

Linear Algebra

Lecture 20



V : finite dimensional vector space over \mathbb{F} .

let $T: V \rightarrow V$ be a linear map.

B : ordered basis for V .

$[T]_B$: matrix representation of T w.r.t. B .

Eigenvalues & eigenvectors of T .

$$T(v_i) = \lambda_i v_i \quad \text{for some } \lambda_i \in \mathbb{F}$$

and $v_i \neq 0$.

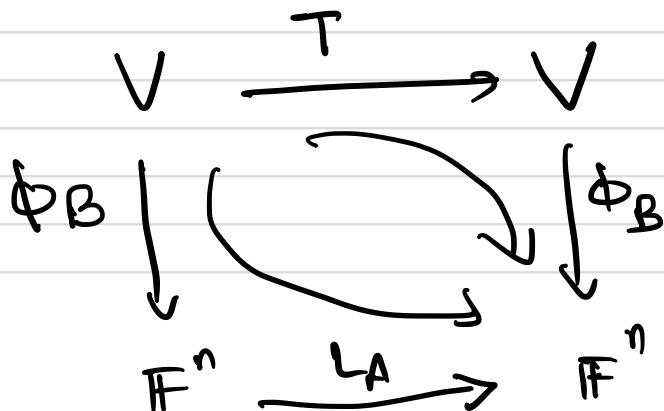
Suppose $\{v_1, \dots, v_n\}$ forms a basis.

Q: Does there exist an ordered basis for V such that (T)

$[T]_{\Gamma}$: a diagonal matrix.

$$[T]_{\Gamma} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \text{where}$$

$\Gamma = \{v_1, \dots, v_n\}$
ordered basis
for V .

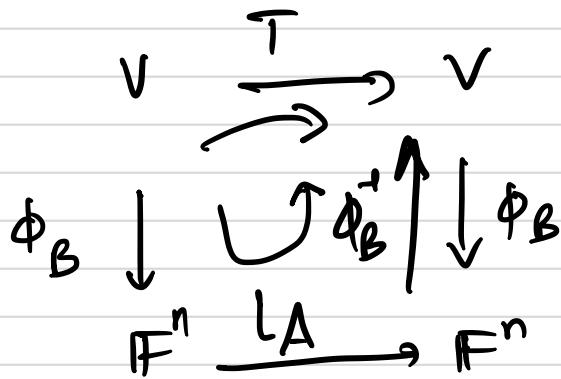


$$\begin{aligned} v &\in V \\ &\downarrow \\ \Phi(v) &= [v]_B \end{aligned}$$

v is an eigenvector of T



$\phi_B(v)$ is an eigenvector of $[T]_B$.



ϕ_B is an isomorphism
(one-one, onto, linear $f \in \mathbb{F}$)

Suppose T is diagonalizable.

$\Rightarrow \exists$ an ordered basis $B = \{v_1, \dots, v_n\}$ for V .

where $T(v_i) = \lambda_i v_i \quad i=1, 2, \dots, n$
for some $\lambda_i \in \mathbb{F}$

$$[T]_B = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

Consider the matrix $Q = \begin{bmatrix} | & | & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix} \in M_{n \times n}(\mathbb{F})$

$$Q^T A Q = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \lambda_n \end{bmatrix}$$

Ex:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Ex: } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \rightarrow \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem: Let T be a linear transformation on a vector space V and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T . If v_1, v_2, \dots, v_k are eigenvectors corresponding to $\lambda_1, \dots, \lambda_k$ resp., then $\{v_1, \dots, v_k\}$ is linearly independent.

Proof: By induction:

Let $n=1$; λ_1, v_1 , clearly $\{v_1\}$ is linearly independent.

Assume $\{v_1, \dots, v_{k+1}\}$ is a linearly ind. set.

To prove: $\{v_1, \dots, v_{k-1}, v_k\}$ is linearly independent.

Let $a_1, \dots, a_n \in \mathbb{F}$ s.t.

$$a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0 \quad \rightarrow (*)$$

Apply $(T - \lambda_k I)$ on both sides,

$$(T - \lambda_k I)(a_1 v_1) + \dots + (T - \lambda_k I)(a_{k-1} v_{k-1})$$

$$+ (T - \lambda_k I)(\cancel{a_k v_k}) = 0$$

$$\Rightarrow a_1 (\lambda_1 - \lambda_k) v_1 + a_2 (\lambda_2 - \lambda_k) v_2 + \dots$$

$$\dots + a_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1} = 0$$

$\therefore \{v_1, \dots, v_{k-1}\}$ are independent

$$a_i (\lambda_i - \lambda_k) = 0 \quad \text{for } i=1, 2, \dots, k-1$$

$$\Rightarrow a_i = 0 \quad \text{for } i=1, 2, \dots, k-1$$

since λ_i 's are distinct.

$$\Rightarrow a_k v_k = 0 \Rightarrow a_k = 0$$

$\Rightarrow \{v_1, \dots, v_{k-1}\}$ is linearly indep.

Corollary: Let T be a linear operator on an n -dimensional vector space V . If T has n distinct eigenvalues, then T is diagonalizable.

[Diagonalizability and factorization §]
[characteristic polynomial.]

Definition: A polynomial $f(t) \in P(\mathbb{F})$ splits over \mathbb{F} if there are scalars $c, \alpha_1, \alpha_2, \dots, \alpha_n$ (not necessarily distinct) in \mathbb{F} s.t.

$$f(t) = c(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_n)$$

e.g. $t^2 - 1 = (t - 1)(t + 1)$

$t^2 + 1$ does not split over \mathbb{R}

If $(t - i)(t + i)$ over \mathbb{C} .

Theorem: The characteristic polynomial of any diagonalizable linear operator splits.

$$(t^2+1) (t^2-1)$$

Definition: Let λ be an eigenvalue of a linear operator with characteristic polynomial $f(t)$. The algebraic multiplicity of λ is the largest positive integer k s.t. $(t-\lambda)^k$ is a factor of $f(t)$.

$$(t-\lambda)^k \mid f(t)$$

Ex. 1 $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \longleftrightarrow$

Ex. 2 $\begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}$

Definition: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T . Define $E_\lambda = \{x \in V : T(x) = \lambda x\}$

$$= N(T - \lambda I)$$

The set E_λ is called the eigenspace of T corresponding to eigenvalue λ .

Thm: Let T be a linear operator on a finite dimensional vector space V and let λ be an eigenvalue of T with algebraic multiplicity m . Then

$$1 \leq \dim(E_\lambda) \leq m$$

Proof: Choose an ordered basis $\{v_1, v_2, \dots, v_p\}$ of E_λ .

Extend this basis to the ordered basis $B = \{v_1, \dots, v_p, v_{p+1}, \dots, v_n\}$ of V .

Denote by $A = [T]_B$

How does this A look like?!

$$\begin{aligned} T(v_1) &= \cancel{\lambda_1} v_1 \\ &= \cancel{\lambda_1} v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n \\ &\vdots \end{aligned}$$

$$\begin{aligned} T(v_p) &= \cancel{\lambda_p} v_p \\ &= 0 \cdot v_1 + \cdots + \cancel{\lambda_p} v_p + 0 \cdot v_{p+1} + \cdots + 0 \cdot v_n \end{aligned}$$

$$A = \left[\begin{array}{c|c} \cancel{\lambda_1} & 0 \\ \vdots & \cancel{\lambda_p} \\ 0 & 0 \end{array} \mid \begin{array}{c} B \\ \hline - \\ C \end{array} \right]$$

The characteristic polynomial of T

$$f(t) = \det(A - tI_n)$$

$$= \det \left[\begin{array}{c|c} (\lambda - t) I_p & B \\ \hline 0 & C - t I_{n-p} \end{array} \right]$$

$$= \det[(\lambda - t) I_p] \times \det[C - t I_{n-p}]$$

$$f(t) = (\lambda - t)^p g(t)$$

$$\Rightarrow 1 \leq p \leq m$$



$\dim(E_x) =$ dimension of eigenspace corresponding to the eigenvalue λ .
= geometric multiplicity of λ .

lemma: let T be a linear operator, and let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T . For each i , let $v_i \in E_{\lambda_i}$. If $v_1 + v_2 + \dots + v_k = 0$ then $v_i = 0$ for all i .

Proof: trivial.

Theorem: Let T be a linear operator on a vector space V . Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T . For each $i = 1, \dots, k$ let S_i be a finite linearly independent subset of E_{λ_i} . Then $S = S_1 \cup S_2 \cup \dots \cup S_k$ is linearly independent in V .

Proof: Suppose for each i

$$S_i = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$$

$$S = \{v_{ij} : j=1, \dots, n_i, i=1, \dots, k\}$$

Consider the scalars $\{a_{ij}\} \subseteq F$

$$\sum_{i=1}^k \sum_{j=1}^{n_i} a_{ij} v_{ij} = 0$$

For each i :

$$w_i = \sum_{j=1}^{n_i} a_{ij} v_{ij} \in E_{\lambda_i}$$

$$\Rightarrow w_1 + w_2 + \dots + w_k = 0$$



Theorem :

Theorem: Let T be a linear operator on a finite dimensional vector space V . such that the characteristic polynomial splits. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T . Then

- (a) T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$ for every $i=1,2,\dots,k$.
- (b) If T is diagonalizable and B_i is an ordered basis for E_{λ_i} , then $B = B_1 \cup B_2 \cup \dots \cup B_k$ is an ordered basis for V consisting of eigenvectors of T .