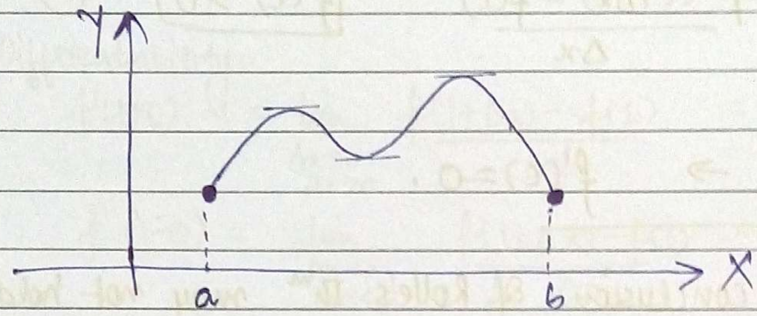
DIFFERENTIAL CALCULUS – ONE VARIABLE

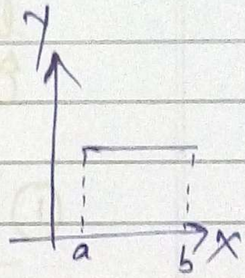
ROLLE'S THEOREM: If a function f is

(a) continuous in $[a,b]$
(b) differentiable in (a,b)
(c) $f(a)=f(b)$

} Sufficient conditions (but not necessary conditions).

then $\exists$ (there exist) $c \in (a,b)$ s.t. (such that) $f'(c)=0$

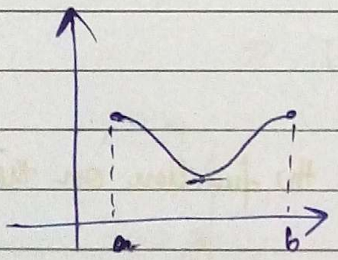


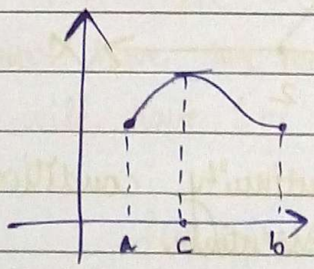


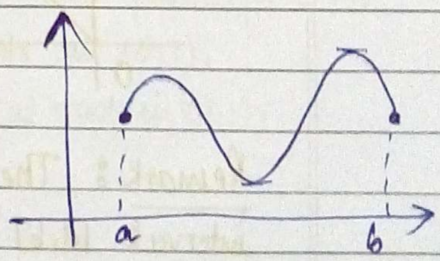
PROOF: Suppose M & m are maximum & minimum of $f(x)$ in $[a,b]$

* Case I :- If $M=m \Rightarrow f(x)=M=m=$ Constant.
$\Rightarrow f'(x)=0 \quad \forall x \in (a,b)$

* Case II :- $M \neq m$. Then at least one of them must be different from equal values of $f(a)$ & $f(b)$







Let $M=f(c)$ be different from equal values $f(a)$ or $f(b)$. Since f is differentiable in (a,b), $f'(c)$ exists.

$f(c+\Delta x) - f(c) \leq 0 \quad \forall \Delta x > 0$ or $\Delta x < 0$

$\Rightarrow \dfrac{f(c+\Delta x) - f(c)}{\Delta x} \leq 0$ for $\Delta x > 0$

and $\dfrac{f(c+\Delta x)-f(c)}{\Delta x} \geq 0$ for $\Delta x<0$

Passing limit as $\Delta x \to 0$, we get

$$\lim_{\substack{\Delta x \to 0 \\ (\Delta x>0)}} \frac{f(c+\Delta x)-f(c)}{\Delta x} = \boxed{f'(c) \leq 0} \quad -(1)$$

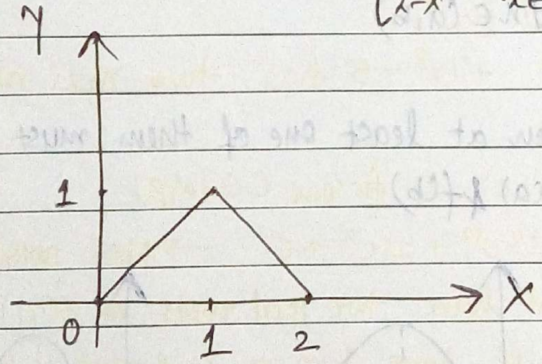
and

$$\lim_{\substack{\Delta x \to 0 \\ (\Delta x<0)}} \frac{f(c+\Delta x)-f(c)}{\Delta x} = \boxed{f'(c) \geq 0} \quad -(2)$$

(1) and (2) $\rightarrow$ $f'(c)=0$.

Remark: The conclusion of Rolle's Th^m may not hold for a function that does not satisfy any of its conditions.

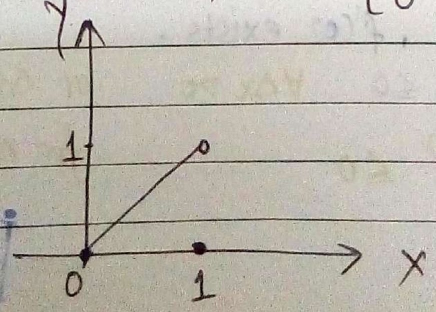
Ex.
$$f(x) = \begin{cases} x & x \in [0,1] \\ 2-x & x \in (1,2] \end{cases}$$



(Not differentiable in $(0,2)$)

Remark: The Continuity conditions for the function on the closed interval $[a,b]$ is essential.

Ex. Consider
$$f(x) = \begin{cases} x &, x \in [0,1) \\ 0 &, x=1 \end{cases}$$



(Not continuous at $x=1$)

Ex. Discuss the applicability of Rolle's Th^m to the function

$$f(x) = \begin{cases} x^2+1 & x \in [0,1] \\ 3-x & x \in (1,2] \end{cases}$$

1) Continuity

$$f(1+0) = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} 3-(1+\Delta x) = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} (2-\Delta x) = 2 = f(1)$$

2) Differentiability.

$$f'(1+0) = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{f(1+\Delta x) - f(1)}{\Delta x} = -1$$

$$f'(1-0) = \lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{f(1+\Delta x) - f(1)}{\Delta x} = 2$$

Since f is not diff. at $x=1 \Rightarrow$ Rolle's Th^m is not applicable here.

Ex. Using Rolle's Th^m, show that the equation $x^{13}+7x^3-5=0$ has exactly one real root in $[0,1]$.

Let $f(x) = x^{13} + 7x^3 - 5$ has two real roots, say α, β in $(0,1)$

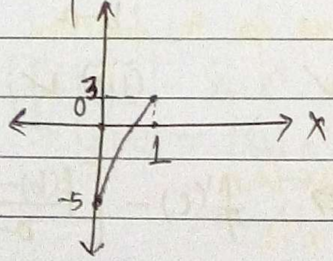
$\Rightarrow f(\alpha) = f(\beta) = 0$

Using Rolle's Th^m : $f'(c) = 0$ for some $c \in (\alpha, \beta)$

$\Rightarrow 13c^{12} + 21c^2 = 0 \longrightarrow$ Not possible for $c>0$

$\Rightarrow$ It cannot have two real roots in $(0,1)$.

$\Rightarrow$ It will have 1 or no real root in $(0,1)$.



$f(0) \cdot f(1) < 0$

$\Rightarrow$ It must have one real root in $(0,1)$.

Hence, Proved.

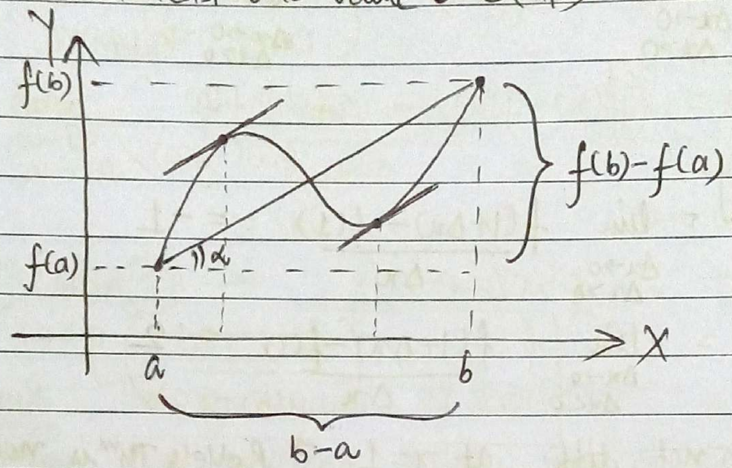
LAGRANGES MEAN VALUE THEOREM :-

If a function is

(a) Cont. in $[a,b]$

(b) diff. in (a,b)

then $\exists$ at least one value $c \in (a,b)$ s.t. $\dfrac{f(b)-f(a)}{b-a} = f'(c)$



$\tan\alpha = \dfrac{f(b)-f(a)}{b-a}$

PROOF! Define $\phi(x) = f(x) - \left[\dfrac{f(b)-f(a)}{b-a}\right]x$

(i) $\phi(a) = f(a) - \left[\dfrac{f(b)-f(a)}{b-a}\right]a = \dfrac{bf(a) - af(a) - af(b) + af(a)}{b-a}$

$= \dfrac{bf(a) - af(b)}{b-a}$

$\phi(b) = f(b) - \left[\dfrac{f(b)-f(a)}{b-a}\right]b = \dfrac{bf(b) - af(b) - bf(b) + bf(a)}{b-a}$

$= \dfrac{bf(a) - af(b)}{b-a}$

$\Rightarrow \phi(a) = \phi(b)$ (ii) ✓ (iii) ✓

By Rolle's Th^m,

$\phi'(c) = 0 \Rightarrow f'(c) - \left[\dfrac{f(b)-f(a)}{b-a}\right] = 0$

i.e $\boxed{f'(c) = \dfrac{f(b)-f(a)}{b-a}}$

GENERALISED MVT : (CAUCHY MEAN VALUE THM):

If $f(x)$ and $g(x)$ are two functions continuous in $[a,b]$ and diff. in (a,b). Also, $g'(x)$ does not vanish anywhere inside the interval, then there exist a point c in (a,b) s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

(Define $g(x)=x$, CMVT $\Rightarrow$ LMVT)

PROOF: $\varphi(x) = (f(x)-f(a)) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)(g(x)-g(a))$

(i) $\varphi(a)=\varphi(b)=0$ (ii) ✓ (iii) ✓

By Rolle's Thm, $\varphi'(c)=0$

$$\Rightarrow \boxed{f'(c) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)g'(c) = 0}$$

Ex. Using MVT, show that
$$|\cos e^x - \cos e^y| \le |x-y| \quad \text{for } x,y \le 0$$

Let $f(t)=\cos e^t$ in $[x,y]$

Use LMVT:

$$\frac{\cos e^x - \cos e^y}{x-y} = f'(c) \qquad c\in(x,y)$$

$$|\cos e^x - \cos e^y| \le |x-y|\,|f'(t)|_{\substack{\text{Max value}\\ t\in[x,y]}}$$

$|f'(t)| = |-\sin e^t \cdot e^t|$

$\Rightarrow |f'(t)| = \underbrace{|\sin e^t|}_{0<\ \le 1}\underbrace{|e^t|}_{0<\ \le 1} \le 1$

$$\Rightarrow \max_{t\in[x,y]} |f'(t)| \le 1$$

$$\Rightarrow \boxed{|\cos e^x - \cos e^y| \le |x-y|}$$

Ex. Using MVT, show that

$$\ln(1+x) \le \frac{x}{\sqrt{1+x}} \quad \text{for } x>0$$

Hint: $f(t) = \ln(1+t) - \frac{t}{\sqrt{1+t}}$ in $[0,x]$

- Using LMVT,

$$\frac{\left(\ln(1+x) - \frac{x}{\sqrt{1+x}}\right) - \left(\ln(1+0) - 0\right)}{x-0} = f'(c) \quad \text{where } c \in (0,x)$$

$$\ln(1+x) - \frac{x}{\sqrt{1+x}} = x\left\{\frac{1}{1+c} - \frac{\left[\sqrt{1+c}\,(1) - \frac{1}{2}\frac{c}{\sqrt{1+c}}\right]}{(1+c)}\right\}$$

$$= \frac{x}{(1+c)}\frac{\left(2\sqrt{1+c} - 2(1+c) + c\right)}{2\sqrt{1+c}}$$

$$= \frac{\cancel{2}x}{\cancel{2}(1+c)^{3/2}}\left(\sqrt{1+c} - 1 - \frac{c}{2}\right)$$

$$= \frac{x}{(1+c)^{3/2}}\left(\sqrt{1+c} - \sqrt{1+c+\frac{c^2}{4}}\right) \le 0$$

Hence, $\ln(1+x) \le \frac{x}{\sqrt{1+x}}$ for $x>0$.

INDETERMINATE FORMS: $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, 0^0, \infty^0, 1^\infty, \infty-\infty$

REMARK: The indeterminate form does not imply that the limit does not exist. In many cases, algebraic cancellations, L'Hospital rule or other methods can be used to evalute the limit.

Ex. $\lim_{x\to 0} \frac{x^2}{x} = 0$ $\qquad \lim_{x\to 0^+} \frac{1}{x^2} = \infty$

L'HOSPITAL RULE: Let the functions $f(x)$ and $\varphi(x)$ in $[a,b]$ satisfy the conditions of CMVT and vanish at the point $x=a$ i.e $f(a)=\varphi(a)=0$. Then if the ratio $\frac{f'(x)}{\varphi'(x)}$ has a limit as $x\to a$, then, there exist $\lim_{x\to a}\frac{f(x)}{\varphi(x)}$ and

$$\boxed{\lim_{x\to a}\frac{f(x)}{\varphi(x)} = \lim_{x\to a}\frac{f'(x)}{\varphi'(x)}}$$

PROOF: Let $x\in[a,b]$ and $x\neq a$ using Cauchy's MVT,

$$\frac{f(x)-f(a)}{\varphi(x)-\varphi(a)} = \frac{f'(\xi)}{\varphi'(\xi)} \quad \text{where } \xi\in(a,x)$$

Since $f(a)=\varphi(a)=0$

$$\Rightarrow \frac{f(x)}{\varphi(x)} = \frac{f'(\xi)}{\varphi'(\xi)}$$

Note that as $x\to a$, $\xi\to a$

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)} = \lim_{x\to a}\frac{f'(\xi)}{\varphi'(\xi)} = \lim_{\xi\to a}\frac{f'(\xi)}{\varphi'(\xi)} = \lim_{x\to a}\frac{f'(x)}{\varphi'(x)}$$

REMARK: The above theorem holds for the case where the functions $f(x)$ and $\varphi(x)$ are not defined at $x=a$, but $\lim_{x\to a} f(x)=0$ & $\lim_{x\to a}\varphi(x)=0$

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Remark! $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to a} \frac{f''(x)}{g''(x)} = \ldots$

Remark! The L'Hospital Rule is also applicable if $\lim\limits_{x\to\infty} f(x)=0$ & $\lim\limits_{x\to\infty} g(x)=0$

Ex. $\lim\limits_{x\to\infty} \frac{\sin\left(\frac{n}{x}\right)}{\left(\frac{1}{x}\right)}$ $\left(\frac{0}{0}\right)$

Applying L'Hospital Rule:-

$= \lim\limits_{x\to\infty} \frac{\cos\left(\frac{n}{x}\right)\cancel{\left(-\frac{n}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = n.$

Th^m Suppose $f(x)=\infty$ and $g(x)=\infty$ as $x\to a$ (or $x\to\pm\infty$). Then

$$\lim_{\substack{x\to a\\(\text{or } x\to\pm\infty)}} \frac{f(x)}{g(x)} = \lim_{\substack{x\to a\\(\text{or } x\to\pm\infty)}} \frac{f'(x)}{g'(x)}$$

provided $\lim\limits_{\substack{x\to a\\(\text{or } x\to\pm\infty)}} \frac{f'(x)}{g'(x)}$ exist

Remark! If the $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist, it does not mean that $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ does not exist.

Ex. $\lim\limits_{x\to\infty} \frac{x+\sin x}{x}$ $\left(\frac{\infty}{\infty}\right)$

L'Hospital Rule:- $\lim\limits_{x\to\infty} \frac{1+\cos x}{1} \to$ does not exist.

$\to \lim\limits_{x\to\infty} \left(1+\frac{\sin x}{x}\right) = 1 + \lim\limits_{x\to\infty} \frac{\sin x}{x} \to 0 = 1.$

Remark: $\left(\frac{0}{0}\right) \frac{f}{g} = \frac{\left(\frac{1}{g}\right)}{\left(\frac{1}{f}\right)} \left(\frac{\infty}{\infty}\right)$

Ex.

$$\lim_{x\to0^+} x^n \ln x \xrightarrow{\text{difficult}} = \lim_{x\to0} \frac{x^n}{\left(\frac{1}{\ln x}\right)} \left(\frac{\infty}{\infty}\right)$$

easy

$$\left(\frac{\infty}{\infty}\right) \lim_{x\to0} \frac{\ln x}{\frac{1}{x^n}} = \lim_{x\to0} \frac{\frac{1}{x}}{\frac{-n}{x^{n+1}}} = 0$$

Remark: The forms

$0^\infty = 0$, $\infty\cdot\infty = \infty$, $\infty+\infty = \infty$, $\infty^\infty = \infty$, $\infty^{-\infty} = 0$ are not indeterminant forms.

Indeterminate form $0\cdot\infty$:

Suppose $f(x)\to0$ and $g(x)\to\infty$ as $x\to a$
then $f(x)\cdot g(x)$ as $x\to a$ is undefined

$$f(x)\cdot g(x) = \frac{f(x)}{\left[\frac{1}{g(x)}\right]} \left(\frac{0}{0}\right) = \frac{g(x)}{\left[\frac{1}{f(x)}\right]} \left(\frac{\infty}{\infty}\right)$$

Indeterminate form $\infty-\infty$:

$$f(x) - g(x) = \frac{\left(\frac{1}{g(x)} - \frac{1}{f(x)}\right)}{\frac{1}{f(x)g(x)}} \left(\frac{0}{0}\right)$$

Indeterminate forms of type $0^0, \infty^0, 1^\infty$:

$$\lim_{x\to a} f(x)^{g(x)}, \quad \begin{cases} f(x)\to0 & g(x)\to0 \\ f(x)\to\infty & g(x)\to0 \\ f(x)\to1 & g(x)\to\infty \end{cases}$$

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Consider $y = f(x)^{g(x)}$

$$\ln y = g(x)\ln f(x) \qquad (0\cdot\infty)$$

$$\lim_{x\to a}(\ln y) = A \quad \text{(say)}$$

$$\ln\left(\lim_{x\to a} y\right) = A$$

$$\boxed{\lim_{x\to a} y = e^{A}}$$

Ex. $\displaystyle\lim_{x\to 0^+} x^x \qquad (0^0)$

Let $y = x^x$

$$\ln y = x\ln x$$

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\left(\frac{1}{x}\right)}$$

L'Hospital's Rule: $\displaystyle\ln\left(\lim_{x\to 0^+} y\right) = \lim_{x\to 0^+}\left(\frac{\frac{1}{x}}{\frac{-1}{x^2}}\right) = \lim_{x\to 0^+}(-x) = 0$

$$\Rightarrow \boxed{\lim_{x\to 0^+} y = e^0 = 1}$$

Ex. $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) \qquad (\infty-\infty)$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x - x^2}{x^2\sin^2 x}\right) \qquad \left(\frac{0}{0}\right)$$

$$= \underbrace{\lim_{x\to 0}\left(\frac{x^2}{\sin^2 x}\right)}_{1}\cdot\lim_{x\to 0}\left(\frac{\sin^2 x - x^2}{x^4}\right) = \lim_{x\to 0}\left(\frac{2\sin x\cos x - 2x}{4x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin 2x - 2x}{4x^3}\right) = \lim_{x\to 0}\frac{2\cos 2x - 2}{12x^2}$$

$$= \lim_{x\to 0}\frac{1}{6}\left(\frac{\cos 2x - 1}{x^2}\right) = \lim_{x\to 0}\frac{1}{6}\left(\frac{-\cancel{2}\sin 2x - 0}{\cancel{2}x}\right)$$

$$= \boxed{-\frac{1}{3}}$$

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Ex. $\lim_{x\to\infty}\left(x+\frac{1}{\ln(1-\frac{1}{x})}\right)$

$$= \lim_{x\to\infty} \frac{\left(\ln(1-\frac{1}{x}) + \frac{1}{x}\right)}{\frac{1}{x}\cdot\ln(1-\frac{1}{x})}$$

$$= \lim_{y\to 0} \frac{\ln(1-y) + y}{y\ln(1-y)} \quad \Bigg| \quad = \lim_{y\to 0} \frac{\frac{1}{1-y}(-1) + 1}{\frac{-y}{1-y} + \ln(1-y)}$$

$$= \lim_{y\to 0} \frac{(-1)\left(\frac{1+1-y}{\cancel{(1-y)}}\right)}{\frac{(1-y)\ln(1-y) - y}{\cancel{(1-y)}}} = \lim_{y\to 0} \frac{(2-y)}{y+(y-1)\ln(1-y)}$$

$$\lim_{y\to 0} \frac{\left(\cancel{-y} - \frac{y^2}{2} - \frac{y^3}{3} \cdots + \cancel{y}\right)}{y\left(-y - \frac{y^2}{2} - \frac{y^3}{3} \cdots\right)} = \lim_{y\to 0} \frac{-0 - \frac{1}{2} - \frac{y}{3}\cdots}{-1 - \frac{y}{2} - \frac{y^2}{3}\cdots}$$

$$= \boxed{+\frac{1}{2}}$$

$$\lim_{x\to 0} \frac{x\ln(1-\frac{1}{x}) + 1}{\ln(1-\frac{1}{x})} = \lim_{x\to 0} \frac{\ln(1-\frac{1}{x}) + \frac{x^2}{x-1}\left(\frac{1}{x^2}\right)}{\frac{x}{x-1}\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x(x-1)} - \frac{1}{(x-1)^2}}{\frac{-1}{x^2(x-1)^2}(2x-1)} = \lim_{x\to 0} \frac{x[(x-1)-x]}{-(2x-1)}$$

$$= \lim_{x\to 0} \frac{x}{2x-1} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

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TAYLOR'S FORMULA:-

Let us assume that a function f has all derivatives upto $(n+1)$th order in some interval containing to point $x=a$.

AIM:- Find a polynomial $P_n(x)$ of degree n, s.t;

$$P_n(a)=f(a),\quad P_n'(a)=f'(a),\quad P_n''(a)=f''(a),\ \ldots\quad P_n^{(n)}(a)=f^{(n)}(a) \qquad \text{(i)}$$

Construction:

ASSUME: $P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \ldots\ldots + C_n(x-a)^n$.

$P_n'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \ldots\ldots + nC_n(x-a)^{n-1}$

$P_n''(x) = 2C_2 + 6C_3(x-a) + \ldots\ldots + C_n\, n.(n-1)(x-a)^{n-2}$

$\vdots$

$P_n^{(n)}(x) = n!\, C_n$

Using condition (i):-

$C_0 = f(a)$ $\qquad P_n'(a) = C_1 = f'(a)$

$C_2 = \dfrac{f''(a)}{2!}$ $\qquad ---- \qquad C_n = \dfrac{f^{(n)}(a)}{n!}$

$$\underbrace{P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots + \frac{(x-a)^n}{n!}f^{(n)}(a)}_{\text{Taylor's polynomial}}.$$

Denote $R_n(x)$ the difference between the values of the given function $f(x)$ and the polynomial P_n.

$$R_n(x) = f(x) - P_n(x) \Rightarrow f(x) = P_n(x) + R_n(x)$$

(Remainder of Taylor's polynomial)

How to evaluate $R_n(x)$?

Let us write the remainder in the form:

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$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} Q$$

We define an auxillary function of t as

$$F(t) = f(x) - f(t) - \frac{(x-t)}{1!} f'(t) - \frac{(x-t)^2}{2!} f''(t) - \cdots - \frac{(x-t)^n}{n!} f^{(n)}(t) - \frac{(x-t)^{n+1}}{(n+1)!} Q$$

$t \in [a, x]$

Note: $F(a) = 0$ $= f(x) - \left[f(a) + \frac{(x-a)}{1!} f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a)\right] - \frac{(x-a)^{n+1}}{(n+1)!} Q$

$= f(x) - [P_n(x) + R_n(x)] = f(x) - f(x) = 0.$

$F(x) = 0$

By Rolle's Th^m,

$F'(c) = 0$ for some $c \in (a, x)$

$$\left[-\frac{(x-t)^n}{n!} f^{n+1}(t) + \frac{(x-t)^n}{n!} Q\right]_{t=c} = 0$$

$$\Rightarrow \frac{(x-c)^n}{n!} \left(Q - f^{(n+1)}(c)\right) = 0$$

$$\Rightarrow \boxed{Q = f^{n+1}(c)}$$

Finally :-

$$\boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(c)}$$

TAYLOR'S FORMULA

OR

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(c)$$

OR

$f(x) = f(a) + (x-a) f'(c) \rightsquigarrow f'(c) = \frac{f(x) - f(a)}{(x-a)} \Rightarrow$ LMVT.

REMARK:

1. $R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}\left[f^{n+1}\left(a+\theta(x-a)\right)\right]$ $0<\theta<1$

$\rightarrow$ equivalent to 'c'

2. Maclaurin's Formula when $a=0$.

3. In the Taylor's Formula, if the remainder $R_n(x) \to 0$ as $n \to \infty$ then

$$f(x) = f(a) + (x-a)f'(a) + \dots + \frac{(x-a)^n}{n!}f^n(a) + \dots \Big\} \text{ TAYLOR'S SERIES}$$

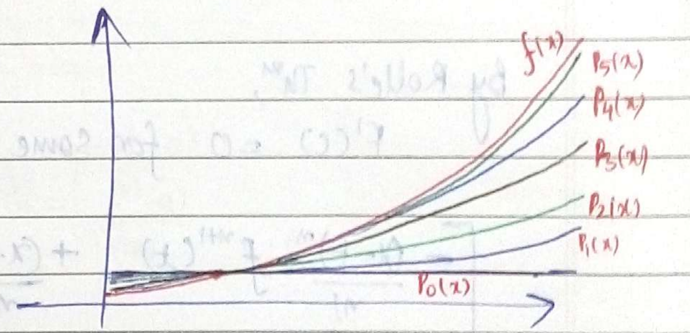
$f(x) = e^x$ $a=0$

$P_0(x) = 1$

$P_1(x) = 1+x$

$P_2(x) = 1+x+\frac{x^2}{2!}$

$P_3(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$



EXAMPLE: Obtain the Taylor's formula for the function $f(x) = \sin x$ about the point $x=0$.

Show that the remainder form goes to zero as $n \to \infty$ and write down the Taylor's series expansion of $f(x)$.

Approximate $\sin 30^\circ$ with the Taylor's polynomial of degree 3 and estimate the error using remainder form.

Verify the error estimate with the exact error.

SOLUTION: TAYLOR'S FORMULA:-

$f(x) = \sin x$ $a=0$ $f'(x) = \cos x$

$P_0(x) = 0$

$P_1(x) = x$

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$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$f''''(x) = \sin x$ $\quad f^{(IV)}(0) = 0$

$f^{(v)}(x) = \cos x$ $\quad f^{(v)}(0) = 1$

$\vdots$

$f^{2n}(x) = (-1)^n \sin x$ $\quad f^{2n}(0) = 0$

$f^{2n+1}(x) = (-1)^n \cos x$ $\quad f^{2n+1}(0) = (-1)^n$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{2n}}{(2n)!} f^{2n}(0) + \frac{x^{2n+1}}{(2n+1)!} f^{2n+1}(0) + \frac{x^{2n+2}}{(2n+2)!} f^{(2n+2)}(c), \quad c \in (0, x)$$

$$\Rightarrow f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{x^{2n+1}}{(2n+1)!}(-1)^n + \frac{x^{2n+2}}{(2n+2)!} f^{(2n+2)}(c)$$

Remainder $\to 0$:

$$|R_n| = \left| \frac{x^{2n+2}}{(2n+2)!} (-1)^{n+1} \sin c \right| \le \left| \frac{x^{2n+2}}{(2n+2)!} \right| = \frac{|x|^{2n+2}}{(2n+2)!}$$

$$* \lim_{n \to \infty} \frac{|x|^{2n+2}}{(2n+2)!}$$

For a fixed x, we can always find a N such that $|x| < N$

Consider $(2n+2) > N$ and do the following:-

$$\frac{|x|^{2n+2}}{(2n+2)!} = \frac{|x|^{2n+2}}{1 \cdot 2 \cdots (2n+2)} = \frac{|x|}{1} \frac{|x|}{2} \frac{|x|}{3} \cdots \frac{|x|}{(N)} \frac{|x|}{(N+1)} \cdots \frac{|x|}{(2n+2)}$$

Let $\frac{|x|}{N} = q < 1$

Then, $$\frac{|x|^{2n+2}}{(2n+2)!} < \frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \frac{|x|}{3} \cdots \frac{|x|}{N-1} \cdot q \cdot q \cdot q \cdots q$$

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$$= \frac{|x|^{N-1}}{(N-1)!} q^{(2n+2)-(N-1)} = \frac{|x|^{N-1}}{(N-1)!} \cdot q^{2n-N+3} \quad q<1$$

As $n \to \infty$

$$\frac{|x|^{2n+2}}{(2n+2)!} \to 0 \quad \text{Hence} \quad \lim_{n\to\infty} |R_n| = 0$$

The Taylor's series expansion is given as:-

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots}$$

Approximation of Sin 30':-

$$\sin 30' = \sin\frac{\pi}{6} = \frac{\pi}{6} - \left(\frac{\pi}{6}\right)^3 \frac{1}{3!}$$

$$= 0.499674179$$

ERROR ESTIMATE:-

$$R_3(x) = \left|\frac{x^4}{4!} f^4(c)\right| \Rightarrow \left| R_3(x)\Big|_{x=\pi/6} \right| = \left|\frac{\pi^4}{6^4} \cdot \frac{1}{4!} \sin c\right|$$

$0 \le \sin c \le \frac{1}{2}$

$\because c \in (0, \pi/6)$

$$\le \frac{\pi^4}{6^4} \cdot \frac{1}{4!} \cdot \frac{1}{2} = \frac{0.00313}{2} = 0.001565$$

ACTUAL ERROR:-

$$\sin\frac{\pi}{6} - 0.499674179 = 0.00032582$$

In this case, $f^{IV}(0) = 0$, so a better error bound may be obtained

$$R_4(x) = \left|\frac{x^5}{5!} f^{(5)}(c)\right| \Rightarrow \left|R_4\left(\frac{\pi}{6}\right)\right| = \left|\left(\frac{\pi}{6}\right)^5 \frac{1}{5!} \cos c\right|$$

$$\le \frac{\pi^5}{6^5} \frac{1}{5!} = 0.000327$$

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EXAMPLE: Find the number of terms that must be retained in the Taylor's polynomial approximation about the point $x=0$ for the function $\cosh x$ in the interval $[0,1]$ such that $|\text{error}| < 0.001$.

$f(x) = \cosh x$ $\qquad \cosh(x) = \dfrac{e^x + e^{-x}}{2}$ $\qquad \cosh 0 = 1$

$f'(x) = \sinh x$

$f''(x) = \cosh x$ $\qquad \sinh x = \dfrac{e^x - e^{-x}}{2}$ $\qquad \sinh 0 = 0$

$\vdots$

$$R_n(x) = \left|\frac{x^{n+1}}{(n+1)!} f^{n+1}(\xi)\right| \qquad \xi \in (0,1)$$

$$= \frac{|x|^{n+1}}{(n+1)!} \left|f^{n+1}(\xi)\right| \leq \frac{1}{(n+1)!}\left|f^{n+1}(\xi)\right|$$

Note that $\left|f^{(n+1)}(\xi)\right| \leq \dfrac{e^{\xi} + e^{-\xi}}{2} < \dfrac{e + e^{-1}}{2} = 1.543$

Now set $\left(\dfrac{e+e^{-1}}{2}\right) \dfrac{1}{(n+1)!} < 0.001 \Rightarrow n \geq 5$

$\Rightarrow$ Minimum six terms are required.

$$P_5(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \frac{x^5}{5!} f^{(v)}(0).$$

$$P_5(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$$

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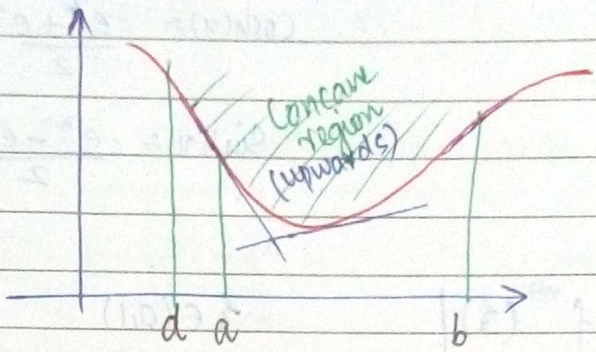
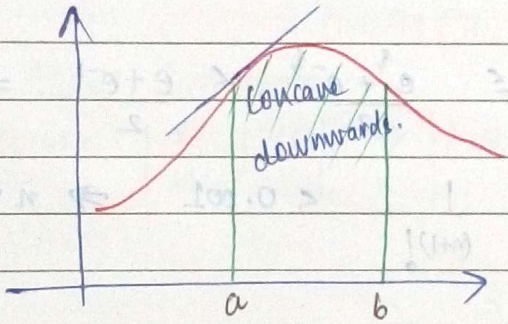
CONVEXITY, CONCAVITY, POINT OF INFLECTION

Def: A curve is convex downwards (or concave upwards) on the interval (a,b) if all points of the curve lie above tangent to it on the interval.

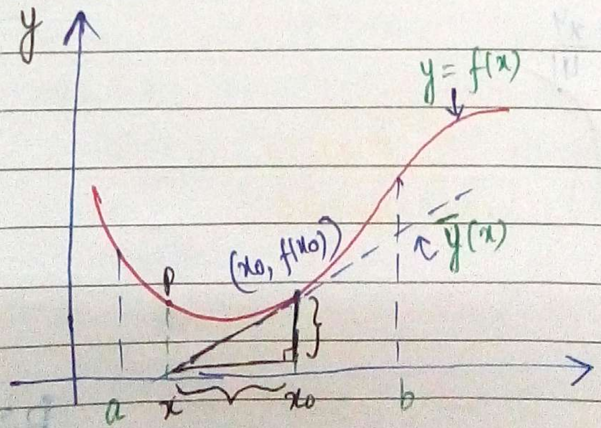


Similarly, we say a curve is convex upwards (concave downwards) on the interval (a,b) if all points of the curve lie below any tangent to it on the interval.



Th. If at all points of an interval (a,b), the second derivative of the function $f(x)$ is positive i.e. $f''(x) > 0$, the curve $y = f(x)$ on this interval is convex downwards.

Proof:



Slope of tangent at x_0 =

$$f'(x_0) = \frac{f(x_0) - \bar{y}(x)}{x_0 - x}$$

$$\Rightarrow \boxed{\bar{y}(x) - f(x_0) = f'(x_0)[x - x_0]}$$

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$$\bar{y}(x) = f(x_0) + f'(x_0)\,[x - x_0]$$

Aim: $(y-\bar{y})(x) \geq 0 \quad \forall\, x \in (a,b)$ [i.e. curve lies above tangent]

Consider: $y - \bar{y} = f(x) - [f(x_0) + f'(x_0)(x - x_0)]$

$$= \underbrace{f(x) - f(x_0)} - f'(x_0)(x - x_0)$$
$$= f'(c_1)(x - x_0) - f'(x_0)(x - x_0) \quad \text{(Applying LMVT)}$$

(c_1 lies b/w x and x_0)

$$y - \bar{y} = [f'(c_1) - f'(x_0)](x - x_0)$$

Applying LMVT again:-

$$y - \bar{y} = f''(c_2)\,(c_1 - x_0)\,(x - x_0)$$

(c_2 lies b/w x_0 & c_1)

Case 1: $x > x_0$

$$\Rightarrow \quad x_0 < c_2 < c_1 < x$$

$$\Rightarrow y - \bar{y} = (>0)(>0)(>0)$$

$$y - \bar{y} > 0$$

Case 2: $x < x_0$

$$\Rightarrow \quad x < c_1 < c_2 < x_0$$

$$\Rightarrow y - \bar{y} = (>0)(<0)(<0)$$

$$y - \bar{y} > 0$$

Thus, if $f''(x) > 0$, curve lies above tangent & is thus convex downwards.

Th. If at all points of an interval (a,b), the second derivative of the function $f(x)$ is negative i.e $f''(x) < 0$, the curve $f(x) = y$ on this interval is concave (convex upwards).

Proof: Same as above.

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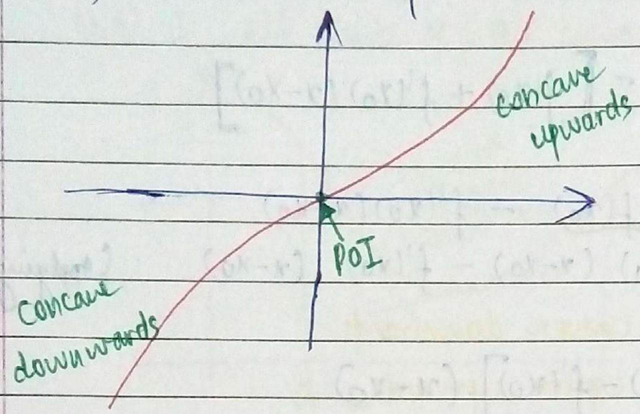
POINT OF INFLECTION :

The point that seperates the convex part of a continuous curve from the concave part is called point of inflection of curve.

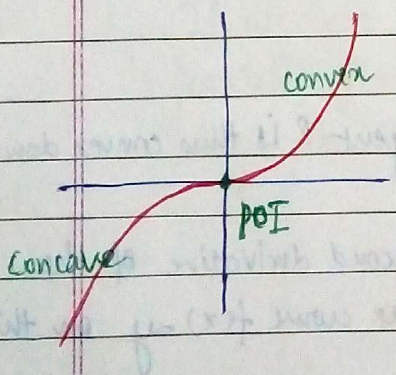


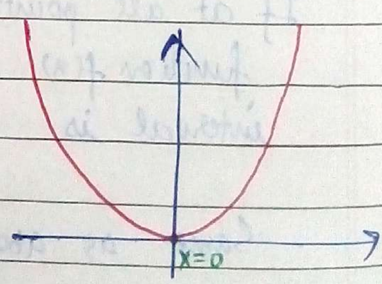
Th. Let a curve be defined by the equation $y=f(x)$. If $f''(a)=0$ or $f''(a)$ does not exist and if the derivative $f''(x)$ changes sign as x passes through a, then the point of the curve with abscissa $x=a$ is the point of inflection.

__Examples.__

1. $y = x^3$
$y'' = 6x$

$y''<0$ for $x<0$ & $y''>0$ for $x>0$ & $y''=0$ for $x=0$
Hence, for $x<0$, the curve is concave (convex upwards)
$x>0$, the curve is convex (convex downwards)
$(0,0)$ ~~x=0~~, is a point of inflection.





2. $y = x^4 \Rightarrow y'' = 4\cdot3\cdot x^2$ (concave upwards)
The curve is convex for $x \in (-\infty,\infty)$.
Also, note that at $x=0$, $y''=0$ but y'' does not change sign

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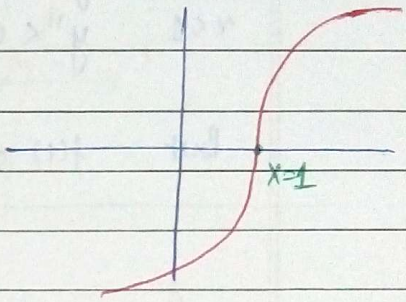
passing through $x=0$, so the curve has no point of inflection.

3. $y = (x-1)^{1/3}$

$y' = \frac{1}{3}(x-1)^{-2/3}$ $\quad (x \neq 1)$

$y'' = \frac{-2}{9}(x-1)^{-5/3} = \frac{-2}{9(x-1)^{5/3}}$ $\quad (x \neq 1)$



for $x<1$, $y''>0$ convex downwards

$x>1$, $y''<0$ convex upwards

$\Rightarrow (1,0)$ is the point of inflection.

Homework:

Q1. Find the interval in which the function $y = x + x^{5/3} + 5/3$ is convex or concave (convex upwards)
(convex downwards)

Q2. Investigate the point of inflection of the function
$$f(x) = \frac{x^2}{x-1}$$

Answers:

A1. $y = x + x^{5/3} + 5/3$

$y'' = 0 + \frac{5}{3} \cdot \frac{2}{3} x^{-1/3} + 0 = \frac{10}{9x^{1/3}}$ $\quad x \neq 0$

for $x<0$, $y''<0$ convex upwards

$x>0$, $y''>0$ convex downwards

$\Rightarrow (0, 5/3)$ is the point of inflection.

A2. $y = \frac{x^2}{x-1}$ $\Rightarrow y' = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2}$

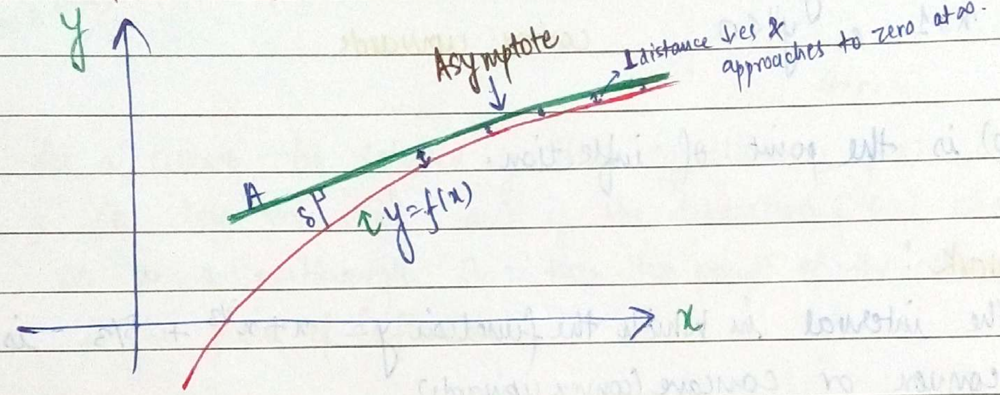
$= \frac{(x-1)^2 - 1}{(x-1)^2} = 1 - \frac{1}{(x-1)^2}$

$$y'' = 2(x-1)^{-3} \quad \text{or} \quad \frac{2}{(x-1)^3} \qquad x \neq 1.$$

$x > 1$, $y'' > 0$ convex downwards
$x < 1$, $y'' < 0$ convex upwards

But $f(1)$ does not exist $\Rightarrow$ There is no point of inflection on the curve.

ASYMPTOTES



Def: A straight line A is called an asymptote to a curve, if the perpendicular distance δ from the variable point M of the curve to this line approaches to zero as the point M reaches to infinity

OR

Tangent to the curve at infinity.

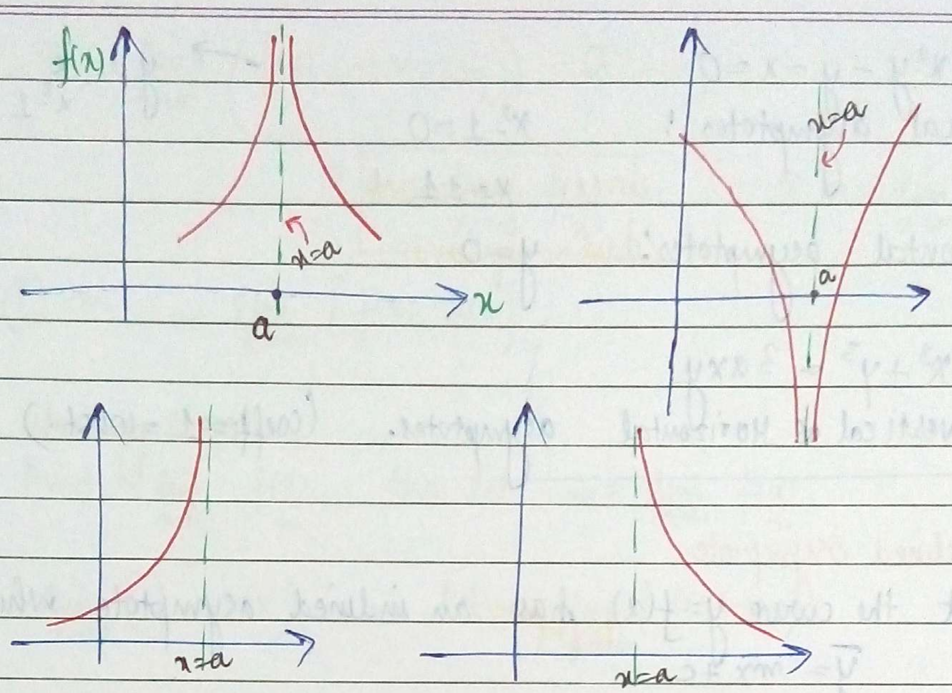
Asymptotes →
- { Vertical asymptote (|| to Y-axis)
- { Horizontal asymptote (|| to x-axis)
- Inclined asymptote

1. **Vertical asymptote:**

If $\lim\limits_{x \to a^+} f(x) = \pm\infty$ or $\lim\limits_{x \to a^-} f(x) = \pm\infty$

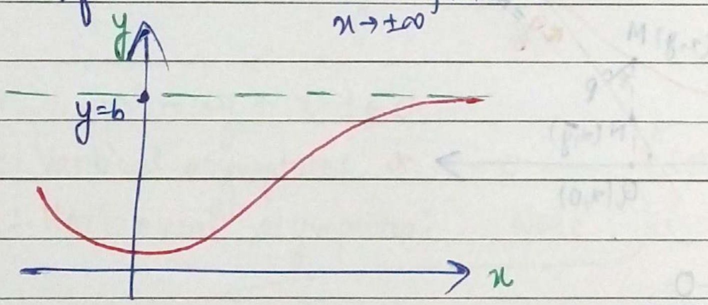
or $\lim\limits_{x \to a} f(x) = \pm\infty$

Then the straight line $x = a$ is an asymptote to the curve $y = f(x)$.



2. **Horizontal asymptote:**

The line $y=b$ is a horizontal asymptote for the function $y=f(x)$ if $\lim_{x\to\pm\infty} f(x) = b$



WORKING TIPS $(f(x,y)=0) \rightarrow$ Algebraic Curves.

- Vertical asymptotes:
  The are obtained by equating to zero the coefficients of the highest power of y in $f(x,y)=0$.

- Horizontal asymptotes:
  The are obtained by equating to the zero the coefficient of the highest power of x in $f(x,y)=0$.

- No vertical or horizontal asymptotes:
  If the coeff. of highest powers of x & y in $f(x,y)=0$ are constants.

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Ex. $x^2y - y - x = 0 \longrightarrow y = \frac{x}{x^2-1}$

1. Vertical asymptotes: $x^2 - 1 = 0$

$x = \pm 1$

2. Horizontal asymptotes: $y = 0$

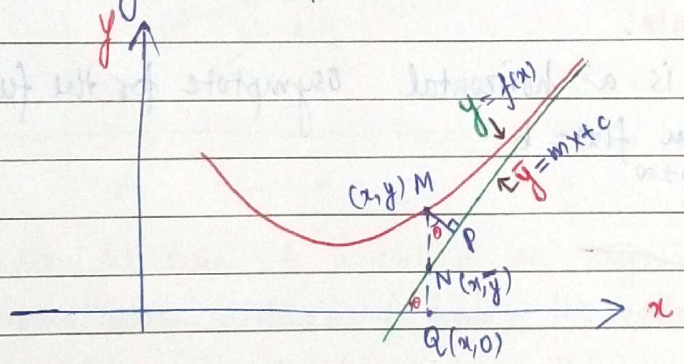
Ex. $x^3 + y^3 = 3axy$

No vertical or Horizontal asymptotes. (coeff. = 1 = const.)

3. Inclined asymptote

Let the curve $y = f(x)$ has an inclined asymptote whose equation is $\bar{y} = mx + c$

Let $M(x, y)$ be a point on the curve and $N(x, \bar{y})$ a point on asymptote



Given $\lim_{x \to \infty} MP = 0$

From ΔMNP

$\cos\theta = \frac{MP}{NM} \Rightarrow NM = \frac{MP}{\cos\theta}, \quad \theta \neq \pi/2$

$\Rightarrow \lim_{x \to \infty} NM = 0$

$NM = |QM - QN| = |y - \bar{y}|$

$NM = |f(x) - (mx + c)|$

$\lim_{x \to \infty} NM = 0 \Rightarrow \boxed{\lim_{x \to \infty} (f(x) - mx - c) = 0}$ — 1

$\Rightarrow \lim_{x \to \infty} x\left(\frac{f(x)}{x} - m - \frac{c}{x}\right) = 0$

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$$\Rightarrow \lim_{x\to\infty}\left(\frac{f(x)}{x} - m - \frac{c}{x}\right) = 0$$

$$\Rightarrow \boxed{m = \lim_{x\to\infty}\frac{f(x)}{x}} \Rightarrow m = \lim_{x\to\infty} f'(x)$$

$$① \Rightarrow \quad c = \lim_{x\to\infty}\left[f(x) - mx\right]$$

Working steps:

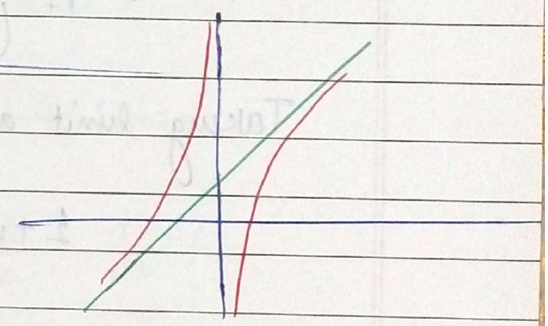
1. Find $\lim_{x\to\pm\infty}\frac{f(x)}{x}$ then let $m = \lim_{x\to\pm\infty}\frac{f(x)}{x}$

2. $\lim_{x\to\pm\infty}\left(f(x) - mx\right)$ let $c = \lim_{x\to\pm\infty}\left(f(x) - mx\right)$

3. $y = mx + c$

Ex. Find the asymptotes of the curve

$$y = \frac{x^2 + 2x - 1}{x}$$



$\longrightarrow x^2 - xy + 2x - 1 = 0$

1. Vertical asymptotes: $x = 0$ (coeff of highest of y is x)
2. Horizontal asymptotes: None. $\left(\lim_{x\to\infty} \longrightarrow x \text{ is const}\right)$

$\longrightarrow$ Also, $\lim_{x\to 0^+}\frac{x^2 + 2x - 1}{x} = -\infty$

$$\lim_{x\to 0^-}\frac{x^2 + 2x - 1}{x} = \infty$$

3. Inclined asymptotes:

$$\lim_{x\to\pm\infty} \frac{y}{x} = \lim_{x\to\pm\infty}\left(\frac{x^2 + 2x - 1}{x^2}\right) = \lim_{x\to\pm\infty}\left(1 + \frac{2}{x} - \frac{1}{x^2}\right) = 1$$

Hence, $m = 1$.

Now, $\lim_{x\to\pm\infty}(y - mx) = \lim_{x\to\pm\infty}(y - x) = \lim_{x\to\pm\infty}\left(2 - \frac{1}{x}\right) = 2$

Hence, $c = 2$.

$\Rightarrow$ Straight line $y = x + 2$ is an inclined asymptote.

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Ex. Find the asymptotes of the curve $x^3+y^3-3axy=0$

No vertical or horizontal asymptotes (coeff of x^3 & y^3 = const)

We want to find $m=\lim_{x\to\pm\infty}\frac{y}{x}$

& $c=\lim_{x\to\pm\infty}(y-mx)$

then $y=mx+c$ is the asymptote.

Rewriting the given eqn as:

$$1+\left(\frac{y}{x}\right)^3-3a\cdot\frac{1}{x}\left(\frac{y}{x}\right)=0$$

Taking limit as $x\to\infty$ and setting $m=\lim_{x\to\infty}\frac{y}{x}$, we get

$$1+m^3=0 \Rightarrow (m+1)(m^2-m+1)=0$$

$\Rightarrow \boxed{m=-1}$ Since m^2-m+1 has no real roots.

Now $c=\lim_{x\to\infty}(y+x)$

Let us take $y+x=p$ then $c=\lim_{x\to\infty}p$.

Subst. $y=p-x$ in the equation.

$$x^3+(p-x)^3-3ax(p-x)=0$$

$$\cancel{x^3}+p^3-\cancel{x^3}+3px^2-3p^2x-3apx+3ax^2=0$$

$$\Rightarrow 3ax^2+3px^2-3xp(p+a)+p^3=0$$

Dividing by x^2:

$$3(a+p)-\frac{3p}{x}(p+a)+\frac{p^3}{x^2}=0$$

Taking lim $x\to\infty$, we get $p=-a$ $\Rightarrow$ $\boxed{y=-x-a}$ is the only asymptote.

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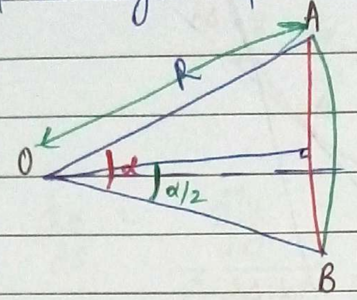
CURVATURE

→ Ratio of the length of arc of a circle to the length of its chord.



$$\lim_{\alpha \to 0} \frac{\text{arc } AB}{\text{cord } AB}$$

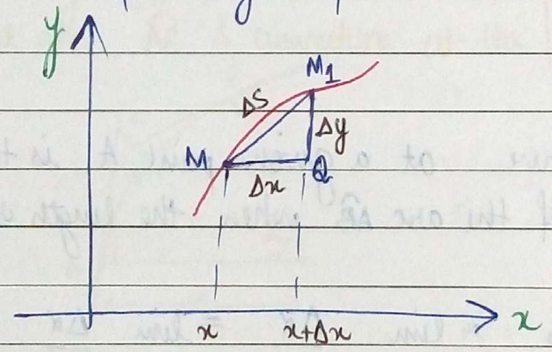
$AP = R\sin\alpha/2$

$\overline{AB} = 2R\sin\alpha/2$

$\widehat{AB} = \alpha R$

$$\lim_{\alpha \to 0} \frac{\widehat{AB}}{\overline{AB}} = \lim_{\alpha \to 0} \frac{\alpha R}{2R\sin\alpha/2} = \lim_{\alpha \to 0} \frac{(\alpha/2)}{\sin(\alpha/2)} = 1$$

→ Rate of change of the arc with respect to abscissa.



$$\lim_{M \to M_1} \frac{\Delta s}{\Delta x} = \frac{ds}{dx} = ?$$

Derivative of s wrt x?

ΔMM_1Q

$$\overline{MM_1}^2 = \Delta x^2 + \Delta y^2$$

$$\Rightarrow \frac{\overline{MM_1}^2}{(\Delta s)^2} (\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\lim_{\overline{MM_1} \to 0} \left(\frac{\overline{MM_1}}{\Delta s}\right)^2 \cdot \left(\frac{\Delta s}{\Delta x}\right)^2 = \lim_{\overline{MM_1} \to 0} \left(1 + \frac{\Delta y^2}{\Delta x^2}\right)$$

$$1 \cdot \lim_{\overline{MM_1} \to 0} \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\boxed{\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

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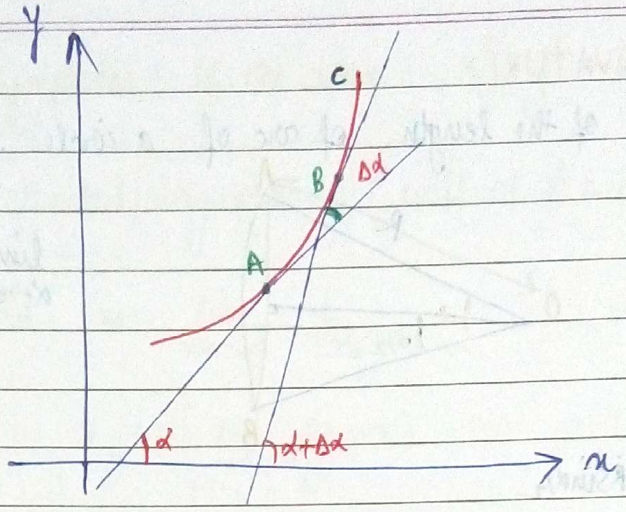
CURVATURE:



1. **Angle of contigence ($\Delta\alpha$)** of the arc AB of a curve C is the angle between the tangents at A and B to the curve C.

2. The average curvature (K_{av}) of an arc $\widehat{AB}$ is the ratio of the corresponding angle of contigence $\Delta\alpha$ to the length of the arc.

$$\boxed{K_{av} = \frac{\Delta\alpha}{\widehat{AB}}}$$

3. The **curvature K** of a curve at a given point A is the limit of the average curvature of the arc $\widehat{AB}$ when the length of the arc approaches to zero.

$$K = \lim_{B\to A} K_{av} = \lim_{B\to A} \frac{\Delta\alpha}{\widehat{AB}} = \lim_{B\to A} \frac{\Delta\alpha}{\Delta S}$$

$$= \lim_{\Delta S\to 0} \frac{\Delta\alpha}{\Delta S} = \frac{d\alpha}{dS}$$

$$\Rightarrow \boxed{K = \left|\frac{d\alpha}{dS}\right|}$$

Calculation of curvature :-

$$y = f(x)$$

$$\frac{d\alpha}{dx} = \frac{d\alpha}{dS} \cdot \frac{dS}{dx} \Rightarrow \frac{d\alpha}{dS} = \frac{\left(\frac{d\alpha}{dx}\right)}{\left(\frac{dS}{dx}\right)}$$

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$$\tan\alpha = \frac{dy}{dx} \Rightarrow \alpha = \tan^{-1}\left(\frac{dy}{dx}\right)$$

$$\frac{d\alpha}{dx} = \frac{1}{\left[1+\left(\frac{dy}{dx}\right)^2\right]} \cdot \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d\alpha}{ds} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\boxed{K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}}$$

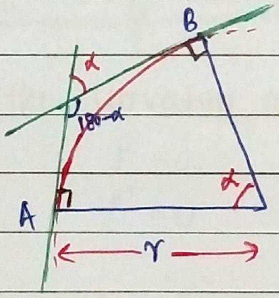
→ $y = f(x)$

→ Cartesian form

$x = f(y)$

$$K = \frac{\frac{d^2x}{dy^2}}{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}$$

Ex. For a given circle of radius r, determine the average curvature of the arc $\widehat{AB}$ & curvature at the point A.



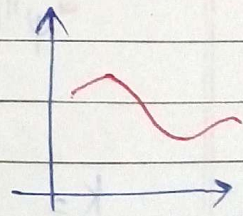
$\widehat{AB} = \alpha r$

$$K_{av} = \frac{\alpha}{\alpha r} = \frac{1}{r}$$

$$K_A = \lim_{B\to A} \frac{1}{r} = \frac{1}{r}$$

PARAMETRIC FORM:

$x = \varphi(t)$ $y = \psi(t)$



$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\psi'(t)}{\varphi'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \cdot \frac{dt}{dx}$$

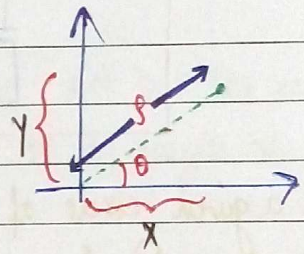
$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\,\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3} = \frac{\left[\varphi'(t)\,\psi''(t) - \psi'(t)\,\varphi''(t)\right]}{[\varphi'(t)]^3}$$

$$K = \frac{\left|\dfrac{\varphi'\psi'' - \psi'\varphi''}{\varphi'^3}\right|}{\left(\sqrt{1+\left(\dfrac{\psi'}{\varphi'}\right)^2}\right)^3}$$

$$\boxed{K = \frac{|\varphi'\psi'' - \psi'\varphi''|}{\left[\varphi'^2 + \psi'^2\right]^{3/2}}}$$

POLAR FORM:

$$\rho = f(\theta)$$



$$x = \rho\cos\theta = f(\theta)\cos\theta = \varphi(\theta)$$
$$y = \rho\sin\theta = f(\theta)\sin\theta = \psi(\theta)$$

$$\varphi'(\theta) = \frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta$$

$$\psi'(\theta) = \frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta$$

$$\varphi''(\theta) = \frac{d^2f}{d\theta^2}\cos\theta - 2\frac{df}{d\theta}\sin\theta - f(\theta)\cos\theta$$

$$\psi''(\theta) = \frac{d^2f}{d\theta^2}\sin\theta + 2\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta$$

$$K = \frac{|\psi''\varphi' - \psi'\varphi''|}{\left[\varphi'^2 + \psi'^2\right]^{3/2}}$$

$$\boxed{K = \frac{\left|2(\rho')^2 - (\rho'')\rho + \rho^2\right|}{\left[(\rho')^2 + \rho^2\right]^{3/2}}}$$

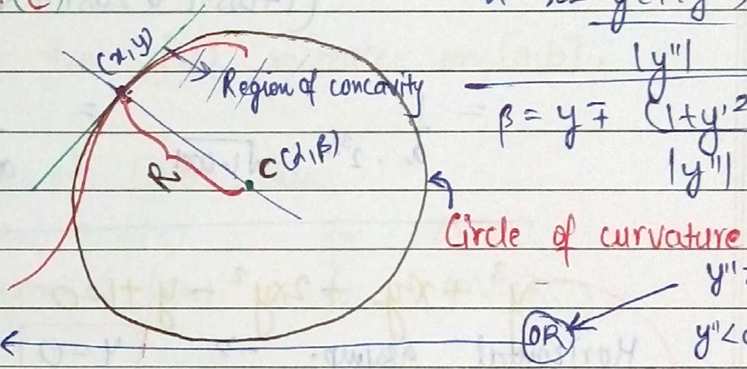
Radius of Curvature (R) :

$$\boxed{R = \frac{1}{K}}$$

$$\boxed{R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1+y'^2\right]^{3/2}}{|y''|}}$$

Centre of Curvature : (C)

$$R = \frac{1}{K}$$



Co-ordinates of centre:-

$$\alpha = x \pm \frac{y'(1+y'^2)}{|y''|}$$

$$\beta = y \mp \frac{(1+y'^2)}{|y''|}$$

$y''>0$ (lower signs)

$y''<0$ (upper signs)

OR

$$\boxed{\alpha = x - \frac{y'(1+y'^2)}{y''}}$$

$$\boxed{\beta = y + \frac{(1+y'^2)}{y''}}$$

Ex. Find the curvature of $\rho = a\theta$ $(a>0)$.

$\rho' = a$

$\rho'' = 0$

$$K = \frac{\left|2(a)^2 + 0\cdot a\theta + (a\theta)^2\right|}{\left[a^2 + a^2\theta^2\right]^{3/2}} = \frac{a^2}{a^3}\,\frac{|2+\theta^2|}{(1+\theta^2)^{3/2}}$$

$$\boxed{K = \frac{1}{a}\left\{\frac{|2+\theta^2|}{(\theta^2+1)^{3/2}}\right\}}$$

Ex. Curvature of the cycloid

$x = a(t - \sin t) = \phi$

$y = a(1-\cos t) = \psi$

$\phi' = a(1-\cos t)$ $\qquad$ $\psi' = a(0+\sin t)$

$\phi'' = a\sin t$ $\qquad$ $\psi'' = a\cos t$

$$K = \frac{|\varphi'\psi'' - \varphi''\psi'|}{[\varphi'^2 + \psi'^2]^{3/2}}$$

$$K = \frac{|a(1-\cos t)\cdot a\cos t - a\sin t \cdot a\sin t|}{(a^2(1-\cos t)^2 + a^2\sin^2 t)^{3/2}}$$

$$= \frac{1}{a}\,\frac{|\cos t - \cos^2 t - \sin^2 t|}{(1+\cos^2 t - 2\cos t + \sin^2 t)^{3/2}} = \frac{1}{a}\,\frac{|\cos t - 1|}{[2(1-\cos t)]^{3/2}}$$

$$= \frac{1}{a\cdot 2^{3/2}\sqrt{1-\cos t}} = \frac{1}{a\cdot 2^{3/2}\cdot 2^{1/2}|\sin t/2|} = \boxed{\frac{1}{4|a||\sin t/2|}} = K$$

Ex. $y^3 + x^2y + 2xy^2 - y + 1 = 0$

Horizontal asymp. $\rightarrow$ $\boxed{y=0}$

$$\left(\frac{y}{x}\right)^3 + \left(\frac{y}{x}\right) + 2\left(\frac{y}{x}\right)^2 - \frac{y}{x^3} + \frac{1}{x^3} = 0$$

$x \to \infty$

$$m^3 + m + 2m^2 = 0$$

$$m = 0, -1, -1$$

$$c = \lim_{x\to\infty} (y - mx)$$

$$c = \lim_{x\to\infty} (y + x)$$

We take $p = y + x$ then $c = \lim_{x\to\infty} p$.

$$\Rightarrow (p-x)^3 + x^2(p-x) + 2x(p-x) - (p-x) + 1 = 0$$

$$p^3 - p^2x - p + x + 1 = 0$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{p^3}{x} - p^2 - \frac{p}{x} + 1 + \frac{1}{x}\right) = 0 \qquad \Rightarrow \lim_{x\to\infty}(-p^2 + 1) = 0$$

$$\lim_{x\to\infty} p = \pm 1$$

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$$\boxed{c = \pm 1}$$

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$\Rightarrow$ Asymptotes : $\boxed{\begin{array}{l} y = -\cancel{x} + 1 \\ y = -x - 1 \end{array}}$

INCREASE and DECREASE OF A FUNCTION (Application of MVT)

A function f is increasing (strictly) on interval I if

$$f(b) > f(a) \quad \forall\ b > a \text{ in } I$$

Th. If f increases on an interval $[a,b]$ then $f'(x) \geq 0$ on this interval.

If $f'(x) > 0$ on (a,b) then $f(x)$ increases on $[a,b]$.

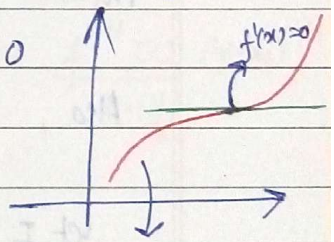
Proof: I. Since f increases on $[a,b]$, then

$$f(x+\Delta x) > f(x) \quad \text{for} \quad \Delta x > 0$$

$$\& \quad f(x+\Delta x) < f(x) \quad \text{for} \quad \Delta x < 0$$

$\Rightarrow \dfrac{f(x+\Delta x) - f(x)}{\Delta x} > 0$ in both cases for any sufficiently small Δx.

$\Rightarrow \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} \geq 0 \quad \Rightarrow f'(x) \geq 0$



rising function.

II. Consider two values x_1 & x_2 such that $x_1 < x_2$

By LMVT

$$f(x_2) - f(x_1) = f'(c)\,(x_2 - x_1), \quad x_1 < c < x_2$$

It is given that $f'(c) > 0$

$\Rightarrow f(x_2) - f(x_1) > 0$

$\Rightarrow f(x)$ is an increasing function.

Q Determine the intervals of monotonicity of the function

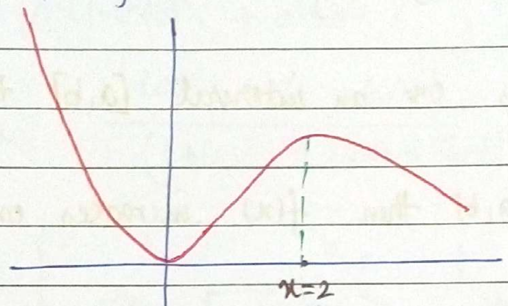
$$f(x) = x^2 e^{-x}$$

$$f'(x) = 2x \cdot e^{-x} - x^2 e^{-x} = x e^{-x}(2-x)$$

If $0 < x < 2$; $f'(x) > 0 \Rightarrow$ function is ↑ in $(0,2)$

If $x < 0$; $f'(x) < 0 \Rightarrow$ ——— ↓ ——— $(-\infty, 0)$

If $x > 2$; $f'(x) < 0 \Rightarrow$ ——— ↓ ——— $(2, \infty)$



LAGRANGES FORM OF REMAINDER (Derivation)

Recall: Taylor's Polynomial.

$$P_n(x) = f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots + \frac{(x-x_0)^n}{n!} f^n(x_0)$$

Also, $R_n(x) = -P_n(x) + f(x)$

Let I be an open interval & let n be a non negative integer and suppose that the function $f: I \to R$ has $(n+1)$ derivatives. Suppose that at the point x_0 in I,

$$f^{(k)}(x_0) = 0 \quad \text{for} \quad 0 \le k \le n$$

Then for each point $x \ne x_0$ in I, there is a point c b/w x & x_0 at which

$$f(x) = \frac{f^{n+1}(c)}{(n+1)!} (x-x_0)^n$$

PROOF:

Define $g(x) = (x-x_0)^{n+1}$ $\quad \forall x \in I$

$\Rightarrow g^k(x_0) = 0$ $\quad 1 \le k \le n$ $\quad$ & $\quad g^{n+1}(x_0) = (n+1)!$

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Let x be a point in I and suppose $x > x_0$. By Cauchy MVT to the functions $f(x)$ & $g(x)$ in $[x_0, x]$, we get

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(x_1)}{g'(x_1)} \Rightarrow \frac{f(x)}{g(x)} = \frac{f'(x_1)}{g'(x_1)} \; ; \; x_0 < x_1 < x \quad -(1)$$

Applying CMVT for f' & g'

$$\frac{f'(x_1)}{g'(x_1)} = \frac{f''(x_2)}{g''(x_2)} \; ; \; x_0 < x_2 < x_1 \quad -(2)$$

From (1) & (2)

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{f''(x_2)}{g''(x_2)}$$

Continuing, we get

$$\frac{f(x)}{g(x)} = \frac{f^{n+1}(x_{n+1})}{g^{n+1}(x_{n+1})} \; ; \; x_0 < x_{n+1} < x_n < x$$

Setting $x_{n+1} = c$

$$\frac{f(x)}{g(x)} = \frac{f^{n+1}(c)}{(n+1)!} \Rightarrow f(x) = \frac{f^{n+1}(c)}{(n+1)!} \cdot (x-x_0)^{n+1}$$

Coming back to $R_n(x) = f(x) - P_n(x)$

Note that

$$R_n(x_0) = R_n'(x_0) = \cdots = R_n^{(n)}(x_0) = 0$$

With the above intermediate result:

$$R_n(x) = \frac{R_n^{n+1}(c)}{(n+1)!}(x-x_0)^{n+1} \qquad x_0 < c < x$$

Also note that

$$R_n^{(n+1)}(x) = f^{(n+1)}(x) - \cancel{P_n^{(n+1)}(x)} \to 0 = f^{(n+1)}(x)$$

$$\therefore \boxed{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}$$

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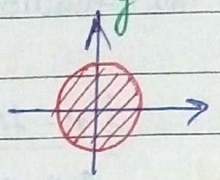
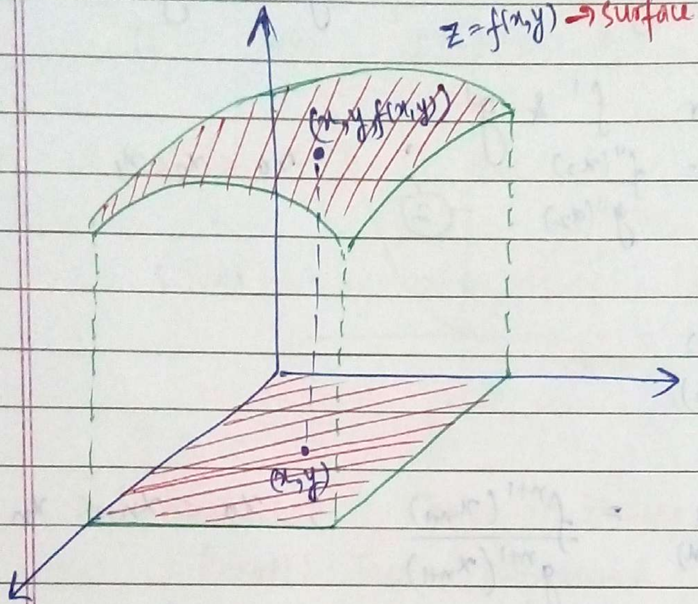
Eg. $Z = \sqrt{1-x^2-y^2}$

Domain: $1-x^2-y^2 \geq 0$
$x^2+y^2 \leq 1$





DEF.

* Distance b/w the 2 points:

$P(x_0, y_0)$ - - - $Q(x_1, y_1)$ $\qquad |PQ| = \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}$

* Neighbourhood of a point:

Consider a point $P(x_0, y_0)$

δ - Neighbourhood of P ($N_\delta(P)$ or $N(P, \delta)$)

$$= \{(x,y) : \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta\}$$

OR

$N_\delta(P) = (x,y) : \{x_0 - \delta \underset{(<)}{\leq} x \underset{(<)}{\leq} x_0 + \delta \;,\; y_0 - \delta \underset{(<)}{\leq} y \underset{(<)}{\leq} y_0 + \delta\}$

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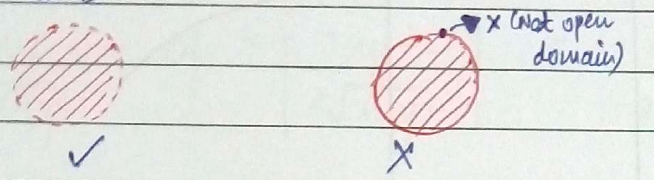
* Open Domain:

A domain D is open, if there exist a number $\delta > 0$ corresponding to every point p in D such that all the points in δ-neighbourhood of p are in D.

Example:



* Bounded Domain:

D is bounded if $\exists$ a number (finite & +ve) M such that D can be enclosed within a circle with radius M & centre at origin.

* Closed Region:

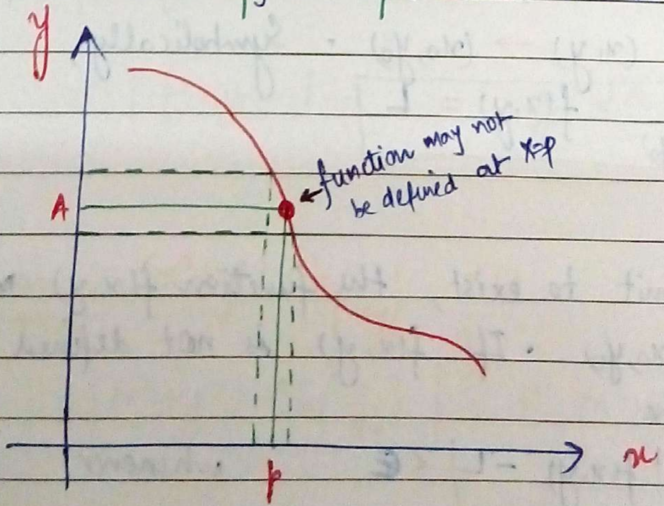
A closed region is a bounded domain with its boundary.

* Bounded functions:

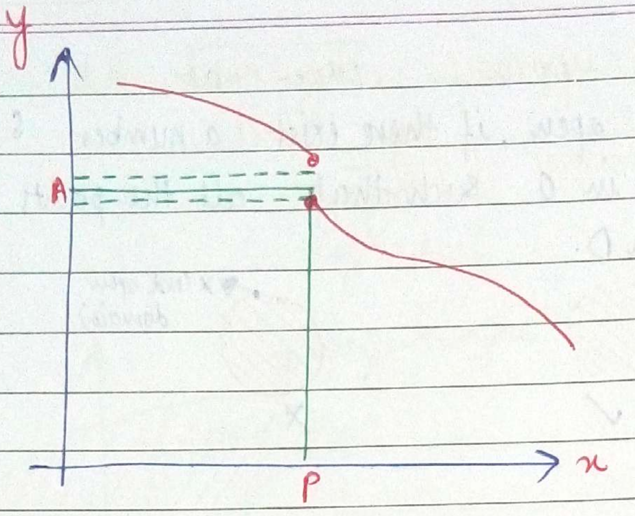
A function $f(x,y)$ defined in same domain D in R^2 is bounded, if there exist a real no. M (finite), such that $|f(x,y)| \leq M$ for all $(x,y) \in D$.

ONE VARIABLE (Recall)

$\lim_{x \to p} f(x) = A$ means that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - A| < \epsilon$ whenever $0 < |x-p| < \delta$



$\lim_{x \to p} f(x) = A$



For a given ϵ, δ does not exist

$\Rightarrow \lim_{x \to p} f(x) \neq A$

$\lim_{x \to p} f(x) = A$ means that every neighbourhood $N_\epsilon(A)$ of A

LIMITS (TWO VARIABLES)

Let $z = f(x,y)$ be a function of two variables defined in a domain D. Let $P(x_0, y_0)$ be a point in D. If for a given real no. $\epsilon > 0$, however small, we can find a real no. $\delta > 0$ such that for every point (x,y) in the δ-neighbourhood of $P(x_0, y_0)$

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

then the real no. L is called the limit of the function $f(x,y)$ as $(x,y) \to (x_0, y_0)$. Symbolically,

$$\boxed{\lim_{(x,y) \to (x_0, y_0)} f(x,y) = L}$$

REMARK:

Note that for limit to exist, the function $f(x,y)$ may or may not be defined at (x_0, y_0). If $f(x,y)$ is not defined at $P(x_0, y_0)$ then we write

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

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EXAMPLE: Using δ-ϵ approach, show that

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{x^2+y^2}}\right) = 0$$

for $(x,y) \neq (0,0)$

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{2xy}{2\sqrt{x^2+y^2}}\right| \leq \frac{x^2+y^2}{2\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2}$$

$< \frac{1}{2}\delta < \epsilon$

as $(x-y)^2 \geq 0$

If we choose $\delta < 2\epsilon$

then $\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

Hence $\boxed{\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0}$

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ϵ-δ Def.

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

given ϵ

$\exists\ \delta$ $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

$\Rightarrow |f(x,y) - L| < \epsilon$

$|f(x,y) - L| \leq \dots\ \underbrace{g\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)}_{g(\delta) < \epsilon} < \epsilon,$

Ex. $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{xy}\right) = 0$

$\sqrt{x^2+y^2} < \delta$

$$\left|(x^2+y^2)\sin\left(\frac{1}{xy}\right)\right| \leq (x^2+y^2) < \delta^2 < \epsilon$$

$\delta^2 < \epsilon$

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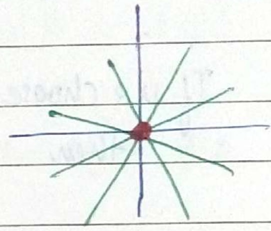
REMARK: Since $(x,y) \to (x_0,y_0)$ in the two dimensional plane, there are ∞ no. of paths joining (x,y) to (x_0,y_0). Since the limit, if exists, is unique, the limit should be the same along all the paths. Thus, the limit cannot be approached by point P along a particular path and finding the limit of $f(x,y)$. If the limit is dependent on a path, then the limit does not exist.

Ex. $$\lim_{(x,y)\to(0,0)} \left(\frac{x^2y}{x^4+y^2}\right)$$



Path: $y = mx$:

$$\lim_{x\to 0} \frac{x^2 \cdot mx}{x^4+m^2x^2}$$

$$= \lim_{x\to 0} \frac{mx}{m^2+x^2} = 0$$

Path: $y = mx^2$:

$$\lim_{x\to 0} \frac{x^2 \cdot mx^2}{x^4+m^2x^4} = \lim_{x\to 0} \frac{m}{1+m^2}$$ } path dependent

For different value of m, we get diff. limits.

$\Rightarrow$ LIMIT DOES NOT EXIST!

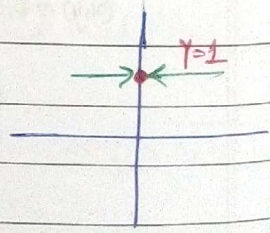
Ex. Show that $$\left[\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}\right]$$ does not exist.

Path: $y = mx$!

$$\lim_{x\to 0} \frac{x \cdot mx}{x^2+m^2x^2} = \lim_{x\to 0} \frac{m}{1+m^2}$$ = Path dependent.

$\Rightarrow$ Limit does not exist.

Ex. Show that $$\left[\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right)\right]$$ does not exist.



$y = 1$. LHL: $$\lim_{x\to 0^-} \tan^{-1}\left(\frac{1}{x}\right) = -\pi/2$$

RHL: $$\lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right) = +\pi/2$$

LHL $\neq$ RHL $\Rightarrow$ limit does not exist.

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* Try to have same degree in denominator while choosing path.

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WORKING WITH LIMITS: Suppose, we have

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$

then

(i) $\lim_{(x,y)\to(x_0,y_0)} [k\,f(x,y)] = kL_1$ for any real constant k.

(ii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L_1 \pm L_2$

(iii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\, g(x,y)] = L_1 . L_2$

(iv) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)/g(x,y)] = \frac{L_1}{L_2}$, provided $L_2 \neq 0$

CONTINUITY:

* A function $z = f(x,y)$ is said to be cont. at a point (x_0,y_0) i

(i) $f(x,y)$ is defined at (x_0,y_0)

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

* A function $f(x,y)$ is said to be cont. at (x_0,y_0) if for a given $\epsilon > 0$ there exist a real no. $\delta > 0$ such that $|f(x,y) - f(x_0,y_0)| < \epsilon$ whenever $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

* If $f(x_0,y_0)$ is defined and $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ exists but $f(x_0,y_0) \neq L$,

then the point (x_0,y_0) is called a point of removable discontinuity.

Convert to polar co-ordinates.
Calculating limit becomes easy.

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Ex. Show that the following functions are continuous:-

(i) $f(x,y) = \begin{cases} \dfrac{2x^4+3y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Change the co-ordinate system.

$x = r\cos\theta \qquad y = r\sin\theta$

$$\left|\frac{2x^4+3y^4}{x^2+y^2} - 0\right| = \left|\frac{2r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2(\cos^2\theta+\sin^2\theta)}\right| = 2r^2\cos^4\theta + 3r^2\sin^4\theta$$

$$< 5r^2 < 5\delta^2 < \epsilon$$

$$\sqrt{x^2+y^2} < \delta$$

$$\boxed{\delta^2 < \frac{\epsilon}{5}}$$

OR

$$\lim_{(x,y)\to(0,0)} \frac{2x^4+3y^4}{x^2+y^2} = \lim_{r\to 0} \frac{2r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2}$$

$$= \lim_{r\to 0} (2r^2\cos^4\theta + 3r^2\sin^4\theta) = 0$$

(ii) $f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & (x,y) \neq (0,0) \\ 1/2 & (x,y) = (0,0) \end{cases}$

Substitute $x+2y = t$

$$= \lim_{t\to 0} \frac{\sin^{-1} t}{\tan^{-1} 2t} = \lim_{t\to 0} \left(\frac{\sin^{-1} t}{t}\right) \cdot \frac{2t}{(\tan^{-1} 2t)} \cdot \frac{1}{2} = 1/2.$$

Ex. $f(x,y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Path: $y = mx$

$$= \lim_{(x,y)\to(0,0)} \frac{(x-mx)^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{1+m^2-2m}{1+m^2} = \lim_{x\to 0} 1 - \frac{2m}{1+m^2}$$

$\Rightarrow$ Limit does not exist. $\Rightarrow$ Path dependent.

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Ex. $f(x,y) = \begin{cases} \dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Way 1: Subst. $\sqrt{x^2+y^2} = t$

$\lim_{t\to 0} \frac{\sin t}{t} = 1 \neq 0$ $\Rightarrow$ Limit does not exist $\Rightarrow$ Not cont.

Way 2: $x = r\cos\theta$, $y = r\sin\theta$

Ex. $f(x,y) = \begin{cases} \dfrac{e^{xy}}{x^2+1} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$\lim_{(x,y)\to(0,0)} \left(\frac{e^{xy}}{x^2+1}\right) = \frac{1}{0+1} = 1$ $\Rightarrow$ Lt. does not exist $\Rightarrow$ Not cont.

Ex. $f(x,y) = \begin{cases} \dfrac{x^4y^4}{(x^2+y^4)^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Path: $y^2 = mx$

$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3} = \lim_{x\to 0} \frac{x^4 \cdot m^2x^2}{(x^2+m^2x^2)^3} = \lim_{x\to 0} \frac{m^2}{(1+m^2)^3}$

$\Rightarrow$ Path dependent

$\Rightarrow$ Lt. doesnot exist $\Rightarrow$ Not cont.

Ex. $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{\tan xy} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Path: $y = mx$

Lt. $= \lim_{x\to 0} \left\{ \left[\frac{x^2(1+m^2)}{x^2}\right] \cdot \left[\frac{mx^2}{\tan(mx^2)}\right] \cdot \frac{1}{m} \right\} = \frac{1+m^2}{m}$ $\Rightarrow$ Path dependent

$\Rightarrow$ Lt. does not exist

$\Rightarrow$ Not cont.

(Product Rule of limits should not be applied here $\because$ one of Lt. does not exist).

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Ex. $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^{\cancel{3}}\cos^3\theta}{\cancel{r^2}} = 0$

$x = r\cos\theta \quad y = r\sin\theta$

Ex. $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$ ⇒ limit does not exist

Ex. $\lim_{(x,y)\to(0,0)} \frac{x^2y}{(x+y)\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r(\cos\theta+\sin\theta)\cdot r} = \lim_{r\to 0} r\left(\frac{\cos^2\theta\sin\theta}{\boxed{\cos\theta+\sin\theta}}\right)$

↳ unbounded

⇒ cannot say limit = 0

Path: $y = -x$ → limit does not exist.

Ex. $\lim_{(x,y)\to(0,0)} \frac{x^2y}{(|x|+|y|)\sqrt{x^2+y^2}} = \lim_{r\to 0} r\left(\frac{\cos^2\theta\sin\theta}{|\cos\theta|+|\sin\theta|}\right) = 0.$

PARTIAL DERIVATIVES

$Z = f(x,y)$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0)$$

$$= \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

$$= \left.\frac{d}{dx} f(x,y_0)\right|_{x=x_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{\Delta y\to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

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Ex. Find the value of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ at the point (x,y) of the following functions.

(i) $f(x,y) = ye^{-x}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{ye^{-(x+\Delta x)} - ye^{-x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{ye^{-x}(e^{-\Delta x} - 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{ye^{-x}\left(\cancel{1} - \Delta x + \frac{(\Delta x)^2}{2!} - \dots \cancel{-1}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} ye^{-x}\left(-1 + \frac{\Delta x}{2!} - \dots\right) = \boxed{-ye^{-x}}$$

$$\boxed{\frac{\partial f}{\partial y} = e^{-x}}$$

(ii) $f(x,y) = \sin(2x+3y)$

$f_x(x,y) = \cos(2x+3y) \cdot 2$

$f_y(x,y) = \cos(2x+3y) \cdot 3$

REMARK.

★ Existence of partial derivative ~~ma~~ do not imply continuity. Continuity & existence of partial derivatives are two diff. things. One does not imply other.

Ex. Show that ~~for~~ $f(x,y) = \begin{cases} (x+y)\sin\frac{1}{(x+y)} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$

is continuous at $(0,0)$ but it's partial derivative do not exist at $(0,0)$.

$$\lim_{(x,y)\to(0,0)} (x+y) \sin\frac{1}{(x+y)}$$

$x+y = t$

$$= \lim_{t \to 0} t \sin\left(\frac{1}{t}\right) = 0.$$

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$$\lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \sin\left(\frac{1}{\Delta x}\right) - 0}{\Delta x}$$

$$= \underbrace{\lim_{\Delta x\to 0} \sin\left(\frac{1}{\Delta x}\right)}_{\text{Limit does not exist.}}$$

Ex. Show that the function $f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

is not cont. at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0,0)$.

Path: $y = mx$.

$$\lim_{x\to 0} \frac{x(mx)}{x^2+2m^2x^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1+2m^2)} = \frac{m}{1+2m^2} = \text{dependent on } m$$

$\Rightarrow$ limit does not exist.

$\Rightarrow$ $f(x,y) \to$ Not continuous at $(0,0)$.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{\Delta x\to 0} \frac{\left[\frac{(x+\Delta x)\,y}{(x+\Delta x)^2+2(y)^2}\right] - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x \cdot 0 - 0}{\Delta x} = 0 \\ \frac{\partial f}{\partial y} &= \lim_{\Delta y\to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0-0}{\Delta y} = 0. \end{aligned}\right\} \text{Partial derivatives exist}$$

Th. Sufficient condition for continuity at (x_0,y_0) :-

One of the first order partial derivative exist and is bounded in the nbd (neighbourhood) of (x_0,y_0) and the other exists at (x_0,y_0).

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Partial derivatives of higher order :-

$$\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}_{f_{xx}} \rightarrow \frac{\partial^2 f}{\partial x^2} \qquad \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}_{f_{yx}} \rightarrow \frac{\partial^2 f}{\partial y\,\partial x} \qquad \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)}_{f_{yy}} \rightarrow \frac{\partial^2 f}{\partial y^2}$$

* If the mixed derivatives f_{xy} & f_{yx} are continuous in an open domain D, then at any point $(x,y) \in D$ $\boxed{f_{yx} = f_{xy}}$

Ex. Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

- $f_x(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0-0}{\Delta x} = 0$

- $f_y(0,0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{0-0}{\Delta y} = 0$

- $f_y(\Delta x, 0) = \lim\limits_{\Delta y \to 0} \dfrac{f(\Delta x, \Delta y) - f(\Delta x, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{\Delta x \cdot \Delta y^3}{(\Delta x + \Delta y^2)\cdot \Delta y}$

$$= \lim_{\Delta y \to 0} \frac{\Delta x \cdot \Delta y^2}{\Delta x + \Delta y^2} = 0$$

- $f_x(0, \Delta y) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, \Delta y) - f(0, \Delta y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta x \cdot \Delta y^3}{(\Delta x + \Delta y^2)\cdot \Delta x} = \Delta y$

$$\boxed{f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

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- $f_{yx}(0,0) = \lim_{\Delta y \to 0} \dfrac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \dfrac{\Delta y - 0}{\Delta y} = 1.$

DIFFERENTIABLITY

ONE VARIABLE:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Let $\varepsilon(\Delta x)$ denote the difference

$$\varepsilon(\Delta x) = \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} - f'(x)\right]$$

$$\lim_{\Delta x \to 0} \varepsilon(\Delta x) = 0$$

This implies

$$f(x+\Delta x) = f(x) + f'(x)\cdot\Delta x + \Delta x \cdot \varepsilon$$

Def. The function $y = f(x)$ is said to be differentiable at the point x if at this point

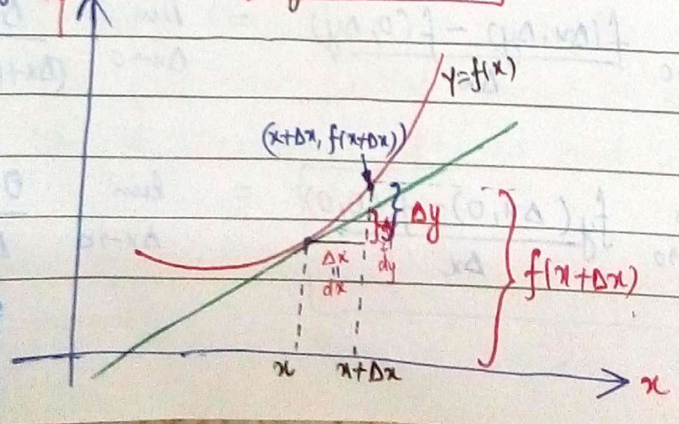
$$\Delta y = f(x+\Delta x) - f(x) = a\,\Delta x + \varepsilon \cdot \Delta x$$

where a is independent of Δx and

$$\lim_{\Delta x \to 0} \varepsilon = 0$$

The value a is the derivative of f at x.

The differential of the dependent variable y, written as dy is defined as $\boxed{dy = f'(x)\cdot\Delta x}$ or $\boxed{dy = f'(x)\,dx}$



* $\Delta x = dx$

But dy is not always Δy

i.e. $dy \neq \Delta y$ (general)

TWO VARIABLES:

The function $Z=f(x,y)$ is said to have a total differential or to be differentiable at the point (x,y) if at the point

$$\boxed{\Delta Z = \underbrace{a\Delta x + b\Delta y}_{\text{differential}} + \epsilon_1 \Delta x + \epsilon_2 \Delta y}$$

where a, b are independent of $\Delta x, \Delta y$ &

ϵ_1 & ϵ_2 are functions of Δx & Δy such that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_1 = 0 \qquad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_2 = 0$$

The linear function of Δx & Δy; $a\Delta x + b\Delta y$ is called the total differential of z at the point (x,y) and is denoted by dZ:

$$dZ = a\Delta x + b\Delta y = a\,dx + b\,dy.$$

{ x & y are independent, $\Delta x = dx$, $\Delta y = dy$, Z is dependent. }

Ex: Show that $Z = x^2 + xy + xy^2$ is differential and write down its total differential.

$$\Delta Z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$\Delta Z = (x+\Delta x)^2 + (x+\Delta x)(y+\Delta y) + (x+\Delta x)(y+\Delta y)^2 - x^2 - xy - xy^2$$

$$= (\Delta x)^2 + 2x\Delta x + x\Delta y + \Delta x\cdot y + \Delta x\cdot\Delta y + x(\Delta y^2 + 2y\Delta y) + (\Delta x)\underbrace{(y+\Delta y)^2}_{(y^2+\Delta y^2+2y\Delta y)}$$

$$= 2x\Delta x + x\Delta y + \Delta x y + 2xy\Delta y + \Delta x\cdot y^2 + \Delta x(\Delta x + \Delta y(1+2y)) + \Delta y(x\Delta y + \Delta x\Delta y)$$

$$= \underbrace{(2x+y+y^2)}_{a}\Delta x + \Delta y\underbrace{(x+2xy)}_{b} + \Delta x\underbrace{(\Delta x + \Delta y(1+2y))}_{\epsilon_1} + \Delta y\underbrace{(x\Delta y + \Delta x\Delta y)}_{\epsilon_2}$$

(ϵ_1 = partial derivative of z wrt x)

total differential:

$$\boxed{dZ = (2x+y+y^2)\,dx + (x+2xy)\,dy}$$

Th: If $Z=f(x,y)$ is differentiable then $f(x,y)$ is continuous and has partial derivatives wrt x & y at the point (x,y) and that $\boxed{a = f_x(x,y)}$ and $\boxed{b = f_y(x,y)}$ (Necessary condition)

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Proof: Let f be differentiable then

$$\lim \left[f(x+\Delta x, y+\Delta y) - f(x,y) \right] = \lim \underset{0}{a\Delta x} + \lim \underset{0}{b\Delta y} + \lim \underset{0}{\varepsilon_1 \Delta x} + \lim \underset{0}{\varepsilon_2 \Delta y}$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x+\Delta x, y+\Delta y) = f(x,y)$$

$\Rightarrow$ f is continuous

$\Delta y = 0$

$$\underbrace{\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}} = \lim_{\Delta x \to 0} \frac{a\Delta x + \cancel{\varepsilon_1 \Delta x}}{\cancel{\Delta x}} \qquad \left\{ \lim_{\Delta x \to 0} \varepsilon_1 = 0 \right\}$$

$$\boxed{f_x(x,y) = a}$$

$$\boxed{b = f_y(x,y)} \qquad \Rightarrow \text{partial derivatives exist}$$

Th. (Sufficient condition for differentiability):-
If the function $z = f(x,y)$ has continuous first order partial derivatives at a point (x,y) then $f(x,y)$ is differentiable at (x,y).

Proof:-

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$
$$= f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x,y)$$

$$\left\langle \text{LMVT} \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(c) \right\rangle$$

$$\therefore = \Delta x \cdot f_x(x+\theta_1 \cdot \Delta x, y+\Delta y) + \Delta y \cdot f_y(x, y+\theta_2 \cdot \Delta y)$$

$$(0 < \theta_1, \theta_2 < 1)$$

$$* \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_x(x+\theta_1 \Delta x, y+\Delta y) = f_x(x,y)$$

$$\Leftrightarrow f_x(x+\theta_1 \Delta x, y+\Delta y) = f_x(x,y) + \varepsilon_1$$

$$\left(\varepsilon_1 \to 0 \text{ when } \Delta x \to 0, \ \Delta y \to 0 \right)$$

$$\Rightarrow f_y(x, y+\theta_2 \cdot \Delta y) = f_y(x,y) + \varepsilon_2$$

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$$\Rightarrow \boxed{\Delta z = f_x(x,y)\cdot\Delta x + \varepsilon_1\cdot\Delta x + f_y(x,y)\cdot\Delta y + \varepsilon_2\cdot\Delta y}$$

$$\Delta z = dz + \varepsilon_1\cdot\Delta x + \varepsilon_2\cdot\Delta y$$

or

$$\frac{\Delta z - dz}{\Delta \rho} = \frac{\varepsilon_1\Delta x}{\Delta\rho} + \frac{\varepsilon_2\cdot\Delta y}{\Delta\rho} \qquad \left\{\Delta\rho = \sqrt{\Delta x^2+\Delta y^2}\right\}$$

$$\lim_{\Delta\rho\to 0}\frac{\Delta z - dz}{\Delta\rho} = \lim_{\substack{\Delta\rho\to 0\\ \Delta x\to 0\\ \Delta y\to 0}}\left[\varepsilon_1\frac{\Delta x}{\Delta\rho} + \varepsilon_2\frac{\Delta y}{\Delta\rho}\right]$$

bounded $\because\ \Delta\rho \geq \Delta x\ ;\ \Delta\rho\geq\Delta y$

$$\Rightarrow \boxed{\lim_{\Delta\rho\to 0}\frac{\Delta z - dz}{\Delta\rho} = 0}$$

Test for differentiability

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3 Sep, 2015

Ex.

$$f(x,y) = \begin{cases}\dfrac{x^3+2y^3}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y) = (0,0)\end{cases}$$

(Continuous, partial derivatives exist, but function is not differentiable.)

I). Continuity.

$x = r\cos\theta \qquad y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0}\frac{r^{\cancel{3}}(\cos^3\theta+2\sin^3\theta)}{\cancel{r^2}} = 0 \quad ✓$$

II)

$$f_x(0,0) = \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{\Delta x^3}{\Delta x^2}-0}{\Delta x} = 1$$

$$f_y(0,0) = \lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{\frac{2\Delta y^3}{\Delta y^2}-0}{\Delta y} = 2.$$

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III) Differentiability

$$\alpha = \lim_{\Delta\rho\to 0} \frac{\Delta Z - dZ}{\Delta \rho}$$

$$\Delta Z = f(0+\Delta x, 0+\Delta y) - f(0,0) = \frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2 + \Delta y^2}$$

$$dZ = f_x \Delta x + f_y \Delta y = \Delta x + 2\Delta y$$

$$\Rightarrow \alpha = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\left(\frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2+\Delta y^2}\right) - (\Delta x + 2\Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{-2\Delta x^2 \Delta y - \Delta x \Delta y^2}{(\Delta x^2 + \Delta y^2)^{3/2}}$$

Path: $\Delta y = m(\Delta x)$

$$= \lim_{\Delta x \to 0} \frac{-2m - m^2}{(1+m^2)^{3/2}} = \text{dependent on } m.$$

$\Rightarrow$ function is not differentiable.

Ex.

$$f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(Continuous, differentiable but partial derivatives are not continuous)

(I) Continuity

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) = 0 = f(0,0)$$ ✓

(II) $$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^{\cancel{2}} \cos\left(\frac{1}{\sqrt{\Delta x^2}}\right)}{\cancel{\Delta x}} = 0$$

$$f_y(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = 0$$

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(III) Differentiability $\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} \to 0 = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta x^2+\Delta y^2)\cos\left(\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\right)}{\sqrt{\Delta x^2+\Delta y^2}}$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sqrt{\Delta x^2+\Delta y^2}\cos\left(\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\right) = 0 \checkmark$$

(function satisfie differentiability condition.)

(IV) $f_x(0,0) = 0 = f_y(0,0)$

for $(x,y) \neq (0,0)$

$$f_x(x,y) = \cancel{-}(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\left(\frac{\cancel{-}1}{\cancel{2}}\frac{1}{(x^2+y^2)^{3/2}} \cdot \cancel{2}x\right) + 2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + 2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$\lim_{\substack{x\to 0 \\ y\to 0}} f_x(x,y) = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x}{\sqrt{x^2+y^2}}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + 0$$

Path: $y=0$

$$= \lim_{x\to 0} \frac{x}{|x|}\sin\left(\frac{1}{|x|}\right)$$

$= \lim_{x\to 0^+} \sin\left(\frac{1}{|x|}\right)$ = does not exist.

i.e $\boxed{\lim_{\substack{x\to 0 \\ y\to 0}} f_x(x,y) \neq 0}$ $\Rightarrow$ partial derivative is not continuous

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Ex:
$$f(x,y) = \begin{cases} \frac{x^2y(x-y)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Find $\frac{\partial^2 f}{\partial x \partial y}$ & $\frac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$

$$\frac{\partial^2 f}{\partial x \partial y} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y}$$

* $$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \frac{0-0}{\Delta x} = 0$$

* $$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \frac{0-0}{\Delta y} = 0$$

* $$f_x(0, \Delta y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, \Delta y) - f(0, \Delta y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta x^2 \Delta y(\Delta x - \Delta y)}{\Delta x^2 + \Delta y^2} - 0}{\Delta x} = 0$$

* $$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x, \Delta y) - f(\Delta x, 0)}{\Delta x} = \lim_{\Delta y \to 0} \frac{\frac{\Delta x^2 \Delta y(\Delta x - \Delta y)}{\Delta x^2 + \Delta y^2} - 0}{\Delta y} = \Delta x$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

Ex. Test differentiability of:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = 0 \end{cases}$$

at the origin.

(I) Continuity

$$\lim_{r \to 0} \frac{r^2 \sin\theta \cos\theta}{|r|} = 0 \quad ✓$$

(II) Partial derivatives.

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$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta y} = 0$$

(III) Differentiability

$$\alpha = \lim_{\Delta \rho \to 0} \frac{\Delta Z - dZ}{\Delta \rho}$$

$$\Delta Z = f(0+\Delta x, 0+\Delta y) - f(0,0) = \frac{\Delta x \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$dZ = f_x \Delta x + f_y \Delta y = 0$$

$$\Delta \rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\alpha = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x \Delta y}{\Delta x^2 + \Delta y^2}$$

Path: $(\Delta y) = m(\Delta x)$

$$\alpha = \lim_{\Delta x \to 0} \frac{m \Delta x^2}{\Delta x^2(1+m^2)} = \frac{m}{1+m^2} = \text{dependent}$$

$\Rightarrow$ $f(x,y)$ is Not diff. at $(0,0)$.

Ex: Find the total differential and the total increment of the function $z = xy$ at the point $(2,3)$ for $\Delta x = 0.1$ & $\Delta y = 0.2$

Total differential $= dz$ total increment $= \Delta z$

General: $\Delta z = (x+\Delta x)(y+\Delta y) - xy = x\Delta y + y\Delta x + \Delta x \Delta y = dz + \Delta x \Delta y$

$$\Delta z = f(2+\Delta x, 3+\Delta y) - f(2,3) = (2+\Delta x)(3+\Delta y) - (2\times 3)$$

$$= \underbrace{2\Delta y + 3\Delta x}_{dz} + \Delta x \Delta y$$

$$\Delta z = dz + \Delta x \Delta y$$

$dz = 2(0.2) + 3(0.1) = 0.7$

$\Delta z = 0.7 + 0.02 = 0.72$

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Ex. Let

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Discuss the continuity of f_{yx} at $(0,0)$

$$f_{yx} = \frac{\partial^2 f}{\partial y\, \partial x} = \lim_{\Delta y \to 0} \frac{f_x(x, y+\Delta y) - f_x(x,y)}{\Delta y}$$

$$f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[\dfrac{(x+\Delta x)^2 y^2}{(x+\Delta x)^2 + y^2} - \dfrac{x^2y^2}{x^2+y^2}\right]}{\Delta x}$$

OR

$$f_x = \frac{d}{dx} f(x,y) = \frac{(x^2+y^2)(2xy^2) - (x^2y^2)(2x)}{(x^2+y^2)^2}$$

$$f_x = \frac{2xy^2\,(\cancel{x^2}+y^2-\cancel{x^2})}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2}$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{(x^2+y^2)^2\, 8xy^3 - 2xy^4 \cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{8xy^3\,((x^2+y^2) - y^2)}{(x^2+y^2)^3} = \frac{8x^3y^3}{(x^2+y^2)^3}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} f_{yx} = ?$$

Path: $y = mx$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{8m^3x^6}{x^6(1+m^2)^3} = \frac{8m^3}{(1+m^2)^3}$$

= dependent on m.
⇒ limit does not exist.

⇒ f_{yx} is not continuous.

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COMPOSITE FUNCTIONS

Consider $Z = f(x,y)$ — (1)

and let

$$\left.\begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned}\right\} - (2)$$

or

$$\left.\begin{aligned} x &= \Phi(u,v) \\ y &= \psi(u,v) \end{aligned}\right\} - (2')$$

The equations (1 & 2) or (1 & 2') are said to define Z as composite function of t or (u & v).

Differentiation of composite functions (Chain Rule)

Let $Z = f(x,y)$ passes continuous partial derivative and let $x = \varphi(t)$ & $y = \psi(t)$ passes ~~continuity~~/continuous derivatives.

Then:

$$\frac{dZ}{dt} = \frac{\partial Z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial Z}{\partial y}\cdot\frac{dy}{dt}$$

Proof: Z passes continuous partial derivatives $\Rightarrow$ Z is continuous & differentiable.

$$\Delta Z = Z_x \Delta x + Z_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

(Z_x & Z_y are independent of Δx & Δy)

$\frac{\partial Z}{\partial x} = f_x$ $\quad \frac{\partial Z}{\partial y} = f_y$

$$\frac{\Delta Z}{\Delta t} = \frac{\partial Z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial Z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}$$

Taking limit $\Delta t \to 0$ $(\Delta x \to 0, \Delta y \to 0)$

$\Rightarrow \epsilon_1 = 0, \epsilon_2 = 0$

$$\boxed{\frac{dZ}{dt} = \frac{\partial Z}{\partial x}\frac{dx}{dt} + \frac{\partial Z}{\partial y}\frac{dy}{dt}}$$

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In the case: $z = f(x,y)$ $x = \phi(u,v)$ $y = \psi(u,v)$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \end{aligned}}$$

Ex: $z = xy$ $x = \cos t$ $y = \sin t$

Find $\frac{dz}{dt}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= y(-\sin t) + x(\cos t)$$

$$= -\sin^2 t + \cos^2 t = \cos 2t$$

Ex: $z = f(x,y)$

~~$x = e^y + e^{-y}$~~ $x = e^u + e^{-v}$ $y = e^{-u} + e^v$

Then show:-

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u}) \quad -(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(e^v) \quad -(2)$$

(1) - (2)

$$\boxed{\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} + e^v)$$

$$= \boxed{x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}}$$

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Derivative of a function defined implicitely

ONE VARIABLE

Let the function y of x be defined by $F(x,y)=0$

Let $z = F(x,y) = 0$

$$\Rightarrow \frac{dz}{dx} = \underbrace{\frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}}=0$$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$ where $F_x = \frac{\partial F}{\partial x}$

$F_y = \frac{\partial F}{\partial y}$; $F_y \neq 0$

TWO VARIABLE

$F(x,y,z)=0$ (x & y are independent variables)

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x}\to 1 + \frac{\partial F}{\partial y}\cancel{\frac{\partial y}{\partial x}}\to 0 + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

$\cancel{\frac{\partial F}{\partial y}}$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$ if $F_z \neq 0$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$ if $F_z \neq 0$.

Ex. Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ if

$$\underbrace{x^2+y^2+z^2-c=0}_{F(x,y,z)=0}$$

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} = \frac{-2\cancel{x}}{2\cancel{z}} = \frac{-x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z} = \frac{-2y}{2z} = -y/z$$

OR $\rightarrow$ $x^2+y^2+z^2-c=0$

$2x+0+2z\cdot z_x=0 \Rightarrow z_x = -x/z$

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Harmonic functions:-

If a function of two variables $f(x,y)$ satisfies the $\boxed{\text{Laplace equation}}$

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

Ex: $f(x,y) = x^3y - y^3x$

$$\frac{\partial f}{\partial x} = 3x^2y - y^3 \quad ; \quad \frac{\partial^2 f}{\partial x^2} = 6xy$$

$$\frac{\partial f}{\partial y} = x^3 - 3y^2x \quad ; \quad \frac{\partial^2 f}{\partial y^2} = -6yx$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Homogeneous function :-

We say an expression in (x,y) is homogeneous of order n, if it can be expressed as

$$x^n f\left(\frac{y}{x}\right)$$

Ex: $f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_ny^n$.

$$f(x,y) = x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots + a_n\left(\frac{y}{x}\right)^n\right]$$ $\rightarrow$ Homogeneous function of order n

Ex: $$f(x,y) = \frac{\sqrt{y}+\sqrt{x}}{y+x} = \frac{\sqrt{x}}{x}\underbrace{\frac{\left(\sqrt{\frac{y}{x}}+1\right)}{\left(\frac{y}{x}+1\right)}}_{g(y/x)} = x^{-1/2}\, g\left(\frac{y}{x}\right)$$ $\rightarrow$ Homogeneous function of order $-\frac{1}{2}$.

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Def. A function $f(x,y)$ is said to be homogeneous of degree n if it satisfies $\boxed{f(tx,ty) = t^n f(x,y)}$

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If $z = f(x,y)$ is a homogeneous function of x & y of order n then

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

$$\boxed{x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z}$$

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Proof.

$$z = f(x,y) = x^n g\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial x} = n x^{n-1} g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= n x^{n-1} g\left(\frac{y}{x}\right) - x^{n-2} y \cdot g'\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^n g\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x^{n-1} g'\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n x^n g\left(\frac{y}{x}\right) - x^{n-1} y g'\left(\frac{y}{x}\right) + x^{n-1} y g'\left(\frac{y}{x}\right)$$

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n x^n g\left(\frac{y}{x}\right) = nz}$$

Ex. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ $x \neq y$

Not homogeneous

Homogeneous function

Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

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~~Eg~~ $\tan u = \frac{x^3+y^3}{x-y} = z$. Homogeneous function

$$= \frac{x^3}{x}\left(\frac{1+(\frac{y}{x})^3}{1-(\frac{y}{x})}\right) = x^2 \underbrace{\left[\frac{1+(\frac{y}{x})^3}{1-(\frac{y}{x})}\right]}_{g(\frac{y}{x})}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2.z$$

$$x.\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2.\tan u$$

$$x.\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\left(\frac{\sin u}{\cos u}\right)\frac{1}{(\sec^2 u)} = 2\sin u\cos u = \sin 2u$$

Ex. $u = z\,e^{ax+by}$

where z is a homogeneous function of order n.

Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Since z is a homogeneous function,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[\frac{\partial z}{\partial x}e^{ax+by} + z.e^{ax+by}.a\right] + y\left[\frac{\partial z}{\partial y}e^{ax+by} + z.e^{ax+by}.b\right]$$

$$= e^{ax+by}\left[\underbrace{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}}_{nz} + az.x + b.zy\right]$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \underbrace{e^{ax+by}.z}_{u}\,(n + ax + by)}$$

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Ex. $Z = xy\, f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ where f and g are continuous & continuously 2 times differentiable functions.

Then.

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = ?$$

$$Z = \underbrace{x^2 \left(\frac{y}{x}\right) f\left(\frac{y}{x}\right)}_{u_1} + \underbrace{g\left(\frac{y}{x}\right)}_{u_2}$$

$u_2 \rightarrow$ order $= 0$

u_1 → Homogeneous functions

order $= 2$

$\Rightarrow Z = u_1 + u_2$

$$x^2 \frac{\partial^2 u_1}{\partial x^2} + 2xy \frac{\partial^2 u_1}{\partial x \partial y} + y^2 \frac{\partial^2 u_1}{\partial y^2} = n(n-1)u_1 = 2(1)(u_1) = 2u_1 \quad \text{— (1)}$$

$$x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = 1(0)u_2 = 0 \quad \text{— (2)}$$

Adding eq (1) & (2)

$$x^2 \frac{\partial^2 (u_1+u_2)}{\partial x^2} + 2xy \frac{\partial^2 (u_1+u_2)}{\partial x \partial y} + y^2 \frac{\partial^2 (u_1+u_2)}{\partial y^2} = 2u_1 + 0$$

$$\Rightarrow \boxed{x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2xy\, f\left(\frac{y}{x}\right)}$$

Ex: If $z = y + f\left(\frac{x}{y}\right)$ where f is cont. diff. function.

Find the value of $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$.

$Z = u_1 + u_2$ $\quad u_1 = x\left(\frac{y}{x}\right)$ $\quad u_2 = f\left(\frac{x}{y}\right)$

$\quad$ Order $= 1$ $\quad$ Order $= 0$

$$x \frac{\partial u_1}{\partial x} + y \frac{\partial u_1}{\partial y} = n_1 u_1 = u_1 = y \quad \text{— (1)}$$

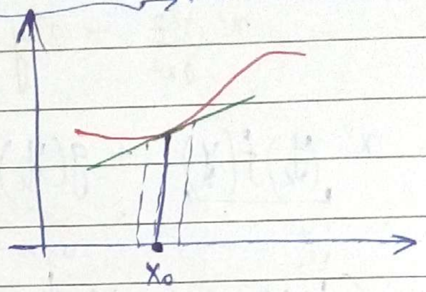
$$x \frac{\partial u_2}{\partial x} + y \frac{\partial u_2}{\partial y} = n u_2 = 0(u_2) = 0 \quad \text{— (2)}$$

$$(1) + (2) \Rightarrow \boxed{x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = y}$$

$$f(x+\Delta x) - f(x_0) = \Delta x \cdot A + \epsilon_1(\Delta x)$$

$$f(x) - f(x_0) = (x-x_0)A + \epsilon_1(x-x_0)$$

$$f(x) = \underbrace{f(x_0) + (x-x_0)A}_{\text{linear function in nbd of } x_0} + \epsilon_1(x-x_0) \rightarrow \text{min order} = 2$$



$$\Rightarrow \varphi(x) = f(x_0) + (x-x_0)A$$

$$f(x,y) = \underbrace{f(x_0,y_0) + (x-x_0)A + (y-y_0)B}_{\text{linear function}} + \epsilon_1(x-x_0) + \epsilon_2(y-y_0)$$

$$\Rightarrow \psi(x,y) = f(x_0,y_0) + (x-x_0)A + (y-y_0)B$$

TAYLOR'S THEOREM:

(x_0, y_0)

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0,y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0,y_0) + R_n$$

$$* \; R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k) \qquad 0<\theta<1$$

Note:- 1. $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) = h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x \partial y}$

2. $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f = h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}$

Proof: $n=2$

Let $x = x_0 + th$

$y = y_0 + tk$

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Define: $\rightarrow$ $\Phi(t) = f(x_0 + th, y_0 + tk)$

- $\Phi'(t) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} \cdot h + \frac{\partial f}{\partial y} \cdot k$

$$\boxed{\Phi'(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0+th, y_0+tk)}$$

- $\Phi''(t) = h\left\{\frac{\partial^2 f}{\partial x^2} \cdot h + \frac{\partial^2 f}{\partial y \partial x} \cdot k\right\} + k\left\{\frac{\partial^2 f}{\partial x \partial y} \cdot h + \frac{\partial^2 f}{\partial y^2} \cdot k\right\}$

$\Phi''(t) = h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}$

$$\boxed{\Phi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0+th, y_0+tk)}$$

- $\Phi'''(t) = h^3 \frac{\partial^3 f}{\partial x^3} + 3h^2k \frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2 \frac{\partial^3 f}{\partial x \partial y^2} + k^3 \frac{\partial^3 f}{\partial y^3}$

$$\boxed{\Phi'''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0+th, y_0+tk)}$$

Taylor's expansion about 0 : $\Phi(t)$:-

$$\Phi(t) = \Phi(0) + t \cdot \Phi'(0) + \frac{t^2}{2!}\Phi''(0) + \frac{t^3}{3!}\Phi'''(\theta t) \qquad 0 < \theta < 1$$

Substitute $t = 1$

$$\Phi(1) = \Phi(0) + \Phi'(0) + \frac{1}{2!}\Phi''(0) + \frac{1}{3!}\Phi'''(\theta)$$

$\Rightarrow$ $f(x_0+h, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) +$
$\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0+\theta h, y_0+\theta k)$

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Ex. Find the quadratic Taylor's polynomial approximation of $f(x,y) = \dfrac{x-y}{x+y}$ about the point $(1,1)$

$f(1,1) = 0$

* $f_x = \dfrac{(x+y)(1) - (x-y)(1)}{(x+y)^2} = \dfrac{2y}{(x+y)^2}$

* $f_x(1,1) = \dfrac{1}{2}$

* $f_y = \dfrac{(x+y)(-1) - (x-y)(1)}{(x+y)^2} = \dfrac{-2x}{(x+y)^2}$

* $f_y(1,1) = -\dfrac{1}{2}$

* $f_{xx} = 2y \dfrac{(-2)}{(x+y)^3} = \dfrac{-4y}{(x+y)^3}$ * $f_{xx}(1,1) = -\dfrac{1}{2}$

* $f_{yy} = (-2x)(-2)\dfrac{1}{(x+y)^3} = \dfrac{+4x}{(x+y)^3}$ * $f_{yy}(1,1) = +\dfrac{1}{2}$

* $f_{xy} = (-2)\left[\dfrac{(x+y)^2 - 2(x+y).x}{(x+y)^4}\right] = \dfrac{-2}{(x+y)^3}(y-x)$

* $f_{xy}(1,1) = 0$

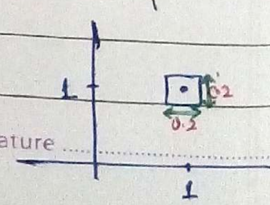
$$P_2(x,y) = f(1,1) + (x-1)f_x(1,1) + (y-1)f_y(1,1) + \frac{1}{2}(x-1)^2 f_{xx}(1,1) +$$

$$\frac{2(x-1)(y-1)}{2!} f_{xy}(1,1) + \frac{1}{2}(y-1)^2 f_{yy}(1,1)$$

$$\boxed{P_2(x,y) = 0 + \frac{(x-1)}{2} - \frac{(y-1)}{2} - \frac{1}{4}(x-1)^2 + \frac{1}{4}(y-1)^2}$$

Ex. Let $f(x,y) = x^2 + xy + y^2$ be linearly approximated by Taylor's ~~polynomial~~ expansion about the point $(1,1)$. Find the maximum error in this linear approximation at a point in the square $|x-1| \le 0.1$

$|y-1| \le 0.1$



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* $f_x = 2x+y$ * $f_x(1,1) = 3$
* $f_y = 2y+x$ * $f_y(1,1) = 3$
* $f_{xx} = 2$
* $f_{yy} = 2$
* $f_{xy} = 1$

$$R_1(x,y) = \frac{1}{2!}\left\{\left[f_{xx}(1,1)(x-1)^2 + 2f_{xy}(1,1)(x-1)(y-1) + f_{yy}(y-1)^2\right]\right\}$$

$$= (x-1)^2 + (x-1)(y-1) + (y-1)^2$$

Max. error $= [R_1(x,y)]_{max} = (0.1)^2 + (0.1)^2 + (0.1)^2 = \boxed{0.03}$

Ex. Obtain Taylor's formula $(n=2)$ for $f(x,y) = \cos(x+y)$ at $(0,0)$

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(\theta x, \theta y)$$

$0 < \theta < 1$

* $f(0,0) = 01$
* 1st order derivatives:-

$f_x = -\sin(x+y) = f_y$ $\quad f_x(0,0) = 0 = f_y(0,0)$

* 2nd order derivatives:-

$f_{xx} = -\cos(x+y) = f_{yy} = f_{xy} = f_{yx}$ $\quad f_{xx}(0,0) = -1 = f_{yy}(0,0) = f_{xy}(0,0)$

* 3rd order derivatives:-

$f_{xxx} = \sin(x+y) = f_{xxy} = f_{xyy} = f_{yyy}$ $\quad f_{xxx}(0,0) = 0$

Taylor's Theorem:-

$$f(x,y) = 1 + \xrightarrow{0}\cancel{\left(x f_x^{(1,1)} + y f_y^{(1,1)}\right)} + \frac{1}{2!}\left(x^2 f_{xx}^{(1,1)} + 2xy\, f_{xy}^{(1,1)} + y^2 f_{yy}^{(1,1)}\right) + \frac{1}{3!}\left(x^3 f_{xxx}(\theta x, \theta y) + 3x^2y\, f_{xxy}(\theta x, \theta y) + 3xy^2 f_{xyy}(\theta x, \theta y) + y^3 f_{yyy}(\theta x, \theta y)\right)$$

$$\boxed{f(x,y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!}\sin(\theta x + \theta y)}$$

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Def.

1. A function $Z = f(x,y)$ has a maximum (or minimum) at the point (x_0, y_0) if at every point in the nbd of (x_0, y_0) the function assumes a smaller value (or larger value) than at the point itself. Such a maximum or minimum is often called relative (or local) maximum or minimum respectively.

2. For a given closed and bounded domain, a function may also attain its greatest value (or least value) on the boundary of the domain.
The smallest and the largest values attained by a function over the entire domain including the boundary are called the absolute (or global) minimum & absolute (or global maximum, respectively.

3. The point (x_0, y_0) is called critical point (or stationary point) of $f(x,y)$ if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

4. A critical point where the function has no minimum or maximum is called a saddle point.

5. Minimum and maximum values together are called extreme values.

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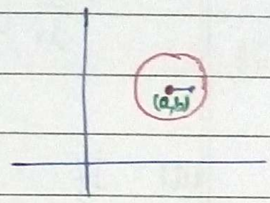
$f(a+h, b+k) = f(a,b) + (h f_x + k f_y)_{(a,b)}$ + higher order terms

$\underline{f(a+h, b+k) - f(a,b)} = \underline{h f_x + k f_y}$ + higher order terms.



Suppose $f_x > 0$

$\Delta f = h f_x + k f_y$ + higher order terms.

Set $k = 0$

$\Delta f = h f_x$ + higher...

for some $h > 0$, $\Delta f > 0$

for some $h <$, $\Delta f < 0$

If f_x is not equal to zero, (a,b) cannot be point of maxima or minima $\because$ Δf varies from -ve to +ve & is not fixed.

$\Rightarrow \underbrace{\boxed{f_x = 0} \quad \& \quad \boxed{f_y = 0}}_{\text{Necessary Condition}}$

10-9-15

Necessary conditions for extremum at (a,b) !

$f_x(a,b) = 0$

$f_y(a,b) = 0$

$$\underset{f(a+h,\, b+k) - f(a,b)}{\Delta f} = \cancel{h f_x}^{0} + \cancel{k f_y}^{0} + \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right) + \text{higher order terms}$$

$f_{xx}(a,b) = r$, $f_{xy}(a,b) = s$, $f_{yy}(a,b) = t$

$$\Delta f = \frac{1}{2}\left[h^2 r + 2hks + k^2 t\right] + \text{higher order terms}$$

$$= \frac{1}{2r}\left[h^2r^2 + 2hk\,rs + k^2tr\right] + \dots$$

$$= \frac{1}{2r}\left[(hr+ks)^2 - k^2s^2 + k^2tr\right] + \dots$$

$$= \frac{1}{2r}\left[(hr+ks)^2 + k^2(rt-s^2)\right] + \dots$$

(a) If $rt-s^2 > 0$

(i) $\Delta f > 0$, if $r>0$ Minimum

(ii) $\Delta f < 0$, if $r<0$ Maxima

(b) If $rt-s^2<0$, if $r>0$

(i) $k=0$, $\Delta f > 0$

(ii) $hr+ks=0$, $\Delta f < 0$

$\hookrightarrow h = -\frac{ks}{r}$ (fix k, get h, observe)

} Changing sign ⇒ No max. or min

Saddle Point.

(c) If $rt-s^2=0$

↳ further investigation required.

∵ ($r>0$ → let, $hr+ks=0$ → let → higher order terms need to be investigated, otherwise, $\Delta f>0$ for $r>0$)

Q Discuss the local extrema of the function $f(x,y) = (4x^2+y^2)e^{-x^2-4y^2}$

$f_x(x,y)=0$, $f_y(x,y)=0$ } Critical points.

$$f_x(x,y) = 8x\,e^{-x^2-4y^2} + (4x^2+y^2)\,e^{(-x^2-4y^2)}(-2x)$$
$$= 2x\,e^{-x^2-4y^2}\left[4-4x^2-y^2\right] \rightarrow \boxed{x=0,\ 4x^2+y^2=4} \ \}\ \mathrm{I}$$

$$f_y(x,y) = 2y\,e^{-x^2-4y^2} + (4x^2+y^2)\,e^{-x^2-4y^2}(-8y)$$
$$= 2y\,e^{-x^2-4y^2}(1-16x^2-4y^2) \rightarrow \boxed{y=0,\ 16x^2+4y^2=1} \ \}\ \mathrm{II}$$

$4x^2+y^2 = 1/4$

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Points:- (i) $x=0, y=0$

(ii) $x=0$ (I) $\rightarrow$ II $\Rightarrow$ $y^2 = \frac{1}{4}$ $\Rightarrow$ $y = \pm\frac{1}{2}$

(iii) $y=0$ (II) $\rightarrow$ I $\Rightarrow$ $x^2 = 1$ $\Rightarrow$ $x = \pm 1$

Required critical points:- $P_1(0,0)$, $P_2(0,\frac{1}{2})$, $P_3(0,-\frac{1}{2})$, $P_4(1,0)$, $P_5(-1,0)$

(r) $f_{xx}(x,y) = 2e^{-x^2-4y^2}[4-12x^2-y^2] + 2e^{-x^2-4y^2}(4x-4x^3-y^2x).(-2x)$

(s) $f_{xy}(x,y) = 2x\, e^{-x^2-4y^2}(-2y) + 2x(4-4x^2-y^2)\, e^{-x^2-4y^2}(-8y)$

(t) $f_{yy}(x,y) = 2e^{-x^2-4y^2}(1-16x^2-12y^2) + 2e^{-x^2-4y^2}(y-16x^2y-4y^3)(-8y)$

* $P_1(0,0)$ $r=8$ $t=2$ $s=0$

$rt-s^2 > 0$ & $r>0$ $\Rightarrow$ Minimum

* $P_2(0,\frac{1}{2})$ $r = 2e^{-1}\{[4-\frac{1}{4}]-0\} = \frac{15}{2e}$

$t = 2e^{-1}(1-\frac{12}{4}) - 16e^{-1}(\frac{1}{2})[\frac{1}{2}-\frac{4}{8}] = \frac{-4}{e}$

$s = 0$

$rt-s^2 < 0$ & $r>0$ $\Rightarrow$ Saddle Point

* $P_3(0,-\frac{1}{2}) \longrightarrow$ Saddle Point

* $P_4(1,0)$

$r = 2e^{-1}(4-12) + 2e^{-1}(4-4)(-2) = \frac{-16}{e}$

$s=0$

$t = 2e^{-1}(1-16) + 0 = \frac{-30}{e}$

$rt-s^2 > 0$ & $r > 0$ $\Rightarrow$ Maximum

* $P_5(-1,0) \longrightarrow$ Maximum

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Ex:

$f(x,y) = y^2 + x^2y + x^4$

How many stationary points? (critical points)

$f_x(x,y) = 2xy + 4x^3 = 2x(y + 2x^2) = 0$

$f_y(x,y) = 2y + x^2 = 0$

1. $x=0, y=0$
2. $y = -2x^2$ ✗ $\quad y = -\frac{x^2}{2}$ ✗

Only 1 critical point.

$f_{xx} = 2y + 12x^2 \quad (r)$

$f_{xy} = 2x \quad (s)$

$f_{yy} = 2 \quad (t)$

$\rightarrow r=0 \quad t=2 \quad s=0$

$\Rightarrow rt - s^2 = 0$

$\Delta f = f(0+h, 0+k) - f(0,0)$

$= k^2 + h^2k + h^4$

$= \left(\frac{k}{2}\right)^2 + h^2k + h^4 + \frac{3k^2}{4} = \left(h^2 + \frac{k}{2}\right)^2 + \frac{3}{4}k^2$

$(h,k) \neq (0,0)$, $\Delta f > 0 \Rightarrow$ Minima

Method of Lagrange multiplication :-

Find the maxima / minima of the function

$u = f(x,y) \quad - (1)$

with the following constraints

$\varphi(x,y) = 0 \quad - (2)$

$\frac{du}{dx} = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} \cdot \frac{dy}{dx}$

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At the point of extremum:

$$\frac{du}{dx} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad -(3)$$

$$\rightarrow \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0 \quad -(4) \Rightarrow \boxed{\frac{dy}{dx} = \frac{-\varphi_x}{\varphi_y}}$$

Assume φ_x, φ_y do not both vanish.

→ Assuming $\varphi_y \neq 0$, multiply (4) by $\lambda = \frac{-f_y}{\varphi_y}$ and add it to equation (3)

$$\frac{\partial f}{\partial x} + \cancel{\frac{\partial f}{\partial y}\frac{dy}{dx}} + \lambda\frac{\partial \varphi}{\partial x} + \cancel{\lambda\frac{\partial \varphi}{\partial y}\frac{dy}{dx}} = 0$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x} = 0} \quad -(5)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y} = 0 \quad -(6)$$

$$\varphi(x,y) = 0 \quad -(7) \quad (=(2))$$

Lagranges Multiplier $= \lambda$

LAGRANGE RULE:-

Define: $F(x,y,\lambda) = f(x,y) + \lambda\,\varphi(x,y)$
(Auxilliary function)

Necessary condition for extremum of F:

$F_x = 0 \Rightarrow f_x + \lambda \varphi_x = 0$

$F_y = 0 \Rightarrow f_y + \lambda \varphi_y = 0$

$F_\lambda = 0 \Rightarrow \varphi(x,y) = 0$

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General Case:

Find extremum of $f(x_1, x_2, \dots x_n)$ and the conditions

$\varphi_i(x_1, x_2, \dots x_n) = 0$, $i = 1, 2, \dots k$

Auxiliary function:-

$F(x_1, x_2, \dots x_n, \lambda_1, \lambda_2, \dots \lambda_k) = f(x_1, x_2, \dots x_n) + \sum_{i=1}^{k} \lambda_i \varphi_i(x_1, x_2, \dots x_n)$

Find the stationary points:-

$F_{x_1} = 0$, $F_{x_2} = 0$, ... $F_{x_n} = 0$
$F_{\lambda_1} = 0$, $F_{\lambda_2} = 0$, ... $F_{\lambda_k} = 0$ } (n+k) equations
(n+k) unknowns

Ex: Find the maximum & minimum of $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$

(I) Interior domain → Interior of circle $x^2 + y^2 < 1$

$f_x = 0 \to 2x = 0$, $f_y = 0 \to 2y = 0$ $\Rightarrow x = 0, y = 0$

(II) Lagrange Multiplier → Max/min. $f(x,y) = x^2 + 2y^2$
Subject to condition $x^2 + y^2 = 1$

$F(x,y,\lambda) = x^2 + 2y^2 + \lambda(x^2 + y^2 - 1)$

$F_x = 0 \Rightarrow 2x + 2\lambda x = 0 \rightarrow 2x(\lambda + 1) = 0$ — (1)

$F_y = 0 \Rightarrow 4y + 2y\lambda = 0 \rightarrow 2y(2 + \lambda) = 0$ — (2)

$F_\lambda = 0 \Rightarrow x^2 + y^2 = 1$ — (3)

$x = 0$: (3) $\Rightarrow y = \pm 1$ & (2) $\Rightarrow \lambda = -2$ → $(0,1), (0,-1)$

$y = 0$: $x = \pm 1$, $\lambda = -1$ → $(1,0), (-1,0)$

$f(\pm 1, 0) = 1$

$\boxed{f(0, \pm 1) = 2}$ → Maximum (on boundary)

$\boxed{f(0,0) = 0}$ → Minimum

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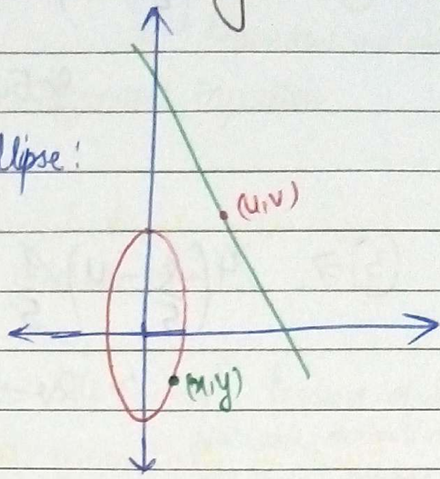
Ex: Find the shortest distance between the line $y = 10 - 2x$ and the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$



Shortest distance b/w the line and ellipse:

Min of

$$f(x,y,u,v) = \sqrt{(x-u)^2 + (y-v)^2}$$

Subject to

$$\varphi_1(x,y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \quad \text{— (1)}$$

$$\varphi_2(u,v) = 2u + v - 10 = 0 \quad \text{— (2)}$$

Auxiliary function :-

$$F(x,y,u,v,\lambda_1,\lambda_2) = (x-u)^2 + (y-v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u+v-10)$$

$$F_x = 0 = 2(x-u) + \frac{\lambda_1 x}{2} \Rightarrow -\lambda_1 x = 4(x-u)$$

$$F_y = 0 = 2(y-v) + \frac{2\lambda_1 y}{9} \Rightarrow -\lambda_1 y = 9(y-v)$$

$$4(x-u)y = 9(y-v)x \quad \text{— (3)}$$

$$F_u = 0 = -2(x-u) + 2\lambda_2 \Rightarrow \lambda_2 = x-u$$

$$F_v = 0 = -2(y-v) + \lambda_2 \Rightarrow \lambda_2 = 2(y-v)$$

$$(x-u) = 2(y-v) \quad \text{— (4)}$$

$$F_{\lambda_1} = 0 \Rightarrow \varphi_1(x,y) = 0 \qquad F_{\lambda_2} = 0 \Rightarrow \varphi_2(u,v) = 0$$

From (3) and (4)

$$8y = 9x \Rightarrow y = \frac{9x}{8}$$

$$(1) \Rightarrow \frac{x^2}{4} + \frac{9^2 x^2}{8^2 \cdot 9} - 1 = 0$$

$$\frac{16x^2 + 9x^2}{64} - 1 = 0$$

$$\Rightarrow x^2 = \frac{64}{25}$$

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$$y^2 = \frac{81}{25}$$

Points $\rightarrow \left(\frac{8}{5}, \frac{9}{5}\right)$ and $\left(\frac{-8}{5}, \frac{-9}{5}\right)$

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(4) $\Rightarrow$ $\left(\frac{8}{5}-u\right) = 2\left(\frac{9}{5}-v\right)$

$8-5u = 18-10v$

$10v-5u = 10 \Rightarrow \quad 2v-u=2$

(3) $\Rightarrow$ $\cancel{4}\left(\frac{8}{5}-u\right)\cancel{\frac{9}{5}} = \cancel{9}\left(\frac{9}{5}-v\right)\cancel{\frac{8}{5}}^{2}$

$\therefore \quad 2v-u=2.$

(2) $\Rightarrow$ $2(2v-2)+v-10=0$

$5v=14 \Rightarrow \quad v=\frac{14}{5} \qquad u=\frac{18}{5}$

for $\left(-\frac{8}{5}, -\frac{9}{5}\right) = (x,y)$, we get corresponding -ve values of u & v

But value of (shortest) distance remains same

$\therefore$ $\Rightarrow$ Shortest distance $\Rightarrow \sqrt{\left(\frac{8}{5}-\frac{18}{5}\right)^2 - \left(\frac{9}{5}-\frac{14}{5}\right)^2}$

$= \sqrt{4+1} = \boxed{\sqrt{5}}$

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DIFFERENTIAL EQUATIONS

Ex.

$x \to$ dependent variable $\quad t \to$ indep.

$$\left(\frac{d^2x}{dt^2}\right)^1 + \frac{dx}{dt} + x = 4$$ } Linear Differential Equation

→ Order 2

→ Degree 1 (Power of highest ordered term)

Ex.

$$x\frac{d^2x}{dt^2} + \frac{dx}{dt} + x^2 = 4 \longrightarrow \text{Non linear Diff. Eq.}$$ → (Product of dependent variable, derivatives with any of them makes diff. eq. non linear.)

Ex

$$t^2\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 4 \longrightarrow \text{Linear Diff. Eq.}$$

Ordinary diff. eq. (No partial derivatives term)

Ex. $\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$ → Partial diff. eq → contains two independent variables.

Solution of a diff. equation:-

Ex

$y = \frac{A}{x} + B$ is a solution of $y'' + \left(\frac{2}{x}\right)y' = 0$ → Family of curve of 2 parameters

$$y' = -\frac{A}{x^2} \qquad y'' = \frac{2A}{x^3}$$

LHS. $\frac{2A}{x^3} + \frac{2}{x}\left(\frac{-A}{x^2}\right) = 0 = RHS.$

FAMILY OF CURVES

An n-parameter family of curves is a set of relations of the form:-

$$\{(x,y) : f(x, y, c_1, c_2, \ldots, c_n) = 0\}$$

Ex.

$$x^2 + y^2 = c \qquad c \in R - R^-$$

(1 parameter) Family of circles.

Ex.

$$(x-c_1)^2 + (y-c_2)^2 = c_3$$ 3 Parameter family of circles.

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Formation of differential equations from a given n-parameters family of curves:-

Ex Obtain the differential equation satisfied by

$$xy = ae^{x} + be^{-x} + x^{2} \quad \text{--- (1)}$$

Diff. wrt x

$$xy' + y = ae^{x} - be^{-x} + 2x \quad \text{--- (2)}$$

Diff. again

$$xy'' + 2y' = ae^{x} + be^{-x} + 2 \quad \text{--- (3)}$$

Using 1,

$$\boxed{xy'' + 2y' = xy - x^{2} + 2}$$

Let $F(x, y, y', y'', \ldots y^{(n)}) = 0$ be an nth order Ordinary differential eq. (ODE)

(i) General Solution : Solution contains n independent arbitrary constants.

(ii) Partiacular Solution : Solution by giving particular values to one or more of the n independent constants.

(iii) Singular Solution : cannot be obtained by any choice of independent arbitrary const.

Ex. $yy' - x(y')^{2} = 1$

General Solution: $y = cx + \frac{1}{c}$

Particular Solution: $y = x + 1$

Singular Solution: $y^{2} = 4x$ {satisfies the equation but cannot be obtained from general solution anyway}.

Explicit : $y = y(x) \rightarrow$ function of x

Ex. $y'' + k^{2}y = 0$ } Solution $\rightarrow y = c_1 \cos kx + c_2 \sin kx$

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why $\left(\frac{dy}{dx}\right) dx \to dy$??

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Implicit Solution : $f(x,y)=0$

Ex. $x+3yy'=0$ } Solution $\to$ $x^2+3y^2=c$

Equation of 1st order and first degree:-

(i) $\frac{dy}{dx} = f(x,y)$

(ii) $M(x,y)\,dx + N(x,y)\,dy = 0$

1. Seperation of variable:

$$f_1(y)\frac{dy}{dx} = f_2(x)$$

Ex. $\frac{dy}{dx} = e^{x-2y} + x^2e^{-2y}$

$$\int e^{2y}\,dy = \int (e^x+x^2)\,dx$$

$$\boxed{\frac{e^{2y}}{2} = e^x + \frac{x^3}{3} + c}$$

2. Equations reducible to Seperable form:

$$\frac{dy}{dx} = f(ax+by+c)$$

Substitute $ax+by+c=v$

$$\Rightarrow a+b\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{1}{b}\left[\frac{dv}{dx}-a\right] = f(v)$$

$$\int \frac{dv}{bf(v)+a} = \int dx \quad +c$$

Ex. $\frac{dy}{dx} = \sec(x+y)$

$x+y=v$

$1+\frac{dy}{dx} = \frac{dv}{dx}$

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$$\frac{dv}{dx} - 1 = \sec v$$

$$\frac{dv}{\sec v + 1} = dx$$

$$\sec v + 1 = \frac{\cos v + 1}{\cos v} = \frac{2\cos^2 v/2}{2\cos^2 v/2 - 1}$$

$$\Rightarrow \frac{(2\cos^2 v/2 - 1)\, dv}{2\cos^2 v/2} = dx$$

$$\Rightarrow \int \left(1 - \frac{1}{2}\sec^2 \frac{v}{2}\right) dv = \int dx$$

$$v - \frac{1}{2}\frac{\tan v/2}{(1/2)} = x + c$$

$$\Rightarrow \boxed{v - \tan v/2 = x + c}$$

Imp. Step → (don't forget to replace it again)

$$\cancel{x} + y - \tan\left(\frac{x+y}{2}\right) = \cancel{x} + c$$

$$\boxed{y = \tan\left(\frac{x+y}{2}\right) + c}$$

3. Homogeneous Equation:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Substitute $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = f(v)$$

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + c$$

Ex: $(x^3 + 3xy^2)\, dx + (y^3 + 3x^2y)\, dy = 0$

$$\frac{dy}{dx} = -\frac{(x^3 + 3xy^2)}{(y^3 + 3x^2y)} = \frac{-\left(1 + 3\left(\frac{y}{x}\right)^2\right)}{\left(\left(\frac{y}{x}\right)^3 + 3\left(\frac{y}{x}\right)\right)}$$

$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

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$$v + x\frac{dv}{dx} = -\frac{1+3v^2}{v^3+3v}$$

$$x\frac{dv}{dx} = -\left[\frac{1+3v^2}{v^3+3v} + v\right] = -\frac{(1+6v^2+v^4)}{v(v^2+3)}$$

$$\frac{(v^3+3v)\,dv}{(1+6v^2+v^4)} = -\frac{dx}{x}$$

* $1+6v^2+v^4 = u$

$$12v + 4v^3 = \frac{du}{dv}$$

$$v^3+3v = \frac{1}{3}\left(\frac{du}{dv}\right)$$

$$\Rightarrow \frac{1}{3}\int\frac{du}{u} = -\int\frac{dx}{x}$$

$$\frac{1}{3}\ln u = -\ln x + c$$

$$\frac{1}{3}\ln(1+6v^2+v^4) = -\ln x + c$$

$$\frac{1}{3}\ln\left(1+6\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^4\right) = -\ln x + c$$

$$\boxed{\ln(x^4+6x^2y^2+y^4) - \frac{1}{3}\ln x = c}$$

4. Equation reducible to homogeneous form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad \text{where } \frac{a}{a'} \neq \frac{b}{b'}$$

Substitute: $x = X+h \qquad y = Y+k$

$$\frac{dy}{dx} = \frac{dY}{dX}\left(\frac{dX}{dx}\right) = \frac{dY}{dX}\cdot 1$$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{aX+bY+\boxed{ah+bk+c}}{a'X+b'Y+\boxed{a'h+b'k+c'}}$$ } Homogeneous if $ah+bk+c=0$, $a'h+b'k+c'=0$ } get h and k

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$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \qquad \frac{a}{a'} = \frac{b}{b'} = \frac{1}{\lambda}$$

$$\frac{dy}{dx} = \frac{ax+by+c}{\lambda(ax+by)+c'} = f(ax+by) \longrightarrow \text{Reducible to Seperable form}$$

Ex:

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

$x = X + h \qquad y = Y + k$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+(Y+k)-3}$$

$\Rightarrow \quad h + 2k - 3 = 0$

$2h + k - 3 = 0 \;] \times 2$

$4h + 2k - 6 = 0$

$-3h + 3 = 0 \Rightarrow h = 1, k = 1$

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} = \frac{1+2(Y/X)}{2+(Y/X)}$$

$$\frac{Y}{X} = v \Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + X\frac{dv}{dx} = \frac{1+2v}{2+v}$$

$$X\frac{dv}{dX} = \frac{1+2v-2v-v^2}{2+v} = \frac{1-v^2}{2+v}$$

$$\frac{(2+v)\,dv}{1-v^2} = \frac{dX}{X}$$

$$\frac{2dv}{1-v^2} + \frac{v\,dv}{1-v^2} = \frac{dx}{x}$$

$1 - v^2 = u \Rightarrow -2v\,dv = du$

(OR)

$$\frac{(2+v)}{(1-v)(1+v)}dv = \frac{dX}{X}$$

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$$2+v = m(1-v) + n(1+v)$$

$m+n=2 \qquad n-m=1$

$2n=3 \Rightarrow n=\frac{3}{2} \qquad m=\frac{1}{2}$

$$\frac{1}{2}\int\frac{dv}{1+v} + \int\frac{3}{2}\frac{dv}{1-v} = \int\frac{dX}{X}$$

$$\frac{1}{2}\ln(1+v) + \frac{3}{2}\frac{\ln(1-v)}{(-1)} = \ln X + \ln c$$

$$\frac{1+v}{(1-v)^3} = X^2c^2$$

where $v = \frac{Y}{X} = \frac{y-1}{x-1}$ and $X = x-1$

$$\Rightarrow \boxed{c^2(x-y)^3 = x+y-2}$$

EXACT DIFFERENTIAL EQUATION:

If M and N are functions of x and y, the equation $Mdx + Ndy$ is called *exact* when there exists a function $f(x,y)$ such that

$$\underbrace{d(f(x,y))}_{\text{Total differential} \to dz} = Mdx + Ndy$$

$$\frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy = Mdx + Ndy$$

($Mdx+Ndy$ = Total differential of some function)

Theorem: The necessary and sufficient condition for the diff. equation $Mdx+Ndy=0$ to be exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

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Proof: <u>Necessary Condition</u>

Let the equation is exact $\Rightarrow \exists$ function $f(x,y)$ such that

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

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$$\Rightarrow \frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \& \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

Assuming the continuity of mixed order derivatives f_{xy} and f_{yx}

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

1 Oct, 15

'$\Leftarrow$' Sufficient condition :

We assume that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

and show that $Mdx + Ndy = 0$ is exact.

$\exists f : \quad df = Mdx + Ndy$.

Let $g(x,y) = \int M dx$ be the partial integral of M such that $\frac{\partial g}{\partial x} = M$

We first prove

$\left(N - \frac{\partial g}{\partial y}\right)$ is a function of y alone.

$$\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

Now we consider :

$$f(x,y) = g(x,y) + \int \left(N - \frac{\partial g}{\partial y}\right) dy$$

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$$df = dg + d\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right)$$

$$= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \cancel{\frac{\partial}{\partial x}}\Big(\underbrace{\int \left(N - \frac{\partial g}{\partial y}\right) dy}_{\text{function of } y \text{ only}}\Big) dx \rightarrow 0 \quad +$$

$$\underbrace{\frac{\partial}{\partial y}\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right)}\, dy$$

$$= \left(\frac{\partial g}{\partial x}\right) dx + \cancel{\frac{\partial g}{\partial y} dy} + \left(N - \cancel{\frac{\partial g}{\partial y}}\right) dy \qquad = M dx + N dy$$

$$df = 0$$

$$f = c \quad \} \text{Solution}$$

Ex: $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0$

$M = x^2 - 4xy - 2y^2 \qquad N = y^2 - 4xy - 2x^2$

$\frac{\partial M}{\partial y} = -4x - 4y \qquad \frac{\partial N}{\partial x} = -4y - 4x$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow$ given diff. eq^n is exact.

$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M dx + N dy$

$\frac{\partial f}{\partial x} = x^2 - 4xy - 2y^2$ and $\frac{\partial f}{\partial y} = y^2 - 4xy - 2x^2$

$\Rightarrow f = \frac{x^3}{3} - 2x^2 y - 2y^2 x + c_1(y)$

OR $f = \frac{y^3}{3} - 2xy^2 - 2x^2 y + c_2(x)$

$\frac{\partial f}{\partial y} =$

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$$\frac{\partial f}{\partial y} = -2x^2 - 4xy + c_1'(y)$$

$$y^2 - \cancel{4xy} - \cancel{2x^2} = -\cancel{2x^2} - \cancel{4xy} + c_1'(y)$$

$$\Rightarrow c_1(y) = \frac{y^3}{3} + c$$

Hence, $f = \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} + c \quad = c_1$ (Some const)

Solution of this (given) differential equation.

Integrating factor:

$Mdx + Ndy = 0$ is not exact

But

$I(x,y)\, Mdx + I(x,y)\, Ndy = 0$

become exact then

$I(x,y)$ is called integrating factor.

Rule I: By Inspection

$d(xy) = ydx + xdy$

$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$ | $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

$d\left(\ln \frac{y}{x}\right) = \frac{xdy - ydx}{xy}$ | $d\left(\ln \frac{y}{x}\right) = \frac{ydx - xdy}{xy}$

$d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{xdy - ydx}{x^2+y^2}$ | $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2+y^2}$

$d(\ln xy) = \frac{ydx + xdy}{xy}$

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Ex. $y(y^2+1)\,dx + x(y^2-1)\,dy = 0$

$M = y(y^2+1) \qquad \frac{\partial M}{\partial y} = 3y^2+1 \qquad N = x(y^2-1) \qquad \frac{\partial N}{\partial x} = y^2-1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Not exact.

$y^2(y\,dx + x\,dy) + (y\,dx - x\,dy) = 0$

Multiplying $\frac{1}{y^2} \rightarrow$ IF

$(y\,dx + x\,dy) + \left(\frac{y\,dx - x\,dy}{y^2}\right) = 0$

$\Rightarrow d(xy) + d\left(\frac{x}{y}\right) = 0$

$\boxed{xy + \frac{x}{y} = c} \rightarrow$ Solution.

Ex. $(y^2e^x + 2xy)\,dx - x^2\,dy = 0$

Multiplying by $\frac{1}{y^2}$

$dx\,e^x + 2\frac{x}{y}\,dx - \frac{x^2}{y^2}\,dy = 0$

$*\ d\left(\frac{x^2}{y}\right) = \frac{2x}{y}\,dx - \frac{x^2}{y^2}\,dy$

$\Rightarrow e^x\,dx + d\left(\frac{x^2}{y}\right) = 0$

Solution : $\boxed{e^x + \frac{x^2}{y} = c}$

Rule II: $M\,dx + N\,dy = 0$ is homogeneous & $Mx + Ny \neq 0$

$I(x,y) = \frac{1}{Mx+Ny}$ is an Integrating factor.

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Ex: $\underbrace{(x^2y - 2xy^2)}_{M}dx - \underbrace{(x^3 - 3x^2y)}_{N}dy = 0$

$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$

$\Rightarrow IF = \frac{1}{x^2y^2}$.

$\underbrace{\left(\frac{1}{y} - \frac{2}{x}\right)}_{M'}dx - \underbrace{\left(\frac{x}{y^2} - \frac{3}{y}\right)}_{N'}dy = 0$

$\frac{\partial M'}{\partial y} = \frac{-1}{y^2} \quad \frac{\partial N'}{\partial x} = \frac{-1}{y^2}$ $\Rightarrow$ It is ~~now~~ an exact equation.

$\frac{\partial f}{\partial x} = \frac{1}{y} - \frac{2}{x} \qquad \frac{\partial f}{\partial y} = \frac{-x}{y^2} + \frac{3}{y}$

$f = \frac{x}{y} - 2\ln x + c_1(y)$

$\frac{\partial f}{\partial y} = \frac{-x}{y^2} - 2\ln x \times 0 + c_1'(y) = \frac{-x}{y^2} + \frac{3}{y}$

$\Rightarrow c_1'(y) = \frac{3}{y}$

$c_1(y) = 3\ln y + c'$

Solution:- $f = \frac{x}{y} - 2\ln x + 3\ln y = c$

Rule III: $Mdx + Ndy = 0$ is of the form
$f_1(x,y)\, y\, dx + f_2(x,y)\, x\, dy = 0$

then $I(x,y) = \frac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$.

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Ex: $(xy\sin xy + \cos xy)\, y\,dx + (xy\sin xy - \cos xy)\, x\,dy = 0$

$$Mx - Ny = xy\left[xy\sin xy + \cos xy - \cancel{x}\, xy\sin xy + \cos xy\right]$$

$$= 2xy\cos xy$$

$\Rightarrow$ $IF = \frac{1}{2xy\cos xy}$

$\rightarrow$ $\left(\tan xy + \frac{1}{xy}\right) y \cdot \frac{dx}{2} + \left(\tan xy - \frac{1}{xy}\right)\frac{x}{2}\, dy = 0$

$$\frac{\partial f}{\partial x} = \frac{y}{2}\tan xy + \frac{1}{2x}$$

$$f = \frac{y}{2}\,\frac{\ln|\sec xy|}{y} + \frac{1}{2}\ln x + c_1(y)$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2}\,\frac{1}{\cos xy}(-\sin xy)\cdot x + 0 + c_1'(y) = \frac{x}{2}\tan xy - \frac{1}{2y}$$

$$c_1'(y) = -\frac{1}{2y}$$

$$c_1(y) = -\frac{\ln y}{2} + c'$$

Solution:- $\frac{1}{2}\ln|\sec xy| + \frac{1}{2}\ln x - \frac{1}{2}\ln y = c''$

or $-\ln|\cos(xy)| + \ln x - \ln y = c'''$

$$\frac{x}{y\cos(xy)} = c$$

$$\boxed{x = c\,y\cos(xy)}$$

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Rule IV:

$$x^a y^b (my\,dx + nx\,dy) + x^r y^s (py\,dx + qx\,dy) = 0$$

$$\text{IF} = x^h y^k \xrightarrow{\text{satisfy}} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{find } h \text{ and } k.$$

{h & k can be obtained by applying the necessary condition of exactness}

Ex.

$$(3x + 2y^2)y\,dx + 2x(2x + 3y^2)\,dy = 0$$

$$x(3y\,dx + 4x\,dy) + y^2(2y\,dx + 6x\,dy) = 0$$

$$\text{If} = x^h y^k \Rightarrow \underbrace{x^h y^{k+1}(3x+2y^2)}_{M}dx + \underbrace{2x^{h+1}y^k(2x+3y^2)}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = 3x^{h+1}(k+1)y^k + 2x^h(k+3)y^{k+2}$$

$$\frac{\partial N}{\partial x} = 4y^k(h+2)x^{h+1} + 6y^{k+2}(h+1)x^h$$

Satisfy Necessary condition:-

$$3x^{h+1}(k+1)y^k + 2x^h(k+3)y^{k+2} = 4y^k x^{h+1}(h+2) + 6y^{k+2}(h+1)x^h$$

$$\Rightarrow 3(k+1) = 4(h+2) \quad \text{and} \quad 2(k+3) = 6(h+1)$$

$$k+1 = \frac{4}{3}(h+2) \qquad k+3 = 3(h+1)$$

$$\Rightarrow \frac{4h}{3} + \frac{5}{3} = 3h$$

$$4h + 5 = 9h$$

$$\boxed{h = 1,\ k = 3}$$

$$\Rightarrow \text{IF} = xy^3$$

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$M = xy^4(3x+2y^2) = 3x^2y^4 + 2xy^6$

$N = 2x^2y^3(2x+3y^2) = 4x^3y^3 + 6x^2y^5$

$\frac{\partial f}{\partial x} = M \Rightarrow f = x^3y^4 + x^2y^6 + c_1(y)$

$\frac{\partial f}{\partial y} = 4y^3x^3 + 6y^5x^2 + c_1'(y) = N$

$\Rightarrow c_1'(y) = 0$

$c_1 = \text{const}.$

Solution:- $\boxed{x^3y^4 + x^2y^6 = c}$

Rule V :

$M(x,y)\,dx + N(x,y)\,dy = 0 \rightarrow$ Not exact.

I.F $= I(x,y)$

$I(x,y)\,M(x,y)\,dx + N(x,y)\,I(x,y)\,dy = 0$

$\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial x}(IN)$ — (1)

This necessary condition must be satisfied.

Assume:-

(i) The IF $\rightarrow$ I is a function of x alone.

(1): $I\frac{\partial M}{\partial y}$ or $I\underbrace{(M_y)}_{\frac{\partial M}{\partial x}} = \underbrace{\frac{dI}{dx}}_{I_x} \cdot N + I\underbrace{(N_x)}_{\frac{\partial N}{\partial x}}$

$\Rightarrow \frac{dI}{dx} = \frac{I(M_y - N_x)}{N}$

$\frac{dI}{I} = \underbrace{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}_{\text{function of } x \text{ alone} \rightarrow \text{say } f(x)} dx$

$\Rightarrow I = e^{\int f dx}$

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Similarly: If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone say $f(y)$

then $I = e^{\int f dy}$

Ex. Solve:- $(x^2+y^2+x)\,dx + xy\,dy = 0$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(y) = \frac{1}{x} = f(x)$$

$$IF = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$\underbrace{(x^3+xy^2+x^2)}_{M'}\,dx + \underbrace{x^2y}_{N'}\,dy = 0$$

$$\frac{\partial f}{\partial x} = x^3+xy^2+x^2 \qquad \frac{\partial f}{\partial y} = x^2y$$

$$f = \frac{x^2y^2}{2} + g(x)$$

$$\frac{\partial f}{\partial x} = \cancel{xy^2} + g'(x) = \cancel{xy^2} + x^3 + x^2$$

$$\Rightarrow g(x) = \frac{x^4}{4} + \frac{x^3}{3} + c'$$

Solution:- $\frac{x^2y^2}{2} + \frac{x^4}{4} + \frac{x^3}{3} = c''$

$$\boxed{3x^4 + 6x^2y^2 + 4x^3 = c}$$

Ex. $\underbrace{(2xy^4e^y + 2xy^3 + y)}_{M}\,dx + \underbrace{(x^2y^4e^y - x^2y^2 - 3x)}_{N}\,dy = 0$

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$$\frac{\partial M}{\partial y} = 2x(4y^3e^y) + 2xy^4e^y + 6xy^2 + 1$$

$$= 2e^y xy^3(4x+y) + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8xy^3e^y + 8xy^2 + 4 = 4(2xy^3e^y + 2xy^2 + 1)$$

$$= 4\cancel{y}\,\frac{M}{y}$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right) = -4y^{-1}$$

$$\Rightarrow IF = e^{\int -4y^{-1}dy} = e^{-4\ln y} = \frac{1}{y^4}$$

$$\Rightarrow \underbrace{\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)}_{MI}dx + \underbrace{\left(x^2e^y - \left(\frac{x}{y}\right)^2 - \frac{3x}{y^4}\right)}_{NI}dy = 0$$

$$\frac{\partial f}{\partial x} = MI \Rightarrow f = 2e^y\frac{x^2}{2} + \frac{2x^2}{2y} + 0 + c_1(y)$$

$$\frac{\partial f}{\partial y} = x^2e^y - \frac{x^2}{y^2} + c_1'(y)$$

$$\Rightarrow c_1'(y) = -\frac{3x}{y^4}$$

$$c_1(y) = \frac{x}{y^3} + c_1$$

Solution :- $\boxed{e^y x^2 + \frac{x^2}{y} + \frac{x}{y^3} = c}$

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Linear Differential Equation:-

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{(linear in } y\text{)}$$

$$\underbrace{dy + P(x)\,dx\,y}_{} = Q(x)\,dx$$

$M = Py \qquad N = 1$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{1}(P-0) = P(x) = \text{function of } x \text{ only.}$$

$$\boxed{IF = e^{\int P(x)\,dx}}$$

$$Py\,e^{\int P dx}\,dx + e^{\int P dx}\,dy = Q(x)\,e^{\int P dx}\,dx$$

$$d\left(y \cdot e^{\int P dx}\right) = Q(x)\,e^{\int P dx}\,dx$$

$$\Rightarrow \boxed{y \cdot e^{\int P dx} = \int Q(x)\,e^{\int P dx}\,dx + c} \rightarrow \text{Solution}$$

i.e $y \cdot (IF) = \int Q(x)\,(IF)\,dx + c$ where $IF = e^{\int P dx}$.

Ex:

$$(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$$

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$$

$\uparrow P \qquad \uparrow Q$

$$IF = e^{\int \frac{2x}{1+x^2}dx} = 1+x^2$$

$$y(1+x^2) = \int 4x^2\,dx + c = \frac{4}{3}x^3 + c$$

Ex.

$$(x+2y^3)\frac{dy}{dx} = y$$

$$\frac{dx}{dy} = 2y^2 + \frac{x}{y} \Rightarrow \frac{dx}{dy} + x\left(-\frac{1}{y}\right) = 2y^2$$

$\uparrow P \qquad \uparrow Q$

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$$IF = e^{\int P dy} = e^{\int -\frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}$$

$$\Rightarrow x\left(\frac{1}{y}\right) = \int \frac{2y^2}{y} dy + c = y^2 + c$$

$$\frac{x}{y} = y^2 + c$$

Equations Reducible to Linear form:-

An equation of the form

$$f'(y) \frac{dy}{dx} + P f(y) = Q$$

Substitute: $f(y) = v \Rightarrow f'(y) \frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dv}{dx} + Pv = Q$$

$$IF = e^{\int P dx}$$

Solution:- $v \cdot e^{\int P dx} = \int Q e^{\int P dx} dx + c$

A special case: Bernoulli's Equation

$$\frac{dy}{dx} + Py = Q y^n \qquad n \neq 0, 1$$

$$y^{-n} \frac{dy}{dx} + P y^{-n+1} = Q$$

Substitute: $y^{-n+1} = v \Rightarrow (1-n) y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dv}{dx} + P(1-n) v = Q(1-n) \rightarrow \text{linear in } v.$$

Ex.

$$(x^2 - 2x + 2y^2) dx + 2xy\, dy = 0$$

$$\frac{dy}{dx} + \left(\frac{x^2 - 2x + 2y^2}{2xy}\right) = 0 \Rightarrow \frac{dy}{dx} + \frac{y}{x} = -y^{-1}\left(\frac{x^2 - 2x}{2x}\right)$$

$$2y \frac{dy}{dx} + \frac{2y^2}{x} = 2 - x$$

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Substitute $y^2 = v$

$$\Rightarrow \frac{dv}{dx} + \frac{2}{x} \cdot v = (2-x)$$

$$IF = e^{\int \frac{2}{x} dx} = x^2$$

$$\Rightarrow v \cdot x^2 = \int (2-x) x^2 \, dx + c$$

$$\boxed{y^2 \cdot x^2 = \frac{2x^3}{3} - \frac{x^4}{4} + c}$$

Ex. Solve: $\frac{dz}{dx} + \frac{z}{x} \ln z = \frac{z}{x} (\ln z)^2$

$$\frac{1}{z} \frac{dz}{dx} + \frac{\ln z}{x} = \frac{(\ln z)^2}{x}$$

$\ln z = t \quad \Rightarrow \frac{1}{z} \frac{dz}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x}$$

$$t^{-2} \frac{dt}{dx} + \frac{t^{-1}}{x} = \frac{1}{x}$$

$t^{-1} = v \quad \Rightarrow \frac{-1}{t^2} \frac{dt}{dx} = \frac{du}{dx}$

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x}$$

$$\text{or} \quad \frac{dv}{dx} + \left(\frac{1}{x}\right) v = \left(-\frac{1}{x}\right)$$

$$IF = e^{\int -\frac{1}{x} dx} = -\frac{1}{x}$$

$$\Rightarrow v \cdot \frac{1}{x} = \int \left(-\frac{1}{x} \cdot \frac{1}{x}\right) dx + c \quad = \frac{1}{x} + c$$

$$v = 1 + cx$$

$$\boxed{\frac{1}{\ln z} = 1 + cx}$$

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Linear diff. eq. of higher order with constant coefficients :-

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = X \quad \text{--- (1)}$$

$a_1, \dots, a_n \rightarrow$ Constant. $\quad X \rightarrow X(x)$

→ General Solution = Complementary function (CF) + Particular Integral (PI)
(n arbitrary const.) (free from const)

* PI: If v be any particular solution, then

$$\frac{d^n v}{dx^n} + a_1 \frac{d^{n-1} v}{dx^{n-1}} + \dots + a_n v = X$$

* CF: It is the general solution of the homogeneous equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = 0 \quad \text{--- (2)}$$

1. Let y_1, y_2 be two linearly independent solutions of (2) [when n=2], then

$c_1 y_1 + c_2 y_2$ is ~~also~~ the general solution of (2) {c_1 & c_2 are arbitrary const.}

Linearly independent → $y_1 \neq c\, y_2$ (c: const)
eg. $y_1 = \sin x$, $y_2 = \cos x$.

$$\frac{d^n}{dx^n}(c_1 y_1 + c_2 y_2) + a_1 \frac{d^{n-1}}{dx^{n-1}}(c_1 y_1 + c_2 y_2) + \dots + a_n (c_1 y_1 + c_2 y_2)$$

$$\Rightarrow c_1 \underbrace{\left(\frac{d^n y_1}{dx^n} + a_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \dots + a_n y_1\right)}_{=0} + c_2 \underbrace{\left(\frac{d^n y_2}{dx^n} + a_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \dots + a_n y_2\right)}_{=0} = 0$$

$$\underset{\text{n arb. const.}}{(u + v)} \rightarrow \text{general solution of (1)}$$

(CF + PI: u = CF, v = PI)

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$$\frac{d^n (u+v)}{dx^n} + a_1 \frac{d^{n-1}(u+v)}{dx^{n-1}} + \dots + a_n(u+v) = X$$

LHS

$$\underbrace{\left(\frac{d^n u}{dx^n} + a_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + a_n u\right)}_{=0\ [CF]} + \underbrace{\left(\frac{d^n v}{dx^n} + a_1 \frac{d^{n-1} v}{dx^{n}} + \dots + a_n v\right)}_{=X\ [PI]}$$

$$= X = \text{RHS}$$

Operator:-

$\frac{d}{dx}y$ — $\left[\frac{d}{dx} \text{ operating on } y\right]$ D

$\frac{d^2 y}{dx^2} \to D^2$

$\frac{d^3}{dx^3} \to D^3$

Product of operators: $(D-\alpha)(D-\beta)y = (D-\beta)(D-\alpha)y$

LHS $(D-\alpha)\left[\left(\frac{d}{dx} - \beta\right)(y)\right] = (D-\alpha)\left(\frac{dy}{dx} - \beta y\right)$

$$= \frac{d}{dx}\left(\frac{dy}{dx} - \beta y\right) - \alpha\left(\frac{dy}{dx} - \beta y\right) = \frac{d^2 y}{dx^2} - (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y$$

$$\boxed{(D-\alpha)(D-\beta)(y) = (D^2 - (\alpha+\beta)D + \alpha\beta)y} \quad -(1)$$

RHS $(D-\beta)\left[\left(\frac{d}{dx} - \alpha\right)y\right] = (D-\beta)\left[\frac{dy}{dx} - \alpha y\right]$

$$= \frac{d}{dx}\left(\frac{dy}{dx} - \alpha y\right) - \beta\left(\frac{dy}{dx} - \alpha y\right) = \frac{d^2 y}{dx^2} - (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y$$

$$\boxed{(D-\beta)(D-\alpha)y = (D^2 - (\alpha+\beta)D + \alpha\beta)y} \quad -(2)$$

From (1) and (2), we get

$$(D-\alpha)(D-\beta)y = (D-\beta)(D-\alpha)y = (D^2 - (\alpha+\beta)D + \alpha\beta)y$$

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In general,

$$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n] y = X$$

$$\boxed{[(D-\alpha_1)(D-\alpha_2) \ldots (D-\alpha_n)] y = X}$$

Solution of $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$

$$(D^2 + a_1 D + a_2) y = 0 \quad \text{——} \ (***)$$

Auxiliary equation $\underline{m^2 + a_1 m + a_2 = 0}$

Case I: Non Repeated Roots :

Suppose α_1 & α_2 are two non-repeated roots.

$$(D-\alpha_1)(D-\alpha_2) y = 0 \quad \text{——} \ (*)$$

A solution $(*)$ would be

$$(D-\alpha_2) y = 0$$

$$\frac{dy}{dx} = \alpha_2 y$$

$$\int \frac{dy}{y} = \int \alpha_2 x \Rightarrow y = e^{\alpha_2 x}$$

Another solution:-

$$(D-\alpha_1) y = 0$$

$$\frac{dy}{dx} = \alpha_1 y \Rightarrow \int \frac{dy}{y} = \int \alpha_1 x \Rightarrow y = e^{\alpha_1 x}$$

General Solution:-

$$\boxed{y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}}$$

Generalisation:-

If $\alpha_1, \alpha_2, \ldots \alpha_n$ are distinct roots of $m^n + a_1 m^{n-1} + \ldots + a_n = 0$

$$(D^n + a_1 D^{n-1} + \ldots + a_n) y = 0$$

auxiliary equation

~~Generalisation~~ (C.F.) $= c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \ldots + c_n e^{\alpha_n x}$

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Case II : Repeated Roots : $(D-\alpha)(D-\alpha)y=0$

Let $(D-\alpha)y = z$

$(D-\alpha)z = 0$

$$z = c_1 e^{\alpha x}$$

$$(D-\alpha)y = c_1 e^{\alpha x}$$

$$\frac{dy}{dx} - \alpha y = c_1 e^{\alpha x} \quad \text{[linear in } y\text{]}$$

$$IF = e^{\int -\alpha dx} = e^{-\alpha x}$$

Solution : $y \cdot e^{-\alpha x} = \int c_1 e^{\alpha x} \cdot e^{-\alpha x} dx + c_2$

$$y \cdot e^{-\alpha x} = c_1 x + c_2$$

$$\Rightarrow \boxed{y = (c_1 x + c_2)\, e^{\alpha x}}$$

Generalisation :-

If a root α is repeated r times; then the solution is

$$y = (c_1 x^{r-1} + c_2 x^{r-2} + \dots + c_r)\, e^{\alpha x}$$

Case III : Complex Roots :

Let $(\alpha + i\beta)$ & $(\alpha - i\beta)$ be two conjugate roots, then the solution will be

$$\bar{c}_1 e^{(\alpha + i\beta)x} + \bar{c}_2 e^{(\alpha - i\beta)x}$$

$$\Rightarrow \bar{c}_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$$

$$\Rightarrow e^{\alpha x}\left[\bar{c}_1(\cos\beta x + i\sin\beta x) + \bar{c}_2(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$

c_1 & $c_2 \rightarrow$ complex no.

$c_1 = \bar{c}_1 + i\bar{c}_2 \quad c_2 = \bar{c}_1 - i\bar{c}_2$

Generalisation :-

If $(\alpha + i\beta)$ and $(\alpha - i\beta)$ are conjugate complex roots, each repeated r times, then :-

$$\boxed{e^{\alpha x}\left[(p_1 + p_2 x + \dots + p_r x^{r-1})\cos\beta x + (q_1 + q_2 x + \dots + q_r x^{r-1})\sin\beta x\right]}$$

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Linear dependence and Independence

Two functions f and g are called linearly dependent on an open interval I if

$$f(x) = c\,g(x) \quad \forall x \text{ in } I \quad \text{for some constant } c$$

$$(\text{OR } g(x) = c f(x))$$

otherwise, they are called linear independent.

WRONSKIAN TEST To test whether two solutions of $y'' + p(x)y' + q(x)y = 0$ are linearly independent.

Define the Wronskian of solutions y_1 & y_2 to be

$$W(x) = y_1(x)\,y_2'(x) - y_1'(x) - y_2(x)$$

$$= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Theorem: Let y_1 & y_2 be solutions of $y'' + p(x)y' + q(x)y = 0$ on an open interval I. Then

1. Either $W(x) = 0 \ \forall x$ in I or $W(x) \neq 0 \ \forall x$ in I
2. y_1 & y_2 are linearly independent on I iff $W(x) \neq 0$ on I

Example:- $y_1(x) = \cos x$ & $y_2(x) = \sin x$, Solution of $y'' + y = 0$

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

So, y_1 and y_2 are linearly independent.

Example:- Consider $y'' + xy = 0$ and its two solutions

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12960}x^9 + \dots$$

$$\forall x \in \mathbb{R}$$

$$y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \frac{1}{45360}x^{10} + \dots$$

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Sol: Note that calculating Wronskian at any non-zero x will be difficult, so we consider $x=0$ for Wronskian

$$W(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Hence, $y_1(x)$ & $y_2(x)$ are linearly independent.

Finding the general solution of a 2^{nd} order homogeneous linear eq^n if we know one solution:

Let y_1 be some known particular solution of the equation

$$y'' + py' + qy = 0 \quad — (1)$$

Aim is to find another solution y_2 of the given equation so that y_1 and y_2 are linearly independent.

Since y_1 & y_2 are two solution, they will satisfy (1) i-e;

$$y_2'' + py_2' + qy_2 = 0 \quad \times y_1$$
$$y_1'' + py_1' + qy_1 = 0 \quad \times y_2$$

Subtract

$$(y_1 y_2'' - y_1'' y_2) + p(y_1 y_2' - y_1' y_2) + q(\cancel{y_1 y_2} - \cancel{y_1 y_2}) = 0 \quad — (2)$$

Recall: $W = y_1 y_2' - y_1' y_2$

$W' = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_1' y_2'} - y_1'' y_2$

$$\Rightarrow \quad W' + pW = 0 \quad — (3)$$

Solving (3):

$$W = Ce^{-\int p\,dx} \quad \{C \neq 0\}$$

$$\Rightarrow \quad \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{1}{(y_1)^2} Ce^{-\int p\,dx}$$

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{1}{y_1^2} Ce^{-\int p\,dx}$$

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$$\frac{y_2}{y_1} = \int \frac{c\,e^{-\int p\,dx}}{y_1^2}\,dx + c'$$

Since we are looking for a (particular) solution, we can set $c' = 0$ and $c = 1$;

$$\boxed{y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx}$$

Example:- Given that $y = x$ is a solution of the differential eq.

$$(1-x^2)y'' - 2xy' + 2y = 0 \qquad 0 < x < 1$$

Find the other linearly indep. solution.

$$p(x) = \frac{-2x}{1-x^2} \qquad y_1 = x$$

$$y_2 = x \int \frac{e^{-\int \left(\frac{-2x}{1-x^2}\right)dx}}{x^2}\,dx$$

$$= x \int \frac{e^{-\ln(1-x^2)}}{x^2}\,dx = x \int \frac{1}{x^2(1-x^2)}\,dx$$

$$= x \int \left[\frac{1}{x^2} + \frac{1}{1-x^2}\right] dx$$

$$= x \int \left[\frac{1}{x^2} + \frac{1}{2(1+x)} + \frac{1}{2(1-x)}\right] dx$$

$$= x \left[-\frac{1}{x} + 2^{-1}\ln(1+x) - 2^{-1}\ln(1-x)\right]$$

$$y_2 = -1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)$$

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C.F.

$f(D)y=0$

Auxiliary equation $f(m)=0$

roots $\alpha_1, \alpha_2, \dots \alpha_n$

Case I: Roots are Real & non-repeated

$$Cf = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \dots + c_n e^{\alpha_n x}$$

Case II: Roots are Real but Repeated

$\alpha_1 = \alpha_2 = \alpha$, $\alpha_3, \alpha_4 \dots \alpha_n$

$$Cf = (c_1 + c_2 x) e^{\alpha x} + c_3 e^{\alpha_3 x} + \dots + c_n e^{\alpha_n x}$$

Case III: Roots are complex and non-repeated

$\alpha \pm i\beta$, $\alpha_3, \alpha_4 \dots \alpha_n$

$$Cf = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{\alpha_3 x} + \dots + c_n e^{\alpha_n x}$$

Case IV: Roots are complex and repeated

~~$Cf = e^{\alpha x}(c_1 + c_2$~~ $\alpha \pm i\beta, \alpha \pm i\beta, \alpha_5, \alpha_6, \dots \alpha_n$

$$Cf = e^{\alpha x}((c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x) + c_5 e^{\alpha_5 x} + \dots + c_n e^{\alpha_n x}$$

Evaluation of C.F.:

Ex. Solve the differential equation

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$$

Operator form:- $\to (D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$

Auxillary eq^n: $(m^4 - 2m^3 + 5m^2 - 8m + 4)\cancel{y} = 0$

Its Roots $\to$ $m = 1, 1, 2i, -2i$

The general solution:-

$$y = (c_1 + c_2 x)e^{1x} + e^{0x}(c_3 \cos 2x + c_4 \sin 2x)$$

$$\boxed{y = (c_1 + c_2)e^x + c_3 \cos 2x + c_4 \sin 2x}$$

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Ex. Suppose roots of the auxiliary eq. are:
$1, 2, 2, 1 \pm 2i, 1 \pm 2i$

The general solution:-
$$y = c_1 e^{x} + (c_2 + c_3 x) e^{2x} + e^{x}\left[(c_4 + c_5 x)\cos 2x + (c_6 + c_7 x)\sin 2x\right]$$

Determination of particular integral:

Diff. eq^n. $\quad f(D)\,y = X$

$$\boxed{P.I. = \frac{1}{f(D)} X}$$

1. General method of getting P.I.

$$\frac{1}{(D-\alpha)} X = e^{\alpha x}\int X e^{-\alpha x}\,dx$$

Proof:-

Let $y = \frac{1}{(D-\alpha)} X$

On operating $(D-\alpha)$ both sides, we get

$$(D-\alpha)\,y = X$$

$$\Rightarrow \frac{dy}{dx} - \alpha y = X \quad \text{(linear eq. in } y\text{)}$$

$$IF = e^{\int -\alpha\,dx} = e^{-\alpha x}$$

$$y \cdot e^{-\alpha x} = \int X e^{-\alpha x}\,dx + c$$

{Const. part will be taken care by CF $\Rightarrow$ don't put in PI}

$$\boxed{y = e^{\alpha x}\int X e^{-\alpha x}\,dx} + c e^{\alpha x}$$

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Ex. Solve $(D^2+a^2)y = \sec ax$

Auxiliary eq^n is $m^2+a^2=0 \Rightarrow m = \pm ai$

$CF = c_1 \cos ax + c_2 \sin ax$

$$PI = \frac{1}{D^2+a^2}\sec ax = \frac{1}{(D-ia)(D+ia)}\cdot \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax$$

Consider $\frac{1}{D-ia}\cdot \sec ax = e^{iax}\int \sec ax\, e^{-iax}\,dx$

$$= e^{iax}\int \sec ax[\cos ax - i\sin ax]\,dx$$

$$= e^{iax}\int [1 - i\tan ax]\,dx$$

$$= e^{iax}\left[x + \frac{i}{a}\ln|\cos ax|\right]$$

Similarly, $\frac{1}{D+ia}\sec ax = e^{-iax}\left[x - \frac{i}{a}\ln|\cos ax|\right]$

$$\Rightarrow PI = \frac{1}{2ia}\left[e^{iax}\cdot x + \frac{i}{a}e^{iax}\ln|\cos ax| - e^{-iax}x + e^{-iax}\frac{i}{a}\ln|\cos ax|\right]$$

$$= \frac{1}{2ia}\left((e^{iax}-e^{-iax})x + \frac{i}{a}(e^{iax}+e^{-iax})\ln|\cos ax|\right)$$

$$\boxed{PI = \frac{x}{a}\sin ax + \frac{1}{a^2}\ln|\cos ax|\cdot\cos ax}$$

General Solution:

$$\boxed{y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\ln|\cos ax|\cdot\cos ax}$$

2. Short Methods for finding PI

*(a) X is of the form e^{ax}

(i) $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ (where $f(a)\neq 0$)

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Proof of :

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad \text{where } f(a) \neq 0$$

Let $f(D) = D^n + C_1 D^{n-1} + \dots + C_{n-1} D + C_n$

Consider $f(D) e^{ax} = [D^n + C_1 D^{n-1} + \dots + C_{n-1} D + C_n] e^{ax}$

$$= [a^n + C_1 a^{n-1} + \dots + C_{n-1} a + C_n] e^{ax}$$

$$f(D) e^{ax} = f(a) e^{ax}$$

Operating both sides by $\frac{1}{f(D)}$

(ii) If $f(a) = 0$, then $f(D)$ must have a factor of type $(D-a)^r$

Then, $\boxed{\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}}$

Ex: $PI = \frac{1}{D^3 - D^2 - D + 1} \cdot e^x$ $\quad$ $f(D) = D^3 - D^2 - D + 1$

$a = 1$ and $f(a) = 0$

$$PI = \frac{1}{(D-1)^2 (D+1)} e^x \longrightarrow \left\{ \left(\frac{1}{D+1}\right) e^x = \frac{1}{(1+1)} e^x \right\}$$

$$= \frac{1}{(D-1)^2} \frac{1}{2} e^x$$

$$= \frac{1}{2} \frac{x^2 e^x}{2!} = \frac{x^2 e^x}{4}.$$

Ex: $PI = \frac{1}{D^2+D+5} \cdot 7 e^{0x} = 7 \frac{1}{D^2+D+5} e^{0x} = \frac{7}{(0+0+5)} = \frac{7}{5}$

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Linear eq^n: $f(D)y = X$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = \sin x$$

$$(D^2 + 5D + 6)y = \sin x$$

$$f(D)y = \sin x \underbrace{= f(x)}_{\text{Non-homogeneous}}$$

Homogeneous eq^n: If $y = y_1$ is a solution, $y = cy_1$ is also a sol

$$f(D)y = 0$$

or $$\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 6xy = 0$$

→ General sol^n of homogeneous eq^n → $c_1y_1 + c_2y_2$

If eq^n is not homogeneous,
General Solution → $\underbrace{(c_1y_1 + c_2y_2)}_{CF} + \underbrace{y_p}_{PI}$

PI

* $\frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2}$

$$D^{-1}Dy = D^{-1}x$$

$$y = \frac{1}{D}x = \frac{1}{D-0}x$$

$AA^{-1} = I$ {Identity}
$DD^{-1} = 1$

$$y = e^{0x}\int x e^{-0x}\,dx = \frac{x^2}{2}$$

* $\frac{dy}{dx} - 2y = x$

$$(D-2)y = x$$

$$y = \frac{1}{D-2}x = e^{2x}\int x \cdot e^{-2x}\,dx$$

$\because$ $$\frac{1}{D-\alpha}f(x) = e^{\alpha x}\int f(x)e^{-\alpha x}\,dx$$

Algebra of operator: $D^2y = DX$
$\Rightarrow Dy = X$

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$f(D)y = X$

$$P.I. = \frac{1}{f(D)} X = \frac{1}{(D-\alpha_1)(D-\alpha_2)} X$$

Use partial fraction

$$\frac{1}{(D-\alpha_1)}\underbrace{\left[\frac{1}{(D-\alpha_2)} X\right]}_{\text{operate this first}} \rightarrow \left(\frac{1}{D-\alpha_1}\right)\left[e^{\alpha_2 x}\int e^{-\alpha_2 x} X\, dx\right]$$

operate this next.

---

2. Short Methods for finding P.I

(b) X is $\cos ax$ or $\sin ax$

$$P.I. = \frac{1}{f(D)}\cos ax = \frac{1}{\varphi(D^2)}\cos ax = \frac{1}{\varphi(-a^2)}\cos ax$$

provided $\varphi(a^2) \neq 0$

$\boxed{\text{Replace } D^2 \text{ by } -a^2}$

Ex. $P.I. = \frac{1}{D^4+D^2+1}\cos 2x = \frac{1}{(D^2)^2+D^2+1}\cos 2x \quad \left[\begin{matrix}(a=2) \\ \Rightarrow -a^2=-4\end{matrix}\right]$

$$= \frac{1}{16-4+1}\cos 2x = \frac{1}{13}\cos 2x.$$

Ex. $P.I. = \frac{1}{D^2-2D+1}\cos 3x \qquad \begin{matrix} a=3 \\ -a^2=-9\end{matrix}$

$$= \frac{1}{-9-2D+1}\cos 3x = \frac{1}{-2(D+4)}\cos 3x$$

Follow formula

Multiply with conjugate ✓

$$= \frac{(D-4)}{-2}\frac{1}{D^2-16}\cos 3x$$

$$= \frac{(D-4)}{-2}\frac{1}{(-9-16)}\cos 3x = \frac{1}{50}\left[\frac{d}{dx}\cos 3x - 4\cos 3x\right]$$

$$= \frac{1}{50}(-3\sin 3x - 4\cos 3x)$$

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If $\varphi(-a^2)=0$

Ex: $\frac{1}{D^2+a^2}\sin ax = \text{img}\left\{\frac{1}{D^2+a^2}\cos ax + i\frac{1}{D^2+a^2}\sin ax\right\}$

$= \text{imag}\left\{\frac{1}{D^2+a^2} e^{iax}\right\}$

Consider → $\frac{1}{D^2+a^2} e^{iax} = \frac{1}{(D-ai)(D+ai)} e^{iax}$

$= \frac{1}{(D-ai)^1}\frac{1}{(2ai)} e^{iax} = \frac{1}{2ai}\frac{x}{\lfloor 1} e^{iax}$

$= \frac{x}{2ia}\{\cos ax + i\sin ax\} = \frac{x}{2a}\sin ax - i\frac{x}{2a}\cos ax$

Hence, $PI = -\frac{x}{2a}\cos ax$

Rules:-

$$\boxed{\frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos ax}$$

$$\boxed{\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax}$$

Ex: Solve $(D^2+4)y = \sin^2 x$

$y = \frac{1}{D^2+4}\sin^2 x = \frac{1}{2}\frac{1}{D^2+4}(1-\cos 2x)$

$= \frac{1}{2}\frac{1}{D^2+4}(e^{0x}) - \frac{1}{2}\frac{1}{D^2+4}\cos 2x$

$= \frac{1}{2}\frac{1}{0+4}e^{0x} - \frac{1}{2}\left(+\frac{x}{2(2)}\sin 2x\right)$

$= \frac{1}{8}(1 - x\sin 2x)$

General Solution:- $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{8}(1 - x\sin 2x)$

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* (c) X is x^m or a polynomial of degree m.
Take out the lowest degree term $f(D)$, so as to reduce it in the form $[1 \pm F(D)]^{\alpha}$.
Take it to numerator and Expand it.

Ex: Find $\frac{1}{D^3-D^2-6D}(x^2+1)$

$$= \frac{1}{-6D\left(1+\frac{D}{6}-\frac{D^2}{6}\right)}(x^2+1) = \frac{-1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$= \frac{-1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2-\dots\right](x^2+1)$$

$$= \frac{-1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}+\dots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[(x^2+1)-\frac{1}{6}(2x)+\frac{7}{36}\cdot 2\right]$$

$\left\{ \because D^3(x^2+1)=0 \;\; \therefore D^n(x^2+1)=0 \right\}$

$$= -\frac{1}{6D}\left(x^2-\frac{x}{3}+\frac{25}{18}\right)$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

* (d) X is $e^{ax}V$, where V is any function of x.

Rule: $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$

Ex: $P.I. = \frac{1}{D^2+3D+2}e^{2x}\sin x$

$$= e^{2x}\frac{1}{(D+2)^2+3(D+2)+2}\sin x = e^{2x}\frac{1}{D^2+7D+12}\sin x$$

$$= e^{2x}\frac{1}{-1+7D+12}\sin x = e^{2x}\frac{1}{7D+11}\sin x = \frac{e^{2x}}{7}\frac{\left(D-\frac{11}{7}\right)}{\left(D^2-\frac{11^2}{7^2}\right)}\sin x$$

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$$= \frac{e^{2x}}{7} \frac{\left(D - \frac{11}{7}\right)}{\left[1 - \frac{11^2}{7^2}\right]} \sin x = \frac{e^{2x}}{-7\left(1 + \frac{121}{49}\right)} \left(\cos x - \frac{11}{7}\sin x\right)$$

$$= \frac{-7\, e^{2x}}{(49+121)} \left(\cos x - \frac{11}{7}\sin x\right) = \boxed{\frac{+e^{2x}(11\sin x - 7\cos x)}{170}}$$

* (e) X is xV

Rule: $\frac{1}{f(D)}(x.V) = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V$

Ex: $P.I. = \frac{1}{D^2 - 2D + 1} x\sin x$

$$= x \frac{1}{D^2-2D+1}\sin x - \frac{(2D-2)}{(D^2-2D+1)^2}\sin x$$

$$= x \frac{1}{-2D}\sin x - \frac{2(D-1)}{(D-1)^4}\sin x$$

$$= \frac{-x}{2}(-\cos x) - 2\frac{1}{(D-1)^3}\sin x$$

OR

$$= x \frac{+1}{-2D}\sin x - \frac{2(D-1)}{(-2D)^2}\sin x$$

$$= \frac{-x}{2}(-\cos x) - 2\frac{(D-1)\sin x}{4(-1)}$$

$$= \frac{x\cos x}{2} + \frac{1}{2}(\cos x - \sin x)$$

$$= \boxed{\frac{1}{2}(x\cos x + \cos x - \sin x)}$$

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Method of variation of parameters

$$(D^2+a_1D+a_2)y = f(x) \text{ or } X \quad (1) \rightarrow CF + PI$$

Let $y = c_1y_1 + c_2y_2$ is the general solution of $(D^2+a_1D+a_2)y = 0$

Both are general solⁿ: $\rightarrow CF + (PI)_1$, $\rightarrow CF + (PI)_2$

We assume that

$y = c_1(x)y_1 + c_2(x)y_2$ is a particular solution of (1), where c_1 & c_2 are functions of x

$$y' = c_1'y_1 + c_1y_1' + c_2'y_2 + c_2y_2'$$

$$y' = c_1y_1' + c_2y_2' + \underbrace{c_1'y_1 + c_2'y_2}_{=0}$$

$$\boxed{c_1'y_1 + c_2'y_2 = 0}$$

$$y'' = c_1'y_1' + c_1y_1'' + c_2'y_2' + c_2y_2''$$

Keep in mind:-
$$y_1'' + a_1y_1' + a_2y_1 = 0$$
and $$y_2'' + a_1y_2' + a_2y_2 = 0$$

Eqⁿ: $y'' + a_1y' + a_2y = X$

$$(c_1'y_1' + c_1y_1'' + c_2'y_2' + c_2y_2'') + a_1(c_1y_1' + c_2y_2') + a_2(c_1y_1 + c_2y_2) = X$$

$$c_1\underbrace{(y_1'' + a_1y_1' + a_2y_1)}_{0} + c_2\underbrace{(y_2'' + a_1y_2' + a_2y_2)}_{0} + c_1'y_1' + c_2'y_2' = X$$

$$\Rightarrow \boxed{c_1'y_1' + c_2'y_2' = X}$$

using Cramer's Rule: $c_1' = \dfrac{\begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\dfrac{y_2X}{W}$

{ $W \neq 0$ ∵ y_1 & y_2 are linearly independent }

$$c_2' = +\frac{y_1X}{W}$$

$$\boxed{c_1 = \int -\frac{y_2X}{W}\,dx}$$

$$\boxed{c_2 = \int \frac{y_1X}{W}\,dx}$$

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General Solution:-

$$y = c_1 y_1 + c_2 y_2 + y_1 \int -\frac{y_2 X\,dx}{W} + y_2 \int \frac{y_1 X\,dx}{W}$$

Ex.

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

Auxiliary eqn: $(D^2-1)y = 0$ or $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$.

CF $\rightarrow$ $c_1 e^x + c_2 e^{-x}$

Let PI $\Rightarrow y = C_1 y_1 + C_2 y_2 = C_1(x) e^x + C_2(x) e^{-x}$

$y' = C_1 y_1' + C_2 y_2' + (C_1' y_1 + C_2' y_2) \rightarrow 0$ $\quad y'$

$y'' = \ldots$

$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$

$C_1 = \int -\frac{y_2 X\,dx}{W} = -\int \frac{e^{-x}}{-2} \frac{2}{1+e^x}\,dx$

$= \int \frac{(+e^{-x})\,dx}{e^x(1+e^{-x})} = -\int \frac{t\,dt}{(1+t)} = -\int 1\,dt + \int \frac{1}{1+t}\,dt$

$= -t + \ln|1+t| = -e^{-x} + \ln|1+e^{-x}|$

$= -e^{-x} + \ln|1+e^x| - \ln e^x$

$= -x - e^{-x} + \ln|1+e^x|$

$C_2 = \int \frac{+y_1 X\,dx}{W} = \int \frac{e^x}{-2} \frac{2\,dx}{1+e^x}$

$= -1 \int \frac{dt}{1+t} = -\ln|1+e^x|$

$\Rightarrow$ PI: $C_1(x) y_1 + C_2(x) y_2 = e^x\left[x + e^{-x} - \ln|1+e^x|\right] + e^{-x}\left[-\ln|1+e^x|\right]$

$= xe^x + 1 - \ln|1+e^x| \{e^x + e^{-x}\}$

General Solution:- $\boxed{e^x\left[x + e^{-x} - \ln|1+e^x| + c_1\right] + e^{-x}\left[-\ln|1+e^x| + c_2\right]}$

If we add const. of integration in PI, no need of calculating CF.

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CAUCHY EULER EQUATIONS:

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n y = X$$

$$(x^n D^n + a_1 x^{n-1} D^{n-1} + \dots + a_n)y = X$$

Substitute:- $x = e^z \Rightarrow \ln x = z$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\frac{dy}{dz} = x\frac{dy}{dx} = xD \equiv D_1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2}$$

$$x^2D^2 \equiv D_1(D_1 - 1)$$

$$\vdots$$

$$x^3D^3 \equiv D_1(D_1-1)(D_1-2)$$

$$\vdots$$

$$\boxed{x^nD^n \equiv D_1(D_1-1)\dots(D_1-n+1)}$$

where $D \equiv \frac{d}{dx}$ and $D_1 \equiv \frac{d}{dz}$ ....

Ex.

$$(x^2D^2 - xD + 2)y = x\ln x$$

$x = e^z \Leftrightarrow z = \ln x$ and $D_1 \equiv \frac{d}{dz}$

$$(D_1(D_1-1) - D_1 + 2)y = z \cdot e^z$$

$$(D_1^2 - 2D_1 + 2)y = z \cdot e^z$$

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CF: AE: $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

$$CF = e^{z}(c_1 \cos z + c_2 \sin z)$$

PI:

$$= \frac{1}{D_1^2 - 2D_1 + 2} z \cdot e^{z}$$

$$= e^{z} \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} z = e^{z} \frac{1}{D_1^2 + 1} z$$

$$= e^{z} (1 + D_1^2)^{-1} z$$

$$= e^{z} (1 - D_1^2 \ldots) z = e^{z} (z - 0 \ldots)$$

$$= z \cdot e^{z} = x \ln x.$$

General Solution :-

$$\boxed{y = x(c_1 \cos(\ln x) + c_2 \sin(\ln x)) + x \ln x}$$

Equations reducible to Euler-Cauchy form :-

Ex-1.

$$\frac{d^3y}{dx^3} - \frac{4}{x}\frac{d^2y}{dx^2} + \frac{5}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 1$$

Multiply x^3 :-

$$x^3 \frac{d^3y}{dx^3} - 4x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} - 2y = x^3$$

$$(x^3 D^3 - 4D^2 x^2 + 5xD - 2)y = x^3$$

$x = e^{z} \Rightarrow z = \ln x$

$$(D_1(D_1-1)(D_1-2) - 4D_1(D_1-1) + 5D_1 - 2)y = e^{3z}$$

$$(D_1^3 - 3D_1^2 + 2D_1 - 4D_1^2 + 4D_1 + 5D_1 - 2)y = e^{3z}$$

$$(D_1^3 - 7D_1^2 + 11D_1 - 2)y = e^{3z}$$

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$D_1 = 2,$

Ans:- $y = c_1x^2 + c_2 x^{(5+\sqrt{21})/2} + c_3 x^{(5-\sqrt{21})/2} - \frac{x^3}{5}$

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Ex.2 $2x^2y\frac{d^2y}{dx^2}+4y^2=x^2\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}$

Hint:- $y=z^2 \Rightarrow \frac{dy}{dx}=2z\frac{dz}{dx}$ or $\frac{dz}{dx}=\frac{1}{2\sqrt{y}}\frac{dy}{dx}$

$x^2\frac{d^2z}{dx^2}-x\frac{dz}{dx}+z=0$

$x=e^t$

$(D_1^2-2D_1+1)z=0 \Rightarrow z=(c_1+c_2t)e^t$

$z=[c_1+c_2(\ln x)]x$

$\boxed{y=(c_1+c_2\ln x)^2x^2}$

Ex.3 $(a+bx)^n\frac{d^ny}{dx^n}+a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}(a+bx)\frac{dy}{dx}+a_ny=X$

Substitute:- $a+bx=v \Rightarrow \frac{dv}{dx}=b$

$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}=b\frac{dy}{dv}$

$\frac{d^2y}{dx^2}=b\left(\frac{d^2y}{dv^2}\cdot\frac{dv}{dx}\right)=b^2\frac{d^2y}{dv^2}$

$\Rightarrow\dots$ $\frac{d^ny}{dx^n}=b^n\frac{d^ny}{dv^n}$

$b^nv^n\frac{d^ny}{dv^n}+a_1v^{n-1}b^{n-1}\frac{d^{n-1}y}{dv^{n-1}}+\dots$

$v^n\frac{d^ny}{dv^n}+\frac{a_1}{b^1}v^{n-1}\frac{d^{n-1}y}{dv^{n-1}}+\dots+\frac{a_{n-1}}{b^{n-1}}v\frac{dy}{dv}+\frac{a_ny}{b^n}=\frac{X}{b^n}$

$v=e^z \Rightarrow z=\ln v$

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Ex:

$$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos(\ln(1+x))$$

$$1+x = v \Rightarrow \frac{dv}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{dy}{dv} \text{ and } \frac{d^ny}{dx^n} = \frac{d^ny}{dv^n}$$

$$v^2\frac{d^2y}{dv^2} + v\frac{dy}{dv} + y = 4\cos(\ln v)$$

$$v = e^z \Rightarrow \ln v = z$$

$$(v^2D^2 + vD + 1)y = 4\cos(\ln v)$$

$$(D_1(D_1-1) + D_1 + 1)y = 4\cos z$$

$$(D_1^2+1)y = 4\cos z$$

CF

$$m^2+1=0$$

$$\Rightarrow m = \pm i$$

$$CF = c_1\cos z + c_2\sin z$$

$$PI = \frac{1}{D_1^2+1}4\cos z = 4\frac{z}{2(1)}\sin(1)z = 2z\sin z$$

$$\Rightarrow y = c_1\cos(\ln v) + c_2\sin(\ln v) + 2(\ln v)\sin(\ln v)$$

$$\boxed{y = c_1\cos[\ln(1+x)] + c_2\sin[\ln(1+x)] + 2\ln(1+x)\sin[\ln(1+x)]}$$

Simultaneous Ordinary differential equations:-

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots y_n)$$

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I. Method of Elimination

Eg. Solve :- $\frac{dx}{dt} = 7x - y$ $\qquad \frac{dy}{dt} = 2x + 5y$

$x, y \rightarrow$ Dependent variables (depends on t)

Denoting $\frac{d}{dt} \equiv D$;

$$① — (D-7)x + y = 0 \quad] \times 2$$
$$② — -2x + (D-5)y = 0 \quad] \times (D-7)$$

Multiplying ① by 2 and operating ② by $(D-7)$ and then adding :-

$$2(D-7)x + 2y = 0$$
$$-2(D-7)x + (D-5)(D-7)y = 0$$

$$(D^2 - 12D + 37)y = 0$$

AE: $m^2 - 12m + 37 = 0$

$$m = \frac{12 \pm \sqrt{144 - 148}}{2} = 6 \pm i$$

CF = $y = e^{6t}(c_1 \cos t + c_2 \sin t)$

$\Rightarrow x = \frac{(D-5)y}{2}$

$$* \; Dy = e^{6t}(-c_1 \sin t + c_2 \cos t) + 6e^{6t}(c_1 \cos t + c_2 \sin t)$$
$$= e^{6t}((6c_1 + c_2)\cos t + (6c_2 - c_1)\sin t)$$

$$Dy - 5y = e^{6t}[(c_1 + c_2)\cos t + (c_2 - c_1)\sin t]$$

$$\Rightarrow x = \frac{1}{2} e^{6t}[(c_1 + c_2)\cos t + (c_2 - c_1)\sin t]$$

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II. Method of Differentiation

Ex. Determine the general solutions for x & y :-

$$\frac{dx}{dt} - y = t \qquad \frac{dy}{dt} + x = 1 \Rightarrow \frac{dy}{dt} = 1 - x$$

$$\frac{d^2x}{dt^2} - \frac{dy}{dt} = 1$$

$$\frac{d^2x}{dt^2} + x = 2$$

$(m^2+1) = 0 \Rightarrow m = \pm i$

$$CF = c_1 \cos t + c_2 \sin t$$

$$PI = \frac{1}{D^2+1}(2) = (1+D^2)^{-1}(2) = (2 - 0 \cdots) = 2$$

General Solution :- $\boxed{x = c_1 \cos t + c_2 \sin t + 2}$

$$\frac{dx}{dt} = -c_1 \sin t + c_2 \cos t$$

$$y = \frac{dx}{dt} - t \Rightarrow \boxed{y = -c_1 \sin t + c_2 \cos t - t}$$

Ex. Solve: $\frac{dy_1}{dx} = y_1 + y_2 + x$

$$\frac{dy_2}{dx} = -4y_1 - 3y_2 + 2x$$

$$\frac{d^2y_1}{dx^2} = \frac{dy_1}{dx} + \frac{dy_2}{dx} + 1 = \frac{dy_1}{dx} + \left(-4y_1 - 3\left(\frac{dy_1}{dx} - y_1 - x\right) + 2x\right) + 1$$

$$\frac{d^2y_1}{dx^2} = -2\frac{dy_1}{dx} - y_1 + 5x + 1$$

$$\frac{d^2y_1}{dx^2} + 2\frac{dy_1}{dx} + y_1 = 5x + 1$$

$$(D^2 + 2D + 1)\, y = 5x + 1$$

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CF: AE: $m^2+2m+1=0 \Rightarrow m=-1$

$$y_1 = (c_1+c_2x)e^{-x}$$

PI $\Rightarrow$ $y = \frac{1}{(D+1)^2}(5x+1) = (1-2D+\dots)(5x+1)$

$$= 5x+1-10 = 5x-9$$

$$\Rightarrow \boxed{y_1 = (c_1+c_2x)e^{-x} + 5x-9}$$

$$\frac{dy_1}{dx} = e^{-x}(c_2) + (-e^{-x})(c_1+c_2x) + 5 = e^{-x}(c_2-c_1-c_2x)+5$$

$$y_2 = \frac{dy_1}{dx} - y_1 - x = e^{-x}(c_2-c_1-c_2x)+5-\left[(c_1+c_2x)e^{-x}+(5x-9)\right]-x$$

$$y_2 = e^{-x}(c_2-c_1-c_2x-c_1-c_2x) - 5x+9+5-x$$

$$\boxed{y_2 = e^{-x}(c_2-2c_1-2c_2x) - 6x+14}$$

III. Method of undetermined coefficients

Consider

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{aligned}\right\} \quad (1)$$

We seek a particular solution:

$$x_1 = \alpha_1 e^{kt}, \quad x_2 = \alpha_2 e^{kt}, \quad \dots, \quad x_n = \alpha_n e^{kt}$$

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(1) $\Rightarrow$ $$k\alpha_1 e^{kt} = (a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n) e^{kt}$$

$$k\alpha_2 e^{kt} = (a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n) e^{kt}$$
$$\vdots$$
$$k\alpha_n e^{kt} = (a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n) e^{kt}$$

$\Rightarrow$
$$\left.\begin{aligned} &(a_{11}-k)\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n \\ &a_{21}\alpha_1 + (a_{22}-k)\alpha_2 + \cdots + a_{2n}\alpha_n \\ &\vdots \\ &a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + (a_{nn}-k)\alpha_n \end{aligned}\right\} \quad (2)$$

To have non-trivial solution of the above system

$$\begin{vmatrix} a_{11}-k & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-k & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & (a_{nn}-k) \end{vmatrix} = 0$$

This equation is called auxiliary eq. of the system (1).

After solving the matrix, we get 'n' values of k.

* Suppose roots of the auxiliary eq. are real and distinct, say $k_1, k_2, \ldots, k_n$.

For each root k_i:

evaluate $\rightarrow$ $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_n^{(i)}$

For the root k_i, obtain the following solution of the system:

$$x_1^{(i)} = \alpha_1^{(i)} e^{k_i t}, \quad x_2^{(i)} = \alpha_2^{(i)} e^{k_i t}, \ldots, x_n^{(i)} = \alpha_n^{(i)} e^{k_i t}$$

We introduced 'n' arbitrary constants $\rightarrow c_i$

General Solution:-

$$x_1 = \sum_{i=1}^{n} c_i \alpha_1^{(i)} e^{k_i t}$$
$$x_2 = \sum_{i=1}^{n} c_i \alpha_2^{(i)} e^{k_i t}$$
$$\vdots$$
$$x_n = \sum_{i=1}^{n} c_i \alpha_n^{(i)} e^{k_i t}$$

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Ex.
$$\frac{dx_1}{dt} = 2x_1 + 2x_2$$

$$\frac{dx_2}{dt} = x_1 + 3x_2$$

$x_1 = \alpha_1 e^{kt}$ $\quad x_2 = \alpha_2 e^{kt}$

$\Rightarrow$ $(2-k)\alpha_1 + 2\alpha_2 = 0 \quad — (1)$

and $\alpha_1 + (3-k)\alpha_2 = 0 \quad — (2)$

$$\begin{vmatrix} 2-k & 2 \\ 1 & 3-k \end{vmatrix} = 0 \Rightarrow (2-k)(3-k) - 2 = 0$$

$k^2 - 5k + 4 = 0$

$\Rightarrow k = 1, 4$ $\quad \cancel{k = \frac{5 \pm \sqrt{25+32}}{2}}$

$\cancel{= 5 \pm \sqrt{57}}$

For $k_1 = 1$,

(1): $\alpha_1 + 2\alpha_2 = 0 \Rightarrow \alpha_1 = -2\alpha_2$

(2): $\alpha_1 + 2\alpha_2 = 0$

Let us choose $\alpha_1 = 1 \Rightarrow \alpha_2 = -1/2$

Solution: $x_1^{(1)} = e^t$ $\quad x_2^{(1)} = -\frac{1}{2}e^t$

For $k_2 = 4$,

$-2\alpha_1 + 2\alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2$

$\alpha_1 - \alpha_2 = 0$

Let us choose $\alpha_1 = \alpha_2 = 1$

Solution: $x_1^{(2)} = e^{4t}$ $\quad x_2^{(2)} = e^{4t}$

General Solution:
$$\boxed{\begin{aligned} x_1 &= c_1 e^t + c_2 e^{4t} \\ x_2 &= -\frac{c_1}{2} e^t + c_2 e^{4t} \end{aligned}}$$

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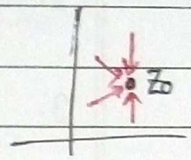
Complex Analysis

Complex Functions :

$f : \underset{\text{Domain}}{D} \to \underset{\text{Range}}{R}$ $\qquad f(z) \in C$

$$w = f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$



Limit : $\displaystyle\lim_{z \to z_0} f(z) = l$

if and only if

for any given $\epsilon > 0$, there exist a positive no. $\delta > 0$ such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$

Ex. $$\lim_{z \to i} \left\{ \underbrace{\frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}}_{\to f(z)} \right\}$$

$$= \lim_{z \to i} \frac{(3z^3 - (2-3i)z^2 + (5-2i)z + 5i)\cancel{(z-i)}}{\cancel{(z-i)}}$$

$$= -3i + (2-3i) + (5-2i)i + 5i = 4i + 4$$

Prove that

$$\lim_{z \to i} \{f(z)\} = 4i + 4 \quad \text{using } \epsilon\text{-}\delta \text{ approach.}$$

$$|f(z) - (4i+4)| <$$

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CONTINUITY: $f(z)$ is continuous at $z = z_0$ if :

1. the limit $\lim_{z \to z_0} f(z)$ exists , say $\lim_{z \to z_0} f(z) = l$.

2. $f(z)$ is defined at z_0 i.e $f(z_0)$ exists.

3. $l = f(z_0)$.

$f(z)$ is said to be continuous at $z = z_0$ if for any $\epsilon > 0$, we can find $\delta > 0$ such $|f(z) - f(z_0)| < \epsilon$ whenever $0 < |z - z_0| < \delta$

Ex. $f(z) = z^2$ is continuous at $z = z_0$.

$$\lim_{z \to z_0} f(z) = z_0^2 = f(z_0)$$

ϵ-δ approach :- $|z^2 - z_0^2| < \epsilon$ when $|z - z_0| < \delta$

$$\begin{aligned}
|z^2 - z_0^2| &= |(z - z_0)(z + z_0)| \\
&= |z - z_0|\,|z + z_0| \\
&= |z - z_0|\,|z - z_0 + 2z_0| \leq |z - z_0|\left[\,|z - z_0| + 2|z_0|\,\right]
\end{aligned}$$

Choose $\delta < 1$

$$\leq \delta\left[1 + 2|z_0|\right] < \epsilon$$

$$\delta = \min\left\{1, \frac{\epsilon}{(1 + 2|z_0|)}\right\}$$

DERIVATIVE: $\boxed{f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}}$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

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Ex: Find the derivative of $f(Z) = Z^2$

$$\lim_{\Delta Z \to 0} \frac{f(Z+\Delta Z) - f(Z)}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{(Z+\Delta Z)^2 - Z^2}{\Delta Z}$$

$$= \lim_{\Delta Z \to 0} \frac{\cancel{Z^2} + \Delta Z^2 + 2Z\Delta Z - \cancel{Z^2}}{\Delta Z} = \lim_{\Delta Z \to 0} (\Delta Z + 2Z)$$

$$= 2Z$$

Ex: $f(Z) = \bar{Z}$ is not differentiable at any Z.

$$\lim_{\Delta Z \to 0} \frac{f(Z+\Delta Z) - f(Z)}{\Delta Z}$$

$$= \lim_{\Delta Z \to 0} \frac{\overline{Z+\Delta Z} - \bar{Z}}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{\overline{\Delta Z}}{\Delta Z}$$

Along x-axis : $\lim_{\Delta Z \to 0} \frac{\overline{\Delta Z}}{\Delta Z} = 1$ as $\overline{\Delta Z} = \Delta Z$

Along y-axis (Imaginary Axis) : $\lim_{\Delta Z \to 0} \frac{\overline{\Delta Z}}{\Delta Z} = -1$ as $\overline{\Delta Z} = -\Delta Z$

$$\frac{\overline{ik}}{ik} = \frac{-i}{i} = -1$$

Theorem: Let f be differentiable at Z_0 then f is continuous at Z_0

$$f(Z_0+\Delta Z) - f(Z_0) = \frac{f(Z_0+\Delta Z) - f(Z_0)}{\Delta Z} \cdot \Delta Z \quad , \Delta Z \neq 0$$

$$\lim_{\Delta Z \to 0} \left[f(Z_0+\Delta Z) - f(Z_0)\right] = \underbrace{\left(\lim \frac{f(Z_0+\Delta Z) - f(Z_0)}{\Delta Z}\right)}_{f'(Z_0)} \left(\lim_{\Delta Z \to 0} \Delta Z\right)$$

$$= f'(Z_0) \cdot 0 = 0$$

Hence, function is continuous at Z_0.

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Analytic function:

* If the derivative $f'(Z)$ exists at all points Z of a domain D, then $f(Z)$ is said to be analytic in D.
* The terms regular & holomorphic are also used for analytic.
* A function $f(Z)$ is said to be analytic at a point Z_0 if there exist a neighbourhood $|Z-Z_0|<\delta$ at all points of which, $f'(Z)$ exists.

CAUCHY RIEMANN EQUATIONS (C-R equation)

A necessary condition that $f(Z) = u+iv$ be analytic in a domain D is that u and v satisfy C-R equations:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \quad -(1) \text{ in D.}$$

Moreover, if the partial derivatives in (1) are continuous in D, then C-R equations are sufficient for analyticity of $f(Z)$ in D.

Proof: NECESSARY CONDITION:

Assume $f'(Z)$ exists at Z.

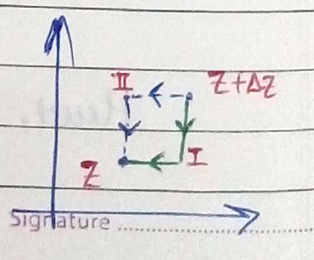
We need to show

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(Z) = \lim_{\Delta Z \to 0} \frac{f(Z+\Delta Z) - f(Z)}{\Delta Z}$$

$$= \lim_{\Delta Z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

{I: first put $\Delta y \to 0$ & then $\Delta x \to 0$}



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Along path I:

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]$$

$$f'(z) = u_x + iv_x \quad \text{— (1)}$$

Along path II:

$$= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{i\Delta y} \right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y} \right]$$

$$f'(z) = -i u_y + (1) v_y = v_y - iu_y \quad \text{— (2)}$$

From (1) & (2),

$$\boxed{u_x = v_y \quad \text{and} \quad v_x = -u_y}$$

Ex. $f(z) = \bar{z}$ ⇒ Check differentiability.

$f(z) = \bar{z} = x - iy$

$u(x,y) = x$ $\quad v(x,y) = -y$

$u_x = 1$ $\quad v_y = -1$ $\quad u_y = 0$ $\quad v_x = 0$

Since $u_x \neq v_y$ ⇒ $f(z)$ is not differentiable.

Ex.

$f(z) = z \, Re(z)$

$= (x+iy)\, x = x^2 + ixy$

$u(x,y) = x^2$ $\quad v(x,y) = xy$

$u_x = 2x$ $\quad v_y = x$ $\quad u_y = 0$ $\quad v_x = y$

CR equations holds only at origin $\left\{ \because 2x = x \;\&\; y = 0 \right\}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; x = 0$

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C-R equations do not hold at any point except $z=0$.
$\Rightarrow$ f is not differentiable at z if $z \neq 0$, but may have a derivative at 0.

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z \, Re(\Delta z)}{\Delta z} = 0$$

$\Rightarrow$ $f(z)$ is differentiable only at origin and analytic nowhere.

Ex.

$$f(z) = |z|^2 = x^2+y^2$$

$u(x,y) = x^2+y^2 \qquad v(x,y) = 0$

$u_x = 2x \qquad v_y = 0$
$u_y = 2y \qquad v_x = 0$
} CR equations hold at origin only.

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta x^2 + \Delta y^2) - 0}{(\Delta x + i\Delta y)} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Delta x - i\Delta y) = 0$$

$$\lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z \, \overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z}.$$

$\Rightarrow$ $f(z)$ is differentiable only at origin and analytic nowhere.

Ex. Show that $f(z) = \sqrt{|xy|}$ is not differentiable at the origin although CR equations are satisfied at the point.

$u(x,y) = \sqrt{|xy|} \qquad v(x,y) = 0$

At the origin
$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Hence, at origin, CR equations are satisfied.

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$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{\sqrt{|xy|}-0}{x+iy}$$

Along the path $y=mx$

$$= \lim_{x\to 0}\frac{\sqrt{|mx^2|}}{(1+im)x} = \lim_{x\to 0}\frac{|x|}{x}\underbrace{\left(\frac{\sqrt{|m|}}{1+im}\right)}_{\text{dependent on } m.}$$

$\Rightarrow f$ is not diff at origin.

Ex. Prove that the function:-

$$f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & z\neq 0 \\ 0 & z=0 \end{cases}$$

is continuous and CR equations are satisfied at the origin yet $f'(z)$ does not exist there.

$$f(z) = \frac{(x^3-y^3)}{x^2+y^2} + i\left(\frac{x^3+y^3}{x^2+y^2}\right) = u+iv$$

$z<\delta$
i.e. $x+iy<\delta$

$$= (x-y)\frac{(x^2+y^2+xy)}{(x^2+y^2)} + i(x+y)\frac{(x^2+y^2-xy)}{(x^2+y^2)}$$

$$= [(x-y)+i(x+y)] + (xy)\frac{[(x-y)-i(x+y)]}{(x^2+y^2)} = (z+iz) + (xy)\frac{[\bar{z}-i\bar{z}]}{|z|^2 \to z\bar{z}}$$

$$= z(1+i) + \frac{xy(1-i)}{z}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(3x^2)-(x^3-y^3)(2x)}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = ?$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(3y^2)-(x^3+y^3)(2y)}{(x^2+y^2)^2} = \frac{3x^2y^2+y^4-2yx^3}{(x^2+y^2)^2}$$

at (0,0):-

$$\frac{\partial u}{\partial x} = \frac{\left(\frac{x^3-0}{x^2+0}\right)-0}{x} = 1 \qquad \frac{\partial u}{\partial y} = \frac{\frac{-y^3}{y^2}-0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \frac{\frac{x^3}{x^2}-0}{x} = 1 \qquad \frac{\partial v}{\partial y} = \frac{\frac{y^3}{y^2}-0}{y} \to 1$$

$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$ $\Rightarrow$ CR eqⁿ are satisfied.

$$f'(z) = \lim_{(x,y)\to(0,0)}\frac{f(z)-0}{z-0}$$

$y=mx$

= Path dependent

$\Rightarrow f'(z)$ does not exist at origin.

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HARMONIC FUNCTION :

$u(x,y)$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$ Laplace equation

Th.: If $f(z) = u+iv$ is analytic in a domain D, then u & v Satisfy laplace equation :-

$\underbrace{u_{xx} + u_{yy} = 0}$ $\qquad v_{xx} + v_{yy} = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

Assuming cont of partial derivatives.

☆ **Th.:** Let u be harmonic on a domain D, then for some v, u+iv defines an analytic function of z in D.

Construction of analytic function:-

Ex: Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic and find v such that $f(z) = u+iv$ is analytic.

$$\frac{\partial u}{\partial x} = -e^{-x}(x\sin y - y\cos y) + e^{-x}(\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(x\sin y - y\cos y) - e^{-x}\sin y - e^{-x}\sin y \quad = u_{xx}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x\sin y + y\cos y + \sin y + \sin y) = u_{yy}$$

$u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic function.

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IMPORTANT NOTES

CR equation:-

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -xe^{-x}\sin y + e^{-x}y\cos y + e^{-x}\sin y$$

$$v = xe^{-x}\cos y - e^{-x}\cos y + e^{-x}\left[y\sin y - \int \sin y\, dy\right] + f(x)$$

$$v = xe^{-x}\cos y + e^{-x}y\sin y + \cancel{e^{-x}\cos y} - \cancel{e^{-x}\cos y} + f(x)$$

$$v = xe^{-x}\cos y + e^{-x}y\sin y + f(x)$$

$$\frac{\partial v}{\partial x} = e^{-x}\cos y - xe^{-x}\cos y - e^{-x}y\sin y + f'(x) = -\frac{\partial u}{\partial y}$$

$$\Rightarrow f'(x) = 0$$

$$f(x) = c$$

Hence, $\boxed{v = xe^{-x}\cos y + e^{-x}y\sin y + c}$

Ex. Find an analytic function $f(z) = u + iv$ given that $v = \frac{x}{x^2+y^2} + \cosh x \cos y$ $\left\{\cosh(x) = \frac{e^x + e^{-x}}{2}\right\}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{-x}{(x^2+y^2)^2}(2y) + \cosh x(-\sin y) = \frac{-2xy}{(x^2+y^2)^2} - \sin y \cosh x$$

$$u = \int \frac{-2xy\, dx}{(x^2+y^2)^2} - \int \sin y \cosh x\, dx = \int \frac{-y\, dt}{(t+y^2)^2} - \sin y \sinh x$$

$x^2 = t$

$$= \frac{+y}{(t+y^2)} - \sin y \sinh x + f(y) = \frac{y}{x^2+y^2} - \sin y \sinh x + f(y)$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} - \cos y \sinh x + f'(y)$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2) - x(2x)}{x^2+y^2} - \sinh x \cos y$$

$$\Rightarrow f'(y) = 0$$

$$f(y) = c.$$

Assume $c = 0$ {we want 1 analytic eq^n}

$$\Rightarrow \boxed{f(z) = \left(\frac{y}{x^2+y^2} - \sin y \sinh x\right) + i\left(\frac{x}{x^2+y^2} + \cosh x \cos y\right)}$$

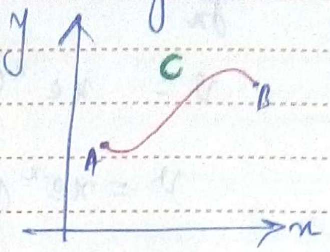
ANUBHAV JAIN NOTES

IMPORTANT NOTES

LINE INTEGRAL

Complex indefinite integrals are called line integrals :

$$\int_C f(z)\,dz$$

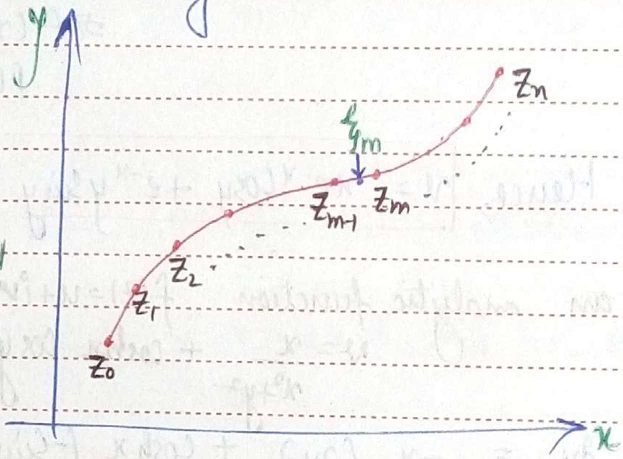


* The integrand $f(z)$ is integrated over a given curve C in a complex plane
* C is called the path of integration
* C may be represented parametrically as $z(t) = x(t) + iy(t)$, $a \le t \le b$

Definition:-

$$\int_C f(z)\,dz = \lim_{n\to\infty}\sum_{m=1}^{n} f(\xi_m)\underbrace{(z_m - z_{m-1})}_{\Delta z_m}$$

If this limit exists, we call function integrable



If c is closed path then the line integral is denoted by $\oint_C f(z)dz$

Basic properties of integration:-

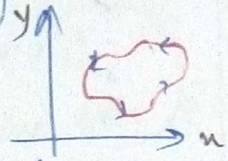
1. Linearity
$$\int [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz$$

2. $$\int_{z_0}^{z} f(z)dz = -\int_{z}^{z_0} f(z)dz$$

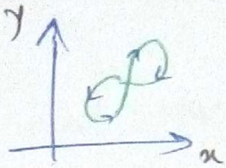
3. $$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \quad ; \; C = C_1 + C_2$$

4. Suppose $f(z)$ is integrable along a curve C having finite length 'L' and suppose there exists a positive no. 'M' such that $|f(z)| \le M$ on C, then
$$\left|\int_C f(z)dz\right| \le ML$$

1) **Simple Closed Curve:** A closed curve that does not intersect (or touch) itself anywhere is called a simple closed curve.



Simple Closed Curve

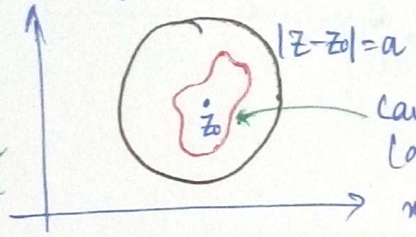


Not a Simple Closed curve.

Simply and Multiply Connected Domains:

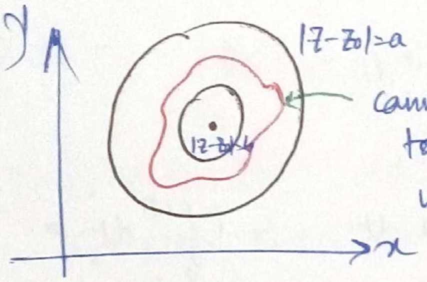
A domain D is called simply-connected if any simple closed curve which lies in D can be shrunk to a point without leaving D. A region which is not simply connected is called multiply-connected.

Simply connected domain

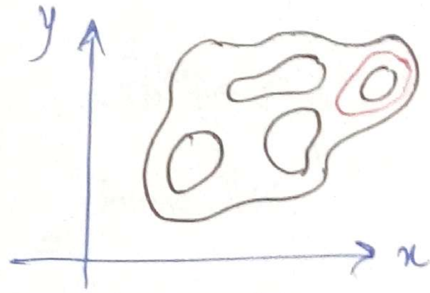


Can be shrunk to a point
(any simply connected curve can be shrunk to a point)

Multiply connected domain



cannot be shrunk to a single point within D.

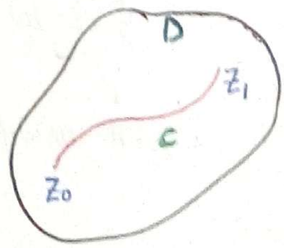


Evaluation of line integrals:-

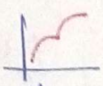
(I) First Method (Restricted to Analytic function)

Let $f(z)$ be analytic in a simply connected domain D. There exist an analytic function $F(z)$ such that $F'(z)=f(z)$ in D, then

$$\int_C f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz = F(z_1)-F(z_0)$$



(II) Second Method (General)



Let C be a piece wise smooth path represented by $z=z(t)$, $a\le t\le b$.

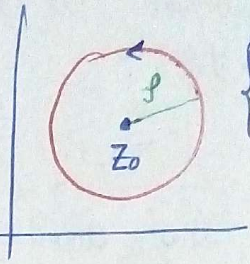
Let $f(z)$ be a continuous function on C, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt \qquad \dot{z}(t) \to \frac{dz(t)}{dt}$$

Ex:

$\oint_C (z-z_0)^m dz$, m is an integer and c is the circle of radius ρ a

centre z



{Convention:- Anticlockwise +ve}

Case I: $m \geq 0$

$f(z) = (z-z_0)^m \rightarrow$ analytic function

$$\boxed{\oint_C (z-z_0)^m dz = 0}$$

{initial & end points are same $F(i) = F(f)$} (initial, final)

$$\boxed{\oint_C \underset{\text{analytic}}{f(z)}\, dz = 0}$$

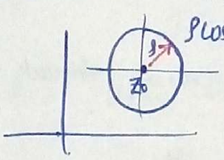
Case II: $m = -1$

$f(z) = \dfrac{1}{z-z_0}$

Parametric form:-

$z(t) = z_0 + \rho(\cos t + i \sin t)$, $0 \leq t \leq 2\pi$

$\underline{z(t) = z_0 + \rho e^{it}}$



$\Rightarrow \oint_C (z-z_0)^{-1} dz = \int_0^{2\pi} (\rho e^{it})^{-1} \cdot \underbrace{\rho i e^{it}}_{\dot{z}(t)} dt$

Not analytic

$= \int_0^{2\pi} \frac{1}{\rho} e^{-it} \cdot \rho e^{it} \cdot i\, dt = i \int_0^{2\pi} dt$

$$\boxed{\oint_C (z-z_0)^{-1} dz = 2\pi i}$$

Case III : $m \leq -2$

$\oint (z-z_0)^m dz = \int_0^{2\pi} (\rho e^{it})^m \rho i e^{it} dt = \rho^{m+1} i \int_0^{2\pi} e^{(m+1)it} dt$

Not analytic

$= \rho^{m+1} \cancel{i} \left| \frac{e^{(m+1)it}}{(m+1)\cancel{i}} \right|_0^{2\pi}$ $\{m \neq -1\}$

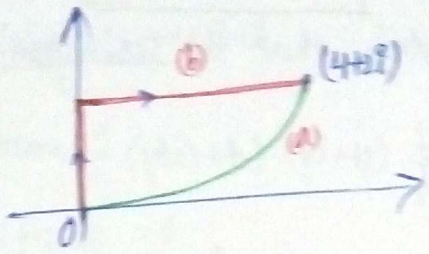
$= \frac{\rho^{m+1}}{m+1} \{e^{(m+1)2\pi} - e^0\} = \frac{\rho^{m+1}}{m+1}(1-1) = \boxed{0}$

$$\Rightarrow \oint_C (z-z_0)^m dz = \begin{cases} 0 & m \in I - \{-1\} \\ 2\pi i & m = -1 \end{cases}$$

Evaluate $\int_C \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along the curve given by

(a) $z(t) = t^2 + it$

(b) the line from $z=0$ to $z=2i$ and then the line $z=2i$ to $z=4+2i$



(a) $$\int_C \bar{z}\,dz = \int \left(\overline{t^2+it}\right) \frac{dz}{dt}\cdot dt = \int_0^2 (t^2 - it)(2t+i)\,dt$$

$$= \int_0^2 \left[2t^3 + t - i(t^2)\right] dt = \left[\frac{t^4}{2} + \frac{t^2}{2} - i\frac{t^3}{3}\right]_0^2$$

$$= \frac{2^4+2^2}{2} - i\frac{2^3}{3} = 10 - \frac{8}{3}i$$

(b) $$\int_C \bar{z}\,dz = \int_{C_1} \bar{z}\,dz + \int_{C_2} \bar{z}\,dz$$

C_1: $z(t) = it$ $\quad t \in [0,2]$

C_2: $z(t) = 2i + t$ $\quad t \in [0,4]$

$$\Rightarrow \int_C \bar{z}\,dz = \int_0^2 (-it)(i)\,dt + \int_0^4 (t-2i)(1)\,dt$$

$$= \left[\frac{t^2}{2}\right]_0^2 + \left[\frac{t^2}{2} - 2it\right]_0^4 = 2 + 8 - 8i = 10 - 8i$$

Ex $\int_C z\,\mathrm{Re}(z)\,dz$

C: $z(t) = t - it^2$ $\quad 0 \le t \le 2$

$$\int_C z\,\mathrm{Re}(z)\,dz = \int_0^2 (t - it^2)\,t\,[1 - 2it]\,dt = \int_0^2 t^2(1 - 2t^2 - 3it)\,dt$$

$$= \int_0^2 \left[t^2 - 2t^4 - 3it^3\right] dt = \left[\frac{t^3}{3} - \frac{2t^4}{5} - \frac{3it^4}{4}\right]_0^2$$

$$= \frac{8}{3} - \frac{32}{5} - \frac{3}{4}i(16) = \frac{40-96}{15} - 12i$$

$-\frac{56}{15}$ $\quad = -\frac{152}{15} - 12i$

5-11-

Cauchy Integral Theorem:-

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D, $\boxed{\oint_C f(z)\,dz = 0}$

Proof: additional assumption that the derivative $f'(z)$ is continuous,

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) \quad -(1)$$

We know from the C-R equations,

$$f'(z) = u_x + iv_x = v_y - iu_y$$

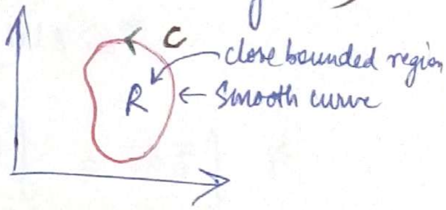
Since $f'(z)$ is cont. then it implies continuity of u_x, v_x, u_y, v_y.

Hence, by GREEN'S THEOREM:

$$\boxed{\oint_C (u\,dx - v\,dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy} \quad -(2)$$

(R is the region bounded by C)

(Transformation btw double integral & line integrals)



Let $F_1(x,y)$ & $F_2(x,y)$ be continuous & have cont. partial derivatives $\frac{\partial F}{\partial y}$ & $\frac{\partial F}{\partial x}$ everywhere in some domain containing R, then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \oint_C F_1\,dx + F_2\,dy$$

Since, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \overset{eq(2)}{\Rightarrow} \oint (u\,dx - v\,dy) = 0$

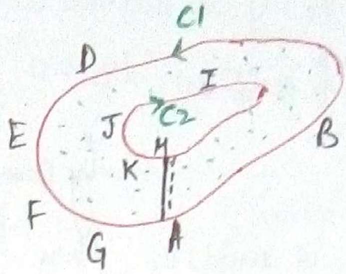
Similarly, $\oint (v\,dx + u\,dy) = 0$

$\Rightarrow eq(1) \rightarrow \oint_C f(z)\,dz = 0$

Remark: Cauchy theorem for multiply connected domain:-

Construct cross-cut.

Then, the region bounded by ABDEFGAHKJIHA is simply connected.

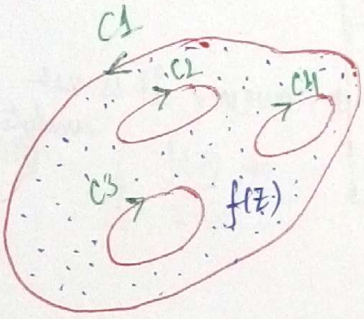


Then by Cauchy theorem:

$$\oint_{AB\cdots HA} f(z)dz = 0$$

$$\oint_{ABCDEFGA} f(z)dz + \cancel{\oint_{AH} f(z)dz} + \oint_{HKJIH} f(z)dz + \cancel{\oint_{HA} f(z)dz} = 0$$

$$\boxed{\oint_{C1} f(z)dz + \oint_{C2} f(z)dz = 0}$$ $\{C_1 \,\&\, C_2$ are in opp. dir$^n\}$



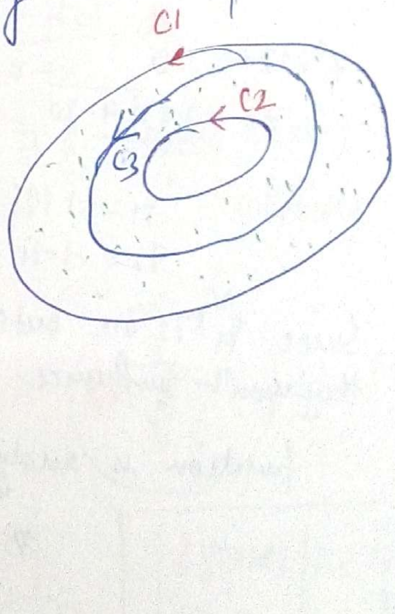
$$\boxed{\oint_{C1} f(z)dz + \oint_{C2} f(z)dz + \oint_{C3} f(z)dz + \oint f(z)dz = 0}$$

Remark: Deformation of path

Let $f(z)$ be analytic in a domain D bounded by two simple closed curves $C1 \,\&\, C2$ and also on $C_1 \,\&\, C2$.

Then,

$$\oint_{C1} f(z)dz = \oint_{C2} f(z)dz = \oint_{C3} f(z)dz$$

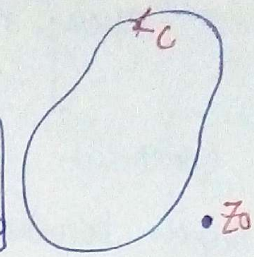


Recall: $\oint_{C} (z-z_0)^m dz = \begin{cases} 2\pi i & m = -1 \\ 0 & m \neq -1, \; m \in \mathbb{Z} \end{cases}$

C: Circle of radius ρ centre z_0

Let C be any closed curve

If z_0 is outside C

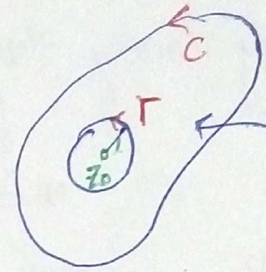


$$\oint_C (z-z_0)^m \, dz = 0$$

[Since z_0 lies outside curve, $f(z)$ is analytic everywhere ⇒ Integral = 0]

by Cauchy Theorem

If z_0 is inside C, then let Γ be a circle of radius ε & with centre $z = z_0$



$f(z)$ is analytic

$$\oint_C (z-z_0)^m \, dz$$

$$= \oint_\Gamma (z-z_0)^m \, dz$$

$$= \begin{cases} 2\pi i & m = -1 \\ 0 & m \neq -1 \end{cases}, \quad m \text{ is int}$$

z_0 lies inside C.

NOTE:

1. $\displaystyle\oint_C \frac{1}{z-z_0} \, dz = 2\pi i$	If z_0 is inside C
2. $\displaystyle\oint_C \frac{1}{(z-z_0)^m} \, dz = 0$	$m = 2, 3, \dots$ If z_0 is inside C

→ However it is not analytic

Ex: Evaluate

$$\int_C \frac{z+4}{z^2+2z+5} \, dz \qquad C: |z+1| = 1$$

Singular points → where function is not defined or function is not analytic.

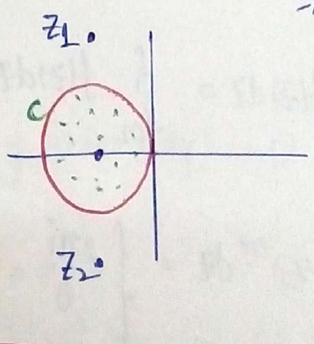
⇒ $z^2+2z+5=0$ gives singular points

$z^2+2z+5 = 0$

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$z_1 = -1+2i$

$z_2 = -1-2i$



Since z_1 & z_2 are outside curve, they won't influence anything.

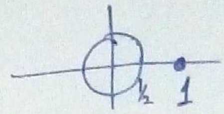
function is analytic in domain

$$\Rightarrow \int_C \frac{z+4}{z^2+2z+5} \, dz = 0$$

Ex.
$$\oint_c \frac{z^2-z+1}{z-1}\,dz \qquad c: |z| = \frac{1}{2}$$

$$= 0$$

by Cauchy Integral Th^m.



Remark: Do not use Cauchy Theorem if $f(z)$ is not analytic

eg. $\oint_c \bar{z}\,dz = 2\pi i \qquad \oint_c \frac{1}{z-z_0}\,dz = 2\pi i$

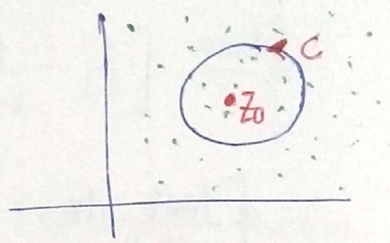
Cauchy Integral formula:-

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed curve C in D that encloses z_0, we have :-

$$\boxed{\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)}$$

or

$$\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-z_0}\,dz = f(z_0)$$



Proof:-

$$\oint_c \frac{f(z)}{z-z_0}\,dz = \oint_c \frac{f(z_0)+f(z)-f(z_0)}{z-z_0}\,dz$$

$$= \underbrace{\oint_c \frac{f(z_0)}{z-z_0}\,dz} + \underbrace{\oint_c \frac{f(z)-f(z_0)}{z-z_0}\,dz}$$

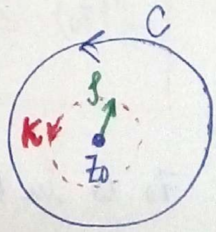
$f(z_0)$ → const. → comes out

$$= 2\pi i\, f(z_0) + \oint_c \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

Since $f(z)$ is continuous, for any $\varepsilon > 0$, $\exists\ \delta$ such that $|f(z)-f(z_0)| < \varepsilon$ whenever $|z-z_0| < \delta$

$$\oint_c \frac{f(z)-f(z_0)}{z-z_0}\,dz = \oint_k \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

(Deformation of path)



$\rho < \delta$

$$\left|\oint_k \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \left(\frac{\varepsilon}{\rho}\right)(2\pi\rho) = 2\pi\varepsilon$$

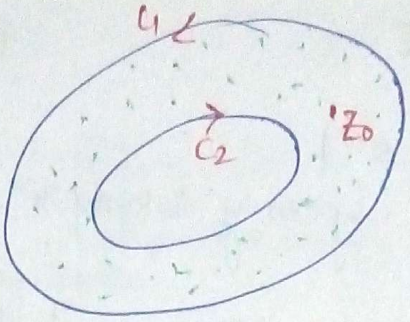
$$\boxed{\left|\oint_c f(z)\,dz\right| < ML}$$

L → length of curve

M: $|f(z)| < M$

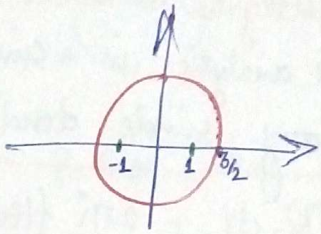
and ε is arbitrary → it can be however small

$$\Rightarrow \boxed{\oint_k \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0}$$



$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}\,dz$$

Ex: $\oint_C \frac{\tan z}{z^2-1}\,dz$ $\qquad C: |z| = 3/2$



$$= \oint_C \frac{\tan z}{(z+1)(z-1)}\,dz$$

$$= \oint_C \frac{\tan z}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]dz$$

$$= \underbrace{\oint_C \frac{\tan z}{2(z-1)}\,dz}_{} - \underbrace{\oint_C \frac{\tan z}{2(z+1)}\,dz}_{}$$

$\frac{\tan z}{2} \rightarrow$ analytic

and $z_0 = -1$

$z_0 = +1$

$$= 2\pi i \,\frac{\tan(+1)}{2} - 2\pi i\,\frac{\tan(-1)}{2} = \boxed{2\pi i \tan 1}$$

Derivative of an analytic function: 6-11-15

If $f(z)$ is analytic inside and on the boundary of C of a simply connected Domain D, then we have:

$$\boxed{f^{n}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

$n = 1, 2, 3, \ldots$

where z_0 is in D ie $\{z_0 \in D\}$

(where C is any simple closed curve in D enclosing the point $z = z_0$.)

Proof:- Using Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-z_0} dz$$

$$f(z_0+\Delta z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-z_0-\Delta z_0)} dz$$

$$f(z_0+\Delta z_0) - f(z_0) = \frac{1}{2\pi i} \oint_c f(z) \left[\frac{1}{z-z_0-\Delta z_0} - \frac{1}{z-z_0} \right] dz$$

$$= \frac{1}{2\pi i} \oint_c f(z) \cdot \frac{\Delta z_0}{(z_0-z_0-\Delta z_0)(z-z_0)} dz$$

$$\Rightarrow \lim_{\Delta z_0 \to 0} \frac{f(z_0+\Delta z_0) - f(z_0)}{\Delta z_0} = \lim_{\Delta z_0 \to 0} \frac{1}{2\pi i} \oint_c f(z) \cdot \frac{1}{(z-z_0-\Delta z_0)(z-z_0)} dz$$

$$\boxed{f'(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^2} dz}$$

Note: * All derivatives of any analytic function $f(z)$ are also analytic function.
* An analytic function is differentiable to n times (n can be even ∞)

Cauchy-Inequality:-

Let $f(z)$ be analytic inside and on a circle C of radius r and center z_0

then $\left| f^{(n)}(z_0) \right| \leq \frac{M \lfloor n}{r^n}$

where M is a constant such that $|f(z)| \leq M$

Proof:- $\left| f^{(n)}(z_0) \right| = \left| \frac{\lfloor n}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \neq \left| \frac{\lfloor n}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$

Using M-L inequality:-

$$\leq \frac{\lfloor n}{\cancel{2\pi}} \frac{M}{r^{n+1}} \cdot \cancel{2\pi} r \qquad \left\{ \begin{array}{l} |z-z_0| = r \\ L = 2\pi r \end{array} \right\}$$

$$\boxed{\left| f^{(n)}(z_0) \right| \leq \frac{M \lfloor n}{r^n}}$$

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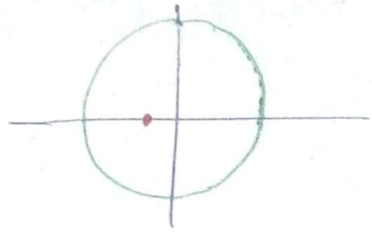
Morera's Theorem:

If $f(z)$ is continuous in a simply connected domain D and $\oint_C f(z)\,dz = 0$ for every closed path in D, then $f(z)$ is analytic.

Ex. Evaluate:

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz \qquad C: |z| = 3$$



$f(z) = e^{2z}$

$z_0 = -1 \qquad n = 3$

$$f^3(-1) = \frac{3!}{2(\pi)i} \oint_C \frac{e^{2z}}{(z+1)^4}\,dz$$

$f'(z) = 2e^{2z}$

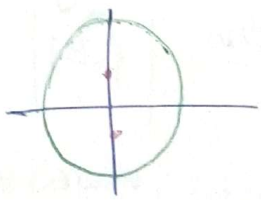
$f''(z) = 4e^{2z} \qquad f'''(z) = 8e^{2z}$ or $f^3(z)$

$$8 \cdot e^{-2} = \frac{6}{2\pi i} \oint \frac{e^{2z}}{(z+1)^4}\,dz$$

$$\Rightarrow \boxed{\frac{8\pi i}{3e^2} = \oint \frac{e^{2z}}{(z+1)^4}\,dz}$$

Ex.

$$\oint_C \frac{e^{zt}}{z^2+1}\,dz \qquad C: |z| = 3$$



$$\frac{e^{zt}}{z^2+1} = \frac{e^{zt}}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$A(z-i) + B(z+i) = e^{zt}$

~~$A = B$ and $A + B = e^{zt}$~~

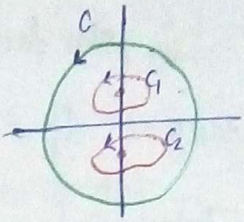
$$\Rightarrow \oint_C \frac{e^{zt}}{z^2+1} = \frac{1}{2i} \oint_C \left\{ \frac{-e^{zt}}{(z+i)} + \frac{e^{zt}}{(z-i)} \right\} dz$$

$$\oint_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0)$$

$$\Rightarrow \frac{1}{2i} \oint_C \left\{ \frac{-e^{zt}}{(z+i)} + \frac{e^{zt}}{(z-i)} \right\} dz = \frac{2\pi i}{i} \left\{ \frac{-e^{-it}}{2} + \frac{e^{it}}{2} \right\}$$

$$= 2\pi i \sin t$$

OR



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$

$$= \oint_{C_1} \frac{e^{zt}}{z^2+1}\,dz + \oint_{C_2} \frac{e^{zt}}{z^2+1}\,dz$$

$$= \oint_{C_1} \frac{\overset{\to f_1(z)}{\left(\frac{e^{zt}}{z+i}\right)}}{(z-i)}\,dz + \oint_{C_2} \frac{\overset{\to f_2(z)}{\left(\frac{e^{zt}}{z-i}\right)}}{(z+i)}\,dz$$

$$= 2\pi i\left(\frac{e^{it}}{i+i}\right) + 2\pi i\left(\frac{e^{it}}{-i-i}\right) = \boxed{2\pi i \sin t}$$

Ex.

$$\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz \qquad C: |z| = 1$$

$f(z) = \sin^6 z$

$$\left(\frac{2!}{2\pi i}\right)\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = f^2(\pi/6) = f''(\pi/6)$$

$$\Rightarrow \oint \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \left(\frac{\cancel{2}\pi i}{\cancel{2}}\right)\cdot\frac{21}{16}$$

$$= \boxed{\frac{21\pi i}{16}}$$

$f'(z) = 6\sin^5 z\cos z$

$f''(z) = 6\cdot 5\sin^4 z\cos^2 z - 6\sin^6 z$

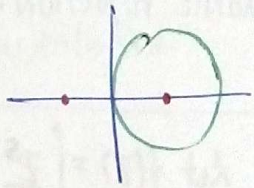
$= 6\sin^4 z(5\cos^2 z - \sin^2 z)$

$f''(\pi/6) = 6\left(\tfrac{1}{2}\right)^4\left[5\cdot\tfrac{3}{4} - \tfrac{1}{4}\right]$

$= \frac{\cancel{6}^3}{16}\cdot\frac{\cancel{14}^7}{\cancel{4}} = \frac{21}{16}$

Ex. Evaluate:

$$\oint_C \frac{3z^2+z}{z^2-1}\,dz \qquad C: |z-1| = 1$$

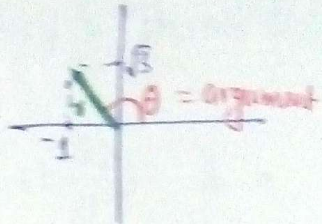


$$= \oint_C \frac{\overset{\to f(z)}{\left(\frac{3z^2+z}{z+1}\right)}}{z-1}\,dz$$

$$= 2\pi i\ \overset{\to f(1)}{\frac{3(1)+1}{1+1}} = \boxed{4\pi i}$$

Q Write $z = -1 + i\sqrt{3}$ in polar form.

$|z| = 2$ $\quad \tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3} \Rightarrow \theta = \pi - \pi/3 = \frac{2\pi}{3}$



Polar form:- $z = 2e^{i\frac{2\pi}{3}}$

$Arg(z) = \frac{2\pi}{3}$

Arg(z) → principle value of argument
↓
$\in (-\pi, \pi]$

* $Arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ → I & IV Quad.
* $Arg(z) = \pi + \tan^{-1}\left(\frac{y}{x}\right)$ → II Quad.
* $Arg(z) = -\pi + \tan^{-1}\frac{y}{x}$ → III Quad.

LIMIT: $|f(z) - l| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

then $\lim_{z \to z_0} f(z) = l$

CONTINUITY: If $\lim_{z \to z_0} f(z) = l = f(z_0)$, then f is continuous at $z = z_0$.

DIFFERENTIABILITY: $f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ → If this limit exists, function is differentiable with value $f'(z)$ same as this limit.

ANALYTIC: Differentiable over a domain

↳ Necessary condition:- satisfy CR equations → $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

↳ Sufficient condition:- continuity of partial derivatives & satisfy CR eqns.

HARMONIC FUNCTION: $u(x,y)$ satisfy Laplace eqⁿ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Q Let $f(z) = \begin{cases} \frac{z^5}{|z|^4} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$

Check whether the CR equations are satisfied at origin. Is f differentiable at 0?

$z = |z|e^{i\theta}$ $\quad \tan\theta = y/x \Rightarrow \theta = \tan^{-1}\frac{y}{x}$

$f(z) = \frac{z^5}{|z|^4} = \frac{|z|^5 e^{i5\theta}}{|z|^4} = |z| e^{i5\theta} = \sqrt{x^2+y^2}\,(\cos 5\theta + i\sin 5\theta)$

$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2}$

$\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$

↳ $\frac{(x+iy)^5}{(x^2+y^2)^2} = \frac{x^5 + 5ix^4y - 10x^3y^2 - 10ix^2y^3 + 5xy^4 + iy^5}{(x^2+y^2)^2}$

$$u = \sqrt{x^2+y^2} \quad \cos 5\theta \quad = \frac{x^5 - 10x^3y^2 + 5xy^4}{(x^2+y^2)^2}$$

$$v = \sqrt{x^2+y^2} \quad \sin 5\theta \quad = \frac{5x^4y - 10x^2y^3 + y^5}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{\sqrt{x^2+y^2}} \cos 5\theta + \sqrt{x^2+y^2}\left(-5\sin 5\theta \frac{\partial \theta}{\partial x}\right)$$

$$= \frac{2x}{\sqrt{x^2+y^2}} \cos 5\theta + \sqrt{x^2+y^2}\,(-5\sin 5\theta)\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{2x\cos 5\theta + 5y\sin 5\theta}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial y} = \frac{2y}{\sqrt{x^2+y^2}} \sin 5\theta + \sqrt{x^2+y^2}\,(5\cos 5\theta)\,\frac{x}{(x^2+y^2)}$$

$$= \frac{2y\sin 5\theta + 5x\cos 5\theta}{\sqrt{x^2+y^2}}$$

OR

$$u_x\Big|_{x=0,y=0} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0}\left[\frac{\frac{x^5+0}{x^4} - 0}{x}\right] = 1$$

$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{\frac{0+y^5}{y^4} - 0}{y} = 1$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

At origin, $u_x = v_y$ and $u_y + v_x = 0$ $\Rightarrow$ CR equations are satisfied.

NOTE

Polar form of CR equations:-

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

ANUBHAV JAIN NOTES

$$\lim_{\Delta z\to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z\to 0} \frac{\Delta z^5}{|\Delta z|^4 \cdot \Delta z} = \lim_{\Delta z\to 0} \frac{\Delta z^4}{|\Delta z|^4} = \lim_{\Delta z\to 0} \frac{\Delta z^4}{|\Delta z|^2 |\Delta z|^2}$$

$$= \lim_{\Delta z\to 0} \frac{\Delta z^4}{\Delta z \cdot \overline{\Delta z} \cdot \Delta z \cdot \overline{\Delta z}} = \lim_{\Delta z\to 0} \frac{\Delta z^2}{\overline{\Delta z}^2} = \lim_{\Delta x\to 0} \left(\frac{\Delta x + im\Delta x}{\Delta x - im\Delta x}\right)^2$$

$\Delta y = m\Delta x$

$\to$ Path dependent

$\Rightarrow$ Not diff.

Q $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$

$$f(z) = \frac{\bar{z}^2}{z} = \frac{(x-iy)^2}{(x+iy)} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3 - 3iyx^2 - 3y^2x + iy^3}{x^2+y^2}$$

$$u = \frac{x^3 - 3y^2x}{x^2+y^2} \qquad v = \frac{y^3 - 3yx^2}{x^2+y^2}$$

$f(0) = \underset{u(0,0)}{0} + i\underset{v(0,0)}{0}$

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x^3 - 0}{x^3} = 0\ 1$$

$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{y^3 - 0}{y^3} = 01$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$u_x = v_y$ and $u_y + v_x = 0$ $\Rightarrow$ CR equations are satisfied.

At origin, $\displaystyle\lim_{\Delta z\to 0} \frac{f(0+\Delta z) - f(0,0)}{\Delta z} = \lim_{\Delta z\to 0} \frac{\overline{\Delta z}^2}{\Delta z^2} = \lim_{\Delta x, \Delta y \to 0,0} \frac{(\Delta x - i\Delta y)^2}{(\Delta x + i\Delta y)^2}$ Path $\Delta y = m\Delta x$ $\displaystyle = \lim_{\Delta x\to 0} \frac{(1-im)^2}{(1+im)^2}$

= Path dependent

$\Rightarrow$ $f(z)$ is not differentiable at origin.

Q Show that the function given by $v = \dfrac{-y}{x^2+y^2}$ is harmonic. Find its conjugate function u and its corresponding analytic function $f(z)$
$u + iv$

$$v_x = \frac{+y}{(x^2+y^2)^2}(2x) \qquad v_{xx} = 2\frac{\left[y(x^2+y^2)^2 - xy\left(2(x^2+y^2)\right)(2x)\right]}{(x^2+y^2)^4}$$

$$= \frac{2y\left[x^2+y^2-4x^2\right]}{(x^2+y^2)^3} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$v_y = (-1)\frac{\left[(x^2+y^2)(1) - y(2y)\right]}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_{yy} = \frac{(x^2+y^2)^2[2y] - 2(x^2+y^2)\cdot 2y\cdot(y^2-x^2)}{(x^2+y^2)^4} = \frac{2y\left[x^2+y^2-2y^2+2x^2\right]}{(x^2+y^2)^3} = \frac{2y\left[3x^2-y^2\right]}{(x^2+y^2)^3}$$

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$V_{xx} + V_{yy} = 0 \Rightarrow$ v is harmonic function.

$u_x = v_y$ and $u_y = -v_x$

$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$ and $\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$

$$u = \int (-x) \frac{(2y\,dy)}{(x^2+y^2)^2}$$

$x^2+y^2 = t$

$2y\,dy = dt$

$$\Rightarrow u = (-x)\,\frac{-1}{(x^2+y^2)} + c = \frac{x}{x^2+y^2} + c(x)$$

$$\underbrace{\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + c'(x) = \frac{y^2-x^2}{(x^2+y^2)^2}}$$

$\Rightarrow c'(x) = 0$

$c(x) = c$

$\therefore$ $u = \frac{x}{x^2+y^2} + c$ → take $c = 0$

$$f(z) = \frac{x}{x^2+y^2} + i\,\frac{(-y)}{x^2+y^2}$$

$$f(z) = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$$

$$\Rightarrow \boxed{f(z) = \frac{1}{x+iy}}$$

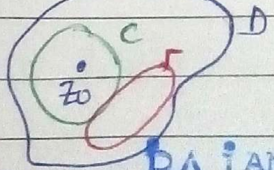
not analytic

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

$$\oint_\Gamma \frac{f(z)}{z - z_0}\,dz = 0$$

Cauchy Integral Thm

analytic



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Integrand → Analytic

⇒ Integration — path independent

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Q. Evaluate $\int_C 3\bar{z}\,dz$ from $z=0$ to $z=4+2i$

path: $z = t^2+it$, $t\in(0,2)$

$f(z) = 3\bar{z} = 3(t^2 - it)$

$dz = (2t+i)\,dt$

C: Not closed curve.

also, $3\bar{z} \xrightarrow{\text{Integrand}}$ Not analytic

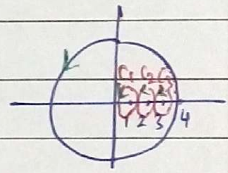
⇒ Integration is path dependent.

$$\int_C 3\bar{z}\,dz = 3\int_0^2 (t^2 - it)(2t+i)\,dt$$

$$= 3\int_0^2 \left[(2t^3+t) - i(t^2)\right] dt = 3\left[\left(\frac{t^4}{2} + \frac{t^2}{2}\right) - i\,\frac{t^3}{3}\right]_0^2$$

$$= 3\left(10 + \frac{8i}{3}\right) = \boxed{30+8i}$$

Q. $\oint_C \dfrac{dz}{(z-1)(z-2)(z-3)}$ C: $|z|=4$



$$= \oint_{C_1} \frac{\frac{1}{(z-2)(z-3)}}{(z-1)}\,dz + \oint_{C_2} \frac{\frac{1}{(z-1)(z-3)}}{(z-2)}\,dz + \oint_{C_3} \frac{\frac{1}{(z-1)(z-2)}}{(z-3)}\,dz$$

$$= 2\pi i\,\frac{1}{(1-2)(1-3)} + 2\pi i\,\frac{1}{(2-1)(2-3)} + 2\pi i\,\frac{1}{(3-1)(3-2)}$$

$$= +\pi i - 2\pi i + \pi i = \boxed{0}$$

OR

$$\oint_C \frac{1}{(z-1)(z-2)(z-3)} = \oint_C \frac{1}{2(z-1)} - \oint_C \frac{1}{(z-2)} + \oint_C \frac{1}{2(z-3)}$$

$$= \frac{2\pi i}{2} - (2\pi i) + \frac{2\pi i}{2} = \boxed{0.}$$

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