Linear Algebra

Lecture 14

Dual Spaces

V, W are $\mathbb{F}$ vector spaces.

$\mathcal{L}(V,W)$: $\mathbb{F}$-vector space of all linear transformations from V to W.

$\mathcal{L}(V,\mathbb{F})$: Dual space

$\shortparallel$

V^*

When V is a finite dimensional vector space over $\mathbb{F}$, then so is V^* and in fact, $\dim(V) = \dim(V^*)$

$\Rightarrow$ V is isomorphic to V^*.

Elements of V^* are called as functionals.

Examples :

1) Consider vector space of continuous real valued functions on $[0, 2\pi]$.

Fix some $g \in V$.

Define $T_g : V \longrightarrow \mathbb{R}$

$$T_g(x(t)) = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, g(t)\, dt \quad \forall x(t) \in V$$

$g(t) = \cos nt$ or $\sin nt$, then

$T_g(x(t))$ computes n^{th} Fourier coefficient.

2) $T : M_{n \times n}(\mathbb{F}) \longrightarrow \mathbb{R}$ $\qquad T \in (M_{n\times n}(\mathbb{F}))^*$

$T(A) = \text{trace}(A)$

3) Let V - finite dimensional vector space over $\mathbb{F}$.

$B = \{x_1, \dots, x_n\}$ is an ordered basis of V.

Every vector $x \in V$,

$$x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Define $f_i : V \longrightarrow \mathbb{R}$ as

$f_i(x) = a_i$ for $i = 1, 2, \dots, n$.

f_i's are called as coordinate function w.r.t. ordered basis B.

In particular. $f_i(x_j) = a_j \delta_{ij}$

Thm: Suppose V is a finite dimensional vector space with ordered basis $B = \{x_1, \dots, x_n\}$. Let f_i, $i = 1, 2, \dots, n$ be the i^{th} coordinate function w.r.t. B. Let $B^* = \{f_1, f_2, \dots, f_n\}$. Then B^* is an ordered basis for V^* and for any $f \in V^*$

$$f = \sum_{i=1}^{n} f(x_i) f_i$$

Proof: $\dim(V^*) = \dim(V) = n$

Let $g \in V^*$ and let

$$g = \sum_{i=1}^{n} f(x_i) f_i$$

for any $1 \leq j \leq n$,

$$g(x_j) = \left(\sum_{i=1}^{n} f(x_i) f_i \right)(x_j) = \sum_{i=1}^{n} f(x_i) f_i(x_j)$$

$$= \sum_{i=1}^{n} f(x_i) \delta_{ij} = f(x_j)$$

$\blacksquare$

Ex:

$V = \mathbb{R}^n$, $F = \mathbb{R}$

$V^* = (\mathbb{R}^n)^*$

Suppose $\{e_1, e_2, \dots, e_n\}$ is an ordered basis for $\mathbb{R}^n$.

$\{f_1, \dots, f_n\}$ is an ordered basis for $(\mathbb{R}^n)^*$ s.t.

$$f_i(e_j) = \delta_{ij} \quad \text{for} \quad \begin{array}{l} i = 1, 2, \dots, n \\ j = 1, 2, \dots, n \end{array}$$

For any $x \in \mathbb{R}^n$, co-ordinate of x w.r.t. B

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Matrix representation of f_1

(Let us assume that ordered basis for $\mathbb{R}$ is $\{1\}$).

$f_1(e_1) = 1$, $f_1(e_2) = 0$, $\dots$, $f_1(e_n) = 0$

$$A_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}_{1 \times n}$$

Theorem: Let V and W be finite dimensional vector spaces over $\mathbb{F}$ with ordered basis β & γ resp. For any linear transformation $T: V \longrightarrow W$, the mapping $T^t: W^* \to V^*$ defined as $T^t(g) = gT$ for all $g \in W^*$. Then T^t is a linear transformation. Further let A be the matrix representation of T w.r.t. β & γ, then A^t is the matrix representation of T^t w.r.t. γ^* and β^*.

Proof:

V	W
$\beta = \{x_1, \ldots, x_n\}$	$\gamma = \{y_1, \ldots, y_m\}$
V^*	W^*
$\beta^* = \{f_1, \ldots, f_n\}$	$\gamma^* = \{g_1, \ldots, g_m\}$

Let $T: V \longrightarrow W$

the matrix representation of T is $A_{m\times n}$.

First we want to prove

$T^t: W^* \longrightarrow V^*$ is a linear transformation.

for any $g \in W^*$

$T^t(g) = gT \in V^*$

Let $a \in \mathbb{F}$, $g_1, g_2 \in W^*$

$$T^t(ag_1+g_2) = (ag_1+g_2)T$$

$$= ag_1T + g_2T$$

$$= a(g_1T) + (g_2T)$$

$$= aT^t(g_1) + T^T(g_2)$$

We want to compute the matrix representation of $T^t : W^* \to V^*$ w.r.t. the ordered bases γ^* and β^* of W^* and V^* respectively.

To find the j^{th} column of the matrix representation of T^t,

$$T^t(g_j) = g_j T = \sum_{s=1}^{n} (g_j T)(x_s) f_s$$

The i^{th} row and j^{th} column entry is given by

$$\begin{aligned}
(g_j T)(x_i) &= g_j(T(x_i)) \\
&= g_j\left(\sum_{k=1}^{m} A_{ki} y_k\right) \\
&= \sum_{k=1}^{m} A_{ki}\, g_j(y_k) \\
&= \sum_{k=1}^{m} A_{ki}\, \delta_{jk} = A_{ji}
\end{aligned}$$

Identification of a finite dimensional vector space V with its double dual V^{**}.

For a vector $x \in V$

define $\hat{x} : V^* \longrightarrow \mathbb{F}$

by $\hat{x}(f) = f(x)$

<u>Check</u>: $\hat{x}$ is a linear transformation.

$$\hat{x}(af_1 + f_2) = (af_1 + f_2)(x)$$
$$= af_1(x) + f_2(x)$$
$$= a\hat{x}(f_1) + \hat{x}(f_2)$$

In fact, we can show that

$$x \longleftrightarrow \hat{x}$$

is an isomorphism.

<u>Lemma</u>: Let V be a finite dimensional vector space, $x \in V$. If $\hat{x}(f) = 0$ for all $f \in V^*$, then $x = 0$.

Proof: Let $x \neq 0$.

We want to show, there exist $f \in V^*$, $\hat{x}(f) \neq 0$.

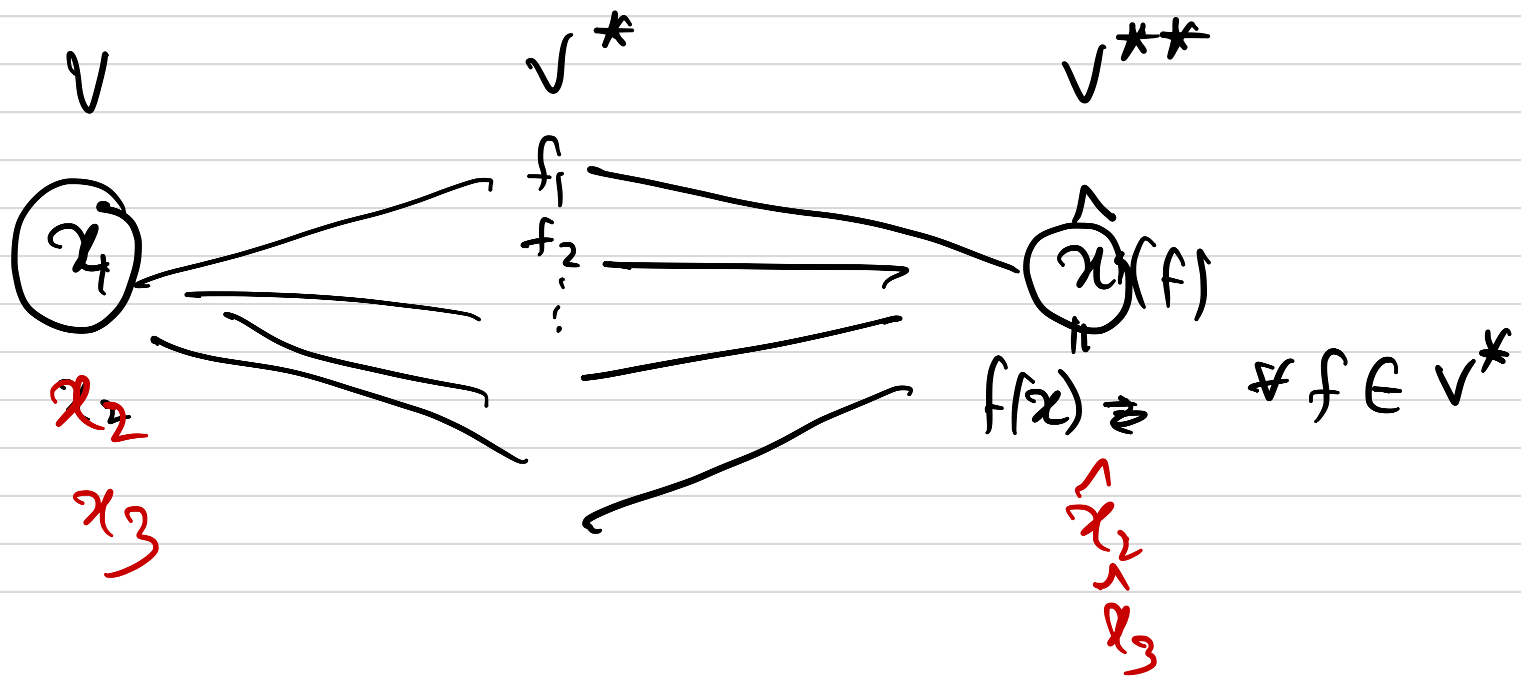
Choose an ordered basis $B = \{x_1, \dots, x_n\}$ of V. Assume without loss of generality $x_1 = x$.

$B^* = \{f_1, f_2, \dots, f_n\}$ is an ordered basis for V^*.

$f_i(x_j) = \delta_{ij}$

In particular $f_1(x_1) = f_1(x) = 1$

$\blacksquare$



Thm: Let V be a finite dimensional vector space over $\mathbb{F}$ and
$\psi : V \longrightarrow V^{**}$
by $\psi(x) = \hat{x}$.
Then ψ is an isomorphism.

Corollary: Let V be a finite dimensional vector space over $\mathbb{F}$. with dual V^*. Then every ordered basis of V^* is a dual basis of some ordered basis in V.

Pf: Let $\{f_1, \dots, f_n\}$ be an ordered basis in V^*.
Let $\{\hat{x}_1, \dots, \hat{x}_n\}$ be the dual basis corresponding to $\{f_1, \dots, f_n\}$ in V^{**}.

$$\hat{x}_i(f_j) = \delta_{ij} = f_i(x_j)$$

$\Rightarrow$ For $\{x_1, \dots, x_n\}$, $\{f_1, \dots, f_n\}$ is the dual basis. ∎

Example:

$V = P_n(\mathbb{F})$: polynomials upto degree n with coefficients in $\mathbb{F}$.

let $c_0, c_1, \ldots, c_n \in \mathbb{F}$ be distinct scalars.

(a) For $0 \le i \le n$, define

$f_i \in V^*$ as $f_i(p(x)) = p(c_i)$

Claim: $\{f_0, f_1, \ldots, f_n\}$ is a basis for V^*.

It is enough to show that $\{f_0, \ldots, f_n\}$ is linearly independent. (Since $\dim V^* = \dim V = n+1$)

Let $\beta_0, \beta_1, \ldots, \beta_n \in \mathbb{F}$ such that

$$\beta_0 f_0 + \beta_1 f_1 + \cdots + \beta_n f_n = \underset{V^*}{0}$$

(zero functional)

$$(\beta_0 f_0 + \beta_1 f_1 + \cdots + \beta_n f_n)\, p(x) = 0$$

choose $p(x) = (x-c_1)(x-c_2)\cdots(x-c_n)$

$$f_1(p(x)) = p(c_1) = 0$$
$$f_2(p(x)) = p(c_2) = 0$$
$$\vdots$$
$$f_n(p(x)) = p(c_n) = 0$$

$$(\beta_0 f_0)\, p(x) = 0$$
$$\beta_0\, p(c_0) = 0$$
$$\Rightarrow \beta_0 = 0$$

Similarly, $\beta_1 = \cdots = \beta_n = 0$

$\Rightarrow \{f_0, f_1, \cdots, f_n\}$ is a linearly independent set in V^* and hence its basis.

(b) $\exists$ polynomials $p_0(x), p_1(x), \cdots, p_n(x) \in V$ s.t. $\{f_0, f_1, \cdots, f_n\}$ is a dual basis of V^* corresponding to $\{p_0(x), \cdots, p_n(x)\}$

$$f_i(p_j) = \delta_{ij}$$
$$\parallel$$
$$p_j(c_i)$$

$\{p_0, p_1, \cdots, p_n\}$ are called Lagrange polynomials.

(c) <u>Claim</u>: for any scalars $a_0, a_1, \cdots, a_n \in \mathbb{F}$, prove that there exists a unique polynomial $q(x)$ of degree n such that

$$q(c_i) = a_i \quad \text{for } i = 1, 2, \cdots, n.$$

In fact, $q(x) = \sum_{i=0}^{n} a_i \, p_i(x)$

Lagrange interpolation formula:

$$p(x) = \sum_{i=0}^{m} p(c_i)\, p_i(x)$$

$$\forall \, p(x) \in V$$

Integrating both sides

$$\int_a^b p(t)\,dt = \int_a^b \sum_{i=0}^{n} p(c_i)\, p_i(t)\,dt$$

$$= \sum_{i=0}^{n} \left[p(c_i) \underbrace{\int_a^b p_i(t)\,dt}_{d_i} \right]$$

$$= \sum_{i=0}^{n} p(c_i)\, d_i$$

<u>Suppose</u>:

$$c_i = a + \frac{i(b-a)}{n} \qquad \text{for } i = 0, 1, \ldots, n$$