$$f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= a \cdot \Delta x + b \cdot \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

then

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x+\Delta x, y+\Delta y) = f(x,y)$$

Thus f is continuous.

Setting $\Delta y = 0$ and deviding by Δx yield the relation

$$\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = a + \epsilon_1$$

$$\Rightarrow \quad \boxed{f_x(x,y) = a}$$

Similarly $\boxed{f_y(x,y) = b}$

THEOREM: (SUFFICIENT CONDITION FOR DIFFERENTIABILITY)

If the function $z = f(x, y)$ has continuous first order partial derivatives at a point (x, y), then $f(x, y)$ is differentiable at (x, y).

PROOF:

$$\Delta z = f(x + \Delta x,\ y + \Delta y) - f(x, y)$$

$$= f(x + \Delta x, y + \Delta y) - f(x,\ y + \Delta y)$$
$$+ f(x, y + \Delta y) - f(x, y)$$

Using mean value theorem

$$\Delta z = \Delta x\ f_x\ (x + \theta_1 \Delta x,\ y + \Delta y)$$
$$+ \Delta y\ f_y\ (x,\ y + \theta_2\ \Delta y)$$

when $0 < \theta_1, \theta_2 < 1$

Since the partial derivatives f_x and f_y are continuous at the point (x, y), we can wri

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_x(x + \theta_1 \Delta x,\ y + \Delta y) = f_x(x, y)$$

$$\Rightarrow f_x(x + \theta_1 \Delta x,\ y + \Delta y) = f_x(x, y) + \epsilon_1$$

$[\epsilon_1 \to 0$ as $\Delta x, \Delta$
$\to$

and

$$f_y(x,\ y + \theta_2 \Delta y) = f_y(x, y) + \epsilon_2$$

$[\epsilon_2 \to 0$ as $\Delta x, \Delta y$

Thus,

$$\Delta z = \Delta x\, f_x(x, y) + \Delta y\, f_y(x, y) + \epsilon_1 \Delta x + \epsilon_2$$

This implies differentiability of f.

To test the differentiability at a point $P(x, y)$, we use

either

$$\Delta z = \underbrace{\frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y}_{dz} + \epsilon_1 \Delta x + \epsilon_2$$

or

$$\frac{\Delta z - dz}{\Delta \rho} = \epsilon_1 \frac{\Delta x}{\Delta \rho} + \epsilon_2 \frac{\Delta y}{\Delta \rho},$$

where $\Delta \rho = \sqrt{\Delta x^2 + \Delta y^2}$

$$\Rightarrow \lim_{\Delta \rho \to 0} \frac{\Delta z - dz}{\Delta \rho}$$

$$= \lim_{\Delta \rho \to 0} \left[\epsilon_1 \left(\frac{\Delta x}{\Delta \rho} \right) + \epsilon_2 \left(\frac{\Delta y}{\Delta \rho} \right) \right]$$

$$= 0$$

Since $\left| \frac{\Delta x}{\Delta \rho} \right| \leq 1$ and $\left| \frac{\Delta y}{\Delta \rho} \right| \leq 1$

and ϵ_1 and ϵ_2 tends to zero as $\Delta \rho \to 0$.

To test the differentiability we show that

$$\boxed{\lim_{\Delta \rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = 0}$$

REMARKS:

① The function may not be differentiable at a point $P(x, y)$, even if the partial derivatives f_x and f_y exists at P.
(Because existence of partial derivatives is a necessary condition)

② A function may be differentiable even if f_x and f_y

are not continuous. (Because continuity of the f_x and f_y is a sufficient condition).

(3) Sufficient conditions of continuity can be relaxed. It is sufficient that one of the partial derivative exist and the other is continuous.