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Teacher's Sign/Remarks

Real Analysis

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Mathematical Analysis

Field $(F, +, \cdot)$

Abelian group under + & .

if $a \in F$, $b \in F \rightarrow a+b \in F$ it means it is closed under +.

A_1 : $a, b \in F \Rightarrow a+b \in F$ (close)

A_2 : $(a+b)+c = a+(b+c)$ (Associative)

A_3 : $a+b = b+a$ (commutative)

A_4 : $a+0 = 0+a = a$ (identity)

A_5 : $a+(-a) = (-a)+a = 0$ (inverse)

M_1 : $a \cdot b \in F$

M_2 : $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M_3 : $a \cdot b = b \cdot a$

M_4 : $a \cdot 1 = 1 \cdot a = a$

M_5 : $a \neq 0$; $\exists b \Rightarrow a \cdot b = b \cdot a = 1$

$\Rightarrow$ $a \cdot (b+c) = a \cdot b + a \cdot c$

$\forall a, b, c \in F$

$a \equiv b \mod n$

$\Rightarrow \quad n \mid a-b \qquad$ (n divides $(a-b)$)

$\Rightarrow \quad a-b = nk \quad , \quad k \in \mathbb{Z}$

R is a relation on A, $\underline{R \subset A \times A}$

$(a,b) \in R \Leftrightarrow aRb$

★ $R \Rightarrow a \equiv b \mod n$

1)- $(a,a) \in R$

$a \equiv a \mod n \quad \forall a$

$\Rightarrow$ Reflexive

2)- $(a,b) \in R$

$a \equiv b \mod n$

$n \mid a-b \Rightarrow n \mid b-a \Rightarrow b \equiv a \mod n$

$\Rightarrow$ symmetric

3)- $(a,b) \in R$,

$(b,c) \in R$

$\Rightarrow (a,c) \in R$

$n \mid a-b$

$n \mid a-c$

$\Rightarrow \quad n \mid ((a-b)+(b-c))$

$\Rightarrow \quad n \mid a-c$

$\Rightarrow$ Transitive

$R \Rightarrow$ equivalence Relation

★ R → Symmetric, Transitive

$(a,b) \in R \Rightarrow (b,a) \in R$

$(a,b) \in R$, $(b,a) \in R$

$\Rightarrow (a,a) \in R$

where is mistake ??

Problem is (It is not true for all a).

⇒ Equivalence Class :

Let R be an equivalence Relation.

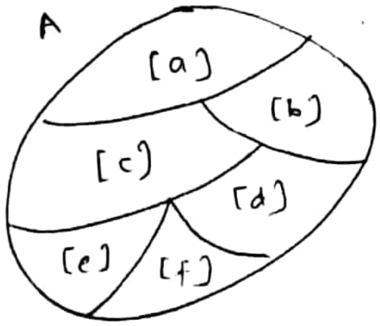
$[a] = \{ b \in A \mid bRa \}$

class a | cla

as $aRa \Leftrightarrow (a,a) \in R$

(Reflexive)

$[a] \neq \phi$ (never empty class as a will be there.)



Two equivalence classes are either disjoint or identical.

eg:- Over $\mathbb{Z}$ $\quad a \equiv b \pmod 3$

$$[0],\ [1],\ [2]$$

$$[0] = \{3m;\ m \in \mathbb{Z}\}$$

$$[1] = \{3m+1;\ m \in \mathbb{Z}\}$$

$$[2] = \{3m+2;\ m \in \mathbb{Z}\}$$

$$[0] \cup [1] \cup [2] = \mathbb{Z}$$

$$A \longrightarrow [0] \cup [1] \cup [2] \ldots\ldots [n-1]$$

$$A = \bigcup_{a \in A} [a]$$

$\Rightarrow$ Law of Trichotomy :

field $F = \mathbb{R}$ set of real no.

$a, b \in \mathbb{R}$

ⓐ $a > b$

ⓑ $a < b$

ⓒ. $a = b$

$\Rightarrow$ Show that $\phi \subset A$

$$x \in \phi \Rightarrow x \in A$$

Let

A

A

Conve

In

Cont

Let A, B be two statements

A and B

A or B

$A \Rightarrow B$

$A \Leftrightarrow B$

negation of statement A.

$A \Leftrightarrow B$

A iff B

iff → if and only if

$A \Leftrightarrow B$

$A \Rightarrow B$ and $B \Rightarrow A$

A: $2+3=5$

B: $7+6=10$

$\forall \xrightarrow{\text{NOT}} \exists$

$\exists \xrightarrow{\text{NOT}} \forall$

Conjunction
AND

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

A and B is true when both A and B is true.

Disjunction
OR

A	B	A or B
T	T	T
T	F	T
F	T	T
F	F	F

A or B is false when both A and B is false.

Negation

A	Neg A
T	F
F	T

Implication $(A \to B)$
i.e if A then B

Hypo. Conclu

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

→ vacuously True

$I \to A \Rightarrow B$

Converse of $I \to B \Rightarrow A$

Inverse $\to \sim A \Rightarrow \sim B$

Contrapositive $\sim B \Rightarrow \sim A$

Implication $\cong$ Contrapositive
equivalent

Converse $\cong$ Inverse

H	C	Original Implication $H \Rightarrow C$	Converse $C \Rightarrow H$	Neg H $\sim H$	Neg C $\sim C$	Inverse $\sim H \Rightarrow \sim C$	Contrapositive $\sim C \Rightarrow \sim H$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

$\boxed{\epsilon - \delta \text{ continuity}}$

⇒ $f(x)$ is cont. at $x = a$.

→ Given $\epsilon > 0$, there exists a $\delta > 0$

s.t.

$|f(x) - f(a)| < \epsilon$ whenever $|x-a| < \delta$

and δ is a fn of ϵ i.e $\delta(\epsilon)$.

→ $\lim_{x \to a} f(x) = l$

⇒ $f(x)$ is not cont. at $x = a$

$\forall \ \delta > 0 \ \exists \ \epsilon > 0$ s.t.

$|f(x) - f(a)| \geq \epsilon$, when $|x-a| < \delta$.

⇒ Negation of $\forall$ becomes some & vice versa.

$a \in \mathbb{R}$

$a > 0$ iff, a is +ve.

a is +ve if $a > 0$

If $a > 0$ then a is +ve.

$a \in \mathbb{R}$

$a = 0$

$a > 0$

or $-a \geq 0$

$a > b \Leftrightarrow a - b$ is +ve.

$a < b \Leftrightarrow b - a$ is +ve.

⇒ **Bounded :**

Let $S \neq \phi$ subset of $\mathbb{R}$.

$\alpha \in \mathbb{R}$ is said to be an upper bound of S.

if $x \leq \alpha \quad \forall \; x \in S$.

this is

$\leq$; $<$ or $=$

Let $S = \{1, 3, 5, 7\}$

So, 7 is upper bound.

→ A set S is said to be bounded above if it has an upper bound.

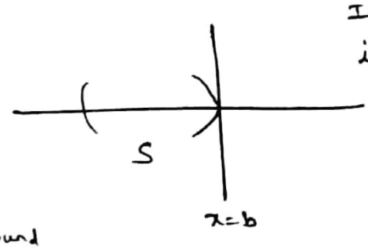
→ Least upper bound (Lub)

→ Supremum (Sup) & Infemum (Inf)

→ Upper bounds are many but lub is only unique.

⇒ An element $b \in \mathbb{R}$ (not neccessarily in S) is said To be an upper bound if

$$x \leq b \quad \forall \; x \in S$$



In this, $b, b+1, b+2, \ldots, b+\alpha$ ($\alpha > 0$) is an upper bound.
Hence, there are infinitely many upper bounds.

7 is an upperbound of S

$$S = \{1, 2, 3, 4, 7\}$$

$$S_1 = \{x \mid x \text{ is } +ve\}$$

$$S_2 = \left\{ x \mid x = \frac{n}{n+1}, \; n \in \mathbb{N} \right\}$$

a is +ve iff $a > 0$

S_1 has no upper bound.

Suppose b is an upper bound of S_1

$$0 < 1 < b$$

$$1 \in S_1.$$

$$0 < b+1 > b$$

as $b+1 > 0$ ✓
$b+1 < b$ ✗

$$b+1 \in S_1$$

So, contradiction $(\Rightarrow \Leftarrow)$

So, No upper bound.

Note: the upperbound may not belong to a set.

→ For S_2,

In S_2,

$$n < n+1 \quad \forall \; n \in \mathbb{N}$$

$$\frac{n}{n+1} < \frac{n+1}{n+1} = 1$$

$$\frac{n}{n+1} < 1 \quad \forall \; n$$

Hence, 1 is an upper bound for S_2.

⟹ **Least Upper Bound (lub) or sup or Supremum:**

An upper bound c of S is said to be a least upper bound (or lub) if

(i)- c is an upperbound for S.

(ii)- If b is another upper bound (other than c) of S then $b > c$.

⟹ Show that the lub is unique for a set S ??

Suppose b and c are two distinct least upper bound. $b \neq c$ (as they are distinct) then

either $b > c$ or $b < c$

c is lub. b will be lub

so, $\Rightarrow\Leftarrow$ $\Rightarrow\Leftarrow$

Hence, $\boxed{b = c}$

⟹ ~~Completeness Axiom in $\mathbb{R}$:~~

A set S is said to be bounded above if it has an upper bound.

A set S is said to be an ~~lower~~ bounded lower if it has an lower bound.

if both upper and lower bound equal, then it is an singleton set.

set.

⟹ Completeness Axiom in $\mathbb{R}$:

1) Any non-empty bounded above set S has a lub. in $\mathbb{R}$

⟹ (The Archimedean Principle):

If b and c are two real numbers with $c > 0$, then there exists a natural number n s.t.

$$nc > b$$

Proof:

Indirect Method: (Indirect means we have to negate below statement)

" If b and c are two real numbers with $c > 0$, then there exists a natural number n s.t.

$$nc > b"$$

i.e suppose there exists real numbers b and $c > 0$ and $nc \leq b \quad \forall\, n \in \mathbb{N}$

$$S = \{x \mid x = nc,\ n \in \mathbb{N}\}$$

S has an upper bound b.

$$S \subset \mathbb{R}$$

Using completeness axiom s.t. $\mathbb{R}$, S has a lub

Let $\text{lub } S = a$

Now, $a - c < a$ (as $c > 0$)

Now, $a-c$ is not an upper bound of S because if it is an upper bound then lub $S = a-c$ (which is not true).

So, $a-c$ is not an upper bound.

$\exists$ an element $mc \in S$ s.t.,

$$mc > a-c$$

$$mc + c > a$$

i.e $c(m+1) > a$

$$c(m+1) \in S$$

which contradicts the lub of S as a.

s

pper

t has

n

$\Rightarrow$ Any +ve real $x, y \in \mathbb{R}$, $x > 0$, $\exists\ n \in \mathbb{N}$ s.t

$$nx > y.$$

Corollary (1): For $b \in \mathbb{R}$, $\exists\ n \in \mathbb{N}$ s.t. $n > b$.

Proof: put $x = 1 > 0$ in Archimedean principle

$na > b$ i.e $nx > y$ $x, y \in \mathbb{R}$, $y > 0$

$$n \cdot 1 > y$$

$$y = b$$

$$n > b$$

Corollary (2): For $b \in \mathbb{R}$ $\exists$ an integer m s.t.

$$m < b$$

Proof:

$$b \in \mathbb{R} \Rightarrow -b \in \mathbb{R}$$

by Corollary 1 $\exists\ n \in \mathbb{N}$

s.t $n > -b$.

$\Rightarrow -n < b \Rightarrow m < b$

set $(m = -n)$

Corollary (3): For $x \in \mathbb{R}$, $\exists$ an integer k s.t.

$$x - 1 \leq k < x.$$

Proof: For $x \in \mathbb{R}$, $\exists\ n \in \mathbb{N}$ s.t. $x < n$ —(i)

$\exists\ m \in \mathbb{Z}$ s.t. $x > m$ —— (ii)

Combining (i) and (ii) -

$$m < x < n$$

Choose largest k from $m, m+1, m+2, \ldots, n$

s.t. $k < x$, $k + 1 \geq x \Rightarrow k \geq x - 1$

(it is possible as the set $\{m, m+1, \ldots, n\}$ is finite)

So, $\boxed{x - 1 \leq k < x}$

eg:- $x = 0.5$
$-2 < .5 < 7$
k from $-2, -1, 0, 1, 2, 3, 4, 5, 6, 7$
$k = 0$

⟹ Rational density theorem :

In between any two real numbers a, b with $a < b$,

$\exists\ r \in \mathbb{Q}$, s.t. $a < r < b$

Eg. For any arbitrary +ve real number ϵ, $\exists$ a natural number n s.t. $\frac{1}{n} < \epsilon$

~~$\lim_{n\to\infty} \frac{1}{n} = 0$~~

$$\left[\ \epsilon > 0,\ \exists\ n \in \mathbb{N} \text{ s.t. } \frac{1}{n} < \epsilon\ \right]$$

$\epsilon > 0$, $a = \frac{1}{\epsilon} \in \mathbb{R}$

Using Archimedean principle,

$\exists\ n \in \mathbb{N}$ s.t. $n > \frac{1}{\epsilon}$

i.e $\frac{1}{n} < \epsilon$

Proof: (Theorem)

$b - a > 0$, his the above result $\boxed{\frac{1}{n} < \epsilon}$

$\exists\ n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$ — (i)

$\Rightarrow \boxed{a < b - \frac{1}{n}}$ $\hookrightarrow$ $[na < nb - 1]$ — (A)

$nb \in \mathbb{R}$

set $x = nb$ in corollary (3)

$\exists\ k \in \mathbb{Z}$ s.t.

$$nb - 1 \le k < nb$$

$na < nb - 1 \le k < nb$ {using (A)}

$\Rightarrow na < k < nb$

$$\frac{na}{n} < \frac{k}{n} < \frac{nb}{n}$$

$\Rightarrow a < \frac{k}{n} < b$

$\hookrightarrow r$

Note : $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$A \to$ Set , $P(A) \to$ Power set of A

$\Rightarrow$ Cantor's Theorem :

Does there exists a surjective map between A and $P(A)$.

$\phi : A \to P(A)$ — No (given by Cantor's theo)

set valued map

— $a \in A$

— $\phi(a)$ is a set

exercise :

Assume onto function & proceed.

For proving Not onto

$y \in B$, no $x \in R$ s.t. $f(x) = y$.

Example 1 : Let b denote any real number and let

*** $S = \{x \mid x$ is a rational number and $x < b\}$

Find Sup S or lub S ??

$\Rightarrow$ $S = \{x \mid x \in \mathbb{Q} : x < b\}$

so, b is an upper bound,

S is bounded above set.

So, it has lub.

So, it has a supremum.

Let $c = \text{lub } S$

$c \leq b$

{ if $c = b$, then b is sup but if $c < b$, then contrad. }

if $c = b$, it is proved and lub $S = b$

and if $c < b$ $\exists$ $c < r < b$

$r \in \mathbb{Q}$

Theorem (A): If a and b are real numbers and $a > 1$, then $\exists$ a ~~rational~~ natural number n s.t.

$$[a^n > b]$$

Proof : Suppose $a^n \leq b$ $\forall$ $n \in \mathbb{N}$ — ↑ negative of above

$S = \{x \mid x = a^n, n \in \mathbb{N}\}$

Then the set S is bounded above by completeness principle, $\rightarrow$ lub S exists.

Let $c = \text{lub} S$

as $a > 1$, $c < ac$ $\Rightarrow$ $\frac{c}{a} < c$

$\exists$ $a^n \in S$ s.t. $a^n > \frac{c}{a}$

{ if $\frac{c}{a} \geq a^n$ $\xrightarrow{\text{then}}$ $\frac{c}{a}$ becomes upper bound as $\frac{c}{a} < ac$ it must be lub S $\Rightarrow\Leftarrow$

$a^n > \frac{c}{a}$

$\Rightarrow$ $a^{n+1} > c$

Now, $a^{n+1} \in S$

which contradicts the definition of upper bound as c is the lub.

⇒ **Corollary:** $S = \{10^n \mid n \in \mathbb{N}\}$

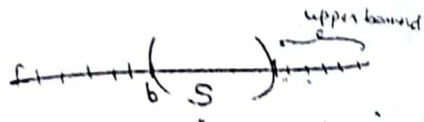
The set S is unbounded above.

Proof: Take $a = 10$ in Th-A

Let $S \neq \phi$ be a set in $\mathbb{R}$.

Def^n: A real number b, where b is not necessarily in S is said to be a lower bound for S if

$[x \geq b \quad \forall\ x \in S.]$



(any number in left also lower bound)

⇒ Greatest Lower Bound (glb / inf / infimum) Of set S:

S is non-empty subset of $\mathbb{R}$.

$\alpha = \text{glb } S$ or $\inf . S$

if (i) - $x \geq \alpha \quad \forall\ x \in S$

(ii) - if β is a lower bound then $\alpha \geq \beta$.

Note: α may not be an element of S.

→ If $\alpha \in S$, $[\text{glb } S \text{ or } \inf S = \min S]$

⇒ Completeness axiom for bounded below set:

Any bounded below subset of $\mathbb{R}$ has a greatest lower bound (glb).

Bounded set: A set S is said to be bounded if it is both bounded from ~~above~~ below and bounded from above.

$\mathbb{R} \cup \{\pm\infty\}$ ← extended real number system

$a + \infty = \infty \qquad a \in \mathbb{R}$

$a - \infty = -\infty$

$a \cdot \infty = \infty \qquad a > 0$

$a \cdot \infty = -\infty \qquad a < 0$

Note: If S is unbounded above set. Then, $\boxed{\sup S = +\infty}$

If S is unbounded below set. Then, $\boxed{\inf S = -\infty}$

Ex: Show that $\text{lub}\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = 1$

★★★

Proof:

$$n < n+1 \quad \forall \quad n \in \mathbb{N}$$

so,

$$\frac{n}{n+1} < \frac{n+1}{n+1}$$

i.e $\frac{n}{n+1} < 1$

$$S = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$$

Hence, 1 is an upper bound of S.

→ Now, we have to check for lub.

Suppose $\text{lub} S = c < 1$

$1 - c > 0$ so, $\exists\ n \in \mathbb{N}$

s.t. $\frac{1}{n} < 1 - c$

$c < 1 - \frac{1}{n} \Rightarrow c < \frac{n-1}{n}$

$\frac{n-1}{n} \in S$ $\quad n \neq 1$

Which is a contradiction.

when $n = 1$

$\frac{1}{1} < 1 - c$

i.e $\boxed{c < 0}$

That means, c is smaller than every element of S which is a contradiction.

upper bound

mber in
lower
bound

S

nded

mber

$S = +\infty$

$S = -\infty$

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Ex ② : An upper bound c of $S \neq \phi$ in $\mathbb{R}$ is the lub of S iff $\forall\ \varepsilon > 0\ \exists\ S_\varepsilon \in S$ (S_ε depends on ε) (don't use $S(\varepsilon)$)

s.t.

$$c - \varepsilon < S_\varepsilon \le c$$

Ex ③ : A lower bound b of ~~$S \neq \phi$~~ a non-empty subset of $\mathbb{R}$ is the glb of S iff $\varepsilon > 0$,

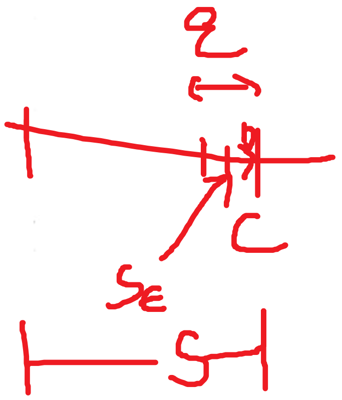
$\forall\ \varepsilon > 0,\ \exists\ S_\varepsilon \in S$ s.t.

$$b \le S_\varepsilon \le b + \varepsilon$$

Ex ④ : Prove that if $0 \le a - b < \varepsilon$ for every $\varepsilon > 0$, then $a = b$.
***

Proof ② : Suppose c is an upper bound of S satisfying the following conditions:

$\forall\ \varepsilon > 0\ \exists\ S_\varepsilon \in S$ s.t. $c - \varepsilon < S_\varepsilon \le c$



Suppose lub $S = b$

If $b < c$,

set $\varepsilon := c - b$, $\exists\ S_\varepsilon$ s.t.

$b = c - \varepsilon < S_\varepsilon$ i.e. b is not an upper bound.

$b < c$ is wrong

so, $b = c$

for converse :

given c is the lub of S to show that for $\forall\ \varepsilon > 0\ \exists\ S_\varepsilon \in S$

s.t.

$$c - \varepsilon < S_\varepsilon \le c$$

c is lub, $\varepsilon > 0$, $c - \varepsilon < c$,

$c - \varepsilon$ is not an upper bound. because if it is an upper bound then $c - \varepsilon$ is the lub which contradicts as c is the lub.

$\exists\ S_\varepsilon \in S$ $\quad S_\varepsilon > c - \varepsilon$, $\Rightarrow \boxed{c - \varepsilon < S_\varepsilon \le c}$

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⇒ Let $A, B \neq \phi$

We say there is a one-to-one correspondence between A and B, if there exists a function

$$f : A \to B$$

(i)- f is one-one (one-to-one). } bijective

(ii)- f is onto.

→ Two sets A and B are said to be equivalent if $\exists$ a one-to-one correspondence between A & B and we denote it by

$$\underline{A \approx B}$$

$\approx$ is an equivalence relation.

Ex:

$A \approx A$

$i(x) = x, \quad \forall\, x \in A, \quad \approx$ is reflexive.

→ $A \approx B \Rightarrow B \approx A$

$f : A \to B$ which is bijective.

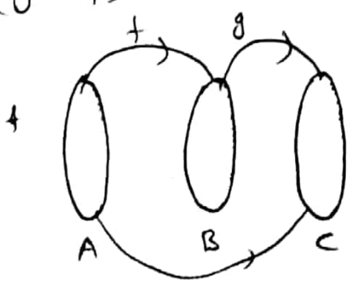
$\exists\ f^{-1} : B \to A$ is also bijective

Hence, $B \approx A$

(iii)- $A \approx B, \quad B \approx C \Rightarrow A \approx C$

$f : A \xrightarrow{\text{bijective}} B, \quad g : B \to C$ is bijective

$(g \circ f) : A \to C$ is bijective.



$(f \circ g)(x) = f(g(x))$

or may be written as

$(x)(f \circ g)$

$J = \mathbb{N} = \{1, 2, 3, \ldots\}$

Whole numbers $= \mathbb{N} \cup \{0\}$

$\mathbb{Z} = I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$J_n = \{1, 2, 3, \ldots n\}$

⇒ **Finite Set :**

A set A is said to be finite if $\exists$ a set J_n s.t.

$$A \approx J_n$$

⇒ **Infinite Set :**

A set which is not finite is said to be infinite.

⇒ **Countable set / denumerable / Enumerable :**

A set is said to be countable / denumerable / enumerable if $A \approx J$.

→ A set A is said to be atmost countable if it is either finite or countable.

→ A set which is (not countable) is called to be uncountable.
i.e it must be infinite set first

→ A infinite set is said to be countable if there is one-one correspondence

$\{1, 2, 3, 4, 5, \ldots\}$

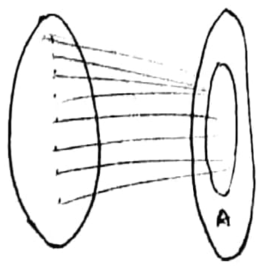
⇒ A sequence in set A is a map $f : \mathbb{N} \to A$

$f(n) = x_n$ $\qquad x_n \in A \;\; \forall n$

$f(1) = x$

$f(2) = x_2$

$f(n) = x_n$



A is countable

$\{x_1, x_2, \ldots, x_n\}$

Ex:

$J : 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \; \ldots$

$I : 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$

$f : J \to I$

$f(n) = \frac{n}{2}$ if n is even.

$\quad\;\; = -\frac{n-1}{2}$ if n is odd.

Ex:

$J \longrightarrow N$

$f(n) = 2n$

or $J \to 2J$

$f(n) = 2n$

i.e $\boxed{J \approx 2J}$

Note:

→ A set A is infinite iff it is equivalent with one of its proper Subsets.

$\Rightarrow$

$Q^+ \to$ set of +ve rational numbers

$= \{ p/q, \; q, p \in J \}$

n^{th} row, denominator n,

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} \\[2mm]
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} \\[2mm]
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} \\[2mm]
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} \\[2mm]
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5}
\end{array}$$

$f(1) = 1$

$f(2) = ?$

$f(3) = 1/2, \; f(5) = 3$

$f(4) = \frac{1}{3}$.

Theorem :

→ (A) The set of all positive rational numbers is countable.

we prove Q^+ & Q^- countable then U countable

$Q^+ = \{ p/q : p, q \in J \}$

A is countable.
B is countable.
$A \cup B$ is countable.

$f : J \to Q^+$
bijective

$Q = Q^+ \cup Q^- \cup \{0\}$

A_α $\alpha \in I$ $I = J$

$\cup A_\alpha$ → arbitrary unions of A_α.

$\cap A_\alpha$ → arbitrary intersection of A_α.

$\cup A_\alpha = \{ x \mid x \in A_\alpha \text{ for some } \alpha \}$

$\cap A_\alpha = \{ x \mid x \in A_\alpha \text{ for all } \alpha \}$

$\bigcup_{i=1}^{\infty} A_i$ → countable union.

$\bigcap_{i=1}^{\infty} A_i$ → countable intersection.

$f : Q^+ \to Q^-$

$f(x) = -x$.

Proof : (A)

By definition of Q^+, every element of Q^+ can be written in the form $\frac{p}{q}$, $p, q \in J$.

Now consider the following arrangements in which the n^{th} row consists of all fractions with denominator n and numerators with $1, 2, 3, 4, \ldots$ successively and the m^{th} column consists of all fractions with numerator m and denominator $1, 2, \ldots$ etc.

$$
\begin{array}{cccccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \frac{7}{1} & \cdots \\[1em]
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \frac{7}{2} & \cdots \\[1em]
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \frac{7}{3} & \cdots \\[1em]
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \frac{7}{4} & \cdots \\[1em]
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \frac{7}{5} & \cdots \\[1em]
\frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \frac{5}{6} & \frac{6}{6} & \frac{7}{6} & \cdots \\[1em]
\frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & \frac{7}{7} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

→ This is a finite set with infinite notation. same as in $\{1, 1, 1, \dots\}$

→ By following the indicated path and omiting those functions which have values already encountered along the path. We define that

letting a f such that

$f(1) = 1$, $f(2) = 2$, $f(3) = 1/2$, $f(4) = 1/3$,

$f(5) = 3$, $f(6) = 4$, $f(7) = 3/2$, $f(8) = 2/3$,

$f(9) = 1/4$, $f(10) = \frac{1}{5}$ etc. (as we have to prove one-one, onto)

An arbitrary rational number a/b is located in a^{th} column and b^{th} row of this arrangement and ~~thereafter~~ therefore is reached after finitely many steps of the type listed above.

$(0,1)$, $(0,1]$, $[0,1)$, $(0,1]$

Theorem : Set of all real numbers between 0 and 1 is uncountable.

***

$$E = \{x : 0 < x < 1\} \quad x \in R$$

is uncountable.

(i)- $T \subseteq S$, S atmost countable
$\Rightarrow T$ is atmost countable.

(ii)- $T \subseteq S$, T is uncountable $\Rightarrow S$ is uncountable.

Proof :

We know that every real numbers can be represented as non-terminating decimal (some are repeating, others are non-repeating). Assume that the set of real numbers between 0 and 1 is countable.

So, let $\exists$ one-to-one correspondance

$$f : \mathbb{N} \to I = \{x \in R \mid 0 < x < 1\}$$

$$f(1) = 0 \cdot a_{11} a_{12} a_{13} \cdots a_{1n} \cdots$$

$$f(2) = 0 \cdot a_{21} a_{22} a_{23} \cdots a_{2n} \cdots$$

$$f(3) = 0 \cdot a_{31} a_{32} a_{33} \cdots a_{3n}$$

$$f(4) = 0 \cdot a_{41} a_{42} a_{43} \cdots a_{4n}$$

Each a_{ij}, satisfies $0 \le a_{ij} \le 9$

(Note : In case where the representations are possible, such that $\frac{1}{4} = 0.24999\ldots = 0.2500$

and $\frac{4}{5} = 0.7999\ldots = 0.800$; $\mathfrak{J}$ choose the repetition of 9's rather than 0's in each case).

This establishes the uniqueness of representation.

Let,

$f(1) = 0.385217343 \ldots$

$f(2) = 0.721003245 \ldots$

$f(3) = 0.90909090 \ldots$

$f(4) = 0.42347347734 \ldots$

$f(5) = 0.26139999 \ldots$

Let us construct a real number $0 < c < 1$

s.t $c \notin f(\mathbb{N})$

Consider $c = 0.c_1 c_2 c_3 c_4 c_5 \ldots c_n \ldots$

Let c_1 be a number between 1 and 9 not equal to a_{11};

c_2 be any number between 1 and 9 not equal to a_{22};

c_3 be any number between 1 and 9 not equal to a_{33};

In general, c_i be any number between 1 and 9 not equal to a_{ii}.

By construction c differs from f(1) in 1st decimal from f(2) in 2nd decimal and in general c differs from f(i) in ith decimal place.

Hence, $f(\mathbb{N})$ fails to include, and so we reject the assumption that the real numbers between 0 and 1 are countable.

i.e set of real numbers between 0 and 1 is uncountable.

→ Four intervals $(0,1)$, $[0,1)$, $(0,1]$, $[0,1]$ are uncountable.

<u>Theorem</u>: Show that $\mathbb{R}$ is uncountable.

as using above $\mathbb{R}$ b/w 0 & 1 is uncountable. so, using

$T \subseteq S$, if T is uncountable, S is uncountable.

So, $\mathbb{R}$ is uncountable. as $(0,1) \subset \mathbb{R}$

As $(0,1)$ is uncountable. so, $\mathbb{R}$ is uncountable.

→ We have prove $(0,1)$ uncountable in exam.

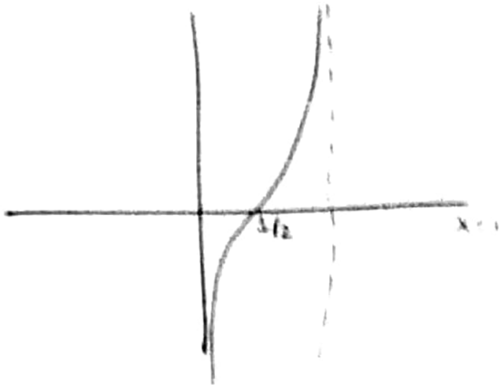
Theorem: There is a one-to-one correspondence from the open interval $(0,1)$ and $\mathbb{R}$ (entire real line).

Proof:

Define a map

$f : (0,1) \to \mathbb{R}$

$f(x) = \dfrac{x - 1/2}{x - x^2}$



$0 < x_1 < x_2 < 1$

$\exists \quad f(x_1) < f(x_2)$

i.e f is strictly increasing.

Theorem: The interval $(0,1)$ and $(0,1]$ have the same cardinality.

Proof:

let $A = \{ \frac{1}{n} \mid n > 1, \ n \text{ is an integer} \}$

If $x \in (0,1) - A$

$f(x) = x$

$f(1/n) = \dfrac{1}{n-1}$, $n > 1$

f is one-to-one and onto

for Onto:

$y = \dfrac{1}{n-1}$

$\left[n = 1 + \dfrac{1}{y} \right]$

$n > 1$

$f\left(\frac{1}{n}\right) = f\left(\frac{1}{n_1}\right)$

$\Rightarrow \boxed{n = n_1}$

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⟹ Well-Ordering Principle :

Every non-empty set of +ve integers contains a least element

Th: If A is an infinite subset of $\mathbb{N}$, then $A \cong \mathbb{N}$.

(i.e. A is countable).

Proof: Since $A \neq \phi$, and $A \subseteq \mathbb{N}$, by well-ordering principle $\exists\ a_1$ (smallest integers) s.t. $a_1 \in A$.

Since A is infinite.

$$A - \{a_1\} \neq \phi.$$

Let a_2 be the smallest element of $A - \{a_1\}$.

Since A is not finite we continue this process to get

$$a_{n+1} \in A - \{a_1, a_2, \dots a_n\}.$$

So, the sequence $a_1, a_2, a_3, \dots, a_n$ contains every element of A, and the mapping

$f : A \to \mathbb{N}$ $\quad$ f is a bijection

$f(a_n) = n$ $\quad$ i.e A is countable.

Theorem: Infinite subset of a countable set is countable.

Proof: Suppose $E \subset A$, and E is infinite.

As A is countable, arrange the elements x of A in a sequence $\{x_n\}$ of distinct (pts) elements.

construct a sequence $\{n_k\}$ as follows.

let n_1 be the smallest +ve integer s.t. $x_{n_1} \in E$.

Having chosen $n_1, n_2, n_3, \dots n_{k-1}$ $\quad (k = 2, 3, \dots)$

let n_k be the smallest integers $> n_{k-1}$

s.t. $x_{n_k} \in E$.

$(k = 1, 2, 3, \ldots)$

$f(k) = x_{n_k}$

i.e. $E \to \mathbb{N}$

$f : \mathbb{N} \to E$

$f(k) = x_{n_k}$

Note:

A_α is countable

then $\bigcup_{n=1}^{\infty} A_n$ is countable.

$\Rightarrow$ Show that the set of Rational numbers is countable.

Proof: We know $\mathbb{Q}^+$ is countable

$\mathbb{Q}^-$ is equivalent to $\mathbb{Q}^+$.

$f : \mathbb{Q}^+ \to \mathbb{Q}^-$

$f(x) = -x$

$g : \mathbb{N} \to \mathbb{Q}^+$ and $f : \mathbb{Q}^+ \to \mathbb{Q}^-$

~~gof~~ $f \circ g : \mathbb{N} \to \mathbb{Q}^-$ is a bijection

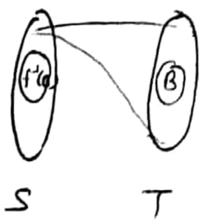
i.e. $\mathbb{Q}^-$ is countable.

$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$.

Hence, $\mathbb{Q}$ is countable.

$\rightarrow$

$f : S \to T$



$f(A)$ is image of A.

$f(A) = \{ f(x) \mid x \in A \}$

$f^{-1}(B) = \{ x \mid f(x) \in B \}$

↳ inverse image of B.

Theorem : $f(A \cap B) \subset f(A) \cap f(B)$

$\Rightarrow$ Let $y \in f(A \cap B)$

$\exists\ x \in A \cap B$ s.t. $f(x) = y$

$x \in A \cap B \Rightarrow x \in A$ and $x \in B$.

$x \in A \Rightarrow f(x) \in f(A)$ $\Rightarrow$ $y \in f(A)$

$x \in B \Rightarrow f(x) \in f(B)$ $\quad$ $y \in f(B)$

Hence, $y \in$ both $f(A)$ and $f(B)$.

So, $y \in f(A) \cap f(B)$

Hence, $\boxed{f(A \cap B) \subset f(A) \cap f(B)}$

$\Rightarrow$ $f(A \cap B) \neq f(A) \cap f(B)$

Proof : Let $S = \{s_1, s_2, s_3\}$

$T = \{t_1, t_2\}$

$A = \{s_1, s_2\}$, $B = \{s_2, s_3\}$

$f : S \to T$

$f(s_1) = t_1$; $f(s_2) = t_2$; $f(s_3) = t_1$.

$A \cap B = \{s_2\}$ & $f(A) = \{t_1, t_2\}$

$f(A \cap B) = \{t_2\}$ $\quad$ $f(B) = \{t_2, t_1\}$

So, $f(A) \cap f(B) = \{t_1, t_2\}$

Hence, $\boxed{f(A \cap B) \neq f(A) \cap f(B)}$

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9/8/18

$(0,1) \rightarrow$ uncountable

$\mathbb{R} \rightarrow$ uncountable

If $\mathbb{R}$ is countable, then $(0,1) \subset \mathbb{R}$

$\Rightarrow (0,1)$ is countable.

which is a contradiction.

$\Rightarrow$ The following statements are equivalent

a).- S is atmost countable.

b).- there exists a surjection of $\mathbb{N}$ onto S.

c).- there exists an injection of S ~~onto~~ into $\mathbb{N}$.

if set S is infinite

$S : \rightarrow N \rightarrow$ bijection

<u>Proof</u> :

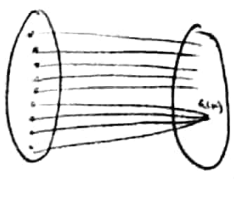
$(a) \Rightarrow (b)$ $\qquad \mathbb{N}_n = \{1, 2, \ldots, n\} \rightarrow$ set of n elements

Given S is atmost countable

If S is finite, $\exists$ a bijection h from set $\mathbb{N}_n$ onto S and we will define a map H on $\mathbb{N}$ by

$H : \mathbb{N} \rightarrow S$

by $\quad H(k) := \begin{cases} h(k) & \text{for } k = 1, 2, \cdots n \\ h(n) & \text{for } k > n \end{cases}$



i.e. H is onto from $\mathbb{N}$ to S.

★ ⇒ If S is countable

$\exists$ bijection $f : \mathbb{N} \to S$

Hence, ~~because~~ $\exists$ a surjection from $\mathbb{N}$ to S.

(b) $\Rightarrow$ (c)

If $H : \mathbb{N} \to_{s \in} S$ is onto/surjection.

define $H_1 : S \to \mathbb{N}$ by letting $H_1(s)$ be the least element in the set

$$H^{-1}(s) = \{ n \in \mathbb{N} :$$

(using well ordering principle).

To show that H_1 is one-to-one

i.e $\quad H_1(s) = H_1(t) \Rightarrow s = t$

Now, if $s, t \in S$.

$$n_{st} = H_1(s) = H_1(t)$$

$$\left.\begin{array}{l} H(n_{st}) = s \\ H(n_{st}) = t \end{array}\right\} \Rightarrow \boxed{s = t}$$

$$s = H(n_{st}) = t$$

Hence, H_1 is one-to-one.

★ (c) $\Rightarrow$ (a)

if $H_1 : S \to \mathbb{N}$ is one-to-one.

then $H_1 : S \to H_1(S)$ is bijective.

$$H_1(S) \subset \mathbb{N}$$

$\mathbb{N}$ is countable.

$\Rightarrow H_1(S)$ is countable.

$\Rightarrow$

$f : \mathbb{N} \to H_1(s)$

$f^{-1} : H_1(s) \to \mathbb{N}$

$H_1 : S \to H_1(s)$

$H_1 : S \to H_1(s) \xrightarrow{f^{-1}} \mathbb{N}$

$f^{-1} \circ H_1 : S \to \mathbb{N}$

Hence, S is countable.

Theorem: Let A be the set of all sequences whose elements are digits 0 and 1.
This set A is uncountable.

The elements of A are sequences like
$(1, 0, 0, 1, 0, 1, 1, \ldots)$

Proof:

Let E be a countable subset of A.
As E is countable, E consists of the sequences $s_1, s_2, s_3, \ldots$

$$E = \{s_1, s_2, s_3, \ldots\}$$

We construct a sequence s as follows: if the n^{th} digit in s_n is 1, we let the n^{th} digit of s be 0, and vice-versa. Then the sequence 's' differs from every element of E in at least one place; $s \notin E$ but $s \in A$.

Hence, $\boxed{E \subsetneq A.}$

→ Any countable subset A is a proper subset of A.

→ We have proved that every countable subset of A is a proper subset of A.

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So, A is uncountable.

(for otherwise A would be a proper subset of A, which is absurd).

<u>09/8/17</u>

$\rightarrow$ [A_m is a sequence of atmost countable sets

$\bigcup_{m=1}^{\infty} A_m$ is atmost countable.]

$\downarrow$

countable union

$\bigcup_{\alpha \in I} A_\alpha \leftarrow$ arbitrary union

<u>Proof</u>:

Given A_m is atmost countable. $\left[A = \bigcup_{m=1}^{\infty} A_m\right]$

So, for each $m \in \mathbb{N}$, let $\phi_m : \mathbb{N} \xrightarrow{\text{onto}} A_m$

Define a map

$$\psi : \mathbb{N} \times \mathbb{N} \to A$$

by $\psi(m,n) = \phi_m(n)$.

To show that ψ is <u>onto</u>.

If $a \in A$, i.e. $a \in A_m$ for some $m \in \mathbb{N}$.

Hence, $\exists$ a least $n \in \mathbb{N}$ s.t. $a = \phi_m(n)$

Therefore,

$a = \psi(m,n)$

ψ is onto.

A, P(A)
if $A = \mathbb{N}$
$P(\mathbb{N})$
$\hookrightarrow$ uncountable

's'

$\Rightarrow$ Show that there is no onto/surjective map from A to P(A).

<u>Proof</u>: Suppose $\exists$ $\psi : A \to P(A)$

Two possibility $a \in \psi(a)$ or $a \notin \psi(a)$

$a \in A$, which is <u>surjective</u>.

$\psi(a) \in P(A)$

$D = \{a \in A \mid a \notin \psi(a)\}$

$D \subset A$, i.e $D \in P(A)$.

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$d : X \times X \to \mathbb{R}$

→ all normal 3 properties

→ $(\mathbb{R}^n, d)$ usual metric

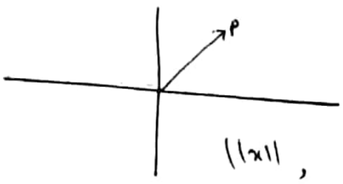
$x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$; $x_j \in \mathbb{R}$ $j = 1, 2, \ldots, n$

$y \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n)$; $y_j \in \mathbb{R}$ $j = 1, 2, \ldots, n$

Dot product or scalar product of x and y.

$$x \cdot y = \sum_{j=1}^{n} x_j y_j$$

$$\|x\| = \sqrt{\sum_{j=1}^{n} x_j^2} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$



$x = (x_1, x_2)$

$y = (y_1, y_2)$

$x \cdot y = x_1 y_1 + x_2 y_2$

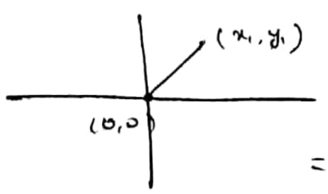
$|x|$ = length of vector from origin to x.

$0 = (0, 0, 0, \ldots, 0)$

$$d(x, y) = \|y - x\|$$

$$= \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$



$$= \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2}$$

$$= \sqrt{x_1^2 + y_1^2}$$

EX: $x, y \in \mathbb{R}$

$$d^*(x,y) = \frac{|x-y|}{1+|x-y|}$$

$$d^*(x,y) < 1 \quad \forall\ x,y$$

$$\rightarrow \quad d^*(x,y) = \frac{|x-y|}{1+|x-y|}$$

$$1+|x-y| > 0$$

$$|x-y| \geq 0$$

So, $d^*(x,y) \geq 0$ (1st condition)

$$d^*(x,y) = 0 \Rightarrow \frac{|x-y|}{1+|x-y|} = 0 \Rightarrow |x-y| = 0.$$

iff $\boxed{x=y}$. (2nd condition)

$$d^*(x,y) = \frac{|x-y|}{1+|x-y|}$$

$$= \frac{|y-x|}{1+|y-x|}$$

$$\boxed{d^*(x,y) = d^*(y,x)}$$

To show that,

$$d^*(x,y) + d^*(y,z) \geq d^*(x,z)$$

i.e **T.P:** $\dfrac{|x-y|}{1+|x-y|} + \dfrac{|y-z|}{1+|y-z|} \geq \dfrac{|x-z|}{1+|x-z|}$

$$\cancel{\frac{|x-y|}{1+|x-y|}} + \cancel{\frac{|y-z|}{1+}}$$

$1+|x-y| + |y-z| \geq 1+|x-y|$ & $1+|y-z|$

so,

$$\frac{1}{1+|x-y|+|y-z|} \leq \frac{1}{1+|x-y|} \text{ or } \frac{1}{1+|y-z|}$$

$$\frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \geq \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} = \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|}$$

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Note :

$m \geq k \geq 0$

so, $\left[\dfrac{m}{1+m} \geq \dfrac{k}{1+k}\right]$

$x = y$

$\Rightarrow f(x) = f(y)$

$f(m) = \dfrac{m}{1+m}$

$f'(m) = \dfrac{(1+m)-m}{(1+m)^2} = \dfrac{1}{(1+m)^2}$

$|x-y| + |y-z| \geq |x-z|$

so,

$$\frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \geq \frac{|x-z|}{1+|x-z|}$$

So,

$$\left[\frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \geq \frac{|x-z|}{1+|x-z|}\right]$$

= Proved.

Q. In $\mathbb{R}$,

① $d_1(x,y) = |x^2 - y^2|$

② $d_2(x,y) = 2|x-y|$

③ $d_3(x,y) = \text{glb}\{2, |x-y|\}$

$\rightarrow$ ①. $d_1(1,-1) = |1^2 - (-1)^2| = 0$

but $1 \neq -1$.

Hence d_1 is not metric.

②. d_2 is metric.

metric obtained by scalar multiplication is metric.

$d_2(x,y) = 2|x-y| \geq 0$

as $|x-y| \geq 0$

$d_2(x,y) = 0$

$2|x-y| = 0$

iff $|x-y| = 0$

i.e iff $[x = y]$

$[d_2(y,x) = 2|y-x| = 2|x-y| = d_2(x,y)]$

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$$d_2(x,y) + d_2(y,z) \geq d_2(x,z)$$

T.P: $2|x-y| + 2|y-z| \geq 2|x-z|$

as we know,

$$|x-y| + |y-z| \geq |x-z|$$

so,

$$2|x-y| + 2|y-z| \geq 2|x-z|$$

Hence, $d_2(x,y) + d_2(y,z) \geq d_2(x,z)$.

So, d_2 is a metric.

So, on generalization any multiple of a metric is also a metric.

3.

W. Rudin

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Q. For $x, y \in \mathbb{R}$, define

① $d_1(x,y) = (x-y)^2$

② $d_2(x,y) = \sqrt{|x-y|}$

③ $d_3(x,y) = |x-2y|$

④ $d_4(x,y) = \dfrac{|x-y|}{1+|x-y|}$ ✓

①.

$d_1(x,y) = (x-y)^2$

as $(x-y)^2 \geq 0$

so, $d_1(x,y) \geq 0$.

$d_1(x,y) = (x-y)^2 = 0$

$x-y = 0$

iff $x=y$

$d_1(y,x) = (y-x)^2 = (x-y)^2 = d_1(x,y)$

$d_1(x,y) + d_1(y,z) \geq d_1(x,z)$

$\cancel{x^2+y^2-2xy+y^2+z^2-2yz \geq d_1(x,z)}$

$d_1(0,1) = 1$

$d_1(1,2) = 1$

$d_1(0,2) = 4$

$\boxed{1+1 \ngeq 4}$

★ subset of a metric space is also a metric space.

Let X be a metric space.

points means they are elements of X. (X, d)

subsets means subsets of X. $Y \subset X$

(X, d)

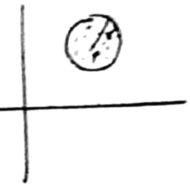
$\Rightarrow (Y, d)$ is also a metric space.

1. Neighbourhood, $r > 0$, $p \in X$ nhbd.

$$N_r(p) = \{ q \in X \mid d(p,q) < r \}$$

This is Neighbourhood at point p with radius r.

p → center
r → radius
all inside points
nhbd



2. Deleted Nhbd

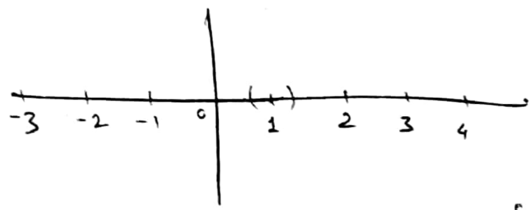
$$N_r(p) = N_r(p) \neq p$$

$$N_r(p) = \{ q, \ 0 < d(p,q) < r \}$$

3. Limit point

A point p is said to be a limit point of E if every nhbd of p, has a point $q \neq p$.

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$



$$N_{1/4}(1) = \{1\}$$

as $d(1, p) < 1/4$

$p = 1$ only

So, Every point of $\mathbb{Z}$ is not a limit point.
It is discrete point.

→ A point p is said to be a limit point if every nhbd $N_r(p) = N(p, r)$ intersecting the set at a point q other than p $q \neq p$.

3. Isolated points

If $p \in E$ and p is not a limit point, it is called an isolated point.

4. E is <u>closed</u> if <u>every limit pt. of E</u> is a point of E.

5. A point p is said to be an interior point if $\exists$ a nbhd $N_r(p)$ s.t. $N_r(p) \subset E$.

6. E is open if every pt. of E is an interior point.

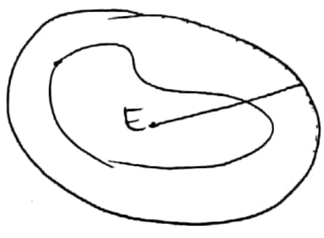
7. $E^c = \{x \mid x \notin E\}$

8. <u>Perfect set</u> :

E is perfect if it is closed and every point of E is a limit of E.

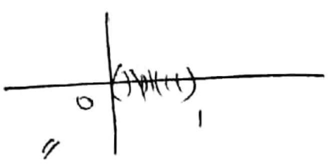
9. <u>Bounded set</u> :

A set E is said to be bounded if $\exists\, q \in X$, $M \in \mathbb{R}$ s.t. $d(p,q) < M$ $\forall\, p \in E$.



10. <u>Dense set</u> :

A set E is said to be dense in X, if every point of X is a limit point of E or point of E or both.



//
Uncountable no. of limit points

(0,1)
↳ 0, 1 is limit point of E but does not belong to E.
So, p may or may not belong to E but q does.

$\mathbb{Q}, \mathbb{R}$

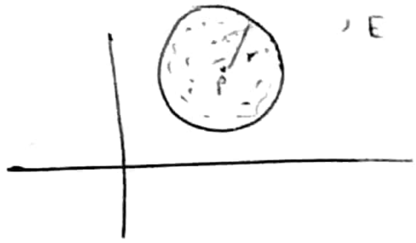
- $\mathbb{Q}$ is dense in $\mathbb{R}$.
- Every real number is limit point of $\mathbb{Q}$.
- Between 2 real no., there is a rational number.

$\Longrightarrow$

<u>Theorem</u>: Every neighbourhood is an open set.

Let E be the neighbourhood of p with radius r & center p.

$E = N_r(p)$



To show that E is open.

i.e. every point of E is an interior point of E.

$\Longrightarrow$

Let $q \in E$

$d(p,q) < r$

$r - d(p,q) > 0$

$h = r - d(p,q) > 0$

$\Rightarrow \boxed{d(p,q) = r - h}$

For all points s

s.t. $d(q,s) < h$

$d(p,s) \leq d(p,q) + d(q,s)$

$= r - h + h < r$

$d(p,s) < r$

$s \in E$.

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→ $N_\epsilon(p) = \{q \in X \mid d(p,q) < \epsilon\}$

→ nbhd of point p

$N(p, \epsilon)$ is the nbhd with centre 'p' and radius ϵ.

$N(p, \epsilon)$ is an open set
i.e every pt. of $N(p, \epsilon)$ is an interior point.

⇒ Let E be a subset of X. p is a limit point of E iff every nbhd of p contains infinitely many points of E.

⇒ Corollary: If the set E is finite, it does not have a limit pt.

Because if it has a limit point, then every nbhd of the point intersects the set into infinite many pts, but which is not possible, as the set is finite.

$N \cap E$ contain infinite points.

→ if the set E doesn't have any limit points then, this is closed (vacuosly true (hypothesis is wrong)).

Hence, a finite set is closed.
It is vacuosly true as there is no limit point.

→ $\{x \in \mathbb{R} \mid x < 0, x = 99\}$

Let us consider

$E = \{ \frac{1}{n}, | n \in \mathbb{N} \}$ ← i.e infinite set.

$E = \{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \}$

→ Here, 0 is a limit point of E

$N(0, \epsilon)$ as any epsilon will cut it.
(i.e intersect other than 0).

→ But . 1 ~~is no~~
But $1, 1/2, 1/3 \dots$ is not a limit point.

→ 0 is a limit point.

Hence, E is not <u>closed</u> as limit pt $0 \notin E$.

<u>Theorem</u> : Let E be a subset of X. p is a limit point of E, then every nhbd of p contains infinitely many points of E.

<u>Proof</u> :
Suppose $\exists$ a deleted nhbd N of p s.t.
N contains only finite number of points $p_1, p_2, \dots, p_n$ of E.

so, $d(p, p_1) > 0$, $d(p, p_2) > 0, \dots, d(p, p_n) > 0$

Let $r = \min\{ d(p, p_1), d(p, p_2), \dots, d(p, p_n) \}$
minimum of finite numbers is finite.

i.e $r > 0$.

Now, $N_r(p)$ will not intersect any point other than p.

Hence, p is not a limit point.

$\Rightarrow \Leftarrow$
This shows that there doesn't exist any deleted nhbd.
Hence proved.

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Example :

let us consider the following subsets of $\mathbb{R}^2$:

(a). The set of all complex numbers z s.t. $|z|<1$.

(b). The set of all complex numbers Z s.t. $|z|\le 1$.

(c). A finite set

(d). A set of integers. ✓

(e). $E = \{ \frac{1}{n} \mid n = 1,2,\ldots \}$ ✓

(f). The set of all complex number ($i e \mathbb{R}^2$).

(g). Open interval $(a,b) = \{ x \mid a<x<b \}$ ✓

(d), (e), (g) ~~is~~ are subset of $\mathbb{R}$.

	Closed	Open	Perfect	Bounded
(a).	No	Yes	No	Yes
(b).	Yes	No	Yes	Yes
(c).	Yes	No	No	Yes
(d).	Yes	No	No	No
(e).	No	No	No	Yes
(f).	Yes	Yes	Yes	No
(g).	No		No	Yes

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$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$

$a_i < b_i \quad \forall\ i = 1, 2, \dots, k.$

$x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k.$

k-cell $= \{x_i,\ a_i \le x_i \le b_i,\ i = 1, 2, \dots, k\}$

bounded

1. Cells are closed interval.

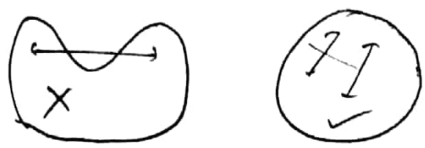
2.

→ A subset $E \subset \mathbb{R}^k$ is said to be convex.

if $\forall\ x, y \in E$

$\lambda x + (1-\lambda) y \in E$

$0 \le \lambda \le 1$



→ Open ball in $\mathbb{R}^k$

$B(x,y) = \{y \in \mathbb{R}^k \mid |y-x| < r\}$

$\bar{B}(x,y) = \{y \in \mathbb{R}^k \mid |y-x| \le r\}$

$B, \bar{B}$ are convex sets.

$\longrightarrow$ Let $y, z \in B(x, r)$

$|y - x| < r, \quad |z - x| < r.$

$\lambda y + (1-\lambda) z \in B(x, r)$

or $\lambda z + (1-\lambda) y \in B(x, r)$.

$\Rightarrow$ $|\lambda y + (1-\lambda) z - x|$

$= |\lambda(y - x) + \lambda x + (1-\lambda) z - x|$

$= |\lambda(y - x) - x(1-\lambda) + (1-\lambda) z|$

$= |\lambda(y - x) + (1-\lambda)(z - x)|$

$\leq \lambda|y - x| + (1-\lambda)|z - x|$

$< \lambda r + (1-\lambda) r$

$< r$

In closed, just change $<$ to $\leq$

Note:

$\longrightarrow$ $\{(x, y) \mid x^2 + y^2 = 1\} \rightarrow$ is not convex.

for convex $\rightarrow$ $\{(x, y) \mid x^2 + y^2 < 1 \text{ or } \leq 1\}$.

$\Rightarrow$

$\mathbb{R}$.

$\{G_\alpha\}$ be a collection of subset of $\mathbb{R}$.

$\{F_\alpha\}$ is a collection of subsets of $\mathbb{R}$.

$\rightarrow$ Suppose $\{G_\alpha\}$ is open $\forall \alpha$.

& $\{F_\alpha\}$ is closed $\forall \alpha$.

$\longrightarrow$ $\bigcup_{\alpha \in I} G_\alpha \longrightarrow$ open $\{$i.e arbitrary union of open set is open$\}$.

& $\bigcap_{\alpha \in I} F_\alpha \rightarrow$ closed

$\longrightarrow$ $\bigcap_{i=1}^{n} G_i$ is open $\{$Finite intersection of open set is open$\}$

this doesn't hold for $n = \infty$

$\bigcup_{i=1}^{n} F_i$ is closed $\{$" union " closed " " closed$\}$

→ $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$

→ $\cap G_n = \{0\}$

→ $\mathbb{R}$, ϕ

$\llcorner$ it is both open and closed.

→ G^c is closed.

F^c is open.

⇒ A collection τ (to subsets of $\mathbb{R}$) $\{G_\alpha\}$ i.e. $\tau = \{G_\alpha\}$ is said to be a <u>topology</u>. if

(a). $\phi, \mathbb{R} \in \tau$.

(b). $\bigcup_{\alpha \in I} G_\alpha \in \tau$.

(c). $\bigcap_{i=1}^{n} G_i \in \tau$.

→ $(\mathbb{R}, \tau)$ is called a topological space.

$X = \{a, b, c\}$.

$\tau = \{\phi, X, \{a\}, \{a, b\}\}$

<u>Theorem</u>: Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α.

Then, $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$

Demorgan's law:

$(A \cup B)^c = A^c \cap B^c$

$\left(\bigcup_\alpha A_\alpha\right)^c = \bigcap_\alpha A_\alpha^c$

<u>Proof</u>: Let $A = \left(\bigcup_\alpha E_\alpha\right)^c$, $B = \bigcap_\alpha E_\alpha^c$.

To show that, <u>$A = B$</u>

→ Let $x \in A = \left(\bigcup_\alpha E_\alpha\right)^c$

$x \notin \bigcup_\alpha E_\alpha$

$A \subset B$ $\quad$ $x \notin E_\alpha$ for any α

$x \in E_\alpha^c$ for any α

set}.

set is open}

" closed}

$\Rightarrow x \in \cap E_\alpha^c$

Similarly for $B \subset A$,

Let $x \in B$

$x \in E_\alpha^c \quad \forall \alpha$

$x \notin E_\alpha \quad \forall \alpha$

$\rightarrow x \notin \cup E_\alpha$.

$\Rightarrow x \in (\cup E_\alpha)^c$.

$A_1 \cup A_2 \cup \cdots A_n$,

$A_{n+1} = \phi$

$A_{n+2} = \phi$.

$\rightarrow A_1 \cup A_2 \cup A_3 \cdots \cup A_n \cup A_{n+1} \cup A_{n+2} \cdots$

$(A_1 \cup A_2 \cup A_3 \cdots \cup A_n \cup \phi \cup \phi)^c$

$= A_1^c \cap A_2^c \cdots \cap A_n^c$

Theorem : A set E is open if E^c is closed.

Proof : Suppose E^c is closed. To show that E is open i.e. every point of E is an interior point.

Let $x \in E \Rightarrow x \notin E^c$.

$\Rightarrow x$ is not a limit pt. of E^c.

Hence, $\exists$ a nhbd N of x s.t. $N \cap E^c = \phi$.

$\Rightarrow x \in N \subset E$ i.e. E is open.

→ Boundary points are not interior points.
because for b.p. we will not get a nhbd which is inside the set.

$\Rightarrow$ Suppose E is open. To show that E^c is closed.

→ i.e every point is a limit point.

Let x be a limit point of E^c.

i.e. every nhbd of x contains points of E^c.

Hence, x is not an interior point of E.

Since E is open → $x \notin E$.

i.e $x \in E^c$.

Hence, E^c is closed.

Corollary : A set F is closed if F^c is open.

(Replace E^c by F above)

Theorem : For any collection $\{G_\alpha\}$ of open sets,

(a). $\bigcup_\alpha G_\alpha$ is open.

(b). For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

(c). For any finite collection $G_1, G_2, \ldots, G_n$ of open sets $\bigcap_{i=1}^{n} G_i$ is open.

(d). For any finite collection $F_1, F_2, \ldots, F_n$ of closed sets, $\bigcup_{j=1}^{n} F_j$ is closed.

Proof :

a). Let $G = \bigcup_{\alpha \in I} G_\alpha$ (arbitrary union of G_α).

TO show that every point of G is an interior point of G (i.e. G is open).

Let $x \in G \Rightarrow x \in G_\alpha$ for some α

if $A_i \subset B_i$ then $\bigcup A_i \subset \bigcup B_i$

As G_α is open for all α.

$\exists\ N_r$ s.t. $x \in N_r \subset G_r$.

$\Rightarrow$ x is an interior point of G.

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Theorem: a)- For any collection $\{G_\alpha\}$ of open sets $\bigcup_\alpha G_\alpha$ is open.

b)- For any collection $\{F_\alpha\}$ of closed sets $\bigcap_\alpha F_\alpha$ is closed.

(We can't drop finiteness in (c) and d). respectively).

c)- For any finite collection $G_1, G_2, \ldots, G_n$ of open sets $\bigcap_{j=1}^{n} G_j$ is open.

d)- For any finite collection $F_1, F_2, \ldots, F_n$ of closed sets $\bigcup_{j=1}^{n} F_j$ is closed.

★ For topology, b & d ✓, a & c ✗.

→ ϕ, $\mathbb{R}$ are both open and closed.

⇒ A topology on $\mathbb{R}$:

A collection $\mathcal{T}$ is said to be a topology on $\mathbb{R}$ if

a)- $\phi, \mathbb{R} \in \mathcal{T}$.

b)- $\{A_\alpha\} \in \mathcal{T} \Rightarrow \bigcup A_\alpha \in \mathcal{T}$.

c)- $\{A_i\}_{i=1}^{n}$ ~~on~~ $\bigcap_{j=1}^{n} A_j \in \mathcal{T}$.

$(\mathbb{R}, \mathcal{T})$ is called a topological space.

⇒ Proof:

a). let $G = \bigcup_{\alpha \in I} G_\alpha$

To show that G is open.

i.e every point of G is an interior point of G.

let $x \in G \Rightarrow x \in G_\alpha$ for some α.

i.e x is an interior point of G_α.

i.e $\exists\, N(x,\epsilon) \subset G_\alpha$.

$\bigcup N(x,\alpha) \subset \bigcup_\alpha G_\alpha = G$ i.e $\boxed{N(x,\alpha) \subset G}$

if $A \subset B$ $\Rightarrow \boxed{\bigcup A \subset \bigcup B}$ i.e arbitrary union is also subset.

b).

$$\left(\bigcap_\alpha F_\alpha \right)^c = \bigcup_\alpha F_\alpha^c$$

F_α is closed.

$\Rightarrow$ F_α^c is open.

$\bigcup_\alpha F_\alpha^c$ is open.

$\Rightarrow$ $\left(\bigcup_\alpha F_\alpha^c \right)^c$ is closed.

i.e $\bigcap_\alpha F_\alpha$ is closed.

c).

Let $G = \bigcap_{i=1}^n G_i$

$x \in G \Rightarrow x \in G_i \quad \forall i$.

but each G_i is open

$\exists$ $r_i > 0$ such that

$N(x, r_i) \subset G_i \qquad i = 1, 2, \ldots, n$

define $r = \min(r_1, r_2, \ldots, r_n)$

$r > 0$

Now, $N(x, r) \subset N(x, r_j) \quad \forall j$

$\Rightarrow$ $N(x, r) \subset G$.

$N(x, r) \subset N(x, r_1)$

$N(x, r) \subset N(x, r_2)$

$\vdots \qquad \vdots$

$N(x, r) \subset N(x, r_n)$

d) -

$$\left(\bigcup_{j=1} F_i \right)^c = \bigcap_{j=1}^n F_i^c$$

F_i^c is open for each i,

$\bigcap_{j=1}^n F_i^c$ is open (proved in (c))

$\Rightarrow \left(\bigcap_{i=1}^{n} F_i^c \right)^c = \bigcup_{i=1}^{n} F_i$ is closed.

$\rightarrow$

let $G_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$

G_n is open subset of $\mathbb{R}$ for each 'n'

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n}\right) = \{0\}$$

Let $x \in \bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n}\right)$

$$\frac{-1}{n} < x < \frac{1}{n} \quad \forall n$$

$$x < \frac{1}{n} \quad \forall n$$

$$\Rightarrow x \leq 0$$

Again, $\frac{-1}{n} < x$

$\Rightarrow 0 \leq x.$ i.e $\boxed{x = 0}$.

$\rightarrow$ Set,

$$F_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$$

$F_1 = \{0\}$, $F_2 = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$, $F_3 = \left\{-\frac{2}{3}, \frac{2}{3}\right\}$. . . .

$$\bigcup_{n=1}^{\infty} F_n = (-1, 1).$$

$\rightarrow$ Let $x \in \bigcup_{n=1}^{\infty} F_n$

i.e $x \in F_n$ for finite n.

$$-1 + \frac{1}{n} \leq x \leq 1 - \frac{1}{n}$$

$$-1 < -1 + \frac{1}{n} \leq x \leq 1 - \frac{1}{n} < 1$$

$$\Rightarrow -1 < x < 1$$

i.e $x \in (-1, 1)$

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Theorem: If X is a metric space and $E \subset X$.

a)- $\overline{E}$ is closed.

b)- $E = \overline{E}$ iff E is closed.

c)- $\overline{E} \subset F$ for every closed set $F \subset X$ s.t. $E \subset F$.

Proof:

a)- If $p \in X \Rightarrow p \in \overline{E}$, the result follows:

i.e. $\overline{E}$ is closed. Suppose $p \in X$ and $p \notin \overline{E} = E \cup E'$

Hence, p is neither a point of E nor a limit point of E. Hence, p has a nbhd which does not intersect E.

$E \subset X$, $\quad p \in X$

Let E' denotes the set of all limit points of E.

$\overline{E} = E \cup E'$

$\downarrow$

closure of a set

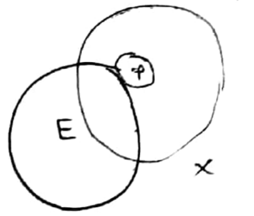
Note: Closure of a set is always closed in a metric space.

i.e E is an interior point of E^c. E^c is open.

$\overline{E} = E \cup E'$

$E' \subset \overline{E}$

$E \subset \overline{E}$



(Not for all, here there exists)

b)- If $E = \bar{E}$, then E is closed by (a).

If E is closed to show that $E = \bar{E}$

i.e $\bar{E} \subset E$

as E is closed, $E' \subset E$

$$E \cup E' \subset E \cup E$$

$$\Rightarrow \quad \bar{E} \subset E$$

and we know $E \subset \bar{E}$ $\Rightarrow$ $\boxed{E = \bar{E}}$

T.P:-
if $A \subset B$
then $A' \subset B'$

~~c)~~-

T.P: if $A \subset B \Rightarrow A' \subset B'$

Let $x \in A'$.

i.e every nbhd of x intersects A,

as $A \subset B$, it intersects B also.

c)-

For every closed set $F \supset E$

$\cancel{\Rightarrow F \supset E'}$

As F is closed.

$F \supset F'$, as $F \supset E$

Hence, $F \supset E'$, $F \cup E \supset E' \cup E$.

$$\Rightarrow \quad F \supset \bar{E}$$

i.e $\boxed{\bar{E} \subset F}$

$: E \cup \bar{E}$

...int

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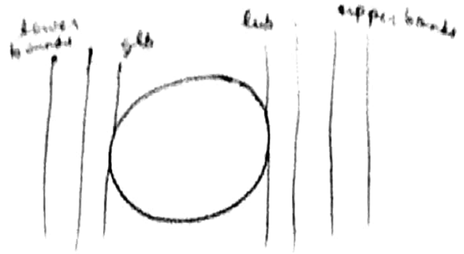
Theorem: Let E be a non-empty set of real numbers which is bounded above



Let $y = \sup E$.

Then, $y \in \bar{E}$.

Hence, $y \in E$ if E is closed.

Proof:

If $y \in E$, then $y \in \bar{E} = E \cup E'$

Assume $y \notin E$.

For any $h > 0$, $\exists$ a point $x \in E$ s.t.

$$y - h < x < y$$

As $x \in E$, y is a least upper bound.

$x \leq y$, but $x \neq y$, $x \in E$

$y \notin E$.

hence $\underline{x < y}$

Suppose $y - h \geq x$ i.e $x \leq y - h$

$y - h < x$ is true

$\Rightarrow$ $y - h$ is an upper bound of E which is smaller than y $\Rightarrow$ contradiction

as $y = \sup E$.

i.e y is a limit point of E.

$\Rightarrow$ $\underline{y \in \bar{E}}$

$y \in E = \bar{E}$.

→ subset of a metric space is metric.

⇨ $E \subset Y \subset X$

$X \to$ metric space

Y is a subset of X.

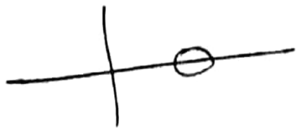
Suppose E is open relative to Y.

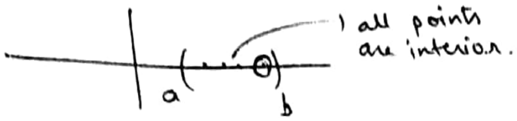
Can you say that E is open relative to X?

Let $Y = \mathbb{R}$. $\quad R \subset \mathbb{R}^2$.

$X = \mathbb{R}^2$. $\quad$ i.e. $a \to (a,0)$

$E = (a,b)$.





(a,b) is open subset of $\mathbb{R}$. but (a,b) is not an open subset of $\mathbb{R}^2$.

Hence, i.e. no nhbd of any point $x \in (a,b)$ contains inside (a,b).

Theo.: Suppose $Y \subset X$. A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X.

$E = Y \cap G$

G is open set in X

$E \subset Y \subset X$

E is open in X.

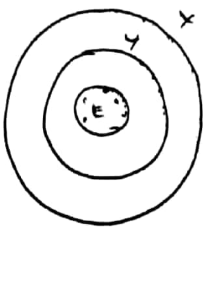
$\forall p \in E$,

$\exists r_p$ s.t.

$d(p,q) < r_p$

$, q \in X_n$

$\Rightarrow q \in E$



→ E is open in Y.

$\forall p \in E \ \exists r_p^t > 0$

s.t. $d(p,q) < r_p^t$

$q \in Y$.

$\Rightarrow q \in E$.

Proof :

Suppose G is open in X and $E = Y \cap G$.

To show that E is open relative to Y.

i.e. every point of E is an interior point relative to Y.

$\forall\ p \in E$, $p \in G$, as G is open every nbhd

V_p of p is contained in G.

$\therefore\ V_p \subset G$

i.e $p \in [V_p \cap Y \subset G \cap Y] = E$

Suppose E is open relative to Y.

to each $p \in E$, there is a positive number r_p

s.t.

$$d(p,q) < r_p, \quad q \in Y \Rightarrow q \in E$$

Let V_p be the set of all $q \in X$ s.t.

$d(p,q) < r_p$, and

define

$$G = \bigcup_{p \in E} V_p, \quad V_p \text{ is open}$$

and arbitrary union of open sets is open.

as G is open subset of X.

since, $p \in V_p$, $\forall\ p \in E$

$$E \subset G \cap Y \quad \text{—(i)}$$

Now, by the choice of V_p,

we have

and

$$V_p \cap Y \subset E$$

for every $p \in E$.

$$\bigcup V_p \cap Y \subset E$$

$$\Rightarrow \quad G \cap Y \subset E \quad \text{—(ii)}$$

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Open covering :-

By open covering, we mean we have to cover a collection of open sets

$\{G_\alpha\}$ is a collection of open sets is called an open covering of E.

if $\boxed{E \subset \bigcup_{\alpha \in I} G_\alpha}$

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$$E \subset \bigcup_{\alpha \in I} V_\alpha$$

$\{V_\alpha\}$ is collection of open subsets of X.

$(0, 1)$

$(-1, 9)$ } covers this

$$I_n = (0, 1/_{n+1})$$

$$(0,1) \not\subset \bigcup_{n=1}^{k} \left(0, \frac{1}{n+1}\right)$$

{ i.e $(0,1)$ is not covered by $\left(0, \frac{1}{n+1}\right)$ }

→ A subset E of a metric space X is said to be compact if every open covering of E has a finite subcovering.

$E \subset \bigcup_{\alpha \in I} V_\alpha$ $\{V_\alpha\}$ is an open covering

then $E \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}$.

set $(0,1)$ is not compact as the open covering

$$I_n = \left(0, \frac{1}{n+1}\right)$$

does not have a finite sub covering $(0,1)$ can't be covered by finite numbers of

$$I_1, I_2, \dots, I_k.$$

$$E \subset Y \subset X$$

$\downarrow$

E is open in Y

but E may not be open in X.

$E = Y \cap G$, G is open relative to X.

Theorem :

Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y.

Proof :

Suppose K is compact relative to X. To show that K is compact relative to Y.

Let $\{V_\alpha\}$ be an open covering relative to Y.

As V_α is open in Y, $\exists$ G_α open in X.

$$\text{s.t. } V_\alpha = Y \cap G_\alpha.$$

Since K is compact, $\exists$ finite indices $\alpha_1, \alpha_2, \dots, \alpha_n$

s.t.

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup G_{\alpha_3} \cdots \cup G_{\alpha_n}.$$

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cdots \cup G_{\alpha_n}.$$

$$K \cap Y \subset (G_{\alpha_1} \cup G_{\alpha_2} \cdots \cup G_{\alpha_n}) \cap Y$$

$$\Rightarrow K \subset (G_{\alpha_1} \cap Y) \cup (G_{\alpha_2} \cap Y) \cup (G_{\alpha_3} \cap Y) \dots \cup (G_{\alpha_n} \cap Y).$$

$$K \subset V_{\alpha_1} \cup V_{\alpha_2} \cdots \cup V_{\alpha_n}.$$

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→ K is compact relative to Y to show that it is compact relative to X.

→ let $\{G_\alpha\}$ be collection of open subsets of X which cover K and put $V_\alpha = Y \cap G_\alpha$.

V_α is open in Y $\quad V_\alpha \subset G_\alpha \quad \forall \alpha$.

$$K \subset V_{\alpha_1} \cup V_{\alpha_2} \cdots \cup V_{\alpha_n} \subset G_{\alpha_1} \cup G_{\alpha_2} \cdots \cup G_{\alpha_n}$$

Theorem :

Compact subsets of metric spaces are closed.

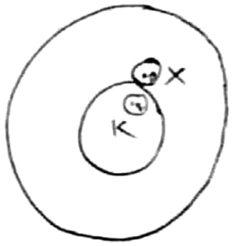
$K \subset X$
compact
$\Rightarrow K$ is closed.

Proof :

Let K be a compact subset of a metric space X.

To show that K^c is open.

Suppose $p \in X$, $p \notin K$



If $q \in K$, let V_q and W_q be the nbhd of p and q of radius $< \frac{1}{2} d(p,q)$

as K is compact.

$\exists\ q_1, q_2, \ldots, q_n \in K$ s.t. $\quad p \in V_{q_1} \cap V_{q_2} \cdots \cap V_{q_n} = V$

$$K \subset W_{q_1} \cup W_{q_2} \cdots \cup W_{q_n} = W$$

$$p \in V \subset K^c$$

K^c is open.

Hence, K is closed.

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Theorem: Closed subsets of compact sets are compact.

Proof:

Let $F \subset K \subset X$.

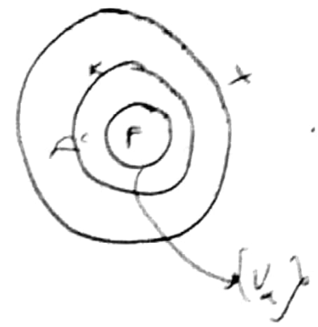
F is closed subset of X.

K is compact.

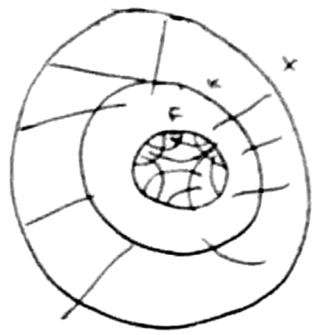
Let $\{V_\alpha\}$ be an open covering of F.

Now adjoin F^c to the collection $\{V_\alpha\}$.

$$\Omega = \{V_\alpha \cup F^c\}$$



Ω is a covering of K.
as K is compact, $\exists$ a finite subcovering Φ which covers K and also covers F as $F \subset K$.



$J = \{V_\alpha\}$

$\{F^c \cup V_\alpha\}$

In Φ, if F^c is there we can remove it still the subcollection is finite.

i.e. F is compact.

$\Rightarrow$ **Corollary:**

F ~~and~~ is closed and K is compact subsets of X.
Then $F \cap K$ is compact.

Proof: $F \cap K \subset K$

K is compact $\Rightarrow$ K is closed.

$F \cap K$ (intersection of closed sets is closed).

$\downarrow$ closed

closed subset of $G \Rightarrow F \cap K$

i.e compact set is compact.

Theorem :- Let $\{K_\alpha\}$ be a collection of compact subset of X with every finitely many intersection of K_α is non-empty, then $\cap K_\alpha \neq \phi$.

i.e any finite interse $\bigcap_{j=1}^{n} K_j \neq \emptyset$, $\bigcap_{j=1}^{n} K_j \neq \phi$ then whole intersec. non-empty

$\rightarrow$

for each α,

$K_{\alpha_1} \cap K_{\alpha_2} \cdots \cdots \cap K_{\alpha_n} \neq \phi$

$\Rightarrow \cap K_\alpha \neq \phi$

Proof :

Fix K_1, from the collection $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^c$.

$\{G_\alpha\}$ is an open covering.

Assume that no point of K_1 belong to K_α.

Hence $\{G_\alpha\}$ is an open covering of K_1.

As, K_1 is compact, $\exists$ finite indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ s.t.

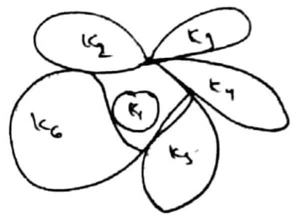
$$K_1 \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$$

$$K_1 \cap K_{\alpha_1} \cap K_{\alpha_2} \cdots \cap K_{\alpha_n} = \phi$$

$\Rightarrow\Leftarrow$ (i.e so, our assumption is wrong)

Hence,

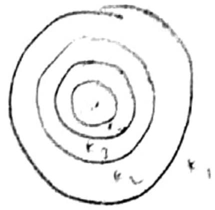
$$\cap K_\alpha \neq \phi$$



⇒ **Corollary:**

If $\{K_n\}$ is a sequence of non-empty compact sets s.t. $K_n \supset K_{n+1}$ $(n=1,2,\cdots)$

then $\bigcap_{n=1}^{\infty} K_n \neq \phi$.



Proof:

$K_n \supset K_{n+1}$

$$\Rightarrow \bigcap_{n=1}^{m} K_n = K_m \neq \phi$$

Theorem: If E is an infinite subset of a compact set K. Then E has a limit point in K.

eg:-

$E \subset K$

$E = \{1/n\}$

$K = [0,1]$

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