$$

with a suitable $t$ depending on $x$ between $x_0$ and $x_1$.

Integrating (1) from $x_0$ to $x_1 = x_0+h$ gives:

$$E = \int_{x_0}^{x_0+h} f(x)\,dx - \frac{h}{2}\left[f(x_0)+f(x_1)\right] = \int_{x_0}^{x_0+h} (x-x_0)(x-x_1)\frac{f''(t(x))}{2}\,dx.$$

Applying weighted mean value theorem ($(x-x_0)(x-x_1)$ does not change sign in $[x_0, x_0+h]$), we get

$$E = \frac{f''(\tilde{t})}{2}\int_{x_0}^{x_0+h}(x-x_0)(x-x_0-h)\,dx.$$

Subst. $x - x_0 = v \Rightarrow dx = dv$

$$= \frac{f''(\tilde{t})}{2}\int_0^h v(v-h)\,dv = \frac{f''(\tilde{t})}{2}\left[\frac{1}{3}h^3 - \frac{h}{2}\cdot h^2\right]$$

$$= -\frac{h^3 f''(\tilde{t})}{12} \qquad \text{where } \tilde{t} \in [x_0, x_1].$$

Error in multiple application:

$$E = \sum_{i=0}^{n-1} \left( -\frac{h^3}{12} f''(\tilde{t}_i) \right)$$

$$= -\frac{h^3}{12} \sum_{i=0}^{n-1} f''(\tilde{t}_i)$$

Using Discrete mean value theorem:

$$= -\frac{h^3}{12} \cdot n \cdot f''(\hat{t})$$ with suitable, unknown $\hat{t}$ between a and b.

$$\boxed{E = -\frac{(b-a)}{12} \cdot h^2 f''(\hat{t})}$$

Error bounds: Let $M_2 = \max\limits_{[x_0, x_n]} |f''(x)|$ Then.

$$\boxed{|E| \leq \frac{(b-a) h^2}{12} M_2}$$

Example: Evaluate the following integral using trapezoidal rule with $n = 2, 4$. Compare with the exact solution.

$$\int_0^1 \frac{dx}{3+2x}.$$

Find the bound on the error. Also find the number of subintervals required if the error is to be less than $5 \times 10^{-4}$.

Sol: i) number of subintervals $= 2$ ie $h = 0.5 = \frac{b-a}{n}$ ($b = 1$, $a = 0$, $n = 2$)

Hence.

$$I_1 = \frac{0.5}{2}\left[f(0) + 2f(0.5) + f(1)\right] = \frac{0.5}{2}\left[\frac{1}{3} + 2 \cdot \frac{1}{4} + \frac{1}{5}\right]$$

$$= 0.25833$$

(ii) number of subinterval $n = 4$:

$$h = \frac{1-0}{4} = \frac{1}{4}.$$

$$x_0 = 0 \quad x_1 = \frac{1}{4} \quad x_2 = \frac{2}{4} \quad x_3 = \frac{3}{4} \quad x_4 = 1.$$

Hence.

$$I_2 = \frac{1}{4}\cdot\frac{1}{2}\cdot\left[f(0) + 2\left(f(1/4) + f(\tfrac{1}{2}) + f(3/4)\right) + f(1)\right]$$

$$= \frac{1}{8}\left[\frac{1}{3} + 2\left\{\frac{2}{7} + \frac{1}{4} + \frac{2}{9}\right\} + \frac{1}{5}\right]$$

$$= 0.25615$$

Exact solution: $\frac{1}{2}\ln(5/3) = 0.25541$

Errors: $E_1 = |0.25541 - 0.25833|$

$= 0.00292$

$E_2 = |0.25541 - 0.25615| = 0.00074.$

Errors bounds: $f(x) = \frac{1}{3+2x}$ $\quad f'(x) = -\frac{2}{(3+2x)^2}$

$$f''(x) = \frac{8}{(3+2x)^3}$$

and $M_2 = \max\limits_{[0,1]} \frac{8}{(3+2x)^3} = \frac{8}{27}.$

Hence. $|\text{Error}| \le \frac{(b-a)h^2}{12}\cdot M_2 = \frac{1}{\cancel{12}_{3}}h^2\cdot\frac{\cancel{8}\,2}{27}$

$$= \frac{2h^2}{81}$$

i) for $h = 1/2$ $\quad |\text{Error}| \le 0.00617.$

ii) for $h = \frac{1}{4}$ $\quad |\text{Error}| \le 0.00154$

given $\epsilon = 5\times10^{-4}$.

$$\Rightarrow \frac{(b-a)h^2}{12}M_2 \le 5\times10^{-4}$$

$$\Rightarrow \frac{(b-a)(b-a)^2}{12n^2}\cdot\frac{8}{27} \le 5\times10^{-4}$$

$$\Rightarrow \frac{1\times8}{12\times27\times5\times10^{-4}} \le n^2$$

$$\Rightarrow 49.38 \le n^2 \Rightarrow n \geq 7.03.$$

Since $n$ is an integer, we require $\underline{\underline{n=8}}$.

## Simpsons 1/3 Rule

$$I = \int_a^b f(x)\,dx \simeq \int_a^b P_2(x)\,dx.$$

Let $X_0 = a,\ x_1,\ x_2 = b$.

$$I \simeq \int_{x_0}^{x_2}\left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)\right]dx$$

$$= \frac{1}{2h^2}f(x_0)\int_{x_0}^{x_2}(x-x_1)(x-x_1+x_1-x_2)\,dx$$

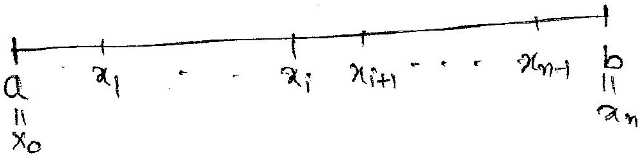
$$- \frac{1}{h^2}f(x_1)\int_{x_0}^{x_2}(x-x_0)(x-x_0+x_0-x_2)\,dx$$

$$+ \frac{1}{2h^2}f(x_2)\int_{x_0}^{x_2}(x-x_0)(x-x_0+x_0-x_1)\,dx.$$

$$= \frac{f(x_0)}{2h^2}\left[\frac{1}{3}(h^3+h^3) - h\cdot 0\right] - \frac{f(x_1)}{h^2}\left[\frac{1}{3}(2h)^3 + \left(\frac{-2h}{2}\right)\cdot(2h)^2\right] + \frac{f(x_2)}{2h^2}\left[\frac{1}{3}\cdot(2h)^3 + \left(-\frac{h}{2}\right)(2h)^2\right].$$

$$I \simeq \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

## Multiple application of Simpson's Rule:




$$b-a = nh$$

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$\approx \frac{h}{3}\{f(x_0)+4f(x_1)+f(x_2)\} + \frac{h}{3}\{f(x_2)+4f(x_3)+f(x_4)\}$$

$$+ \cdots + \frac{h}{3}\{f(x_{n-2})+4f(x_{n-1})+f(x_n)\}$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5\cdots}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)\right].$$

$\square$.

**Error:** Single application

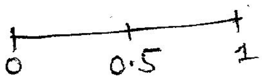
$$E = -\frac{h^5}{90} f^{(4)}(\xi) \qquad \xi \in (a,b) \text{ or } \xi \in (x_0, x_2)$$

multiple application

$$E = -\frac{b-a}{180} h^4 f^{(4)}(\xi) \qquad \xi \in (a,b) \text{ or } \xi \in (x_0, x_n).$$

**Example:** Evaluate $\int_0^1 \frac{dx}{3+2x}$ using Simpson's rule with $n=2,4$. Compare with the exact solution.

**Sol:** $n=2$:




$$I \approx \frac{h}{3}\left[f(0)+4f\left(\tfrac{1}{2}\right)+f(1)\right] = \frac{0.5}{3}\left[\frac{1}{3}+4\cdot\frac{1}{4}+\frac{1}{5}\right]$$

$$= 0.25556.$$

For $n=4$:

$$\begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{array}$$

$$h = \frac{1}{4}$$

$$I \approx \frac{h}{3}\left[f(0) + 4\left\{f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right)\right\} + 2f\left(\tfrac{1}{2}\right) + f(1)\right]$$

$$= 0.25542.$$ Ans.

---

**Discrete mean value theorem:**

Let $f \in C^0[a,b]$ and let $x_j$ be $n+1$ points in $[a,b]$ and $c_j$ be $(n+1)$ constants, all having the same sign. Then there exists $\xi \in [a,b]$ such that

$$\sum_{j=0}^{n} c_j f(x_j) = f(\xi) \sum_{j=0}^{n} c_j$$

In particular, if $c_j = 1 \ \forall \ j = 0, 1, \dots n$.

Then

$$\boxed{\frac{1}{n+1} \sum_{j=0}^{n} f(x_j) = f(\xi)}$$

**Weighted mean value theorem:** Assume $f$ and $g$ are continuous in $[a,b]$. If $g$ never changes sign in $[a,b]$. Then

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx. \qquad c \in [a,b].$$

# Transcendental and polynomial equations (Determination of the roots)

Problem:

Determination of the root of an equation of the form

$$f(x)=0.$$

1. Fixed point iteration method
2. Bisection method
3. Newton's method or Newton-Raphson method
4. Secant method.

## Fixed point iteration method:

We transform $f(x)=0$ into the form

$$x = g(x)$$

and then setup an iteration method:

$$x_{n+1} = g(x_n) \qquad n = 0, 1, \cdots \qquad (*)$$

A solution of $(*)$ is called a fixed point of $g$.

The choice of $g$ is not unique.

## Convergence of fixed point iteration:

Let $x = s$ be a solution of $x = g(x)$ and suppose that $g$ has a continuous derivative in some interval $J$ containing $s$. Then if $|g'(x)| \leq K < 1$ in $J$ the iteration scheme $x_{n+1} = g(x_n)$ converges for any $x_0$ in $J$.

Example: Find a solution of $f(x) = x^3+x-1 = 0$ by fixed point iteration method.

Sol: $x^3+x-1=0 \Rightarrow x(x^2+1) = 1 \Rightarrow x = \frac{1}{1+x^2} =: g(x)$

iteration method becomes:

$$x_{n+1} = \frac{1}{1+x_n^2}$$

Note that $g'(x) = \frac{2x}{(1+x^2)^2}$

or $|g'(x)| = \frac{2|x|}{(1+x^2)^2} < 1$. $\forall x$.

So we have convergence for any $x_0$.

take $x_0 = 0$.

$x_1 = \frac{1}{1+0} = 1$

$x_2 = \frac{1}{2} = 0.5$

$x_3 = 0.800$

$x_4 = 0.6098$

$x_5 = 0.7290$ $\quad x_6 = 0.6530$ $\quad x_7 = 0.7011$

$x_8 = 0.6705$ $\quad x_9 = 0.6899$ $\quad x_{10} = 0.6775$

$x_{11} = 0.6854$ $\quad x_{12} = 0.6804$ $\quad x_{13} = 0.6836$

$x_{14} = 0.6815$ $\quad x_{15} = 0.\underline{68}29$ $\quad x_{16} = 0.\underline{682}0$

**Bisection Method:** The bisection method is based on the following property:

Theorem for zeros of continuous functions: Given a continuous function $f: [a,b] \to \mathbb{R}$ such that $f(a)f(b) < 0$, then $\exists$ $\alpha \in (a,b)$ such that $f(\alpha) = 0$.

Algorithm: Starting from $I_0 = [a,b]$ so that $f(a)f(b) < 0$, the bisection method generates a sequence of subinterval $I_k = [a^{(k)}, b^{(k)}]$ $k \geq 0$ with $I_k \subset I_{k-1}$, $k \geq 1$. and enjoy the property that $f(a^{(k)})f(b^{(k)}) < 0$.

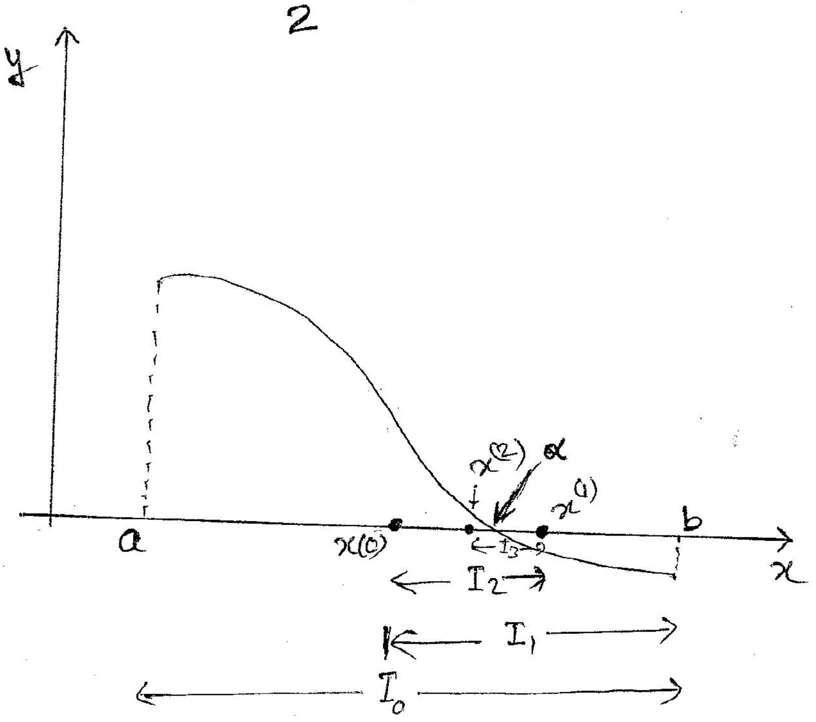
Precisely we set $a^{(0)} = a$ & $b^{(0)} = b$ and

$x^{(0)} = \frac{a^{(0)} + b^{(0)}}{2}$ then for $k \geq 0$:

set $a^{(k+1)} = a^{(k)}$, $b^{(k+1)} = x^{(k)}$ if $f(x^{(k)})f(a^{(k)}) < 0$;

set $a^{(k+1)} = x^{(k)}$, $b^{(k+1)} = b^{(k)}$ if $f(x^{(k)})f(b^{(k)}) < 0$;

set $x^{(k+1)} = \frac{a^{(k+1)} + b^{(k+1)}}{2}$.




Note that $|I_k| = \frac{|I_0|}{2^k}$, $k \geq 0$

$= \frac{b-a}{2^k}$, $k \geq 0$.

Denoting $e^{(k)} = x^{(k)} - \alpha$. (absolute error at step k)

It follows $|e^{(k)}| < \frac{|I_k|}{2} = \frac{(b-a)}{2^{k+1}}$, $k \geq 0$.

$\Rightarrow \lim_{k \to \infty} |e^{(k)}| = 0$

\# The bisection method is therefore globally convergent.

<u>Example:</u> Perform five iterations of the bisection method to obtain the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0.$$

<u>Sol:</u> Since $f(0) = 1$ & $f(1) = -3$

& $f(0)f(1) < 0$,

the smallest positive root lies in the interval. $(0,1)$.

take $a_0 = 0$ & $b_0 = 1$.

$$x_0 = \frac{0+1}{2} = 0.5$$

$f(a_0)f(x^{(0)}) < 0$

1st iter. $\underbrace{a_1 = 0}_{f>0}$ $\underbrace{b_1 = 0.5}_{f<0}$ $\underbrace{x_1 = 0.25}_{f<0}$ $f(0)f(x_1) < 0$

2nd iter. $\underbrace{a_2 = 0}_{f>0}$ $\underbrace{b_2 = 0.25}_{f<0}$ $\underbrace{x_2 = 0.125}_{f>0}$ $f(b_2)f(x_2) < 0$

3rd iter. $\underbrace{a_3 = 0.125}_{f>0}$ $\underbrace{b_3 = 0.25}_{<0}$ $\underbrace{x_3 = 0.1875}_{f>0}$ $f(b_3)f(x_3) < 0$

4th iter. $\underbrace{a_4 = 0.1875}_{f>0}$ $\underbrace{b_4 = 0.25}_{f<0}$ $\underbrace{x_4 = 0.21875}_{<0}$ $f(a_4)f(x_4) < 0$

The root lies in $(0.1875, 0.21875)$

The approximate root is taken as the mid point of this interval that is, $x_5 = \underline{\underline{0.203125}}$.

## Newton Raphson method:

Let $x_K$ be an approximation to the root of $f(x)=0$.

Let $\Delta x$ be an increment in $x$ such that $x_K+\Delta x$ is an exact root.

Then $$f(x_K+\Delta x)=0$$

Expanding Taylor series about the point $x_K$, we get

$$f(x_K)+\Delta x\, f'(x_K)+\frac{1}{\lfloor 2}\Delta x^2 f''(x_K)+\cdots=0$$

Neglecting second and higher powers of $\Delta x$, we get

$$f(x_K)+\Delta x\, f'(x_K)\simeq 0$$

$$\Rightarrow \quad \Delta x \simeq -\frac{f(x_K)}{f'(x_K)}$$

Hence we get the iteration method

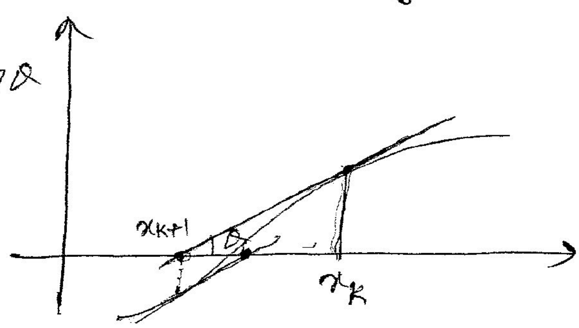
$$x_{K+1}=x_K+\Delta x = x_K-\frac{f(x_K)}{f'(x_K)}, \quad K=0,1,\ldots$$

$$\boxed{x_{K+1}=x_K-\frac{f(x_K)}{f'(x_K)}}$$

Alternatively: function is approximated by its tangent line. Then $x$-intercept gives better approximation of the root.

From the figure: $$\frac{f(x_K)-0}{(x_K-x_{K+1})}=f'(x_K)=\tan\alpha$$

$$\Rightarrow \quad \boxed{x_{K+1}=x_K-\frac{f(x_K)}{f'(x_K)}}$$



Examples: Perform four iterations of the Newton-Raphson method to find the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0.$$

Solution: The smallest positive root lies in the interval $(0,1)$. Take the initial approximation as $x_0 = 0.5$. We have

$$f(x) = x^3 - 5x + 1 = 0$$

$$f'(x) = 3x^2 - 5$$

N-R method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow x_{k+1} = x_k - \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5}$$

$$= \frac{2x_k^3 - 1}{3x_k^2 - 5} \qquad k = 0, 1, \ldots$$

$x_0 = 0.5$

$x_1 = 0.1765$

$x_2 = 0.2016$

$x_3 = 0.2016$

$x_4 = 0.2016$

$x_0 = 0.5$

$x_1 = 0.1764705882$

$x_2 = 0.2015680743$

$x_3 = 0.2016396750$

$x_4 = 0.2016396757$

Example: Apply Newton-Raphson's method to determine a root of the equation

$$f(x) = \cos x - x e^x = 0$$

such that $|f(x^*)| < 10^{-8}$, where $x^*$ is the approximation to the root. Take the initial approximation as $x_0 = 1$.

Solution: $x_0 = 1$.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{(\cos x_k - x_k e^{x_k})}{(-\sin x_k - e^{x_k} - x_k e^{x_k})}$$

| K | $x_K$ | $f(x_K)$ |
|---|---|---|
| 0 | 1 | |
| 1 | 0.6531 | −.4606 |
| 2 | 0.5313 | − 0.0418 |
| 3 | 0.5179--- | $-4.6413 \times 10^{-4}$ |
| 4 | 0.5178--- | $-5.9268 \times 10^{-8}$ |
| 5 | 0.5178--- | $-8.88 \times 10^{-16}$ |

Secant method: Though Newton's method is very powerful, it requires evaluation of the derivative.
This suggests the idea of replacing derivatives $f'(x_n)$ by the difference quotient.

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

The method becomes now

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})}(x_n - x_{n-1})} \quad n = 0, 1, 2, \ldots$$

This is known as secant method.