DIFFERENTIAL EQUATIONS:

An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Examples:

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 = e^t \quad \text{— (i)}$$

$$y(y^2+1)dx + x(y^2-1)\,dy \quad \text{— (ii)}$$

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2 \quad \text{— (iii)}$$

Mathematical classifications:

ORDINARY DIFF. EQUATIONS: → Involves derivatives w.r.t single independent variables

PARTIAL DIFF. EQUATIONS: → Involves partial derivatives (more than one independent variables

ORDER OF A DIFFERENTIAL EQUATION: → The order of the highest order derivative involved

DEGREE OF A DIFFERENTIAL EQUATION: → The degree of the highest order derivative involved.

i) ODE, order - 4 degree 1

ii) ODE, order - 1 degree 1

iii) PDE, order - 3 degree 2.

LINEAR AND NONLINEAR DIFFERENTIAL EQUATION:

A differential equation is called linear if

i) every dependent variable and every derivative occur in the first degree only, and

ii) no products of dependent variables and/or derivatives occur.

If not linear than it is called nonlinear.

Note: Every linear equation is of first degree, but every first degree equation may not be linear.

$$\frac{d^2y}{dx^2} + y \cdot \frac{dy}{dx} + y = 0 \qquad \text{1st degree but nonlinear}$$

SOLUTION OF A DIFFERENTIAL EQUATION:

Any relation between the dependent and independent variables which satisfies the differential equation is called a solution or integral of the differential equation.

Ex. $y = \frac{A}{x} + B$ is a solution of

$$y'' + \left(\frac{2}{x}\right)y' = 0$$

Check! $y' = -\frac{A}{x^2} \Rightarrow y'' = \frac{2A}{x^3}$

Subst. in the equation: $\underline{0 = 0}$.

Note: It should be noted that a solution of a differential equation does not involve the derivatives of the dep. variable w.r.t the indep. variable or variables.

Family of curves: An n-parameter family of curves is a set of relations of the form

$$\{(x,y) : f(x,y,c_1,c_2,\cdots c_n) = 0\}$$

Example:

i) set of concentric circles

$x^2+y^2 = c$ $\rightarrow$ one parameter family if c takes non-negative real values

ii) Set of circles:

$(x-c_1)^2 + (y-c_2)^2 = c_3$ $\rightarrow$ three parameters family if c_1, c_2 takes all real values and c_3 takes all non-negative real values.

Note: Solution of a differential equation is a family of curves.

Formation of differential equations from a given n-parameters family of curves:

From a given family of curves containg n arbitrary constants, we can obtain an nth order differential equation whose solution is the given family:

- Differentiate the given equation n times to get n additional equations containing those arbitrary constants.
- Eliminate n arbitrary constants from the $(n+1)$ equations.
- obtain a differential equation of the nth order.

Ex: Obtain the differential equation satisfied by

$$xy = ae^{x} + be^{-x} + x^2$$

where a & b are arbitoary constants.

Sol: Given family of curves:

$$xy = ae^{x} + be^{-x} + x^2 \quad \text{—}(1)$$

Differentiating w.r.t x, we get

$$xy' + y = ae^{x} - be^{-x} + 2x$$

Differentiating again:

$$xy'' + 2y' = ae^{x} + be^{-x} + 2$$

Using (1) we get

$$xy'' + 2y' = xy - x^2 + 2$$

which is the desired differential equation.

Remark: Observe that the number of arbitoary constants in a solution of a differential equation depends upon the order of the differential equation. It is evident from the above example that a general solution (defined later) of an nth order differential equation will contain n arbitoary constants.

General, particular, and Singular Solution

Let $F(x, y, y', y'', \cdots y^{(n)}) = 0$ be an nth order ordinary differential equation.

i) General solution: solution containing n-independent arbitrary constants.

ii) Particular solution: solution by giving particular values to one or more of the n-independent constants.

iii) Singular solution: cannot be obtained by any choice of independent arbitrary constant.

Example: a) $y = (x+c)^2$ is the general solution of

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0 \quad -①$$

b) $y = x^2$ is a particular solution of ① $(c=0)$

c) $y = 0$ is a singular solution.

Ex: Consider $yy' - x(y')^2 = 1$

General solution: $y = cx + \frac{1}{c}$

Particular solution: $y = x + 1 \qquad (c=1)$

Singular solution: $y^2 = 4x$.

Explicit & Implicit solutions:

Explicit : $y = y(x)$

Implicit : $F(x, y) = 0$

Example: $y'' + k^2 y = 0$

solution: $y = C_1 \cos kx + C_2 \sin kx$

↳ explicit solution

Example: $x + 3yy' = 0$

Solution: $x^2 + 3y^2 = C$

↳ Implicit solution

Equation of first order and first degree :

We shall consider two standard forms of differential equations

i) $\dfrac{dy}{dx} = f(x, y)$

ii) $M(x, y)\, dx + N(x, y)\, dy = 0$.

Solution methods:

- **Separation of variables:** If a differential equation can be written in the form

$$f_1(y)\,\frac{dy}{dx} = f_2(x) \qquad \text{— ①}$$

then we say variables are separable in the given differential equation.

Solution of ①:

$$\int f_1(y)\,dy = \int f_2(x)\,dx + C \quad \text{(how?)}$$

Example: $\frac{dy}{dx} = e^{x-2y} + x^2 e^{-2y}$

$$\Rightarrow e^{2y}\frac{dy}{dx} = e^x + x^2$$

Integrating both side:

$$\frac{e^{2y}}{2} = e^x + \frac{x^3}{3} + c_1$$

or $\boxed{e^{2y} = 2e^x + \frac{2}{3}x^3 + C}$

Equation reducible to separation of variables:

Consider $\frac{dy}{dx} = f(ax+by+c)$ —①

OR $\frac{dy}{dx} = f(ax+by)$

Subst. $ax+by+c = v$ OR $ax+by = v$

$$\Rightarrow a + b\cdot\frac{dy}{dx} = \frac{dv}{dx}$$

Then (1) reduces to

$$\frac{1}{b}\left[\frac{dv}{dx} - a\right] = f(v)$$

$$\Rightarrow \frac{dv}{dx} = bf(v) + a$$

$$\Rightarrow \int \frac{dv}{bf(v)+a} = \int dx$$

Example: $\frac{dy}{dx} = \sec(x+y)$

Sol: Let $x+y = v \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Then the diff. eq. becomes:

$$\frac{dv}{dx} = \sec v + 1 \qquad \text{(separable form)}$$

$$= \frac{1+\cos v}{\cos v} = \frac{2\cos^2\frac{v}{2}}{2\cos^2\frac{v}{2} - 1}$$

$$\Rightarrow \int \left[1 - \frac{1}{2}\sec^2\left(\frac{v}{2}\right)\right] dv = \int dx$$

$$\Rightarrow v - \tan\left(\frac{v}{2}\right) = x + C$$

Subst. $v = x+y$: $\boxed{y - \tan\left(\frac{x+y}{2}\right) = c}$

Homogeneous equations: A differential equation of first order and first degree is said to be homog. if it is of the form or can be put in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{—} (1)$$

<u>Solution:</u> $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$(1) \Rightarrow v + x\frac{dv}{dx} = f(v)$$

$$\Rightarrow x\frac{dv}{dx} = f(v) - v \quad (\text{separable form})$$

$$\Rightarrow \int \frac{dv}{f(v)-v} = \int \frac{dx}{x} + C.$$

Example: $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$

<u>Sol:</u> $$\frac{dy}{dx} = -\frac{x^3+3xy^2}{y^3+3x^2y} = -\frac{1+3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3 + 3\left(\frac{y}{x}\right)}$$

Subst. $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{1+3v^2}{v^3+3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^4 - 6v^2 - 1}{v^3+3v}$$

$$\Rightarrow -\int \frac{4(v^3+3v)}{v^4+6v^2+1}\cdot dv = \int 4\cdot\frac{dx}{x}$$

$$\Rightarrow -\ln(v^4+6v^2+1) = 4\ln x + \ln C \quad (x>0)$$

$$\Rightarrow x^4 c\,(v^4 + 6v^2 + 1) = 0$$

$$\Rightarrow c x^4\left(\frac{y^4}{x^4} + 6\frac{y^2}{x^2} + 1\right) = 0$$

$$\Rightarrow \boxed{c\,(y^4 + 6y^2x^2 + x^4) = 0}$$

Equation reducible to homogeneous form:

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad , \text{ where } \frac{a}{a'} \neq \frac{b}{b'} \ (*) \qquad —① $$

Take $\left.\begin{array}{l} x = X+h \\ y = Y+k \end{array}\right\} \quad —②$

where X & Y are new variables and h & k are constants to be so chosen that the resulting equation in X and Y becomes homogeneous

② $\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dY}{dX}$

$$\left[\begin{array}{l} y(x) = Y(X) + k \\ \text{or } Y(X) = y(x) + k \\ \dfrac{dY}{dX} = \dfrac{d}{dX}\left[y(x)+k\right] = \dfrac{dy}{dx} \cdot \underbrace{\dfrac{dx}{dX}}_{=1} \end{array}\right]$$

① $\Rightarrow$

$$\frac{dY}{dX} = \frac{aX+bY+ah+bk+c}{a'X+b'Y+a'h+b'k+c'} \qquad —③$$

In order to make ③ homog. choose h and k such that

$$\left.\begin{array}{l} ah+bk+c = 0 \quad \text{and} \\ a'h+b'k+c' = 0 \end{array}\right\} \text{(alway possible because } ab'-a'b \neq 0\text{)}$$

Getting h & k we have $X = x-h$ & $Y = y-k$

$$\Rightarrow \frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} = \frac{a+b\left(\frac{Y}{X}\right)}{a'+b'\left(\frac{Y}{X}\right)} \qquad \text{homogeneous in } X \,\&\, Y$$

(*) In case $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{1}{\lambda} \Rightarrow a' = \lambda a$ & $b' = \lambda b$

Subst. $\dfrac{dy}{dx} = \dfrac{ax+by+c}{\lambda(ax+by)+c'} = f(ax+by)$ (Can be solved by subst $ax+by = u$)

Ex: $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$ — ①

Sol. Take $x = X+h$ & $y = Y+K$ so that $\dfrac{dy}{dx} = \dfrac{dY}{dX}$.

$$\frac{dY}{dX} = \frac{X+2Y+(h+2K-3)}{2X+Y+(2h+K-3)} \quad \text{— ②}$$

Choose h, K so that $\left.\begin{array}{l} h+2K-3=0 \\ 2h+K-3=0 \end{array}\right\} \Rightarrow h=1 \ \& \ K=1.$

So from ① $X = x-1$ $Y = y-1$

② $\Rightarrow \dfrac{dY}{dX} = \dfrac{X+2Y}{2X+Y}$

Take $Y = vX \Rightarrow \dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$$X\frac{dv}{dX} = \left(\frac{1+2v}{2+v}\right) - v = \frac{1-v^2}{2+v}$$

$$\Rightarrow \frac{dX}{X} = \left[\frac{1}{2}\left(\frac{1}{1+v}\right) + \frac{3}{2}\left(\frac{1}{1-v}\right)\right] dv$$

Integrating:

$$\Rightarrow \ln X + \ln C = \frac{1}{2}\left[\ln(1+v) - 3\ln(1-v)\right]$$

$$\Rightarrow 2\ln(XC) = \ln\left(\frac{1+v}{(1-v)^3}\right) \Rightarrow X^2C^2 = \frac{1+v}{(1-v)^3}$$

sub: $v = \dfrac{y-1}{x-1}$

$$\boxed{C^2(x-y)^3 = x+y-2}$$

Exact Differential Equations

If M and N are functions of x and y, the equation $M dx + N dy = 0$ is called exact when there exists a function $f(x,y)$ such that

$$d(f(x,y)) = M dx + N dy$$

or

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M dx + N dy$$

Theorem: The necessary and sufficient condition for the differential equation

$$M dx + N dy = 0$$

to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad — ①$$

<u>Proof</u>: The condition is necessary '⇒'

Let the equation be exact, then

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M dx + N dy$$

Equating coefficients of dx & dy, we get:

$$M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y}$$

Assuming f to be continuous upto 2nd order partial derivatives, we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the equation is exact then M & N satisfy ①.

Now we show that the condition ① is sufficient.

We assume the $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and show that the equation

$M\,dx + N\,dy$ is exact.

That means we find a function $f(x,y)$ such that

$$df = M\,dx + N\,dy.$$

Let $g(x,y) = \int M\,dx$ be the partial integral of M such that

$$\frac{\partial g}{\partial x} = M.$$

We first prove that $\left(N - \frac{\partial g}{\partial y}\right)$ is a function of y only.

Consider $\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y}$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x} \qquad \left(\text{assuming } \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Now consider:

$$f = g(x,y) + \int \left(N - \frac{\partial g}{\partial y}\right) dy \quad \text{and then}$$

$$df = dg + d\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right) = \frac{\partial g}{\partial x}\cdot dx + \frac{\partial g}{\partial y}\,dy + \overbrace{\frac{\partial}{\partial x}\left(\underbrace{\int\left(N - \frac{\partial g}{\partial y}\right) dy}_{\text{function of } y}\right) dx}^{=0}$$

$$+ \frac{\partial}{\partial y}\left(\int\left(N - \frac{\partial g}{\partial y}\right) dy\right) dy$$

$$= \frac{\partial g}{\partial x}\,dx + \cancel{\frac{\partial g}{\partial y}\,dy} + N\,dy - \cancel{\frac{\partial g}{\partial y}\,dy}$$

$$= M\,dx + N\,dy.$$

$\Rightarrow$ The given differential equation is exact.

Remark: The solution of an exact differential equation $Mdx + Ndy = 0$ can be written as

$$f = c$$

i.e.,

$$\underset{(y\text{ const.})}{\int M\,dx} + \int \underbrace{\left(N - \frac{\partial g}{\partial y}\right)}_{\text{function of } y \text{ alone}} dy = c$$

OR

$$\underset{(y\text{ const})}{\int M\,dx} + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$$

Example: Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Sol: $M = x^2 - 4xy - 2y^2 \qquad N = y^2 - 4xy - 2x^2$

$$\frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x} \implies \text{the equation is exact.}$$

Hence, there exists a function $f(x,y)$ such that

$$d(f(x,y)) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M\,dx + N\,dy$$

$$\Rightarrow \frac{\partial f}{\partial x} = x^2 - 4xy - 2y^2 \quad \& \quad \frac{\partial f}{\partial y} = y^2 - 4xy - 2x^2$$

Int. of 1st w.r.t. x

$$\Rightarrow f = \frac{x^3}{3} - 2x^2y - 2xy^2 + c_1(y)$$

On differentiation w.r.t y:

from above.

$$\frac{\partial f}{\partial y} = -2x^2 - 4xy + c_1'(y) = y^2 - 4xy - 2x^2$$

$$\Rightarrow c_1'(y) = y^2 \Rightarrow c_1(y) = \frac{y^3}{3} + c_2$$

Hence: $f = c_3 \Rightarrow \dfrac{x^3}{3} - 2x^2y - 2xy^2 + \dfrac{y^3}{3} + c_2 = c_3$

$$\Rightarrow \boxed{x^3 - 6xy(x+y) + y^3 = c}$$

Example: Show that the differential equation

$$(3xy+y^2)\,dx + (x^2+xy)\,dy = 0$$

is not exact and hence it cannot be solved by the method discussed above.

Sol: Check: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$3x+2y \neq 2x+y$$

So the given equation is not exact.

However, if we proceed with the method given above, we get

$$\frac{\partial f}{\partial x} = 3xy+y^2 \qquad \frac{\partial f}{\partial y} = x^2+xy$$

$$\Rightarrow \quad f = \frac{3}{2}x^2y + y^2x + f_1(y)$$

$$\frac{\partial f}{\partial y} = \frac{3}{2}x^2 + 2yx + f_1'(y) = x^2+xy$$

$$\Rightarrow \quad f_1'(y) = \underbrace{-\frac{x^2}{2} - xy}_{\text{depends on } x \,\&\, y.}$$

(Not possible to solve)

Thus, there is no $f(x,y)$ exists and hence it can not be solved in this way.

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If M and N are functions of x and y, the equation $M\,dx + N\,dy = 0$ is called exact when there exists a function $f(x,y)$ such that

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Theorem: The necessary and sufficient condition for the differential equation

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to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{— ①}$$

<u>Proof</u>: The condition is necessary '⇒'

Let the equation be exact, then

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M\,dx + N\,dy$$

Equating coefficients of dx & dy, we get:

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That means we find a function $f(x,y)$ such that

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We first prove that $\left(N - \frac{\partial g}{\partial y}\right)$ is a function of y only.

Consider $\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y}$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x} \qquad \left(\text{assuming } \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Now consider:

$$f = g(x,y) + \int \left(N - \frac{\partial g}{\partial y}\right) dy. \text{ and then}$$

$$df = dg + d\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right) = \frac{\partial g}{\partial x}\cdot dx + \frac{\partial g}{\partial y}\,dy + \overbrace{\frac{\partial}{\partial x}\left(\underbrace{\int\left(N - \frac{\partial g}{\partial y}\right) dy}_{\text{function of } y}\right) dx}^{=0}$$

$$+ \frac{\partial}{\partial y}\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right) dy$$

$$= \frac{\partial g}{\partial x}\,dx + \cancel{\frac{\partial g}{\partial y}\,dy} + N\,dy - \cancel{\frac{\partial g}{\partial y}\,dy}$$

$$= M\,dx + N\,dy.$$

$\Rightarrow$ The given differential equation is exact.

Remark: The solution of an exact differential equation $Mdx+Ndy=0$ can be written as

$$f=c$$

i.e.,

$$\int_{(y\text{ const.})} M\,dx + \int \underbrace{\left(N-\frac{\partial g}{\partial y}\right)}_{\text{function of } y \text{ alone}} dy = c$$

OR

$$\int_{(y\text{ const})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c.$$

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Sol: $M=x^2-4xy-2y^2 \qquad N=y^2-4xy-2x^2$

$$\frac{\partial M}{\partial y} = -4x-4y = \frac{\partial N}{\partial x} \Rightarrow \text{the equation is exact.}$$

Hence, there exists a function $f(x,y)$ such that

$$d(f(x,y)) = \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy = M\,dx+N\,dy$$

$$\Rightarrow \frac{\partial f}{\partial x} = x^2-4xy-2y^2 \quad \& \quad \frac{\partial f}{\partial y} = y^2-4xy-2x^2$$

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$$\Rightarrow f = \frac{x^3}{3}-2x^2y-2xy^2+c_1(y)$$

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$$\frac{\partial f}{\partial y} = -2x^2-4xy+c_1'(y) = y^2-4xy-2x^2$$

$$\Rightarrow c_1'(y)=y^2 \Rightarrow c_1(y)=\frac{y^3}{3}+c_2$$

Hence: $f=c_3 \Rightarrow \dfrac{x^3}{3}-2x^2y-2xy^2+\dfrac{y^3}{3}+c_2 = c_3$

$$\Rightarrow \boxed{x^3-6xy(x+y)+y^3=c}$$

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Sol: Check: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$3x+2y \neq 2x+y$$

So the given equation is not exact.

However, if we proceed with the method given above, we get

$$\frac{\partial f}{\partial x} = 3xy+y^2 \qquad \frac{\partial f}{\partial y} = x^2+xy$$

$$\Rightarrow \quad f = \frac{3}{2}x^2y + y^2x + f_1(y)$$

$$\frac{\partial f}{\partial y} = \frac{3}{2}x^2 + 2y\,x + f_1'(y) = x^2+xy$$

$$\Rightarrow \quad f_1'(y) = \underbrace{-\frac{x^2}{2} - xy}_{\text{depends on } x \,\&\, y.}$$

depends on x & y. (Not possible to solve)

Thus, there is no $f(x,y)$ exists and hence it can not be solved in this way.

Exact Differential Equations: Integrating Factors

If an equation of the form $M\,dx + N\,dy = 0$ is not exact, it is sometimes possible to choose a function of x & y such that after multiplying all terms of the equation, it becomes exact. Such a multiplier is called an integrating factor. That is, if $I(x,y)$ is an integrating factor then the differential equation

$$I(x,y)\,M(x,y)\,dx + I(x,y)\,N(x,y)\,dy = 0$$

becomes exact.

Note: Although an equation of the form $M\,dx + N\,dy = 0$ always has integrating factor(s), there is not general rule of finding them. We now discuss some methods of finding integrating factors.

Rule I: By inspection

This method is based on recognition of some standard exact differentials that occur frequently in practice.

i) $d(xy) = y\,dx + x\,dy$

ii) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$ or $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

iii) $d\left(\ln \frac{y}{x}\right) = \frac{x\,dy - y\,dx}{xy}$ or $d\left(\ln \frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy}$

iv) $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$ or $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2+x^2}$

v) $d(\ln xy) = \frac{y\,dx + x\,dy}{xy}$

Ex. Solve the differential equation

$$y(y^2+1)\,dx + x(y^2-1)\,dy = 0$$

(Check! it is not exact D.E.)

Sol: Rewriting:

$$y^2(y\,dx + x\,dy) + y\,dx - x\,dy = 0$$

Divinding it by y^2: (I.F.)

$$y\,dx + x\,dy + \frac{y\,dx - x\,dy}{y^2} = 0$$

$$\Rightarrow d(xy) + d\left(\frac{x}{y}\right) = 0$$

$$\Rightarrow xy + \frac{x}{y} = c \quad \Rightarrow \boxed{xy^2 + x = cy}$$

Ex. Solve $(y^2e^x + 2xy)\,dx - x^2\,dy = 0$

(Check! it is not exact D.E.)

We know that

$$d\left(\frac{x^2}{y}\right) = \frac{2x}{y}\,dx - \frac{x^2}{y^2}\,dy$$

Diving the given equation by y^2, we get:

$$\left(e^x + \frac{2x}{y}\right)dx - \frac{x^2}{y^2}\,dy = 0$$

$$\Rightarrow d(e^x) + d\left(\frac{x^2}{y}\right) = 0$$

$$\Rightarrow \boxed{e^x + \frac{x^2}{y} = c}$$

Rule II: $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$.

In this case $I(x,y) = \dfrac{1}{Mx+Ny}$ is an integrating factor.

Example: $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ — (1)

Sol.

$$M = x^2y - 2xy^2 \qquad N = -(x^3 - 3x^2y)$$

$$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2$$

$$= x^2y^2 \neq 0.$$

$$I.F. = \frac{1}{x^2y^2}$$

Multiplying (1) by I.F.

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad — (2)$$

Now equation (2) is an exact differential equation (check!)

If u is the exact differential of (2) then:

$$\frac{\partial u}{\partial x} = \frac{1}{y} - \frac{2}{x} \Rightarrow u = \frac{x}{y} - 2\ln x + \varphi(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{x}{y^2} + \varphi'(y) = -\frac{x}{y^2} + \frac{3}{y} \Rightarrow \varphi'(y) = \frac{3}{y}$$

$$\Rightarrow \varphi(y) = 3\ln y + c_1$$

$$\Rightarrow \boxed{\frac{x}{y} - 2\ln x + 3\ln y + c_1 = c_2}$$

or $$\boxed{\frac{x}{y} - 2\ln x + 3\ln y = c}$$

Rule III: $Mdx+Ndy=0$ is of the form $f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0$ then $\dfrac{1}{Mx-Ny}$ is an integrating factor provided $Mx-Ny\neq 0$

Ex: Solve

$$(xy\sin xy+\cos xy)\,y\,dx+(xy\sin xy-\cos xy)\,x\,dy=0$$

$$M=(xy\sin xy+\cos xy)\,y \qquad N=(xy\sin xy-\cos xy)\,x$$

$$Mx-Ny=(xy\sin xy+\cos xy)\,xy-(xy\sin xy-\cos xy)\,xy$$

$$=2xy\cos xy\neq 0$$

$$IF=\frac{1}{2xy\cos xy}$$

Multiplying the given equation by I.F.

$$\frac{1}{2}\left(y\tan xy+\frac{1}{x}\right)dx+\frac{1}{2}\left(x\tan xy-\frac{1}{y}\right)dy=0$$

it must be exact (check!)

Solution: $\boxed{\dfrac{x}{y}\sec xy=c}$

Rule IV: An integrating factor for an equation of the form

$$x^a y^b (my\,dx + nx\,dy) + x^r y^s (py\,dx + qx\,dy) = 0$$

is $x^h y^k$ where h & k can be obtained by applying the condition that after multiplication by $x^h y^k$ the equation must become exact. Here a, b, m, n, r, s, p, q are constants.

Example: Solve $(3x + 2y^2)y\,dx + 2x(2x + 3y^2)\,dy = 0$

This equation can be rewritten as

$$x(3y\,dx + 4x\,dy) + y^2(2y\,dx + 6x\,dy) = 0$$

multiplying the integrating factor $x^h y^k$, we get

$$(3x^{h+1}y^{k+1} + 2x^h y^{k+3})\,dx + (4x^{h+2}y^k + 6x^{h+1}y^{k+2})\,dy = 0$$

If it is exact we must have

$$3(k+1)x^{h+1}y^k + 2(k+3)x^h y^{k+2} = 4(h+2)x^{h+1}y^k + 6(h+1)x^h y^{k+2}$$

This is satisfied if

$$3(k+1) = 4(h+2) \quad \&$$
$$2(k+3) = 6(h+1)$$

Solving these we get $h = 1,\ k = 3$.

Integrating factor is xy^3.

Solution: $\boxed{x^3y^4 + x^2y^6 = c}$

Rule V : Most general approach :

The idea is to multiply the given differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

by a function $I(x,y)$ and then try to choose $I(x,y)$ so that the resulting equation

$$I(x,y)\,M(x,y)\,dx + I(x,y)\,N(x,y)\,dy = 0 \quad \text{— ①}$$

becomes exact.

The above equation is exact if and only if

$$\frac{\partial (IM)}{\partial y} = \frac{\partial (IN)}{\partial x} \quad \text{— } (*)$$

If a function I satisfying $(*)$ can be found then the given equation ① will be exact. However solving $(*)$ is very difficult so we consider some special cases.

i) An integrating factor I that is either as function of x alone or
ii) a function of y alone.

In the case i), the equation $(*)$ reduces to

$$IM_y = IN_x + NI_x \Rightarrow I_x = \frac{IM_y - IN_x}{N}$$

If $\dfrac{M_y - N_x}{N}$ is a function of x only, say $f(x)$ then

$I(x) = e^{\int f(x)\,dx}$ is an integrating factor. (by solving $\frac{dI}{I} = f(x)\,dx$)

In the case ii) If $\frac{1}{M}(N_x - M_y)$ is a function of y alone, say $f(y)$

then $I(y) = e^{\int f(y)\,dy}$ is an I.F.

Example: Solve $(x^2+y^2+x)\,dx + xy\,dy = 0$ —①

$$M = x^2+y^2+x \qquad N = xy$$

$$\frac{\partial M}{\partial y} = 2y \quad \& \quad \frac{\partial N}{\partial x} = y$$

$$\therefore \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y-y) = \frac{1}{x}$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = x.$$

Multiplying ① by x:

$$(x^3+xy^2+x^2)\,dx + x^2y\,dy = 0 \quad \text{This must be an exact D.E.}$$

Solution: $(3x^4 + 6x^2y^2 + 4x^3) = c$

Ex: Solve $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

$$M = 2xy^4e^y + 2xy^3 + y \qquad N = x^2y^4e^y - x^2y^2 - 3x.$$

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -8xy^3e^y - 8xy^2 - 4$$

$$= -4(2xy^3e^y + 2xy^2 + 1)$$

$$= -\frac{4}{y}\cdot(2xy^4e^y + 2xy^3 + y) = -\frac{4}{y}\cdot M$$

$$\Rightarrow \text{I.F.} = e^{\int -\frac{4}{y}dy} = y^{-4}$$

Solution: $\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$

Exact Differential Equations (Summary)

Necessary and sufficient condition of $M(x,y)dx + N(x,y)dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Integrating Factors

Rule I: By Inspection

Example:

$$d(xy) = ydx + xdy, \quad d(\ln xy) = \frac{ydx + xdy}{xy} \quad etc.$$

Rule II: $Mdx + Ndy = 0$ **is homogeneous and** $Mx + Ny \neq 0$ **then** $I(x,y) = \dfrac{1}{(Mx + Ny)}$ **is an integrating factor**

Rule III: $Mdx + Ndy = 0$ **is of the form** $f_1(xy)ydx + f_2(xy)xdy = 0$ **and** $Mx - Ny \neq 0$ **then** $\dfrac{1}{(Mx - Ny)}$ **is an integrating factor**

Rule IV: $Mdx + Ndy = 0$ **is of the form** $x^a y^b(mydx + nxdy) + x^r y^s(pydx + qxdy) = 0$ **then** $I(x,y) = x^h y^k$ **may be taken as an integrating factor, where** h, k **are obtained so that the differential equation after multiplication by** $I(x,y)$ **becomes exact**

Rule V: Most general approach

If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is function of x alone say $f(x)$, then $I(x) = e^{\int f(x)dx}$ is an I.F.

If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is function of y alone say $f(y)$, then $I(y) = e^{\int f(y)dy}$ is an I.F.

Linear Differential Equation:

A first order differential equation is called linear if it can be written in the form:

$$\frac{dy}{dx} + P(x)\,y = Q(x) \qquad (\text{linear in } y)$$

Rewritten as

$$\underbrace{dy + Py\,dx}_{\text{Compare with } M dx + N dy \text{ to get}} = Q(x)\,dx \qquad \text{— ①}$$

$$M = Py \qquad N = 1$$

Observe that $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{1}(P-0) = P$ (function of x alone)

Hence I.F. $= e^{\int P dx}$

multiplying ① by $e^{\int P dx}$

$$e^{\int P dx}\,dy + Py\,e^{\int P dx}\,dx = Q(x)\,e^{\int P dx}\cdot dx$$

$$\Rightarrow \quad d\left(e^{\int P dx}\cdot y\right) = Q\,e^{\int P dx}\,dx$$

Integrating:

$$e^{\int P dx}\cdot y = \int Q\,e^{\int P dx}\,dx + C$$

OR

$$\boxed{y.\,IF = \int Q.\,I.F.\,dx + C}$$

Note: Sometimes a differential equation cannot be put in the form $\frac{dy}{dx} + P(x)\,y = Q(x)$ which is linear in y, but in the form

$$\frac{dx}{dy} + P_1(y)\,x = Q_1(y)$$

which is linear in x, then

$$I.F. = e^{\int P_1 dy}$$

and the solution

$$x \cdot I.F. = \int Q_1\, I.F.\, dy + c$$

Ex. Solve $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\,y = \frac{4x^2}{1+x^2} \quad (\text{linear in } y)$$

$$I.F. = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$$

Solution: $y \cdot I.F = \int Q \cdot I.F\, dx + C \Rightarrow y\cdot(1+x^2) = \int 4x^2\,dx + c$

$$\Rightarrow \boxed{y(1+x^2) = \frac{4}{3}x^3 + c}$$

Ex. Solve $(x+2y^3)\frac{dy}{dx} = y \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2$

$$I.F. = e^{\int -\frac{1}{y}dy} = e^{-\ln y} = \frac{1}{y}$$

$$\Rightarrow x\cdot\frac{1}{y} = \int 2y^2 \cdot \frac{1}{y}\,dy + c$$

$$= \int 2y\,dy + c$$

$$\Rightarrow \boxed{\frac{x}{y} = y^2 + c}$$

Equation reducible to linear form:

An equation of the form

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \qquad \text{— (1)}$$

Putting $f(y) = v \Rightarrow f'(y)\frac{dy}{dx} = \frac{dv}{dx}$

Equation (1) reduces to:

$$\frac{dv}{dx} + Pv = Q \qquad (\text{linear in } v)$$

A special case: Bernoulli's Equation

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \qquad \text{— (2)}$$

where P & Q are constants or function of x and n is a constant except 0 & 1 is called Bernoulli's differential equation.

Note that equation (2) can be written as

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$$

Subst: $\frac{1}{y^{n-1}} = v \Rightarrow (1-n)\,y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{1}{(1-n)}\cdot\frac{dv}{dx} + Pv = Q$$

$$\Rightarrow \frac{dv}{dx} + P(1-n)v = Q(1-n) \qquad (\text{linear in } v)$$

Ex. $(x^2-2x+2y^2)\,dx + 2xy\,dy = 0$

Sol: Rewriting:

$$2xy\frac{dy}{dx} + x^2 - 2x + 2y^2 = 0$$

or $$2y\frac{dy}{dx} + \frac{2y^2}{x} = \frac{2x-x^2}{x}$$

Subst. $y^2 = v \Rightarrow 2y\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dv}{dx} + \frac{2}{x}\cdot v = (2-x) \qquad \text{(linear in } v)$$

$$\text{I.F.} = e^{\int \frac{2}{x}\cdot dx} = x^2$$

$$\Rightarrow v\cdot x^2 = \int (2-x)\,x^2\,dx + C$$

$$\Rightarrow \boxed{y^2x^2 = \frac{2}{3}x^3 - \frac{x^4}{4} + C}$$

Ex: $\frac{dy}{dx} - y\tan x = -y^2\sec x$

Sol: Dividing by y^2:

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\cdot\tan x = -\sec x$$

putting $\frac{1}{y} = v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow -\frac{dv}{dx} - v\tan x = -\sec x \Rightarrow \frac{dv}{dx} + v\tan x = \sec x. \qquad \text{(linear in } v)$$

$$\text{I.F.} = e^{\int \tan x\,dx} = e^{\ln \sec x} = \sec x$$

Solution: $$v\cdot\sec x = \int \sec^2 x\,dx + C$$

$$\Rightarrow v\cdot\sec x = \tan x + C$$

$$\Rightarrow \boxed{y^{-1}\sec x = \tan x + C}$$

Ex. Solve $\frac{dz}{dx} + \frac{z}{x}\ln z = \frac{z}{x}(\ln z)^2$, $x>0, z>0$

Sol: Dividing by z:

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\ln z = \frac{1}{x}(\ln z)^2$$

Subst. $\ln z = t \Rightarrow \frac{1}{z}\cdot\frac{dz}{dt} = \frac{dt}{dx}$

$$\Rightarrow \frac{dt}{dx} + \frac{1}{x}t = \frac{1}{x}t^2$$

$$\Rightarrow \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{t}\cdot\frac{1}{x} = \frac{1}{x}$$ Bernoulli's equation

Subst. $\frac{1}{t} = v \Rightarrow -\frac{1}{t^2}\frac{dt}{dx} = \frac{dv}{dx}$

$$\Rightarrow -\frac{dv}{dx} + \frac{1}{x}\cdot v = \frac{1}{x}$$

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

Solution: $v\cdot\frac{1}{x} = -\int \frac{1}{x}\cdot\frac{1}{x}dx + C$

$$\frac{v}{x} = \frac{1}{x} + C$$

$$\Rightarrow v = 1 + Cx$$

$$\Rightarrow \boxed{(\ln z)^{-1} = 1 + Cx}$$

C.F.

$f(D)y = 0$

Auxiliary equation $\quad f(m) = 0$

roots $\alpha_1, \alpha_2, \dots \alpha_n$.

Case I: Roots are real and non-repeated

$$C.F. = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + \dots + C_n e^{\alpha_n x}$$

Case II: Roots are real but repeated, say.

$\alpha_1 = \alpha_2 = \alpha$; $\alpha_3, \alpha_4 \dots \alpha_n$

$$C.F. = (C_1 + C_2 x)e^{\alpha x} + C_3 e^{\alpha_3 x} + \dots + C_n e^{\alpha_n x}$$

Case III: Roots are complex and non-repeated

$\alpha \pm i\beta, \alpha_3, \alpha_4 \dots \alpha_n$

$$C.F. = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) + C_3 e^{\alpha_3 x} + \dots + C_n e^{\alpha_n x}$$

Case IV: Roots are complex and repeated

$\alpha \pm i\beta, \alpha \pm i\beta, \alpha_5, \alpha_6, \dots \alpha_n$

$$C.F. = e^{\alpha x}\left((C_1 + C_2 x)\cos \beta x + (C_3 + C_4 x)\sin \beta x\right) + C_5 e^{\alpha_5 x} + \dots + C_n e^{\alpha_n x}.$$

Evaluation of C.F.:

Ex. Solve the differential equation

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$$

In operator form:

$$(D^4 - 2D^3 + 5D^2 - 8D + 4)\,y = 0$$

Auxiliary equation: $m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$

Its roots: $m = 1, 1, 2i, -2i$

The general solution:

$$y = (C_1 + C_2 x)e^x + C_3 \cos 2x + C_4 \sin 2x.$$

Ex. Suppose roots of the auxiliary eq. are

$$1, 2, 2, 1 \pm 2i, 1 \pm 2i$$

GENERAL SOLUTION:

$$y = C_1 e^x + (C_2 + C_3 x)e^{2x} + e^x\left[(C_4 + C_5 x)\cos 2x + (C_6 + C_7 x)\sin 2x\right]$$

Determination of particular integral:

Diff. Eq. $f(D)\,y = X$

$$\boxed{P.I. = \frac{1}{f(D)}.X}$$

1. General method of getting P.I.

$$\boxed{\frac{1}{(D-\alpha)}X = e^{\alpha x}\int X e^{-\alpha x}\,dx}$$

Proof: Let $y = \dfrac{1}{D-\alpha}X$

On operating $D-\alpha$ both sides, we get

$$(D-\alpha)\,y = X$$

$$\Rightarrow \frac{dy}{dx} - \alpha y = X \qquad \text{(linear equation in } y\text{)}$$

$$I.F. = e^{\int -\alpha\,dx} = e^{-\alpha x}$$

$$\Rightarrow y \cdot e^{-\alpha x} = \int X e^{-\alpha x}\,dx + c$$

$$\Rightarrow \boxed{y = e^{\alpha x}\int X e^{-\alpha x}\,dx} + c\,e^{\alpha x}$$

Ex. Solve $(D^2+a^2)y = \sec ax$

Auxiliary equation: $m^2+a^2=0 \Rightarrow m = \pm ai$

$$C.F. = C_1 \cos ax + C_2 \sin ax$$

$$P.I. = \frac{1}{D^2+a^2} \sec ax = \frac{1}{(D-ia)(D+ia)} \cdot \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax$$

Consider $\frac{1}{D-ia} \sec ax = e^{iax} \int \sec ax \, e^{-iax} dx$

$$= e^{iax} \int \sec ax \left[\cos ax - i \sin ax\right] dx$$

$$= e^{iax} \int \left(1 - i \frac{\sin ax}{\cos ax}\right) dx$$

$$= e^{iax}\left[x + \frac{i}{a} \ln|(\cos ax)|\right]$$

Similarly $\frac{1}{D+ia} \sec ax = e^{-iax}\left[x - \frac{i}{a} \ln|(\cos ax)|\right]$

Hence,

$$P.I. = \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a} \ln(\cos ax)\right\} - e^{-iax}\left\{x - \frac{i}{a} \ln(\cos ax)\right\}\right]$$

$$= \frac{1}{2ia}\left[x\left(e^{iax} - e^{-iax}\right) + \frac{i}{a} \ln|(\cos ax)| \left\{e^{iax} + e^{-iax}\right\}\right]$$

$$= \frac{x}{a} \cdot \sin ax + \frac{1}{a^2} \ln|(\cos ax)| \cdot \cos ax.$$

GENERAL SOLUTION:

$$y = C_1 \cos ax + C_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a^2} \ln|(\cos ax)| \cos ax$$

2. Short Methods for finding P.I. (Proofs: Shanti Narayan)

- X is of the form $\boxed{e^{ax}}$

i) $$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$ where $f(a) \neq 0$

ii) If $f(a) = 0$, then $f(D)$ must have a factor of the type $(D-a)^r$.

Then, $$\boxed{\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{\lfloor r} e^{ax}}$$

Ex. $$\text{P.I.} = \frac{1}{D^3 - D^2 - D + 1} \cdot e^x$$

$$= \frac{1}{(D-1)^2 (D+1)} e^x$$

$$= \frac{1}{(D-1)^2} \frac{1}{2} e^x$$

$$= \frac{1}{2} \cdot \frac{x^2}{\lfloor 2} \cdot e^x = \frac{1}{4} x^2 e^x.$$

Ex. $$\text{P.I.} = \frac{1}{D^2 + D + 5} \cdot 7$$

$$= \frac{1}{D^2 + D + 5} \cdot 7 e^{0x}$$

$$= 7 \cdot \frac{1}{D^2 + D + 5} \cdot e^{0x} = \frac{7}{5}.$$

Proof of:

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad \text{where } f(a) \neq 0$$

Let $f(D) = D^n + C_1 D^{n-1} + \cdots + C_{n-1} D + C_n$

Consider

$$f(D) e^{ax} = \left[D^n + C_1 D^{n-1} + \cdots + C_{n-1} D + C_n\right] e^{ax}$$

$$= \left[a^n + C_1 a^{n-1} + \cdots + C_{n-1} a + C_n\right] e^{ax}$$

$$f(D) e^{ax} = f(a) e^{ax}$$

Operating both side by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} \left(f(a) e^{ax}\right)$$

$$\Rightarrow e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$

$$\Rightarrow \boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$

• X is $\cos ax$ or $\sin ax$

$$\boxed{P.I. = \frac{1}{f(D)}\cos ax = \frac{1}{\varphi(D^2)}\cos ax = \frac{1}{\varphi(-a^2)}\cos ax}$$

provided $\varphi(-a^2) \neq 0$

$\boxed{\text{Replace } D^2 \text{ by } -a^2}$

Ex. $$P.I. = \frac{1}{D^4+D^2+1}\cos 2x = \frac{1}{(D^2)^2+D^2+1}\cos 2x$$

$$= \frac{1}{16-4+1}\cos 2x = \frac{1}{13}\cos 2x$$

Ex. $$P.I. = \frac{1}{D^2-2D+1}\cos 3x$$

$$= \frac{1}{-9-2D+1}\cos 3x = \frac{1}{-2D-8}\cos 3x$$

$$= -\frac{1}{2}\cdot\frac{1}{D+4}\cdot\cos 3x$$

$$= -\frac{1}{2}\,\frac{D-4}{D^2-16}\cos 3x$$

$$= \frac{1}{50}(D-4)\cos 3x$$

$$= \frac{1}{50}\cdot(-3\cdot\sin 3x - 4\cos 3x)$$

$$= -\frac{1}{50}\left(3\sin 3x + 4\cos 3x\right).$$

If $\varphi(-a^2) = 0$:

Ex. $\frac{1}{D^2+a^2} \sin ax$

$$= \text{imag}\left\{ \frac{1}{D^2+a^2} \cos ax + i \frac{1}{D^2+a^2} \sin ax \right\}$$

$$= \text{imag}\left\{ \frac{1}{D^2+a^2} \cdot e^{iax} \right\}$$

Consider. $\frac{1}{D^2+a^2} e^{iax} = \frac{1}{(D-ai)(D+ia)} \cdot e^{iax}$

$$= \frac{1}{D-ai} \cdot \frac{1}{2ia} e^{iax}$$

$$= \frac{1}{2ia} \cdot \frac{x}{1} \cdot e^{iax}$$

$$= \frac{x}{2ia} \{\cos ax + i \sin ax\}$$

$$= \frac{x}{2a} \sin ax - i \frac{x}{2a} \cos ax$$

P. I. $= -\frac{x}{2a} \cos ax$.

Rules:

$$\boxed{\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax}$$

$$\boxed{\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax}$$

Ex. Solve $(D^2+4)y = \sin^2 x$

$$C.F. = C_1 \cos 2x + C_2 \sin 2x.$$

$$P.I. = \frac{1}{D^2+4} \sin^2 x$$

$$= \frac{1}{D^2+4} \cdot \frac{1}{2}(1-\cos 2x)$$

$$= \frac{1}{2} \cdot \left[\frac{1}{4} - \frac{1}{D^2+4} \cdot \cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} - \frac{x}{2\cdot 2} \sin 2x\right]$$

$$= \frac{1}{8}\left[1 - x \sin 2x\right]$$

General solution:

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8}\left[1 - x \sin 2x\right].$$

- X is x^m or a polynomial of degree m.

Take out the lowest degree term from $f(D)$, so as to reduce it in the form

$$[1 \pm F(D)]^{\alpha}.$$

Take it to numerator and expand it.

Ex. Find $\dfrac{1}{D^3-D^2-6D}\cdot(x^2+1)$

$$= \frac{1}{-6D\left(1+\frac{D}{6}-\frac{D^2}{6}\right)}(x^2+1)$$

$$= -\frac{1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2-\cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}\cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[(x^2+1)-\frac{1}{6}(2x)+\frac{7}{36}\cdot 2\right]$$

$$= -\frac{1}{6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

- X is $e^{ax}V$, where V is any function of x.

Rule: $$\boxed{\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V.}$$

Ex.

$$\text{P.I.} = \frac{1}{D^2+3D+2}\, e^{2x}\sin x$$

$$= e^{2x}\cdot \frac{1}{(D+2)^2+3(D+2)+2}\cdot \sin x$$

$$= e^{2x}\cdot \frac{1}{D^2+7D+12}\sin x$$

$$= e^{2x}\cdot \frac{1}{7D+11}\sin x$$

$$= e^{2x}\,\frac{7D-11}{49D^2-121}\sin x$$

$$= e^{2x}\cdot \frac{7D-11}{-170}\sin x$$

$$= -\frac{e^{2x}}{170}\left(7\cdot\cos x - 11\sin x\right)$$

$$= \frac{e^{2x}}{170}\left(11\sin x - 7\cos x\right).$$

- X is xV

Rule $$\boxed{\frac{1}{f(D)}(x\cdot V) = x\cdot\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V}$$

Ex.

$$P.I. = \frac{1}{D^2-2D+1}\cdot x\sin x$$

$$= x\frac{1}{D^2-2D+1}\sin x - \frac{(2D-2)}{(D^2-2D+1)^2}\sin x$$

$$= x\frac{1}{-2D}\sin x - \frac{(2D-2)}{(-2D)^2}\sin x$$

$$= +\frac{x}{2}\cos x - \frac{(2D-2)}{4(-1)}\sin x$$

$$= \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

$$= \frac{1}{2}(x\cos x + \cos x - \sin x).$$

$$f(D)y = X$$

$$P.I. = \frac{1}{f(D)} X$$

1. General rule $\quad \frac{1}{D-\alpha} X = e^{\alpha x} \int X e^{-\alpha x}\, dx$

2. Short Methods:

a) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad ; \; f(a) \neq 0$

a') If $f(a) = 0$; $\quad \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{\lfloor r} e^{ax}$;

$$f(D) = (D-a)^r \, \Phi(D)$$

b) $\frac{1}{\Phi(D^2)} \cos ax = \frac{1}{\Phi(-a^2)} \cos ax \quad ; \; \Phi(a^2) \neq 0$

b') $\frac{1}{\Phi(D^2)} \sin ax = \frac{1}{\Phi(-a^2)} \sin ax \; ; \; \Phi(-a^2) \neq 0$

b'') $\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$

b''') $\frac{1}{D^2+a^2} \sin ax = \frac{x}{2a} \sin ax$

c) $\frac{1}{f(D)} \left(e^{ax} V\right) = e^{ax} \frac{1}{f(D+a)} V$

d) $\frac{1}{f(D)} (xV) = x \cdot \frac{1}{f(D)} V - \frac{f'(D)}{\{f(D)\}^2} V$

Method of variation of parameters

Consider the following second order non-homogeneous linear equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x) \tag{1}$$

Let $y = c_1y_1 + c_2y_2$, with c_1 and c_2 as arbitrary constants, be the general solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$$

We assume that

$$y = C_1y_1 + C_2y_2 \tag{2}$$

is the general solution of the non-homogeneous equation (1), where C_1 and C_2 are functions of x to be so chosen that (1) is satisfied.

Differentiating (2) we get

$$y^{'} = C_1y_1^{'} + C_2y_2^{'} + \underbrace{C_1^{'}y_1 + C_2^{'}y_2}_{=0} \tag{3}$$

For simplicity, in order to find C_1 and C_2 we assume that

$$C_1^{'}y_1 + C_2^{'}y_2 = 0 \tag{4}$$

Differentiating (3) again,

$$y^{''} = C_1y_1^{''} + C_2y_2^{''} + C_1^{'}y_1^{'} + C_2^{'}y_2^{'} \tag{5}$$

Substituting y, y' and y'' in (1) we get

$$C_1\left(y_1^{''} + a_1y_1^{'} + a_2y_1\right) + C_2\left(y_2^{''} + a_1y_2^{'} + a_2y_2\right) + C_1^{'}y_1^{'} + C_2^{'}y_2^{'} = f(x)$$

$$\implies \quad C_1^{'}y_1^{'} + C_2^{'}y_2^{'} = f(x) \tag{6}$$

Solving the equations (4) and (6):

$$C_1^{'} = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2^{'} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix}} = -\frac{y_2f(x)}{W}$$

Here W is called Wronskian. It is non-zero because y_1 and y_2 are linearly independent. Similarly

$$C_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f(x)}{W}$$

After integrating:

$$C_1 = \int -\frac{y_2 f(x)}{W} dx + d_1 \quad \text{and} \quad C_2 = \int \frac{y_1 f(x)}{W} dx + d_2$$

Hence the general solution of the non-homogeneous equation

$$\boxed{y = d_1 y_1 + d_2 y_2 + y_1 \int -\frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx}$$

Example: Apply the method of variation of parameters to solve

$$\frac{d^2 y}{dx^2} - y = \frac{2}{1+e^x} \tag{7}$$

Solution:

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

Let $y = C_1 e^x + C_2 e^{-x}$ be the general solution of the given equation.

$$y' = C_1 e^x - C_2 e^{-x} + \underbrace{C_1' e^x + C_2' e^{-x}}_{=0}$$

$$y'' = C_1 e^x + C_2 e^{-x} + C_1' e^x - C_2' e^{-x}$$

Substituting in (7)

$$C_1' e^x - C_2' e^{-x} = \frac{2}{1+e^x}$$

The Wronskian

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Hence

$$C_1 = -\frac{1}{2} \int -e^{-x} \frac{2}{1+e^x} dx + d_1 = \int \frac{e^{-x}}{1+e^x} dx + d_1$$

Substitute $e^x = z \Rightarrow e^x dx = dz$

$$C_1 = \int \frac{1}{z^2(1+z)} dz + d_1 = \int \frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z} dz + d_1$$

$$C_1 = -\frac{1}{z} - \ln z + \ln(1+z) + d_1 = -e^{-x} - x + \ln(1+e^x) + d_1$$

Similarly

$$C_2 = -\frac{1}{2}\int e^x \frac{2}{1+e^x} dx + d_1 = -\ln(1+e^x) + d_2$$

The general solution of the differential equation

$$\boxed{y = d_1 e^x + d_2 e^{-x} - 1 - xe^x + (e^x - e^{-x})\ln(1+e^x)}$$

Cauchy-Euler Equations

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X \tag{8}$$

or

$$(x^n D^n + a_1 D^{n-1} + \cdots + a_n) y = X \tag{9}$$

is called Euler-Cauchy equation.

Working Rule: To solve equation (8) we change the variable from x to z by putting $x = e^z$ i.e. $z = \ln(x)$.

$$z = \ln(x) \Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$$

We define a new operator

$$x\frac{d}{dx} \equiv \frac{d}{dz} \equiv D_1$$

Again

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D_1(D_1 - 1)y$$

Thus we have the following formulas for $D \equiv \frac{d}{dx}$ and $D_1 \equiv \frac{d}{dz}$

$$\begin{aligned}
&xD = D_1 \\
&x^2 D^2 = D_1(D_1 - 1) \\
&x^3 D^3 = D_1(D_1 - 1)(D_1 - 2) \\
&\vdots \\
&x^n D^n = D_1(D_1 - 1)(D_1 - 2)\ldots(D_1 - n + 1)
\end{aligned}$$

Substituting these operator relations in the equation (9), we obtain a linear differential equation with constant coefficient

$$f(D_1)y = Z, \quad \text{where Z becomes a function of z only}$$

Example 1.

$$(x^2D^2 - xD + 2)y = x\ln x \tag{10}$$

Let $x = e^z$ so that $z = \ln x$ and $D_1 \equiv \frac{d}{dz}$ then the equation (10) becomes

$$[D_1(D_1 - 1) - D_1 + 2]\, y = ze^z$$

Auxiliary equation $m^2 - 2m + 2 = 0$ and its roots are $m = 1 \pm i$
Hence

C.F. $= e^z\left[c_1 \cos(z) + c_2 \sin(z)\right] = x\left[c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))\right]$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D_1^2 - 2D_1 + 2} ze^z = e^z \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} z \\ &= e^z \frac{1}{D_1^2 + 1} z = e^z(1 + D_1^2)^{-1} z = e^z z = x\ln(x)\end{aligned}$$

General solution

$$\boxed{y = x\left[c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))\right] + x\ln(x)}$$

Equations reducible to Euler-Cauchy form There can be several forms of equation which can be reduced to Euler-Cauchy form

Example 1: Solve

$$\frac{d^3y}{dx^3} - \frac{4}{x}\frac{d^2y}{dx^2} + \frac{5}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 1$$

Solution: $y = c_1x^2 + c_2x^{(5+\sqrt{21})/2} + c_3x^{(5-\sqrt{21})/2} - x^3/5$

Example 2:

$$2x^2y\frac{d^2y}{dx^2} + 4y^2 = x^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

Hint: $y = z^2$

Solution:

$$y = z^2 \Rightarrow \frac{dy}{dx} = 2z\frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\left(\frac{dz}{dx}\right)^2 + 2z\frac{d^2z}{dx^2}$$

Substituting these values in the differential equation we get

$$x^2\frac{d^2z}{dx^2} - x\frac{dz}{dx} + z = 0$$

or

$$[x^2D^2 - xD + 1]z = 0$$

Substitute $x = e^t \Leftrightarrow \ln x = t$

$$\Rightarrow x\frac{dz}{dx} = \frac{dz}{dt} \Rightarrow xD \equiv D_1$$

Similarly

$$x^2D^2 = D_1(D_1 - 1)$$

Then the equation becomes

$$[D_1^2 - 2D_1 + 1]z = 0 \Rightarrow z = [c_1 + c_2t]e^t$$

$$\Rightarrow z = [c_1 + c_2\ln(x)]x$$

$$\Rightarrow \boxed{y = (c_1 + c_2\ln(x))^2x^2}$$

Example 3: A differential equation of the form

$$(a+bx)^n\frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_ny = X$$

can be reduced to Euler-Cauchy equation by putting

$$a + bx = v \Rightarrow \frac{dv}{dx} = b$$

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = b\frac{dy}{dv}$$

Again

$$\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dv^2} \text{ or in general } \frac{d^ny}{dx^n} = b^n\frac{d^ny}{dv^n}$$

Substituting these derivatives in the equation, we get

$$v^n \frac{d^n y}{dv^n} + \frac{a_1}{b} v^{n-1} \frac{d^{n-1} y}{dv^{n-1}} + \cdots + \frac{a_{n-1}}{b^{n-1}} v \frac{dy}{dv} + \frac{a_n}{b^n} y = \frac{X}{b^n}$$

which is an standard Euler-Cauchy equation.

Example 4: solve

$$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\ln(1+x)$$

Solution: Let $(1+x) = v \Rightarrow \frac{dv}{dx} = 1$.
Hence $\frac{dy}{dx} = \frac{dy}{dv}$ and $\frac{d^2y}{dx^2} = \frac{d^2y}{dv^2}$ and the differential equation becomes

$$v^2 \frac{d^2 y}{dv^2} + v\frac{dy}{dv} + y = 4\cos\ln v$$

Put $v = e^z \Rightarrow \ln(v) = z$ and let $D_1 \equiv \frac{d}{dz}$

$$[D_1(D_1 - 1) + D_1 + 1]\, y = 4\cos z$$

$$\left(D_1^2 + 1\right) y = 4\cos z$$

$$\begin{aligned} \text{C.F.} = c_1\cos(z) + c_2\sin(z) &= c_1\cos(\ln v) + c_2\sin(\ln v) \\ &= c_1\cos(\ln(1+x)) + c_2\sin(\ln(1+x)) \end{aligned}$$

$$\text{P.I.} = 2z\sin z = 2\ln(v)\sin(\ln(v)) = 2\ln(1+x)\sin(\ln(1+x)).$$

The general solution

$$\boxed{y = c_1\cos(\ln(1+x)) + c_2\sin(\ln(1+x)) + 2\ln(1+x)\sin(\ln(1+x)).}$$

SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n)$$

I. METHOD OF ELIMINATION

EXAMPLE: SOLVE

$$\frac{dx}{dt} = 7x - y$$

$$\frac{dy}{dt} = 2x + 5y$$

Denoting $\frac{d}{dt} \equiv D$:

$$(D-7)x + y = 0 \quad \text{--- (i)}$$

$$-2x + (D-5)y = 0 \quad \text{--- (ii)}$$

multiplying (i) by 2 and operating (ii) by $(D-7)$ and then adding the two equations:

$$2y + (D-7)(D-5)y = 0$$

$$\Rightarrow (D^2 - 12D + 37)y = 0$$

auxiliary equation $m^2 - 12m + 37 = 0$

Its roots are $6 \pm i$

$$\text{C.F.} = \boxed{y = e^{6t}(c_1 \cos t + c_2 \sin t)}$$

$$\Rightarrow \quad Dy = 6e^{6t}(c_1 \cos t + c_2 \sin t) + e^{6t}(-c_1 \sin t + c_2 \cos t)$$

$$= e^{6t}\left[(6c_1 + c_2)\cos t + (6c_2 - c_1)\sin t\right]$$

$$\text{(ii)} \Rightarrow \quad x = \frac{1}{2}\left[Dy - 5y\right]$$

$$\Rightarrow \quad \boxed{x = \frac{1}{2} e^{6t}\left[(c_1 + c_2)\cos t + (c_2 - c_1)\sin t\right]}$$

II.

METHOD OF DIFFERENTIATION

EXAMPLE: Determine the general solutions for x and y for

$$\frac{dx}{dt} - y = t \quad \text{—(i)}$$

$$\frac{dy}{dt} + x = 1 \quad \text{—(ii)}$$

Differentiating (i) w.r.t. t and replacing $\frac{dy}{dt}$ from (ii), we get

$$\frac{d^2x}{dt} - (1 - x) = 1$$

$$\Rightarrow \quad \frac{d^2x}{dt} + x = 2$$

$$\text{C.F.} = C_1 \cos t + C_2 \sin t$$

$$\text{P.I.} = \frac{1}{D^2+1} \cdot 2$$

$$= 2.$$

$$\Rightarrow \boxed{x = C_1 \cos t + C_2 \sin t + 2}$$

$$(i) \Rightarrow \quad y = \frac{dx}{dt} - t$$

$$\boxed{y = -C_1 \sin t + C_2 \cos t - t}$$

<u>EXAMPLE</u>: Solve $\dfrac{dy_1}{dx} = y_1 + y_2 + x$ — (i)

$$\frac{dy_2}{dx} = -4y_1 - 3y_2 + 2x \quad \text{— (ii)}$$

Differentiating (i) : $\dfrac{d^2y_1}{dx} = \dfrac{dy_1}{dx} + \dfrac{dy_2}{dx} + 1$

$$\Rightarrow \frac{d^2y_1}{dx} = \frac{dy_1}{dx} + \left(-4y_1 - 3y_2 + 2x\right) + 1$$

$$= \frac{dy_1}{dx} - 4y_1 - 3\underbrace{\left(\frac{dy_1}{dx} - y_1 - x\right)}_{\text{from (1)}} + 2x + 1$$

$$= -2\frac{dy_1}{dx} - y_1 + 5x + 1$$

$$\Rightarrow \frac{d^2y_1}{dx} + 2\frac{dy_1}{dx} + y_1 = 5x + 1.$$

auxiliary equation $m^2 + 2m + 1 = 0$

$$\Rightarrow m = -1, -1.$$

$$C.F. = (c_1 + c_2 x)\, e^{-x}$$

$$P.I. = \frac{1}{D^2+2D+1} \cdot 5x+1$$

$$= (D+1)^{-2}(5x+1)$$

$$= (1-2D+\cdots)(5x+1)$$

$$= (5x+1) - 2(5)$$

$$= 5x-9$$

$$\boxed{y_1 = (c_1 + c_2 x)e^{-x} + 5x - 9}$$

form(i)

$$y_2 = \frac{dy_1}{dx} - y_1 - x$$

$$= -c_1 e^{-x} + c_2\left(-x e^{-x} + e^{-x}\right) + 5 - c_1 e^{-x} - c_2 x e^{-x} - 5x + 9 - x$$

$$= -2c_1 e^{-x} - 2c_2 x e^{-x} + c_2 e^{-x} - 6x + 14$$

$\Rightarrow$

$$\boxed{y_2 = e^{-x}\left(-2c_1 - 2c_2 x + c_2\right) - 6x + 14}$$

III Method of undetermined coefficients:

Consider

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}\right\} \quad -(1)$$

We seek a particular solution.

$$x_1 = \alpha_1 e^{kt}, \quad x_2 = \alpha_2 e^{kt}, \quad \ldots, x_n = \alpha_n e^{kt}$$

It is required to determine the constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ and k in such a way that the functions $\alpha_1 e^{kt}, \alpha_2 e^{kt}, \ldots \alpha_n e^{kt}$ satisfy the above system of differential equations.

(1) $\Rightarrow$

$$\begin{aligned} k\alpha_1 e^{kt} &= (a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n)e^{kt} \\ k\alpha_2 e^{kt} &= (a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n)e^{kt} \\ &\vdots \\ k\alpha_n e^{kt} &= (a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n)e^{kt} \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} (a_{11}-K)\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n &= 0 \\ a_{21}\alpha_1 + (a_{21}-K)\alpha_2 + \cdots + a_{2n}\alpha_n &= 0 \\ &\vdots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + (a_{nn}-K)\alpha_n &= 0 \end{aligned}\right\} \quad \text{— (2)}$$

For a nontrivial solution of the above system

$$\begin{vmatrix} a_{11}-K & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21}-K & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & (a_{nn}-K) \end{vmatrix} = 0$$

This equation is called auxiliary equation of the system(1)

Suppose the roots of the auxiliary equation are real and distinct say, $K_1, K_2, \cdots K_n$.

For each root K_i:

evaluate

$$\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_n^{(i)}$$

For the root K_i, Obtain the following solution of the system:

$$x_1^{(i)} = \alpha_1^{(i)} e^{K_i t}, \quad x_2^{(i)} = \alpha_2^{(i)} e^{K_i t}, \ldots, x_n^{(i)} = \alpha_n^{(i)} e^{K_i t}$$

$i = 1, 2, \ldots n.$

<u>General solution</u>:

$$x_1 = \sum_{i=1}^{n} C_i \alpha_1^{(i)} e^{K_i t}$$

$$x_2 = \sum_{i=1}^{n} C_i \alpha_2^{(i)} e^{K_i t}$$

$$\vdots$$

$$x_n = \sum_{i=1}^{n} C_i \alpha_n^{(i)} e^{K_i t}$$

EXAMPLE: $\frac{dx_1}{dt} = 2x_1 + 2x_2$

$$\frac{dx_2}{dt} = x_1 + 3x_2$$

auxiliary equation:

$$\begin{vmatrix} 2-K & 2 \\ 1 & 3-K \end{vmatrix} = 0$$

$$\Rightarrow 6 - 5K + K^2 - 2 = 0$$

$$\Rightarrow K^2 - 5K + 4 = 0$$

$$\Rightarrow (K-4)(K-1) = 0 \Rightarrow K_1 = 1, \; K_2 = 4.$$

For: $K_1 = 1$:

Solving the system:

$$\alpha_1 + 2\alpha_2 = 0$$

Choose $\alpha_1 = 1 \Rightarrow \alpha_2 = -1/2$

Solution: $x_1^{(1)} = e^t \qquad x_2^{(1)} = -\frac{1}{2}e^t$

For $K_2 = 4$: system:

$$-2\alpha_1 + 2\alpha_2 = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = 1 \text{ (choose)}$$

Solution $x_1^{(2)} = e^{4t} \qquad x_2^{(2)} = e^{4t}$

General solution:

$$x_1 = c_1 e^t + c_2 e^{4t}$$

$$x_2 = -\frac{1}{2}c_1 e^t + c_2 e^{4t}$$