Class Test 5 Solution

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Q1 (a) True.

Let $\bar{a} \in R/I$ then $\bar{a}^2 = \bar{a}$

$\Rightarrow \bar{a}(\bar{a}-\bar{1}) = \bar{0}$

Since I is a prime ideal R/I is an integral domain which implies either $\bar{a} = \bar{0}$ or $\bar{a} = \bar{1}$.

Thus $|R/I| = 2$.

(b) False

(0) is a prime ideal in $\mathbb{Q}[x]$ but not a maximal ideal.

Since $\mathbb{Q}[x]$ is a PID every non-zero prime ideal is maximal in $\mathbb{Q}[x]$.

(c) False

$\mathbb{Z}[x]$ an UFD but not a PID as the ideal $(2,x) \subseteq \mathbb{Z}[x]$ can't be generated by a single elt.

(d) $N(1+5i) = 26$ which neither a prime nor the square of a prime. Thus it is not a Gaussian prime

False.

(e) False.

$2x \in \mathbb{Z}[x]$ is reducible but it is irreducible over $\mathbb{Q}[x]$ as 2 is an unit in $\mathbb{Q}$.

Q2. A maximal ideal of $\mathbb{Z}_8 \oplus \mathbb{Z}_{30}$ is of the form either $I \oplus \mathbb{Z}_{30}$ or $\mathbb{Z}_8 \oplus J$ where I and J are maximal ideals of $\mathbb{Z}_8$ and $\mathbb{Z}_{30}$ respectively. $\mathbb{Z}_8$ has only one maximal ideal i.e $2\mathbb{Z}_8$ and $\mathbb{Z}_{30}$ has 3 maximal ideals namely

$2\mathbb{Z}_{30}$, $3\mathbb{Z}_{30}$, $5\mathbb{Z}_{30}$.

Thus the maximal ideals are

$$2\mathbb{Z}_8 \oplus \mathbb{Z}_{30}, \qquad \mathbb{Z}_8 \oplus 3\mathbb{Z}_{30}$$

$$\mathbb{Z}_8 \oplus 2\mathbb{Z}_{30}, \qquad \mathbb{Z}_8 \oplus 5\mathbb{Z}_{30}.$$

Remark: Many students has written that maximal ideals of $\mathbb{Z}_{30}$ is $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$ which is wrong. I have deducted 1 mark for that.

Q3. $\mathbb{Z}/5\mathbb{Z}[x] \Big/ (x^3 + 2x^2 + ax + 3)$ will be a field if the poly $f(x) = x^3 + 2x^2 + ax + 3$ is irreducible over $\mathbb{Z}/5\mathbb{Z}$.

Since $f(x)$ is of deg 3 thus it is enough to check if $f(x)$ has any root over $\mathbb{Z}/5\mathbb{Z}$.

we can check that for $a = \bar{0}, \bar{1}, \bar{2}$ in $\mathbb{Z}/5\mathbb{Z}$ $f(x)$ doesn't have any root.

Thus for $a = \bar{0}, \bar{1}, \bar{2}$ the quotient ring is a field.

Q4. $17 \equiv 1 \pmod 4$

& $7 \equiv 3 \pmod 4$, and $17 = (1+4i)(1-4i)$

Now take $A + iB = \begin{cases} 7(1+4i)(1-4i) & \text{or} \\ 7(1+4i)(1+4i) & \text{or} \\ 7(1-4i)(1-4i) \end{cases}$

Then multiply each of them by the units ± 1, $\pm i$ of $\mathbb{Z}[i]$ we get total 12 representations

Q5 $\alpha = 27 - 23i$, $\beta = 8 + i$

Then $\dfrac{\alpha}{\beta} = \dfrac{193}{65} - \dfrac{211}{65} i$

Taking closest integer we get

$$\frac{\alpha}{\beta} = (3 - 3i) - \frac{2}{65}(1 + 8i)$$

$$\Rightarrow \quad \alpha = (3 - 3i)\beta - 2i$$

Here $N(2i) < N(\beta)$.