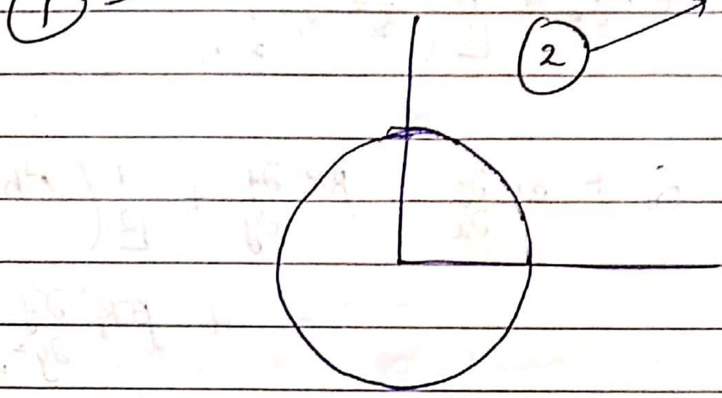
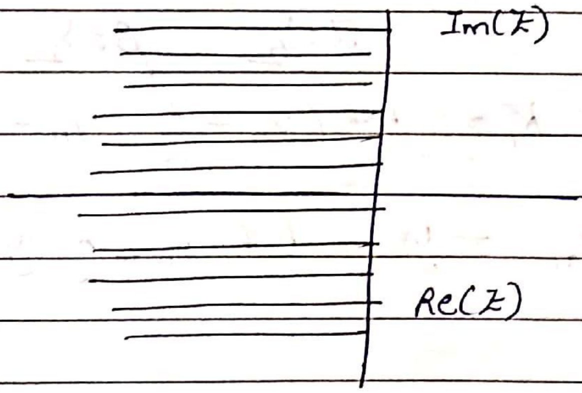
Routh-Hurwitz Criterion:

$f(\xi) = 0$ $\qquad\qquad$ $\xi = \dfrac{1+z}{1-z}$



After making the transformation:

$$P(z) = 0$$



Z-plane

i) It transforms interior of the unit circle to the left half of the z-plane.

Interior on $|\xi| = 1$

$\longleftrightarrow$ left half plane $z \leq 0$

ii) The unit circle $|\xi| = 1$

$\longleftrightarrow$ imaginary axis

iii) $z = 0 \longleftrightarrow \xi = 1$

$$P(z) = b_0 z^k + b_1 z^{k-1} + b_2 z^{k-2} + \cdots + b_k = 0 \quad \text{—— } (3)$$

Denote:

$$D : \begin{vmatrix} b_1 & b_3 & b_5 & \cdots & b_{2k-1} \\ b_0 & b_2 & b_4 & \cdots & b_{2k-2} \\ 0 & b_1 & b_3 & \cdots & b_{2k-3} \\ 0 & b_0 & b_2 & \cdots & b_{2k-4} \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & b_k \end{vmatrix}$$

$$b_j > 0 \quad \forall j$$

Routh-Hurwitz Criterion states:

The real parts of the roots of (3) are negative iff The principal minor of D are Positive.

$k = 1$: $b_0 > 0$, $b_1 > 0$

$k = 2$: $b_0 > 0$, $b_1 > 0$, $b_2 > 0$

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Stability of linear multi-step Method:

$$\rho(E)\, u_{j-k+1} - h\sigma(E)\, u'_{j-k+1} = 0$$

Let $\xi_1, \xi_2, \ldots, \xi_k$ be the roots of $\rho(\xi)=0$

For consistent method : $\xi_1 = 1$

$$\rho(1) = 0$$

$1, \xi_2, \ldots, \xi_k \longrightarrow$ are the roots of $\rho(\xi)$

Def: i) The linear multi-step method is stable if $|\xi_j| < 1$ for $j \neq 1$

ii) If $|\xi_j| > 1$ for some j then unstable

iii) Conditionally stable if ξ_j's are simple and if more than one of these roots have modulus unity.

Stability of this l.m.s.m :-

$$u' = \lambda u$$

Applying l.m.s.m to this test equation

$$u_{j+1} = \sum_{i=1}^{k} a_i u_{j-i+1} + h\lambda \sum_{i=0} b_i u_{j-i+1}$$

$$u_{j+1} = \sum_{i=1}^{k} a_i u_{j-i+1} + h\lambda \sum_{i=0}^{k} u_{j-i+1}$$

Exact Solution:

$$u(t_{j+1}) = \text{Method} + T_{j+1}$$

$$\epsilon_j = u_j - u(t_j)$$

Error equation:

$$\bar{h} = \lambda h$$

$$\epsilon_{j+1} = \sum_{i=1}^{k} a_i \epsilon_{j-i+1} + \bar{h} \sum_{i=0}^{k} b_i \epsilon_{j-i+1} - T_{j+1}$$

$$|\rho(E) - \bar{h}\sigma(E)| \epsilon_{j-k+1} + T_{j+1} = 0$$

a linear k-step method

Assume Error:

$$\epsilon_j = \alpha \xi^j$$

$$\boxed{\rho(\xi) - \bar{h}\sigma(\xi) = 0}$$

characteristic equation $h \to 0$

Distinct & Real:

$$\epsilon_j = c_1 \xi_{1h}^j + c_2 \xi_{2h}^j + \cdots + c_k \xi_{kh}^j$$

or repeating roots:

$$\epsilon_j = (c_1 + c_2 j)\xi_{1h}^j + c_3 \xi_{3h}^j + \cdots + c_k \xi_{kh}^j$$

$c_1, c_2, c_3, \ldots, c_k$; acts constant

For $h \to 0$ characteristic equation reducing to $\rho(\xi) = 0$

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$$\xi_{1h} = \xi_i \left[1 + \bar{h} k_i + O(\bar{h}^2) \right]$$

for $i = 1, 2, \dots k$

$k_i \longrightarrow$ growth parameters

$$\rho\left[\xi_i + \bar{h} k_i \xi_i + O(\bar{h})^2\right] - \bar{h}\,\sigma\left[\xi_i + \bar{h} k_i \xi_i + O(\bar{h})^2\right] = 0$$

Taylor Series about ξ_i

$$\rho(\xi_i) + \bar{h} k_i \xi_i (\rho'(\xi_i)) - \bar{h}\,\sigma(\xi_i) + O(\bar{h}^2) = 0$$

since $\rho(\xi_i) = 0$

$$\boxed{k_i = \frac{\sigma(\xi_i)}{\xi_i\, \rho'(\xi_i)}} \quad \forall i$$

$$\xi_{ih}^{j} = \xi_i^{j} \left[1 + \bar{h} k_i + O(\bar{h}^2) \right]$$

$$\boxed{\xi_1 = 1} \qquad \boxed{\rho(1) = 0}$$

$$k_1 = \frac{\sigma(1)}{\xi_1 \cdot \rho'(1)} = 1$$

$$\xi_{1h} = \xi_1 \left[1 + \bar{h} \cdot 1 + O(\bar{h}^2) \right]$$

$$\xi_1 \left[1 + \bar{h} + O(\bar{h})^2 \right] = e^{\bar{h}}$$

$$\xi_{1h} = 1$$

$$\xi_{1h} = e^{\bar{h}} = e^{\lambda h}$$

$$\boxed{u' = \lambda u}$$

$\xi_{1h} \longrightarrow$ Principal Root

$[\xi_{2h}, \xi_{3h}, \dots, \xi_{kh}]$

$\hookrightarrow$ extraneous roots

Find the growth factors associated with the linear multi-step method given below:

$$u_{j+1} = u_{j-1} + \frac{h}{3}\left(u'_{j+1} + 4u_j' + u_{j-1}' \right)$$

Apply this l.m.s.m to the test equation:

$$u' = \lambda u$$

$$\rho(\xi) = \xi^2 - 1 \qquad \sigma(\xi) = \frac{1}{3}(\xi^2 + 4\xi + 1)$$

$$\rho(\xi) = 0$$

$$\xi_{1,2} = \pm 1$$

$$K_1 = \frac{\sigma(1)}{\xi_1 \rho'(1)} = \frac{2}{1\cdot 2} = 1$$

$$K_2 = \frac{\sigma(-1)}{\xi_2 \rho'(-1)} = \frac{2/3}{-2} = \frac{-1}{3}$$

Now using K_i, the roots ξ_{ih} are approximation as follows

$$\xi_{1h} = \xi_1 \left(1 + \lambda h K_1 + \cdots\right] = \left(1 + \lambda h + \cdots\right)$$
$$= e^{\lambda h} + O(h^2)$$

$$\xi_{2h} = \xi_2 \left[1 + \lambda h K_2 + \cdots\right] = -1\left[-1 \mp \frac{\lambda h}{3} + \cdots\right]$$
$$= -e^{-\lambda h/3} + O(h^2)$$

$$y_j = c_1 \xi_{1h}^j + c_2 \xi_{2h}^j$$

Now the solution of the difference equation (or the l.m.s.m) is

$u' = \lambda u$

$$y_j = c_1 e^{j\bar{h}} + c_2 (-1)^j e^{-\frac{1}{3} j\bar{h}}$$

Check the Routh-Hurwitz theorem:

The characteristic equation is:

$$\left(1 - \frac{\bar{h}}{3}\right)\xi^2 - \frac{4\bar{h}}{3}\xi - \left(1 + \frac{\bar{h}}{3}\right) = 0$$

Substitute

$$\xi = \frac{1+z}{1-z}$$

$$\left(1 - \frac{\bar{h}}{3}\right)(1+z)^2 - \frac{4\bar{h}}{3}(1+z)(1-z) - \left(1 + \frac{\bar{h}}{3}\right)(1-z)^2 = 0$$

$$\Rightarrow \quad \frac{\bar{h}}{3} z^2 + 2z - \bar{h} = 0$$

$$b_0 z^2 + b_1 z + b_2 = 0$$

We need,

$$\frac{\bar{h}}{3} > 0, \quad 2 > 0, \quad -\bar{h} > 0$$

$$\Rightarrow \quad \bar{h} > 0 \text{ and } -\bar{h} > 0 \quad \Longleftrightarrow$$

So, the Routh-Hurwitz criterion is not satisfied for any value of $\bar{h}$.

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For what values of the parameter λh, the R-H criterion is satisfied ~~to~~ for the given l.m.s.m:

$$u_{j+1} = u_j + \frac{h}{12}\left(5u_{j+1}' + 8u_j' - u_{j-1}'\right)$$

Apply the l.m.s.m. to the test equation: $u' = \lambda u$, we obtain the characteristic equation as:

$$\left(1 - \frac{5}{12}\lambda h\right)\xi^2 - \left(1 + \frac{2}{3}\lambda h\right)\xi + \frac{\lambda h}{12} = 0$$

Use $\xi = \frac{1+z}{1-z}$

$$\left(1 - \frac{5}{12}\bar{h}\right)(1+z)^2 - \left(1 + \frac{2}{3}\lambda h\right)(1+z)(1-z)$$

$$+ \frac{\lambda h}{12}(1-z)^2 = 0$$

$$b_0 z^2 + b_1 z + b_2 = 0$$

where $b_0 = 2 + \frac{\bar{h}}{3}$, $b_1 = 2 - \bar{h}$, $b_2 = -\bar{h}$

>0 >0 >0

when $\lambda < 0$:

$$\xi_{ih}^{i} = \xi_i^j \left[1 + \bar{h} K_i + O(\bar{h})^2\right]^j$$

$$= \xi_i^j \cdot \exp(\lambda h K_i \cdot j)$$

$$i = 1, 2, \ldots, k$$

$\xi^j \cdot e^{j\lambda h K_i}$

when $\xi_1 = 1$, $K_1 = 1$

and $\xi_{1h} = e^{j\lambda h} + O(\bar{h})^2$ $\qquad u' = \lambda u$

Thus ξ_{1h} approximate the solution of the differential equation $u' = \lambda u$

$$\xi_{1h} = \text{principal root}$$

The $(k-1)$ roots are called as extraneous roots.

i) if any of the roots ξ_{ih}, $i = 1, 2, \ldots, k$ is such that $|\xi_{ih}| > 1$, then the error $|\epsilon_j|$ grows unboundedly.

ii) If there is a multiple root of magnitude unity, then $|\epsilon_j|$ grows unboundedly.

iii)

Definition:

The linear multi-step method (*) when applied to the test equation $u' = \lambda u$, $\lambda < 0$ is ~~such that~~ said to be absolutely stable if

$$|\xi_{ih}| \le 1 \qquad i = 1, 2, \ldots$$

The region of absolute stability is the set of points in the $h\lambda$- plane for which the method is absolutely stable.

Ex: The 4^{th} order A-M method

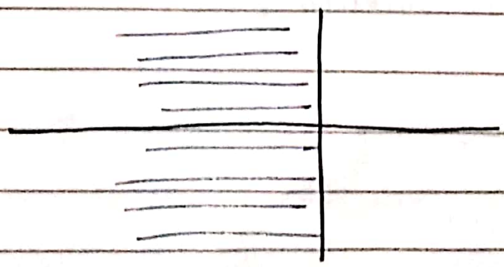
$$u_{j+1} = u_j + \frac{h}{12}\left(5u'_{j+1} + 8u'_j - u'_{j-1}\right)$$

is absolutely stable as $(-6, 0)$

Definition: 2 :

A l.m.s.m when applied to the diff. equation $u' = \lambda u$, λ is a complex constant with negative real part is said to be A-stable if all solutions of $\rho(E)\, u_{j-k+1} - h\sigma(E)\, u'_{j-k+1} = 0$ tend to zero as $j \to \infty$

The region of Absolute-stability is the entire left half plane of λh plane.



$-\infty < \overline{\lambda h} < 0$

Case: when λ is real,
this interval is $(-\infty, 0)$

Definition 3:
The l.m.s.m is s.t.b weakly stable if there is more than one simple root of $\rho(\xi)=0$ on the unit circle.

And strongly stable if not.

Ex:
2^{nd} order Nystrom method:

$$y_{j+1} = y_{j-1} + 2hf_j$$

on $u' = \lambda u \Rightarrow y_{j+1} = y_{j-1} + 2h\lambda y_j$

ch. eq: $\xi^2 - 2\bar{h}\xi - 1 = 0$

$$\xi_{1h} = e^{\bar{h}} \; ; \; \xi_{2h} = -e^{-\bar{h}}$$

$$y_j = c_1 e^{j\bar{h}} + c_2 (-1)^j e^{-j\bar{h}}$$

i) $\lambda > 0$: ξ_{2h} discrete

ii) $\lambda < 0$: ξ_{2h} oscillates with increasing amplitude.
For $\bar{h} < 0$, the method is unstable.

iii) For $\bar{h} = 0$, this method is

iv)

Definition 4:

The l.m.s.m is s.t.b. Relatively stable if

$$|\xi_{in}| \leq |\xi_{1n}| \quad , i = 1, 2, 3, \ldots k$$

The region

Boundary Value Problems

2-point BVP

$$y'' + ay' + by = c(x) \quad \text{in } [a, b]$$

i) $y(a) = r_1$
ii) $y(b) = r_2$ — Dirichlet BC's

or

i) $\left.\frac{dy}{dx}\right|_{x=a} = r_3$
ii) $\left.\frac{dy}{dx}\right|_{x=b} = r_4$ — Neumann BC's

i) $a_0 y(a) + a_1 y'(a) = r_5$
ii) $b_0 y(b) + b_1 y'(b) = r_6$

$\}$ Type-III condition or Robin boundary condition

Second order ODE —

$$f(x, y(x), y'(x), y''(x)) = 0$$

Linear 2nd order equation:

$$-u''(x) + p(x)\cdot u'(x) + q(x)\cdot u(x) = r(x) \qquad [0, b]$$

p, q and r are functions of x are continuous in $[0, b]$.

1. and if $q > 0$ $\forall$ x in $[a, b]$ then the solution exists and is unique.

General —

$$u'' = f(x, u, u') \qquad x \in [a, b] \quad — (1)$$

Existence & uniqueness of the solution :

i) $f(x, u, u')$ is continuous w.r.t x, u and u' in $-\infty < x, u, u' < \infty$

ii) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}$ exist and continuous

iii) $\frac{\partial f}{\partial u} > 0$ and $\left|\frac{\partial f}{\partial u'}\right| \leq \omega, \quad \omega > 0$

Then, the BVP (1) has a unique existing solution.

B.V.P.

finite difference method

using initial value solver (shooting and matching technique)

<u>Finite differences:</u>

$$\frac{dy}{dx} \approx \begin{cases} \frac{y_{j+1} - y_j}{h} + O(h) & \text{Forward diff.} \\ \frac{y_j - y_{j-1}}{h} + O(h) & \text{Backward diff.} \\ \frac{y_{j+1} - y_{j-1}}{2h} + O(h^2) & \text{Central diff.} \end{cases}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_j} \approx \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + O(h^2)$$

Now, suppose given –

$$u'' = f(x, u, u')$$

Question :-

$$-u'' + P(x)\,u' + Q(x)\,u = r(x) \qquad [a, b]$$

type I $\qquad u(a) = r_1, \quad u(b) = r_2$

Step - I

u_0

$x_0 = a \quad x_1 \quad x_2 \quad \cdots \quad x_{j-1} \quad x_j \quad x_{j+1} \quad \cdots \quad x_{n-1} \quad x_n = b$

$$-\left[\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right] + P(x_j)\left[\frac{u_{j+1} - u_{j-1}}{2h}\right] + Q(x_j)\cdot u_j = r_j$$

$j = 1, 2, 3, \ldots, n-1$

$j = 0 \longrightarrow u_0 = r_1$

At $j = 1$

$$f_1(u_2, u_1, u_0) = r_1$$

$j = 2$

$$f_2(u_3, u_2, u_1) = r_2$$

$\vdots$

$j = n-1$

$$f_{n-1}(u_n, u_{n-1}, u_{n-2}) = r_{n-1}$$

At $j = n \longrightarrow u_n = r_n$

$$\begin{bmatrix} \ddots & & 0 & 0 \\ & \ddots & & 0 \\ 0 & & \ddots & \\ 0 & 0 & & \ddots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \end{bmatrix}$$

Tridigonal (rest all elements 0)

2 point B.V.P.

linear, Finite difference method

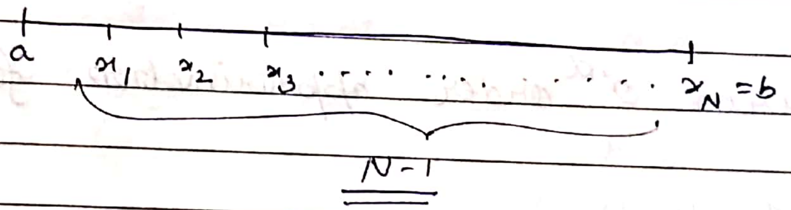
$$-u'' + p(x)u' + q(x)\cdot u = r(x) \qquad , \quad a < x < b$$

Type-I B.C.'s

i) $u(x=a) = r_1$

ii) $u(x=b) = r_2$

Step 1:



Step 2:

$$-u''(x) + p(x)u'(x) + q(x)\cdot u(x) = r(x) \Big|_{x=x_j}$$

$$-\left[\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}\right] + p(x_j)\cdot\left[\frac{u_{j+1} - u_{j-1}}{2h}\right] +$$

$$q(x_j)\cdot u_j = r_j$$

$1 < j < N-1$

$$A_j u_{j-1} + B_j u_j + C_j u_{j+1} = \frac{h^2}{2} r(x_j)$$

$1 \leq j < N-1$

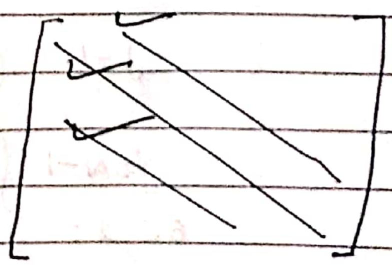
Thomas Algorithm:

$j = 1 \quad u_0$

$\vdots$

$j = N-1 \quad u_N \big|_{x=b}$

$j=0$
$j=1$
$\vdots$
$j=N-1$
$j=N$



$$A_j = \frac{-1}{2}\left(1 + \frac{h}{2} p_j\right) \quad ; \quad B = \left(1 + \frac{h^2}{2} q_j\right)$$

$$C = \frac{-1}{2}\left(1 - \frac{h}{2} p_j\right)$$

$$\boxed{T\underline{x} = \underline{b}}$$

Type 3 boundary conditions:

{ Robin 2 point BVP }

$$a_0 u(0) - a_1 u'(0) = r_1$$
$$b_0 u(b) + b_1 u'(b) = r_2$$

With 2^{nd} order approximation for the derivative

Step 1:

$x = a = x_0 \; (j=0)$ $\qquad\qquad$ $x = b$

BC1;

$$a_0 u(0) - a_1 u'(0) = \gamma_1$$

$$a_0 u_0 - a_1 \left[\frac{u_{0+1} - u_{0-1}}{2h}\right] = \gamma_1$$

B.C.1 : $\quad 2h a_0 u_0 - a_1 u_1 + a_1 u_{-1} = 2h\gamma_1$

$u_1 \quad u_0$

$x_{-1} \quad x_0$

$j = 1$

$\vdots$

$j = N-1$

B.C.2 : $\quad b_0 u(b) + b_1 u'(b) = \gamma_2$

$$\therefore \quad b_0 u_N + b_1\left(\frac{u_{N+1} - u_{N-1}}{2h}\right) = \gamma_2$$

$$2hb_0 u_N + b_1(u_{N+1} - u_{N-1}) = 2h\gamma_2$$

$$-b_1 u_{N-1} + 2hb_0 u_N + b_1 u_{N+1} = 2h\gamma_2$$

$$-\left(\frac{u_{-1} - 2u_0 + u_1}{h^2}\right) + b_0\left(\frac{u_1 - u_{-1}}{2h}\right) + 2_0 u_0 = \gamma_0$$

$$\boxed{Tx = b}$$

$$u'' = u + x \quad ; \qquad u(0) = 0$$
$$u(1) = 0$$
$$h = 0.25$$

Step-I

$u_0 = 0$, $u_4 = 0$

$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $1 = x_4$

$j = 1, 2, 3$

Step-II

$$\left. u'' = u + x \right|_{x_j}$$

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} - u_j = x_j$$

$$u_{j-1} - 2u_j + u_{j+1} - h^2 u_j = h^2 x_j$$

$$u_{j-1} - (h^2+2)u_j + u_{j+1} = h^2 x_j$$

$j = 1, 2, 3$

$j=1$

$$u_0 - 2.0625 u_1 + u_2 = \frac{1}{16}\cdot\frac{1}{4} = \frac{1}{64}$$

$j=2$

$$u_1 - 2.0625 u_2 + u_3 = \frac{1}{16}\cdot\frac{2}{4} = \frac{1}{32}$$

$j=3$

$$u_2 - 2.0625 u_3 + u_4 = \frac{1}{16}\cdot\frac{3}{4} = \frac{3}{64}$$

$$\begin{pmatrix} -\left(2+\frac{1}{16}\right) & 1 & 0 \\ 1 & -\left(2+\frac{1}{16}\right) & 1 \\ 0 & 1 & -\left(2+\frac{1}{16}\right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{64} \\ \frac{1}{32} \\ 3/64 \end{pmatrix}$$

→ If the size of this matrix is large then it would be called sparse matrix. {having large no. of zeroes}.

$h = 0.1$

Exercise: Solve it numerically.

$u_1 = -0.034885$

$u_2 = -0.056326$

$u_3 = -0.050037$

Exercise:

$$\left.\begin{aligned} u'' &= x u; \\ u(0) + u'(0) &= 1 \\ u(1) &= 1 \end{aligned}\right\} \quad h = 1/3$$

Use Central difference approximation. Obtain the 2nd order solution of the derivation.

Soln:

$$u_{j-1} - 2u_j + u_{j+1} = h^2 u_j x_j$$

$j=0$ B.C. 1:

$j = 1, 2$

$u_3 = 1$

0	$\frac{1}{3}$	$2/3$	1
x_0	x_1	x_2	x_3

$$u_0 + \left(\frac{u_1 - u_{-1}}{2h}\right) = 1$$

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At $j=0$ the difference equation

$$u_{-1} - 2u_0 + u_1 = h^2 x_0 u_0$$

$$\boxed{u_{-1} = 2u_0 - u_1}$$
$$= 2hu_0 - 2h + u_1$$

$-2u_0 + 3u_1 = 1$

$u_0 - \frac{55}{27}u_1 + u_2 = 0$ $\qquad$ $u_1 - \frac{55}{27}u_2 = 1$

$u_0 = -0.98795$ $\qquad$ $u_1 = -0.3253012$

$u_2 = 0.32503012$

2 point B.V.P. (Linear o.d.e)

$$\boxed{-y'' + xy = 0}$$

$h = \frac{1}{4}$ $\qquad$ $u(0) = ?$ $\qquad$ $u(01) = -1$

$0 \quad \frac{1}{4} \quad \frac{2}{4} \quad \frac{3}{4} \quad 1$

Central diff.

$$-\left(\frac{y_{j-1} - 2y_j + y_{j+1}}{h^2}\right) + x_j \cdot y_j = 0$$

$j = 1, 2, 3$

$$-y_{j-1} + (2 + x_j h^2)y_j - y_{j+1} = 0$$

$j=1$: $\quad -y_0 + \left(2 + \frac{1}{4} \cdot \frac{1}{16}\right)y_1 - y_2 = 0$

$j=2$: $\quad -y_1 + \left(2 + \frac{1}{2} \cdot \frac{1}{16}\right)y_2 - y_3 = 0$

$j=3$: $\quad -y_2 + \left(2 + \frac{3}{4} \cdot \frac{1}{16}\right)y_3 - y_4 = 0$

$j=4$: $\quad y_4 = -1$

$j=0$: $\quad$ B.C. 1 $\quad y' = 1$ at $x = 0$

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$$\frac{y_1 - y_{-1}}{2h} = 1 \Rightarrow \boxed{y_{-1} = y_1 - 2h}$$

For the difference equation to be valid at $j=0$

$$-y_{-1} + 2y_0 - y_1 = 0$$

$$\Rightarrow \boxed{2y_0 - y_1 = y_{-1}}$$

$$2y_0 - 2h = 2y_0 - y_1$$

$$2y_1 - 2y_0 = 2h \Rightarrow \boxed{y_1 - y_0 = h}$$

$j=0$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & \left(2+\frac{1}{x_1}\right) & -1 & 0 \\ 0 & -1 & \left(2+\frac{1}{x_2}\right) & -1 \\ 0 & 0 & -1 & \left(2+\frac{1}{x_3}\right) \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} h \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Thomas Algorithm $\boxed{T_{n \times n} x = b}$

$$|T| \neq 0 \quad ; \quad T = UL$$

$$Tx = b \; ; \; (UL)x = b \qquad \text{Use } Lx = z$$

$$Uz = b$$

$$Lx = z$$

$$T = ULx$$

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$$

$i = 1, 2, \ldots n$

where $a_1 = 0$, $c_n = 0$

$$\begin{bmatrix} b_1 & c_1 & & \\ a_2 & b_2 & c_2 & \\ & a_3 & b_3 & c_3 \\ & & \ddots & \\ & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$T = LU$

$$L = \begin{bmatrix} l_{11} & & & \\ \beta_2 & l_{22} & & \\ & \beta_3 & \ddots & \\ & & \ddots & \\ & & \beta_n & l_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & \alpha_1 & & \\ & 1 & \alpha_2 & \\ & & 1 & \ddots \\ & & & 1 & \alpha_{n-1} \\ & & & & 1 \end{bmatrix}$$

$|T| = |L||U|$ Find l_{ii}, $\alpha_1, \ldots, \alpha_{n-1}$, $\beta_2, \ldots, \beta_n$

$\boxed{Tx = b}$

$LUx = b \Rightarrow Lz = b$

$\underline{Ax = b}$ $|A| \neq 0$

$\boxed{x = A^{-1}b}$ $Ax = b \rightarrow$ Iteratively

$\underline{x}^{(n+1)} = \underline{H x}^{(n)} + d$

$\hookrightarrow$ Iterative Matrix $x^{(0)}$ should be $\downarrow$ guessed properly will not always converge to a solution.

$|\underline{x}^{(n+1)} - \underline{x}^{(n)}| < \underline{\epsilon}$

eigen values of $|H|$

spectral radius of $(H) < 1$ then the solution will converge.

Non-linear Boundry Value problem:

$u'' = f(x, u, u')$

$u(a) = \gamma_1$ $u(b) = \gamma_2$

In matrix form, this system of equation,

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

Comparing the entries of matrix A, we have

$$\omega_1 = b_1 \qquad \omega_1 \alpha_1 = c_1 \qquad \Rightarrow \qquad \alpha_1 = \frac{c_1}{\omega_1}$$

$$\beta_i = a_i \quad , \quad i = 1, 2, 3, \dots n$$

We have

$$Ax = d$$

Letting $\quad Ux = z \quad$ and $\quad Lz = d$

$$\begin{bmatrix} \omega_1 & & & \\ \beta_2 & \omega_2 & & \\ & \ddots & \ddots & \\ & & \ddots & \ddots \end{bmatrix}$$

Now,

$$Ux = z$$

$$\begin{bmatrix} 1 & \alpha_1 & & \\ & 1 & \alpha_2 & \\ & & 1 & \alpha_3 \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

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Non-linear 2 point Boundry Value Problem:

$$u'' = f(x, u, u') \qquad a < x < b$$

such that; $a_0 u(a) - a_1 u'(a) = r_1$

$b_0 u(b) + b_1 u'(b) = r_2$

→ Finite difference formulation leads to

$\underline{F}(\underline{X}) = 0$ where

$$\underline{F} = [f_1 \ f_2 \ \dots \ f_n]^T$$

$$f(x) = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots$$

Given: x_0 $\qquad \{x_n\}_{n=1}^{\infty}$

$x_n \longrightarrow x$

Let: $x_{n+1} = \phi(x_n)$

$|\phi'| < 1 \quad \forall \ x$ in which the root lies

Newton's Method:

$$\underline{X}^{(n+1)} = \underline{X}^{(n)} - \frac{F(\underline{X}^{(n)})}{J[\underline{F}(\underline{X}^{(n)})]}, \quad n = 0, 1, 2, \dots$$

$\underline{X}^{(0)}$ is given

$$|\underline{X}^{(s+1)} - \underline{X}^{(s)}| < G$$

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We have $f_1(u_1, u_2) = 0$ & $f_2(u_1, u_2) = 0$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}_{\underline{x} = \underline{x}^{(n)}}$$

$$\underline{x}^{(n+1)} = H\underline{x}^{(n)} + \underline{d}$$

Example:

$$u'' = \frac{3}{2}u^2$$

$u(0) = 4$, $u(1) = 1$, $h = \frac{1}{3}$

$u_0 = 4$ $\qquad u_3 = 1$

$0 \qquad \frac{1}{3} \qquad \frac{2}{3} \qquad 1$

$$(u_{j-1} - 2u_j + u_{j+1}) = \frac{3}{2}h^2u_j^2$$

$j = 1, 2, \ldots$

$j=1$: $\quad u_0 - 2u_1 + u_2 = \frac{3}{2}\left(\frac{1}{3}\right)^2 \cdot u_1^2$

$$4 - 2u_1 + u_2 = \frac{1}{6}u_1^2$$

or $f_1(u_1, u_2) = u_1^2 + 12u_1 - 6u_2 - 24 = 0$ — (i)

$j=2$: $\quad u_1 - 2u_2 + u_3 = \frac{3}{2} \cdot \frac{1}{9}u_2^2$

$$u_1 - 2u_2 + 1 = \cancel{\tfrac{3}{2}}\frac{1}{6}u_2^2$$

or $f_2(u_1, u_2) u_2^2 - 6u_1 + 12u_2 - 6 = 0$ — (ii)

$$\begin{pmatrix} u_1^{(s+1)} \\ u_2^{(s+1)} \end{pmatrix} = \begin{pmatrix} u_1^{(s)} \\ u_2^{(s)} \end{pmatrix} - \begin{bmatrix} 2u_1+12 & -6 \\ -6 & 2u_2+12 \end{bmatrix}^{-1}_{(u_1^{(s)},\, u_2^{(s)})} \begin{bmatrix} f_1(u_1^{(s)}, u_2^{(s)}) \\ f_2(u_1^{(s)}, u_2^{(s)}) \end{bmatrix}$$

$u_1^{(0)} = 2$, $u_2^{(0)} = 1.5$ $\longrightarrow$ guess

1st $u_1^{(1)} = 2.30147$, $u_2^{(1)} = 1.470588$

2nd $u_1^{(2)} = 2.29504$, $u_2^{(2)} = 1.467949$

3rd $u_1^{(3)} = 2.2950399$, $u_2^{(3)} = 1.4679474$

Solve:

$$u'' = \frac{1}{2}(1 + x + u)^3$$

$$u'(0) - u(0) = -\tfrac{1}{2} \qquad h = \tfrac{1}{2}$$

$$u'(1) + u(1) = 1$$

— Use N-R method to solve the resulting algebraic system of equations.

$u_0^{(0)} = 0.001$ $\quad u_1^{(0)} = -0.1$ $\quad u_2^{(0)} = 0.001$

$$u_{j-1} - 2u_j + u_{j+1} = \frac{h^2}{2}(1 + x_j + u_j)^3$$

$0 \quad \tfrac{1}{2} \quad 1$

Also Solve using Iterative Process!!

$$\begin{cases} u_1^2 + 12u_1 - 6u_2 - 24 = 0 & \text{— (1)} \\ u_2^2 - 6u_1 + 12u_2 - 6 = 0 & \text{— (2)} \end{cases}$$

Iterative

$$X^{(n+1)} = HX^{(n)} + b$$

$$12u_1 - 6u_2 = -u_1^2 + 24$$

$$-6u_1 + 12u_2 = -u_2^2 + 6$$

$$\begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1^2 + 24 \\ -u_2^2 + 6 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{(s+1)} = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix}^{-1} \begin{bmatrix} -u_1^{2(s)} + 24 \\ -u_2^{2(s)} + 6 \end{bmatrix}$$

$$u_1^{(0)} = 1.5, \quad u_2^{(0)} = 2$$

$$\underline{F}(\underline{u}) = 0$$

$$\underline{u}^{(s+1)} = \underline{P}\,\underline{f}(u^{(s)}) + \underline{q}$$

Shooting Method: (Initial Value Problem Solver)

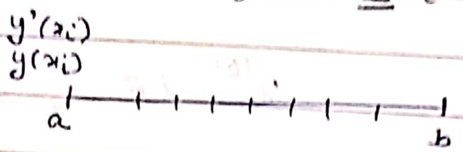
Linear BVP:

$$\left.\begin{aligned} -y'' + p(x)y' + q(x)y &= r(x) \\ \text{such that: } \quad a_0 y(a) - a_1 y'(a) &= \gamma_1 \\ b_0 y(b) + b_1 y'(b) &= \gamma_2 \end{aligned}\right\}, \quad a < x < b \qquad \text{—— } (1)$$

Ex: $y(a) = \gamma_1$
$y(b) = \gamma_2$

IVP

$$-y'' + py' + qy = r$$
$$y(a) = \gamma_1$$
$$y'(a) = \underline{s} \text{ (assume)}$$



Shooting Method:

General 2 point BVP: (linear)

$$-y'' + p(x)y' + q(x)y = r(x), \qquad a < x < b$$

s.t. :

$$a_0 y(a) - a_1 y'(a) = \gamma_1$$
$$b_0 y(b) + b_1 y(b) = \gamma_2$$

$$y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{y_{C.F.}}, + \underbrace{y_3(x)}_{y_{PI}}$$

Soln. of the homogenous equation

3 IVPs :

Approximate solution for $y(x)$ as

$$y(x) = A\phi_0(x) + B\phi_1(x) + \phi_2(x)$$

1^{st} IVP:

$$-\phi_0'' + P(x)\phi_0' + q(x)\phi_0 = 0$$
$$\phi_0(0) = 1 \qquad \phi_0'(0) = 0$$

2^{nd} IVP:

$$-\phi_1'' + P(x)\phi_1' + q(x)\phi_1 = 0$$
$$\phi_1(0) = 0 \qquad \phi_1'(0) = 1.$$

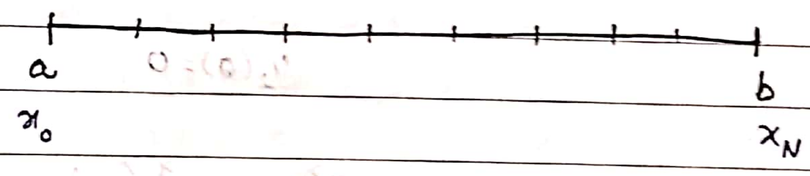
3^{rd} IVP:

$$-\phi_2'' + P(x)\phi_2' + q(x)\phi_2 = r(x)$$
$$\phi_2(0) = 0 \qquad \phi_2'(0) = 0$$

$\phi_2 \quad \phi_2'$

$\phi_1 \quad \phi_1'$

$\phi_0 \quad \phi_0'$



$\phi_i(x_j)$

$\phi_i'(x_j)$

~~B.C.(1)~~ ~~e.g.~~

$$y = A\phi_0 + B\phi_1 + \phi_2$$
$$y(x_i) = A\phi_0(x_i) + B\phi_1(x_i) + \phi_2(x_i)$$

B.C.1: $a\left[A\phi_0(0) + B\phi_1(0) + \phi_2(0)\right] - a_1\left[A\phi_0'(0) + B\phi_1'(0) + \phi_2'(0)\right] = r_1$

B.C.2:

$$b_0 y(b) + b_1 y'(b) = r_2$$

Solve:

$$u'' = u - 4xe^{x}$$

$$-u'(0) + u(0) = 1 \qquad 0 < x < 1$$
$$u(1) + u'(1) = e$$

$a_0 = 1 \quad a_1 = 1 \quad b_0 = 1 \quad b_1 = 1 \quad \gamma_1 = 1 \quad \gamma_2 = e$

IVP1:

$$u(x) = Au_0(x) + Bu_1(x) + u_2(x)$$

$$u_0'' - u_0 = 0 \qquad u_0(0) = 1 \qquad u_0'(0) = 0$$

IVP2:

$$u_1'' - u_1 = 0$$
$$u_1(0) = 0 \qquad u_1'(0) = 1$$

IVP3:

$$u_2'' - u_2 = -4xe^{x}$$

$$u_2(0) = 0 \qquad u_0'(0) = 0$$

$$u_2(x) = \frac{1}{2}e^{-x} - e^{x}\left(x^2 - x + \tfrac{1}{2}\right)$$

$$u_0(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$u_1(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

1st B.C. : $A - B = -1$

2nd B.C. : $Ae + Be = e$

$\Rightarrow A + B = 1$

$\Rightarrow A = 0, \quad B = 1$

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$u(x) = u_1(x) + u_2(x)$

$\therefore \quad u(x) = \frac{1}{2}e^{-x} - e^{x}\left(x^2 - x + \frac{1}{2}\right) + \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$u(x) = x(1-x)e^{x}$

Alternate Method:

When the BVP is non-homogenous, then it is sufficient to solve two IVP.

$$-u_1'' + p(x)u_1' + q(x)u_1 = r(x) \quad \text{—— (1)}$$

$$-u_2'' + p(x)u_2' + q(x)u_2 = r(x) \quad \text{—— (2)}$$

With suitable I.C. at $x = a$

General solution is

$$u(x) = \lambda u_1(x) + (1-\lambda) u_2(x)$$

BCs:

Type 1:

$u_1(a) = \gamma_1 \quad ; \quad u_1'(0) = 0$

$u_2(a) = \gamma_1 \quad ; \quad u_2'(a) = 1$

$u(b) = \gamma_2$ given

After solving the IVPs: $u_2(b) = \gamma_2 = \lambda u_1(b) + (1-\lambda) u_2(b)$

$\Rightarrow \quad \lambda = (\gamma_2 - u_2(b)) / (u_1(b) - u_2(b))$

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Type 2 BC:

$$u'(0) = \gamma_1$$
$$u'(b) = \gamma_2$$

I.Cs for the IVP 1: $u_1(0) = 0$
$u_2'(0) = \gamma_1$

I.Cs for the IVP 2: $u_2(0) = 1$
$u_2'(0) = \gamma_1$

$$\lambda = \frac{\gamma_2 - u_2'(b)}{u_1'(b) - u_2'(b)}$$

Type 3: BCs:

$$\left.\begin{aligned} a_0 u(0) - a_1 u'(0) &= \gamma_1 \\ b_0 u(b) + b_1 u'(b) &= \gamma_2 \end{aligned}\right\}$$

Take $u_1(a) = 0$, $u_1'(0) = -\gamma_1 / a_1$

$u_2(0) = 1$, $u_2'(a) = \dfrac{a_0 - \gamma_1}{a_1}$

$$\lambda = \frac{\gamma_2 - [b_0 u_2(b) + b_1 u_2'(b)]}{[b_0 u_1(b) + b_1 u_1'(b)] - [b_0 u(b) + b_1 u_2'(b)]}$$

Ex:

$$u'' = u - 4xe^{x}, \quad 0 < x < 1$$

$$u(0) - u'(0) = -1 \qquad h = \frac{1}{4}$$
$$u(1) + u'(1) = -e$$

Solve the resulting IVPs using the 4^{th} order R-K method.

IVP1:

$$u_1'' - u_1 = -4xe^{x}$$
$$u_1(0) = 0 \Rightarrow u_1'(0) = -\frac{\gamma_1}{a_1} = 1$$

IVP2:

$$u_2'' - u_2 = -4xe^{x}$$
$$u_2(0) = 1, \Rightarrow u_2'(0) = \frac{a_0 - \gamma_1}{a_1} = 2$$

Using 4^{th} order R-K method:

$$\lambda = 1.05228$$

$$u(x) = \lambda u_1(x_i) + (1-\lambda) u_2(x_i)$$
$$i = 0, 1, 2, 3, 4$$

$u(0.25) = 0.17506$
$u(0.5) = 0.33568$
$u(0.75) = 0.31527$
$u(1.0) = -0.07609$

$$u(x) = x(1-x)e^{x}$$

Non-Linear 2-point B.V.P. :

$u'' = f(x, u, u')$, using the shooting method

$a < x < b$

Type (i)

B.C.s: $u(a) = \gamma_1$; $u(b) = \gamma_2$

Convert this to an I.V.P first

IVP:

$$u'' = f(x, u, u')$$

$$u(x = a) = \gamma_1 \quad \text{given}$$

$$u'(x = a) = s \quad \text{(guess)}$$

$x_0 = a$, x_1, x_2, x_3, ... $x_N = b$

Use the 4^{th} order R-K Method, Reduce this to a system of 1^{st} order equations

Construct

$$u_c(x = b) - u(b)$$

$$\phi(s) = u(b, s) - \gamma_2$$

Aim: $\phi(s) \longrightarrow 0$

Rather we find root of $\phi(s) = 0$,

use (i) N-R Method or (ii) Secant Method

such that $s^k \ni \phi(s^k) = 0$

Secant Method:

$$s^{(k+1)} = s^{(k)} - \left[\frac{s^{(k)} - s^{(k-1)}}{\phi(s^{k}) - \phi(s^{k-1})}\right]\phi(s^{k})$$

B.V.P.

$u'' = f(x, u, u')$; $\underline{\underline{u(0) = \gamma_1}}$, $u'(0) = \gamma_2$

I.V.P.   I.V.P

$u'' = f(x, u, u')$
$u(0) = \gamma_1$
$u'(0) = s^{0}$

$u'' = f(x, u, u')$
$u(a) = \gamma_1$
$u'(0) = s^{1}$

$u_c(b)$ $u_c(b)$

$\phi(s^{0}) = u_c(b, s^{0}) - \gamma_2$ $\phi(s^{1}) = u_c(b, s^{1}) - \gamma_2$

Obtain $s^{(2)}$ check $\phi(s^{(2)}) \to 0$?

Newton-Raphson Method:

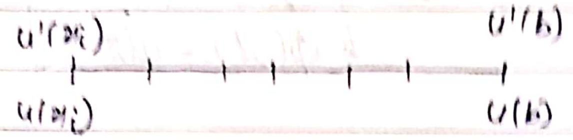
$$\phi(s) = 0$$

$$s^{(k+1)} = s^{(k)} - \frac{\phi(s^{(k)})}{\phi'(s^{(k)})} \qquad k = 0, 1, 2, \ldots$$

Boundry Conditions of 2^{nd} kind and $\phi(s)$

B.V.P. : $u'' = f(x, u, u')$
$u'(a) = \gamma_1$, $u'(b) = \gamma_2$

IVP: $u'' = f(x, u, u')$, $u(a) = s$
$u'(a) = \gamma_1$



$$\boxed{\phi(s) = u'(b, s) - \gamma_2}$$

B.Cs of third kind:

$$a_0 u(a) - a_1 u'(a) = \gamma_1$$
$$b_0 u(b) + b_1 u'(b) = \gamma_2$$

Assume either $u(a)$ or $u'(a)$

Let $u'(a) = s$, then $u(a) = (\gamma_1 + a_1 s)/a_0$

or $u(a) = s \Rightarrow u'(a) = \left(\dfrac{a_0 s - \gamma_1}{a_1}\right)$

$$\phi(s) = b_0 u(b, s) + b_1 u'(b, s) - \gamma_2 = 0$$

Finding $\phi'(s^k)$ (Newton-Raphson Method)

$$u'' = f(x, u, u')$$

such that $\left.\begin{aligned} a_0 u(a) - a_1 u'(a) &= \gamma_1 \\ b_0 u(b) + b_1 u'(b) &= \gamma_2 \end{aligned}\right\}$

$$\phi(s) = b_0 u(b, s) + b_1 u'(b, s) - \gamma_2 = 0$$

Denote $u_s = u(x, s)$

$$u_s' = u'(x, s)$$
$$u_s'' = u''(x, s)$$

Then the IVP : $u_0'' = f(x, u_s, u_s')$ — (1)

$$\left.\begin{aligned} u_s(0) &= \frac{1}{a_0}(a_1 s + r_1) \\ u_s'(a) &= s \end{aligned}\right\} \quad \text{— (2)}$$

Diff (1) w.r.t 's'.

$$\frac{\partial}{\partial s}(u_s'') = \frac{\partial f}{\partial x} \cdot \cancel{\frac{\partial x}{\partial s}}^{\,0} + \frac{\partial f}{\partial u_s} \cdot \frac{\partial u_s}{\partial s} + \frac{\partial f}{\partial u_s'} \cdot \frac{\partial u_s'}{\partial s} \quad \text{— (3)}$$

Diff. (ii) w.r.t 's'.

$$\left.\begin{aligned} \frac{\partial}{\partial s} u_s(a) &= \frac{1}{a_0} \cdot a_1 = \frac{a_1}{a_0} \\ \frac{\partial}{\partial s} u_s'(0) &= 1 \end{aligned}\right\} \quad \text{— (4)}$$

Let $v = \frac{\partial u_s}{\partial s}$, then diff. wrt 'x'.

$$v' = \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u_s}{\partial s}\right) = \frac{\partial}{\partial s} u_s'$$

Likewise $v'' = \frac{\partial}{\partial x} v' = \frac{\partial}{\partial s}(u_s'')$

from (3) & (4).

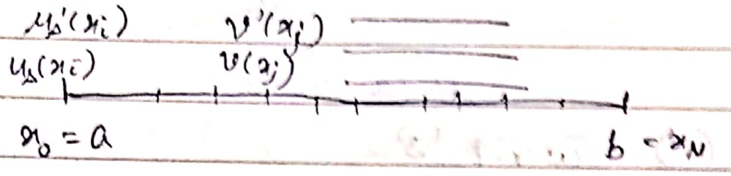
$$v'' = \frac{\partial f}{\partial u_s} \cdot v + \frac{\partial f}{\partial u_s'} \cdot v' \quad \text{and} \quad \text{— (5)}$$

$$v(0) = \frac{a_1}{a_0} \quad ; \quad v(0) = 1 \quad \Big\} \rightarrow (5,6)$$

Eqs (5,6) is called the first variational equation

Eqn (1,2)
Eqn (5,6)

The IVP (5,6) is solved step by step along with (1,2)



Compute $v(b)$ and $v'(b)$

$$\phi(s) = b_0 u_s(b, s) + b_1 u_s'(b, s) - \gamma_2$$

$$\frac{d\phi}{ds} = b_0 v + b_1 v'(b)$$

$$s^{k+1} = s^k - \frac{\phi(s^k)}{\phi'(s^k)} \longrightarrow \{s^k\}$$

If we have the first kind boundary conditions:
$a_0 = 1$, $a_1 = 0$, $b_0 = 1$, $b_1 = 0$

and $\phi(s) = u_s(b) - \gamma_2$

we have $v(a) = \frac{0}{1} = 0$

$$v'(a) = 1$$

$$v'' = f_u \cdot v + f_{u'} \cdot v'$$

$$\frac{d\phi}{ds} = v(b)$$

If we have 2^{nd} kind Boundary Condition:

$$v(a) = 0 \qquad v'(0) = 1$$
$$v'' = f_u \cdot v + f_{u'} \cdot v'$$

$$\phi(s) = u_s'(b) - \gamma_2$$
$$\frac{d\phi}{ds} = v'(b)$$

Problem:

Use shooting method to solve the BVP

$$u'' = 2uu', \qquad 0 < x < 1$$
$$u(0) = 0.5 \qquad u(1) = 1 \qquad h = 0.25$$

Use the 3^{rd} order Taylor Series method:

$$u_{j+1} = u_j + hu_j' + \frac{h^2}{\lfloor 2} u_j'' + \frac{h^3}{\lfloor 3} u_j''' \quad \text{to}$$

solve the corresponding IVP and the Secant Method for the iteration. Iterate until the error is less than 0.005. Compare your solution with the exact solution $u(x) = \frac{1}{2-x}$.

choose $s^{(0)} = 0.5$ and $s^{(1)} = 0.1$

Solution:

u_0'	u_1'	u_2'	u_3'	u_4'
u_0	u_1	u_2	u_3	u_4
$x_0 = 0$	$x_1 = \frac{1}{4}$	$x_2 = \frac{1}{2}$	$x_3 = \frac{3}{4}$	1

IVP1

$$u'' = 2uu'$$
$$u(0) = 0.5$$
$$u'(0) = s^0 = 0.5$$

IVP2

$$u'' = 2uu'$$
$$u(0) = 0.5$$
$$u'(0) = s^{(1)} = 0.1$$

$$u_{j+1} = u_j + hu_j' + \frac{h^2}{\lfloor 2} u_j'' + \frac{h^3}{\lfloor 3} u_j'''$$

$$u_j''' = 2[u_j u_j'' + u_j'^2]$$

$$u_{j+1}' = u_j' + h u_j'' + \frac{h^2}{\underline{|2}} u_j''' + \frac{h^3}{\underline{|3}} u_j''''$$

$u(0.25) = u_1 = 0.64323$	$u_1 = 0.52844 \quad u_1' = 0.12875$
$u'(0.25) = u_1' = 0.65625$	
	$u_2 = 0.56534 \quad u_2' = 0.16830$
$u(0.5) = u_2 = 0.83875$	
$u'(0.5) = u_2' = 0.92817$	$u_3 = 0.61407 \quad u_3' = 0.22437$
$u(0.75) = u_3 = 1.13074$	$u_4 = 0.67991 \quad u_4' = 0.30698$
$u'(0.75) = u_3' = 1.45289$	
$u(1) = u_4 = 1.62699$	
$u'(1) = u_4' = 2.63844$	

$s^0 = 0.5$, $\quad \phi(s^0) = u(1, s^0) - 1.0 = 0.62699$

$s^1 = 1$ $\quad \phi(s^{(1)}) = u(1, s^{(1)}) - 1 = -0.32089$

$$s^{(2)} = s^{(1)} - \left[\frac{s^{(1)} - s^{(0)}}{\phi(s^1) - \phi(s^0)}\right] \phi(s^{(1)})$$

$$s^{(2)} = 0.23519$$

$u'' = 2uu'$, $\quad u(0) = 0.5 \quad u'(0) = s^{(2)} = 0.23519$

Solving this IVP,

$u_4 = 0.9546$

$u_4' = 0.8639$

$\phi(s^{(1)}) = -0.04536$

$s^{(2)} = 0.25751$

$u_4 = 1.00394$

$\phi(s^{(2)}) = 0.00394 < \epsilon$

Let us again consider the same problem, but here use 4th Order R-K Method to solve the IVP and Newton - Raphson for the iteration.

$$u'' = 2uu' \quad , \quad 0 < x < 1$$
$$u(0) = 0.5 \quad , \quad u(1) = 1.0$$

u_0	u_1	u_2	u_3	u_4
0	1/4	1/2	3/4	1

$u = p$

$u' = p' = q$

$u'' = p' = 2qr$

$p(x_i), \ q(x_i)$

1st variational equation,

$v'' = f_u \cdot v + f_{u'} \cdot v'$

$v'' = 2u' \cdot v + 2u \cdot v'$

$$\begin{bmatrix} r \\ s \end{bmatrix}' = \begin{bmatrix} s \\ 2qr + 2ps \end{bmatrix}$$

$r(x_i) \quad s(x_i)$

$\phi(1, s^0) = |u(1, s^0) - \gamma_2| < \epsilon$

$\phi'(s^0) = v(b) = v(1)$

Let $s^{(0)} = 0.5$ then $s^{(1)} = s^{(0)} - \dfrac{\phi(s^0)}{\phi'(s^0)}$

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We have for:

$n=0: \quad -y_{-1} + 2y_0 - y_1 = -1/16$

$n=1: \quad -y_0 + \frac{129}{69} y_1 - y_2 = -1/16$

$n=2: \quad -y_1 + \frac{65}{32} y_2 - y_3 = -1/16$

$n=3: \quad -y_2 + \frac{131}{64} y_3 - y_4 = -1/16$

$y_{-1} = -\frac{1}{2} y_0 + y_1 - \frac{1}{2} \qquad y_n = 1$

$y_4 = 1$

$$\begin{bmatrix} 3/2 & -2 & 0 & 0 \\ -1 & 129/69 & -1 & 0 \\ 0 & -1 & 65/32 & -1 \\ 0 & 0 & -1 & 131/64 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = -\frac{1}{16} \begin{bmatrix} 9 \\ 1 \\ 1 \\ -15 \end{bmatrix}$$

$y_0 = -19.8523 \qquad y_1 = -14.6050$

$y_2 = -8.5294 \qquad y_3 = -4.6361$

Q. Find the solution of the boundary value problem:

$$x^2 y'' - 2y + x = 0$$

$$y(2) = y(3) = 0$$

Using the shooting method. Use the 4^{th} order Runge-Kutta method with $h = 0.25$ to solve the initial value problem. Compare with the exact solution.

$$y(x) = \frac{19x^2 - 5x^3 - 36}{38} x$$

$$y(x) = \phi_0(x) + \mu_1 \phi_1(x)$$

μ_1 is a parameter to be determined:

I: $$x^2\phi_0'' - 2\phi_0 + x = 0$$

$$\phi_0(2) = 0 \qquad \phi_0'(2) = 0$$

II: $$x^2\phi_1'' - 2\phi_1 = 0$$
$$\phi_1(2) = 0 \quad , \quad \phi_1''(2) = 1$$

The second boundary conditions gives:
$$y(3) = \phi_0(3) + \mu_1 \phi_1(3) = 0$$
$$\Rightarrow \mu_1 = \frac{-\phi_0(3)}{\phi_1(3)}$$

The equivalent system of first order IVPs are:

I: $$\phi_0 = w^{(1)}$$
$$\phi_0' = w^{(1)'} = v^{(1)}$$

$$\therefore \phi_0'' = v^{(1)'} = \frac{2w^{(1)}}{x^2} - \frac{1}{x}$$

$$\begin{bmatrix} w^{(1)} \\ v^{(1)} \end{bmatrix}' = \begin{bmatrix} v^{(1)} \\ \frac{2}{x^2} w^{(1)} - \frac{1}{x} \end{bmatrix}$$

I. Cs.
$$\begin{bmatrix} w^{(1)}(2) \\ v^{(1)}(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

II: $$\phi_1(x) = w^{(2)}$$
$$\phi_1''(x) = v^{(2)}$$

$$\therefore \begin{bmatrix} W^{(2)} \\ V^{(2)} \end{bmatrix}' = \begin{bmatrix} V^{(2)} \\ \frac{2}{x^2} W^{(2)} \end{bmatrix}$$

ICs

$$\begin{bmatrix} W^{(2)}(2) \\ V^{(2)}(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Applying the 4^{th} order Runge-Kutta method with $h = 0.25$ in I we get

$n=0$: $$\begin{bmatrix} W_1^{(1)} \\ V_1^{(1)} \end{bmatrix} = \begin{bmatrix} -1.5648 \times 10^{-2} \\ -0.1183 \end{bmatrix}$$

$n=1$: $$\begin{bmatrix} W_2^{(1)} \\ V_2^{(1)} \end{bmatrix} = \begin{bmatrix} -5.8336 \times 10^{-2} \\ -0.2266 \end{bmatrix}$$

$n=2$: $$\begin{bmatrix} W_3^{(1)} \\ V_3^{(1)} \end{bmatrix} = \begin{bmatrix} -0.1272 \\ -0.3185 \end{bmatrix}$$

$n=3$: $$\begin{bmatrix} W_4^{(1)} \\ V_4^{(1)} \end{bmatrix} = \begin{bmatrix} -0.2222 \\ -0.4259 \end{bmatrix}$$

Similarly, from II we get

$n=0$: $$\begin{bmatrix} W_1^{(2)} \\ V_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0.2511 \\ 1.0133 \end{bmatrix}$$

$n=1$: $$\begin{bmatrix} W_2^{(2)} \\ V_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0.5083 \\ 1.0466 \end{bmatrix}$$

$n=2:$ $\begin{bmatrix} W_3^{(2)} \\ V_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0.7755 \\ 1.0929 \end{bmatrix}$

$n=3:$ $\begin{bmatrix} W_4^{(2)} \\ V_4^{(2)} \end{bmatrix} = \begin{bmatrix} 1.0555 \\ 1.1481 \end{bmatrix}$

$$\mu = \frac{-W_4^{(1)}}{W_4^{(2)}} = 0.21057$$

The solution values are given by

x_n	y_n	$y(x_n)$
x_1	0.037838	0.0378289
x_2	0.0487040	0.0416842
x_3	0.0354676	0.0354366
x_4	0	0

$$y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0''$$

Q Use the shooting method to find the solution of the BVP. $y'' = 6y^2$ $y(0)=1$, $y(0.5)= 4/9$

Assume the initial approximation:

$y'(0) = \alpha_0 = -1.8$

$y'(0) = \alpha_1 = -1.9$

and find the solution of the IVP using the 4^{th} order R-K Method with $h=0.1$. Improve the value of $y'(0)$ using the secant method.

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IVPs :

I: $y'' = 6y^2$

$y(0) = 1$, $y'(0) = -1.8$

II: $y'' = 6y^2$

$y(0) = 1$, $y'(0) = -1.9$

and obtain the solution values at $x = 0.5$

$$g(\alpha_0) = y(\alpha_0 : b) - 4/9$$

$$g(\alpha_1) = y(\alpha_1 : b) - 4/9$$

The secant method gives:

$$\alpha_{n+1} = \alpha_n - \left[\frac{\alpha_n - \alpha_{n-1}}{g(\alpha_n) - g(\alpha_{n-1})}\right] g(\alpha_n)$$

$n = 1, 2, 3, \ldots$

x	$y(0)=1$, $\alpha_0 = -1.8$	$y(0)=1$, $\alpha_1 = -1.9$	$y(0)=1$, $\alpha_{sc} = -1.998853$	$y(0)=1$, $y'(0) = -2$
0.1	0.8468373	0.8366594	0.8265893	0.8264724
0.2	0.7372285	0.7158495	0.6947327	0.6944873
0.3	0.6605514	0.6261161	0.5921678	0.5917743
0.4	0.6162643	0.5601089	0.5108485	0.5102787
0.5	0.5824725	0.5130609	0.4453193	0.444[illegible]5383

Numerical Solutions of Partial Differential Equations

2^{nd} order p.d.e. of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y) \quad \text{— (1)}$$

where A, B, C are functions of x, y, u, u_x & u_y then the eq.(i) is called a quasi-linear P.d.e.

$$u = u(x,y)$$

$$x \in \mathbb{R} \qquad y \in \mathbb{R}$$

$$x \in [a,b] \qquad y \in [c,d]$$

$$u(x,y) = c$$

$$g(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$$

A linear partial differential equation may be written as

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0 \quad \text{— (2)}$$

where all $A, B, \ldots, G$ are functions of x, y

if $G \equiv 0 \quad \forall (x,y) \in \mathbb{R}^2$ or $D \subset \mathbb{R}^2$

Solution of equation (2) is called an integral surface,
$\phi(x, y, u) = 0 \quad$ — (3)

Rea.

Reduction to its canonical form (Standard)

1) Linear P.d.e.
2) Solution $\phi(x, y, u) = 0$
   └ Integral surface

Characteristic curves:
If on the integral surface, there exists curves across which the partial derivatives u_{xx}, u_{xy}, u_{yy} are discontinuous or indeterminate then these curves are called the characteristic curves.

$$\xi = \xi(x, y) \quad , \quad \eta = \eta(x, y)$$

$$u(x, y) : \longleftrightarrow u(\xi, \eta)$$

This transform should be one - one and onto.

eq (2) : $\quad u(x, y) \longrightarrow u(\xi, \eta)$

$$u_x, u_y \longrightarrow u_\xi, u_\eta$$

$$u_{xx}, u_{xy}, u_{yy} \longrightarrow u_{\xi\xi}, u_{\eta\eta}, u_{\xi\eta}$$

(2) becomes:

$$\begin{cases} u_x = (u_\xi \cdot \xi_x + u_\eta \cdot \eta_x) \\ u_y = (u_\xi \cdot \xi_y + u_\eta \cdot \eta_y) \end{cases}$$

$$(A\xi_x^2 + 2B\xi_x \cdot \xi_y + C\xi_y^2) u_{\xi\xi} + 2(A\xi_x \cdot \eta_x + B\eta_y \cdot \xi_x + B\xi_y \cdot \eta_x + C\xi_y \cdot \eta_y) u_{\xi\eta} + (A\eta_x^2 + 2B\eta_x \eta_y + C\eta_y^2) u_{\eta\eta} + \dots + \dots + \dots = 0$$

Setting the coefficients of $u_{\xi\xi}$, $u_{\xi\eta}$, $u_{\eta\eta}$ to zero and simplifying the set

$$A(\xi_x/\xi_y)^2 + 2B(\xi_x/\xi_y) + C = 0$$

$$A(\eta_x/\eta_y)^2 + 2B(\eta_x/\eta_y) + C = 0$$

or

$$A\left(\frac{dy}{dx}\right)^2 - 2B\left(\frac{dy}{dx}\right) + C = 0$$

$\xi(x,y) = C$

$\eta(x,y) = C$

$\xi_x + \xi_y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\eta_x/\eta_y$

$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$

$$\frac{dy}{dx} = \frac{2B \pm \sqrt{4B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} \Rightarrow \begin{array}{l}\xi(x,y) = C \\ \eta(x,y) = C\end{array}$$

$B^2 - AC$:
- $= 0$: Parabolic
- > 0 : Hyperbolic
- < 0 : Elliptical

$(x, y) \longleftrightarrow \xi(x,y) = c$ &

(ξ, η)

$\eta(x,y) = c'$

Canonical form (1) Parabolic eqn: $B^2 - AC = 0$

Tr. eqn: $u_{\xi\xi} = F_1(\xi, \eta, u, u_\xi, u_\eta)$

$u_{\eta\eta} = F_2(\xi, \eta, u, u_\xi, u_\eta)$

(2) Hyperbolic : $B^2 - AC > 0$

Tr. eqn : $u_{\xi\xi} - u_{\eta\eta} = F_{\xi}(\xi, \eta, u, u_{\xi}, u_{\eta})$

or $u_{\xi\eta} = F_{u}(\xi, \eta, u, u_{\xi}, u_{\eta})$ — (6)

(3) Elliptical : $B^2 - AC < 0$ (Pure BVP)

Tr. eqn : $u_{\xi\xi} + u_{\eta\eta} = F_{\eta}(\xi, \eta, u, u_{\xi}, u_{\eta})$ — (7)

Examples :

1. Parabolic :

$u = u(t, x)$

$u_t = k . u_{xx}$ $\quad t \geq 0$

$\downarrow$ conductivity $\quad x \in [a, b] \subset \mathbb{R}$

$$\frac{\partial u}{\partial t} = k . \frac{\partial^2 u}{\partial x^2}$$

$A = k, \quad B = 0, \quad C = 0$

$B^2 - AC = 0$

1 - Dimensional heat conduction equation.

2. Hyperbolic

$u = u(x, t)$

$u_{tt} = c^2 u_{xx}$ $\qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

1-Dimensional Wave propagation

$A = 1, \quad B = 0, \quad C = -c^2$

$B^2 - AC > 0$

3. Elliptical : Laplace equation

$$u = u(x, y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

$$A = 1, \quad B = 0, \quad C = 1$$

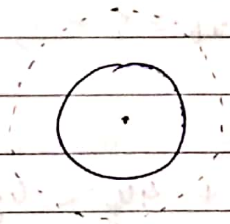
$$B^2 - AC < 0$$

Classify the p.d.e.

1. $$\frac{\partial^2 u}{\partial x^2} + 2x \cdot \frac{\partial^2 u}{\partial x \partial y} + (1-y^2)\frac{\partial^2 u}{\partial y^2} = 0$$

$$A = 1, \quad B = x, \quad C = 1 - y^2$$

$$B^2 - AC = x^2 + y^2 - 1 \quad \begin{cases} = 0 \\ > 0 \\ < 0 \end{cases}$$



$x^2+y^2=1$

$\forall$ (x, y) on this circle,
↳ Parabolic

$x^2+y^2>1$
↳ Hyperbolic

$x^2+y^2<1$
↳ Elliptic

2. $u_{tt} + 4u_{tx} + 4u_{xx} + 2u_x - u_t = 0$

$u = u(t, x)$

i) Find the characteristic curve
ii) Classify the eqn. by reducing this to its standard form

$$A = 1 \qquad B = 2 \qquad C = 4$$

$$B^2 - AC = 4 - 4 = 0 \quad \text{— Parabolic}$$

classmate

Date

Page

$$\frac{dx}{dt} = \frac{B}{A} = 2 \Rightarrow x - 2t = c$$

Take $\xi(x,t) = x - 2t = c$

$\eta(x,t) = t$

$$(x,t) \longleftrightarrow (\xi, \eta)$$

$$\left.\begin{aligned} \xi &= x - 2t = c \\ \eta &= t = c \end{aligned}\right\}$$

$u = u(x,t)$

$u = u(\xi, \eta)$

$$u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x = u_\xi \cdot 1 + u_\eta \cdot 0$$

$$\therefore u_x = u_\xi$$

$$u_t = u_\xi \cdot \xi_t + u_\eta \cdot \eta_t$$

$$= u_\xi(-2) + u_\eta \cdot (1) = -2u_\xi + u_\eta = u_t$$

$u_{xx} = ?$, $u_{xt} = ?$, $u_{tt} = ?$

The transformed eqn:

$$u_{\eta\eta} + 4u_\xi - u_\eta = 0$$

or

$$u_{\eta\eta} = u_\eta - 4u_\xi = F(\xi, \eta, u, u_\xi, u_\eta)$$

canonical form of the given equation

Ex:

$$u_{tt} + (5 + 2x^2)u_{xt} + (1+x^2)(4+x^2)u_{xx} = 0$$

$\longrightarrow$ Classify this equation

I.B.V.P. $\qquad$ E $\qquad$ B.V.P.

Parabolic $\qquad$ Hyperbolic $\qquad$ Ellibtic

P.	H.	E
$u_t = t u_{xx}$	$u_{tt} = c^2 u_{xx}$	$u_{xx} + u_{yy} = 0$
1-D Heat conduction problem	1-D Wave propogation Problem	2-D Potential Problem
1 — Intial Condition	2 - Intial Conditions	
2 — Boundry Condition	2 - Boundary Conditions	2 conditions wrt x & y each

1. Heat Conduction Equation :

$u_t = k u_{xx}$:

Intial Condition: $u(x, t=0) = f(x)$

Boundary Conditions: 1^{st} type

i) $u(x=a, t) = g(t)$

ii) $u(x=b, t) = h(t)$

2^{nd} type

i) $\left.\frac{\partial u}{\partial x}\right|_{(a,t)} = g'(t)$

ii) $\left.\frac{\partial u}{\partial x}\right|_{(b,t)} = h'(t)$

2. Wave propogration Equation :

$u_{tt} = c^2 u_{xx}$

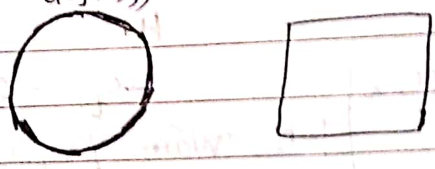
Intial Conditions: $u(t=0, x) = f_1(x)$

$\frac{\partial u}{\partial t}(t=0, x) = f_2(x)$

3. Laplace Equation:

$$\nabla^2 u = 0$$

$u = f(x,y)$



Elliptical Equation : (Numerical Solution)

Laplace Equation:

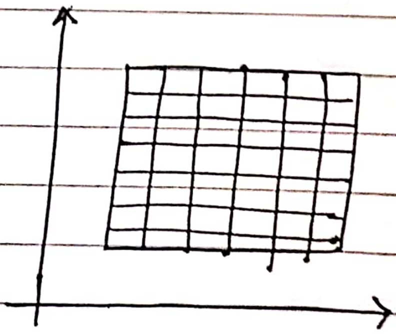
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{— Laplace Equation}$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{— Poisson's Equation}$$

Numerical Procedure

Pre-processive | Processive | Post Processive



$x \in [0,b]$	$x \in [0,b]$
$y \in [a,b]$	$y \in [c,d]$
Square	rectangle

$x_0 = a$

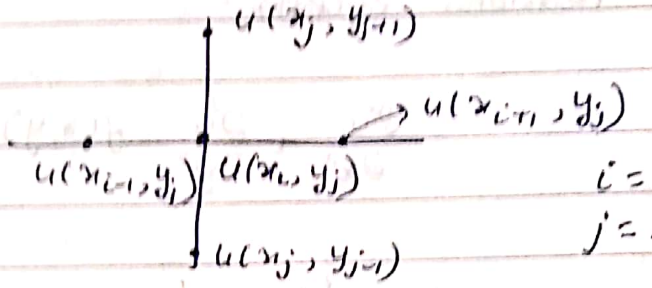
$x_1 = 0 + h = x_0 + h$

$x_2 = 0 + 2h = x_0 + 2h = x_1 + h$

$x_{j+1} - x_j = h \qquad y_{k+1} - y_k = k$

$$u(x,y)\Big|_{(x_i, y_j)} = u_{ij} = u(x_i, y_j)$$

$f(x_i) = f_i$



$i = 1, 2, \ldots, N$

$j = 1, 2, \ldots, N$

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \right)\Bigg|_{(x_i, y_j)}$$

$$\left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \right) + \left(\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} \right) = 0$$

$$+ O(h^2) \qquad\qquad + O(k^2)$$

If $h = k$ $\Rightarrow$ then we have square mesh

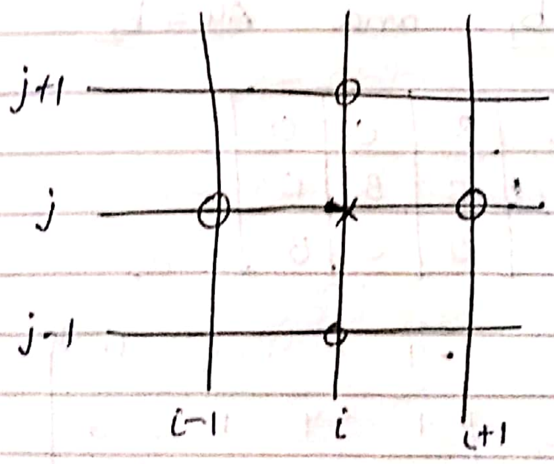
$$(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + (u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) + O(h^2, k^2) = 0$$

Standard 5-Point formula.

or $u_{ij} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} \right)$

$\forall \quad i = 1, 2, \ldots, N$

$j = 1, 2, \ldots N$



Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$

and Possion's Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{—— (2)}$$

$$\Omega = \{(x,y) \mid 0 < x, y < \}$$

Boundary Conditions:

$u(x,y) = g(x,y)$ for all $(x,y) \in \Omega$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} \qquad h = k$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = 0 \quad \text{—— (3)}$$

and

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = h^2 f_{ij} \quad \text{—— (4)}$$

$Au = b_1$ and $Au = b_2$

$$A = \begin{bmatrix} B & C & 0 \\ C & B & C \\ 0 & C & B \end{bmatrix}$$

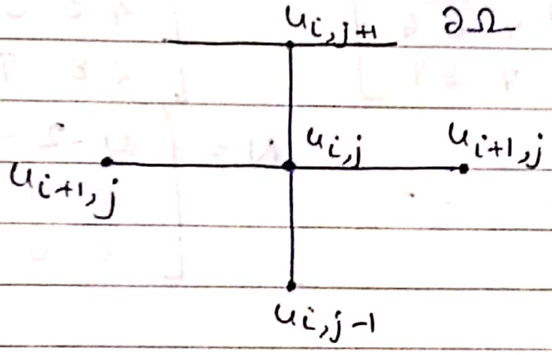
where

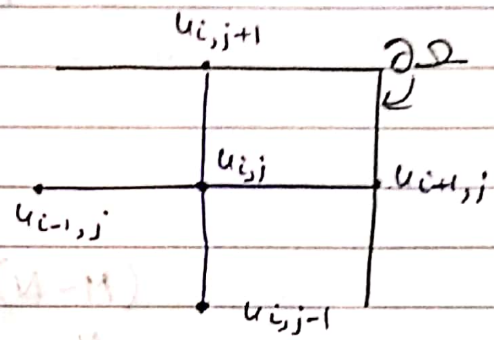
$$B = \begin{bmatrix} -4 & 1 & \cdots & & 0 \\ 1 & -4 & 1 & \cdots & 0 \\ \vdots & 1 & -4 & 1 & \cdots 0 \\ 0 & \vdots & & 1 & -4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

Boundary Conditions:

a) Dirichlet conditions:

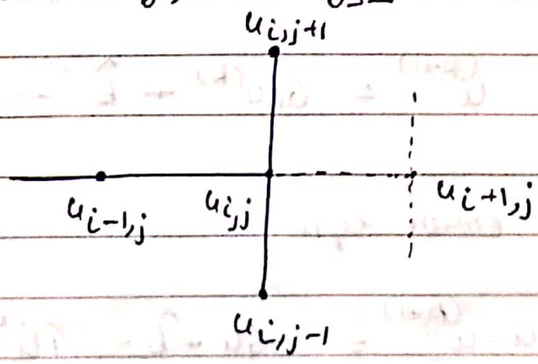




$u_{i,j+1}$ is known $\qquad$ $u_{i,j+1}$, $u_{i+1,j}$ are known

b) Neumann Condition:

$$\frac{\partial u}{\partial n} = v(x,y) \quad \partial\Omega$$



$$\frac{u_{i+1,j} - u_{i-1,j}}{2h} = v_{ij}$$

Laplace's equation

$$-2u_{i-1,j} - u_{i,j+1} - u_{i,j+1} + 4u_{ij} = 2h v_{ij}$$

Poisson's equation:

$$-2u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + 4u_{ij} = 2h v_{ij} - h^2 f_{ij}$$

Solution Procedure:

$$Au = b \quad \text{--- (1)}$$

Consider splitting

$$A = M - N$$

where M is invertible

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $M = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$

$N = \begin{bmatrix} 0 & -2 & -3 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

$$(M-N)u = b$$

$$Mu = Nu + b$$

$$u = (M^{-1}N)u + M^{-1}b$$

$$u = Gu + \hat{b}$$

$$Mu^{(k+1)} = Nu^{(k)} + b$$

or $$u^{(k+1)} = (M^{-1}N)u^{(k)} + M^{-1}b$$

or $$u^{(k+1)} = Gu^{(k)} + \hat{b} \quad \text{--- (3)}$$

Recursive error eqns

$$u - u^{(k+1)} = Gu + \hat{b} - Gu^{(k)} - \hat{b}$$

$$= \quad u - u^{(k+1)} = G(u - u^{(k)})$$

$$e_{k+1} = G.e_k$$

$$e_k = u - u^{(k)}$$

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Jacobi:

$$A = D - L - U \quad \text{— (5)}$$
$$(D - L - U)u = b$$

$$\Rightarrow \quad u = D^{-1}(L+U)u + D^{-1}b$$
$$u^{(k+1)} = D^{-1}(L+U)u^{(k)} + D^{-1}b \quad \text{— (6)}$$

$$u^{(k+1)} = G u^{(k)} + \hat{b} \qquad \text{where} \quad G = D^{-1}(L+U)$$

In scalar form,

$$u_i^{(k+1)} = -\sum_{j=1, i\neq j}^{n} \frac{a_{ij}}{a_{ii}} u_i^{(k)} + \frac{b_i}{a_{ii}} \quad , i = 1,2,\dots,n$$

Laplace Equation

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)}\right]$$

Poisson's Equation

$$u_{ij}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)} - h g_{ij}\right]$$

Gauss Seidel:

$$u_{ij}^{(k+1)} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} u_j^{(k+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} u_j^{(k)} + \frac{b_i}{a_{ii}}$$

$$\text{—} \quad i = 1,2,3,\dots,n \quad \text{— (8)}$$

$$(D-L)u^{(k+1)} = Uu^{(k)} + b$$
$$\Rightarrow \quad u^{(k+1)} = (D-L)^{-1}Uu^{(k)} + (D-L)^{-1}b$$
$$= G_{GS}u^{(k)} + \hat{b} \quad / \quad G_{GS} = (D-L)^{-1}U$$

Laplace Equation:

$$u_{ij}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} + u_{i+1,j}^{(k)} + u_{i,j+1}^{(k)} \right]$$

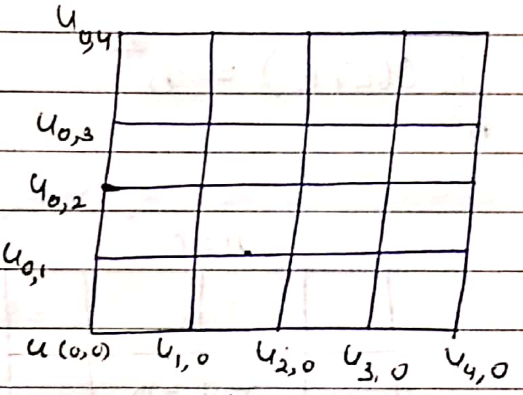
Poisson's Equation:

$$u_{ij}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} + u_{i+1,j}^{(k)} + u_{i,j+1}^{(k)} - h^2 f_{ij} \right]$$

Solve $\nabla^2 u = 0$

Subjected to the Boundary conditions $u = 1$ on all sides take $h = k = \frac{1}{4}$

$$\underbrace{u(0,y)=1 \quad u(1,y)=1}_{\forall y} \qquad \underbrace{u(x,0)=1 \quad u(x,1)=1}_{\forall x}$$



$\{x_0, x_1, x_2, x_3, x_4\}$ $\{y_0, y_1, y_2, y_3, y_4\}$

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$$

$$u_{1,1} = \frac{1}{4}[u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0}]$$

$$= \frac{1}{4}[u_{2,1} + 1 + u_{1,2} + 1]$$

$$\therefore u_{1,1} = \frac{1}{4}[u_{2,1} + u_{1,2} + 2]$$

Like wise write all the other 8 equations

Suppose $u_{i,j}$ is written in a new form:

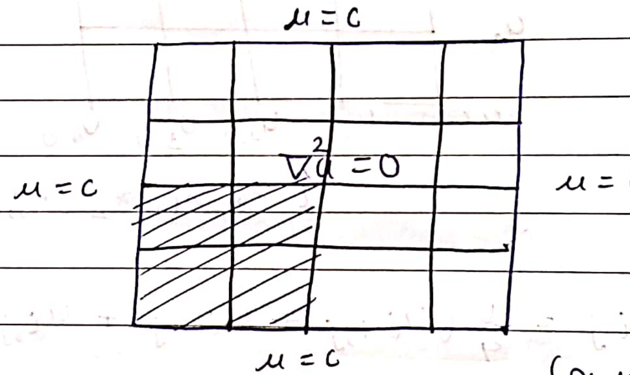
Daigonal 5-point formula
$$\left\{ u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1}\right]\right.$$

To find the local truncation error

$$u_{i+1,j} = u(x+\Delta x, y) = \text{Taylor series}$$

similarly for $u_{i-1,j}$, $u_{i,j+1}$, $u_{i,j-1}$

$$\frac{1}{h^2} O(LTE) = h^2$$



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial x^2} = 0$$

$$(x, y) \longleftrightarrow (y, x)$$

Large Systems: Then

$\nabla^2 u = (0$ or $f(x,y))$ symmetry of the domain can be used to solve the entire system.

$$Ax = b \quad \times \text{ Direct}$$

~~Instead iterative methods are used such as Gauss Seidel & Gauss Jacobi.~~

Jacobi Method: {Iterative Method}

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right]$$

done for $n = 0, 1, 2, \ldots$

$u_{i,j}^{(0)}$: guess value

Gauss Seidel Method:

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)}, u_{i,j-1}^{(n)}\right]$$

$n = 0, 1, 2, 3, \ldots$

$u_{i,j}^{(0)}$: guess

Convergence : $\left| u_{i,j}^{(n+1)} - u_{i,j}^{(n)} \right| < \epsilon$

$$Ax = b$$

$$A = L + D + U$$

$$X^{(n+1)} = HX^{(n)} + b^*$$

$$H = -D^{-1}(L+U)$$

$$(L + D + U)x = b$$

$$Dx = -(L+U)x + b$$

$$b^* = D^{-1} \cdot b$$

$$X^{(n+1)} = HX^{(n)} + b^*$$

converges only when the spectral radius of H is less than one. $\rho(H) < 1$

$$(D+L+U)x = b$$
$$(D+L)x = -Ux + b$$
$$H = -(D+L)^{-1}U$$
$$b^{*} = (D+L)^{-1}b$$

Gauss Seidel

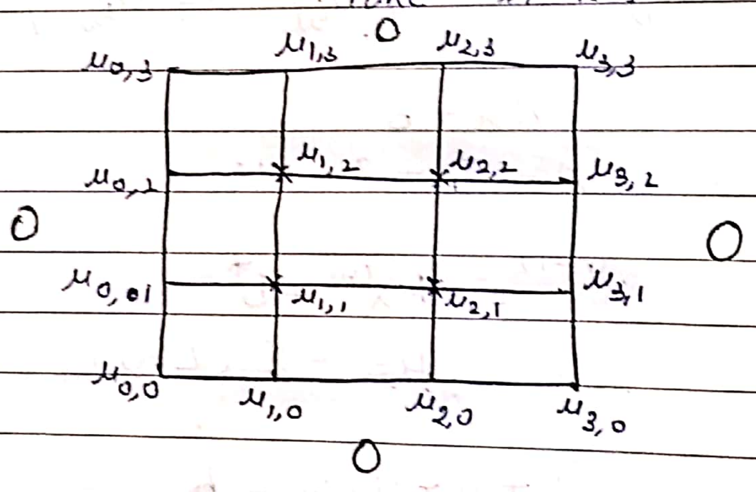
<u>solve:</u>

$$\nabla^2 u = f(x,y)$$

$$\nabla^2 u = -10(x^2+y^2+10)$$

$u=0$ on the side with sides $x=0,\ y=0,\ x=3,\ y=3$

Take $h=k=1$



$$u_{1,1} = \frac{1}{4}\left[0 + u_{2,1} + 0 + u_{1,2}\right] = \frac{1}{4}(u_{2,1} + u_{1,2})$$

$$u_{2,1} = \frac{1}{4}\left[u_{1,1} + 0 + 0 + u_{2,2}\right] = \frac{1}{4}\left[u_{1,1} + u_{2,2}\right]$$

$$u_{1,2} = \frac{1}{4}\left[0 + u_{1,1} + u_{2,2} + 0\right] = \frac{1}{4}(u_{1,1} + u_{2,2})$$

$$u_{2,2} = \frac{1}{4}\left[\right.$$

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$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{k^2}=-10(x_i^2+y_i^2+10)$$

$$u_{ij}=\frac{1}{4}(u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1})+\frac{10}{4}(x_i^2+y_i^2+10)\times h^2$$

$$u_{1,1}=\frac{1}{4}(u_{1,2}+u_{2,1}+120)$$

$$u_{1,2}=\frac{1}{4}(u_{2,2}+u_{1,1}+150)$$

$$u_{2,1}=\frac{1}{4}(u_{2,2}+u_{1,1}+150)$$

$$u_{2,2}=\frac{1}{4}(u_{1,2}+u_{2,1}+150)$$

Take $u_{1,1}^{(0)}=0$ $u_{2,1}^{(0)}=0$ $u_{1,2}^{(0)}=0$ $u_{2,2}^{(0)}=0$

6^{th} iteration:

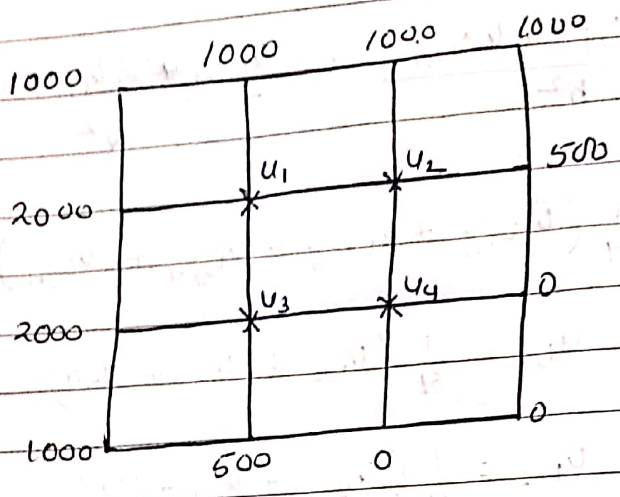
$u_{1,2}=75=u_{2,1}$

$u_{1,1}=67.5$

$u_{2,2}=82.5$

Gauss Seidel Method

Question:



Solve $\nabla^2 u = 0$ in the above system.

$$u_1 = \frac{1}{4}\left[2000 + 1000 + u_2 + u_3\right] = \frac{1}{4}\left[3000 + u_2 + u_3\right]$$

$$u_2 = \frac{1}{4}\left[500 + 1000 + u_1 + u_4\right] = \frac{1}{4}\left[1500 + u_1 + u_4\right]$$

$$u_3 = \frac{1}{4}\left[2000 + 500 + u_1 + u_4\right] = \frac{1}{4}\left[2500 + u_1 + u_4\right]$$

$$u_4 = \frac{1}{4}\left[0 + 0 + u_2 + u_3\right] = \frac{1}{4}(u_2 + u_3)$$

$$u_1^{(n+1)} = \frac{1}{4}\left[3000 + u_2^{(n)} + u_3^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[1500 + u_1^{(n)} + u_4^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[2500 + u_1^{(n)} + u_4^{(n)}\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[u_3^{(n)} + u_2^{(n)}\right]$$

Let $\left(u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, u_4^{(0)}\right) = (0, 0, 0, 0)$

$$u_1^{(1)} = \frac{1}{4}(3000) = 750$$

$$u_2^{(1)} = \frac{1}{4}(1500) = \frac{750}{2} = 375.r$$

$u_3^{(1)} = 625 \qquad u_4^{(1)} = 0$

Gauss - Jacobi

After 10^{th} iteration:

$u_1 = 1208 \qquad u_2 = 791.5 \qquad u_3 = 1041.5$

$u_4 = 458$

For Gauss-Seidel approximation:-

$$u_1^{(n+1)} = \frac{1}{4}\left[3000 + u_2^{(n)} + u_3^{(n)}\right]$$

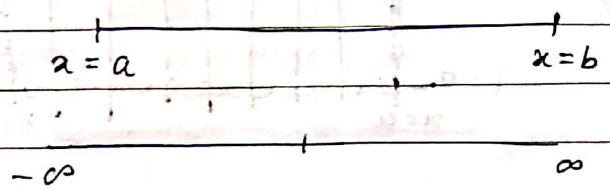
$$u_2^{(n+1)} = \frac{1}{4}\left[1500 + u_1^{(n+1)} + u_4^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[2500 + u_1^{(n+1)} + u_4^{(n)}\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[u_2^{(n+1)} + u_3^{(n+1)}\right]$$

Solution of 1-D Wave Equation:- (Hyperbolic equation)

$u = u(t, x)$



$$\frac{\partial^2 u}{\partial t^2} = c \cdot \frac{\partial^2 u}{\partial x^2}$$

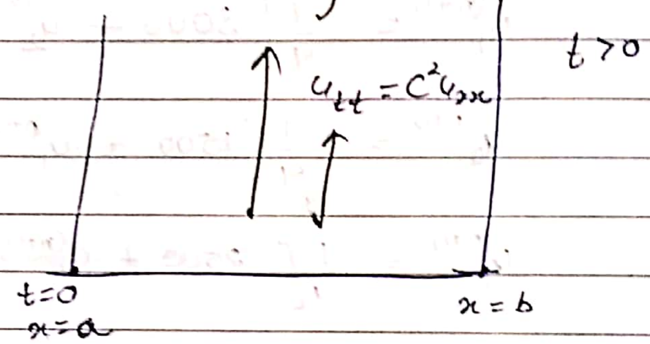
2 (Initial Conditions): $\left.\begin{aligned} u(x, t=0) &= f(x) \\ \left.\frac{\partial u}{\partial t}\right|_{(x, t=0)} &= g(x) \end{aligned}\right\} \forall\ x \in [a, b]$

2 Boundry Conditions:

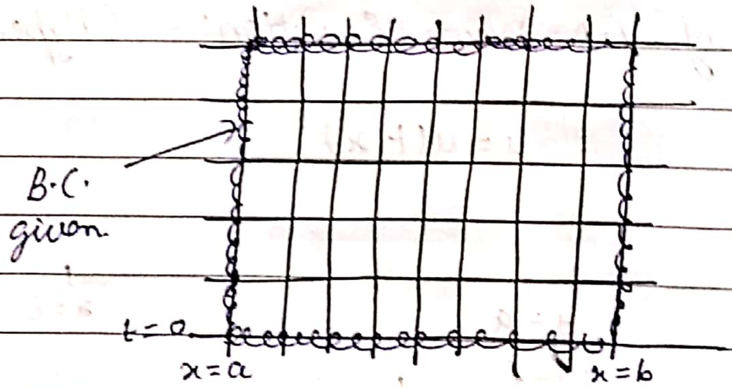
$$\left.\begin{aligned} u(x=a, t) &= h^*(t) \\ u(x=b, t) &= k^*(t) \end{aligned}\right\} \quad \forall\ t>0$$

Type I (Dirichlet B.C.)

$$\left.\begin{aligned} \frac{\partial u}{\partial x}(x=a, t) &= h^*(t) \\ \frac{\partial u}{\partial x}(x=b, t) &= k^*(t) \end{aligned}\right\} \quad \text{Type II} \quad \forall\ t>0$$



$\Delta x = h \qquad \Delta t = k$



$$u_{tt} = c^2 u_{xx} \Big|_{(x_i, t_j)} \qquad x_i = x_0 + ih$$
$$t_j = t_0 + jk$$

$$u_{tt}\Big|_{(x_i, t_j)} = c^2 u_{xx}\Big|_{(x_i, t_j)}$$

$$\frac{u_i^{j-1} - 2u_i^j + u_i^{j+1}}{k^2} + O(k^2) = c^2\left(\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} + O(h^2)\right)$$

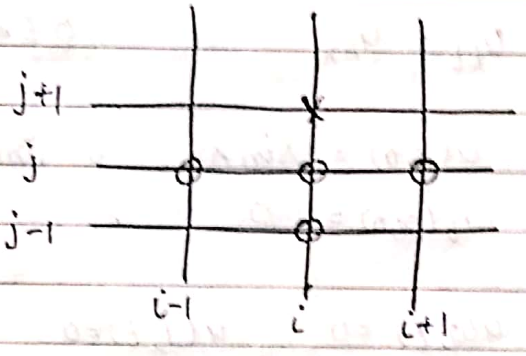
or

$$u_i^{j-1} - 2u_i^j + u_i^{j+1} = \frac{c^2k^2}{h^2}\left(u_{i-1}^j - 2u_i^j + u_{i+1}^j\right) + O(h^2, k^2)$$

$$\boxed{\alpha = \frac{ck}{h}} \Longrightarrow \text{Mesh ratio parameter}$$

$$u_i^{j+1} = \left(-u_i^{j-1} + 2u_i^j\right) + \alpha^2\left(u_{i-1}^j - 2u_i^j + u_{i+1}^j\right)$$

$$u_i^{j+1} = -u_i^{j-1} + 2(1-\alpha^2)u_i^j + \alpha^2\left(u_{i-1}^j + u_{i+1}^j\right)$$



Properties of this method:

i) Explicit Method

ii) $O(k^2, h^2)$

$$\frac{\partial u}{\partial t}(t=0, x) = g(x)$$

$$\frac{u_i^1 - u_i^0}{k} + O(k) = g(x_i) \quad \{\text{Forward difference approximation}\}$$

$$\Rightarrow u_i^1 = u_i^0 + k g_i$$

Truncation Error: $\frac{k^2}{12}\left[k^2 \cdot \frac{\partial^4 u_i^j}{\partial t^4} - h^2 \frac{\partial^4 u_i^j}{\partial x^4}\right]$

Order is: $\frac{1}{h^2} \; O(LTE)$

$\Rightarrow \; O(k^2, h^2)$ { 2^{nd} order w.r.t time and space }

Solve:

$$u_{tt} = u_{xx} \qquad 0 \le x \le 1$$

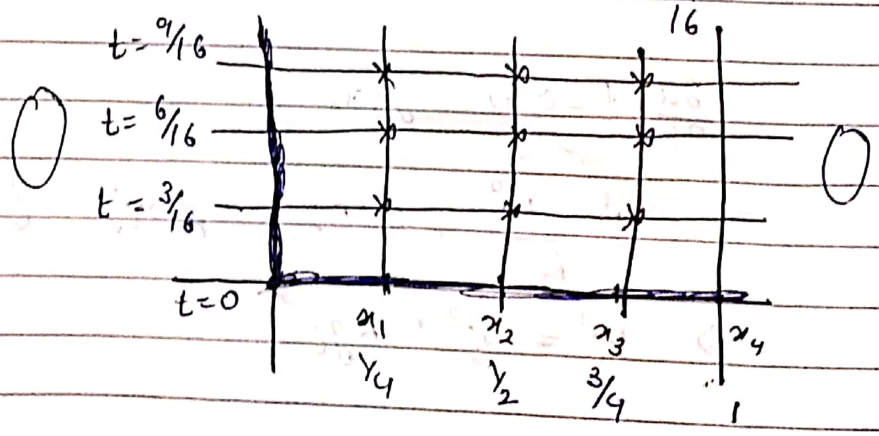
s.t.:

$$\left.\begin{aligned} u(x,0) &= \sin \pi x \\ u_t(x,0) &= 0 \end{aligned}\right\} \text{Initial Conditions}$$

$$\left.u(0,t) = 0 \qquad u(1,t) = 0 \qquad \forall\, t \ge 0\right\} \text{Boundry Conditions.}$$

$h = 1/4 \qquad \alpha = 3/4 \qquad \alpha = \frac{ck}{h} = \frac{1 \times k}{1/4} = \frac{3}{4}$

$$k = \frac{3}{16}$$



$j=0 \qquad i=1,2,3.$

$$u_1^1 = \frac{7}{8}u_1^0 + \frac{9}{16}(u_0^0 + u_2^0) - u_1^{-1}$$

$$u_2^1 = \frac{7}{8}u_2^0 + \frac{9}{16}(u_1^0 + u_3^0) - u_2^{-1}$$

$$u_3^1 = \frac{7}{8}u_3^0 + \frac{9}{16}(u_2^0 + u_4^0) - u_3^{-1}$$

$u_m^{-1} = u_m^1$

as $\frac{u_i^1 - u_i^{-1}}{2k} = 0$

$u_1^1 = 0.59065$
$u_2^1 = 0.83525$
$u_3^1 = 0.59061$

$j=4$: $\quad u_1^5 = -0.6884 = u_3^5$

$u_2^5 = -0.97354$

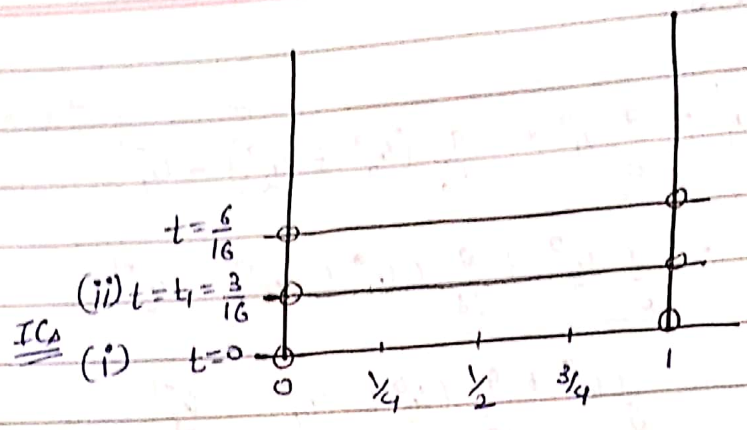
Ex:

$$u_{tt} = k.u_{xx} \qquad : k=1$$

s.t. I.C.s
i) $u(x, t=0) = \sin \pi x \qquad , 0 \le x \le 1$
ii) $\frac{\partial u}{\partial t}(x, t=0) = 0 \qquad , \quad ''$

B.C.s
i) $u(x=a, t) = 0$
ii) $u(x=b, t) = 0$ $\Big\}\ \forall\ t \geq 0$

Use Explicit Method. Take $h = 1/4$; (mesh ratio parameter) $r = 3/4$
Integrate for 5 times steps.
Compare with the exact solution $u(x,t) = (\sin \pi x)(\cos \pi t)$



$$u_i^{j+1} = 2(1-r^2)u_i^j + r^2(u_{i+1}^j + u_{i-1}^j) - u_i^{j-1}$$

or $$u_i^{j+1} = \frac{7}{8}u_i^j + \frac{9}{16}(u_{i-1}^j + u_{i+1}^j) - u_i^{j-1}$$

$j = 0,1,2,3,4,5$ $\qquad\qquad$ $i = 1,2,3$

$j=0$:

To eliminate fictious unknown.

$$u_1^1 = \frac{7}{8}u_1^0 + \frac{9}{16}(u_0^0 + u_2^0) - u_1^{-1}$$

$$u_2^1 = \frac{7}{8}u_2^0 + \frac{9}{16}(u_1^0 + u_3^0) - u_2^{-1}$$

$$u_3^1 = \frac{7}{8}u_3^0 + \frac{9}{16}(u_2^0 + u_4^0) - u_3^{-1}$$

I.C.2

$$\frac{\partial u}{\partial t}(x, t=0) = 0$$

Central difference

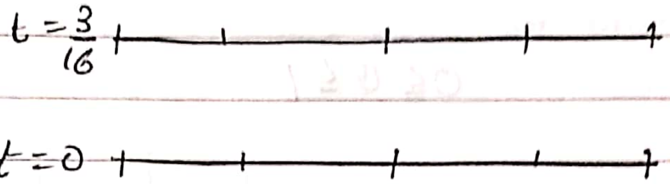
$$\frac{u_i^1 - u_i^{-1}}{2k} = 0 \quad \forall i$$

$$u_i^1 = u^{-1}$$

$$u_1^1 = \frac{7}{8}\sin\frac{\pi}{4} + \frac{9}{16}\left(0 + \sin\frac{2\pi}{4}\right) - u_1^1$$

$$\Rightarrow u_1^1 = 0.59065$$

Similarly, $u_2^1 = 0.83525$ $u_3^1 = 0.59061$



$j = 1$

$u_1^2 = 0.27951$ $u_2^2 = 0.39528$ $u_3^2 = 0.27951$

$j = 2;\ j = 3;\ j = 4$

$u_1^5 = u_3^5 = -0.6884$; $u_2^5 = -0.97354$

<u>Exact:</u>

$u_1^5 = u_3^5 = -0.69352$ $u_2^5 = -0.98079$

i) Explicit
ii) $O(k^2, h^2)$
iii) 3-level method

<u>Implicit Method</u>

$$u_{tt} = c^2 u_{xx}$$

Explicit $\delta_t^2 u_i^j = r^2 \delta_x^2 u_i^j$

$$\delta_t^2 u_i^j = u_i^{j-1} - 2u_i^j + u_i^{j+1}$$

$$\delta_x^2 u_i^j = u_{i-1}^j - 2u_i^j + u_{i+1}^j \qquad r = \frac{ck}{h}$$

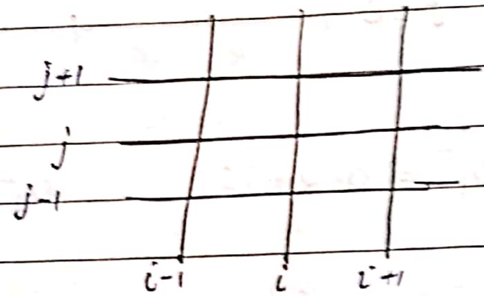
Implicit

$$\delta_t^2 u_i^j = r^2 \delta_x^2 \left[\theta u_i^{j+1} + (1-2\theta) u_i^j + \theta u_i^{j-1} \right]$$

$O(k^2, h^2)$

3-level $\qquad 0 \le \theta \le 1$

$\theta = \frac{1}{2}$



Ex: Use the implicit method with $\theta = \frac{1}{2}$, solve $u_{tt} = u_{xx}$ in $0 < x < 1$, such that $u(x,0) = \sin \pi x$, $\left.\frac{\partial u}{\partial t}\right|_{(x,0)} = 0$, $u(0,t) = 0 = u(1,t)$

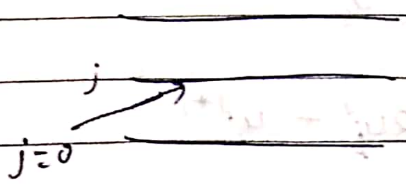
$h = \frac{1}{4}, \; k = \frac{3}{16}$

$$\delta_t^2 u_i^j = \frac{r^2}{2} \delta_x^2 \left(u_i^{j+1} + u_i^{j-1} \right)$$

$$\left(u_i^{j-1} - 2u_i^j + u_i^{j+1} \right) = \frac{9}{32} \left(\left(u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1} \right) + \left(u_{i-1}^{j-1} - 2u_i^{j-1} + u_{i+1}^{j-1} \right) \right)$$

$$-\frac{9}{32} u_{i-1}^{j+1} + \frac{25}{16} u_i^{j+1} - \frac{9}{32} u_{i+1}^{j+1} = 2u_i^j + \frac{9}{32} u_{i-1}^{j-1} - \frac{25}{16} u_i^{j-1} + \frac{9}{32} u_{i+1}^{j-1}$$

$i = 1, 2, 3, \ldots$



$u_i^1 = u_i^{-1} \quad \forall \; i = 1, 2, 3 \ldots$

$j = 0$:

$$\begin{bmatrix} 25/8 & -9/16 & 0 \\ -9/16 & 25/8 & -9/16 \\ 0 & -9/16 & 25/8 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} 2u_1^0 \\ 2u_2^0 \\ 2u_3^0 \end{bmatrix}$$

j=0: $u_1^1 = u_3^1 = 0.66709$, $u_2^1 = 0.85855$

j=1: $u_1^2 = u_3^2 = 0.33533$, $u_2^2 = 0.47422$

j=2: $u_1^2 = u_3^3 = -0.0315$ $u_2^3 = -0.04425$

Parabolic Equation:

1-D Heat Conduction

$$u = u(t, x)$$

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \quad a < x < b$$

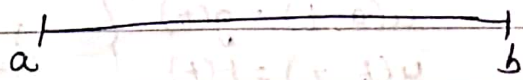
$[a, b]$

I.C.: $u(x, t=0) = f(x) \quad \forall \ x \in [a, b]$

B.Cs: $u(x=0, t) = g(t) \quad \forall \ t > 0$

$u(x=b, t) = h(t) \quad \forall \ t > 0$

1. Forward Time Central Space (FTCS) or Schmidt Method:

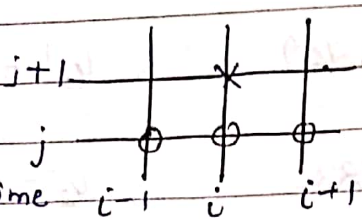


$$\frac{u_i^{j+1} - u_i^j}{k} + O(k) = \alpha\left(\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2}\right) + O(h^2)$$

$$\therefore u_i^{j+1} = u_i^j + \frac{\alpha k}{h^2}\left(u_{i-1}^j - 2u_i^j + u_{i+1}^j\right)$$

Mesh ratio parameter $(r) = \frac{\alpha k}{h^2}$

$$u_i^{j+1} = r u_{i-1}^j + (1-2r)u_i^j + r u_{i+1}^j + O(k, h^2)$$



2-level w.r.t time

$O(k, h^2)$

2. BTCS (Laasonen's Method)
3. CTCS (Richardson's Method)

Parabolic Equation

Ex: 1-D heat conduction via thin rod

$$\frac{\partial u}{\partial t} = K \cdot \frac{\partial^2 u}{\partial x^2}$$

$$a < x < b$$

such that, Initial Conditions:

$$u(x, t=0) = f(x)$$

Boundary Conditions:

Type-I
$$\left.\begin{aligned} u(a, t) &= g(t) \\ u(b, t) &= p(t) \end{aligned}\right\}$$

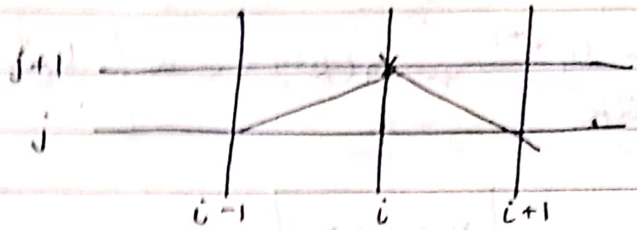
$x = a$ ——— $x = b$

FTCS (Schmidt Method)

$$\frac{u_i^{j+1} - u_i^j}{k} + O(k) = \alpha \left[\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} \right] + O(h^2)$$

Define $\boxed{r = \frac{\alpha k}{h^2}}$ $\longrightarrow$ mesh ratio

$$u_i^{j+1} = (1-2r)\,u_i^j + r\,(u_{i-1}^j + u_{i+1}^j) + O(k, h^2)$$



$i = 1, 2, 3, \ldots, M$

$j = 1, 2, 3, \ldots, N$

BTCS: (Laasonen's Method)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

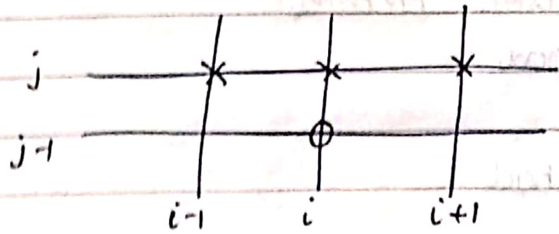
$$\frac{u_i^j - u_i^{j-1}}{k} + O(k) = \alpha\left[\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2}\right] + O(h^2)$$

$\boxed{r = \frac{\alpha k}{h^2}}$ → Mesh ratio parameter

$i = 1, 2, \ldots, N$

$j = 1, 2, \ldots, M$

$$(1-2r)\,u_i^j - r\,(u_{i-1}^j + u_{i+1}^j) = u_i^{j-1}$$



system of eqns

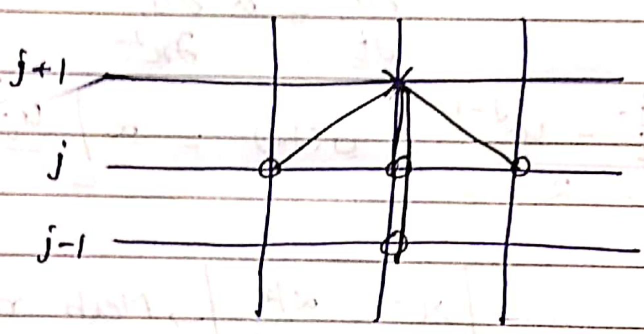
i) Implicit ii) $O(k, h^2)$ iii) 2-step method

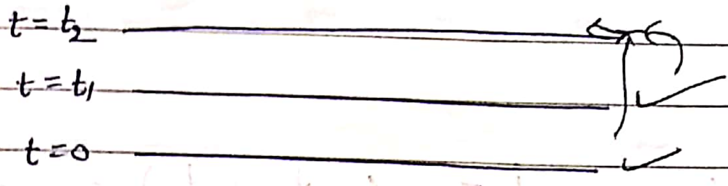
CTCS: (Richardson Method)

$$\frac{u_i^{j+1} - u_i^{j-1}}{2k} + O(k^2) = \alpha\left[\frac{u_{i-1}^{j} - 2u_i^{j} + u_{i+1}^{j}}{h^2}\right]$$

$$\boxed{r = \frac{\alpha k}{h^2}}$$

$$u_i^{j+1} = u_i^{j-1} + r\left[u_{i-1}^{j} - 2u_i^{j} + u_{i+1}^{j}\right]$$





Crank-Nicolson Method:

i) Implicit Method

ii) $O(h^2, k^2)$

iii) 2 - step method

$$u_t = u_{xx} \qquad \alpha = 1$$

Procedure:

i) Evaluate this at the grid point $\left(mh, \left(n+\frac{1}{2}\right)k\right)$

ii) replace u_{xx} by the mean of its forward difference approximation at n^{th} and $(n+1)^{th}$ time levels.

$$u_t\Big|_{(mh,\,(n+\frac{1}{2})k)} = u_{xx}\Big|_{(mh,\,(n+\frac{1}{2})k)}$$

or

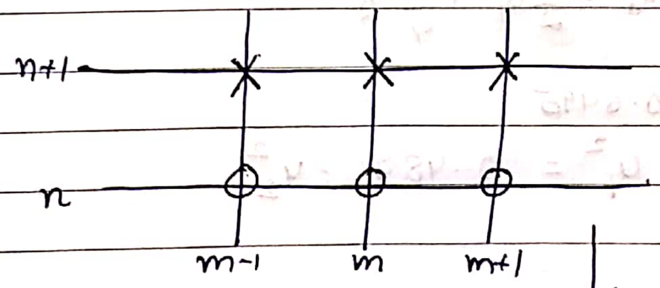
$$\frac{1}{2(k/2)}\,\delta_t u_m^{n+1/2} = \frac{1}{2h^2}\left\{\delta_x^2 u_m^{n+1} + \delta_x^2 u_m^n\right\}$$

$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{1}{2}\left[\frac{u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}}{h^2} + \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}\right]$$

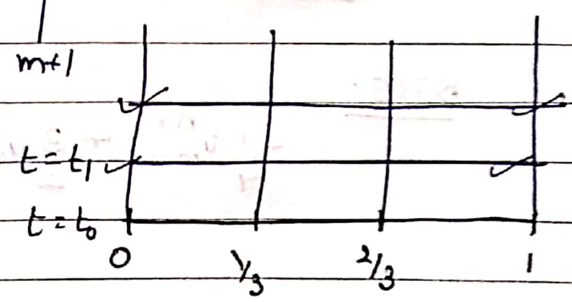
$$r = \frac{\alpha k}{h^2}$$

$$-r u_{m-1}^{n+1} + (2+2r)u_m^{n+1} - r u_{m+1}^{n+1} = r u_{m-1}^n + (2-2r)u_m^n + r u_{m+1}^n$$

$m = 1, 2, 3, \ldots P$
$n = 1, 2, 3, \ldots, Q$



Dirichlet Boundary Conditions



Solve: $u_t = u_{xx}$

such that $u(x,0) = \sin \pi x$, $0 < x < 1$

$u(0,t) = u(1,t) = 0$

$h = \frac{1}{3}$, $k = \frac{1}{36}$

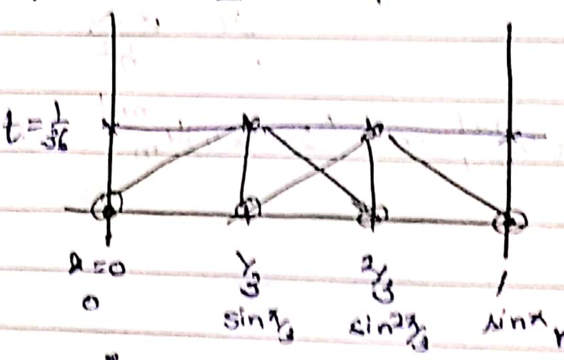
$u_{exact}(x,t) = e^{-\pi^2 t} \sin \pi x$

<u>Schmidt Method:</u>

$$u_m^{n+1} = (1-2\lambda)u_m^n + \lambda(u_{m-1}^n + u_{m+1}^n)$$

$$\lambda = \frac{1/36}{1/9} = \frac{1}{4}$$

$$u_m^{n+1} = \frac{1}{4}u_{m-1}^n + \frac{1}{2}u_m^n + \frac{1}{4}u_{m+1}^n$$



$m = 1, 2$

$n = 0, 1, 2, 3 \ldots$

$n = 0$, $m = 1, 2$

$$u_1^1 = \frac{1}{4}u_0^0 + \frac{1}{2}u_1^0 + \frac{1}{4}u_2^0$$

$$= 0.6495$$

$$u_1^2 = 0.4871 = u_2^2$$

<u>BTCS:</u>

$$-\frac{1}{4}u_{m-1}^{n+1} + \frac{3}{2}u_m^{n+1} - \frac{1}{4}u_{m+1}^{n+1} = u_m^n$$

$m = 1, 2$

$n = 0, 1, 2, 3$

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$n=0, \; m=1$

$$-\frac{1}{4}u_0^1 + \frac{3}{2}u_1^1 - \frac{1}{4}u_2^1 = u_1^0$$

$n=0, \; m=2:$

$$-\frac{1}{4}u_1^1 + \frac{3}{2}u_2^1 - \frac{1}{4}u_3^1 = u_2^0$$

$$u_1^2 = u_2^2 = 0.5542$$

If CTCS is used then $u_1^2 = u_2^2 = 0.5239$

Crank-Nicolson Method:

$$-\frac{1}{8}u_{m-1}^{n+1} + \frac{5}{4}u_m^{n+1} - \frac{1}{8}u_{m+1}^{n+1} = \frac{1}{8}u_{m-1}^{n} + \frac{3}{4}u_m^{n} + \frac{1}{8}u_{m+1}^{n}$$

$m = 1, 2$
$n = 0, 1, 2, 3$

$n=0, \; m=1$

$$\frac{5}{4}u_1^1 - \frac{1}{8}u_2^1 = \frac{3}{4}u_1^0 + \frac{1}{6}u_2^0 = \frac{7\sqrt{3}}{16}$$

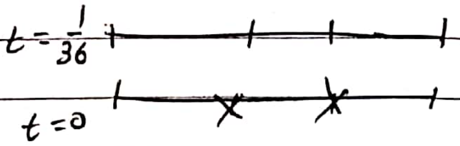
$n=0, \; m=1:$

$$-\frac{1}{8}u_1^1 + \frac{5}{4}u_2^1 = \frac{1}{8}u_1^0 + \frac{3}{4}u_2^0 = \frac{7\sqrt{3}}{16}$$

$$u_1^1 = u_2^1 = 0.6736$$

$n=1, \; m=1,2$

$$u_1^2 = u_2^2 = 0.5239$$



Solve:

$$u_t = u_{xx} \qquad u(x,0) = 1$$

$$u_x(0,t) = u(0,t)$$