Sequence and Series

In Real Analysis, when we study the questions of a more analytic nature, i.e. limit, continuity, differentiation etc, we need the concept of convergence of a sequence.

Deft A ~~fa~~ mapping $f: \mathbb{N} \to \mathbb{R}$ is said to be a real sequence or a sequence in $\mathbb{R}$.

Remark:

(1) If $X: \mathbb{N} \to \mathbb{R}$ is a sequence, we will usually denote the value of X at n by the symbol x_n instead of $X(n)$.

(2) Let $X: \mathbb{N} \to \mathbb{R}$ ~~by~~ be a sequence defined by

$$x_n = (-1)^n.$$

Here the sequence is $\{-1, 1, -1, 1, - - -\}$ which has infinitely many terms ~~th~~ that alternate between -1 and 1. But the set of values $\{(-1)^n \mid n \in \mathbb{N}\}$ is equal to $\{-1, 1\}$.

So, for a sequence, ~~$\{x_n \mid n \in \mathbb{N}\}$~~ the ordering induced by the ordering of natural numbers is important; Therefore, we distinguish notationally between the sequence $(x_n \mid n \in \mathbb{N})$, whose infinitely many terms have an ordering, and the set of values $\{x_n \mid n \in \mathbb{N}\}$ in the range of the sequence ~~to~~ that are not ordered.

Examples:

1) Let $c \in \mathbb{R}$, then the sequence ~~B:=~~ $C := (c, c, c, \ldots)$, all of whose terms equal c, is called the constant sequence c.

2) $C := \cancel{(c^n}, c(c, c^2, c^3, \ldots, c^n, \ldots)$ is a sequence.

If $c = 1/2$, then the set of values is bounded.

If $c = 2$, then the set of values in the sequence is unbounded.

3) Sometimes, it is convenient to specify $f(1)$ and write $f(n)$ in terms of $f(m)$, $m < n$.

We ~~can~~ can write the sequence of odd natural numbers $(2n-1 \mid n \in \mathbb{N})$ by

$$x_1 = 1, \quad x_n = x_{n-1} + 2, \; n \geq 2.$$

Fibanocci sequence is one such celebrated sequence defined by:-

$$\cancel{F}: x_1 = 1, \; x_2 = 1, \; x_{n+1} = x_n + x_{n-1} \; \forall \, n \geq 3.$$

Bounded Sequence:

A sequence $(x_n \mid n \in \mathbb{N})$ is said to be bounded if there are real numbers m and M such that $m \leq x_n \leq M \;\; \forall n \in \mathbb{N}$.

For a real sequence $(x_n \mid n \in \mathbb{N})$ which is bounded above, ~~the~~ the range has the supremum by the supremum property of $\mathbb{R}$. This is called the supremum of ~~$\{f(x$~~ $(x_n \mid n \in \mathbb{N})$ and is denoted by $\sup((x_n))$.

For a real sequence unbounded above, we define

$$\sup((x_n)) = \infty.$$

Similarly, we can define the infimum of bounded and unbounded ~~sequ~~ sequences.

The limit of a Sequence:-

Let's start with a sequence $\{\frac{1}{n} \mid n \in \mathbb{N}\}$. The terms of the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}, \ldots\}$ are all positive numbers which are becoming smaller and smaller, that is they are approaching the number 0 as $n \to \infty$. So, '0' should be the limit of the sequence. By Archimedean property, we note that for $\varepsilon > 0$, there is a natural number m_ε such that $\frac{1}{m_\varepsilon} < \varepsilon$. Also for $n \geq m_\varepsilon$,

$$0 < \frac{1}{n} \leq \frac{1}{m_\varepsilon} < \varepsilon.$$

So, $$\left|0 - \frac{1}{n}\right| < \varepsilon \quad \forall n \geq m_\varepsilon.$$

Defn: A sequence (x_n) in $\mathbb{R}$ is said to converge to a real number x (or x is said to be a limit of (x_n)) if for every $\varepsilon > 0$, there exists a natural number k (depending on ε) such that

$$|x_n - x| < \varepsilon \quad \forall n \geq k$$

i.e., $$x - \varepsilon < x_n < x + \varepsilon \quad \forall n \geq k,$$

i.e., all terms except at most a finite number of terms lie in the ε-nbhd of x.

If a sequence has a limit, we say that the sequence is <u>convergent</u>. Otherwise, we say that the sequence is <u>divergent</u>.

<u>Example</u>

1) Let $x_n = (\frac{1}{n})$. Then we have seen that 0 is a limit of $(\frac{1}{n} \mid n \in \mathbb{N})$.

2) Let $x_n = (b, b, b, \ldots)$. Then it is obvious that the real number b is a limit of x_n.

3) Let $x_n = (\frac{n^2+1}{n^2} \mid n \in \mathbb{N})$. Then, note that

$$x_n = 1 + \frac{1}{n^2}, \forall n \in \mathbb{N}.$$

Let us choose $\varepsilon > 0$.
~~Then~~ Then $\left| 1 + \frac{1}{n^2} - 1 \right| < \varepsilon$ if $\frac{1}{n^2} < \varepsilon$ i.e., if $n > \frac{1}{\sqrt{\varepsilon}}$.
We choose k to be $> \frac{1}{\sqrt{\varepsilon}}$. $\therefore |x_n - 1| < \varepsilon \;\; \forall n \geq k$.

4) Let $0 < b < 1$, and $x_n = (b^n \mid n \in \mathbb{N})$.

Then, $|b^n - 0| < \varepsilon$ if $n \ln b < \ln \varepsilon$ i.e., if $n > \frac{\ln \varepsilon}{\ln b}$ (as $\ln b < 0$)

Let k be a natural number such that $k > \frac{\ln \varepsilon}{\ln b}$ (we can get such a natural number using Archimedean property of $\mathbb{R}$). Then,

$$|x_n - 0| < \varepsilon \quad \forall n \geq k.$$

5) Let $-1 < b < 0$. Then, $|b^n - 0| < \varepsilon \iff |b|^n < \varepsilon$.

So, $|b^n - 0| < \varepsilon$ if $n \ln |b| < \ln \varepsilon$ i.e., if $n > \frac{\ln \varepsilon}{\ln |b|}$.

We can again, show that $(b^n \mid n \in \mathbb{N})$ is convergent.

6) $b = 1$, then $(b^n \mid n \in \mathbb{N})$ is a constant sequence.

For, $b=-1$, choose; if possible, x to be a limit point of $(b^n \mid n \in \mathbb{N})$. Choose $\varepsilon = 1/4$. Then, we ~~can~~ cannot get a natural number k such that

$|b^n - x| < 1/4 \quad \forall\, n \geq k$ where $b^n = -1$ if n is odd

$= 1$ if n is even.

The questions now arise is—

a) In the previous examples, do the sequence have more than one limits?

2) Are the sequences $(b^n \mid n \in \mathbb{N})$ for $b > 1$ and $b < -1$ convergent?

We obtains the answers of these questions from the ~~foll~~ fallowing results.

A) Uniqueness of Limits:— A sequence in $\mathbb{R}$ can have at most one limit.

Proof: Let (x_n) be a convergent sequence and let x and x' be two limits of (x_n). Then for each $\varepsilon > 0$, there are natural numbers K_1 and K_2 such that $|x_n - x| < \varepsilon/2 \;\forall\, n \geq K_1$

and $|x_n - x'| < \varepsilon/2 \;\forall\, n \geq K_2$

$\therefore |x - x'| \leq |x_n - x| + |x_n - x'| < \varepsilon$

Since $\varepsilon > 0$ is an arbitary positive number, $x = x'$.

B) A convergent sequence of real numbers is bounded.

Proof: Suppose $\lim (x_n) = x$. Choose $\varepsilon = 1$.

that $|x_n - x| < 1 \ \forall n \geq K$.

$\therefore |x_n| \leq |x_n - x + x| \leq |x_n - x| + |x|$

$< 1 + |x| \ \forall n \geq K$.

Let $M := \text{Max}\{|x_1|, \ldots, |x_{K-1}|, 1 + |x|\}$.

~~Then~~, Then, $|x_n| \leq M \ \forall n$.

Remark: If $b > 1$, then by Archimedean property, ~~we~~ for a real number M, we get a natural number m such that $m > \frac{\ln M}{\ln b}$. So, $m \ln b > \ln M$ (as $\ln b > 0$)

So, $b^m > M$.

So, ~~$(b^m$~~ $(b^n \mid n \in \mathbb{N})$ is unbounded.

So, $(b^n \mid n \in \mathbb{N})$ is a divergent sequence.

Similarly, for $b < -1$, $(b^n \mid n \in \mathbb{N})$ is a divergent sequence.

~~Tails~~ Tails of Sequences:-

The convergence (or divergence) of a sequence depends only on the 'ultimate behaviour' of the terms. For any sequence, if we drop the first m elements, then the resulting sequence converges if and only if the origin sequence converges. and the limits are the same.

Def: If $X = (x_n \mid n \in \mathbb{N})$ is a sequence of real numbers and if m is a given natural number, then the m-tail of X is the sequence

$X_m := (x_{m+n} \mid n \in \mathbb{N}) = (x_{m+1}, x_{m+2}, \ldots)$

~~For, example,~~

For example, the 5-tail of the Fibonacci sequence is $(8, 13, 21, 34, \ldots)$.

<u>Theorem</u>: Let $X = (x_n | n \in \mathbb{N})$ be a sequence of real numbers and let m be a natural number. Then the m-tail $X_m = (x_{m+n} | n \in \mathbb{N})$ of X converges if and only if X converges. In this case, $\lim X_m = \lim X$.

<u>Methods to find limits of sequences:-</u>

A). A sequence (x_n) is said to be a <u>null sequence</u> if $\lim x_n = 0$.

Let (x_n) be a sequence of real numbers, $x \in \mathbb{R}$. If (a_n) is a null sequence and if for some constant $C > 0$ and some $m \in \mathbb{N}$, we have

$$|x_n - x| \leq C\, a_n \quad \forall n \geq m,$$

then $\lim x_n = x$.

B) Let $X = (x_n)$ and $Y = (y_n)$ be ~~sequ~~ sequences of real numbers that converge to x and y respectively, and let $c \in \mathbb{R}$. Then the sequences $X+Y := (x_n + y_n)$, $X - Y := (x_n - y_n)$, $X \cdot Y = (x_n \cdot y_n)$, $cX = (c x_n)$ converge to $x+y$, $x-y$, xy, cx respectively.

C) If $X = (x_n)$ converges to x and $Z = (z_n)$ is a sequence of real numbers such that $z_n \neq 0 \; \forall n \geq m$ for a natural number m and Z converges to $z \neq 0$, then the ~~quotient~~ quotient sequence $X/Z := \left(\frac{x_n}{z_n}\right)$, $n \geq m$, converges to $\frac{x}{z}$..

D) If $X = (x_n)$ converges to x, then the sequence $|X| := (|x_n|)$ converges to $|x|$.

Remark:- The converse of D) is not true as $(x_n) = ((-1)^n)$ is not convergent but $(|x_n|) = (1)$ is convergent.

Example:-

1) Let $z_n = (a^n + b^n)^{1/n}$, $0 < a < b$

$$z_n = (a^n + b^n)^{1/n} = b.\left(1 + \left(\frac{a}{b}\right)^n\right)^{1/n}.$$

Let ~~$c < 1$~~ $c := \frac{a}{b}$. Then, $0 < c < 1$ $\therefore (1 + c^n)^{1/n} > 1$

Let $(1 + c^n)^{1/n} = 1 + d_n$.

$$\therefore 1 + c^n = (1 + d_n)^n \geq 1 + n d_n.$$

$$\therefore 0 \leq d_n \leq \frac{c^n}{n}.$$

Now, $(c^n \mid n \in \mathbb{N})$ converges to 0 as $c < 1$,

$\left(\frac{1}{n} \mid n \in \mathbb{N}\right)$ converges to 0, so

$\left(\frac{c^n}{n}\right)$ converges to 0.

Now, using the fallowing results, we can show that $\lim d_n = 0$ So, $\lim z_n = b.1 = b$.

Theorem:- If $X =$ ~~(x_n)~~ (x_n) is a convergent sequence of real numbers and if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $n \geq m$, then $\lim x_n \geq 0$.

Proof:- Let $\lim x_n = x$ and if possible assume that $x < 0$

Let us choose $\varepsilon := -\frac{x}{2}$.

Then, $0 < x - \varepsilon < x < x + \varepsilon < 0$.

As $\lim x_n = x$, so there exists a natural number k_1 such that $x_n \in (x-\varepsilon, x+\varepsilon)\ \forall n \geq k_1$.

Let $k_2 := \max\{m, k_1\}$.

Then, by given hypothesis, $x_n \geq 0\ \forall n \geq k_2$ and from above $x_n < 0\ \forall n \geq k_2$, which is a contradiction.

So, $\lim x_n = x \geq 0$.

Cor. A) Let (x_n) and (y_n) be two convergent sequences and $x_n \geq y_n\ \forall n \geq m$ for a natural number m, then

$$\lim x_n \geq \lim y_n.$$

B) Let (x_n) be a convergent sequence and $a \leq x_n \leq b\ \forall n \in \mathbb{N}$. Then $a \leq \lim x_n \leq b$.

~~San~~ Sandwich Theorem or Squeeze Theorem:-

In the previous results, we have assumed that the given sequence is convergent and then we get some information about the limit. What will happen if the convergence of the sequence is not known. For example, we consider (d_n) in the previous example.

Theorem: Let (x_n), (y_n) and (z_n) be 3 sequences of real numbers and there is a natural number m such that

$$x_n \leq y_n \leq z_n \quad \forall n \geq m.$$

If $\lim x_n = \lim z_n = l$, then the sequence (y_n) is convergent and $\lim y_n = l$.

Proof: Let $\epsilon > 0$. Then there exist natural numbers k_1 and k_2 such that

$$l-\epsilon < x_n < l+\epsilon \quad \forall n \geq k_1,$$

$$l-\epsilon < z_n < l+\epsilon \quad \forall n \geq k_2$$

Let $k = \max\{k_1, k_2, m\}$. Then for all $n \geq k$,

$$l-\epsilon < x_n < y_n < z_n < l+\epsilon.$$

$\therefore$ (y_n) is convergent and $\lim y_n = l$.

Example: 1) Let us consider the sequence $\left(\frac{\sin n}{n}\right)$

Here $(\sin n)$ is not convergent, so we can't apply the previous results except Squeeze Theorem.

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

We know that $\lim\left(\frac{1}{n}\right) = 0$, so $\lim\left(-\frac{1}{n}\right)$

$$= -\lim\left(\frac{1}{n}\right) = 0.$$

Then by Squeeze Theorem, $\cancel{\lim}\left(\frac{\sin n}{n}\right)$ is convergent and $\lim\left(\frac{\sin n}{n}\right) = 0$

2) Let $(x_n) = \left(n^{1/n^2}\right)$

Here $\frac{1}{n^2} \leq \frac{1}{n}$ $\therefore$ $n^{1/n^2} \leq n^{1/n}$ $\therefore$ $1 \leq n^{1/n^2} \leq n^{1/n}$.

Let $(y_n) = \left(n^{1/n}\right) \geq 1$.

Let $n^{1/n} = 1 + c_n$, where $c_n \geq 0$

$$\therefore (1+c_n)^n = n$$

$$\Rightarrow n \geq 1 + n c_n$$

$$\Rightarrow 0 \leq c_n \leq 1 - 1/n$$

$\therefore$ $(1 - 1/n)$ converges to 1, so by Squeeze Theorem,

$\therefore (c_n) \to 0$ $\therefore (y_n) \to 1$

$\therefore$ Again, using Squeeze Theorem, we get

$(x_n) \to 1$.

Few more important techniques:-

A) Let the sequence $(x_n \mid n \in \mathbb{N})$ be convergent. Then the sequence $(|x_n| \mid n \in \mathbb{N})$ of absolute values is also convergent and

$$\lim(|x_n|) = |\lim(x_n)|.$$

B) Let $X = (x_n)$ be a sequence of non-negative real numbers. If X is convergent, then so is $(\sqrt{x_n})$ and $\lim(\sqrt{x_n}) = \sqrt{\lim(x_n)}$.

C) Let (x_n) be a sequence of positive real numbers such that $\lim \frac{x_{n+1}}{x_n} = l$.

(i) If $0 \le l < 1$, then $\lim(x_n) = 0$,

(ii) If $l > 1$, then $\lim(x_n) = \infty$ (will be defined later).

D) Let (x_n) be a sequence of positive real numbers such that $\lim \sqrt[n]{x_n} = l$.

(i) If $0 \le l < 1$, then $\lim x_n = 0$,

(ii) If $l > 1$ then $\lim x_n = \infty$

Proof of (B):- Here $x \geq 0$

Case-1 Let $x = 0$. Then, for a given $\varepsilon > 0$, there is a natural number m such that $|x_n - 0| < \varepsilon^2$ $\forall n \geq m$.

$\therefore \sqrt{|x_n|} < \varepsilon \Rightarrow |\sqrt{x_n} - 0| < \varepsilon$ $\forall n \geq m$.

$\therefore (\sqrt{x_n})$ is convergent and $\lim \sqrt{x_n} = 0 = \sqrt{x}$

Case-2 Let $x > 0$. $\therefore \sqrt{x} > 0$

$$\sqrt{x_n} - \sqrt{x} = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

As $x > 0$, so there exists a natural number k such that $|x_n - x| < \frac{x}{2}$ $\forall n \geq k$, i.e.

$$-\frac{x}{2} < x_n - x \quad \forall n \geq k, \text{ i.e.}$$

$$x_n > \frac{x}{2} \quad \forall n \geq k$$

So, $\sqrt{x_n} + \sqrt{x} > \sqrt{\frac{x}{2}} + \sqrt{x}$

$$\therefore \sqrt{x_n} - \sqrt{x} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}} < \frac{x_n - x}{\sqrt{\frac{x}{2}} + \sqrt{x}}$$

$\therefore |\sqrt{x_n} + \sqrt{x}| < \varepsilon$ for $n \geq k_1$ for some natural number k_1

Remark:- In C), if $\lim \frac{x_{n+1}}{x_n} = 1$ ~~and~~ or

D) if $\lim \sqrt[n]{x_n} = 1$, then we can't conclude.

Divergent Sequences

Def:

1) A real ~~seque~~ sequence $(x_n | n \in \mathbb{N})$ is said to diverge to ∞ if for ~~a~~ every positive number M, there ~~exist~~ exists a natural number k ~~such~~ such that

$$x_n > M \quad \forall n \geq k.$$

2) A real sequence $(x_n | n \in \mathbb{N})$ is said to

diverge to $-\infty$ if for every positive number M, there is a natural number k such that

$$x_n < -M \quad \forall n \geq k.$$

3) A real sequence $(x_n | n \in \mathbb{N})$ is said to be a properly divergent sequence if it either diverges to ∞, or diverges to $-\infty$.

4) A bounded sequence that is not convergent is said to be an oscillatory sequence of finite oscillation.

5) An unbounded sequence that is not properly divergent is said to be an oscillatory sequence of infinite oscillation.

Monotone Sequences

Let us consider two examples first:-

A) Let us define a sequence (x_n) by

$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2x_n} \quad \forall n \geq 1.$$

How to find whether (x_n) converges or not.

Now, note that

$$x_{n+1}^2 - x_n^2 = 2x_n - 2x_{n-1} \quad \forall n \geq 2$$

$$= 2(x_n - x_{n-1})$$

Also, $-x_2 + x_1 = \sqrt{2} - \sqrt{2\sqrt{2}} = \sqrt{2} - \sqrt{2}\, 2^{1/4}$

$$= \sqrt{2}(1 - 2^{1/4}) < 0$$

$\therefore x_2 > x_1$

$\therefore$ By Principle of induction, $x_{n+1}^2 - x_n^2 > 0$.

$\Rightarrow (x_{n+1} - x_n)(x_{n+1} + x_n) > 0$

Since each $x_n > 0$, so $x_{n+1} > x_n$.

Also, $x_n^2 = 2x_{n-1} \Rightarrow 2x_{n-1} > x_{n-1}^2$

$\Rightarrow x_{n-1}(2 - x_{n-1}) > 0$

As $x_{n-1} > 0 \ \forall n \geq 2$, so $x_{n-1} < 2$.

Therefore, (x_n) is ~~an~~ monotonically increasing and bounded sequence.

B) Let ~~(x_n)~~ (x_n) be a sequence where

$x_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \ \forall n \in \mathbb{N}$.

Then, $x_{n+1} - x_n = \frac{1}{n+1} > 0 \ \therefore x_{n+1} > x_n$.

Now, $x_{2^n} = 1 + 1/2 + \cdots + \frac{1}{2^n}$

$= 1 + 1/2 + (1/3 + 1/4) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right)$,

$> 1 + 1/2 + (1/4 + 1/4) + \cdots + \left(\frac{1}{2^n} \cdots + \frac{1}{2^n}\right)$

$= 1 + 1/2 + \cdots + 1/2 = 1 + \frac{n}{2}$.

So, (x_n) is a monotonically increasing and unbounded sequence.

We shall see, from the following results, that the sequence in A) is convergent whereas the sequence in B) is divergent.

<u>Def:</u> Let $X = (x_n)$ be a sequence of real numbers.

a) X is said to be a <u>monotone ~~convergence~~ increasing sequence</u> if $x_{n+1} \geq x_n \ \forall n \in \mathbb{N}$.

b) X is said to be a <u>monotone decreasing</u>

sequence if $x_{n+1} \le x_n \; \forall n \in \mathbb{N}$,

X is said to be a <u>monotone sequence</u> if it is either a monotone increasing sequence or a monotone decreasing sequence.

<u>Monotone Convergence Theorem:-</u>

A. A monotone increasing sequence, if bounded above, is convergent and it converges to the least upper bound.

B. A monotone decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound.

C. A monotone increasing sequence that is unbounded above, diverges to ∞.

D. A monotone decreasing sequence that is unbounded below, diverges to $-\infty$.

<u>Proof:-</u> We shall prove A and C. The proof of B and D will be similar.

<u>Proof of A.</u> Let M be the least upper bound of the set $\{x_n \mid n \in \mathbb{N}\}$. Then for every $\varepsilon > 0$, there exists a natural number m such that

$$u - \varepsilon < x_m \le u \quad \text{---} \; (i)$$

As, $(x_n \mid n \in \mathbb{N})$ is monotone increasing, so

$$x_n \ge x_m \quad \forall n \ge m \quad \text{---} \; (ii)$$

So, combining (i) and (ii), we get

$$u - \varepsilon < x_m \le x_n \le u < u + \varepsilon. \quad \forall n \ge m.$$

$\therefore (x_n)$ is convergent and $\lim x_n = M$.

Proof of C:- As (x_n) is unbounded above, so for every $M>0$, $\exists$ a natural number m such that

$$x_m \geq M.$$

So, $x_n \geq x_m \geq M \ \forall n \geq m$.

So, (x_n) diverges to $+\infty$.

Applications:-

1) Let the sequence $(x_n | n \in \mathbb{N})$ be defined by

$$x_n := \left(1+\frac{1}{n}\right)^n$$

Then (x_n) is a monotone increasing sequence, bounded above.

Let us consider $n+1$ positive numbers $1+\frac{1}{n}$, $1+\frac{1}{n}, \ldots, 1+\frac{1}{n}$ (n times) and 1

Applying A.M. $\geq$ G.M., we get

$$\frac{n\left(1+\frac{1}{n}\right)+1}{n+1} \geq \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$$

$$\Rightarrow \left(1+\frac{1}{n+1}\right)^{n+1} \geq \left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow x_{n+1} \geq x_n.$$

So, (x_n) is a monotone increasing sequence.

~~Also,~~ Note that $n! \geq 2^{n-1}$.

So, $x_n = (1+1/n)^n$

$$= 1+1+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+ \cdots + \frac{n(n-1)\cdots 2.1}{n!}\cdot\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}(1 - 1/n) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots \frac{2}{n}\cdot\frac{1}{n}$$

$$< 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}$$

$$= 2 + \frac{1}{2}\cdot\frac{1 - (\frac{1}{2})^{n-2}}{1 - 1/2} \leq 2 + 1 = 3.$$

Also, $x_n \geq 2$. $\therefore$ (x_n) is bounded above.

By Monotone convergence theorem, (x_n) is a convergent sequence. Its limit is denoted by $\underline{e}$.

2) Let $y_n := (1 + \frac{1}{n})^{n+1}$.

Let us consider $n+2$ positive numbers $1 - \frac{1}{n+1}$, $1 - \frac{1}{n+1}, \cdots, 1 - \frac{1}{n+1}$ ($(n+1)$-times) and 1.

Applying A.M. $\geq$ G.M., we get

$$\frac{(n+1)(1 - \frac{1}{n+1}) + 1}{n+2} \geq (1 - \frac{1}{n+1})^{\frac{n+1}{n+2}},$$

$$\text{or, } \left(\frac{n+1}{n+2}\right)^{n+2} \geq \left(1 - \frac{1}{n+1}\right)^{n+1}$$

$$\text{or, } \left(\frac{n+1}{n}\right)^{n+1} \geq \left(\frac{n+2}{n+1}\right)^{n+2}$$

$$\text{or, } \left(1 + \frac{1}{n}\right)^{n+1} \geq \left(1 + \frac{1}{n+1}\right)^{n+2}$$

$$\text{or, } y_n \geq y_{n+1}$$

$\therefore$ (y_n) is a monotone decreasing sequence.

Also, $y_n = 1 + \frac{n+1}{n} + \frac{(n+1)n}{2!}\frac{1}{n^2} + \cdots + \frac{1}{n^{n+1}}$

> 1

So, (y_n) is bounded. So, by Monotone convergence theorem, (y_n) is convergent.

$\therefore$ ~~lim~~ $y_n = x_n\left(1+\frac{1}{n}\right) = x_n + x_n \cdot \frac{1}{n}$

$\therefore \lim y_n = \lim x_n + \lim x_n \cdot \lim \frac{1}{n}$

$= e$.

Subsequences and Balzano–Weierstrass Theorem

We start with the following example.

Let $x_n := c^{1/n}$ for $c>1$.

Here $x_{n+1}^{n+1} = c = x_n^{n} \Rightarrow x_n = x_{n+1}^{\frac{n+1}{n}} = x_{n+1} \cdot x_{n+1}^{1/n} \geq x_{n+1}$

$x_n \geq 1$

So, (x_n) is a monotone decreasing and bounded sequence and $x_n \geq 1 \Rightarrow \lim x_n \geq 1$.

$= x$

Let us ~~cons~~ consider $y_n := x_{2n}$.

Then, $y_{n} = x_{2n} = c^{\frac{1}{2n}} = \left(c^{1/n}\right)^{1/2} = (x_n)^{1/2}$

$\Rightarrow y_n^2 = x_n$

Taking limit both sides (here we assume that (y_n) is also convergent and $\lim y_n = \lim x_n$),

$\Rightarrow x^2 = x$

$\Rightarrow x = 0$ or 1.

$\Rightarrow x = 1$

Here (r_n) is a subsequence of (x_n). Subsequences are important in ~~eas~~ establishing the convergence or divergence of the sequence.

<u>Def^n:</u> Let (x_n) be a real sequence and (r_n) be a strickly increasing sequence of natural numbers, i.e., $r_1 < r_2 < \cdots < r_n < \cdots$. Then the sequence $(x_{r_n})_n$ is said to be a <u>subsequence</u> of the sequence $(x_n)_n$.

<u>Example:</u>

1) Let $x_n = \frac{1}{n}$ and $r_n = 2n$ for all $n \in \mathbb{N}$.

Then, $(x_{r_n}) = \left\{\frac{1}{2}, \frac{1}{4}, \cdots\right\}$ is a subsequence of (x_n).

2) Let $x_n = \frac{1}{n}$ and $r_n = 2n-1$ ~~for~~ for all $n \in \mathbb{N}$.

Then, $(x_{r_n}) = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots\right\}$ is another subsequence of (x_n).

<u>Theorem:</u> If a sequence of real numbers $X = (x_n)$ converges to x, then any subsequence (x_{r_n}) also converges to x.

<u>Proof:</u> Let $\varepsilon > 0$. Then there exist a natural number m such that

$$|x_n - x| < \varepsilon \quad \forall n \geq m.$$

Since (r_n) is a strickly increasing sequence of natural numbers, there exist a natural number k such that $r_n \geq m$ $\forall n \geq k$.

So, $|x_{r_n} - x| < \varepsilon \quad \forall n \geq k$

So, (x_{r_n}) ~~con~~ is convergent and converges to x.

Remark:- If there exists two different subsequences (x_{r_n}) and (x_{k_n}) of a sequence (x_n) such that (x_{r_n}) and (x_{k_n}) converge to two different limits, ~~then~~ the sequence (x_n) is not convergent.

Example:-

1) Let $X := ((-1)^n)$.

Then X has two subsequences (x_{2n}) and (x_{2n-1}), which converge to two different limits 1 and -1 respectively. So, X is divergent.

2) Let $X := (\sin n)_n$.

Observe that $\sin x \geq 1/2 \quad \forall x \in [\pi/6, \frac{5\pi}{6}]$.

Let us denote $[\pi/6, \frac{5\pi}{6}]$ by I_1.

So, $f(x) \geq 1/2 \quad \forall x \in I_1$

Since $\frac{5\pi}{6} - \pi/6 = \frac{4\pi}{6} = 2/3 \pi > 2$, so I_1 contains at least two natural numbers. Let n_1 be the first natural number. Similarly, let $I_k := [\pi/6 + 2\pi(k-1), \frac{5\pi}{6} + 2\pi(k-1)]$,

$f(x) \geq 1/2 \quad \forall x \in I_k$,

and each I_k contains at least two natural numbers. Let n_k be the first among them. ~~Then~~ This way choose a subsequence $X_1 := (\sin n_k)_k$

and we get $\sin n_k \geq 1/2 \quad \forall k \in \mathbb{N}$.

Again, observe that $\sin x \leq -1/2 \quad \forall x \in I'_k := [\frac{7\pi}{6} +$

$2\pi(k-1), \frac{11\pi}{6} + 2\pi(k-1)]$

Length of each $I_k' > 2$, so it contains at least two natural numbers. Let n_k be the first one. Then

$X_2 := (\sin n_r)_r$ is a subsequence of X and

$\sin n_r \leq -\frac{1}{2} \ \forall r \in \mathbb{N}$.

If $(\sin n)_n$ is a convergent sequence, then X_1 and X_2 are also convergent subsequences and they have the same limit.

Though, if X_1 is a convergent sequence, then its limit $\geq \frac{1}{2}$ and if X_2 is a convergent sequence, then its limit $\leq -\frac{1}{2}$. So, $X = (\sin n)$ is not a convergent sequence.

The following result is also useful to establish the divergence of a sequence. This is based on a careful negation of the definition of the limit of a sequence. The proof is trivial.

<u>Theorem:</u> Let $X = (x_n)$ be a sequences of real numbers. Then the following are equivalent:

(i) The sequence $X = (x_n)$ does not converge to $x \in \mathbb{R}$.

(ii) There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0$

(iii) There exists an $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

We can prove the divergence of a sequence by studying the properties of subsequences. Can we guess the convergence of a sequence by studying properties of proper subsequences? There are some partial answers, the following is one such:—

Theorem:— If the subsequences (x_{2n}) and (x_{2n-1}) of a sequence (x_n) converge to the same limit x, then the ~~so~~ sequence (x_n) is convergent and $\lim(x_n) = x$.

Proof:— Let $\varepsilon > 0$. Then there exist natural numbers k_1 and k_2 such that

$$|x_{2n} - x| < \varepsilon \quad \forall n \geq k_1,$$

$$|x_{2n-1} - x| < \varepsilon \quad \forall n \geq k_2.$$

Let $k := \max\{k_1, k_2\}$. Then for all $n \geq k$,

$$|x_{2n} - x| < \varepsilon \text{ and } |x_{2n-1} - x| < \varepsilon.$$

So, for all $n \geq 2k-1$, $|x_n - x| < \varepsilon$.

$\therefore$ (x_n) is convergent and $\lim(x_n) = x$.

Question:— A un-bounded sequences may not necessarily have a convergent subsequence. For example:—

$x_n := (-1)^n n$. Each of its subsequence is unbounded, so is not convergent. We can ask the following question:

Can a bounded sequence have a convergent subsequence?

Let's start with a bounded sequence (x_n). Then $(x_n) \in I = [a, b]$ $\forall n \in \mathbb{N}$.

Let $c_1 = \frac{a+b}{2}$. Let $I_1' := [a, c_1]$, $I_1'' := [c, b]$

Then at least one of I_1' and I_1'' contains infinitely many terms of (x_n). Let $I_1 := [a_1, b_1]$ be such an interval.

Let $c_2 = \frac{a_1+b_1}{2}$ and $I_2' := [a_1, c_2]$ and $I_2'' := [c_2, b_1]$

Then at least one of I_2' and I_2'' contains infinitely many terms of (x_n). Let $I_2 := [a_2, b_2]$ be such an interval.

Continuing this way, we obtain a nested sequence of closed and bounded intervals (I_n) such that $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = \inf\left\{\frac{b-a}{2^n} : n \in \mathbb{N}\right\} = 0$.

So, by Nested Interval property, there exists a unique point $x \in \bigcap_{n=1}^{\infty} I_n$.

Now, we construct a subsequence $\{x_{n_k}\}_k$ whose limit will be x.

I_1 contains infinitely many terms of (x_n), so using the Well-ordering property, there is a smallest natural number n_1 such that $x_{n_1} \in I_1$. ~~Then, we~~ Then ~~we~~ again as I_2 contains infinitely many terms of (x_n), so there is a natural number $n_2 (> n_1)$ ~~an~~ such that $x_{n_2} \in I_2$.

This way we obtain a subsequence (x_{n_k}) of (x_n) with the property that $x_{n_k} \in I_k$. It can be shown easily that (x_{n_k}) is convergent and $\lim(x_n) = x$.

We have proven one celebrated result of Real