Tutorial 6 & 7 Discussion

Tut 6

Q 12. Find all automorphism of $\mathbb{R}$.

Ans Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be an automorphism.

$$\phi(1) = 1.$$

Let $n \in \mathbb{Z}$ then $\phi(n) = \phi(\underbrace{1+1+\cdots+1}_{n\text{-time}})$

$$= \phi(1) + \cdots + \phi(1)$$

$$= n.$$

$\therefore \quad \phi(n) = n \quad \forall n \in \mathbb{Z}.$

$1/m \in \mathbb{Q}.$

$$1 = \phi(1) = \phi\left(m \cdot \frac{1}{m}\right) = \underbrace{\phi(1/m) + \cdots + \phi(1/m)}_{m\text{-times}}$$

$$= m\,\phi(1/m).$$

$\Rightarrow \quad \phi(1/m) = 1/m.$

$\therefore \phi(n/m) = n\phi(1/m) = n/m.$

$\phi(x) = x \quad \forall x \in \mathbb{Q}.$

<u>WTS</u> ϕ is order preserving.

[i.e if $x \leq y \Rightarrow \phi(x) \leq \phi(y)$.]

Let $x \in \mathbb{R}$ s.t $x \geq 0$.

Then $x = (\sqrt{x})^2$

$\Rightarrow \phi(x) = (\phi(\sqrt{x}))^2 \geq 0.$

Therefore if $a, b \in \mathbb{R}$ and $a \leq b$.

then $b - a \geq 0 \Rightarrow \phi(b-a) \geq 0$

$\Rightarrow \phi(b) \geq \phi(a).$

$\therefore$ ϕ is order preserving map.

<u>WTS</u> ϕ is the identity map.

Suppose ϕ is not identity map.

i.e $z \in \mathbb{R}$ s.t $\phi(z) \neq z$.

<u>Case 1</u>. $\phi(z) < z$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$,

$\exists\ r \in \mathbb{Q}$ s.t $\phi(z) < r < z$.

Since $\phi(r) = r \Rightarrow r = \phi(r) < \phi(z)$

$\therefore\ \phi(z) < r < \phi(z)$.

which is a contradiction.

<u>case 2</u> $z < \phi(z)$.

Again applying the same argument we get a contradiction.

$\therefore\ \phi(x) = x \quad \forall x \in \mathbb{R}$.

Therefore ϕ is the identity map.

Tut 6

Q5. WTS $\mathbb{Q}[\sqrt{2}, \sqrt{3}] \subseteq \mathbb{Q}[\sqrt{2}+\sqrt{3}] = K$

$$(\sqrt{2}+\sqrt{3})^2 = 3+2+2\sqrt{6} \in K.$$

$$\Rightarrow \sqrt{6} \in K.$$

Tut-7

Q5. Maximal ideal of $\mathbb{Z}[x]$.

$$I = (f(x), p).$$

$$\mathbb{Z}[x]/(x) \cong \mathbb{Z}.$$

$$\mathbb{Z}[x]/(p,x) \cong \mathbb{Z}_p.$$

$$\mathbb{Z}[x]/(p, f(x)) \cong \frac{\mathbb{Z}/p\mathbb{Z}[x]}{(\bar{f}(x))}.$$

The maximal ideals of $\mathbb{Z}[x]$ are of the form $(p, f(x))$ where p is a prime number and $f(x)$ is a poly in $\mathbb{Z}[x]$ which irreducible modulo p.

$$I = (p, x).$$

Consider $I = (2, x^2+3)$.

$$\frac{\mathbb{Z}[x]}{I} = \frac{\mathbb{Z}[x]}{(2, x^2+3)} \cong \frac{\mathbb{Z}[x]/(2)}{(2, x^2+3)/(2)}$$

$$\cong \frac{\mathbb{Z}/2\mathbb{Z}[x]}{(x^2+\bar{1})}$$

$x^2+3 \in \mathbb{Z}[x]$

$\hookrightarrow$ image of of this poly in $\mathbb{Z}/2\mathbb{Z}[x]$

$(2, x^2+3) \ni h(x).$

$$h(x) = 2.h_1(x) + \underbrace{(x^2+3)h_2(x)}.$$

$(2, x^2+3)/(2).$

$$\downarrow$$

(x^2+3)

$\underline{x^2+1+2}$

Let M be a maximal ideal of $\mathbb{Z}[x]$.

Assume $\boxed{M \cap \mathbb{Z} \neq 0.}$

$\mathbb{Z}/M\cap\mathbb{Z}$ injects into $\mathbb{Z}[x]/M$.

$\mathbb{Z}[x]/M$ is a field.

therefore $\mathbb{Z}[x]/M$ is an int domain so $\mathbb{Z}/M\cap\mathbb{Z}$ is also int domain

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook] \arrow[dr, bend left] & \mathbb{Z}[x] \arrow[d] \\ & \mathbb{Z}[x]/M \end{tikzcd}$$

so $M \cap \mathbb{Z}$ is a prime ideal $\mathbb{Z}$.

so $M \cap \mathbb{Z} = (p)$. where p is a prime no.

$$\mathbb{Z}[x] \xrightarrow{\varphi} \mathbb{Z}/p\mathbb{Z}[x]$$ is surj ring hom.

$$a_0 + a_1 x + \cdots + a_n x^n \longmapsto \bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_n x^n$$

Let $M' = \varphi(M)$

Note that $\mathbb{Z}[x]/M \cong \dfrac{\mathbb{Z}/p\mathbb{Z}[x]}{M'}$.

Since $\dfrac{\mathbb{Z}/p\mathbb{Z}[x]}{M'}$ is a field

$\therefore M' = (f_0(x))$ where $f_0(x) \in \mathbb{Z}/p\mathbb{Z}[x]$ is irreducible.

$\therefore M = (p, f(x))$ s.t $f \equiv f_0 \pmod{p}$