- Ch-1 : Infinite series.
- Ch-2 : ODE.
- Ch-3 : Tensor
- Ch-4 : Sturm - Liouville theory, self-adjoint and orthogonal polynomials.

★ INFINITE SERIES

An infinite series is the sum of the terms of a sequence $u_1, u_2, u_3, \ldots,$ i.e, Σu_n.
ordinary add up of infinite no. of terms doesn't necessarily have a meaning. We thus associate sequence of partial sums of the series $\{S_n\}$ where, $S_n = u_1 + u_2 + \cdots + u_n$.

i.e, $S_1 = u_1$, $S_2 = u_1 + u_2$, ... etc.

→ The infinite series Σu_n is said to be convergent if the sequence $\{S_n\}$ converges. Thus if $\{S_n\}$ tends to a limit, then this limit is sum of infinite series i.e,

$$\Sigma u_n = \lim S_n.$$

if $\{S_n\}$ diverges, Σu_n diverges.

if $\{S_n\}$ oscillates, Σu_n oscillates.

→ Necessary conditions for convergence:-

Statement : A necessary condition for convergence of an infinite series Σu_n is that $\lim_{n\to\infty} u_n = 0$.

Proof: Let $S_n = u_1 + u_2 + \cdots + u_n$
Suppose Σu_n converges. Then $\{S_n\}$ also converges.
Let $\lim_{n\to\infty} S_n = S$. Now $u_n = S_n - S_{n-1}$.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \Rightarrow S-S=0$$

Hence, for a convergent series

$$\boxed{\lim_{n\to\infty} u_n = 0}$$

Note:
i) A series may diverge even if $\lim_{n\to\infty} u_n = 0$
ii) If $\lim u_n \neq 0$, series diverges.

Cauchy's General Principal of convergence.

Statement :- A series Σu_n converges iff to each $\epsilon > 0$, $\exists$ a positive integer m such that.

$$|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \epsilon \quad \forall\, n \geq m, \ \forall\, p \geq 1.$$

Proof : Σu_n converges iff the sequence $\{S_n\}$ converges. By the Cauchy general principle for sequence, a sequence $\{S_n\}$ converges iff to each $\epsilon > 0$, $\exists$ a positive integer m, such that $|S_{n+1} - S_n| < \epsilon$, $\forall\, n \geq m$ & $\forall\, p \geq 1$

$\Rightarrow |u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \epsilon$, $\forall\, n \geq m$ & $\forall\, p \geq 1$.

$\Rightarrow$ **Examples :-** 1) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots \dots$ is not convergent.

$$u_n = \frac{n}{n+1}, \quad \lim_{n\to\infty} u_n = \lim \frac{1}{1+1/n} = 1 \neq 0$$

$\therefore$ series diverges.

2.) $\Sigma \frac{1}{n}$ doesn't converge, even though $\lim \frac{1}{n} = 0$.

if possible, let $\Sigma \frac{1}{n}$ converge.

Then for any given ϵ, say $\epsilon = \frac{1}{4}$, $\exists$ a positive integer, 'm' such that

$$\left| \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p} \right| < \epsilon, \quad \forall\, n \geq m \text{ and } p \geq 1.$$

For $n = m$ & $p = m$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p} = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m} > m \cdot \frac{1}{2m} = \frac{1}{2}$$

$\therefore$ Thus contradiction arises. Hence $\Sigma \frac{1}{n}$ diverges.

☆ Standard results :

1.) If $\Sigma u_n = u$, then $\Sigma c u_n = cu$, c is n-independent number.

Note : Behaviour of a series as regards convergence is not altered by :-

(i) the alteration, addition or omission of a finite no. of terms.

(ii) multiplication of all terms by a non-zero number.

2.) If $\sum u_n = u$, $\sum v_n = v$, then $\sum w_n = u+v$, where $w_n = v_n + u_n$.

Note: If any two of 3 series $\sum u_n$, $\sum v_n$, $\sum w_n$ are convergent, then third is also convergent.

3.) If a series $\sum u_n$ converges to the sum u, then so does any series obtained from $\sum u_n$ by grouping the terms in brackets without altering the order of the terms.

Note: Converse is not always true.
e.g., $(1-1)+(1-1)+\cdots$ is convergent, but
$1-1+1-1+1-\cdots$ is divergent.

★ Series of positive terms:

Here $\{s_n\}$.

$u_1+u_2+u_3+\cdots+u_n$.

$$\{s_n\} = \{u_1, u_1+u_2, u_1+u_2+u_3, \cdots\}$$

Theorem: $\sum u_n$ converges iff $\{s_n\}$ is bounded above.

Theorem: Geometric series $1+r+r^2+\cdots$, converges for $r<1$ and diverges otherwise.

★ A comparison series :-

Th: $\sum \frac{1}{n^p}$ is convergent iff $p>1$. $\sum \frac{1}{n^p}$ iff $p>1$

★ Comparison Test:

1.) If $\sum u_n$ and $\sum v_n$ are two positive-term series, and $u_n \le k v_n$ $(k \ne 0)$ $\forall$ $n \ge m$, then
 i) $\sum u_n$ converges if $\sum v_n$ converges.
 ii) $\sum v_n$ diverges if $\sum u_n$ diverges.

2.) (Limit form).
If $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, where $l \ne 0, \infty$ then both series converge or diverge together.

Scanned by CamScanner

3.) If $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$, $\forall\, n \geq m$, then same conclusion as before.

Problem:- $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$ is convergent. To show??

Ans:- $\frac{1}{2!} = \frac{1}{2}$; $\frac{1}{3!} = \frac{1}{6} < \frac{1}{2^2}$; ... $\frac{1}{n!} < \frac{1}{2^{n-1}}$.

So we can say:-

$$\underbrace{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}}_{\text{this is a GP of common ratio } r = \frac{1}{2} < 1.} > 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}.$$

and so convergent.

$\Rightarrow$ Here, given series is convergent.

★. Cauchy's Root Test :-

1.) If $\lim_{n\to\infty} (u_n)^{1/n} = l$, then $\sum u_n$

i) converges if $l < 1$

ii) diverges if $l > 1$

test fails if $l = 1$.

Problem:- Show that, $u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ converges.

Soln:- clearly, $u_n > 0 \;\forall\; n$.

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-\frac{1}{n}\cdot n^{3/2}} \Rightarrow \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} \Rightarrow \frac{1}{e} < 1.$$

$\therefore$ $\sum u_n$ converges.

★. D'Alembert's Ratio Test:-

1.) If $\sum u_n$ is positive term series and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then the series (i) converges if $l < 1$.

(ii) diverges if $l > 1$.

(iii) test fails if $l = 1$.

Problem:- ~~Check the~~ Find the values of x, for which the power series $\sum a_n x^n$ converges and diverges, where

$\sum a_n x^n$

$$a_n = \frac{n^2-1}{n^2+1}.$$

Solⁿ:-

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{x^{n+1}}{x^n} = \lim_{n\to\infty} \frac{(n+1)^2-1}{(n+1)^2+1} \cdot \frac{n^2+1}{n^2-1} \cdot x = x$$

$\therefore$ By D'Alemberts Test, $\sum a_n x^n$ converges for $x<1$, diverges for $x>1$, Test fails for $x=1$.

For $x = 1$,

$$\lim_{n\to\infty} a_n = 1 \neq 0 \Rightarrow \text{series diverges for } x=1$$

$\therefore$ $\sum a_n x^n$ converges for $x<1$,
diverges for $x \geq 1$.

$\lim_{n\to\infty} (u_n)^{1/n} = l$ <1 >1 $=1$

★ Raabe's Test :-

Statement :- If $\sum u_n$ is positive term series and

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$$

then, the series i) converges if $l>1$.
ii) diverges if $l<1$.

→ Test fails for $l=1$.

$\frac{(n^2+2n)(n^2+1)}{(n^2+2n+2)(n^2-1)}$ $x > 1$

Raabe's test

$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$

$l>1$ conv
$l<1$ div
$l=1$ fail

Ex 6: Test the convergence of the series.

$$\frac{\alpha}{\beta} + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \cdots$$

Solⁿ:

$$u_n = \frac{(1+\alpha)(2+\alpha)(3+\alpha)\cdots(n-1+\alpha)}{(1+\beta)(2+\beta)\cdots\cdots(n-1+\beta)}, \quad n = 2, 3, \ldots$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} \Rightarrow \frac{(n+\alpha)}{(n+\beta)} \Rightarrow 1. \Rightarrow \text{D'Alembert's ratio test fails.}$$

Raabe's test:-

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) \Rightarrow n\left[\frac{(n+\beta)}{(n+\alpha)} - 1\right]$$

$$\Rightarrow n\left[\frac{(\beta-\alpha)}{(n+\alpha)}\right]$$

$$\Rightarrow \beta - \alpha$$

∴ By Raabe's test, the series converges if $\beta - \alpha > 1$, i.e., $\beta > \alpha + 1$. and diverges if $\beta < \alpha + 1$. Test fails for $\beta = \alpha + 1$.

∴ For $\beta = \alpha + 1$, the series becomes.

$$\frac{\alpha}{\alpha+1} + \frac{1+\alpha}{2+\alpha} + \frac{1+\alpha}{3+\alpha} + \cdots\cdots = \frac{\alpha}{1+\alpha} + \sum_{n=2}^{\infty} \frac{1+\alpha}{n+\alpha}.$$

consider, $\Sigma V_n \neq \cancel{\alpha}$, where $V_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{(1+\alpha)}{(n+\alpha)} \cdot n = 1+\alpha \neq 0, \infty.$$

Since, $\Sigma \frac{1}{n}$ diverges, given series diverges.

Thus, series converges for $\beta > \alpha + 1$ and diverges to $\beta \leq \alpha + 1$.

Scanned by CamScanner

★ Logarithmic Test:

Statement: If $\sum u_n$ is a positive term series and $\lim_{n\to\infty}\left(n\ln\frac{u_n}{u_{n+1}}\right)=l$, then the series converges for $l>1$, and diverges for $l<1$.

Exp:

$$1+\frac{x}{1!}+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\dots\dots$$

Sol^n:

$$\cancel{u_x = \frac{(n-1)^{n-1}x^{n-1}}{(n-1)!}} \qquad u_n = \frac{n^n x^n}{n!}.$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\cdot x$$

$$\Rightarrow \frac{(n+1)^{n+1}}{n^n}\cdot\frac{1}{(n+1)}\cdot x \Rightarrow \left(\frac{n+1}{n}\right)^n\cdot x.$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n x \Rightarrow ex.$$

∴ By ratio test, series converges for $ex<1$, i.e, for $x<\frac{1}{e}$ and diverges for $x>\frac{1}{e}$. Test fails for $x=\frac{1}{e}$.

for, $x=\frac{1}{e}$: $1+\frac{1/e}{1!}+\frac{2^2(1/e)^2}{2!}+\dots\dots$, $u_n = \frac{n^n(1/e)^n}{n!}$, $n=1,2,\dots$

$$\frac{u_n}{u_{n+1}} = \frac{n^n}{n!}\cdot\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{1}{e^n}\cdot e^{n+1} = \left(\frac{n}{n+1}\right)^n e.$$

Now, $\lim_{n\to\infty}\left(n\ln\left(\frac{u_n}{u_{n+1}}\right)\right) = \lim_{n\to\infty}\left[n\ln\left(\left(\frac{n}{n+1}\right)^n e\right)\right]$

$$\Rightarrow n\left[1+n\ln\left(\frac{n}{n+1}\right)\right] \Rightarrow n\left[1-n\ln\left(1+\frac{1}{n}\right)\right]$$

$$\Rightarrow \quad \boxed{\therefore \ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots \text{ for } |x|<1}$$

$$\Rightarrow n\left[1-\frac{n}{n}+\frac{n}{2n^2}\right] \Rightarrow \boxed{\frac{1}{2}} < 1$$

Now, By logarithmic test, series diverges for $x=1/e$.

∴ Given series converges for $x<1/e$ and diverges for $x\geq 1/e$.

★ Gauss Test :

If $\sum u_n$ is a positive term series, and $\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^{p}}$, $\alpha > 0$, $p > 1$, $\{\gamma_n\}$ is a bounded sequence.

then,

i) for $\alpha \neq 1$, $\sum u_n$ converges for $\alpha > 1$ and diverges for $\alpha < 1$.

whatever β may be :-

ii) for $\alpha = 1$, $\sum u_n$ converges for $\beta > 1$ and diverges for $\beta \leq 1$.

Ex : $\frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots$

Sol^n :

$$u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n+1)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} = \left(1 + \frac{1}{2n+2}\right)^2$$

$$\frac{u_n}{u_{n+1}} \Rightarrow 1 + \frac{1}{(2n+2)^2} + \frac{1}{n+1} \Rightarrow 1 + \frac{1}{n+1} + \frac{1}{4(n+1)^2}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} = \left(1 + \frac{3}{2n}\right)^2 \left(1 + \frac{1}{n}\right)^{-2}$$

$$\frac{u_n}{u_{n+1}} = \left(1 + \frac{3}{2n} + \frac{9}{4n^2} + \cdots\right)\left(1 - \frac{2}{n} + \frac{3}{n^2} - \cdots\right)$$

$$\frac{u_n}{u_{n+1}} = \left(1 + \frac{3}{n} + \frac{9}{4n^2} + \cdots\right)\left(1 - \frac{2}{n} + \frac{3}{n^2} - \cdots\right)$$

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{1}{n^2}\left(\frac{-3}{4} + \frac{1}{2n} - \cdots\right)$$

So, given series diverges.

Ex : HYPERGEOMETRIC SERIES :-

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha \cdot (\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2 \cdot \gamma \cdot (\gamma+1)} x^2 + \cdots$$

for all positive values of x and $\alpha, \beta, \gamma > 0$.

Sol^n:- $1 + \sum u_n$ is a positive term series.

$$u_n = \frac{\alpha(\alpha+1)(\cdots(\alpha+n-1).\beta(\beta+1)\cdots(\beta+n-1)}{1\cdot 2\cdot 3\cdots n\,.\,\gamma.(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)} x^n$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{(n+1).(\gamma+n)}{(\alpha+n)(\beta+n)}\frac{1}{x} = \frac{1}{x} \qquad n=1,2,3,\ldots$$

$\therefore$ By ratio test, series converges for $x<1$, diverges for $x>1$ and test fails for $x=1$.

For $x=1$,

Gauss test :- $$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)} = \left(1+\frac{\gamma+1}{n}+\frac{\gamma}{n^2}\right)\left(1+\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}\right)^{-1}.$$

$$\frac{u_n}{u_{n+1}} = 1 + \frac{-\alpha-\beta+\gamma+1}{n} + \frac{(\alpha+\beta-1)(\alpha+\beta-\gamma)-\alpha\beta}{n^2} + \cdots$$

$$(1+x)^{-1} = 1-x+x^2-x^3+\cdots$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots$$

$\therefore$ Hence by Gauss test, for $x=1$, series converges for $1-\alpha-\beta+\gamma>1$ or $\gamma>\alpha+\beta$.

and diverges for $\gamma \le \alpha+\beta$.

Thus, for pos. α, β, γ, x, the hypergeometric series,

i) converges for $x<1$.

ii) diverges for $x>1$

iii) for $x=1$, converges for $\gamma>\alpha+\beta$, and diverges for $\gamma \le \alpha+\beta$.

★ Series with arbitrary terms :-

1.) **Alternating series** :- $u_1 - u_2 + u_3 - u_4 + \dots \pm u_n > 0$

⊙ **Liebnitz Test** :- If (i) $u_{n+1} < u_n$, $\forall n$

(ii) $\lim_{n\to\infty} u_n = 0$

then $\sum u_n$ converges.

Ex :- $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$ $(p>0)$

Sol^n :- $u_n = \frac{1}{n^p}$, $n = 1, 2, \dots$; $u_{n+1} \le u_n$ $\forall n$, $\lim_{n\to\infty} u_n = 0$

$\therefore$ By Liebnitz test, $\sum u_n$ converges.

Absolute convergence :- $\sum u_n$ is said to be absolutely convergent if $\sum |u_n|$ converges.

Result :- Every absolutely convergent series is convergent.

Ex :- Show that for any fixed 'x', the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is convergent.

Sol :- $u_n = \frac{\sin(nx)}{n^2}$. $|u_n| \le \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges.

$\therefore$ $\sum |u_n|$ converges, i.e., series $\sum u_n$ is absolutely convergent.

$\Rightarrow$ $\sum u_n$ converges.

Ex :- Show that the series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ converges absolutely for all 'x'.

$\rightarrow$ $|u_n| = \frac{|x|^n}{n!} < \frac{|x|^n}{2^{n-1}}$

$\rightarrow$

Taylor's Theorem :- (for finite Series)

If i) $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ are continous in $[a,b]$.

ii) $f^n(x)$ exists in (a,b)

then, $\exists\ \xi \in (a,b)$ such that.

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots + \frac{(b-a)^n}{n!} f^n(\xi).$$

$b = a+h$:-

$$f(a+h) = f(a) + h \cdot f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^n(a+\theta h)$$

$, \quad 0<\theta<1.$

$a=x$: $f(x+h) = f(x) + h \cdot f'(x) + \ldots + \frac{h^n}{n!} f^n(x+\theta h).$

If n^{th} term,

$$\frac{x^n}{n!} f^n(a+\theta h) \xrightarrow[(n\to\infty)]{} 0$$

$\forall\ x \in (c,d) \subseteq (a,b)$

then we have infinite series.

$$f(x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \ldots$$

with radius of convergence $(d-c)$.

$\rightarrow f(x) = e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} e^x \rightarrow 0$ as $n\to\infty$

$x \in (-\infty, \infty)$.

$\rightarrow f(x) = \sin x$

$f'(x) = \cos x = \sin\left(\frac{\pi}{2} + x\right)$

$f''(x) = \cos\left(\frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} \cdot 2 + x\right)$

$\vdots$

$f^n(x) = \sin\left(\frac{\pi}{2} \cdot n + x\right)$

$$\left| \frac{x^n}{n!} f^n(\theta x) \right| = \left| \frac{x^n}{n!} \sin\left(\frac{n\pi}{2} + \theta x\right) \right|.$$

⊙ Taylor's expansion around $x=0$:-

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \Rightarrow \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} (-1)^n ??$$

⊙ $\ln(1+x) = \cancel{x - x + \frac{x^2}{2}} \; x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Ex:- Show that $\lim_{n\to\infty} \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} \cdot x^n = 0$, where $|x|<1$, $m \in \mathbb{R}$.

Sol^n:- Consider the series $\sum u_n$, $u_n = \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^n$.

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \left|\frac{m-n}{n}\right| |x| = |x| < 1 \Rightarrow \sum |u_n| \text{ is convergent.}$$

$\Rightarrow \sum u_n$ absolutely converge $\Rightarrow \sum u_n$ converges.

$\therefore \lim_{n\to\infty} u_n = 0$.

*. Tests for arbitrary terms series:-

1.) Abel's test:-

If $b_n > 0$, monotonic function and if the series $\sum u_n$ is convergent, so then the series $\sum b_n u_n$ is convergent.

Ex:- Check convergence of the series.

$$0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - \cdots$$

consider the series $\sum u_n : 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots$ (1)

and the sequence $\{b_n\} : \left\{0, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \cdots\right\}$ (2)

$\because$ $u_{n+1} \le u_n$ and $\lim_{n\to\infty} u_n = 0$, by Leibnitz test, series $\sum u_n$ is convergent.

Again, $\{b_n\}$ is positive m, i.{monotonous increase}

$\therefore$ By Abel's test $\sum b_n u_n$ is convergent.

2) Dirichlet's test :-

If b_n is a positive monotonous decrease function with limit zero, and if, for the series Σu_n the sequence $\{S_n\}$ of the partial sums is bounded, the series $\Sigma b_n u_n$ is convergent.

Dirichlet's Theorem :- {Rearrangement of terms} :-

A series obtained from an absolutely convergent series by rearrangement of terms converge absolutely has the same sum as the original.

★. Series of function :- $\{f_n(x)\}$.

Consider the sequence $\{f_n(x)\}$, where f_n is defined as $I = [a, b]$.

→ Pointwise convergence :-

★ Abel's test :-

If i) $\{b_n\}$ is positive, monotonic and bounded sequence.
ii) Σu_n is convergent.
then $\Sigma b_n u_n$ is convergent.

★ Sequence & Series of function :-

$\{f_n(x)\}$, $x \in I \in [a, b]$.

⊙ Point-wise- convergence :-

The sequence $\{f_n(x)\}$ is said to be pointwise-convergence to $f(x)$ on $[a, b]$, if for each $\xi \in [a, b]$, the corresponding sequence $\{f_n(\xi)\}$ of numbers converges to $f(\xi)$.

Then, we write, $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in [a, b]$.

Scanned by CamScanner

• For pointwise convergence, the properties such as boundedness, continuity, integrability, differentiability etc are not always transferred to ~~some~~ sum function.

Ex: • Boundedness :- The G.P, $1+x+x^2+\dots$ converges to $(1-x)^{-1}$. since, $-1<x<1$. Each term is bounded but sum is not ~~always~~ bounded at $x=1$.

• Continuity :- Consider $\sum_{n=0}^{\infty} f_n(x)$, $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$.

At, $x=0$, $f_n(0) = 0 \ \forall n \Rightarrow f(0)=0$.

for $x \neq 0$, this is a G.P of common ratio, $\frac{1}{1+x^2}$, so that its sum function, $f(x) = \dfrac{a}{1-r} = \dfrac{x^2}{1-\frac{1}{(1+x^2)}} = 1+x^2$

$\therefore f(x) = 0, \ x=0$
$= 1+x^2, \ x \neq 0$

$\rightarrow f(x)$ is discontinous at $x=0$ but each $f_n(x)$ is continous.

• Differentiability :- $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$, $x \in \mathbb{R}$.

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = 0 \Rightarrow f'(x) = 0 \Rightarrow f'(0) = 0$

But, $f_n'(x) = \sqrt{n}\cos nx \Rightarrow f_n'(0) = \sqrt{n} \to \infty$ as $n \to \infty$.

Thus, at $x=0$, $\{f_n'(x)\}$ diverges, but the limit function exists, $f'(0)=0$.

$\Rightarrow$ limit of differentials is not equal to differential of limit.

• Integrability :- $f_n(x) = nx(1-x^2)^n$, $0 \le x \le 1$, $n = 1,2,3,\dots$

At, $x=0$, $f_n(0) = 0$, $\forall n \Rightarrow f(0) = 0$.

for $0 < x \le 1$, $\lim_{n\to\infty} f_n(x) = 0$, since $0 \le 1-x^2 < 1$.

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = 0$, $0 \le x \le 1$

$$\Rightarrow \int_0^t f_0(x)\,dx = 0, \qquad t>0.$$

Again, $$\int_0^t f_n(x)dx = \int_0^t nx(1-x^2)^n dx = -\frac{n}{2}\cdot\left[\frac{(1-x^2)^{n+1}}{n+1}\right]_{x=0}^{1} = \frac{n}{2n+2}.$$

$$\Rightarrow \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n}{2n+2} = \frac{1}{2}.$$

$$\Rightarrow \lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1\left[\lim_{n\to\infty} f_n(x)\right]dx.$$

$\therefore$ Limit of integral is not equal to integral of limit.

We thus, introduce the notion of uniform convergence.

⊙ Uniform Convergence:

A sequence of functions $\{f_n(x)\}$ is said to converge uniformly on $[a,b]$ to a function $f(x)$, if for any $\epsilon>0$, $\exists$ an integer N, depending on ϵ only (not dependent on x) such that, $\forall\, x \in [a,b]$, $|f_n(x)-f(x)|<\epsilon$, $\forall\, n\geq N$.

Pointwise-convergence:-

$\{f_n(\xi)\}$ converges to $f(\xi)$ for each $\xi\in[a,b]$ for pointwise convergence.

$x=\xi$, for any $\epsilon>0$, $\exists\, N(\xi)$ s.t.

$\forall\, \xi\in[a,b]$, $|f_n(\xi)-f(\xi)|<\epsilon$, $\forall\, n\geq N(\xi)$.

Note: Uniform convergence $\Rightarrow$ Pointwise convergence but not vice-versa.

Cauchy criteria for uniform convergence :-

$\{f_n(x)\}$ converges uniformly on $[a,b]$ iff for each $\epsilon>0$, and $\forall\ x \in [a,b]$, $\exists$ an integer N such that,

$$|f_{n+p}(x) - f_n(x)| < \epsilon \ , \forall\ n \geq N \ , \quad p \geq 1 .$$

★ Weierstrass M-test :-

A series of functions $\Sigma f_n(x)$ will converge uniformly (and absolutely) on $[a,b]$ if $\exists$ a convergent series ΣM_n of positive numbers such that,

$$|f_n(x)| \leq M_n \ , \forall\ x \in [a,b].$$

Ex:- Show that the series $\Sigma \frac{x}{n^p + x^2 n^q}$ converges uniformly over any finite interval $[a,b]$, where $q>0$, $p>1$.

Sol^n:-

$$|f_n(x)| = \left|\frac{x}{n^p + x^2 n^q}\right| < \frac{\alpha}{n^p}, \quad \alpha \geq \max\{|a|, |b|\}.$$

The series $\Sigma \frac{\alpha}{n^p}$ converges for $p \geq 1$.

Hence, by M-test, $\Sigma f_n(x)$ converges uniformly on $[a,b]$.

Note:- Fundamental properties like cont., diff, etc are usually transferred from series to function.

★ Power Series:-

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots = \sum_{n=0}^{\infty} a_n x^n$$

is called a power series (in x) and a_n's are coefficient.

→ If a definite positive number R exists such that $\Sigma a_n x^n$ converges for $|x|<R$ and diverges for $|x|>R$, then R is called radius of convergence of power series. and $(-R,R)$ is interval of convergence

The behavior of the series at $x = R$ depends entirely upon the character of the sequence $\{a_n\}$

Results :- If $\overline{\lim} |a_n|^{1/n} = \frac{1}{R}$, then $\sum a_n x^n$ is convergent for $|x| < R$ and diverges for $|x| > R$.

Proof :-

$$\overline{\lim_{n\to\infty}} |a_n x^n|^{1/n} = \frac{|x|}{R}$$

Hence, by Cauchy's root test, the series $\sum a_n x^n$ is absolutely for $|x|>R$ and divergent for $|x|>R$

Def^n:-

$$R = \frac{1}{\overline{\lim_{n\to\infty}} |a_n|^{1/n}}$$

$R = 0 \Rightarrow$ where convergent except at $x=0$.

$R = \infty \Rightarrow$ everywhere convergent.

Ex :- Find the radius of convergence,

1.) $1 + 2x + 3x^2 + \cdots + \cdots = \sum_{n=0}^{\infty} a_n x^n$, $a_n = n+1$

$$A = \overline{\lim_{n\to\infty}} |a_n|^{1/n} = \overline{\lim_{n\to\infty}} (n+1)^{1/n}.$$

$$\Rightarrow \ln(A) \Rightarrow \lim_{n\to\infty} \frac{1}{n} \ln(n+1) = \lim_{n\to\infty} \frac{1}{n+1} = 0 \Rightarrow A = 1.$$

$\therefore R = 1$.

2.) $$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2} \Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2} = 1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \cdots$$

$$R = \frac{1}{\overline{\lim} |a_n|^{1/n}} = \lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right|,$$ provided limit exists.

Ex :- Find the radius of convergence?

1.) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$; $R = \infty$. $\quad \lim_{n\to\infty} \frac{1}{((n-1)!)^{1/n}}$

2.) $1 + x + 2!x^2 + 3!x^3 + \cdots$; $R = 0$ $\quad \frac{1}{(n-1)^{1/n}(n-2)^{1/n}(n-3)^{1/n} \cdots 2^{1/n} \cdot 1}$

3.) $\frac{1}{2} \cdot x + \frac{1}{2} \cdot \frac{3}{5} x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 5 \cdot 8} x^3 + \cdots$; $R = 3/2$

4.) $x + \frac{x^2}{2^2} + \frac{2!\,x^3}{3^3} + \cdots$; $R = e$

5.) $\sum_{n=0}^{\infty} x$ $\quad \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1)}$ $\quad A \Rightarrow \frac{1}{n} \ln(1 \cdot 2 \cdots (n-1))$ $\quad A = (n-1)$ $\quad \ln(A) = \frac{1}{n} \ln(n-1) = \frac{1}{n-1}$ $\quad A = 1$

SERIES Solⁿ OF 2^{nd} ORDER LINEAR ODE :-

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x).y = 0 \quad \text{---} (1)$$

$x \in (a,b)$.

We will find a series solⁿ of (1) around $x = x_0$, $\{x_0 \in (a,b)\}$. in the form :-

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{\rho + n}, \quad \rho \text{ may be '0' some cases.}$$

Nature of the series (i.e., whether $\rho = 0$ or $\rho \neq 0$) depends on whether pt $x = x_0$ is an ordinary pt. or singular pt.

→ Ordinary Pt. :-

A point $x = x_0$ is ordinary pt. of (1) if it is not a pole of $p(x)$ and $q(x)$, otherwise it is singular pt.

Singular Pt.

Regular

A singular point $x = x_0$ is called regular singular point if it is a pole of $p(x)$ order $\not> 1$, and is a pole of $q(x)$ of order $\not> 2$, otherwise it is irregular.

Irregular.

{we will not find the solⁿ around the pt.}

1.) $(1 - x^2)y'' - xy' + 4y = 0$

$p(x) = \frac{-x}{1-x^2} \Rightarrow \frac{1}{p(x)} = 0 \Rightarrow x = \pm 1$ of order 1 each.

$q(x) = \frac{4}{1-x^2} \Rightarrow \frac{1}{q(x)} = 0 \Rightarrow x = \pm 1$, of order 1 each.

$\therefore x = \pm 1$ are regular singular pt.

To check whether $x = \infty$ is singular pt, we have to substitute $x = \frac{1}{\xi}$, and check whether $\xi = 0$ is singular pt. of transformed ODE.

$\Rightarrow \quad x = \frac{1}{\xi} \Rightarrow \frac{dx}{d\xi} = \frac{-1}{\xi^2}$; $\frac{dy}{dx} = \frac{dy}{d\xi} \cdot \frac{d\xi}{dx} = -\xi^2 \frac{dy}{d\xi}$;

$$\frac{d^2y}{dx^2} = \frac{d}{d\xi}\left(\frac{dy}{dx}\right) \cdot \frac{d\xi}{dx} = -\xi^2 \frac{d}{d\xi}\left(-\xi^2 \frac{dy}{d\xi}\right) = \xi^2\left[\xi^2 \frac{d^2y}{d\xi^2} + 2\xi \frac{dy}{d\xi}\right]$$

ODE (1) transforms to :-

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} - \frac{1}{\xi}\left(-\xi^2 \frac{dy}{d\xi}\right) + 4y = 0$$

$$\Rightarrow \quad \frac{d^2y}{d\xi^2} + \left(\frac{2\xi^2+1}{\xi^3}\right)\frac{dy}{d\xi} + \frac{4y}{\xi^4} = 0$$

$p(\xi) = \frac{2\xi^2+1}{\xi^3} \Rightarrow \xi = 0$ is a pole of $p(\xi)$ of order 3.

→ Now we will learn how to find sol^n around ordinary pt :-

$$(1-x^2)y'' - xy' + 4y = 0. \quad (1)$$

Suppose we want a sol^n around $x=0$.

Assume sol^n:- $y \Rightarrow \sum_{r=0}^{\infty} a_r x^r$. (2)

Substituting (2) into (1),

$$(1-x^2)\sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - x\sum_{r=0}^{\infty} r a_r x^{r-1} + 4\sum_{r=0}^{\infty} a_r x^r = 0 \quad (3)$$

Remember, (3) is identity $\Rightarrow$ coeff. of x^r must be '0' for all 'r'.

($\because$ Try to find relation b/w a_r's.)

$$(1-x^2)y'' - xy' + 4y = 0 \quad \text{—— (1)}$$

$x=0$, is an ordinary pt.

Assume, $y = \sum_{r=0}^{\infty} a_r x^r$, $a_0 \neq 0$. —— (2)

Substitute (2) into (1),

$$(1-x^2)\sum a_r r(r-1)x^{r-2} - x\sum_{r=0}^{\infty} r a_r x^{r-1} + 4\sum_{r=0}^{\infty} a_r . x^r = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r . r(r-1)x^{r-2} + \sum_{r=0}^{\infty} a_r[4-r-r(r-1)]x^r = 0$$

$$\Rightarrow \sum_{r=2}^{\infty} a_r . r(r-1)[x^{r-2} - x^r] - \sum_{r=1}^{\infty} r a_r x^r + 4\sum_{r=0}^{\infty} a_r x^r = 0.$$

{ put, $r' = r-2$, for 1st term
, $r'' = r-1$ for 2nd term }

$$\Rightarrow \sum_{r'=0}^{\infty} a_{r'+2}(r'+2)(r'+1)[x^{r'} - x^{r'+2}] - \sum_{r''=0}^{\infty}(r''+1)a_{r''+1}x^{r''+1} + 4\sum_{r=0}^{\infty} a_r x^r = 0.$$

[Dropping prime] $\to$ (′)

$$\Rightarrow \sum_{r=0}^{\infty}\left\{(r+2)(r+1)a_{r+2}x^{r+2} - (r+1)a_{r+1}x^{r+1} + [4+(r+1)(r+2)]a_r x^r\right\} = 0$$

⁂ Coefficients of x^0:-

$2a_2 + 4a_0 = 0 \Rightarrow a_2 = -2a_0$, $a_0 \neq 0$.

coefficient of x^1:-

$-a_1 + 6a_3 + 4a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{2}$

coefficient of x^r:-

$r = 2, 3, 4, \ldots$ → replace $r \to r-2$ in 1st term & $r \to r-1$ in 2nd term.

$$-r(r-1)a_r - ra_r + (r+2)(r+1)a_{r+2} + 4a_r = 0.$$

$$a_{r+2} = \frac{r^2-4}{(r+2)(r+1)}a_r \Rightarrow \boxed{a_{r+2} = \frac{r-2}{r+1}.a_r} \quad r = 2, 3, \ldots$$

General sol^n is :-

$$y(x) = a_0\left[1+\frac{a_2}{a_0}x^2+\frac{a_4}{a_0}x^4+\cdots\right] + a_1\left[x+\frac{a_3}{a_1}x^3+\cdots\right]$$

Ex6. Find two linearly independent sol^n and hence write the general sol^n around $x=0$ of the eq^n :-

$$(1-x^2)y'' - xy' + 4y = 0$$

Sol^n :- Assume sol^n : $y = \sum_{r=0}^{\infty} a_r x^r$, $a_0 \neq 0$

[$x=0$ is ordinary point].

$$a_{r+2} = \frac{r-2}{r+1}a_r, \quad r=0,1,\ldots$$

$r=0$: $a_2 = -2a_0$, $r=3$: $a_5 = \frac{1}{4}\cdot\left(-\frac{1}{2}\right)a_1$

$r=1$: $a_3 = -\frac{1}{2}a_1$.

$\therefore$ sol^n in $y = a_0\left[1+\frac{a_2}{a_0}x^2+\frac{a_4}{a_0}x^4+\cdots\right] + a_1\left[x+\frac{a_3}{a_1}x^3+\frac{a_5}{a_1}x^5+\cdots\right]$

$$y = a_0[1-2x^2] + a_1\left[x-\frac{x^3}{2}-\frac{x^5}{8}-\cdots\right]$$

$$y = a_0 y_1(x) + a_2 y_2(x)$$

where, $y_1(x) = 1-2x^2$

$y_2(x) = x - \frac{x^3}{2} - \frac{x^5}{8} - \cdots$

are 2 linear independent sol^n.

$\therefore$ Frobenius Method. (Sol^n around regular singular pt.).

Indicial Eq^n :-

coeff. of $x^{\rho-1}$: $9\rho(\rho-1) - 12\rho = 0$

$$\Rightarrow \rho\,(3\rho-7) = 0$$

Two values of ρ do not differ by integer $\Rightarrow \boxed{\rho = 0, 7/3}$

$$3\sum a_r(\rho+r)(3\rho+3r-7)x^{\rho+r-1} - \sum a_r[9(\rho+r)(\rho+r-1)-4]x^{\rho+r} = 0$$

x^{ρ}: $3a_1(\rho+1)(3\rho-4) = a_0[9\rho(\rho-1)-4]$

$x^{\rho+r}$: $3a_{r+1}(\rho+r+1)(3\rho+3r-4) = a_r[9(\rho+r)(\rho+r-1)-4]$

$r=0,1,2,\dots$

$\therefore$ Next we put $\rho=0$ and $\rho=7/3$.

★ FROBENIUS METHOD

→ $9x(1-x)y'' - 12y' + 4y = 0$ —— (1)

$x=0$ is regular singular pt.

Assume solⁿ: $y = \sum_{r=0}^{\infty} a_r x^{\rho+r}$, $a_0 \neq 0$ —— (2).

ρ is called exponent.

Substituting (2) in (1):-

→ $9x(1-x)\sum_{r=0}^{\infty} a_r(\rho+r)(\rho+r-1)x^{\rho+r-2} - 12\sum_{r=0}^{\infty} a_r(\rho+r)x^{\rho+r-1} + 4\sum_{r=0}^{\infty} a_r x^{\rho+r} = 0.$

→ $9\sum(\rho+r)(\rho+r-1)[x^{\rho+r-1} - x^{\rho+r}]$ ...

→ $\sum[-9(\rho+r)(\rho+r-1)+4]a_r x^{\rho+r} - \sum[9(\rho+r)(\rho+r-1) - 12(\rho+r)]a_r x^{\rho+r-1} = 0$

→ $3\sum a_r(\rho+r)(3\rho+3r-7)x^{\rho+r-1} - \sum a_r[9(\rho+r)(\rho+r-1)-4]x^{\rho+r} = 0$ —(3)

Indicial Eqⁿ:-

coefficient of lowest power of x

$r=0$ i.e., $x^{\rho-1}$: $\rho(3\rho-7)=0$

$\rho = 0, 7/3$

which do not differ by integer.

$(r=1)$

Next:- x^{ρ}:

$3a_1(\rho+1)(3\rho-4) = a_0 \cdot [9\rho(\rho-1)-4]$

Next, $x^{\rho+r}$: $3a_{r+1}(\rho+r+1)(3\rho+3r-4) = a_r\left[9(\rho+r)(\rho+r-1)-4\right]$

$\therefore \rho = 0$ $\quad r = 0,1,2,\ldots \quad (4)$.

From (4),

$$a_{r+1} = \frac{1}{3}\,\frac{9r(r-1)-4}{(r+1)(3r-4)}\,a_r \quad ; \; r = 0,1,\ldots \quad (5).$$

$\therefore a_1 = \frac{1}{3}a_0$; $a_2 = \frac{2}{3}\cdot\frac{1}{3}a_0$;

one solution is :-

$$y_1(x) = 1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \ldots = 1 + \frac{1}{3}x + \frac{2}{9}x^2 + \ldots$$

$\therefore \rho = 7/3$

From (4), $a_{r+1} = \frac{3r+8}{3r+10}\cdot a_r \quad ; \; r = 0,1,2,\ldots$

$\therefore a_1 = \frac{4}{5}a_0 \quad ; \quad a_2 = \frac{8}{10}\cdot\frac{11}{13}a_0$

Another sol^n is :-

$$y_2(x) = x^{7/3}\left[1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \ldots\right] = x^{7/3}\left[1 + \frac{8x}{10} + \frac{8\cdot 11}{10\cdot 13}x^2 + \ldots\right]$$

$$\boxed{y(x) = A\,y_1(x) + B\,y_2(x)}$$

★ FROBENIUS RULES :-

Rule 1 :- Sol^n of a second order linear ODE around a regular singular point, say $x=0$ is of the form.

$$y = \sum_{r=0}^{\infty} a_r x^{\rho+r}, \quad a_0 \neq 0,$$

where ρ is a rational number, called exponent of ODE at $x=0$. ρ has always 2 values ρ_1, ρ_2.

Rule 2 :- Two cases to be considered separately :

Case 1 :- $\rho_1 \neq \rho_2$ and $|\rho_1 - \rho_2| \neq$ integer.

In this case two linearly independent sol^n are

$$y_{\rho_i}(x) = \sum_{r=0}^{\infty} a_r x^{\rho_i + r}, \quad i = 1,2,\ldots$$

Case 2 :- $|\rho_1 - \rho_2| =$ integer

In this case, one solⁿ is $y_1(x) = \sum_{r=0}^{\infty} a_r x^{\rho_1 + r}$;

where either $\rho_1 = \rho_2$ or ρ_1 is lowest.

Other solⁿ is :-

$$y_2(x) = \left.\frac{\partial y}{\partial \rho}\right|_{\rho=\rho_1} , \text{ where } y = \sum_{r=0}^{\infty} a_r \cdot x^{\rho + r}.$$

Ex :- Find two linearly independently solⁿ of :-

Solⁿ :- $xy'' + y' + mxy = 0$, around $x = 0$.

$x = 0$ is regular singular point.

Assume solⁿ : $y = \sum_{r=0}^{\infty} a_r x^{\rho + r}$, $a_0 \neq 0$.

$\rightarrow x \cdot \sum_{r=0}^{\infty} a_r(\rho+r)(\rho+r-1)x^{\rho+r-2} + \sum_{r=0}^{\infty} a_r(\rho+r)x^{\rho+r-1} + mx \sum_{r=0}^{\infty} a_r x^{\rho+r} = 0$

$\rightarrow$

$\sum a_{r+2}(\rho+r+2)(\rho+r+1)x^{\rho+r+1} + \sum a_{r+2}(\rho+r+2)x^{\rho+r+1} + m \sum_{r=0}^{\infty} a_r x^{\rho+r+1} = 0$

$\rightarrow \sum \left[a_{r+2}(\rho+r+2)^2 + m a_r\right] x^{\rho+r+1} = 0.$

$\sum a_r (\rho+r)^2 x^{\rho+r-1} + m \sum a_r x^{\rho+r+1} = 0.$

Indicial eqⁿ :- coeff. of $x^{\rho-1}$: $\rho^2 = 0 \Rightarrow \rho = 0, 0$ equal values.

— " — x^{ρ} : $a_1 (\rho+1)^2 = 0 \Rightarrow a_1 = 0$ $[\because \rho + 1 \neq 0]$

$x^{\rho+r-1}$: $a_r = -\frac{m}{(\rho+r)^2} a_{r-2}$; $r = 2, 3, \ldots$

$\therefore a_1 = a_3 = a_5 = \ldots = 0.$

$a_2 = -\frac{m}{(\rho+2)^2} a_0$; $a_4 = (-1)^2 \frac{m^2}{(\rho+2)^2(\rho+4)^2} a_0$;

$a_6 = (-1)^3 \frac{m^3}{(\rho+2)^2(\rho+4)^2(\rho+6)^2} \cdot a_0 \Rightarrow$

$$\boxed{y(x) = a_0 x^{\rho}\left[1 + (-1)^1 \frac{m}{(\rho+2)^2} x^2 + (-1)^2 \frac{m^2}{(\rho+2)^2(\rho+4)^2} x^4 + \cdots\right]} \longrightarrow (7)$$

For $\rho = 0$, one sol^n is :-

$$y_1(x) = 1 + (-1)^1 \frac{m}{2^2} \cdot x^2 + (-1)^2 \frac{m^2}{2^2 \cdot 4^2} x^4 + (-1)^3 \cdot \frac{m^3}{2^2 \cdot 4^2 \cdot 6^2} \cdot x^6 + \cdots$$

$\therefore$ Other sol^n is :-

$$y_2(x) = \left.\frac{\partial y}{\partial \rho}\right|_{\rho=\rho_1=0}, \text{ where } y = \cancel{\sum_{r=0}^{\infty} a_r x^{\rho+r}} \quad \text{---} (7).$$

$$\cancel{\frac{\partial y}{\partial \rho} = \sum_{r=0}^{\infty} a_r (\rho+r) x^{\rho+r-1}}$$

$$\cancel{\left.\frac{\partial y}{\partial \rho}\right|_{\rho=0} = \sum_{r=0}^{\infty} a_r \cdot r \rho^{r-1}}$$

$$\cancel{y_2(x) = \sum_{r=0}^{\infty} a_r \cdot r \rho^{r-1}}.$$

$$\frac{\partial y}{\partial \rho} = a_0 x^\rho \ln(x) \left[1 + (-1)^1 \frac{m}{(\rho+2)^2} x^2 + (-1)^2 \cdot \frac{m^2}{(\rho+2)^2 (\rho+4)^2} x^4 + \cdots \right]$$

$$+ a_0 x^\rho \left[(-1)^1 \frac{m}{(\rho+2)^2} \cdot \left(\frac{-2}{\rho+2}\right) x^2 + (-1)^2 \cdot \frac{m^2}{(\rho+2)^2 (\rho+4)^2} \left\{ \frac{-2}{\rho+2} - \frac{2}{\rho+4} \right\} + \cdots \right]$$

$$\therefore \left[z = \frac{1}{(\rho+2)^2} = \ln(z) \Rightarrow -2 \ln(\rho+2) \Rightarrow \frac{1}{z} \cdot \frac{\partial z}{\partial \rho} = \frac{-2}{\rho+2} \Rightarrow \frac{\partial z}{\partial \rho} \right]$$

$$\frac{-2}{(\rho+2)^2} \cdot \frac{1}{(\rho+2)}$$

e.g., Putting, $\rho = 0$, another sol^n is :-

$$y_2(x) = \ln(x) \cdot y_1(x) - \sum_{n=1}^{\infty} (-1)^n \cdot \frac{m^n \cdot x^{2n}}{(n!)^2 \cdot 2^{2n}} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

where, $y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot m^n \cdot x^{2n}}{(n!)^2 \cdot 2^{2n}}$.

→ Above problem is case of ρ = same.

Ex:- $2x^2y'' + xy' - (1+x^2)y = 0$; sd^n around $x=0$ {regular singular point}.

Assume sol^n : $y = \sum_{r=0}^{\infty} a_r . x^{\rho+r}$, $a_0 \neq 0$.

$$\Rightarrow 2\sum_{r=0}^{\infty} a_r(\rho+r)(\rho+r-1)x^{\rho+r} + \sum_{r=0}^{\infty} a_r(\rho+r).x^{\rho+r} - (1+x^2)\sum a_r x^{\rho+r} = 0$$

$$\Rightarrow \sum\left[2a_r(\rho+r)(\rho+r-1) + a_r(\rho+r) - a_r\right]x^{\rho+r} - \sum a_r x^{\rho+r+2} = 0$$

$$\Rightarrow \sum\left[a_r(\rho+r)(2\rho+2r-1) - a_r\right]x^{\rho+r} - \sum a_{r-2}x^{\rho+r} = 0$$

$$\Rightarrow a_r(\rho+r)(2\rho+2r-1) - a_r - a_{r-2} = 0$$

$$\Rightarrow a_r \Rightarrow \frac{1}{[(\rho+r)(2\rho+2r-1) - 1]} . a_{r-2} \quad ; r = 2, 3, \dots$$

Indicial eq^n:- x^ρ : $\rho = 1, -1/2$, two values do not differ by integer.

$x^{\rho+1}$: $a_1 = 0$.

$x^{\rho+r}$:

$$a_r = \frac{a_{r-2}}{(\rho+r-1)(2(\rho+r)+1)} \quad ; r = 2, 3, \dots$$

$\therefore a_1 = a_3 = a_5 = \dots = 0.$

$$a_2 = \frac{a_0}{(\rho+1)(2\rho+5)} \xrightarrow[\rho=-1/2]{\rho=1} \begin{cases} a_0/2\times7 \\ a_0/\frac{1}{2}\times4 \end{cases}$$

$$a_4 = \frac{a_2}{(\rho+3)(2\rho+9)} \xrightarrow[\rho=-1/2]{\rho=1} \begin{cases} \dfrac{a_0}{2.4.7.11} \\ \dfrac{a_0}{\frac{1}{2}.\frac{5}{2}.4.8} \end{cases}$$

The l.i sol are.

$$\therefore y_1(x) = x\left[1 + \frac{x^2}{2.7} + \frac{x^4}{2.7.4.11} + \dots\right]$$

$$y_2(x) = x^{-1/2}\left[1 + \frac{x^2}{\frac{1}{2}.4} + \dots\right]$$

$$y(x) = A\,Y_1(x) + B\,Y_2(x).$$

Legendre's Equation:-

$$(1-x^2)\,y'' - 2xy' + n(n+1)y = 0 \quad , \quad n \in \mathbb{R}, \; x \in (-1,1) \qquad \text{——— (i)}$$

$\therefore$ n is called degree of legendre eqn.

$x = \pm 1$ are two regular singular pts.

$x = \infty$ is also a regular singular pt. [show it ??].

→ Solution around $x=0$ (ordinary pt.)

Assume soln as $y = \sum_{r=0}^{\infty} a_r x^r$; $a_0 \neq 0$.

Substitute in eqn (i).

$$(1-x^2) \sum a_r \cdot r(r-1)x^{r-2} - 2x \sum a_r \cdot r x^{r-1} + n(n+1) \sum a_r x^r = 0$$

$$\sum a_r \cdot r(r-1)x^{r-2} - 2\sum a_r r \cdot x^r + \sum \left[n(n+1)a_r - a_r r(r-1) \right] x^r = 0$$

$$\rightarrow \sum a_{r+2} (r+2)(r+1)x^r - 2\sum a_r \cdot r \cdot x^r + \sum \left[a_r x^r \left(n(n+1) - r(r-1) \right) \right] = 0$$

$$\Rightarrow (r+1)(r+2)a_{r+2} - 2r \cdot a_r + a_r \left[n(n+1) - r \cdot (r-1) \right] = 0$$

$$\Rightarrow (r+1)(r+2)a_{r+2} + a_r \left[n(n+1) - r(r-1) - 2r \right] = 0 \qquad -r^2+r-2r = -r^2-r$$

$$\Rightarrow a_{r+2} \Rightarrow -\frac{\left[n(n+1) - r \cdot (r+1) \right]}{(r+1)(r+2)} a_r$$

$$\boxed{a_{r+2} \Rightarrow \frac{(r-n)(r+n+1)}{(r+1)(r+2)} a_r} \; ; \quad r = 0, 1, 2, \dots$$

$$\therefore \frac{a_2}{a_0} = -\frac{n(n+1)}{1\cdot 2} , \quad \frac{a_4}{a_0} = \frac{n(n-2)(n+1)(n+3)}{1\cdot 2\cdot 3\cdot 4}$$

soln:- $y(x) = a_0\left[1 + \frac{a_2}{a_0}x^2 + \frac{a_4}{a_0}x^4 + \cdots \right] + a_1\left[x + \frac{a_3}{a_1}x^3 + \cdots \right]$

$y(x) = a_0 y_1(x) + a_1 y_2(x) \longrightarrow$ General soln.

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+3)(n-2)}{4!}x^4 - \cdots \equiv P_n(x)$$

Legendre function of I^{st} kind

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n+3)(n+4)}{5!}x^5 - \cdots \equiv Q_n(x)$$

Legendre function of 2^{nd} Kind.

H.W. Show that $P_n(x)$ is convergent and hence find radius of convergent and $Q_n(x)$ is divergent under some specific condition (or in general).

★ Hypergeometric Equation:-

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0 \quad \text{—— (1)}$$

$x = 0, 1, \infty$ are regular singular point.

Sol^n around $x=0$:-

$\gamma - (\alpha+\beta+1)(1-\xi)$
$\gamma - (\alpha+\beta+1) + (\alpha+\beta+1)\xi$

Assume sol^n $y(x) = \sum_{r=0}^{\infty} a_r x^{\rho+r}$, $a_0 \neq 0$

$$\rightarrow x(1-x)\sum_{r=0}^{\infty} a_r(\rho+r)(\rho+r-1)x^{\rho+r-2} + [\gamma - (\alpha+\beta+1)x]\sum_{r=0}^{\infty} a_r(\rho+r)x^{\rho+r-1} - \alpha\beta\sum a_r x^{\rho+r} = 0$$

$$\sum a_r(\rho+r)(\rho+r-1+\gamma)x^{\rho+r-1} = \sum a_r(\rho+r+\alpha)(\rho+r+\beta)x^{\rho+r} \quad \text{— (3)}$$

Indicial eq^n: $x^{\rho-1} = a_0\rho(\rho-1+\gamma) = 0$

$$\rho = 0,\ 1-\gamma$$

coefficient of $x^{\rho-1} = 0$ ⟶ lowest power of x.

Case I:- $1-\gamma \neq$ integer or zero.

$$x^\rho : \quad a_1(\rho+1)(\rho+\gamma) = a_0(\rho+\alpha)(\rho+\beta)$$

$$\rightarrow a_1 = \frac{(\rho+\alpha)(\rho+\beta)}{(\rho+1)(\rho+\gamma)} a_0$$

$x^{\rho+r}$: $\quad a_{r+1} = \dfrac{(\rho+r+\alpha)(\rho+r+\beta)}{(\rho+r+1)(\rho+r+\gamma)} \cdot a_r \quad ; \quad r = 0, 1, 2, \ldots \longrightarrow (4)$

$\underline{\rho = 0}$

$\therefore a_1 = \dfrac{\alpha \cdot \beta}{1 \cdot \gamma} a_0 \quad ; \quad a_2 = \dfrac{\alpha(\alpha+1)\beta \cdot (\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} a_0, \ldots$

One sol^n around $x = 0$ is

$y_1(x) \equiv F(\alpha, \beta; \gamma; x) = 1 + \dfrac{\alpha\beta}{1 \cdot \gamma} x + \dfrac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma \cdot (\gamma+1) \cdot 1 \cdot 2} x^2 + \ldots$

This is hypergeometric series, absolutely convergent for $|x| < 1$.

$\underline{\rho = 1-\gamma}$

from (4), $\quad a_{r+1} = \dfrac{(r+1-\gamma+\alpha)(r+1-\gamma+\beta)}{(r+2-\gamma)(r+1)} a_r \quad ; \quad r = 0, 1, 2, \ldots \hookrightarrow (5)$

compare (5) with (4) :-

(4) for $\rho = 0$ $\xrightarrow[\gamma \to 2-\gamma]{\alpha \to 1-\gamma+\alpha, \ \beta \to 1-\gamma+\beta}$ (5)

Thus, another sol^n around $x = 0$ is

$y_2(x) = x^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta, 2-\gamma, x)$.

Sol^n around $x = 1$:-

put $1 - x = \xi$ in (1) $\Longrightarrow$

$$\xi(1-\xi)\frac{d^2y}{d\xi^2} + \left[(\alpha+\beta+1-\gamma) - (\alpha+\beta+1)\xi\right]\frac{dy}{d\xi} - \alpha\beta y = 0 \quad \text{---} (2).$$

Now compare (2) with (1) :-

here replace $\gamma \to (\alpha+\beta+1-\gamma)$

$\therefore$ sol^n of (1) around $x = 1$ are :-

$y_3(x) = F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x)$

$y_4(x) = (1-x)^{\gamma-\alpha-\beta} \cdot F(\gamma-\beta, \gamma-\alpha, \gamma-\alpha-\beta+1, 1-x)$

$\rightarrow$ provided $\gamma - \alpha - \beta \neq$ integer or 0.

Solⁿ around $x = \infty$.

(1) $\xrightarrow{x = \frac{1}{\xi}}$ $\xi^2(\xi-1)\frac{d^2y}{d\xi^2} + \xi\left[(2-\nu)\xi + (\alpha+\beta-1)\right]\frac{dy}{d\xi} - \alpha\beta y = 0$ — (3)

Assume solⁿ of (3) around ~~$x=0$~~ $\xi = 0$ in the form

$$y = \sum_{r=0}^{\infty} a_r \xi^{\rho+r}, \quad a_0 \neq 0. \quad — (4).$$

Substitute (4) into (3) :-

$$\sum a_r(\rho+r)(\rho+r-1)\left[\xi^{\rho+r+1} - \xi^{\rho+r}\right] + \sum a_r(\rho+r)\left[(2-\nu)\xi^{\rho+r+1} + (\alpha+\beta-1)\xi^{\rho+r}\right]$$
$$- \alpha\beta \sum a_r \xi^{\rho+r} = 0.$$

Indicial Eqⁿ:-

ξ^{ρ} : $\rho^2 - \rho(\alpha+\beta) + \alpha.\beta = 0 \Rightarrow \rho = \alpha, \beta$

Assume, $\alpha - \beta \neq$ integer or zero.

We now put $y = \xi^{\alpha} w(\xi)$ in (3).

$$\xi(1-\xi)\frac{d^2w}{d\xi^2} + \left[(1+\alpha-\beta) - \{\alpha + (\alpha-\nu+1) + \}\xi\right]\frac{dw}{d\xi} - \alpha(\alpha-\nu+1)w = 0$$ — (4)

→ compare (1) with (4) :

$$(1) \xrightarrow{\beta \to \alpha-\nu+1,\ \gamma \to 1+\alpha-\beta} (4)$$

∴ solⁿ of (4) around $\xi = 0$.

$w_1 = F(\alpha, \alpha-\nu+1, 1+\alpha-\beta, \xi)$

$w_2 = \xi^{\beta-\alpha} \cdot F(\beta, 1+\beta-\nu, 1+\beta-\alpha, \xi)$

Since, $y = \xi^{\alpha} . w$, $\xi = 1/x$

solⁿ of (1) around $x = \infty$ are

$$y_5 = \frac{1}{x^{\alpha}} \cdot F(\alpha, \alpha-\nu+1, 1+\alpha-\beta, 1/x)$$

$$y_6 = \frac{1}{x^{\beta}} \cdot F(\beta, 1+\beta-\nu, 1+\beta-\alpha, 1/x)$$

∴ provided $\beta - \alpha \neq$ integer or zero.

Scanned by CamScanner

Bessel Equation:-

Bessel equation of order ν.

$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \qquad (\nu \geq 0) \quad —(1).$$

$x=0$ is a regular singular point.

Solution around $x=0$,

Assume solⁿ : $y(x) = \sum_{r=0}^{\infty} a_r x^{\rho+r}$; $a_0 \neq 0$ —(2).

$\rightarrow x^2 \cdot (\rho+r)(\rho+r-1) \sum a_r x^{\rho+r-2} + x(\rho+r) \sum a_r x^{\rho+r-1} + (x^2-\nu^2) \sum a_r x^{\rho+r} = 0$

$\rightarrow \sum (\rho+r)(\rho+r-1) a_r x^{\rho+r} + \sum (\rho+r) a_r x^{\rho+r} - \nu^2 \sum a_r x^{\rho+r} + \sum a_r x^{\rho+r+2} = 0$

$\rightarrow \sum \left((\rho+r)(\rho+r-1) + (\rho+r) - \nu^2\right) a_r x^{\rho+r} + \sum a_r x^{\rho+r+2} = 0.$

$\rightarrow \sum \left[(\rho+r)^2 - \nu^2\right] a_r x^{\rho+r} + \sum a_r x^{\rho+r+2} = 0$

$\rightarrow$ Indicial eqⁿ:-

x^ρ : $\rho^2 - \nu^2 = 0 \Rightarrow \rho = \pm\nu.$

Case I :- $\nu \neq n/2$, $n = 0, 1, 2, \ldots\ldots$

$x^{\rho+1}$: $r=1$. $\quad [(\rho+1)^2 - \nu^2] a_1 x^{\rho+1} = 0$

$\Rightarrow a_1 = 0.$

Recurrence relⁿ:-

$x^{\rho+r}$: $[-(\rho+r)^2 + \nu^2] a_r = a_{r-2}$; $r = 2, 3, \ldots$

$\therefore a_1 = a_3 = a_5 = \ldots\ldots = 0.$

and, $a_2 = \dfrac{a_0}{\nu^2 - (\rho+2)^2}$; $a_4 = \dfrac{a_0}{(\nu^2 - (\rho+4)^2)(\nu^2 - (\rho+2)^2)}$

$$a_6 = \frac{a_0}{[\nu^2 - (\rho+6)^2][\nu^2 - (\rho+4)^2][\nu^2 \pm (\rho+2)^2]}.$$

simplifying,
$$a_2 = \frac{a_0}{(\nu+\rho+2)(\nu-\rho-2)}$$

$$a_4 = \frac{a_0}{(\nu+\rho+2)(\nu+\rho+4)(\nu-\rho-2)(\nu-\rho-4)}.$$

→ for $\rho = +\nu$,
$$a_2 = -\frac{a_0}{4\cdot(\nu+1)}$$

$$a_4 = +\frac{a_0}{4\cdot 8\cdot(\nu+1)(\nu+2)}$$

Thus, $a_2, a_4, a_6,$ etc. are all finite provided $\nu \neq -1, -2, \ldots$.

→ for $\rho = -\nu$,
$$a_2 = -\frac{a_0}{4\cdot\cancel{(\nu+1)}(1-\nu)}$$

$$a_4 = \frac{a_0}{4\cdot 8\cdot(1-\nu)(2-\nu)}$$

Thus, $a_2, a_4, a_6 \ldots$ all finite points $\nu \neq 1, 2, 3, \ldots$

→ Thus, when ν is not an integer or zero, then two linearly independent solⁿ are :-
$$y_1(x) = x^{\nu}\left[1 - \frac{x^2}{4(\nu+1)} + \frac{x^4}{4\cdot 8\cdot(\nu+1)(\nu+2)} - \ldots\ldots\right] \quad \text{---} (5)$$

$$y_2(x) = x^{-\nu}\left[1 - \frac{x^2}{4(1-\nu)} + \frac{x^4}{4\cdot 8\cdot(1-\nu)(2-\nu)} - \ldots\ldots\right] \quad \text{---} (6)$$

→ **Bessel Eqⁿ of order ν:**

$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \quad ; \quad \nu \geq 0 \quad \text{---} (1)$$

Solution around $x=0$ (regular singular point).

$$y(x) = \sum_{r=0}^{\infty} a_r x^{\rho+r}, \quad a_0 \neq 0 \Rightarrow \text{Indicial eq}^n :- \rho^2-\nu^2 = 0 \text{ ---}(2)$$

$$\rho = \pm\nu.$$

$$a_r = \frac{a_{r-2}}{\nu^2-(\rho+r)^2} \quad ; \quad r = 2, 3, \ldots$$

We see that $a_1 = 0$ then $a_3 = a_5 = \cdots = 0$.
So only even power of x exists.

$\rho = \nu$: $$y_1(x) = x^{\nu}\left[1 - \frac{x^2}{4\cdot(\nu+1)} + \frac{x^4}{4\cdot 8\cdot(\nu+1)(\nu+2)} - \cdots\right] \quad —(3)$$

$\rho = -\nu$: $$y_2(x) = x^{-\nu}\left[1 - \frac{x^2}{4\cdot(1-\nu)} + \frac{x^4}{4\cdot 8\cdot(1-\nu)(2-\nu)} - \cdots\right] \quad —(4)$$

Above are two linearly independent sol^n provided ν is not an integer or zero.

⊙ Bessel function of first kind :-

$$J_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\nu+r+1)} \left(\frac{x}{2}\right)^{2r+\nu} ; \quad \nu \geq 0$$

So, we obtain general sol^n as,
$$y(x) = A\cdot y_1(x) + B\cdot y_2(x).$$

$B = 0$; $A = \dfrac{1}{2^\nu\,\Gamma(\nu+1)} \Rightarrow y(x) = J_\nu$ [Prove, H.W].

$A = 0$; $B = \dfrac{1}{2^\nu\,\Gamma(\nu+1)} \Rightarrow y(x) = J_{-\nu}(x)$ [Prove, H.W].

Thus, when ν is not an integer or zero, G.S is,
$$y(x) = C\cdot J_\nu(x) + D\cdot J_{-\nu}(x).$$

Ex:- Show that $J_\nu = (-1)^\nu J_{-\nu}$, if ν is integer

Case I:- $\nu \to$ positive integer

$$J_{-\nu} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-\nu+r+1)} \cdot \left(\frac{x}{2}\right)^{2r-\nu}$$

Note that, $\Gamma(r) = \infty$ and so $\dfrac{1}{\Gamma(r)} = 0$, for $r \leq 0$.
Since, ν is pos. integer only and sums through integral values, all the coefficients for, $r = 0, 1, 2, \ldots, \nu-1$ must vanish.

$$\therefore J_{-\nu}(x) = \sum_{r=\nu}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-\nu+r+1)} \left(\frac{x}{2}\right)^{2r+\nu}$$

[put $r' = r - \nu$]
$$= \sum_{r=0}^{\infty} \frac{(-1)^{r'+\nu}}{(r'+\nu)!\,\Gamma(r'+1)} \left(\frac{x}{2}\right)^{2(r'+\nu)-\nu}$$

Dropping prime

$$\Rightarrow (-1)^{\nu} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(\nu+r+1)} \left(\frac{x}{2}\right)^{2r+\nu}$$

$$\Rightarrow J_{-\nu}(x) = (-1)^{\nu} J_{\nu}(x).$$

Case 2:- ν is negative integer say $\nu = -k$, k is pos.

By cond 1)

$$J_{K} = (-1)^{\nu} J_{k}$$

Bessel function of 2^{nd} kind (Neumann function):-

$$Y_{\nu}(x) = \begin{cases} \dfrac{J_{\nu}(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} & ; \ \nu \neq \text{int} \\ \lim\limits_{k \to \nu} \dfrac{J_{k}(x)\cos(k\pi) - J_{-k}(x)}{\sin(k\pi)} & ; \ \nu = \text{int} \end{cases}$$

Two linearly independent sol^n of Bessel eq^n of order ν are

$$y(x) = A J_{\nu}(x) + B Y_{\nu}(x). \quad \text{——} (1)$$

Proof:- case i) ν is not integer.

$\sin(\nu\pi) \neq 0$ and Y_{ν} is linear combination of J_{ν} and $J_{-\nu}$ which are themselves linearly independent and sol^n. Hence J_{ν} and Y_{ν} must be linear ind. sol^n.

Thus (1) gives general sol^n of Bessel eq^n of order ν.

case ii) ν is integer.

$$Y_{\nu}(x) = \lim_{k \to \nu} \frac{J_{k}(x)\cos(k\pi) - J_{-k}(x)}{\sin(k\pi)}.$$

in above $cond^n$ $\frac{0}{0}$ forms comes so we have to apply L-hopital rule:

$$Y_{\nu}(x) = \lim_{k \to \nu} \frac{-\pi J_k \sin(k\pi) + \cos(k\pi)\cdot\frac{\partial}{\partial k}(J_k) - \frac{\partial}{\partial k}(J_{-k})}{\pi\cos(k\pi)}$$

$$= \frac{1}{\pi}\left[\frac{\partial}{\partial k}(J_k) - (-1)^{\nu}\frac{\partial}{\partial k}(J_{-k})\right]_{k=\nu} \quad \text{——} (2)$$

To show Y_ν is a solⁿ, when ν is integer.
we observe,

$$x^2 \frac{\partial^2 J_K}{\partial x^2} + x \cdot \frac{dJ_K}{dx} + (x^2 - K^2) J_K = 0 \quad \text{—(3)}$$

$$x^2 \frac{d^2 J_{-K}}{dx^2} + x \frac{dJ_{-K}}{dx} + (x^2 - K^2) J_{-K} = 0 \quad \text{—(4)}$$

Now, differentiating (3) & (4) w.r.t. K,

$$x^2 \cdot \frac{d^2}{dx^2}\left(\frac{\partial}{\partial K} J_K\right) + x \cdot \frac{d}{dx}\left(\frac{\partial}{\partial K} J_K\right) + (x^2 - K^2)\left(\frac{\partial}{\partial K} J_K\right) - 2K J_K = 0 \quad \text{—(5)}$$

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial}{\partial K} J_{-K}\right) + x \cdot \frac{d}{dx}\left(\frac{\partial}{\partial K} J_{-K}\right) + (x^2 - K^2)\left(\frac{\partial}{\partial K} J_{-K}\right) - 2K J_{-K} = 0 \quad \text{—(6)}$$

$\Rightarrow$ (5) $-$ $(-1)^K$.(6) and letting $K \to \nu$, and using $J_{-\nu} = (-1)^\nu J_\nu$ for integer ν,

$$x^2 \cdot \frac{d^2}{dx^2}(\pi Y_\nu(x)) + x \cdot \frac{d}{dx}(\pi Y_\nu) + (x^2 - \nu^2)(\pi Y_\nu) = 0$$

{from (2) for Y_ν }.

$\rightarrow$ So, Y_ν is a solⁿ of Bessel eqⁿ.

Of course when ν is integer, J_ν and Y_ν are l.i
Here G.S is

$$\boxed{Y = A J_\nu + B Y_\nu.}$$

Ex:- one sol^n of hypergeometric eq^n around $x=\infty$ in terms of . find the sol

Ex:- find G.S of Legendre eq^n of degree 'n' ~~within pr~~ (n is non-neg. integ). around $x=\infty$.

Sol^n:- $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$x \in (-1,1)$

$n \in R$

$x=$ regular singular pt. {proved by transforming $x = \frac{1}{\xi}$}.

so put

let $y = \sum_{r=0}^{\infty} a_r x^{\rho+r}$.

$\rightarrow$ $\cancel{(1-x^2)}$ $\left(1-\frac{1}{\xi^2}\right)y'' - \frac{2}{\xi}y' + n(n+1)y = 0$

$(\xi^2-1)y'' - 2\xi y' + n(n+1)\xi^2 y = 0$

$\cancel{\xi} = \cancel{\pm 1}$ $\rightarrow$ ~~regular~~

$\Rightarrow n+1, -n$.

$x = \frac{1}{\xi}$

$\rightarrow$ &

$\frac{dx}{dy} \Rightarrow \frac{-1}{\xi^2}\cdot\frac{d\xi}{dy}$

$\frac{dy}{d\xi} \Rightarrow -\frac{1}{\xi^2}\cdot\frac{dy}{dx}$

$\frac{d^2y}{d\xi^2} \Rightarrow \frac{dy}{dx}\left(\frac{-2}{\xi^3}\right) + \frac{1}{\xi^2}\frac{d}{d\xi}\left(\frac{dy}{dx}\right)$

$-\frac{d^2y}{d\xi^2}$

$+\frac{1}{\xi^2}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Legendre Polynomial $P_n(x)$:-

Q.) If n^{th} degree polynomial $P_n(x)$ is a solution of Legendre equation, show that :-

$$P_n(x) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)!}{2^n \cdot r! \cdot (n-r)! \, (n-2r)!} \cdot x^{n-2r},$$

where the integer p is $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Sol^n :- Substituting, $y = \sum_{r=0}^{\infty} a_r \cdot x^r$ in legendre eq^n,

we have,

$$a_{r+2} = \frac{(r-n)(r+n+1)}{(r+1)(r+2)} a_r .$$

$$y(x) = a_n \left[x^n + \frac{a_{n-2}}{a_n} . x^{n-2} + \frac{a_{n-4}}{a_n} . x^{n-4} + \dots \right]$$

$$\frac{a_{n-2}}{a_n} = -\frac{(n-1).n}{2 \cdot (2n-1)} \; ; \quad \frac{a_{n-4}}{a_{n-2}} = \frac{(n-3)(n-2)}{4 \cdot (2n-3)} \; ; \dots$$

$$\therefore y(x) = a_n \left[x^n - \frac{n(n-1)}{2 \cdot (2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} \cdot x^{n-4} - \dots \right]$$

choosing; $a_n = \frac{(2n)!}{2^n (n!)^2}$ by putting $r = 0$.

$$\Rightarrow \frac{(2n)!}{2^n (n!)^2} \cdot \frac{n(n-1)}{2 \cdot (2n-1)} = \frac{(2n-2)!}{2^n (n-2)! \, (n-1)!}$$

$$\Rightarrow \frac{(2n)!}{2^n \cdot (n!)^2} \cdot \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} = \frac{(2n-4)!}{2^n \cdot 2! \cdot (n-4)! \, (n-2)!}$$

to determine 'p' :- for even 'n', Last term must be 1.

$$\therefore n - 2p = 0$$
$$p = n/2$$

Similarly for odd 'n' :- Last term must be x.

$$\therefore n - 2p = 1$$
$$p = \frac{n-1}{2}$$

$\therefore$ Hence, $P_n(x) = \sum \dots$

Scanned by CamScanner

• Rodrigue's formula for $P_n(x)$:-

$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

Sol^n:- Proof :-

$$P_n(x) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)!}{2^n \cdot r!\,(n-r)!\,(n-2r)!} x^{n-2r},$$

where p is $\frac{n}{2}$ or $\frac{n-1}{2}$ according to n is even or odd resp.

$$\frac{d^n}{dx^n}\left(x^{2n-2r}\right) = (2n-2r)(2n-2r-1) \ldots \{(2n-2r)-n+1\} x^{n-2r}$$

$$y = x^n$$
$$y_1 = n \cdot x^{n-1}$$
$$y_2 = n \cdot (n-1) \cdot x^{n-2}$$
$$\vdots \qquad \vdots$$
$$y_r = n(n-1) \ldots (n-r+1)\, x^{n-r}.$$

⊙ INTERESTING PROPERTIES :-

Orthogonal property of legendre polynomials

$$\int_{-1}^{1} P_m(x) \cdot P_n(x)\, dx = 0, \quad m \neq n$$
$$= \frac{2}{2n+1}, \quad m = n.$$

Proof :- $$\frac{d}{dx}\left[(1-x^2)\frac{dP_m(x)}{dx}\right] + m(m+1)P_m = 0 \quad \text{—(1)}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n = 0 \quad \text{—(2)}$$

$$\rightarrow \cancel{n(n+1)P_n \frac{d}{dx}\left[(1-x^2)\frac{dP_m(x)}{dx}\right] - m(m+1)P_m \frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] = 0}$$

(2). P_m − (1). P_n :-

$$P_m \cdot \frac{d}{dx}\left[(1-x^2) \cdot \frac{dP_n}{dx}\right] - P_n \cdot \frac{d}{dx}\left[(1-x^2) \cdot \frac{dP_m}{dx}\right] + P_m \cdot P_n \left[n(n+1) - m(m+1)\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)\left\{P_m \cdot \frac{dP_n}{dx} - P_n \cdot \frac{dP_m}{dx}\right\}\right] + P_m \cdot P_n\left[n(n+1) - m(m+1)\right] = 0 \quad \text{---(3)}$$

Integrate (3) wrt x in $[-1,1]$

$$\left[n(n+1) - m(m+1)\right]\int_{-1}^{1} P_n \cdot P_m \cdot dx = 0 \quad \text{--- (4)}$$

[$\because$ first term disappears due to $(1-x^2)$ term]

for $m \neq n$:- $\int_{-1}^{1} P_m \cdot P_n \, dx = 0$

Proof:- To prove, $\int_{-1}^{1} P_n^2(x) \, dx = \frac{2}{2n+1}$.

$u_n = (x^2-1)^n$, $u_n^{(m)} = \frac{d^m}{dx^m}(x^2-1)^n$

Rodrigue's formula for P_n : $P_n(x) = \frac{1}{2^n \cdot n!} u_n^{(n)}$.

$$\therefore \int_{-1}^{1} P_n^2(x) \, dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} u_n^{(n)} \cdot u_n^{(n)} \, dx$$

$$= \frac{1}{2^{2n} \cdot (n!)^2}\left[\underbrace{u_n^{(n)} \cdot u_n^{(n-1)}\Big|_{-1}^{1}}_{0} - \int_{-1}^{1} u_n^{(n+1)} \cdot u_n^{(n-1)} \, dx\right]$$

$$\rightarrow \int_{-1}^{1} P_n(x)^2 dx = -\int_{-1}^{1} u_n^{(n+1)} \cdot u_n^{(n-1)} \, dx$$

$$= (-1)^2 \int_{-1}^{1} u_n^{(n+2)} \cdot u_n^{(n-2)} \, dx$$

$\vdots$

$$= (-1)^n \int_{-1}^{1} u_n^{(2n)} \cdot u_n^{(0)} \, dx.$$