Linear Algebra

Lecture 19

Diagonalization.

Let V be a finite dimensional vector space. and let $T: V \longrightarrow V$ be linear.

1. Does there exist an ordered basis B of V s.t. $[T]_B$ = matrix representation of T w.r.t. B is a diagonal matrix??
2. If yes, how to compute/find B??

<u>Definition</u>: A linear operator T on a finite-dimensional vector space V is called diagonalizable if there exists an ordered basis B of V s.t. $[T]_B$ = matrix representation of T w.r.t. B is a diagonal matrix. A square matrix A is called diagonalizable if the linear transformation, L_A, corresponding to A is diagonalizable.

Let V be a finite dimensional vector space and $B = \{v_1, \cdots, v_n\}$ be an ordered basis of V. Let $T : V \to V$ be a linear transformation.

$[T]_B$ is a diagonal matrix. $[T]_B = D$

$$T(v_j) = \sum_{i=1}^{n} D_{ij} v_i = D_{jj} v_j$$
$$= \lambda_j v_j$$

where $\lambda_j = D_{jj}$.

$$T(v_j) = \lambda_j v_j = D_{jj} v_j$$

$$[T]_B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

__Defn__: Let T be a linear operator on a vector space V. A non-zero vector $v \in V$ is called an eigenvector of T if $\exists$ a scalar (from $\mathbb{F}$) λ such that $T(v) = \lambda v$.
λ is called the corresponding eigenvalue.

Let $A \in M_{n \times n}(\mathbb{F})$. A non-zero vector $v \in \mathbb{F}^n$ is called an eigenvector of A if v is an eigenvector of the linear transformation L_A induced by the matrix A.

In other words, $Av = \lambda v$ for some some $\lambda \in \mathbb{F}$.

Example:

Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$; $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$L_A(v_1) = Av_1 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2v_1$$

$$Av_2 = 5v_2$$

$$B = \{v_1, v_2\}$$

Example: $V = \mathbb{R}^2$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

T rotates every vector of $\mathbb{R}^2$ in anticlockwise direction by an angle $\pi/2$.

No eigenvectors $\rightarrow$ no eigenvalues $\rightarrow$ no diagonalizability of T.

Example:

Let $C^\infty(\mathbb{R})$ denote the vector space of functions over $\mathbb{R}$ which are infinitely many times continuously differentiable.

$T(f) = f'$

Let f be a nonzero function in $C^\infty(\mathbb{R})$.

For f to be an eigenvector of T,

$$T(f) = \lambda f \quad \text{for some scalar } \lambda.$$

$$\begin{array}{c} T(f) \\ \parallel \\ f' \end{array}$$

$f' = \lambda f$

$\Rightarrow f = ce^{\lambda x}$ for some $c \in \mathbb{R}$

$e^{\lambda x} \in C^{\infty}(\mathbb{R})$ is an eigenvector of T and λ is corresponding eigenvalue. ∎

Theorem: Let $A \in M_{n\times n}(\mathbb{F})$. Then a scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.

Proof: A scalar λ is an eigenvalue of A if and only if there exists $v \neq 0$ s.t.

$$Av = \lambda v$$

$\Leftrightarrow (A - \lambda I_n) v = 0$

$\Leftrightarrow A - \lambda I_n$ is not invertible. $(v \neq 0)$

$\Leftrightarrow \det(A - \lambda I_n) = 0$ ∎

Defn: Let $A \in M_{n\times n}(\mathbb{F})$. Then the polynomial $f(t) = \det(A - t I_n)$ is called the characteristic polynomial of A.

Definition: Let T be a linear operator on an n-dimensional space V with an ordered basis β. We define the characteristic polynomial $f(t)$ of T to be the characteristic polynomial of $A = [T]_\beta$.

Example:

$T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$

$T(f(x)) = f(x) + (x+1)f'(x)$

T is linear.

$B = \{1, x, x^2\}$

$T(1) = 1$

$T(x) = 2x + 1$

$T(x^2) = 3x^2 + 2x$

$$[T]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Characteristic eq^n of T = Characteristic eq^n of $[T]_B$ = $\det([T]_b - tI_n) = 0$

$\Rightarrow \lambda = 1, 2, 3$ are the eigenvalues of $[T]_B$ or T.

Eigenvectors of T.

$\det\left[[T]_B - \lambda I_3\right] = 0$

$\left[[T]_B - \lambda I_3\right]$ is not invertible.

For $\lambda = 1$

$\left([T]_B - I_3\right)$ is not invertible.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\} = N\left([T]_B - I_3\right)$$

Let V be a finite dimensional vector space.

$T : V \longrightarrow V$ be linear.

B : ordered basis.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_B"'] & V \arrow[d, "\phi_B"] \\ \mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^n \end{tikzcd}$$

$A = [T]_B$: matrix representation of T w.r.t. the ordered basis B.

For $v \in V$, $\phi_B(v) = [v]_B$

Take $v \in V$ be an eigenvector of T corresponding to the eigenvalue λ.

$$T(v) = \lambda v$$

Note that $\phi_B(v) = [v]_B$

To show: $\phi_B(v)$ is an eigenvector of A.

Note:

$\phi_B(v) \in \mathbb{F}^n$.

$$A\,\Phi_B(v) = L_A\,\phi_B(v) = \phi_B T(v)$$
$$= \phi_B(\lambda v)$$
$$= \lambda\,\Phi_B(v)$$

$\Rightarrow A\,\phi_B(v) = \lambda\,\phi_B(v)$

$\Rightarrow \phi_B(v)$ is an eigenvector of A.

Converse is obviously true

$\blacksquare$

Example:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Corresponding to $\lambda = 1$,

$$A - \lambda I_3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$