Group Action

Lecture 15

Application of Group Action :

Thm (Fixed point congruence) :

Let G be a finite p-group acting on a finite set X. Then

$$|X| \equiv |\text{Fixed pt}| \pmod{p}$$

Defn $x \in X$ is said to be a fixed pt. if $gx = x \quad \forall g \in G$

ker of a gp action $= \{ g \in G \mid gs = s \ \forall s \in A \}$

Pf: Let the different orbits of X are represented by $x_1, \dots, x_t$

Then $X = O(x_1) \sqcup O(x_2) \sqcup \dots \sqcup O(x_t)$

Then $|X| = |O(x_1)| + \cdots + |O(x_t)|$ — $(*)$

$\because |O(x_i)| = [G : stb(x_i)]$

$$= \frac{|G|}{|stb(x_i)|}$$

$=$ either power of p or 1.

($\because$ G is p-gp)

If $|O(x_i)| = 1 \Rightarrow G = stb(x_i)$

$\Rightarrow x_i$ is a fixed pt.

Going mod p the eqn. $(*)$ we get

$|X| = |\text{Fixed pts}| \pmod{p}$.

Thm Any non-abelian gp of order 6 is isomorphic to S_3.

Pf: Let G be a non-abelian gp of order 6. By Cauchy's Thm G has subgps of order 2 and order 3 say H and K respectively.

let $H = \langle a \rangle$ and $K = \langle b \rangle$ s.t $|a| = 2$ and $|b| = 3$.

Note that H can not be a normal subgp of G as G is not abelian.

[Because if H is normal subgp of G in that case G become abelian.]

H & K bothes are normal $H \cap K = \{1\}$, $G \cong H \times K$.

Note $[G:H] = 3$, ie there are 3 left cosets of H in G.

Consider the gp action

$$G \times G/H \longrightarrow G/H.$$

↳ set of all left cosets of H

$$(g, hH) \longmapsto ghH$$

The permutation representation of this gp action is

$\phi : G \longrightarrow S_3$ which is a gp homo.

$$\ker \phi = \{ g \in G \mid \sigma_g = Id \}$$

$$= \{ g \in G \mid ghH = hH \}.$$

If $g \in \ker\phi$ then $gH = H \Rightarrow g \in H$.

$\ker\phi \subseteq H$

$\therefore$ $\ker\phi$ is either 1 or H.

But $\ker\phi$ can not be equal to H as H is not a normal subgp of G.

$\therefore$ $\ker\phi = \{1\}$.

Therefore $G \cong S_3$.

Propn Let G be a finite p-group. Any subgp of G with index p is a normal subgp.

Pf: Consider the gp action Let H be a subgp of index p.

$$G \times G/H \longrightarrow G/H$$

$$(g, hH) \longmapsto ghH.$$

The permutation representation of the gp action is

$$\phi : G \longrightarrow S_p$$
$$g \longmapsto \sigma_g \quad \text{where } \sigma_g(hH) = ghH$$

<u>WTS</u> $\ker \phi = H$

Let $\ker \phi = K$

Now $G/K \cong$ Subgp of S_p.

$$\Rightarrow \quad [G:K] \mid p!$$

$\therefore [G:K] = 1$ or a power of p

which implies $[G:K] = 1$ or p.

note that any $g \in K$ then $\sigma_g(H) = H$
$gH = H$

$\Rightarrow g \in H$.

$\Rightarrow K \subset H$.

$\therefore [G:K] > 1$. Thus $[G:K] = p$.

$$K \subset H \subset G.$$

$$\underset{\substack{\shortparallel \\ p}}{[G:K]} = \underset{\substack{\shortparallel \\ p}}{[G:H]}\,[H:K]$$

$$\Rightarrow \quad [H:K] = 1$$

$$\Rightarrow \quad K = H.$$

Therefore H is a normal subgp of G.

<u>Propn</u>. Let G be a finite gp with $|G| > 1$ and p be the smallest prime factor of $|G|$. Any subgp of G with index p is normal subgp of G.

H be a subgp of index p.

$$G \times G/H \longrightarrow G/H.$$
$$(g, hH) \longmapsto ghH$$
$$\phi : G \longrightarrow S_p.$$

WTS $\ker\phi = H$.