

VECTOR SPACES: A non-empty set V with two operations

$$(i) \forall u, v \in V, u+v \in V$$

$$(ii) \forall u \in V, \lambda u \in V \quad \lambda \in \mathbb{R}.$$

and some axioms hold like a) $u+v = v+u$

b) :

c) :

Ex: . Space \mathbb{R}^n , \mathbb{R}_m

- Polynomial space
- Matrix space
- Function space

LINEAR COMBINATION: A vector $v \in V$ is a linear combination of u_1, u_2, \dots, u_m from V if there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}$ such that

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_m u_m.$$

Ex: Express $v = (3, 7, -4)^T$ in \mathbb{R}^3 as a linear combination of the vectors $u_1 = (1, 2, 3)^T$ $u_2 = (2, 3, 7)^T$ $u_3 = (3, 5, 6)^T$. We seek scalars x, y, z such that

$$v = x u_1 + y u_2 + z u_3$$

$$\Rightarrow \begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} + z \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}.$$

OR

$$x + 2y + 3z = 3$$

$$2x + 3y + 5z = 7$$

$$3x + 7y + 6z = -4$$

Solve them
using Gauss Elimination to get

$$x = 2, y = -4, z = 3.$$

Thus $v = 2u_1 - 4u_2 + 3u_3$.

Remark: Expressing a vector as a linear combination of other given vectors is equivalent to solving system of linear equation. This system may not have solution or it may have unique or infinitely many solutions.

Q: In the above example if u_1, u_2 & u_3 are linearly independent then what can you say about uniqueness of the expression $v = xu_1 + yu_2 + zu_3$.

SPANNING SETS: If every $v \in V$ is a linear combination of vectors u_1, u_2, \dots, u_m , then vectors u_1, u_2, \dots, u_m are said to span V .

Remark: 1. Suppose u_1, u_2, \dots, u_m span V . Then, for any vector w , then set w, u_1, u_2, \dots, u_m also spans V .

2. Suppose u_1, u_2, \dots, u_m span V and suppose u_k is a linear combination of some of the other u 's. Then the u 's without u_k also span V .

3. Suppose u_1, u_2, \dots, u_m span V and suppose one of the u 's is the zero vector. Then the u 's without zero vector also span V .

SUBSPACES: $W \subset V$. Then W is a subspace of V if W is itself a vector space.

Ex: Solution space of homogeneous system:

Consider $Ax = \theta$ where $A \in \mathbb{R}^{n \times n}$ $x \in \mathbb{R}^{n \times 1}$ $\theta \in \mathbb{R}^{n \times 1}$

Every solution x can be viewed as a vector in \mathbb{R}^n .

This implies that the set of solutions is a subset of \mathbb{R}^n .

Check whether the solution set is a vector space (subspace of \mathbb{R}^n).
Let x_1 & x_2 be solutions of $Ax = \theta$ that is $Ax_1 = \theta$ & $Ax_2 = \theta$

Then $A(x_1 + x_2) = Ax_1 + Ax_2 = 0$

$\Rightarrow (x_1 + x_2)$ is also a solution of $Ax = 0$.

Further, λx_1 is also a solution of $Ax = 0$ for any $\lambda \in \mathbb{R}$.

\Rightarrow The solution space of $Ax = 0$ is a subspace of \mathbb{R}^n .

THEOREM: Row equivalent matrices have the same row space.

IDEA OF PROOF:

Let M is the matrix obtained by applying one of the following elementary row operations on A :

- i) Interchange of R_i and R_j .
- ii) Replace ~~row~~ R_i by λR_i .
- iii) Replace R_j by $\lambda R_i + R_j$.

Then each row of M is a row of A or a linear combination of rows of A . Hence the row space of M is contained in the row space of A .

On the other hand, we can apply the inverse elementary row operation on M to obtain A ; hence the row space of A is contained in the row space of M .

$\Rightarrow A$ and M have the same row space.

Example: Consider the following two sets of vectors in \mathbb{R}^4 :

$$u_1 = (1, 2, -1, 3) \quad u_2 = (2, 4, 1, -2) \quad u_3 = (3, 6, 3, -7)$$

$$w_1 = (1, 2, -4, 11) \quad w_2 = (2, 4, -5, 14).$$

Let $U = \text{span}(u_i)$ and $W = \text{span}(w_i)$, show that

$$U = W.$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$

(25)

$$\text{R}_2 \leftarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix}$$

$$B \sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

We see that

$$\text{Span}(A) = \text{Span}(B)$$

Basis and Dimension:

- A set $S = \{u_1, u_2, \dots, u_n\}$ of vectors is a basis of V , if,
 - \Downarrow S is linearly independent \Downarrow S spans V .
 - No of elements in S (basis) is called dimension of V .

Note: The vector space $\{0\}$ is defined to have dimension zero.

EXAMPLES of Basis:

a) Vector space \mathbb{R}^n : Consider the following n vectors.

$e_1 = (1, 0, 0, \dots, 0)^T, e_2 = (0, 1, 0, \dots, 0)^T, \dots, e_n = (0, 0, \dots, 1)^T$
 These vectors are linearly independent and forming the basis.

We have:

$$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n. \text{ Here } v = (v_1, v_2, \dots, v_n)$$

b) Vector space of all $r \times s$ matrices (M):

Ex: 2×3 matrices:

consider: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

These vectors are linearly independent and span M .

Hence the dimension of $M = rs$.

Note:

Bases in the example a) & b) above are called usual or standard basis.

c) Vector space $P_n(t)$ of all polynomials of degree $\leq n$:

The set $S = \{1, t, t^2, \dots, t^n\}$ of $(n+1)$ polynomials is a basis of $P_n(t)$.

Some useful theorems:

Th: Let V be a vector space of finite dimension n . Then:

- any $(n+1)$ or more vectors in V are linearly dependent
- Any linearly independent set $S = \{u_1, u_2, \dots, u_n\}$ with n elements is a basis of V .
- any spanning set $\{v_1, v_2, \dots, v_n\}$ of V with n elements is a basis of V .

Th: Suppose S spans a vector space V . Then: (27)

- i) any maximum number of linearly independent vectors in S form a basis of V .
- ii) Suppose one deletes from S every vector that is a linear combination of preceding vectors in S . Then the remaining vectors form a basis of V .

Th: Let V be a vector space of finite dimension and let $S = \{u_1, u_2, \dots, u_r\}$ be a set of linearly independent vectors in V . Then S is a part of a basis of V , that is, S may be extended to a basis of V .

Ex: 1. The following set in \mathbb{R}^4

$(1, 1, 1, 1)^T, (0, 1, 1, 1)^T, (0, 0, 1, 1)^T, (0, 0, 0, 1)^T$
forms a basis of \mathbb{R}^4 .

Because $\dim(\mathbb{R}^4) = 4$ & these vectors are linearly independent.

2. Consider any four vectors in \mathbb{R}^3 , say

$(1, 2, 1), (0, 1, 3), (3, 4, 0), (6, 1, 8)$.

These vectors must be linearly dependent since $\dim(\mathbb{R}^3) = 3$.

Dimension of the solution space of a homogeneous system $Ax=0$:

Ex: Solve the system of equations:

$$x_1 + 3x_2 + 3x_3 - x_4 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 2x_3 + 14x_4 - 3x_5 = 0$$

$$4x_1 + 12x_2 + 2x_3 + 16x_4 + x_5 = 0$$

Sol: Augmented matrix

$$[A|b] = \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 0 \\ 2 & 6 & -2 & 14 & -3 & 0 \\ 4 & 12 & 2 & 16 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 0 \\ 0 & 0 & 8 & -16 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$x_5 = 0 \quad x_4 = \alpha_1 \quad x_2 = \alpha_2$$

$$x_3 = 2\alpha_1 \quad x_1 = -5\alpha_1 - 3\alpha_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha_1 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \alpha_1, \alpha_2 \in \mathbb{R}.$$

The solution space is spanned by two linearly independent vectors.

$[5, 0, 2, 1, 0]^T \neq [3, 1, 0, 0, 0]^T$. Hence the dimension of the solution space is $2 (= 5 - 3) = [\text{no. of unknowns} - \text{rank}(A)]$

Theorem: The dimension of the solution space W of a homogeneous system $Ax=0$ is $n-r$, where n is the number of unknowns and r is the rank of the coeff. matrix A .

Note: Solution space of $Ax=0$ is also called null space of A .

Theorem: Solutions of non-homogeneous linear system are of the

form:

$$x = x_0 + x_h$$

where x_0 is any fixed solution of $Ax=b$ and x_h runs through all the solutions of the corresponding homogeneous system $Ax=0$.

Ex:

$$[A|b] \leftrightarrow \left[\begin{array}{cccc|cc} 1 & 3 & 3 & -1 & 2 & 17 \\ 2 & 6 & -2 & 14 & -3 & -19 \\ 4 & 12 & 2 & 16 & 1 & 7 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|cc} 1 & 3 & 3 & -1 & 2 & 17 \\ 0 & 0 & 8 & -16 & 7 & 53 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array} \right]$$

Choosing $x_4 = \alpha_1$ & $x_2 = \alpha_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \\ 3 \end{bmatrix} + \alpha_1 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = x_0 + \underbrace{x_h}_{\text{Homogeneous part}}$$

LINEAR MAPPING (LINEAR TRANSFORMATION)

Let X and Y be any two vector spaces. A mapping $F: X \rightarrow Y$ is called a linear mapping or linear transformation if it satisfies the following two conditions:

- for any two vectors $x, v \in X$, $F(x+v) = F(x) + F(v)$
- for any scalar k and vector $x \in X$, $F(kx) = kF(x)$.

Remark:

1) Note that for $k=0$: $F(0) = 0$. Thus every linear mapping takes the zero vector into the zero vector.

2) The two conditions above can be combined into one:

$$F(k_1x + k_2v) = k_1F(x) + k_2F(v).$$

Example: Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with

$$F(x_1, y_1, z_1) = (x_1, y_1, 0)$$

Let $u = (a, b, c)$ & $v = (a', b', c')$

then,

$$F(u+v) = F(a+a', b+b', c+c')$$

$$= (a+a', b+b', 0)$$

$$= (a, b, 0) + (a', b', 0)$$

$$= F(u) + F(v)$$

and for any constant k .

$$F(ku) = F(ka, kb, kc) = (ka, kb, 0) = k(a, b, 0)$$

This F is linear.

$$= kF(u).$$

Ex: Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $F(x, y) = (x+1, y+2)$.

Check whether F is linear or not?

Sol: It is not linear since $F(0, 0) = (1, 2) \neq 0$.

Matrices as linear mapping (transformation)

Let $X = \mathbb{R}^n$ & $Y = \mathbb{R}^m$. Then any real $m \times n$ matrix A gives a transformation of \mathbb{R}^n into \mathbb{R}^m ,

$$y = Ax$$

Since,

$$A(u+x) = Au + Ax$$

$$\& A(\lambda x) = \lambda Ax$$

This is a linear transformation.

KERNEL and IMAGE of a Linear mapping:

Let $F: X \rightarrow Y$ be a linear mapping.

$$\text{Ker } F = \{x \in X : F(x) = 0\}$$

$$\text{IM } F = \{y \in Y : \text{there exists } x \in X \text{ for which } F(x) = y\}.$$

Theorem: Let $F: X \rightarrow Y$ be a linear mapping. Then the kernel of F is a subspace of X and image of F is a subspace of Y .

Theorem: Suppose x_1, x_2, \dots, x_m span a vector space X and

Suppose $F: X \rightarrow Y$ is linear. Then $F(x_1), F(x_2), \dots, F(x_m)$ span $\text{Im } F$.

IDEA: Let $y \in \text{Im } F$. Then $\exists x \in X : F(x) = y$.

$$\text{Also: } x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m$$

Therefore,

$$y = F(x) = \alpha_1 F(x_1) + \alpha_2 F(x_2) + \dots + \alpha_m F(x_m)$$

\Rightarrow The vectors $F(x_1), F(x_2), \dots, F(x_m)$ span $\text{Im } F$.

Ex: $F(x_1, y_1, z) = (x_1, y_1, 0)$

$$\text{IM } F = \{(a, b, c) : c=0\} = xy \text{ plane}$$

$$\text{ker } F = \{(a, b, c) : a=0, b=0\} = z\text{-axis}$$

KERNEL & IMAGE OF MATRIX MAPPINGS:

Let us consider $A: \mathbb{R}^4 \rightarrow \mathbb{R}^3$. with

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

Take usual basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ of \mathbb{R}^4 .

Then: Ae_1, Ae_2, Ae_3, Ae_4 spans the image of A .

$$Ae_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}; Ae_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}; Ae_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}; Ae_4 = \begin{bmatrix} a_4 \\ b_4 \\ c_4 \end{bmatrix}.$$

Thus the image of A is precisely the column space of A .

The kernel of A consists all vectors x for which $Ax=0$.

This implies that Kernel of A is the solution space of the homogeneous system $AX=0$, called the null space of A .

RANK and Nullity of a Linear Mapping: Let $F: X \rightarrow Y$ be a linear mapping. Then

$$\text{rank}(F) = \dim(\text{Im } F) \quad \& \quad \text{nullity}(F) = \dim(\text{ker } F).$$

Theorem: Let X be of finite dimension and let $F: X \rightarrow Y$ be a linear map. Then $\text{rank}(F) + \text{nullity}(F) = \dim X$

Proof: Think about matrix mapping: $r + (n-r) = n$ (Page 28 & 12)

Ex: Let $F: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear mapping defined by

$$F(x_1, y_1, z_1, t) = (x-y+z+t, 2x-2y+3z+4t, 3x-3y+4z+5t)$$

Find a basis and dimension of a) the image of F b) Kernel of F .

Sol: We know that the vectors

$$F(1, 0, 0, 0) = (1, 2, 3) \quad F(0, 1, 0, 0) = (-1, -2, -3)$$

$$F(0, 0, 1, 0) = (1, 3, 4) \quad F(0, 0, 0, 1) = (1, 4, 5).$$

Span $\text{Im } F$.

Now consider

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This $(1, 2, 3) \neq (0, 1, 1)$ form a basis of $\text{Im } F$ and
 $\dim(\text{Im } F) = 2$ and $\text{rank}(F) = 4 - 2 = 2$.

b) Set $F(x_1, y_1, z_1, t) = 0 \Rightarrow x-y+z+t=0$

$$2x-2y+3z+4t=0$$

$$3x-3y+4z+5t=0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variables are $y \& t$

$$\dim(\text{Ker } F) = 2 \text{ or nullity}(F) = 2.$$

Let $t = \alpha_1 \& y = \alpha_2$

$$z = -2\alpha_1 \quad x = \alpha_2 + 2\alpha_1 - \alpha_1$$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus $(1, 0, -2, 1) \& (1, 1, 0, 0)$ form a basis for $\text{Ker } F$.

MATRIX REPRESENTATION OF A LINEAR OPERATOR!

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Coordinates of a vector:

Let X be an n -dimensional vector space (over \mathbb{R}) with basis $\{x_1, x_2, \dots, x_n\}$. Then any vector $x \in X$ can be expressed uniquely as a linear combination of the basis vectors i.e.,

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

The scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ are called the coordinates of x with respect to $S = \{x_1, x_2, \dots, x_n\}$.

Notation: $[x]_S = [\alpha_1, \alpha_2, \dots, \alpha_n]^T$.

Suppose $F: X \rightarrow Y$ is a linear mapping. Let X and Y be vector spaces of dimensions m and n with bases $S = \{x_1, x_2, \dots, x_m\}$ and $S' = \{y_1, y_2, \dots, y_n\}$ respectively.

The vectors $F(x_1), F(x_2), \dots, F(x_m)$ belongs to Y so

$$F(x_1) = a_{11} y_1 + a_{12} y_2 + \dots + a_{1n} y_n$$

$$F(x_2) = a_{21} y_1 + a_{22} y_2 + \dots + a_{2n} y_n$$

$$\vdots$$

$$F(x_m) = a_{m1} y_1 + a_{m2} y_2 + \dots + a_{mn} y_n$$

The transpose of the above matrix of coefficients, denoted by $[F]_S^{S'}$ is called the matrix representation of F relative to the bases S and S' .

Theorem: For any vector $x \in X$, $[F]_S^{S'} [x]_S = [F(x)]_{S'}$.

Ex: Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ y-z \end{bmatrix}$$

Determine the matrix of the linear transformation T , with respect to the ordered basis:

$$X = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ in } \mathbb{R}^3 \text{ and } Y = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} \text{ in } \mathbb{R}^2$$

[JEE, P3. 41]

Sol: $T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(1) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(1)$

$$T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(1)$$

$$T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(1) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(0)$$

Therefore the matrix of the linear transformation T w.r.t the given basis

is $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Ex: Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be linear operator defined by $F(x,y) = (2x+3y, 4x-5y)$

- Find the matrix representation of F relative to $S = \{(1,2)^T, (2,5)^T\}$
- " " $E = \{(1,0)^T, (0,1)^T\}$.

Ex: Consider $A = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}$ as linear operator from \mathbb{R}^2 to \mathbb{R}^2 .

Find the matrix representation of A relative to the basis

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\} \quad \& \quad E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$