$\lim_{x\to c} f = L = \lim_{x\to c} h$, then $\lim_{x\to c} g = L$.

Application:- Let ~~f(x)~~ $h(x) := x\sin\left(\frac{1}{x}\right)$.

As $\sin\left(\frac{1}{x}\right)$ has the property that

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1, \text{ so}$$

for $x \geq 0$, $\quad -x \le x\sin\left(\frac{1}{x}\right) \le x$ and

for $x < 0$, $\quad -x \geq \sin\left(\frac{1}{x}\right) \geq x$.

$$\therefore \quad -|x| \le x\sin\left(\frac{1}{x}\right) \le |x|.$$

Since $\lim_{x\to 0} |x| = 0 = \lim_{x\to 0}(-|x|)$, by Squeeze theorem, $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

We can prove the previous example by the following result too.

Theorem:- Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$. Let c be a limit point of A. If f is bounded on a deleted neighbourhood of and $\lim_{x\to c} g(x) = 0$, then

$$\lim_{x\to c} (f\cdot g)(x) = 0$$

Proof:- Since f is bounded on a deleted neighbourhood of c, so there exist $B>0$ and $\delta_1 > 0$ such that

$$|f(x)| < B \quad \forall\, x\in A \text{ and } 0<|x-c|<\delta_1$$

We also have a $\delta_2 > 0$ such that

$$|g(x)| < \varepsilon/B \quad \forall\, x\in A \text{ and } 0<|x-c|<\delta_2.$$

Let $\delta := \min\{\delta_1, \delta_2\}$ $\therefore$ $|f(x)g(x)| < \varepsilon \quad \forall\, x\in A$ and $0<|x-c|<\delta$

$$\therefore \lim_{x \to c} f(x) \cdot g(x) = 0.$$

Continuity

The continuous functions are the most important class of functions in Real Analysis. The term 'continuous' has been used since the time of Newton to refer to the motion of bodies or to describe an unbroken curve, but it was not made precise until the nineteenth century. Bolzano and Cauchy identified continuity as a very significant property of functions and proposed defition, Since the concept of continuity is tied to that of limit, finally Karl Weierstrass in 1870 gave the proper understanding to the idea of continuity.

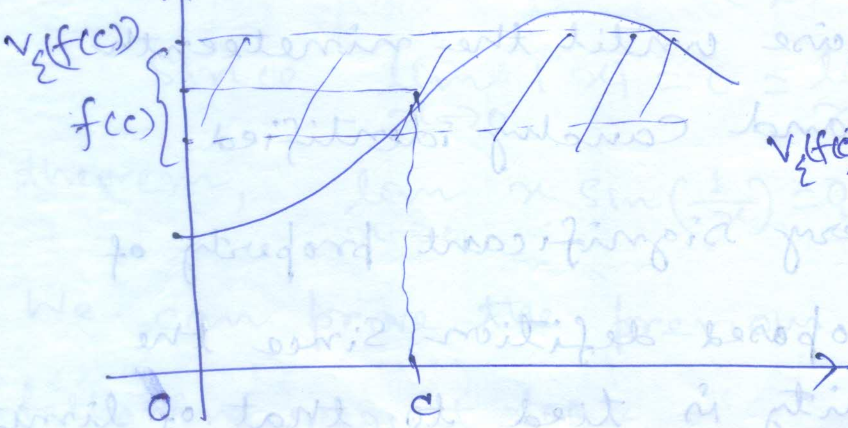
Defn:- Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, and let $c \in A$. We say that f is continuous at c if given any number $\varepsilon > 0$, there exists $\delta > 0$ such that if x is any point of A and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$

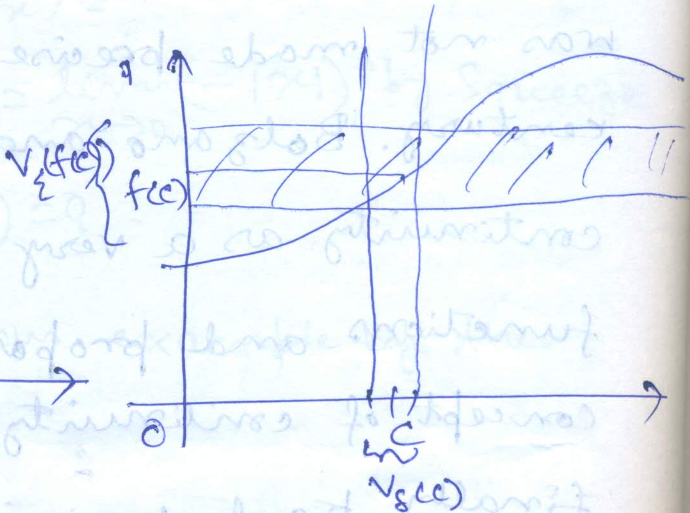
Equivalently, $f: A \to \mathbb{R}$ is said to be continuous at $c \in A$ if given ε- nbhd $N_\varepsilon(f(c))$ of $f(c)$, there exists a δ- nbhd $N_\delta(c)$ of c such that $f(A \cap N_\delta(c)) \subset N_\varepsilon(f(c))$.

~~If f fails to be continuous at c, then we say that f is discont~~ discontinuous at c.

Equivalently, $f : A \to \mathbb{R}$ is said to be continuous at $c \in A$ if for $\varepsilon > 0$ ~~there~~ there exists $\delta > 0$ such that

$$|f(c+h) - f(c)| < \varepsilon \text{ for all } h \text{ satisfying } |h| < \delta \text{ and } c+h \in A.$$





Remark

a) Let $f : A \to \mathbb{R}$ be a function, ~~which is continuous at~~ Let $c \in A$. Let c be a limit point of A. Then f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$.

Proof: Let f be continuous at c. Then for $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \forall x \in N_\delta(c) \cap A.$$

$$\therefore |f(x) - f(c)| < \varepsilon, \ \forall x \in (N_\delta(c) - \{c\}) \cap A.$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Conversely, if $\lim_{x \to c} f(x) = f(c)$, then for $\varepsilon > 0$ there is

$\delta > 0$ such that $\forall x \in (N_\delta(c) - \{c\}) \cap A$ implies

$$f(x) \in N_\varepsilon(f(c))$$

$\therefore |f(x) - f(c)| < \varepsilon \quad \forall x \in N_\delta(c) \cap A.$

$\therefore$ f is continuous at c.

b) If c is not a limit point (or c is an isolated point) of A, then f is continuous at c.

<u>Proof:</u> Since c is an isolated point of A, so there is a nbhd $N_\delta(c)$ s.t. $N_\delta(c) \cap A = \{c\}$.

Let $\varepsilon > 0$, then $x \in N_\delta(c) \cap A \Rightarrow |f(x) - f(c)|$

$$= |f(c) - f(c)| = 0 < \varepsilon.$$

So, f is continuous at c.

<u>Defn:</u>

a) Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. ~~Let A⊂D.~~ f is said to be <u>continuous on A</u> if f is continuous at every point of A.

b) If f fails to be continuous at c, then we say that f is <u>discontinuous at c</u>.

<u>Examples:</u>

i) Let $f(x) := b \quad \forall x \in \mathbb{R}$.

Then $\lim_{x \to c} f(x) = b$ (we have seen in the chapter 'Limit')

$$= f(c)$$

$\therefore$ f is continuous on R

ii) Let $f(x) := x \ \forall x \in \mathbb{R}$.

We have seen that $\lim_{x\to c} f(x) = \lim_{x\to c} x$

$$= c$$

$$= f(c)$$

$\therefore f$ is continuous on $\mathbb{R}$.

iii) Similarly, the function $f(x) := x^2$ is continuous on $\mathbb{R}$.

iv) Let $f(x) := \frac{1}{x} \ \forall x \in \mathbb{R} - \{0\} =: A$.

Then we have observed that for $c \in \mathbb{R}-\{0\}$,

$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{1}{x} = \frac{1}{c} = f(c)$.

Then f is continuous on A.

Even if we extend f to another function ϕ defined by $\phi(x) := f(x) = \frac{1}{x} \ \forall x \in \mathbb{R} - \{0\}$

$$= c, \ x = 0.$$

Then as $\lim_{x\to 0} \phi(x) = \lim_{x\to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$, so

ϕ is not continuous at $x = 0$.

iv) Let $A := \mathbb{R}$ and let f be Dirichlet's ~~discontinuous~~ 'discontinuous function' defined by

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let c be a rational number. By Density Theorem, there exist a sequence (x_n) of irrational numbers s.t. $\lim(x_n) = c$.

Here $\lim f(x_n) = 0$ but $f(c) = 1$. ——— (A)

Similarly if c is an irrational number, then there is a sequence (y_n) of rational numbers s.t. $\lim(y_n) = c$.

Here $\lim f(y_n) = 1$ but $f(c) = 0$. ——— (B)

In both cases (A) and (B), $(x_n) \to c$ but

$$(f(x_n)) \not\to f(c)$$

and $(y_n) \to c$ but

$$(f(y_n)) \not\to f(c).$$

Then by the ~~follow~~ following sequential criterion, f is not continuous at any point in $\mathbb{R}$.

Sequential Criterion for Continuity:-

A function $f: A(\subset \mathbb{R}) \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Proof: Proof is similar to the proof of the 'sequential criterion for limit'.

The following Discontinuity criterion is a consequence of the last theorem.

Discontinuity Criterion:-

Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$, $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A such that (x_n) converges to c, but the sequence $(f(x_n))$ doesnot converge to $f(c)$.

<u>Application:-</u> Thomae's Function.

Let $A = \{x \in \mathbb{R} : x > 0\}$. We define $h : A \to \mathbb{R}$ by

$$h(x) := 0 \quad \text{if } x \text{ is an irrational number}$$

$$:= \frac{1}{n} \quad \text{if } x \text{ is a rational number and } x = \frac{m}{n} \text{ where } m \text{ and } n \text{ are natural numbers having no common factor except } 1.$$

Then, let x be a rational number with $x = \frac{m}{n}$ and $(m, n) = 1$. Then, $h(x) = \frac{1}{n}$.

Let (x_n) be a sequence of irrational numbers in A that converges to x. Then $h(x_n) = 0$ for all n, so $\lim h(x_n) = 0 \neq h(x)$. So, by Divergence Criterion, h is not ~~con~~ continuous at every rational number.

Let y be an irrational number and $\varepsilon > 0$. Then by Archimedean property there is a natural number n_0 such that $\frac{1}{n} < \varepsilon \ \forall n \geq n_0$.

Let $m < n_0$. Then there is a natural no. n_m such that $\frac{n_m}{m} > y + \varepsilon$.

So, $\forall n \geq n_m$, $\frac{n}{m} > y + \varepsilon$.

So, there are only a finite number of rationals with denominators less than n_0,

Choose $\delta > 0$ such that $(y-\delta, y+\delta)$ doesnot contain any rational number with ~~deo~~ denominator less than n_0,

$\therefore$ $|y' - y| < \delta \Rightarrow |h(y') - h(y)| < \varepsilon$.

So, h is continuous at every rational number y.

<u>Remark:-</u>

a) Let $f: A \to \mathbb{R}$ be a continuous function, Let c be a limit point of A and $c \notin A$. Then we can extend f to a continuous function $F: A \cup \{c\} \to \mathbb{R}$ defined by

$$F(x) := \begin{cases} f(x), & x \in A \\ \lim\limits_{x \to c} f(x), & x = c \end{cases}$$

provided, $\lim\limits_{x \to c} f(x)$ exists.

For example:- $f(x) := x \sin\frac{1}{x}$, $x \neq 0$

0 is a limit point of the domain $\mathbb{R} - \{0\}$ of f but $0 \notin \mathbb{R} - \{0\}$. As $\lim\limits_{x \to 0} x \sin\frac{1}{x} = 0$, so we can get another continuous function

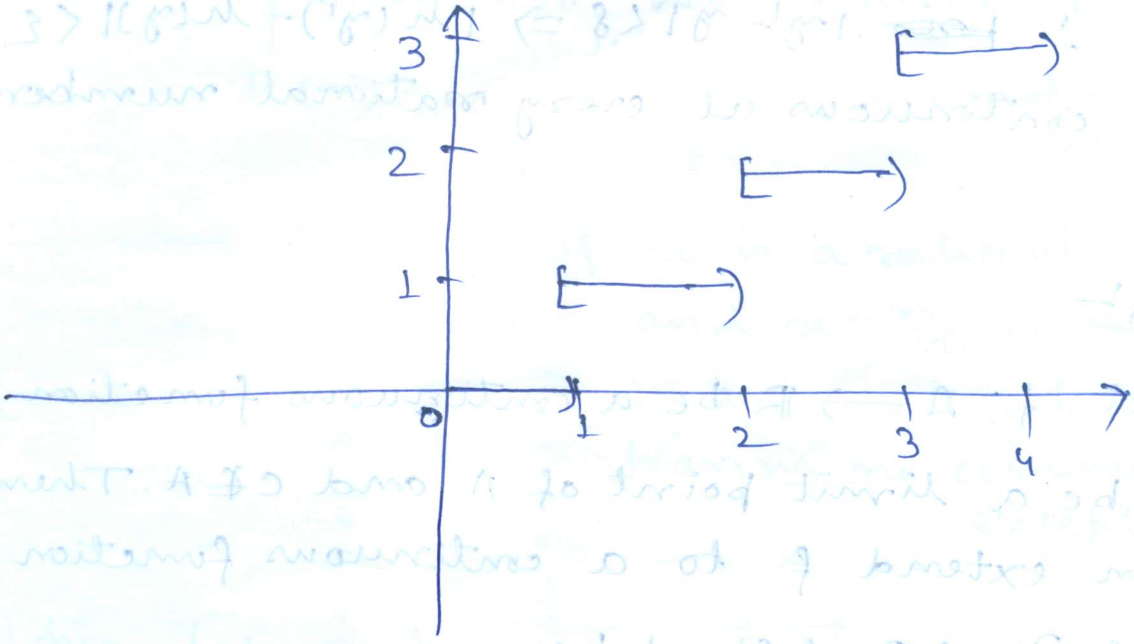
$$F(x) := \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

b) In Remark a) if $\lim\limits_{x \to c} f(x)$ does not exist, then we can't extend f on $A \cup \{c\}$.

For example:- $f(x) = \sin\frac{1}{x}$. By Divergence Criterion of limit, we proved $\lim \sin\frac{1}{x}$ doesnot exist.

<u>One famous example:-</u>

Let $f(x) := [x]$, here $[x]$ denotes the greatest integer n such that $n \le x$.



At points other than integers, f is continuous.
At integers the limit doesn't exist as $\lim\limits_{x \to n-} f(x) = n$ where as $\lim\limits_{x \to n+} f(x) = n+1$.

So, the function f is not continuous at integers.

Combination of Continuous Functions

<u>Theorem:-</u> Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $b \in \mathbb{R}$. Suppose that $c \in A$ and that f and g are continuous at c.

(a) Then $f+g$, $f-g$, fg, bf are continuous at c.

(b) If ~~...~~ $h: A \to \mathbb{R}$ is continuous at $c \in A$ and if $h(x) \neq 0$ for all $x \in A$, then the quotient f/h is continuous at c.

<u>Remark:-</u> Similar result is true when we consider continuity on A.

Composition of continuous functions:-

Theorem:- Let $A, B \subseteq \mathbb{R}$ and ~~let~~ let $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

Proof:- Let $\varepsilon > 0$. Then as g is continuous at $f(c)$, there is $\delta > 0$ such that $|y - f(c)| < \delta$ and $y \in B$ implies ~~$|g \circ f|$~~ $|g(y) - g(f(c))| < \varepsilon$. --- (A)

Since f is continuous at c, there is $\delta_1 > 0$ such that $|x - c| < \delta_1$ and $x \in A$ implies $|f(x) - f(c)| < \delta$.
--- (B)

Combining (A) and (B), we get —

$$|x - c| < \delta_1 \text{ and } x \in A \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon.$$

$$\Rightarrow |g \circ f(x) - g \circ f(c)| < \varepsilon.$$

$\therefore$ $g \circ f$ is continuous at c.

Cor:-

Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and f is continuous at $c \in A$. Then

A) $|f|$ defined by $|f|(x) := |f(x)|$ is continuous at c.

B) Let $f(x) \geq 0$ $\forall x \in A$. Then $\sqrt{f}$ defined by $\sqrt{f}(x) := \sqrt{f(x)}$ is continuous at c.

Proof: A) We have the following inequality —

$$||x| - |c|| < |x-c|.$$

Let $\varepsilon > 0$, Then, $\delta = \varepsilon$ gives

$$||x| - |c|| < |x-c| < \varepsilon.$$

So, g: $\mathbb{R} \to \mathbb{R}$ defined by $x \longrightarrow |x|$ is continuous at each $c \in \mathbb{R}$. So, by the last theorem $|f| = g \circ f$ is continuous at c.

B) Let $g: \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by

$$g(x) = \sqrt{x}.$$

Then, $|\sqrt{x} - \sqrt{c}| = \left|\frac{(\sqrt{x}+\sqrt{c})(\sqrt{x}-\sqrt{c})}{\sqrt{x}+\sqrt{c}}\right| \leq \frac{|x-c|}{\sqrt{c}}$

$< \varepsilon$

if $|x-c| < \varepsilon\sqrt{c}$

$\therefore$ g is continuous at each $c \in \mathbb{R}_{\geq 0}$.

So, $\sqrt{f} = g \circ f$ is continuous at c.

Cor: Let $A \subseteq \mathbb{R}$, $f, g: A \to \mathbb{R}$ be continuous at $c \in A$. Then

$\text{Max}(f,g)$ and $\text{Min}(f,g)$ are continuous at c.

Proof: Note that

$$\text{Max}(f,g)(x) = \frac{1}{2}\left(f(x) + g(x) + |f(x) - g(x)|\right)$$

and $\text{Min}(f,g)(x) = \frac{1}{2}\left(f(x) + g(x) - |f(x) - g(x)|\right)$

Then the proof follows from the above corollary.

~~Examples!~~

Remark:- All the above results are hold for continuity on A too.

Examples:-

a) A polynomial function $p(x)$ of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \text{ where } a_n \neq 0$$

is a continuous function on $\mathbb{R}$.

b) Let $p(x)$ and $q(x)$ be two polynomial functions such that there are at most a finite number of real roots of q. Let $x \in \mathbb{R} - \{\alpha_1, \cdots, \alpha_m\}$ imply $q(x) \neq 0$. Then we define the rational function r by

$$r(x) := \frac{p(x)}{q(x)} \quad \text{for } x \in \mathbb{R} - \{\alpha_1, \cdots, \alpha_m\}.$$

Then r is continuous on $\mathbb{R} - \{\alpha_1, \cdots, \alpha_m\}$.

c)

$$\sin x - \sin c = 2 \sin \frac{x-y}{2} \cos \frac{x+y}{2}.$$

$$\therefore |\sin x - \sin c| \leq 2 \left| \sin \frac{x-c}{2} \right| \leq \frac{2}{2} \cdot |x-c|$$

(as $|\sin z| \leq |z|$)

$$\therefore |x - c| < \varepsilon \Rightarrow |\sin x - \sin c| < \varepsilon.$$

$\therefore$ $\sin x$ is continuous on $\mathbb{R}$

Similarly $\cos x$ is continuous on $\mathbb{R}$.

d) ~~tan~~ $\tan x := \frac{\sin x}{\cos x}$ is continuous on

$$\mathbb{R} - \{x \mid \cos x = 0\} = \mathbb{R} - \{n\pi + \frac{\pi}{2} \mid n \in \mathbb{Z}\}.$$

Example:– A function $f: \mathbb{R} \to \mathbb{R}$ is said to be additive if $f(x+y) = f(x) + f(y)$ for all x and y in $\mathbb{R}$. Prove that if f is continuous at some point x_0, then it is continuous at every point in $\mathbb{R}$.

Answer:– As f is continuous at x_0, so for $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x_0+h) - f(x_0)| < \epsilon \quad \forall |h| < \delta.$$

$$\Rightarrow |f(x_0) + f(h) - f(x_0)| < \epsilon \quad \forall |h| < \delta$$

$$\Rightarrow |f(h)| < \epsilon \quad \forall |h| < \delta.$$

Let x be ~~any po~~ any point of $\mathbb{R}$.

Then, $|f(x+h) - f(x)| = |f(x) + f(h) - f(x)|$

$$= |f(h)| < \epsilon \quad \forall |h| < \delta.$$

So, f is continuous at every point of $\mathbb{R}$.

Properties of Continuous Functions

Theorem (Neighbourhood property):–

Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ be a continuous function, Let $c \in A$, If $f(c) \neq 0$, then there exists a suitable $\delta > 0$ such that for all $x \in N_\delta(c) \cap A$, $f(x)$ keeps the same sign as $f(c)$.

Proof:– Without loss of generality, we assume that $f(c) > 0$

Let $\varepsilon = \frac{f(c)}{2}$. Since f is continuous at c, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \forall x \in N_\delta(c) \cap A.$$

$$\therefore f(c) - \varepsilon < f(x) < f(c) + \varepsilon \quad \forall x \in N_\delta(c) \cap A.$$

$$\Rightarrow f(x) > f(c) - \varepsilon = f(c) - \frac{f(c)}{2} = \frac{f(c)}{2} > 0 \text{ for all } x \in N_\delta(c) \cap A.$$

<u>Note:</u>- This property is called the 'sign preserving property' of a continuous function. It is a local property of a continuous function.

<u>Theorem:</u>- Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function and S be an open subset of $\mathbb{R}$. Then $f^{-1}(S)$ is an open subset of $\mathbb{R}$.

<u>Proof:</u>- Let $s \in f^{-1}(S)$. Then $f(s) \in S$. As S is an open set, so there exists an $\varepsilon > 0$ such that

$$(f(s) - \varepsilon, f(s) + \varepsilon) \subset S.$$

As f is continuous at s, there exists $\delta > 0$ such that $x \in (s-\delta, s+\delta) \Rightarrow f(x) \in (f(s)-\varepsilon, f(s)+\varepsilon) \subset S$.

$$\Rightarrow (s-\delta, s+\delta) \subset f^{-1}(S).$$

So, s is an interior point of S. So, S is an open set in $\mathbb{R}$.

<u>Note:</u>- 1) It is immediate that a function $f: \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every

open subset S of $\mathbb{R}$, $f^{-1}(S)$ is an open subset of $\mathbb{R}$. This is taken as the definition for a continuous function between two topological spaces.

2) If $f: \mathbb{R} \to \mathbb{R}$ is a continuous function and S is an open subset of $\mathbb{R}$. Then $f(S)$ is not necessarily an open set. For example, $f(x) = x^2$ and $S = (-1, 1)$ is an open subset of $\mathbb{R}$. Then $f(S) = [0, 1)$ is not an open subset of $\mathbb{R}$.

3) It can be shown ~~to~~ that ~~for~~ a function $f: \mathbb{R} \to \mathbb{R}$ is continuous if and only if for a closed set S, $f^{-1}(S)$ is closed.

~~Conti~~

Continuous functions enjoy many more nice properties, specially if we put more conditions on the domain set.

Defn: A function $f: A \to \mathbb{R}$ is said to be a **bounded** function if there exist $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$, i.e., ~~when~~ when the range is a bounded set in $\mathbb{R}$. Otherwise f is said to be an **unbounded** function on A.

Example:-

a) $A = (0, \infty)$, $f(x) := x$ or $1/x$ are unbounded.

b) $A = (0, 1)$, $f(x) := 1/x$. Then f is unbounded on A.

~~c) A~~ In the previous examples, ~~f is either~~ the domain set A is either unbounded or open sets.

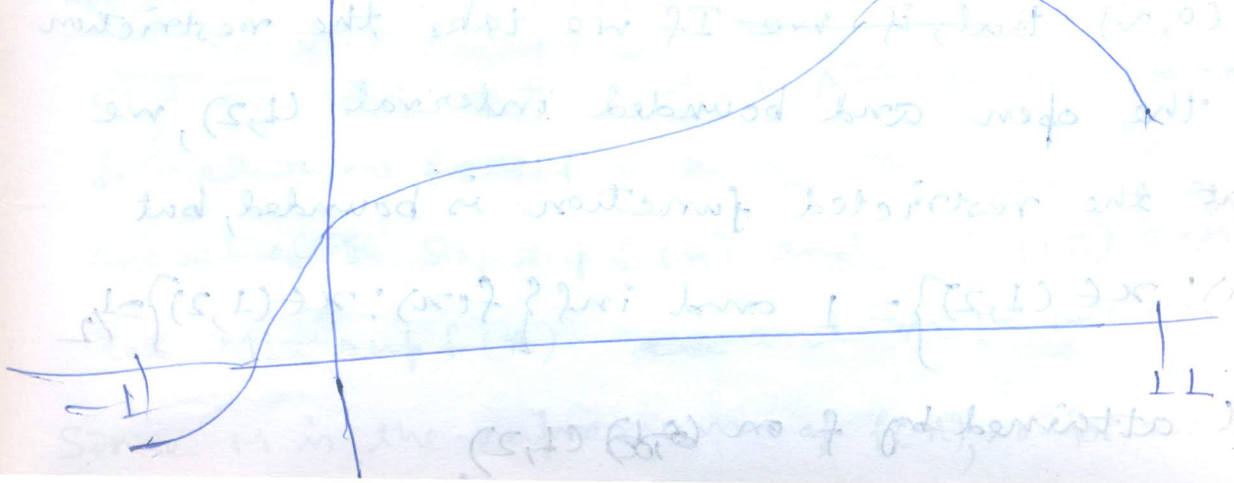
<u>Theorem</u>: Let A be a closed and bounded set. Let $f: A \to \mathbb{R}$ be a continuous function. Then ~~f(x)~~ f is a bounded function.

<u>Proof</u>: Suppose that f is not bounded on A. Then for any $n \in \mathbb{N}$, there is a number $x_n \in A$ such that $|f(x_n)| > n$. Since (x_n) is a bounded sequence, by Balzano–Weierstrass theorem, (x_n) has a convergent subsequence (x_{n_r}). Let (x_{n_r}) converge to x. Since f is continuous on A, so $(f(x_{n_r}))$ converges to $f(x)$. So, $(f(x_{n_r}))$ is a bounded sequence which contradicts the fact that

$$|f(x_{n_r})| > n_r \geq r \quad \forall r \in \mathbb{N}.$$

So, we have arrived at a contradiction. So, f is a bounded function.

<u>Example</u>:



Here $A = [-1, 1]$ is a closed and bounded interval. And f attains maximum and minimum on $[-1, 1]$.

~~Def~~: Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. We say that f has an absolute (global) maximum on A if there is a point $x^* \in A$ such that

$$f(x^*) \geq f(x) \quad \text{for all } x \in A.$$

We say that f has an absolute (global) minimum on A if there is a point $x_* \in A$ such that

$$f(x_*) \leq f(x) \quad \text{for all } x \in A.$$

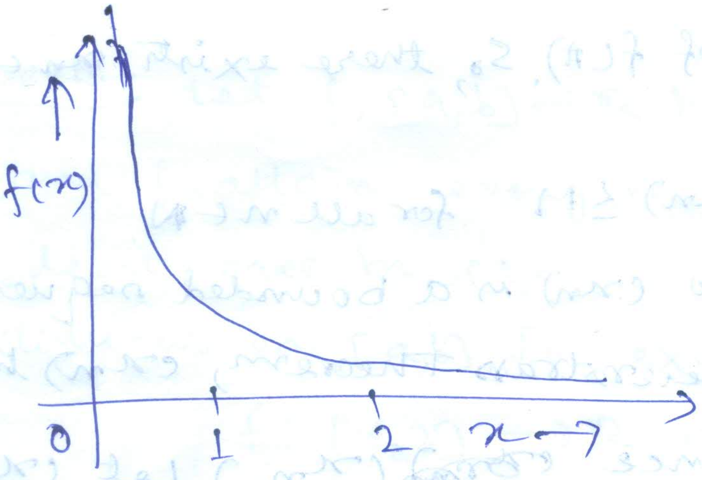
We ~~also~~ also say that x^* is an absolute maximum point for f on A, and that x_* is an absolute minimum point ~~for~~ for f on A.

Example:-

1) A function $f: (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{x}.$$

Here f is not bounded as f is not bounded above. Here $0 = \inf\{f(x) : x \in (0, \infty)\}$ but 0 is not attained by f on $(0, \infty)$. ~~But, if we~~ If we take the restriction of f on the open and bounded interval $(1, 2)$, we see that the restricted function is bounded, but $\sup\{f(x) : x \in (1, 2)\} = 1$ and $\inf\{f(x) : x \in (1, 2)\} = \frac{1}{2}$ are not attained by f on ~~$(0, \infty)$~~ $(1, 2)$



If we take the restriction of f on $[1,2]$, then $\sup\{f(x) : x \in [1,2]\} = 1 = f(1)$

and $\inf\{f(x) : x \in [1,2]\} = 1/2 = f(2)$. So, f has both absalute maximum and absalute minimum when it is restricted to the set $[1,2]$.

2) $f : [0,\infty) \longrightarrow \mathbb{R}$ is defined by

$f(x) = x^2$.

Here f is continuous but not bounded above. So, f doesn't attain absalute maximum.

<u>Maximum-Minimum Theorem:-</u>

Let A be a closed and bounded set and $f : A \rightarrow \mathbb{R}$ be a continuous function. Then f has an absalute maximum and absolute minimum on A.

<u>Proof:</u> By the previous theorem, since A is a closed and bounded set and $f : A \rightarrow \mathbb{R}$ is a continuous function, so $f(A)$ is a non-empty bounded subset of $\mathbb{R}$. So, $\sup f(A)$ and $\inf f(A)$ exist.

Let $M := \sup f(A)$ and $m := \inf f(A)$

Since M is the supremum of $f(A)$, so for $n \in \mathbb{N}$, $M - 1/n$

is not an upper bound of $f(A)$. So, there exist $x_n \in A$ such that

$$M - \frac{1}{n} < f(x_n) \leq M \quad \text{for all } n \in \mathbb{N}.$$

Since A is bounded, so (x_n) is a bounded sequence, so by the Balzano–Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_r}). Let (x_{n_r}) converge to c. Since A is a closed set, so $c \in A$. As (x_{n_r}) converges to c, so by sequential criterion, $(f(x_{n_r}))$ converges to $f(c)$.

Now, $M - \frac{1}{n_r} \leq f(x_{n_r}) \leq M$, then by Squeeze Theorem, as $\lim_{r\to\infty} (M - \frac{1}{n_r}) = M$, so

$$\lim_{r\to\infty} f(x_{n_r}) = M.$$

So, $f(c) = M$.

Similarly, it can shown that there exist a point $d \in A$ such that $f(d) = m$.

Intermediate Value Theorem

Balzano's Intermediate Value Theorem assures us that a continuous function on an interval takes on (at least once) any number that lies between two of its values.

Balzano's Intermediate Value Theorem

Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exist a point $c \in I$ between a and b such that $f(c) = k$.

Remark: Let $f: [a, b] \to \mathbb{R}$ be a function with $f(a) \neq f(b)$ and f attain every value between $f(a)$ and $f(b)$ at least once in (a,b). Still f may not be continuous on $[a,b]$. For example:

$f: [0,2] \to \mathbb{R}$ is defined by

$$f(x) = x, \quad 0 < x \leq 1$$

$$= 3 - x, \quad 1 < x \leq 2.$$

Here f is not ~~contiue~~ continuous at $x = 1$.

Proof: Let $a < b$ and $g(x) := k - f(x)$. Then $g(a) < 0 < g(b)$. Then the proof follows from the following special case directly.

Location of Roots Theorem:

Let $I := [a, b]$ and let $f: I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then there exists a number $c \in (a,b)$ such that $f(c) = 0$.

Proof: Without loss of generality, we assume that $f(a) < 0 < f(b)$. We will generate a sequence of intervals by successive bisections.

Let $I_1 := [a_1, b_1]$ where $a_1 = a$ and $b_1 = b$. Let p_1 be the midpoint $p_1 = \frac{1}{2}(a_1 + b_1)$. If $f(p_1) = 0$, then we take $c := p_1$ and we are done.

If $f(p_1) > 0$ then we set $a_2 := a_1$ and $b_2 := p_1$.

If $f(p_2) < 0$ then we set $a_2 := p_1$ and $b_2 := b_1$.

In either case, we have got $I_2 := [a_2, b_2]$, $I_2 \subset I_1$ and

$f(a_2) < 0$ and $f(b_2) > 0$.

Continuing this bisection process, suppose that we have got the intervals $I_1, I_2, \ldots, I_k$ such that $I_1 \supset I_2 \supset \cdots \supset I_k$ and $f(a_k) < 0$ and $f(b_k) > 0$. We take $p_k := \frac{1}{2}(a_k + b_k)$. If $f(p_k) = 0$, we take $c := p_k$ and we are done.

If $f(p_k) > 0$, we set $a_{k+1} := a_k$, $b_{k+1} := p_k$ and if $f(p_k) < 0$, we set $a_{k+1} := p_k$ and $b_{k+1} := b_k$.

In either case, we let $I_{k+1} := [a_{k+1}, b_{k+1}]$. Then $I_{k+1} \subset I_k$ and $f(a_{k+1}) < 0$, $f(b_{k+1}) > 0$.

If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done.

Otherwise we get a nested sequence of closed and bounded intervals $I_n := [a_n, b_n]$ such that

$f(a_n) < 0$ and $f(b_n) > 0$ for all $n \in \mathbb{N}$.

Again, note that $|I_n| = b_n - a_n = \frac{b-a}{2^{n-1}}$.

So, by Nested Interval Property, $\bigcap_{n=1}^{\infty} I_n$ is exactly a point, say c.

Now, $c \in I_n$ for each n and $\lim(b_n - a_n) = 0$, so $\lim(a_n) = c = \lim(b_n)$.

Since f is continuous at c, so $\lim(f(a_n)) = f(c) = \lim(f(b_n))$.

Since each $f(a_n) < 0$, so $f(c) \le 0$

and each $f(b_n) > 0$, so $f(c) \geq 0$ $\Rightarrow f(c) = 0$.

So, c is a root of f.

Applications of Location of roots theorem:-

A) Show that the equation $x = \cos x$ has a solution in the interval $[0, \pi/2]$. Use the bisection method to find an approximate solution with error less than 10^{-2}.

Solution:- Here $f(x) = x - \cos x$.

$\therefore f(0) = 0 - \cos 0 = -1 < 0$ and $f(\pi/2) = \pi/2 - \cos\pi/2 = \pi/2 > 0$.

$\therefore$ There exists $c \in (0, \pi/2)$ such that $f(c) = 0$.

Here at n-th step, if we get p_n, then

$$|p_n - c| \leq \frac{1}{2}\cdot(b_n - a_n) = \frac{1}{2^n}(\pi/2 - 0) = \frac{\pi}{2^{n+1}}.$$

$$\therefore \frac{\pi}{2^{n+1}} < 10^{-2} \Leftrightarrow 2^{n+1} > \frac{\pi}{(100)^{-1}} \Leftrightarrow (n+1)\ln 2 > \cancel{100\pi}\ \ln(100\pi)$$

$$\Leftrightarrow n+1 > \frac{\ln(100\pi)}{\ln 2}$$

$$\Leftrightarrow n > \frac{\ln(100\pi)}{\ln 2} - 1$$

$$= 8.3 - 1$$

$$= 7.3$$

So, we take n to be 8.

n	a_n	b_n	p_n	$f(p_n)$
1	0	$\pi/2$	.7854	.078370
2.	0	0.7854	.3927	$-.6073 < 0$
3.	.3927	.7854	.5890	$-.24251 < 0$

B) Let $f: [a,b] \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each x in I. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for each $x \in I$.

Solution:- Here $f(I)$ is bounded.

Let $m := \inf f(I)$. Then $\exists\ c \in I$ such that $m = f(c)$ (by Maximum Minimum Theorem).

$\therefore m > 0$.

$\therefore f(x) \geq m$ for all $x \in I$.

C) Show that every polynomial of odd degree with real co-efficients has at least one real root.

Solution:- Let $p(x) := a_{2n+1} x^{2n+1} + a_{2n} x^{2n} + \cdots + a_0$ (here $a_{2n+1} \neq 0$).

$$= x^{2n+1}\left(a_{2n+1} + \frac{a_{2n}}{x} + \cdots + \frac{a_0}{x^{2n+1}}\right)$$

$$= x^{2n+1} \cdot r(x) \text{ where}$$

$$r(x) := a_{2n+1} + \frac{a_{2n}}{x} + \cdots + \frac{a_0}{x^{2n+1}}$$

$\therefore r(x) \to a_{2n+1}$ as $x \to \infty$ and $\to -\infty$,

$\therefore$ ~~$p(x) \to a_{2n+1} x$~~

$\therefore p(x) \to \infty$ if $a_{2n+1} > 0$ and $x \to \infty$

or

if $a_{2n+1} < 0$ and $x \to -\infty$.

Also, $p(x) \to -\infty$ if $a_{2n+1} < 0$ and $x \to \infty$

or

$a_{2n+1} > 0$ and $x \to -\infty$

$\therefore p(x) > 0$ for some x and $p(x) < 0$ for some x.

$\therefore p(x_1) = 0$ for some x_1 by Intermediate Value Theorem.

Preservation Theorem:-

Corollary of Intermediate Value Theorem:-

Let ~~I be an~~ $I = [a,b]$ be a closed and bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ be such that

$$\inf f(I) \leq k \leq \sup f(I),$$

then there exists a number $c \in I$ such that $f(c) = k$.

Proof: It follows from Maximum-Minimum Theorem that there are points c_* and c^* in I such that

$$f(c_*) = \inf f(I) \text{ and } f(c^*) = \sup f(I).$$

So, $f(c_*) \leq k \leq f(c^*)$.

So, $\exists$ $c \in [c_*, c^*]$ such that $f(c) = k$.

Preservation of Intervals Theorems:-

A) Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.

B) Let I be a closed and bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is a closed and bounded interval.

Proof: A) Let $\alpha < \beta$ with $\alpha, \beta \in f(I)$. Then there exists $a, b \in I$ such that $f(a) = \alpha$ and $f(b) = \beta$.

Then by Balzano's Intermediate Value Theorem, for $k \in (\alpha, \beta)$, $\exists$ $c \in [a,b]$ such that $f(c) = k$.

So, $k \in f(I)$ $\therefore [\alpha, \beta] \subset f(I)$. So, $f(I)$ is an interval.

B) As I is a closed and bounded, set, so ~~there exis~~ $f(I)$ is bounded. Let $m := \inf f(I)$ and $M := \sup f(I)$. So, by Maximum-Minimum theorem, $m, M \in f(I)$. $\therefore f(I) \subset [m, M]$.

By Balzano's Intermediate Value Theorem, $[m, M] \subset f(I)$

So, $f(I) = [m, M]$, a closed and bounded interval.

Note: The preservation theorem holds only for closed and bounded intervals. For example:-

a) $f : (0,1) \to \mathbb{R}$ is defined by $f(x) = 1/x$.

Here $(0,1)$ is open and bounded but $f(I) = (1, \infty)$ is open and unbounded.

b) $f : [0, \infty) \to \mathbb{R}$ is defined by $f(x) = \dfrac{1}{1+x^2}$.

Here $[0, \infty)$ is closed and unbounded interval, but $f(I) = (0, 1]$ is not closed.