Group Action

Lecture 13

Propn Let G be a group acting on a set A. For each $s \in A$ the number of elts in $O(s)$ is the index of the stabilizer $[G : G_s]$.

Pf: Define $\phi : G/G_s \longrightarrow O(s)$

G/G_s = Set of all left cosets of G_s.

$$\phi(gG_s) = gs$$

WTS ϕ is well defined

Let $g_1 G_s = g_2 G_s$. WTS $\phi(g_1 G_s) = \phi(g_2 G_s)$

$g_2^{-1} g_1 \in G_s$

$\Rightarrow g_2^{-1} g_1 = g$ for some $g \in G_s$.

$\Rightarrow g_1 = g_2 g \Rightarrow g_1 s = g_2 g s = g_2 s$.

$\therefore\ \Phi$ is well defined.

Clearly Φ is surjective.

<u>WTS</u> Φ is injective.

Let $\Phi(g_1 G_s) = \Phi(g_2 G_s)$

$\Rightarrow g_1 s = g_2 s$

$\Rightarrow g_2^{-1} g_1 s = s$.

$\Rightarrow g_2^{-1} g_1 \in G_s$.

$\Rightarrow g_1 G_s = g_2 G_s$.

$$\boxed{\begin{array}{c} G \times A \to A \\ (g_2^{-1}, g_1 s) \\ \downarrow \\ g_2^{-1}(g_1 s) \\ \parallel \\ (g_2^{-1} g_1) s \end{array}}$$

Hence Φ is injective.

Thus $|O(s)| = [G : G_s]$.

Let G be a gp and A be a set.
G is acting on the set A

$$G \times A \xrightarrow{\psi} A$$
$$(g, s) \longmapsto gs$$

This gp action can be viewed in a different way also. For each fixed $g \in G$ we get a map σ_g defined by

$$\sigma_g : A \longrightarrow A \text{ by}$$
$$\sigma_g(a) = ga = \psi(g, a)$$

elt corresponding to the gp action

and it has the following properties:

① For each fixed $g \in G$, σ_g is a bijection of the set A i.e. σ_g is a permutation of A.

$$G = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R} \right\},$$

$$A = \mathbb{R}^2$$

$$g = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \in G$$

$$\sigma_g : A \longrightarrow A$$

$$\sigma_g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\sigma_g \begin{pmatrix} x \\ y \end{pmatrix} = g \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

g = rotation by $\pi/2$

$\sigma_g \rightsquigarrow$ rotate all the elts of $\mathbb{R}^2$ by $\pi/2$ angle.

$\sigma_g : A \longrightarrow A$

$$\sigma_g(a) = ga$$

<u>WTS</u> σ_g is 1-1.

Let $\sigma_g(a_1) = \sigma_g(a_2)$ <u>WTS</u> $a_1 = a_2$.

$$ga_1 = ga_2$$

$$\Rightarrow g^{-1}(ga_1) = g^{-1}(ga_2)$$

$$\Rightarrow (g^{-1}g)(a_1) = (g^{-1}g)(a_2)$$

$$\Rightarrow a_1 = a_2.$$

Thus σ_g is injective.

σ_g is surjective as $\sigma_g(g^{-1}a) = a$.

$\therefore \sigma_g$ is a bijection i.e a permutation of A.

(2) The map $\phi : G \to S_A$

$$\phi(g) = \sigma_g$$

is a group homomorphism (Ex.)

<u>WTS</u> $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$

<u>WTS</u> $\sigma_{g_1 g_2} = \sigma_{g_1} \sigma_{g_2}$

The above gp homomorphism ϕ is called the permutation representation associated to a gp action.

<u>Group acting on themselves by left multiplication:</u>

$$G \times G \longrightarrow G$$
$$(g, a) \longmapsto ga$$

Here G is a gp and G is acting on itself by left multiplication.

Thm (Cauchy): Let G be a finite group and p be a prime number such that $p \mid |G|$. Then G has an elt of order p i.e G has a subgp of order p.

Pf: Consider the set

$$A = \left\{ (x_1, x_2, \ldots, x_p) \in G \times \cdots \times G \;\middle|\; x_1 x_2 \cdots x_p = 1 \right\}$$

Let $|G| = n$ then $|A| = n^{p-1}$.

Since $p \mid |G|$ $\therefore$ $p \mid |A|$.

Let $\sigma = (1\ 2\ 3 \cdots p) \in S_p$.

and $H = \langle \sigma \rangle \subseteq S_p$, $|H| = p$.

Define a gp action

$$H \times A \longrightarrow A$$

$$\left(\sigma^{i}, (x_1, x_2, \ldots, x_p)\right) \longmapsto \left(x_{\sigma^i(1)}, \ldots, x_{\sigma^i(p)}\right)$$

Rough work

eg $\sigma = (1\,2\,3)$, $\sigma^2 = (1\,3\,2)$

$$\left(\sigma, (x_1, x_2, x_3)\right) \longmapsto \left(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right)$$

$$= (x_2, x_3, x_1).$$

$$\left(\sigma^2, (x_1, x_2, x_3)\right) \longmapsto (x_3, x_1, x_2).$$

Note $|O(s)| = [H : H_s]$

$$= \frac{|H|}{|H_s|}$$

$\therefore$ $|O(s)|$ is either 1 or p as $|H| = p$.

Suppose that $|O(s)| = 1$.

i.e $O((x_1, x_2, \dots, x_p)) = \{(x_1, \dots, x_p)\}$.

$\quad\quad\quad\quad\; \shortparallel$

Then $(x_{\sigma(1)}, x_{\sigma(2)}, \dots x_{\sigma(p)}) = (x_1, \dots, x_p)$

$\Rightarrow \quad x_1 = x_2 = \dots = x_p = x$ (say)

and $\quad x^p = 1$.

Note $\quad A = O(s_1) \sqcup \dots \sqcup O(s_k)$

$p \mid n^{p-1} = |A| = [H : H_s] + \dots + [H : H_s]$.

Observe $O((1, 1, \dots, 1)) = \{(1, 1, \dots, 1)\}$

$\{(1, 1, \dots, 1)\}$ is one elt s.t $|O(1, \dots, 1)| = 1$

So there must be at least $p-1$ elts.

in A s.t the cardinality of orbit is 1.

$\exists$ at least $p-1$ many elts s.t

$x^p = 1$.