Stability analysis of single step method

Recall:

Consider a single step method

$$u_{n+1} = u_n + h\,\Phi(t_n, u_n, f_n, h)$$

Consider $y_{n+1} = y_n + h\,\Phi(t_n, y_n, f(t_n, y_n), h) + \tau_{n+1}$

consistency error

If we define

$$u^*_{n+1} = y_n + h\,\Phi(t_n, y_n, f(t_n, y_n), h)$$

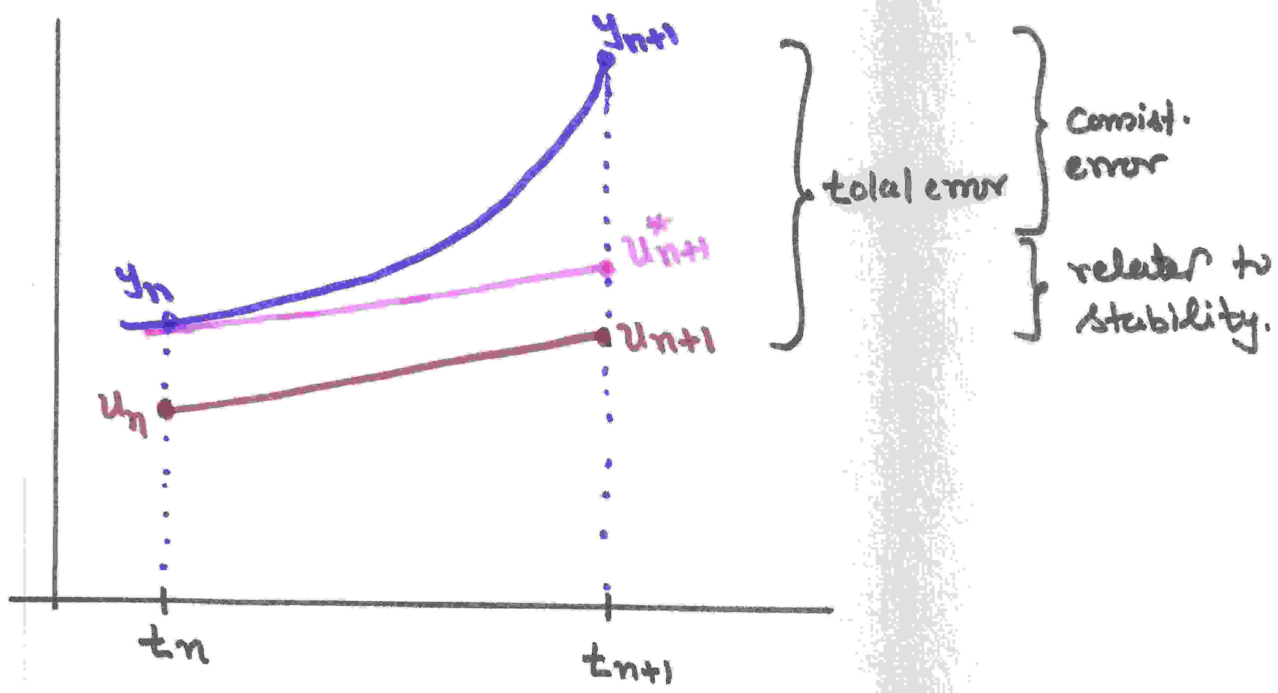
then

$$\tau_{n+1} = y_{n+1} - u^*_{n+1}$$

Now define the error at the node t_{n+1} as

$$e_{n+1} = y_{n+1} - u_{n+1}$$

$$= \underbrace{(y_{n+1} - u^*_{n+1})}_{\text{consistency error}} + \underbrace{(u^*_{n+1} - u_{n+1})}_{\text{stability is concerned.}}$$



We study stability of the single step method to to test equation

$$y' = \lambda y \qquad y(t_0) = y_0 \qquad \text{——— (1)}$$

Here λ is some constant.

Consider the exact solution of the test problem

$$y(t) = y_0 e^{\lambda(t-t_0)}$$

$$\Rightarrow \quad \frac{y(t_{j+1})}{y(t_j)} = \frac{y_0 e^{\lambda(t_{j+1}-t_0)}}{y_0 e^{\lambda(t_j-t_0)}}$$

$$= e^{\lambda(t_{j+1}-t_j)}$$

$$\Rightarrow \quad \boxed{y(t_{j+1}) = e^{\lambda h} y(t_j)}$$

A single step method when applied to (1) will lead to a first order difference equation

$$u_{j+1} = E(\lambda h) u_j \qquad j = 0, 1, 2 \ldots$$

for example: Consider Euler method

$$u_{j+1} = u_j + h f_j$$

$$= u_j + h\lambda u_j$$

$$= \underbrace{(1+\lambda h)}_{E(\lambda h)} u_j$$

Let $\varepsilon_j = u_j - y(t_j)$

then $\varepsilon_{j+1} = u_{j+1} - y(t_{j+1})$

$$= E(\lambda h) u_j - e^{\lambda h} y(t_j)$$

$$= E(\lambda h)\left(\varepsilon_j + y(t_j)\right) - e^{\lambda h} y(t_j)$$

$$= \underbrace{\left[E(\lambda h) - e^{\lambda h}\right] y(t_j)}_{\text{Consist. error}} + \underbrace{E(\lambda h)\,\varepsilon_j}_{\substack{\text{error propagated} \\ \text{form the step } t_j \text{ to } t_{j+1}}}$$

Furthermore,

$$\varepsilon_{j+2} = \left[E(\lambda h) - e^{\lambda h}\right] y(t_{j+1}) + E(\lambda h)\,\varepsilon_{j+1}$$

$$= \left[E(\lambda h) - e^{\lambda h}\right] e^{\lambda h} y(t_j) + E(\lambda h)\left[\left\{E(\lambda h) - e^{\lambda h}\right\} y(t_j) + E(\lambda h)\,\varepsilon_j\right]$$

$$= \left[E(\lambda h) - e^{\lambda h}\right]\left[e^{\lambda h} y(t_j) + E(\lambda h) y(t_j)\right] + E^2(\lambda h)\,\varepsilon_j$$

$$= \underbrace{\left[\left(E(\lambda h)\right)^2 - e^{2\lambda h}\right] y(t_j)}_{\text{Consist. error}} + \underbrace{E^2(\lambda h)\,\varepsilon_j}_{\text{propagated error}}$$

At the step $j+K$, the second term is of the form $E^K(\lambda h)\,\varepsilon_j$. As $j \to \infty$, meaningful results are obtained, only if propagation error decays or remains at least bounded.

Def: we call a single step method

- **absolutely stable** if $|E(\lambda h)| < 1$, $\lambda < 0$ or $Re(\lambda) < 0$
- **relatively stable** if $|E(\lambda h)| < e^{\lambda h}$, $\lambda > 0$

The interval for λh satisfying this condition are called interval of stability.

Discuss the absolute stability of the Euler Method:

Euler method on the test equation $y' = \lambda y$ gives:

$$
\begin{aligned}
u_{j+1} &= u_j + h f_j \\
&= u_j + h\lambda u_j \\
&= (1+\lambda h) u_j =: E(\lambda h) u_j
\end{aligned}
$$

i) if λ is real and $\lambda < 0$: Euler method is absolutely stable

if $|1+\lambda h| < 1 \Rightarrow -1 < 1+\lambda h < 1$

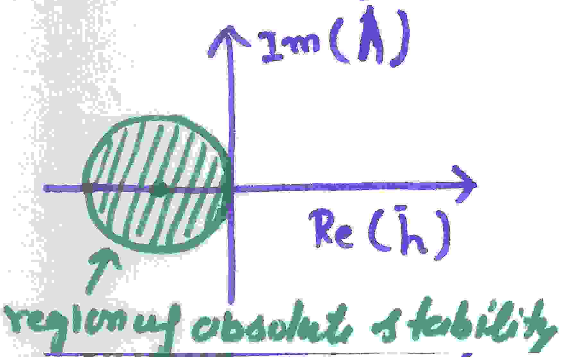
$$
\begin{aligned}
&\Rightarrow -2 < \lambda h < 0 \\
&\Rightarrow \lambda h \in (-2, 0)
\end{aligned}
$$

ii) if λ is complex and $Re(\lambda) < 0$:

Let $\bar{h} = \lambda h = x + iy$ then

$|1+\lambda h| = |1+x+iy| = \sqrt{(x+1)^2 + y^2}$

$\Rightarrow \sqrt{(x+1)^2+y^2} < 1 \Rightarrow (1+x)^2 + y^2 < 1$



Remark: As $E(\lambda h) = 1 + \lambda h < e^{\lambda h}$, $\lambda > 0$

Hence the method is always relatively stable.

It is always easy to verify the relative stability.

Absolute stability of Backward Euler method:

Application of the BEM to the test equation gives:

$$u_{j+1} = u_j + h f_j$$

$$= u_j + h \lambda u_{j+1}$$

$$\Rightarrow u_{j+1} = \left(\frac{1}{1 - \lambda h} \right) u_j$$

$$\Rightarrow E(\lambda h) = \frac{1}{1 - \lambda h}$$

I] When λ is real & $\lambda < 0$: The condition

$$\left| \frac{1}{1 - \lambda h} \right| < 1$$

is always satisfied.

Hence the method is absolutely stable for $-\infty < \lambda h < 0$.

II] λ is complex and $Re(\lambda) < 0$: Let $\lambda h = x + iy$ then

$$\frac{1}{|1 - \lambda h|} < 1 \Rightarrow |1 - \lambda h| > 1$$

$$\Rightarrow |(1 - x) - iy| > 1 \Rightarrow (1 - x)^2 + y^2 > 1$$

Since $Re(\lambda h) = x < 0$, this condition is always satisfied.

Hence the region of absolute stability is the entire left half of λh-plane. (Unconditionally stable)

Runge-Kutta Method of second order:

Runge-Kutta second order method: (one varient)

$$k_1 = f(t_j, u_j)$$

$$k_2 = f(t_j + h, u_j + h k_1)$$

$$u_{j+1} = u_j + \frac{h}{2}(k_1 + k_2)$$

Application to the test equation gives:

$$k_1 = \lambda u_j$$

$$k_2 = \lambda(u_j + h\lambda u_j) = \lambda u_j (1 + h\lambda)$$

$$\Rightarrow \quad u_{j+1} = u_j + \frac{h}{2}\left[\lambda u_j + \lambda u_j(1+h\lambda)\right]$$

$$= \left[1 + \frac{\lambda h}{2} + \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{2}\right] u_j$$

$$u_{j+1} = \underbrace{\left[1 + \lambda h + \frac{\lambda^2 h^2}{2}\right]}_{=: E(\lambda h)} u_j$$

For λ real & $\lambda < 0$:

$$|E(\lambda h)| < 1 \Rightarrow \left|\left[1 + \lambda h + \frac{\lambda^2 h^2}{2}\right]\right| < 1$$

$$\Rightarrow \quad -1 < 1 + \lambda h + \frac{\lambda^2 h^2}{2} < 1$$

$$\Rightarrow -1 < \frac{1}{2}\left[2 + 2\lambda h + \lambda^2 h^2\right] < 1$$

$$\Rightarrow \underbrace{-2 < \left[(1+\lambda h)^2+1\right]}_{\text{always satisfied}} < 2$$

last inequality gives:

$$(1+\lambda h)^2 + 1 < 2$$

$$\Rightarrow (1+\lambda h)^2 < 1$$

$$\Rightarrow -1 < (1+\lambda h) < 1$$

$$\Rightarrow -2 < \lambda h < 0 \Rightarrow \lambda h \in (-2, 0).$$

The stability intervals or regions of all the second order methods are same.

Ex:

Determine the interval of absolute stability of the implicit Runge-Kutta method

$$y_{n+1} = y_n + \frac{h}{4}(3K_1 + K_2)$$

$$K_1 = f\left(x_n + \frac{h}{3}, y_n + \frac{h}{3}K_1\right)$$

$$K_2 = f(x_n + h, y_n + hK_1)$$

when applied to the test equation $y' = \lambda y$, $\lambda < 0$.

Ans: $\lambda h \in (-6, 0)$