28th February

Recall

1) Darboux Integral. ⟵ $\int_a^b f(x)\,dx$

2) Riemann Integral. ⟵

Fundamental Theorem of Integral Calculus.

i) $f \in \mathcal{R}[a,b]$, if $\exists$ a differentiable function F

$F'(x) = f(x)$

then $\int_a^b f(x)\,dx = F(b) - F(a)$.

(ii) $\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$ $F(x) = \int_a^x f(t)\,dt$

f is $R[a,b]$ then f is continuous

Moreover, if f is $C[a,b]$
then $F(x) = \int_a^x f(t)\,dt$
diff $F(x)$ & $F'(x) = f(x)$.

(i) $f_n(x) = \begin{cases} 1 & n < q_k \\ 0 & o/w \end{cases}$ $\{q_k\}$'s are rationals.

(ii) $f_n(x) = \begin{cases} n, & 0 < x < 1/n \\ 0, & 1/n < x < 1 \end{cases}$

$\int f_n = 1$ but $\int f = 0$

$m(E) = 0$

f is riemann integrable if and only if there discontinuities have have lebesgue measure 0

E is the set of discontinuities of f.

$f \in R[a,b] \iff m(E) = 0$

<u>Proof</u>: $\omega_\delta(x) = \sup \{ |f(y) - f(z)|, y, z \in (x-\delta, x+\delta) \cap [a,b] \}$, $\delta > 0$

Oscillation of f at x.

$E_n = \{ x \in E \mid \omega_\delta(x) \geq \frac{1}{n} \}$,

then $E = \bigcup_{n \geq 1} E_n$.

Suppose f is R integrable,

T.S. $m(E) = 0$

If $m(E_n) = 0 \; \forall n$ then $m(E) = 0$.

Let $m(E_N) > 0$ for some N.

ee a partition $x_0 = a < x_1 \ldots < x_n = b$

$U(P,f) - L(P,f)$

$= \sum_{i=1}^{n} (M_i - m_i)(x_{i-1} - x_i) \geq \sum_{(x_{i-1}, x_i) \cap E_N \neq \phi} (M_i - m_i)(x_{i-1} - x_i)$

$\geq \sum_{[x_{i-1}, x_i] \cap E_n \neq \phi} (\omega_\delta(x))(x_i - x_{i-1})$

$\geq \frac{1}{n} \sum (x_i - x_{i-1}) > \frac{\alpha}{n} > 0$

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Suppose $m(E)=0$, T.S f is R-Integrable

If $m(E)=0$, then $m(E_n)=0 \ \forall n$.

Each E_n is compact.

$\Rightarrow E_n \subset \bigcup_{k=1} (a_k, b_k)$ s.t.

$$\sum_{k=1} (b_k - a_k) < \epsilon.$$

$I = \bigcup_{k=1}^{l} (a_k, b_k)$, I^c is compact.

$x \in I^c$, $(c_k, d_k)_{k=1}^{m}$

s.t. $\omega_\delta(x) < \frac{1}{n}$.

$$U(P,f) - L(P,f) = \sum (b_k - a_k)(M_k - m_k) + \sum (d_k - c_k)(M_k' - m_k')$$

$\leq 2M\epsilon$, $\leq \frac{(b-a)}{n}$

$\longrightarrow 0$ as $\epsilon \to 0$, $n \to \infty$.

$\Rightarrow U(P,f) - L(P,f) \leq 2M\epsilon + \frac{(b-a)}{n}$

Recall: $f \in R[a,b] \Leftrightarrow m(E)=0$

E = set of discontinuity of f.

Measurable: A function f is said to be measurable if for every $\alpha \in \mathbb{R}$ $\{x : f(x) < \alpha\}$ is measurable

Let f, g is measurable then $\forall \alpha \in \mathbb{R}$

$\{x : f(x) > \alpha\}$ is measurable

Now consider

$$\{f+g > \alpha\} = \bigcup_{r \in \mathbb{Q}} \{f > \alpha - r\} \cap \{g > r\}$$

$$fg = \frac{1}{4}\{(f+g)^2 - (f-g)^2\}$$

If $\{f_n\}$ is a sequence of measurable function

then $\sup\{f_n\}$ and $\inf f_n$ are also measurable

Now $\limsup f_n = \inf_{k} \sup_{n \geq k} f_n$

Similarly

$\liminf f_n$

Littlewood's Three Principle

① Every measurable set of finite measure is almost finite union of closed intervals.

② Every measurable ~~set~~ function is almost a continuous function. (Lusin's Theorem)

③ Every pointwise convergent sequence of measurable function is almost uniformly convergence (Egorov's Theorem)

① Suppose E is a measurable set with finite measure then $\exists$ F (finite union of closed sets) S.T. $m(E \Delta F) < \varepsilon$, for any given $\varepsilon > 0$.

Proof: E is measurable. we can find a sequence of closed intervals $\{I_j\}$ S.T. $E \subset \bigcup_j I_j$

& $\sum_{j=1}^{\infty} m(I_j) < m(E) + \varepsilon/2$

Now $m(E) < \infty \Rightarrow \sum_{j=1}^{\infty} m(I_j)$ is a convergent series.

Choose N S.T. $\sum_{j=N+1}^{\infty} m(I_j) < \varepsilon/2$

Let $F = \bigcup_{j=1}^{N} I_j$.

$$m(E \setminus F) + m(F \setminus E)$$

$$m(E \setminus F) \leq \sum_{j=N+1}^{\infty} m(I_j) + m\left(\bigcup_{j=1}^{\infty} I_j \setminus E\right)$$

$$\leq \varepsilon/2 + \left(\sum_{j=1} m(I_j) - m(F)\right)$$

Egorov's Theorem (Third Principles)

$\{f_n\}$ be a sequence of measurable function S.T. $f_n \to f$ (pointwise) on a measurable set E. $(m(E_n) < \infty)$. Then for $\varepsilon > 0$, we can find a closed set A_ε such that $m(E \setminus A_\varepsilon) < \varepsilon$ & $f_k \to f$ uniformly on A_ε.

Proof: $f_n \to f$ in E pointwise.

For fix pair of k & n define

$$E_k^n = \{x \in E \mid |f_j(x) - f(x)| \tfrac{1}{n} \ \forall j > k\}$$

& $E_k^n \uparrow E$, we can

Fix n: $E_k^n \subseteq E_{k+1}^n$

find k_n S.T.

$$m(E \setminus E_{k_n}^n) < \frac{1}{2^n}$$

By construction of E_k^n

$$\{x \in E \mid |f_j(x) - f(x)| < \tfrac{1}{n}\} \ \forall x \in E_k^n, \ j > k_n$$

Choose N S.T. $\sum_{n=N}^{\infty} \frac{1}{2^n} < \varepsilon/2$ and

$$\hat{A}_2 = \bigcap_{n \geq N} E_{k_n}^n$$

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$$\boxed{\tilde{A}_\epsilon = \bigcap_{n \geq N} E^n_{k_n}}$$

$$m(E \setminus \hat{A}_\epsilon) \leq \sum_{n \geq N}^{\infty} m(E \setminus E^n_{k_n})$$

$$= \sum_{n \geq N}^{\infty} \frac{1}{2^n} < \epsilon/2$$

For $\delta > 0$ chose $n > N$ s.t. $\frac{1}{n} < \delta$.

$x \in A_\epsilon \Rightarrow x \in E^n_{k_n}$

$\Rightarrow |f_n(x) - f(x)| < \frac{1}{n} < \delta$

$\forall x \in \hat{A}_\epsilon$

$\Rightarrow f_n \longrightarrow f$ uniformly on $\hat{A}_\epsilon$

Find a closed set A_ϵ s.t. $m(\hat{A}_\epsilon \setminus A_\epsilon) < \epsilon/2$

$m(E \setminus A_\epsilon) \leq m(E \setminus \hat{A}_\epsilon) + m(\hat{A}_\epsilon \setminus A_\epsilon)$

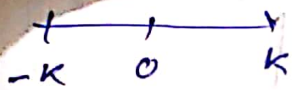
$\leq \epsilon/2 + \epsilon/2 = \epsilon$

Every non negative measurable function can be approximated by an increasing sequence of step f^ns | simple f^ns.

$\exists\ \phi_k \uparrow f$

$\downarrow$

sequence of simple f^n



$Q_K = [-K, K]$

$$F_K(x) = \begin{cases} f(x), & x \in Q_K, \ f(x) \le K \\ K & x \in Q_K \ \ f(x) > K \\ 0 & \text{o/w} \end{cases}$$

As $K \to \infty$ $F_K(x) \to f(x)$ (pointwise)

range of $F_K(x)$ is $[0, K]$

for fixed $K, j \geq 1$ define

$$E_{l,j} = \left\{ x \in Q_K \;\middle|\; \frac{l}{j} < F_K(x) \le \frac{l+1}{j} \right\}.$$

$$0 \le l < K.j$$

Define $F_{K,j}(x) = \sum_l \frac{l}{j} \chi_{E_{l,j}}(x)$

Note that $0 \le F_K(x) - F_{K,j}(x) \le \frac{1}{j}$

$\forall x$

$\boxed{\phi_K = F_{K,K}}$ $\Rightarrow |F_K(x) - \phi_K(x)| \le \frac{1}{K}$

as $K \to \infty$

$\phi_K \longrightarrow f.$

Lusin's Theorem

$$f^{*}(x) = f^{+}(x) - f^{-}(x)$$

$$f^{+}(x) = \max\{f(x), 0\}$$

$$f^{-}(x) = \max\{-f(x), 0\}$$

$$\{\phi_k^1\} \longrightarrow f^{+} \geq 0$$

$$\{\phi_k^2\} \longrightarrow f^{-} \geq 0$$

$$\phi_k^1 - \phi_k^2 \longrightarrow f$$

Suppose f is a measurable function in a finite measure set E. Then $\exists$ a closed set $F_\varepsilon \subset E$ s.t. $f|_{F_\varepsilon}$ is continuous & $m(E \setminus F_\varepsilon) < \varepsilon$ for any given $\varepsilon > 0$.

Proof: Let $\{f_n\}$ be a sequence of simple functions converging pointwise to f. We can find E_n such that $m(E_n) < \frac{1}{2^n}$ & f is continuous outside E_n.

By Egorov's Theorem we can find a closed set $A_{\varepsilon/3}$ s.t. $m(E \setminus A_{\varepsilon/3}) < \varepsilon/3$ & $f_n \to f$ uniformly on $A_{\varepsilon/3}$. Define

$$F' = A_{\varepsilon/3} \setminus \bigcup_{n \geq N} E_n \qquad \left(N \text{ being chosen s.t. } \sum_{n \geq N} \frac{1}{2^n} < \varepsilon/3 \right)$$

$f|_{F'}$ is continuous ($\because$ f_n (Cont.) $\longrightarrow f$ uniformly on F')

Find closed set F_ε s.t. $m(F \setminus F_\varepsilon) < \varepsilon/3$

$$m(E \setminus F_\varepsilon) \leq m(E \setminus A_{\varepsilon/3}) + m(A_{\varepsilon/3} \setminus F') + m(F' \setminus F_\varepsilon)$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

6th March

Lebesgue Integral

Simple functions

$$\phi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x)$$

Definition of Integral must be independent of choice of ϕ.

Canonical Representation of ϕ.

is such that a_k's are not 0, E_k's are disjoint $(a_k \neq 0)$

ϕ takes finitely many distinct non zero values $c_1, c_2, \ldots c_n$

$$E_k = \{x \mid \phi(x) = c_k\}$$

$$\phi(x) = \sum_{k=1}^{M} c_k \chi_{E_k}(x)$$

$\Rightarrow$ E_k's are disjoint.

$$\int_{\mathbb{R}} \phi \, dx = \sum_{k=1}^{M} c_k \, m(E_k)$$

$$\int_E \phi \, dx = \int_{\mathbb{R}} \phi \chi_E \, dx.$$

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pendent of choice of representation of ϕ
If $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}(x)$ is any representation of ϕ, then $\int \phi \, dx = \sum_{k=1}^{N} a_k \, m(E_k)$

② Linearity $\int (a\psi + b\phi) = a\int \psi + b\int \phi$ when ψ & ϕ are simple

③ Additivity, E & F are disjoint measurable set then

$$\int_{E \cup F} \phi = \int_E \phi + \int_F \phi.$$

④ Monotonicity
if $\phi \geq \psi$
then $\int \phi \geq \int \psi$.

⑤ Triangle Inequality

$$\left| \int \phi \, dx \right| \leq \int |\phi| \, dx$$

$$\phi = \sum a_k \chi_{E_k}$$

$$|\phi| = \sum |a_k| \chi_{E_k}$$

$$\left| \int \phi \, dx \right| = \left| \sum a_k m(E_k) \right|$$
$$\leq \sum |a_k| \, m(E_k) = \int |\phi| \, dA$$

$$\chi_{E \cup F} = \chi_E + \chi_F$$

E_k's are disjoint but a_k's are not distinct.

$$a_k = a$$

$$E_a' = \bigcup_{a_k = a} E_k$$

$$m(E_a') = \sum_{a_k = a} m(E_k)$$

$$\int \phi \, dx = \sum_{a_k = a} a \, m(E_a')$$

$$= \sum_{k=1}^{N} a_k \, m(E_k).$$

Suppose E_k's are not disjoint, (but a_k's are distinct.)

$$\bigcup_{k=1}^{N} E_k = \bigcup_{j=1}^{n} E_j^*$$

$$a^* = \sum_{E_j^* \subset E_k} a_k$$

$$\int \phi \, dx = \sum_{j=1}^{n} a^* m(E_j^*) = \sum \sum_{E_j^* \subset E_k} a_k \, m(E) \quad (*)$$

Tutorial

$f : [a, b] \longrightarrow \mathbb{R}$

Result: $\varepsilon > 0$, $\exists$ patition P of $[a, b]$ s.t.
$U(P, f) - L(P, f) < \varepsilon$

Result: f is R. integrable on $[a, b]$ iff
The set of discontinuous pts has measure zero. $E \subset [a, b]$, $m(E) = 0 \Rightarrow$ R integrable

① $f : [0, 1] \longrightarrow \mathbb{R}$

$$f(x) = \frac{1}{2^n}, \quad \frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}$$

$$E = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$$

$$m(E) = 0$$

$\Rightarrow f$ is R. integrable

$$\int_0^1 f \, dx = \sum_{n=1}^{\infty} \int_{1/2^{n+1}}^{1/2^n} \frac{1}{2^n} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^{2n}}$$

$$= \frac{1}{\cancel{8}2} \left(\frac{1}{1 - 1/4} \right)$$

$$\int_0^1 \frac{1}{6} dx = \frac{2}{3}$$

3 (1) $f(x) = \begin{cases} \sin\left(1/\sin x\right), & x \neq 0, \pi, 2\pi \\ 0 & x = 0, \pi, 2\pi \end{cases}$

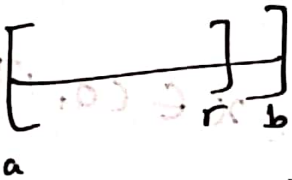
$f: [0, 2\pi]$

$f(x) = \begin{cases} x, & x \in [0,1] \cap \mathbb{Q} \\ x^2, & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$

$f: [0,1] \to \mathbb{R}$ $\quad (0,1)$.

$(0,1)$ - The f^n is discontinuous at each pt. $\therefore m(E) = 0$

(2) $f: [a,b] \longrightarrow \mathbb{R}$ is R-integrable on $[a,r]$ $\forall$ $[a, r < b$.



$\int_a^b f = \lim_{r \to b^-} \int_b^r f$

Proof: Let $\varepsilon > 0$ arbitrary $\quad |f| \leq M$ on $[a,b]$

$r = b - \frac{\varepsilon}{4M} > a$

f is R-integrable on $[a, r]$.

So $\exists$ a partition P of $[a, r]$ s.t.

$U(P,f) - L(P,f) < \varepsilon/2$

Now consider $Q = P \cup \{b\}$ of $[a,b]$

$\Rightarrow$ ~~U(P,f)-L(P,f)~~ $U(Q,f) - L(Q,f) < \varepsilon$

$\Rightarrow U(P,f) - L(P,f) + \left(\sup_{[r,b]} f - \inf_{[r,b]} f\right)(b - r)$

$\leq \cancel{U(P_1, P) - L(f_1}$

$\leq U(P, f) - L(P, f) + 2M(b-r)$

$\leq \varepsilon/2 + 2M \frac{\varepsilon}{4M} = \varepsilon$

$\left|\int_a^b f - \int_a^r f\right| = \left|\int_r^b f\right| \leq \int_r^b |f| \leq M(b-r)$

$= M \frac{\varepsilon}{4M} = \frac{\varepsilon}{4} < \varepsilon$

$\lim_{\varepsilon \to 0} \int_a^r f = \int_a^b f$

$\Rightarrow \lim_{r \to b^-} \int_a^r f = \int_a^b f$

$f : [0,1] \longrightarrow \mathbb{R}$

$f(x) = \begin{cases} \sin(1/x) & , x \in (0,1] \\ 0 & , x = 0. \end{cases}$

$\int_0^1 \sin\left(\frac{1}{x}\right) dx = \lim_{r \to 0^+} \int_r^1 \sin 1/x \, dx$

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$f : [a,b] \to \mathbb{R}$

F is continuous on $[a,b]$, diff. on (a,b).

$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$.

F is antiderivative of f.

Let $f : [a,b] \xrightarrow{int} \mathbb{R}$

$$F(x) = \int_a^x f(t)\,dt \text{ cont.}$$

f is continuous at $c \in [a,b]$

F is diff at c s.t. $F'(c) = f(c)$

(5) $\int_0^1 f(x)\,dx$

$$f(x) = \begin{cases} -\cos(1/x) + 2x\sin(1/x), & (0,1] \\ 0, & x = 0. \end{cases}$$

$$F(x) = \begin{cases} x^2\sin(1/x), & [0,1) \\ 0, & x = 0 \end{cases}$$

F is diff on $[0,1]$ but F' is not cont. at 0

$F(1) - F(0) = \sin 1$

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f is continuous on $[0,x]$

$f(x) \neq 0$, $\forall x > 0$

$[f(x)]^2 = 2\int_0^x f(t)\,dt$, P.T. $f(x) = x$

$$F(x) = \int_0^x f(t)\,dt$$

$\therefore$ F is diff for all $x > 0$

$2f'(x)f(x) = 2f(x)$ $\quad$ $\underline{f'(x) = 1}$ $\Rightarrow f(x) = x \;\forall x > 0$

$f(x) = 0$ or

$$\frac{F(x) - F(0)}{x} = F'(c)$$

for some $c \in (0, x)$

(8) $f : [a, b] \longrightarrow \mathbb{R}$ is cont.

$\int_a^b f = 0 \Rightarrow \exists\, c \in [a, b]$ s.t. $f(c) = 0$

$$F(x) = \int_a^x f(t)\,dt$$

$F(a) = 0$

$F(b) = 0$

$\exists\, c$ s.t. $F'(c) = 0$

$\Rightarrow f(c) = 0$ since f is cont.

14th March

Recall: Fatou's lemma, $f_n \geq 0$, $f_n \to f$

$$\int f \leq \liminf \int f_n$$

Problem: Suppose f is integrable i.e. $\int f < \infty$. Then prove that $f < \infty$ a.e.

Proof: Take $E_K = \{x \mid f(x) \geq k\}$. Then $E_K \searrow E = \bigcap_{k=1}^{\infty} E_K$

True if $\exists k$ s.t. $m(E_K) < \infty$.

$$\Rightarrow E_K \searrow E = \bigcap_{k=1}^{\infty} E_K = \{x \mid f(x) = \infty\}$$

$$\text{Now } \int f \geq \int f \chi_{E_K} \geq k\, m(E_K) \quad \forall k \geq 1$$

$$\Rightarrow m(E_K) = 0$$

$$m(E_K) \to m(E)$$

$$\Rightarrow \lim_{K\to\infty} m(E_K) = m(E) = 0$$

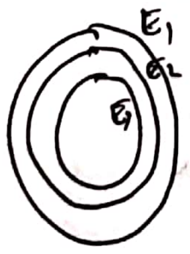
$$\lim_{K\to\infty} m(E_K) = 0$$

$$\Rightarrow f < \infty \text{ a.e.}$$

Prove: E_K is a decreasing sequence of measurable sets s.t. $E_K \searrow E$ then $m(E_K) \to m(E)$ provided $\exists$ K s.t. $m(E_K) < \infty$

$m(E_1) < \infty$

Proof:



$$E_1 = E \cup \left(\bigcup_{k=1}^{\infty} G_K\right)$$

where $G = E_K \setminus E_{K+1}$, $k \geq 1$

$$m(E_1) = m(E) + \lim_{n\to\infty} \left(\sum_{k=1}^{n} m(E_K) - m(E_{K+1})\right)$$

$$= m(E) + \lim_{n\to\infty} \left(m(E_1) - m(E_n)\right)$$

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$m(E_1) = m(E) + m(E_1) - \lim_{n\to\infty} (m(E_n))$

$\hookrightarrow m(E_1) < \infty$

$\Rightarrow \quad m(E) = \lim_{n\to\infty} (m(E_n))$

Consequence of Fatuou's Lemma

Monotone Convergence Theorem.

If $\{f_n\}$ be a sequence of measurable functions s.t. $f_n \nearrow f$, then

$$\int f_n \longrightarrow \int f.$$

$$f_{n+1}(x) \geq f_n(x) \quad \forall n = 1, 2, .$$

Proof $\quad f_n \leq f$

$$\Rightarrow \int f_n \leq \int f$$

$$\Rightarrow \limsup \int f_n \leq \int f \leq \liminf \int f_n \quad \text{—①}$$

We know

$$\liminf \int f_n \leq \limsup \int f_n \quad \text{—②}$$

$$① \,\&\, ② \Longrightarrow \limsup \int f_n = \liminf \int f_n = \int f.$$

If $\{f_n\}$ be a sequence of ~~monoto~~ measurable fn s.t. $f_n \searrow f$ then

$$\int f_n \longrightarrow \int f$$

$$f_{n+1} \leq f_n$$

$$g_n = f_1 - f_n$$

$$g_n \nearrow f_1 - f$$

$\lim \int f_1 - f = \int f_1 - f$

(Using the 1st version of monotone convergence theorem.)

$\Rightarrow \lim \int f = f$

Corollary: Suppose $\sum a_k(x)$, $a_k(x) \geq 0$ is a series of measurable functions

Then $\int \sum_{k=1}^{\infty} a_k(x) = \sum_{k=1}^{\infty} \int a_k(x)$

Moreover, if $\sum_{k=1}^{\infty} \int a_k(x) < \infty$

$\Rightarrow \sum_{k=1}^{\infty} a_k(x) < \infty$ a.e.

Proof.

$f_n = \sum_{k=1}^{n} a_k(x)$, since $a_k(x) \geq 0$ $f_n \uparrow$

$f = \sum_{k=1}^{\infty} a_k(x)$ $f_n \uparrow f$

Applying $\int f_n \longrightarrow \int f$

$\lim_{n \to \infty} \int \sum_{k=1}^{n} a_k(x) = \int \sum_{k=1}^{\infty} a_k(x)$

$\Rightarrow \lim_{n \to \infty} \sum_{k=1}^{n} \int a_k(x) = \int \sum_{k=1}^{\infty} a_k(x)$

$\Rightarrow \sum_{k=1}^{\infty} \int a_k(x) = \int \sum_{k=1}^{\infty} a_k(x)$.

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Borel - Cantelli Lemma

$\{E_k\}$ is a countable sequence of measurable sets s.t. $\sum_{k=1}^{\infty} m(E_k) < \infty$

Define $E = \limsup(E_n)$

$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n} E_k$$

$$= \{x \mid x \in E_k \text{ for infinitely many } k\text{'s}\}$$

Then $m(E) = 0$

Proof: $a_k(x) = \chi_{E_k}(x)$, $x \in E$

$\Leftrightarrow x \in E_k$ for infinitely many E_k's.

$$\Leftrightarrow \sum_{k=1}^{\infty} a_k(x) = \infty$$

$$\sum_{k=1}^{\infty} \int a_k(x) = \sum_{k=1}^{\infty} m(E_k) < \infty$$

$$\Rightarrow \sum_{k=1}^{\infty} a_k(x) < \infty \text{ a.e.}$$

$$\Rightarrow m(E) = 0$$

20th March

Integration of general functions

f is a real valued function. We say f is Lebesgue integrable if $|f|$ is Lebesgue integrable

$$f^+(x) = \max\{f(x), 0\}$$
$$f^-(x) = \min\{-f(x), 0\}$$

Then $f = f^+ - f^-$

$$f^+ \leq |f|$$
$$f^- \leq |f|$$

Define $\int f = \int f^+ - \int f^-$

$$f = f_1 - f_2$$
$$= g_1 - g_2$$
$$\Rightarrow f_1 + g_2 = g_2 + f_2$$

To show:

$$\int f_1 - \int f_2 = \int g_1 - g_2$$
$$\Rightarrow f_1 + g_2 = g_2 + f_2$$
$$\Rightarrow \int f_1 + g_2 = \int g_1 + f_2$$
$$\Rightarrow \int f_1 + \int g_2 = \int g_2 + \int f_2$$

Proposition: Let F be integrable on $\mathbb{R}^D$, Then

(i) $\exists$ a ball B of finite measure such that $\int_{B^c} f < \varepsilon$

(ii) $\exists\ \delta$ s.t. $\int_E f < \varepsilon$ whenever $m(E) < \delta$.

(Absolute continuity of Lebesgue Integral)

Proof: (i) Consider $B_N = \{|x| < N\}$, define $f_N(x) = f \cdot \chi_{B_N}$ $\boxed{f \geq 0}$

$$f_N \leq f_{N+1}, \quad f_N \longrightarrow f$$

$\int |f - f_N| < \varepsilon$ for large N.

$$\Rightarrow \int f - f\chi_{B_N} < \varepsilon$$

$$\Rightarrow \int f(1 - \chi_{B_N}) < \epsilon$$

$$\Rightarrow \int f \chi_{B_N^c} < \epsilon$$

$$\Rightarrow \int_{B_N^c} f < \epsilon$$

② Assume $f \geq 0$, $E_N = \{x \mid f(x) \leq N\}$

$f_N = f\chi_{E_N}$

$f_N \leq f_{N+1}$, By monotone convergence thm, $\exists$ large N S.T.

$$\int_{\mathbb{R}} |f - f_N| < \epsilon.$$

Pick a δ S.T. $N\delta < \epsilon$.

$m(E) < \delta$, then
$$\int_E f = \int f - f_N + \int_E f_N$$
$$\leq \epsilon + N m(E)$$
$$\leq \epsilon + N\delta$$
$$\leq \epsilon + \epsilon$$
$$\leq 2\epsilon.$$

Dominated Convergence Theorem (DCT)

Let $\{f_n\}$ be a sequence of measurable functions such that $f_n(x) \longrightarrow f(x)$ a.e.

Suppose $|f_n(x)| \leq g(x)$ & g is integrable

Then $\int |f_n - f| \longrightarrow 0$

$(\because \int f_n \to \int f)$

Proof.

$$E_N = \{x \mid |x| \leq N \;\&\; g(x) \leq N\}$$

Apply 1st part of the previous propⁿ to get a large N S.T. $\int_{E_N^c} g < \epsilon$.

Consider ~~fn~~ $f_n \chi_{E_N}$, easy to see that $|f_n \chi_{E_N}| < N$

Apply Bounded Convergence Theorem (BCT)

to get
$$\int_{E_N} |f - f_n| < \epsilon \quad \text{for large } n.$$

$$\int f_n - f = \int_{E_N} f_N - f + \int_{E_N^c} f_N - f$$

$$= \cancel{\int f_N - f}$$

$$\le \epsilon + \Big| \int_{E_N^c} \overset{\le g}{f_N} - \overset{\le g}{f} \Big|$$

$$\le \epsilon + 2 \Big| \int_{E_N^c} g \Big|$$

$$\le \epsilon + 2\epsilon$$

$$\le 3\epsilon.$$

$f = u + iv$. ?

$u \le |f|$

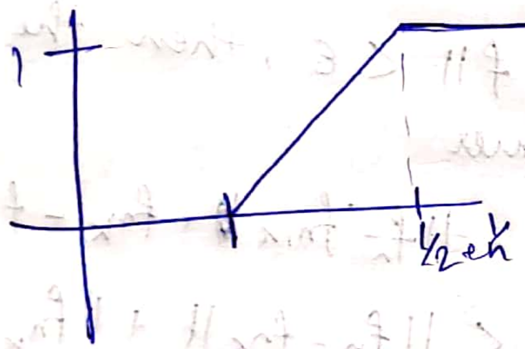
$v \le |f|$

$C[a,b]$, L^1-norm

Q: Whether $C[a,b]$ is complete

Complete: A metric space is said to be complete if every cauchy sequence is convergent in that metric space.

$$f_n(x) = \begin{cases} 0 & 0 \le x \le 1/2 \\ n(x - 1/2) & \text{if } 1/2 \le x \le 1/2 + 1/n \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} \le x \le 1. \end{cases}$$



Suppose there is a function $f \in C[0,1]$

S.T. $d(f_n, f) \longrightarrow 0$ or $\|f_n - f\|_1 \longrightarrow 0$

$$\int_0^{1/2} |f(x)| = \int_0^{1/2} |f - f_n| \le \int_0^1 |f_n - f| \longrightarrow 0$$

$$\Rightarrow \int_0^{1/2} |f(x)|\, dx$$

$$\Rightarrow f(x) = 0 \ \forall\ x \in [0, 1/2)$$

If $1/2 < x \le 1$, then choose n_0 S.T. $\frac{1}{2} + \frac{1}{n_0} \le x$, Then for all $n \ge n_0$.

$$\int_x^1 |1 - f(x)| dx = \int_x^1 |f_n(x) - f(x)|\, dx \le \int_0^1 |f_n(x) - f(x)|\, dx \to 0$$

$$\Rightarrow f(x) = 1 \ \forall\ x \in (1/2, 1]$$

$f(x) = 1 \quad \forall \ 1/2 < x \leq 1$

$f \notin C[0,1]$

$C[a,b]$ is not complete.

Riesz Fischer Theorem $L^1(\mathbb{R})$ is complete

Proof: Consider $(f_n) \longrightarrow$ a cauchy sequence

$\|f_n - f_m\| < \varepsilon$ whenever $m, n \geq N_0$

If we can constuct a subsequence (f_{n_k}) of (f_n) such that

$$\|f_{n_k} - f\| < \varepsilon,$$ then the

proof will be over

$$\|f_n - f\| = \|f_n - f_{n_k} + f_{n_k} - f\|$$

$$\leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\|$$

$$\leq \varepsilon + \varepsilon = 2\varepsilon$$

For each $\varepsilon = \frac{1}{2^k}$, choose $N(\varepsilon)$ s.t.

whenever $n_k \geq N(2^{-k})$, then

$$\|f_n - f_{n_k}\| < \frac{1}{2^k}.$$

Now

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})(x)$$

Let $g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$

$$\Rightarrow \quad f(x) \le g(x) \le \sum_{k=1}^{\infty} |f_{n_k}(x)|$$

$$\int |f(x)| \le \int |g(x)| \le \int |f_{n_1}| + \int \sum \int |f_{n_{k+1}}(x) - f_{n_k}(x)| \le \|f_{n_1}\| + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$$

$\Rightarrow$ By MCT f is integrable

$f < \infty$

$$f_{n_k} = f_{n_1}(x) + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j})$$

for each k

$$f - f_{n_k} \le g$$

Apply DCT $\|f_{n_k} - f\| \longrightarrow 0$

As $n_k \longrightarrow \infty$.