Lebesgue criteria for Riemann Integrability

Recall $J \to$ interval, f bdd on $[a,b]$

$$\omega_f(J) = \sup_x \{ f(x) : x \in J \cap [a,b] \} - \inf_x \{ \quad " \quad \}$$

$$\omega_f(x_0) = \inf_I \{ \omega_f(I) : I \text{ is open interval containing } x_0 \}$$

Lemma Suppose f is bdd on $[a,b]$. Then f is continuous at $x_0 \in [a,b]$ iff $\omega_f(x_0) = 0$

Proof: $\Rightarrow$ Suppose f is cont. at $x = x_0$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. whenever

$x \in (x_0 - \delta, x_0 + \delta)$, $\quad f(x) - f(x_0) < \varepsilon$

i.e. $f(x) < \varepsilon + f(x_0)$

$$\Rightarrow \sup \{ f(x) : x \in (x_0 - \delta, x_0 + \delta) \} \le f(x_0) + \varepsilon \quad \text{--- (1)}$$

$||^{ly}$, $\quad f(x) > f(x_0) - \varepsilon$

$$\Rightarrow \inf\{f(x) : x \in (x_0-\delta, x_0+\delta)\} \geq f(x_0) - \varepsilon \quad — ②$$

① & ② $\Rightarrow$

$$\omega_f\left((x_0-\delta, x_0+\delta) \cap [a,b]\right) \leq 2\varepsilon$$

$$\Rightarrow \omega_f(x_0) \leq \omega_f\left[(x_0-\delta, x_0+\delta) \cap [a,b]\right] \leq 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\omega_f(x_0) = 0$$

Converse part- Suppose $\omega_f(x_0) = 0$

Let $\varepsilon > 0$. There must exist an open interval I which contains x_0 s.t.

$0 \leq \omega_f(I) < \varepsilon$. Now I is open.

$x_0 \in I$, $\exists$ $\delta > 0$ s.t.

$$(x_0-\delta, x_0+\delta) \subseteq I.$$

Thus, $\exists$ $\delta > 0$ s.t. if $x \in (x_0-\delta, x_0+\delta)$, then $-\varepsilon < f(x) - f(x_0) < \varepsilon$.

$$\left[\begin{aligned} & f(x) - f(x_0) \\ & < \sup_{x \in (x_0-\delta, x_0+\delta)} f(x) - \inf_{x \in (x_0-\delta, x_0+\delta)} f(x) \checkmark \\ & = \omega_f(x_0-\delta, x_0+\delta) \\ & \leq \omega_f(I) < \varepsilon \end{aligned} \right] \checkmark$$

Lemma 2 :- For $n \geq 1$, the set

$$D_n = \left\{ x \in [a,b] \mid \omega_f(x) \geq \frac{1}{n} \right\}$$ is

a closed set. (D_n is closed + bdd. $\Rightarrow$ D_n's are compact)

Proof :- We have to show that D_n^c is open.

$\boxed{x \in D_n^c}$

$\Rightarrow \omega_f(x) < \frac{1}{n}$.

$\exists$ an open interval I containing x such that $\omega_f(I) < \frac{1}{n}$.

Since I is open, $\exists\ \delta > 0$ s.t.

$(x-\delta, x+\delta) \cap [a,b] \subset I \cap [a,b]$

Let $z \in (x-\delta, x+\delta)$

$\omega_f(z) \leq \omega_f(x-\delta, x+\delta) \leq \omega_f(I) < \frac{1}{n}$.

$\Rightarrow \quad z \in D_n^c$

$\Rightarrow \quad (x-\delta, x+\delta) \subseteq D_n^c$

$\Rightarrow \quad D_n^c$ is open

$\Rightarrow \quad D_n$ is closed.

Lebegue's Criteria for R- integrability

Let f be a bounded function on $[a,b]$ & let D denote the set of discontinuities of f. Then f is Riemann integrable iff $m(D)=0$.

Proof: $\Leftarrow$ Suppose $m(D)=0$.

$$D = \bigcup_{n \geq 1} D_n$$

$$m(D) = 0 \Rightarrow m\left(\bigcup_{n \geq 1}^{\infty} D_n\right) = 0$$

$\Rightarrow m(D_n) = 0 \quad \forall n \geq 1.$

T.S. f is Riemann integrable.

f is R-integrable if
(1) $\forall \varepsilon > 0, \exists P$
$U(P,f) - L(P,f) < \varepsilon$
$\Longleftrightarrow$
(2) $\forall \varepsilon, \exists$ step functions ϕ & ψ s.t.
$\phi \leq f \leq \psi$
& $\int \psi - \phi < \varepsilon$

$|f| \leq B$ on $[a,b]$.

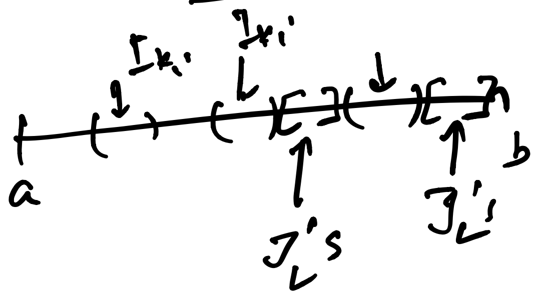
Let $\varepsilon > 0$,

$m(D_n) = 0$ & D_n is compact

There must be finite collection of open sets $\{I_{k_i}\}$, s.t.

$D_n \subset \bigcup_{i=1}^{m} I_{k_i}$, $\boxed{\sum_{i=1}^{m} \ell(I_{k_i}) < \frac{\varepsilon}{4B}}$

$[a,b] - \bigcup_{i=1}^{m} I_{k_i}$ are finite union of closed intervals $J_1, J_2, \ldots, J_L$



$[a,b] = J_1 \cup J_2 \cup \cdots \cup J_L \cup I_{k_1} \cup I_{k_2} \cdots \cup I_{k_m}$

Since $D_n \subset \bigcup I_{k_i}$,

$w_f(x) < \frac{1}{n} \quad \forall\ x \in J_1 \cup J_2 \cup \cdots \cup J_L$

(J_i's are compact)

<u>Now</u>, $\exists$ an open interval I_x containing x

s.t. $w_f(I_x) < \frac{1}{n}$. ✓

$\Rightarrow$ $\{I_x\}_{x \in J_i}$ collection of open interval

must cover J_i

Since each J_i is compact, we take

$\{J_i'\}_{i=1}^{L'}$ whose union is $J_1 \cup J_2 \cup \cdots \cup J_L$

$w_f(J_i') < \frac{1}{n}$. ✓

<u>Define</u>,

$\phi = \inf f$ on $I_{k_1}, I_{k_2}, \ldots, I_{k_m}, J_1', J_2', \ldots, J_{L'}'$

$\psi = \sup f$ on " "

$\underline{\phi \leq f \leq \psi}$ ✓

$$\int_a^b (\psi - \varphi)\, dx$$

$$= \int_{\bigcup_{i=1}^{l'} J_i'} \psi - \varphi \quad + \int_{\bigcup_{i=1}^{k_n} I_{k_i}} \psi - \varphi$$

$$\leq \frac{1}{n} \sum_{i=1}^{l'} l(J_i') \quad + 2B \sum_i l(I_{k_i})$$

$$\leq \frac{1}{n}(b-a) \quad + 2B \times \frac{\varepsilon}{4B}$$

$$\leq \varepsilon$$

by choosing n sufficiently large.

$\Rightarrow$ f is Riemann integrable.

<u>Converse Part</u> :- f is Riemann integrable

then, $m(D_n) = 0$ $\forall\, n \geq 1$.

$n = N$ & $\varepsilon > 0$

T.S. $m(D_N) = 0$, that means,

for every $\varepsilon > 0$, $m(D_N) < \varepsilon$.

Since, f is Riemann integrable on $[a,b]$, we have step functions ϕ, ψ s.t.

$\phi \le f \le \psi$ with

$$\int_a^b (\psi - \phi)\, dx < \frac{\varepsilon}{N}.$$

Consider the following partition:-

$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$

Collection: $\{ (x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n) \}$

C_1

$(x_{k-1}, x_k) \cap D_N \neq \phi$ then put (x_{k-1}, x_k) in C_1

C_2

$(x_{k-1}, x_k) \cap D_N = \phi$ then put (x_{k-1}, x_k) in C_2

Every pt. in D_N is either in C_1 or a member of the set $\{a, x_1, x_2, \ldots, x_n = b\}$

So,

$$\int_a^b (\psi - \phi)\, dx$$

$$= \sum_{C_1} (\psi - \phi)(x)\,(x_k - x_{k-1}) + \sum_{C_2} \cdots$$

$< \varepsilon/N$.

$\sum_{C_1}$, some pt of D_N in each subinterval of this collection C_1

i.e each subinterval in C_1 contains a pt x with $\omega_f(x) \geq \frac{1}{N}$.

Thus, in these subintervals,

$\psi - \varphi \geq \sup f - \inf f \geq \frac{1}{N}$.

$$\sum_{C_1} \leq \sum_{C_1} + \sum_{C_2} < \frac{\varepsilon}{N} \quad \text{--- (A)}$$

$$\sum_{C_1} (\psi - \varphi)(x_k - x_{k-1})$$

$$\geq \frac{1}{N} \sum_{C_1} (x_k - x_{k-1}) \quad \text{--- (B)}$$

So, (A) & (B) $\Rightarrow$

$$\frac{1}{N} \sum_{C_1} (x_k - x_{k-1}) < \frac{\varepsilon}{N}.$$

$$\Rightarrow \sum_{C_1} (x_k - x_{k-1}) < \varepsilon.$$

$$\Rightarrow m(D_N) \leq \sum_{C_1} (x_k - x_{k-1}) < \varepsilon$$

$$\Rightarrow m(D) = 0$$

- Dirichlet function

$$f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

f is not Riemann integrable.

$D = [0,1]$

$m(D) = 1 \neq 0$, by the previous thm f is not Riemann integrable.

$E_1 = [0,1] \cap \mathbb{Q} \rightarrow$ measurable $\rightarrow m(E_1) = 0$

$E_2 = [0,1] \cap \mathbb{Q}^c \rightarrow$ " ; $m(E_2) = 1$

$$f(x) = \begin{cases} 1 & \text{on } x \in E_1 \\ 0 & \text{on } x \in E_2 \end{cases}$$

2nd Case
0 on E_1
1 on E_2

$$\int_{[0,1]} f(x)\, dm$$

$$= \int_{E_1} f(x)\, dm + \int_{E_2} f(x)\, dm$$

$$= 1 \cdot m(E_1) + 0 \cdot m(E_2)$$

$$= 1 \cdot 0 + 0 \cdot 1$$

$$= 0$$

2nd Case
$$0 \cdot 0 + 1 \cdot 1$$
$$= \underline{1}$$

Return to Lebesgue theory :-

Recall BCT ✓

Cor :- $f \geq 0$ is bdd & supported on a set of finite measure E

$\int_E f = 0$ then $f = 0$ a.e. ✓

Thm Suppose f is Riemann integrable on $[a,b]$. Then, f is measurable & $\overset{R}{\int_{[a,b]}} f(x)\,dx = \overset{L}{\int_{[a,b]}} f(x)$.

Proof :- By defn Riemann integrable functions are bdd, say $|f(x)| \le M$.

We may construct two sequences of step functions $\{\varphi_k\}$ & $\{\psi_k\}$ s.t.

$$\varphi_1 \le \varphi_2 \le \cdots \cdots \le f \le \cdots\cdots \le \psi_2 \le \psi_1$$

$$|\varphi_k| \le M \quad \& \quad |\psi_k(x)| \le M'. \checkmark$$

$$\lim_{k\to\infty} \overset{R}{\int_{[a,b]}} \varphi_k(x) = \lim_{k\to\infty} \overset{R}{\int_{[a,b]}} \psi_k(x) = \overset{R}{\int_{[a,b]}} f(x)\,dx. \longrightarrow (*)$$

We also have

$$\overset{R}{\int_{[a,b]}} \varphi_k(x) = \overset{L}{\int_{[a,b]}} \varphi_k(x) \quad \& \quad \overset{R}{\int_{[a,b]}} \psi_k(x) = \overset{L}{\int_{[a,b]}} \psi_k(x) \longrightarrow (**)$$

Let, $\check{\varphi}(x) = \lim_{k\to\infty} \varphi_k(x)$

& $\check{\psi}(x) = \lim_{k\to\infty} \psi_k(x)$.

Then, $\check{\varphi} \le f \le \check{\psi}$.

Here, $\check{\varphi}$ & $\check{\psi}$ are measurable functions.

$$\lim_{k\to\infty} \overset{L}{\int_{[a,b]}} \varphi_k(x) = \overset{L}{\int_{[a,b]}} \check{\varphi}(x)\,dx \quad \text{by BCT}$$

$$\lim_{k\to\infty} \overset{L}{\int_{[a,b]}} \psi_k(x) = \overset{L}{\int_{[a,b]}} \check{\psi}(x)\,dx \quad ''$$

from (*) & (**)

$$\overset{L}{\int_{[a,b]}} (\check{\psi} - \check{\varphi})\,dx = 0$$

& $\hat{\psi} - \hat{\varphi} \geqslant 0$ since $\psi_k - \varphi_k \geqslant 0$.

by <u>Corollary</u>, of <u>BCT</u>,

$$\hat{\psi} - \check{\phi} = 0 \quad \text{a.e.}$$

$$\check{\psi} = \check{\phi} \quad \text{a.e.}$$

<u>Recall:</u> $\check{\phi} \leq f \leq \hat{\psi}$

$\Rightarrow \hat{\psi} = \check{\phi} = f$

$\Rightarrow$ f is measurable.

$\phi_k \to f$, then,

$$\lim_{k\to\infty} \int^{L}_{[a,b]} \phi_k(x)\,dx = \int^{L}_{[a,b]} f(x)\,dx$$

<u>Then</u>, (*) & (**),

$$\int^{L}_{[a,b]} f(x)\,dx = \lim_{k\to\infty} \int^{L}_{[a,b]} \phi_k(x)\,dx$$

$$= \lim_{k\to\infty} \int^{R}_{[a,b]} \phi_k(x)\,dx$$

$$= \int^{R}_{[a,b]} f(x)\,dx.$$

① $f(x) = \frac{1}{2^n}$, $\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$

$f(0) = 0$ $\quad n = 0, 1, 2, \ldots$

$$\int_0^1 f(x)\,dx = 2/3$$

② $f(x) = \begin{cases} \frac{1}{x}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$

$[r, 1]$, $r > 0$, this function f is cont. & therefore integrable on $[r, 1]$. Since f is unbounded it cannot be Riemann integrable.

3. (iv) $f(x) = \begin{cases} x, & x \in [0,1] \cap \mathbb{Q} \\ x^r, & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$

4) $$\lim_{h \to 0+} \frac{1}{2h} \int_{c-h}^{c+h} f(x)\,dx = f(c)$$

$$\left(\lim_{h \to 0} \frac{1}{h} \int_a^{a+h} f(x)\,dx = f(a) \quad \text{if is cont. at } a \right)$$

$$f(c) = \frac{1}{2h} \int_{c-h}^{c+h} f(c)\,dx$$

$$\frac{1}{2h} \int_{c-h}^{c+h} f(x)\,dx - f(c)$$

$$= \frac{1}{2h} \int_{c-h}^{c+h} \left(f(x) - f(c)\right) dx$$

Let $\varepsilon > 0$ f is cont at c.

$\exists\, \delta > 0$ s.t. $|f(x) - f(c)| < \varepsilon$ for $c - \delta < x < c + \delta$.

If $h < \delta$, $h \to 0^+$

$0 < h < \delta$,

$$\left| \frac{1}{2h} \int_{c-h}^{c+h} \left(f(x) - f(c)\right) dx \right|$$

$$\leq \frac{1}{\cancel{2h}} \sup_{x \in [c-h,\, c+h]} |f(x) - f(c)| \times \cancel{2h}$$

$\leq \quad \varepsilon \quad$ Since, $[c-h, c+h] \subseteq (c-\delta, c+\delta)$

Now $\varepsilon > 0$ is arbitrary.

$$\Rightarrow \lim_{h \to 0+} \frac{1}{2h} \int_{c-h}^{c+h} f(x)\,dx = f(c) \quad \square$$

(*) If F is cont. on $[a,b]$ & diff (a,b) with $F' = f$ when $f : [a,b] \to \mathbb{R}$ is Riem. int. Then

$$\int_a^b f(x) = F(b) - F(a).$$

$a \quad b>a \quad b$

* $F(x) = \int_a^x f(t)\,dt$, f is Riemann integrable on $[a,b]$

F is cont.

If f is cont., then $F' = f$

4. $f(x)^2 = 2\int_0^x f(t)\,dt$

T.S. $f(x) = x \ \forall x \geq 0$

<u>Sol:</u> f is cont. $[0,x] \to \mathbb{R}$

$f(0) = 0$ ✓

$f^2 =$

$F(x) = \int_0^x f(t)\,dt$

$F'(x) = f(x) \quad \forall x \geq 0.$ ✓

$f^2(x) = 2\int_0^x f(t)\,dt = 2F(x)$

$2f(x)\,f'(x) = 2f(x)$ $F(x) \geq 0$

$\Rightarrow f'(x) = 1$

$\Rightarrow f(x) = x$

$\Rightarrow$ since $f(0) = 0$

f^2 is diff.
f is cont.
but f is not diff.