Sylow's Theorem

Lecture 16

Sylow's Thm :

If G is a gp of order p^2 then

$$G \cong \mathbb{Z}_{p^2} \quad \text{or} \quad G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}.$$

<u>Defn</u>. Let G be a gp and p be a prime number such that $p \mid |G|$

(1) A gp of order p^α for some $\alpha \geq 1$ is called a p-group.

Subgps of G which are p-groups are called p-subgroups of G.

(2) If G is a group of order $p^\alpha m$ where $p \nmid m$ then a subgp of order p^α is called a Sylow-p subgp of G.

(3) n_p = denotes the no. of Sylow p-subgps of G.

<u>Thm</u>: Let G be a group of order $p^{\alpha}m$ where $p \nmid m$ and p is a prime no.

(1) Sylow p-subgps exists i.e $\exists$ a subgp of order p^{α} in G.

[There exists subgps of order p^k for all $1 \leq k \leq \alpha$ in G]

(2) If P is Sylow p-subgp of G and Q is any p-subgp of G then $\exists$ $g \in G$ s.t $Q \subseteq gPg^{-1}$

i.e Q is contained in some conjugate of P.

In particular any two Sylow p-subgps are conjugate in G.

<u>Discussion</u> P be a Sylow p-subgp of G. $|P| = p^{\alpha}$

Let $g \in G$. consider gPg^{-1}

<u>Q</u> Is $gPg^{-1} \subset G$ a subgp of G?

$$gPg^{-1} = \{ gxg^{-1} \mid x \in P \}.$$

$$gPg^{-1} \ni 1 = g1g^{-1} \text{ where } 1 \in P.$$

$$(gp_1g^{-1})(gp_2g^{-1}) = gp_1\underline{g^{-1}g}p_2g^{-1}$$

$$= g(p_1p_2)g^{-1}$$

$$\in gPg^{-1}$$

The inverse of gpg^{-1} is $gp^{-1}g^{-1}$.

$$(gpg^{-1})(gp^{-1}g^{-1}) = 1.$$

$$\parallel$$

$$gpg^{-1}gp^{-1}g^{-1} = 1.$$

gPg^{-1} is a subgp of G.

$$|gPg^{-1}| = |P| = p^{\alpha}.$$

$\Rightarrow$ If P is a Sylow p-subgp then gPg^{-1} is also a Sylow p-subgp.

Suppose $\exists$ only one Sylow p-subgp then $gPg^{-1} = P \quad \forall g \in G$.

$\Rightarrow$ P is a normal subgp of G.

Let P' be any Sylow p-subgp of G.

i.e P' is a p-subgp of G.

By 2nd statement $\exists\, g \in G$ s.t

$$P' \subseteq gPg^{-1}$$

$$\Rightarrow \quad P' = gPg^{-1}$$

Let in G there are r Sylow p-subgps are there say

$P_1, P_2, P_3, \ldots, P_r$.

Then $\quad P_i = g_i P_1 g_i^{-1}$ for some $g_i \in G$.

$|G| = 24 = \underline{\underline{2}}^3 \cdot \underline{\underline{3}}$

$|P| =$

Sylow 2-subgp. $|P| = 8$.

Sylow $\underline{\underline{3}}$-subgp $|Q| = 3$.

(3) n_p = no. of Sylow p-subgps in G.

$n_p \equiv 1 \pmod{p}$

and $n_p \mid m$

$|G| = p^\alpha m$. s.t $p \nmid m$.

Example. Group of order 15.

$$15 = 3 \cdot 5$$

n_3 = # of Sylow 3-subgp of G.

$n_3 \mid m$ and $n_3 \equiv 1 \pmod 3$

$$\Downarrow$$

$$n_3 = 1.$$

n_5 = # of Sylow 5-subgp of G

$n_5 \mid 3$ and $n_5 \equiv 1 \pmod 5$

$$\Downarrow$$

$$n_5 = 1$$

Let H and K are Sylow 3 and Sylow 5-subgps respectively.

$|H| = 3$, $|K| = 5$

Let $H = \langle x \rangle$ s.t $|x| = 3$ and

$K = \langle y \rangle$ s.t. $|y| = 5$.

$H \trianglelefteq G$, $K \trianglelefteq G$, $H \cap K = \{1\}$,

and $|HK| = 15$. $G = HK$,

$\therefore G \cong H \times K \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

$$\cong \mathbb{Z}/15\mathbb{Z}.$$

i.e G is a cyclic gp of order 15.

<u>Example</u> Let G be a gp of order 21.

$21 = 3.7$

$n_3 \mid 7$ and $n_3 \equiv 1 \pmod 3$

$\Rightarrow n_3 = 1$ or 7.

$n_7 \mid 3$ and $n_7 \equiv 1 \pmod 7$, $\Rightarrow n_7 = 1$

There exists only one Sylow 7 subgp

Say H which is normal.

Let $H = \langle x \rangle$ s.t $|x| = 7$.

<u>Case 1</u>. Let $n_3 = 1 \Rightarrow \exists$ only one Sylow 3-subgp of G.

Let $K = \langle y \rangle$, $|y| = 3$.

and $K \triangleleft G$. $H \cap K = \{1\}$,

$|HK| = 21 \Rightarrow G = HK$.

$G \cong H \times K \cong \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/21\mathbb{Z}$.

i.e G is a cyclic gp of order 21.

Case 2. Let $n_3 = 7$ i.e $\exists$ 7 Sylow 3-subg
Let K be a Sylow 3-subgp and $K = \langle y \rangle$.
$\langle x \rangle = H$ is a normal subgp of G.
$|x| = 7$

$$y x y^{-1} \in H = \{1, x, x^2, \dots, x^6\}.$$

let $y x y^{-1} = x^r$ where $1 \leq r \leq 6$.

Now
$$y^2 x y^{-2} = y(y x y^{-1}) y^{-1}$$
$$= y x^r y^{-1}$$
$$= \underbrace{y x y^{-1} \, y x y^{-1} \cdots y x y^{-1}}_{r\text{-times}}$$
$$= x^{r^2}$$

$$x = y^3 x y^{-3} = x^{r^3} \Rightarrow x^{r^3 - 1} = 1.$$

$\boxed{\text{as } |y| = 3}$

$$\Rightarrow r^3 \equiv 1 \pmod 7.$$
$$\Rightarrow r = 1, 2, 4$$

If $r = 1$ then $yxy^{-1} = x$ $\Big\} \to G = \langle x, y \mid |x| = 7, |y| = 3, yxy^{-1} = x \rangle$

$\Rightarrow G$ is abelian (Check!).

and in this case $G \cong \mathbb{Z}_{21}$.

If $r = 2$ then $yxy^{-1} = x^2$.

$$G = \langle x, y \mid |x| = 7, |y| = 3, yxy^{-1} = x^2 \rangle.$$

If $r = 4$, then $yxy^{-1} = x^4$.

but if $yxy^{-1} = x^2$

$$\Rightarrow y^2 x y^{-2} = x^4$$

But we are back to the previous case since both y & y^2 are gen of K

If G is a gp of order 21 then

$$G \cong \mathbb{Z}/21\mathbb{Z}$$

or

$$G \cong \langle x, y \mid |x| = 7,\ |y| = 3,\ yxy^{-1} = x^2 \rangle.$$