Lecture - 4
13/1/17

Dimension of a subspace.

Let $V \to$ finite dimensional vector space of dimension n.

Let $W \subset V$ be a subspace.

Thm. If W ~~...~~ is a subspace of a vector space V of dimension n and if dimension of W is m, then $m \leq n$. i.e $\dim W \leq \dim V$. If $\dim W = \dim V$, then $W = V$.

Take $V = \mathbb{R}^3$. $\dim \mathbb{R}^3 = 3$.

$W \subset \mathbb{R}^3$.

$\dim W = 0$; $W = \{(0,0,0)\} \Rightarrow W$ is the origin.

$\dim W = 1$. W consists of all lines passing through origin.

$\dim W = 2$. W consists of all planes ~~...~~ containing the origin.

$\dim W = 3$. $W = \mathbb{R}^3$.

Ex1. let $W = \left\{ (x, y, z) ; \begin{array}{l} ax + by + cz = 0 ; \\ a, b, c \in \mathbb{R} . \end{array} \right\} \subset \mathbb{R}^3$.

To find a basis & dimension of W.

Solution. $ax + by + cz = 0$.

No. of free variables = No of unknowns − no. of equations in reduced form

$z = z_0, \quad y = y_0$.

$x = -\frac{1}{a}(by + cz)$.

$\therefore x = -\frac{b}{a} y_0 - \frac{c}{a} z_0$.

$$(x, y, z) = \left(-\frac{b}{a} y_0 - \frac{c}{a} z_0, y_0, z_0\right)$$

$$= y_0 \underbrace{\left(-\frac{b}{a}, 1, 0\right)}_{\underset{\sim}{w_1}} + z_0 \underbrace{\left(-\frac{c}{a}, 0, 1\right)}_{\underset{\sim}{w_2}}.$$

$\therefore \{w_1, w_2\} = \left\{ \left(-\frac{b}{a}, 1, 0\right), \left(-\frac{c}{a}, 0, 1\right) \right\}$.

span W $\longrightarrow$ (A)

$\because$ every element of W can be expressed as a l. c. of $\underset{\sim}{w_1}$ & $\underset{\sim}{w_2}$.

$c_1 \underset{\sim}{w_1} + c_2 \underset{\sim}{w_2} = \underset{\sim}{0}$.

$c_1 \left(-\frac{b}{a}, 1, 0\right) + c_2 \left(-\frac{c}{a}, 0, 1\right) = (0, 0, 0)$

$-\frac{b}{a} c_1 - \frac{c}{a} c_2 = 0$.

$c_1 = 0$

$c_2 = 0$ $\Big\rangle$ $\underset{\sim}{w_1}, \underset{\sim}{w_2}$ are l. i. $\longrightarrow$ (B)

From (A) & (B) we can say $\{w_1, w_2\}$ is basis of W & dim. of $W = 2$.

Note: The subspace W ~~is~~ whose elements are solutions of the equation $ax+by+cz=0 \longrightarrow (1)$, is called the solution space for eq. (1) (null space).

Dim. of solution space is called **nullity**. Note:

$$\text{nullity} = n - r.$$

n = no. of unknowns.

r = rank of the coeff. matrix.

(Note: nullity = no. of free variables).

Ex 2. Find the solution space (null space) for the system.

$$x+2y-z=0.$$
$$2x+5y+2z=0.$$
$$x+4y+7z=0.$$
$$x+3y+3z=0.$$

Hence find a basis & dimension of the solution space.

Sol. Coeff. matrix.

$$A=\begin{bmatrix}1&2&-1\\2&5&2\\1&4&7\\1&3&3\end{bmatrix}\xrightarrow[\substack{R_3\to -R_1+R_3\\ R_4\to -R_1+R_4}]{R_2\to -2R_1+R_2}\begin{bmatrix}1&2&-1\\0&1&4\\0&2&8\\0&1&4\end{bmatrix}\xrightarrow[R_4\to -R_2+R_4]{R_3\to -2R_2+R_3}\begin{bmatrix}1&2&-1\\0&1&4\\0&0&0\\0&0&0\end{bmatrix}$$

The equivalent system of equation is

$$x+2y-z=0$$
$$y+4z=0.$$

Let. $z = c$.

$y = -4\cancel{z} = -4c$.

$x = z - 2y = c - 2x - 4c = 9c$.

$(x, y, z) = (9c, -4c, c)$

$= \underline{c(9, -4, 1)}$.

$\because$ every (x, y, z) which is a solution of the given system is a multiple of $(9, -4, 1)$ we can say, $(9, -4, 1)$ spans the solution space of W. $(9, -4, 1)$ being a single vector is l. i.

So, $(9, -4, 1)$ is a basis for the solution space. Hence nullity $= 1$.

<u>Ex-3</u> Let W be the subspace of $\mathbb{R}^5$ generated by the vectors $(1, 3, -2, 5, 4)$, $(1, 4, 1, 3, 5)$, $(1, 4, 2, 4, 3)$, $(2, 7, -3, 6, 13)$.

Find a basis and dimension of W.

Extend the basis of W to a basis for the whole space $\mathbb{R}^5$.

$$\begin{array}{c} \underset{\sim}{w_1} = \\ \underset{\sim}{w_2} = \\ \underset{\sim}{w_3} = \\ \underset{\sim}{w_4} = \end{array} \begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The vectors $\underset{\sim}{w}_1, \underset{\sim}{w}_2, \underset{\sim}{w}_3, \underset{\sim}{w}_4$ are not l.i.

But the vectors,

$\underset{\sim}{w}_1 = (1, 3, -2, 5, 4)$, $\underset{\sim}{w}_2' = (0, 1, 3, -2, 1)$,

$\underset{\sim}{w}_3' = (0, 0, 1, 1, -2)$ are l.i.

and will also span the subspace which is spanned by $\underset{\sim}{w}_1, \underset{\sim}{w}_2, \underset{\sim}{w}_3, \underset{\sim}{w}_4$.

$\therefore \{w_1, w_2', w_3'\}$ is a basis of W.

$\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$

$\therefore$ dim W = 3.

$$L\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\} = L\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$$

$$= L\{\underset{\sim}{v}_1, \underset{\sim}{v}_3\}$$

$\overset{\underset{\sim}{w}_1}{(1, 3, -2, 5, 4)}$, $\overset{\underset{\sim}{w}_2'}{(0, 1, 3, -2, 1)}$, $\overset{\underset{\sim}{w}_3'}{(0, 0, 1, 1, -2)}$

cannot form a basis for $\mathbb{R}^5$.

With ~~(1, 3,~~ $\underset{\sim}{w}_1, \underset{\sim}{w}_2', \underset{\sim}{w}_3'$ add.

$\underset{\underset{\sim}{e}_4}{(0, 0, 0, 1, 0)}$, $\underset{\underset{\sim}{e}_5}{(0, 0, 0, 0, 1)}$

Then, $(w_1, w_2', w_3', e_4, e_5)$ is a basis of W.

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

because, $w_1, w_2', w_3', e_4, e_5$ being non-zero rows of an echelon matrix, are l.i.

$\underline{\text{Thm.}}$ Let V be a V sp. of dim. n and $S = \{v_1, v_2, \ldots, v_m\}$ $(m \leq n)$ be a subset of V. If S is l. i, then either S is a basis of V or it can be extended to a basis of V.

$\mathbb{R}^3$: $\underset{\tilde{v}_1}{(1, 2, 1)}, \underset{\tilde{v}_2}{(0, 1, 3)} \in \mathbb{R}^3$ and are l. i., but do not form a basis of $\mathbb{R}^3$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

But if you add $\underset{\tilde{v}_3}{(0, 0, 50)}$, then $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ is a basis for $\mathbb{R}^3$.

$\underline{\text{Note:}}$ If $v_3' = (0, 2, 6)$, then, $\{v_1, v_2, v_3'\}$ is not a basis

$\underline{\text{Thm.}}$ If V be an n dimensional vector space. Then any set of N vectors where $N > n$ is l. d.

$$c_1\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + c_2\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} + c_3\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} + c_4\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + c_5\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$c_1 - c_2 + c_4 + 3c_5 = 0$$
$$2c_1 + 3c_3 + c_4 + 2c_5 = 0$$
$$3c_1 + 2c_2 + c_3 + 2c_4 - c_5 = 0$$
$$4c_1 + c_2 + 2c_3 + 3c_4 = 0$$

Thm 1. ~~If~~ A subset of a l.i set is l.i.

Thm 2. A superset of a l.d. set is l.d.

(You can verify from the idea of echelon form of a matrix)

$$\begin{array}{c} \tilde{v}_1 \to \\ \tilde{v}_2 \to \\ \tilde{v}_3 \to \end{array} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues and eigenvectors of a SQUARE matrix.

Suppose A is an $n \times n$ matrix. A number λ (real/complex) is said to be an eigen-value of A, and $\underset{\sim}{v}$ the corresponding eigenvector, if $A\underset{\sim}{v} = \lambda \underset{\sim}{v}$ holds.

Note: $A\underset{\sim}{v} = \lambda\underset{\sim}{v}$.

or. $A\underset{\sim}{v} - \lambda\underset{\sim}{v} = \underset{\sim}{0}$.

or. $(A - \lambda I_n)\underset{\sim}{v} = \underset{\sim}{0}$

$\because$ $\underset{\sim}{v}$ is non-zero, $\therefore$ $|A - \lambda I_n| = 0 \longrightarrow (*)$.

This is called ~~of~~ characteristic (ch.) equation for the matrix A. Its roots are the eigenvalues. L.H.S. of $(*)$ is a polynomial of degree n, called ch. polynomial.

Note 1. No. of eigenvalues of a matrix
= order of the square matrix.
(You have to take multiplicity of the eigenvalue into account).

Note 2. Corresponding to ~~any~~ e-value there may be many e-vectors.

Note 3. Collection of e-values of A is called ~~spectrum~~ spectrum of A. Magnitude of the largest e-value is called the spectral radius of A. Suppose $-1, 2, -5$ are the e-values of a matrix A. Then spectral radius of A is, $|-5| = 5$.

Ex. Find the e-value of a matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Solve $|A - \lambda I| = 0$ or, $|\lambda I - A| = 0$.

$|\lambda I_2 - A| = 0 \Rightarrow \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right| = 0$.

or, $\begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = 0$. $\lambda^2 = 0$; $\lambda = 0, 0$.

$\therefore$ 0 is the e-value of A.

To find e-vector: $(\lambda I - A)\underset{\sim}{v} = \underset{\sim}{0}$.

$$(\lambda I - A)\,\underset{\sim}{v} = \underset{\sim}{0}$$

$$\Rightarrow \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x_2 = 0$, $x_1 \to$ arbitrary.

$\Rightarrow x_2 = 0$

$\therefore$ eigenvectors corr. to $\lambda = 0$ are of the form $(a, 0)$.

$\therefore$ Eigenvectors of a matrix together with the zero-vector forms a vector space. That is known as e-space corresponding to the ~~e~~-value λ.

E-space corresponding to $\lambda = 0$.

$$= \{(a,0) : a \in \mathbb{R}\}.$$

Note. $|A| = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0.$

__Thm.__ 0 is an e-value of a matrix if and only if $\det A = 0$.

__Thm.__ A matrix is non-invertible if and only if 0 is an e-value of A.

Ex 2. Find the e-values & corresp. e-~~values~~ vectors of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

sol: $|\lambda I - A| = 0$.

$$\Rightarrow \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda-2)(\lambda-3) + 2(\lambda-2) = 0.$$

$$(\lambda-2)(\lambda^2 - 3\lambda + 2) = 0.$$

or, $(\lambda-2)(\lambda-2)(\lambda-1) = 0$.

or, $(\lambda-2)^2(\lambda-1) = 0 \Rightarrow \lambda = 2, 2, 1$.

e-vectors corresponding to $\lambda = 2$.

$$\begin{pmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\cancel{2x_1 + 3x_3 = 0}$

$\therefore 2x_1 + 2x_3 = 0$.

or, $x_1 + x_3 = 0$

$x_3 = c$, $x_2 = b$, $x_1 = -c$.

$\therefore$ e-vectors corr. to $\lambda = 2$ are of the form $(x_1, x_2, x_3) = (-c, b, c)$ $\begin{pmatrix} b \neq 0 \\ \& c \neq 0 \end{pmatrix}$.

e-space corr. to $\lambda = 2$.

$$= \left\{ (-c, b, c) : b, c \in \mathbb{R} \right\}.$$

$$(x_1, x_2, x_3) = (-c, b, c).$$

$$= c\underbrace{(-1, 0, 1)}_{\underset{\sim}{w_1}} + b\underbrace{(0, 1, 0)}_{w_2}.$$

$$\cancel{X}\underset{\sim}{w_1} + Y\underset{\sim}{w_2} = \underset{\sim}{0}.$$

$$\therefore X(-1, 0, 1) + Y(0, 1, 0) = (0, 0, 0).$$

$$-X = 0, \quad Y = 0, \quad X = 0.$$

$\therefore$ $\underset{\sim}{w_1}, \underset{\sim}{w_2}$ are l.i.

$\&$ each $(x_1, x_2, x_3) = (-c, b, c)$ can be expressed as a l.c. of $\underset{\sim}{w_1}, \underset{\sim}{w_2}$.

$\therefore$ $\underset{\sim}{w_1}, \underset{\sim}{w_2}$ form a basis for e-space corr. to $\lambda = 2$. $E_{\lambda=2}$

$\therefore$ dim. of e-space $= 2$
$E_{\lambda=2}$

<u>Exercise:</u> Find the dimension of $E_{\lambda=1}$.

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