Date 17/11/2017

Lecture 28

Q) Solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $x>0$, $t>0$

subject to the condn's

$\boxed{U(0,t)=0}$, $U(x,0) = \begin{cases} 1, & 0<x<1 \\ 0, & x \geq 1. \end{cases}$

$U(x,t)$ is bounded.

To find $U(x,t) = ?$

Soln :- Taking the Fourier sine transform of both sides of the given p.d. eqn w.r.t x, we get

$$F_s\left\{\frac{\partial U}{\partial t}\right\} = F_s\left\{\frac{\partial^2 U}{\partial x^2}\right\}$$

Recall

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\,dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin\omega x\,d\omega.$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial U}{\partial t}\sin(\lambda x)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 U}{\partial x^2}\sin(\lambda x)\,dx$$

then if

$$u = u(\lambda, t) = F_s\{U(x,t)\}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty U(x,t)\sin\lambda x\,dx.$$

$\longrightarrow (*)$

this becomes

$$\frac{\partial}{\partial t}\underbrace{\left(\sqrt{\frac{2}{\pi}}\int_0^\infty U(x,t)\sin\lambda x\,dx\right)}_{=u} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 U}{\partial x^2}\sin\lambda x\,dx$$

$$\Rightarrow \frac{du}{dt} = \sqrt{\frac{2}{\pi}}\left[\underbrace{\left[\frac{\partial U}{\partial x}\sin\lambda x\right]_0^\infty}_{(=0)} - \lambda\int_0^\infty \frac{\partial U}{\partial x}\cos(\lambda x)\,dx\right]$$

(why?)

$$\Rightarrow \frac{du}{dt} = \sqrt{\frac{2}{\pi}} \cdot \left[-\lambda \left\{ U \cos \lambda x \right\}_{x=0}^{\infty} - \lambda^2 \int_0^\infty U \sin(\lambda x)\, dx \right]$$

$$\Rightarrow \frac{du}{dt} = \sqrt{\frac{2}{\pi}}\, \lambda \underbrace{U(0,t)}_{(=0)} - \lambda^2 u$$

$$\Rightarrow \frac{du}{dt} + \lambda^2 u = 0 \longrightarrow (2)$$

assuming U & $\frac{\partial U}{\partial x} \to 0$

as $x \to \infty$

Now,

$$F_s\{U(x,0)\}$$

$$= u(\lambda, 0)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{U(x,0)} \sin \lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 1 \cdot \sin(\lambda x)\, dx.$$

$$\boxed{\begin{aligned} &\frac{du}{dt} = -\lambda^2 u \\ &\Rightarrow \frac{du}{u} = -\lambda^2 dt \\ &\quad (\text{on } \textstyle\int) \\ &\Rightarrow \ln u = -\lambda^2 t + \lg c \\ &\Rightarrow u(\lambda, t) = c\, e^{-\lambda^2 t} \end{aligned}}$$

$$\therefore\ u(\lambda,0) = \sqrt{\frac{2}{\pi}} \int_0^1 \sin(\lambda x)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos(\lambda x)}{\lambda} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{(1-\cos\lambda)}{\lambda} \longrightarrow (3)$$

We have

$$u(\lambda,t) = c\, e^{-\lambda^2 t}$$

$$\therefore\ u(\lambda,0) = c \cdot 1 = c$$

$$\Rightarrow\ \boxed{c = \sqrt{\frac{2}{\pi}}\, \frac{(1-\cos\lambda)}{\lambda}} \quad (\text{from } (3))$$

$$\therefore\ u(\lambda,t) = c \cdot e^{-\lambda^2 t}$$

$$= \boxed{\sqrt{\frac{2}{\pi}}\, \frac{(1-\cos\lambda)}{\lambda}\, e^{-\lambda^2 t}}$$

Then taking the inverse Fourier sine transform, we find the reqd. sol^n

$$U(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(\lambda,t) \sin(\lambda x)\, d\lambda$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{1-\cos\lambda}{\lambda}\right) e^{-\lambda^2 t} \sin(\lambda x)\, d\lambda.$$

Physically, this can (solve?) represent the temperature in a solid.

8) Syllabus for the end-sem

Marks: 50, Time: 3 hours

1) Laplace Transform & its properties (special f^ns etc), application to boundary value problems, (i.e. solving p.d. eq^n) & o.d. eq^n & system of o.d. eq^n

2) Fourier series, Dirichlet's criterion, half-range, even-odd f^n, Parseval's theorem, complex form for Fourier series, series sol^n. using F.S, F.I, F.T

Fourier Integral representation, convolution.

Fourier sine or Cosine Integral representation, Parseval's th

Fourier Transform, Fourier Sine or Cosine Transform, Various properties, convolution, Dirac delta (etc.)
1st shifting, 2nd shifting
solving Boundary value Problems (i.e. solving p.d. eqn)

Q1) The steady state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad y > 0$$

solve for $u(x,y)$.

$$u(0,y) = 0, \quad u(\pi, y) = e^{-y}, \quad y > 0$$

$$\boxed{\left.\frac{\partial u}{\partial y}\right|_{y=0}} = 0, \quad 0 < x < \pi$$

Sol^n:- The domain of the variable y & the prescribed cond^n at y=0 indicate that the Fourier cosine transform is suitable for the problem.

We define

$$\boxed{\mathcal{F}_c\{u(x,y)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,y)\cos(\alpha y)\,dy = U(x,\alpha)}$$

$$\therefore \mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \mathcal{F}_c(0)$$

$$\Rightarrow \frac{d^2 U}{dx^2} - \alpha^2 U(x,\alpha) - \sqrt{\frac{2}{\pi}}\,\underbrace{u_y(x,0)}_{(=0)} = 0$$

$u_y = \frac{\partial u}{\partial y}$

We know,

$$\mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\Rightarrow \frac{d^2 U}{dx^2} - \alpha^2 U = 0$$

Since, the domain of x is a finite interval, one choose to write the solⁿ of the o.d. eqⁿ as

A·E
$m^2 - \alpha^2 = 0$
$\Rightarrow m = \pm\alpha$

$$U(x, \alpha) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x) \quad \rightarrow (1)$$

Now, $\mathcal{F}_c(u(0,y)) = \mathcal{F}_c(0)$

$$\Rightarrow U(0, \alpha) = 0 .$$

we know,

$$\mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\Rightarrow \frac{d^2 U}{dx^2} - \alpha^2 U = 0$$

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Now, $\mathcal{F}_c(u(0,y)) = \mathcal{F}_c(0)$

$$\Rightarrow U(0, \alpha) = 0.$$

$\& \ F_c\{u(\pi, y)\} = F_c\{e^{-y}\}$

$\Rightarrow \ U(\pi, \alpha) = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\alpha^2}\right)$

$\Rightarrow 0 = c_1 \cdot 1 + c_2 \cdot 0$

$\Rightarrow c_1 = 0.$

we know,

$F_c\{e^{-bx}\}$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \cos(\alpha x)\, dx$

$= \sqrt{\frac{2}{\pi}}\left(\frac{b}{b^2+\alpha^2}\right)$

$(b = 1)$

$\therefore U(x, \alpha)$

$= c_2 \sinh(\alpha x)$

$\underset{x=\pi}{\therefore} U(\pi, \alpha) = c_2 \sinh(\alpha \pi)$

$\Rightarrow c_2 = \frac{U(\pi, \alpha)}{\sinh(\alpha\pi)} = \sqrt{\frac{2}{\pi}} \frac{1}{(1+\alpha^2)\sinh(\alpha\pi)}$

$\therefore U(x, \alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sinh(\alpha x)}{(1+\alpha^2)\sinh(\alpha\pi)}$

$$\therefore u(x,y) = \mathcal{F}_C^{-1}\{U(x,\alpha)\}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sinh(\alpha x)}{(1+\alpha^2)\sinh(\alpha \pi)} \cos(\alpha y)\, d\alpha.$$

(solve it if possible)

Note :- Had $u(x,0)$ been given in the above example rather than $u_y(x,0)$, then the sine transform could have been appropriate.