Sequences and Series

14-09-18.

$f: \mathbb{N} \to X$

$f(1), f(2), \ldots f(n)$

$f(n) = x_n$ $\{x_n\}$ sequence

$x_1, x_2, \ldots x_n$

for $x_j \in A$ $x_j \in A$

$\{x_n\}$ is a sequence in A.

$f(n) = \frac{1}{n}$ $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

→ The set of values in $\{x_n\}$ is called the range of the sequence

eg. $S_n = \frac{1}{n}$ infinite sequence. $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$. bounded.

eg. $\{i^n\}$, $i = \sqrt{-1}$.

$\{i, -1, -i, 1\}$. range is finite bounded.

$|i^n| \leq 1$ $\forall n$.

⇒ $\{p_n\}$ Convergence

A sequence $\{p_n\}$ converges in X (metric space) if $\varepsilon > 0$ $\exists$ an integer N s.t $n \geq N$ $d(p_n, p) < \varepsilon$.

$p_n \to p$ or $\lim_{n \to \infty} p_n = p$.

eg: $p_n = \{\frac{1}{n}\}$

$\lim_{n \to \infty} \frac{1}{n} = 0 \in \mathbb{R}$ $\therefore$ $\{\frac{1}{n}\}$ Converges in $\mathbb{R}$.

eg. Let X be set of positive real number.

$\{\frac{1}{n}\}$ is a sequence in X.

$\therefore$ $\{\frac{1}{n}\}$ does not converges in X.

So, Convergence of a sequence depends on both the sequence as well the metric space where it is defined.

So we write p_n Converges in X.

eg: $\{i^n\}$ doesn't converges.

eg: $p_n = n^2$ - unbounded, diverges, infinite

eg: $S_n = 1 + \frac{(-1)^n}{n}$ $\quad S_n \to 1$ bounded and have infinite range.

eg: $S_n = 1$ $\quad (n = 1, 2, 3, \ldots)$.

Converge to 1, is bounded and have finite range.

Theorem: Let $\{p_n\}$ be a sequence in X(metric space).

(a) p_n converges to $p \in X$ iff every nbhd of p contains all but finitely many of the terms $\{p_n\}$.

(b) If $p \in X$, $p' \in X$ and $\{p_n\}$ converges to p and p' then ~~p_n~~ $p = p'$.

(c) If $\{p_n\}$ converges then $\{p_n\}$ bounded

(d) If $E \subset X$ and p is a limit pt. of E then $\exists$ a sequence $\{p_n\}$ in E s.t $\lim_{n \to \infty} p_n = p$.

Proof: (a) Suppose $p_n \to p$

~~For fix $\epsilon > 0$, $V = d(p, q) < \epsilon$, $q \in X$, $q \in V$~~

Let V be a nbhd of p.

For some $\epsilon > 0$, $d(q, p) < \epsilon$, $q \in X \Rightarrow q \in V$.

Corresponding to this ϵ, $\exists$ integer N

$n \geq N \Rightarrow d(p_n, p) < \epsilon \Rightarrow p_n \in V$.

$\Rightarrow$ Suppose every nbhd of p containing all but finitely many of terms $\{p_n\}$.

Fix $\epsilon > 0$, let $V = \{p \in X \mid d(p, q) < \epsilon\}$

i.e V is a nbhd of p. By assumption $\exists$ N (corresponding to this V) s.t

$p_n \in V$ if if $n \geq N$. 14-09-18.

$d(p, p_n) < \varepsilon$ for $n \geq N$

$\Downarrow$

$\lim_{n\to\infty} p_n = p$.

(b) $p_n \to p$

$p_n \to p'$ $\Rightarrow$ $p = p'$

$p_n \to p$ means $\varepsilon > 0$ $\exists$ N_1 (integer) s.t $d(p_n, p) < \frac{\varepsilon}{2}$, $n \geq N_1$.

$p_n \to p'$ means $\varepsilon > 0$ $\exists$ N_2 (integer) s.t $d(p_n, p') < \frac{\varepsilon}{2}$, $n \geq N_2$.

$d(p, p') \leq d(p, p_n) + d(p_n, p')$

Let ~~N~~ be the max (N_1, N_2)

$N = \max(N_1, N_2)$

for $n \geq N$

$d(p, p') \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon$.

$\Rightarrow$ $d(p, p') = 0$. $\Rightarrow$ $p = p'$

(c) Given $p_n \to p$.

to show that p_n is bounded.

Choose $\varepsilon = 1$, $\exists$ N (integer) s.t $n \geq N$

$d(p_n, p) < 1$

$r = \max\{d(p_1, p), d(p_2, p), \ldots d(p_{N-1}, p), 1\}$.

Then $d(p_n, p) \leq r$ for $n = 1, 2, \ldots$

(d) As $p \in X$ is a limit point of E, for each +ve integer n, there is a point $p_n \in E$ s.t. $d(p_n, p) < \frac{1}{n}$.

By Archimedean principle

$\varepsilon > 0$, choose N s.t $\varepsilon N > 1$

If $n > N$, $\frac{1}{n} < \frac{1}{N} < \varepsilon$

$\Rightarrow$ $d(p_n, p) < \varepsilon$.

27-09-12

Let $\{p_n\}$ be a sequence in a metric space X.

$f: N \to X$

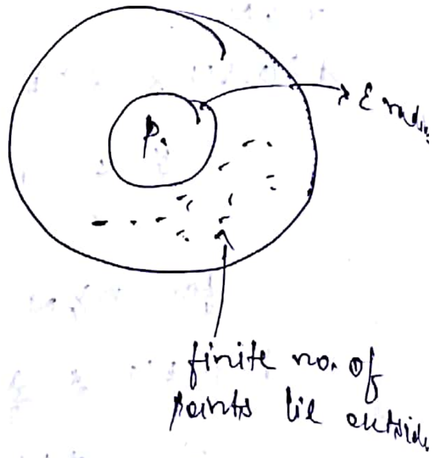
$f(n) = p_n$ $\qquad p_1, p_2, p_3, \ldots$

$(\frac{1}{n}) \to$ bounded $\qquad (n) \to$ unbounded.

$\{p_n\}$ (a sequence) is said to be convergent to an element p in X if $\varepsilon > 0$, $\exists$ a number N s.t. $n \geq N$

$$d(p, q) < \varepsilon.$$

$$\lim_{n\to\infty} p_n = p.$$



<u>Thm:</u> Suppose $\{s_n\}$, $\{t_n\}$ are complex sequences, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$ then

(a) $\lim_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t$

(b) $\lim_{n\to\infty} c s_n = c s$ $\qquad c \in R$

$\lim_{n\to\infty}(c + s_n) = c + s.$

(c) $\lim_{n\to\infty}(s_n t_n) = s t$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ provided $s_n \neq 0$ $(n = 1, 2, \ldots)$ and $s \neq 0$.

Scanned by CamScanner

28-09-18

Proof: $\varepsilon > 0$, $\exists$ integers N_1, N_2 s.t

(a) $n \geq N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$

$n \geq N_2 \Rightarrow |t_n - t| < \varepsilon/2$

$|(s_n + t_n) - (s+t)| = |(s_n - s) + (t_n - t)|$

$\leq |s_n - s| + |t_n - t|$

Let $N = max(N_1, N_2)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$.

(b) $|cs_n - cs| = \varepsilon \quad \forall n \geq N$

$= |c(s_n - s)|$

$= |c| \, s_n - s$

$\varepsilon > 0 \quad \exists N$ s.t $n \geq N$

$|s_n - s| = \frac{\varepsilon}{|c|} \quad c \neq 0$

(c) $|s_n t_n - st| = (n-1)(t_n - t) + s\varepsilon(t_n - t) + t(s_n - s)$,

$|s_n t_n - st| \leq (s_n - s)(t_n - t) + t \cdot + t(s_n - s) + s(t_n - t)|$

$\leq |s_n - s||t_n - t| + s|t_n - t| + |t(s_n - s)|$.

Given $\varepsilon > 0$, $\exists$ integers N_1, N_2 s.t $n \geq N_1 \Rightarrow |s_n - s| < \varepsilon$

$n \geq N_2 \Rightarrow |t_n - t| < \sqrt{\varepsilon}$

Take ~~Root~~ $N = max(N_1, N_2)$ i.e. $n \geq N$ $|(s_n - s)(t_n - t)| < \varepsilon$

~~i.e. $|t(s_n - s)| \to 0$~~ i.e $\lim_{n \to \infty} (s_n - s)(t_n - t) = 0$

$|s_n - s| < \varepsilon$ ~~$t \neq 0$, $n \to \infty$~~ $\forall \varepsilon > 0$, $n \geq N$

~~$\lim_{n \to \infty} \frac{1}{s_n} = \frac{1}{s}$.~~

$|t(s_n - s)| \to 0$

$s|t_n - t| + |t(s_n - s)|$

$|t_n - t| < \frac{t}{|s|}$ $\qquad |s_n - s| < \frac{t}{|t|}$

$\frac{t}{|s|}$ i.e $|s(t_n - t)| \to 0$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ $\quad s \neq 0$ if $s_n \neq 0$ $(n = 1, 2, \ldots)$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| = \frac{|s_n - s|}{|s_n s|} \qquad \frac{0}{s^2} \to 0.$$

As $s_n \to s$, $\epsilon = \frac{|s|}{2}$ $\exists$ k_1

s.t

$|s_n - s| < \frac{|s|}{2}$ for $n \geq k_2$

$\Rightarrow$ $||s_n| - |s|| \leq |s_n - s| < \frac{|s|}{2}$

$|s| - |s_n| < \frac{|s|}{2}$

$\Rightarrow$ $-|s_n| < \frac{|s|}{2} - |s|$ $\qquad |s_n s| \geq \frac{|s|^2}{2}$

$\Rightarrow$ $-|s_n| < -\frac{|s|}{2}$ $\quad \Rightarrow |s_n| \geq \frac{|s|}{2}$ $\quad (n \geq k)$.

Now

$\epsilon > 0$ $\exists$ k_2 integer s.t for $n \geq k_2$

$|s_n - s| < \frac{\epsilon}{2}|s|^2$.

Let $k = \max(k_1, k_2)$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s_n - s|}{|s_n s|} < \frac{\frac{\epsilon}{2}|s|^2}{|s|^2/2} = \epsilon.$$

Thm: Suppose $x_n \in R^k$ $(n = 1, 2, 3, \ldots)$ and $x_n = (\alpha_{1n}, \alpha_{2n}, \ldots \alpha_{k,n})$

(a) then $\{x_n\}$ converges to $x = (\alpha_1, \alpha_2, \ldots \alpha_n)$ iff

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \qquad (1 \leq j \leq k)$$

(b) Suppose $\{x_n\}$, $\{y_n\}$ are sequences in R^k, $\{\beta_n\}$ is a sequence of real numbers, and $x_n \to x$, $y_n \to y$

28-09-18

$\beta_n \to \beta$. Then $\lim_{n\to\infty} (x_n + y_n) = x + y$,

$\lim_{n\to\infty} x_n \cdot y_n = x \cdot y$ $\quad \lim_{n\to\infty} \beta_n x_n = \beta x$.

$\to$ $x_n = (\alpha_{1,n}, \alpha_{2,n}, \dots \alpha_k)$

$y = (\gamma_1, \gamma_2, \dots \gamma_k)$

$|x - y| = \sqrt{(\alpha_1-\gamma_1)^2 + (\alpha_2-\gamma_2)^2 + \dots + (\alpha_n-\gamma_n)^2}$

$|x| = \left(\sum_{j=1}^{n} |\alpha_i|^2\right)^{1/2}$

Proof:- (a) If $x_n \to x$ given $\varepsilon > 0$ $\exists$ an integer N s.t $n \geq N$ $\Rightarrow$ $|x_n - x| < \varepsilon$.

$|x_n - x| = |(\alpha_{1,n} - \alpha_1, \alpha_{2,n} - \alpha_2, \dots \alpha_{k,n} - \alpha_k)|$

$= \left(\sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2\right)^{1/2}$

$|\alpha_{j,n} - \alpha_j| \leq |x_n - x| < \varepsilon$ for $n \geq N$

i.e $|\alpha_{j,n} - \alpha_j| < \varepsilon$ for $n \geq N$

$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$ $\quad j = 1, 2, \dots k$

Converse part

Suppose $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$ $(1 \leq j \leq k)$

$\varepsilon > 0$ $\exists$ a N s.t $n \geq N$ implies

$|\alpha_{j,n} - \alpha_j| < \frac{\varepsilon}{\sqrt{k}}$ $(1 \leq j \leq k)$

hence $n \geq N$

$|x_n - x| = \left\{\sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2\right\}^{1/2}$

$< \left\{\frac{\varepsilon^2}{k} \cdot k\right\}^{1/2} = \varepsilon$.

Subsequences

Let $\{p_n\}$ be a sequence. Consider a sequence of +ve integers $\{n_k\}$ with the properties $n_1 < n_2 < n_3 < \dots < n_k$.

\# $\{p_{n_i}\}$ is called a subsequence of $\{p_n\}$.

The limit of the subsequence is said to be subsequential limit.

$p_n = (-1)^n$

$$(p_{2n}) = (p_2, p_4, p_6, \dots) = (1, 1, 1, \dots)$$

$$(p_{2n+1}) = (-1, -1, -1, \dots).$$

* A sequence $\{p_n\}$ converges to a point p iff every subsequence converges to some limit.

Thm (a): If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a pt. of X.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence (Bolzano-Weierstrass thm).

Proof:-

(a) Let E be the range of the sequence $\{p_n\}$. If E is finite then there is a pt. $p \in E$ and a sequence $\{n_i\}$ with $n_1 < n_2 < \dots$ such that

$$p_{n_1} = p_{n_2} = p_{n_3} = \dots = p \qquad p_{n_i} \to p.$$

If E is infinite, E has a limit pt. in X. Let p be a limit point in X. Choose n_1 s.t $d(p_{n_1}, p) < 1$. Having choosen $n_1, n_2, n_3, \dots n_{k-1}$

(This is possible as p is a limit pt.)

3-10-18

As every nbhd of p contains infinitely many points of E, $\exists$ $n_j > n_{j-1}$ s.t. $d(p, p_{n_j}) < \frac{1}{j}$

$$\Rightarrow \lim_{n \to \infty} p_{n_j} = p$$

(b) Since every bounded sequence in $\mathbb{R}^k$ is contained in a k-cell, which is compact subset of $\mathbb{R}^k$, by (a) every bounded sequence in $\mathbb{R}^k$ is ~~convergent~~ contains a convergent subsequence.

Thm.: The subsequential limits of a sequence $\{p_n\}$ in a metric space X forms a closed subset of X.

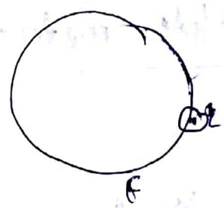
Proof: Let E^* denote the set of all sequential limits of the sequence $\{p_n\}$. Let q be a limit point of E^*. To show that $q \in E^*$. Choose n_1 s.t. $p_{n_1} \neq q$. (if no such n_1 exist then E^* contains single point only i.e closed).

$\delta = d(p, q) \neq 0$.

Choose $n_1, n_2, \dots, n_{i-1}$

Since q is a limit point of E^*, $\exists$ $x \in E^*$ s.t. $d(x, q) < \frac{\delta}{2^i}$. Since $x \in E^*$ $\exists$

$n_i > n_{i-1}$ s.t. $d(x, p_{n_i}) < \frac{\delta}{2^i}$.



$$d(q, p_{n_i}) \leq d(q, x) + d(x, p_{n_i})$$
$$< 2 \cdot \frac{\delta}{2^i}$$
$$= 2^{1-i} \delta \quad , i = 1, 2, 3, \dots$$

$p_{n_i} \to p$

$n_i > n_{i-1}$

$$\Rightarrow p_{n_i} \to q.$$

eg: $s_n = 1 + (-1)^n$

$E^* = \{0, 1\}$.

Cauchy Sequence.

(1.) Let X be a metric space. A sequence $\{p_n\}$ in X is said to be Cauchy sequence if $\exists$ an integer N s.t for $n \geq N$, $m \geq N$

$$d(p_n, p_m) < \epsilon.$$

$\Rightarrow$ eg. $p_n = \frac{1}{n}$ → Cauchy sequence

(2.) Let $E \subset X$, $S = \{d(p,q) ; p,q \in E\}$.

Sup S always exists.

diameter of E = sup S.

i.e $\boxed{\text{diam } E = \text{Sup } S.}$

$d(p,q) \leq \text{diam } E \quad \forall\, p,q \in E.$

$E_N = \{p_N, p_{N+1}, p_{N+2}, \dots\}$

$d(p_{N+i}, p_{n+j}) < \epsilon \quad \forall\, i,j = 1,2,3,\dots$

- A sequence $\{p_n\}$ is Cauchy iff $\lim_{N\to\infty} \text{diam } E_N = 0$.

i.e given $\epsilon > 0$, $N > 0$, $\text{diam } E_n < \epsilon$, $n \geq N$.

Complete metric space : A metric space is said to be complete if every Cauchy sequence converges in

$X = (0,1)$ $\quad p_n = \frac{1}{n}$

Limit pt. (not in X). so not complete.

$\Rightarrow$ R^k is a complete metric space.

Thm: (a) If $\bar{E}$ is the closure of a set E in a metric space X, then $\text{diam } E = \text{diam } \bar{E}$.

(b) If K_n is a sequence of compact set in X s.t,

$K_n \supset K_{n+1}$ $(n=1,2,\dots)$ and if $\lim_{n\to\infty} \dim K_n = 0$ then

$\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

(a) To show that $\dim \bar{E} = \dim E$.

$E \subset \bar{E}$

$\dim E \leq \dim \bar{E}$ (i)

for $\varepsilon > 0$, choose $p \in \bar{E}$, $q \in \bar{E}$. By definition of $\bar{E}$.

There are points p', q' in E such that

$d(p,p') < \varepsilon$, $d(q,q') < \varepsilon$

$d(p,q) \leq d(p,p') + d(p',q') + d(q',q)$

$< \varepsilon + d(p',q') + \varepsilon$

$\leq 2\varepsilon + \dim E$.

$\Rightarrow \dim \bar{E} \leq \dim E + 2\varepsilon$. — (ii)

by (i) and (ii) $\boxed{\dim E = \dim \bar{E}}$

(b) $\{K_n\}$ is a sequence of compact sets. ~~Koo~~

$K_n \supset K_{n+1}$ $(n=1,2,3,\dots)$

we know $\bigcap_{n=1}^{\infty} K_n \neq \phi = K$

to show that K contains exactly one point.

Suppose K contains more than one point then diameter of K.

$\dim K > 0$. But for each n, $K_n \supset K$.

$\Rightarrow \dim K_n \geq \dim K$.

which contradict to $\lim_{n\to\infty} \dim K_n = 0$

$\dim K_n < \varepsilon$, $n \geq N$, $m \geq N$. Choose ~~$\varepsilon_1 > 2\varepsilon$~~ $\varepsilon_1 = \dim K > 0$.

Thm: If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, (b) then $\{p_n\}$ converges to some point of X.

$\Rightarrow$ (a) Every convergent sequence is a Cauchy sequence. (any metric space)

(c) in R^k, every Cauchy sequence converges.

Proof (a) Suppose $p_n \to p$ in X. i.e given $\epsilon > 0$ $\exists$ N s.t

$n \geq N$ $\quad d(p_n, p) < \epsilon$

$m \geq N$ $\quad d(p_m, p) < \epsilon$

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < 2\epsilon$$

as soon as $n \geq N$, $m \geq N$ i.e $\{p_n\}$ is a Cauchy sequence.

Proof (b): Let $\{p_n\}$ be a Cauchy sequence in the compact set X. For $N = 1, 2, 3, \ldots$ Let E_N be the set consisting of $p_N, p_{N+1}, p_{N+2}, \ldots$ i.e $E_N = \{p_N, p_{N+1}, p_{N+2}, \ldots\}$.

$$\lim_{n\to\infty} \operatorname{diam} E_n = \lim_{n\to\infty} \operatorname{diam} \bar{E}_n = 0$$

Being $\bar{E}_N$ a closed subset of the compact set X, it is compact.

$$E_N \supset E_{N+1}$$

$$\Rightarrow \bar{E}_N \supset \bar{E}_{N+1}$$

$\bigcap_{N=1}^{\infty} \bar{E}_N$ consisting of one point.

i.e $\exists$ a unique $p \in X$ s.t $p \in \bar{E}_N$ $\forall N$.

$\lim_{N\to\infty} \bar{E}_N = 0$, means, $\epsilon > 0$ $\exists$ N_0 s.t

Scanned by CamScanner

$\operatorname{diam} \bar{E}_N < \varepsilon$ for $N \geq N_0$ 05-10-18

Since $p \in \bar{E}_N$, $\Rightarrow$ $d(p,q) < \varepsilon$ $\forall$ $q \in \bar{E}$.

Hence $\exists$ $p_n \to q$.

i.e $d(p, p_n) < \varepsilon$, $n \geq N_0$.

i.e $p_n \to p$.

Proof (c): Let $\{p_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. For $N = 1, 2, 3, \ldots$

Let $E_N = \{p_N, p_{N+1}, p_{N+2}, \ldots\}$

As $\lim_{N \to \infty} \operatorname{diam} E_N = 0$, for $\varepsilon = 1$, $\exists$ some N s.t $\operatorname{diam} E_N < 1$.

range of $\{p_n\}$ is the union of E_N and the finite set $\{p_1, p_2, \ldots p_{N-1}\}$.

Hence $\{p_N\}$ is bounded.

Since every bounded subset of $\mathbb{R}^k$, its closure is Compact in $\mathbb{R}^k$, $\{p_n\}$ Converges in $\mathbb{R}^k$.

$\rightarrow$ $\{p_n\} = i^n$ $|i^n| \leq 1$ $\lim_{n \to \infty} i^n$ does not exist.

A real sequence s_n is said to be monotonically increasing if

$s_n \leq s_{n+1}$ $(n = 1, 2, 3, \ldots)$

monotonically decreasing if $s_n \geq s_{n+1}$ $(n = 1, 2, 3, \ldots)$

A sequence is said to be monotonic if either it is monotonically increasing or monotonically decreasing.

Thm: Suppose $\{s_n\}$ is monotonic then $\{s_n\}$ converges iff it is bounded.

Proof: Suppose $\{s_n\}$ is monotonically increasing. i.e $s_n \leq s_{n+1}$.

Let E ~~be~~ $= \{s_1, s_2, s_3, \ldots\}$, E is the range of $\{s_n\}$.

If s_n is bounded, then E has a lub or supremum.

Let us denote it s. i.e $s_n \leq s$ $(n = 1, 2, 3, \ldots)$

For $\epsilon > 0$, $\exists N$ s.t. $S - \epsilon < S_N \leq S$. — (i)

Since $\{S_n\}$ is increasing $n \geq N \Rightarrow S_n \geq S_N$.

From (i) $S - \epsilon < S_N \leq S$ $\quad n \geq N$

i.e. $S_n \to S$.

⇛ Application

Q1. Let $\{y_n\}$ be a sequence defined by

$$y_1 := 1, \quad y_{n+1} := \frac{1}{4}(2y_n + 3) \qquad n \geq 1.$$

Show that $\lim_{n\to\infty} y_n = 3/2$

$y_1 = 1$.

$$y_2 = \frac{1}{4}(2y_1 + 3) = \frac{5}{4}.$$

$y_1 < y_2 < 2$.

To show that $y_n < 2 \quad \forall n$

$y_1 < 2$, $y_2 < 2$. Suppose $y_k < 2$ for some k then

$y_{k+1} = \frac{1}{4}(2y_k + 3) < \frac{1}{4}(2 \cdot 2 + 3) = \frac{7}{4} < 2$.

By method of induction $y_n < 2 \quad \forall n \in N$.

Show that $y_n < y_{n+1} \quad \forall n$.

$$\lim_{n\to\infty} y_{n+1} = \lim \frac{1}{4}\left(2 \lim_{n\to\infty} y_n + 3\right).$$

So bounded. $\Rightarrow$ limit exists

$$y = \frac{1}{4}(2y + 3).$$

$$\Rightarrow \frac{y}{2} = \frac{3}{4} \Rightarrow y = \frac{3}{2}.$$

Q2. If $s_1 = \sqrt{2}$, $s_{n+1} = \sqrt{2+\sqrt{s_n}}$. $(n=1,2,3,\ldots)$ DS-10-18

Show that $\{s_n\}$ converges and then $s_n < 2$ for $n=1,2,3,\ldots$

$s_2 = \sqrt{2+\sqrt{2}}$ $s_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$

$\{s_n\}$ converges to s.

$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} \inf s_n = s.$

$\{s_n\}$ be a bounded sequence.

$M_n = \sup\{s_n, s_{n+1}, s_{n+2}, \ldots\}.$

Since $\{s_{n+1}, s_{n+2}, \ldots\} \subset \{s_n, s_{n+1}, s_{n+2}, \ldots\}.$

$\sup\{s_{n+1}, \ldots\} \leq \sup\{s_n, s_{n+1}, \ldots\}.$

$M_{n+1} \leq M_n.$

$\{M_n\}$ is a decreasing sequence.

$\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} M_n$

$M_n = \inf\{s_n, s_{n+1}, \ldots\}$

$M_n \leq M_{n+1}$

$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \inf s_n$

eg. $s_n = (-1)^n$

$M_n = \sup\{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, \ldots\}$

$= \sup\{-1, 1, -1, 1, \ldots\}$

$= 1.$

$M_n = \inf\{(-1)^n, (-1)^{n+1}, \ldots\}$

$= \inf\{-1, 1, -1, 1, \ldots\}$

$= -1.$

Thm

$\{s_n\}$ infinite sequence.

Suppose there is a real number s satisfying ①

for every $\varepsilon > 0$ there is N for $n \geq N$, $s_n < s + \varepsilon$

(2) for every $\varepsilon > 0$ and $M > 0$ there is $n \geq M$ s.t

$s_n > s - \varepsilon$. Then $s = \lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} s_n$.

Some special sequence.

1. If $0 \leq x_n \leq s_n$ for $n \geq N$, where N is some fixed number and if $s_n \to 0$, then $x_n \to 0$.
2. If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$
3. If $p > 0$, then $\lim_{n\to\infty} (p)^{1/n} = 1$
4. $\lim_{n\to\infty} n^{1/n} = 1$
5. If $p > 0$ and $\alpha \in R$ then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$
6. If $|x| < 1$ then $\lim_{n\to\infty} |x| = 0$

A series $\sum_{n=1}^{\infty} a_n$ is called an infinite series.

$s_n = a_1 + a_2 + \dots + a_n$. ($n^{th}$ partial sum)

$\sum_{n=1}^{\infty} a_n < \infty \Leftrightarrow \{s_n\}$ converges.

$\sum_{n=1}^{\infty} \frac{1}{n}$ then show for $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ ~~diverge~~

$\Rightarrow$ $\sum a_n$ converges iff for every $\varepsilon > 0$, there is an integer N s.t $\sum_{k=n}^{m} |a_k| \leq \varepsilon$ if $m \geq n \geq N$.

This is Cauchy criterion of convergence. 10-10-18

Thm: If $\sum_{n=1}^{\infty} a_n < \infty$, then $\lim_{n\to\infty} a_n = 0$

Proof: $S_n = a_1 + a_2 + \dots + a_n$

$S_{n-1} = a_1 + a_2 + \dots + a_{n-1}$

$S_n - S_{n-1} = a_n$

$\lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} a_n$

$\Rightarrow$ ~~[illegible]~~ $S - S = \lim_{n\to\infty} a_n \Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}$

Converse of this theorem is not true.

eg. $\sum_{n=1}^{\infty} \frac{1}{n}$ $\qquad \lim_{n\to\infty} \frac{1}{n} = 0$

divergent.

1. Comparision Test

If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is fixed positive integer and if $\sum_{n=1}^{\infty} c_n < \infty \Rightarrow \sum a_n < \infty$

$|a_n| \geq b_n > 0$; $\sum b_n \to$ divergent $\Rightarrow \sum a_n$ divergent

for $n \geq N_0$

2. Ratio test The series $\sum a_n$

(a) Converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

for $n \geq N_0$

(b) diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

3. Root test.

given $\sum_{n=1}^{\infty} a_n$, put ~~find~~ $\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$ then if $\alpha < 1$ $\longrightarrow$ Converges

$\alpha > 1$ $\longrightarrow$ divergen

$\alpha = 1$ $\longrightarrow$ test fails.

4. Alternating series test

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

Suppose the sequence a_n of real no.s satisfies the following conditions :-

(a) alternating +ve and -ve sign

(b) $|a_n| \leq |a_{n-1}|$ $\forall n \in N$

(c) $\lim_{n \to \infty} a_n = 0$

then $\sum_{n=0}^{\infty} a_n$ is convergent.

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is Convergent.

✓ If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum a_n$ Converges absolutely.

✓ If $\sum a_n < \infty$ but $\sum |a_n| = \infty$ then $\sum a_n$ is Conditionally convergent.

$f : E \to Y$, X, Y are metric spaces and $E \subset X$.

Let p be a limit pt. of E. We write $f(x) \to q$ as $x \to p$, or $\lim_{x \to p} f(x) = q$. if there is pt q in Y with the following properties. For every $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t $d_Y(f(x), q) < \varepsilon$ for every $0 < d_X(x, p) < \delta$.

i.e

$\lim_{x \to p} f(x) = q$ $\varepsilon > 0$, $\exists$ $\delta > 0$ $d_Y(f(x), q) < \varepsilon$ whenever

$0 < d_X(x, p) < \delta$.

→ f is continuous at $x = p$ if $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t $d_Y(f(x), f(p)) < \varepsilon$, whenever $d_X(x, p) < \delta$.

→ $\lim_{x \to p} f(x) = q$, $\lim_{x \to p} g(x) = q_1$

then $\lim_{x \to p} (f(x) + g(x)) = q + q_1 = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$.

$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{q}{q_1}$; $q_1 \neq 0$

$\lim_{x \to p} \alpha f(x) = \alpha \lim_{x \to p} f(x)$; α is scalar.

$E \subset X$ $f : E \to \mathbb{R}^k$

f is a vector valued metric.

$f(x) \in \mathbb{R}^k$

$f(x) = (f_1(x), f_2(x), f_3(x) \dots, f_k(x))$

Scanned by CamScanner

$\Rightarrow f$ is continuous iff $f_i(x)$ is continuous. $i = 1, 2, \ldots, k$

Thm: Let X, Y, E, f and p be as in definition.

\# Then $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E s.t $p_n \neq p$, $\lim_{n \to \infty} p_n = p$.

Proof :- Suppose $\lim_{x \to p} f(x) = q$. Choose a sequence $\{p_n\}$ ~~such~~ satisfying $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$. Let $\epsilon > 0$ be given, then $\exists$ $\delta > 0$ s.t $d_Y(f(x), q) < \epsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$.

As $p_n \to p$, $\exists$ N s.t $n \geq N$ $0 < d_X(p_n, p) < \delta$. Thus for $n \geq N$ we have $d_Y(f(p_n), q) < \epsilon$.

i.e $\lim_{n \to \infty} f(p_n) = q$. for $\{p_n\}$ in E with $\lim_{n \to \infty} p_n = p$.

<u>Converse</u>

$\lim_{n \to \infty} f(x) = q$ for $\{p_n\}$ in E with $p_n \neq p$ $\lim_{n \to \infty} p_n = p$.

To show that $\lim_{x \to p} f(x) = q$.

Suppose $\lim_{x \to p} f(x) \neq q$ then $\exists$ $\epsilon > 0$ s.t for every $\delta > 0$ there exists a pt $x \in E$ (depending on δ), for which $d_Y(f(x), q) \geq \epsilon$ but $0 < d_X(x, p) < \delta$.

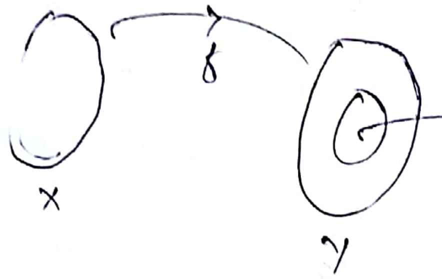
Taking $\delta_n = \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$ $0 < d_X(p_n, p) < \frac{1}{n}$ and $d_Y(f(p), q) \geq \epsilon$. which contradict the assumption.

Scanned by CamScanner

24/10/18

$f : \mathbb{R} \to \mathbb{R}$ f is said to be continuous at $x = a$ if given $\varepsilon > 0$ $\exists\ \delta(\varepsilon)$ s.t $|f(x) - f(a)| < \varepsilon$, $|x - a| < \delta$.

#



V is open subject of Y.

$f^{-1}(V)$ is open in X $\forall$ V open in Y.

inverse image of set $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$

$C \subset X$

$f(C) = \{ f(x) \mid x \in C \}$.

image of a set

A mapping $f : X \longrightarrow Y$ is continuous in X iff $f^{-1}(V)$ is open in X for every open set V in X.

Proof :- Suppose f is continuous in X and V is open in Y we have to show $f^{-1}(V)$ is open in X, i.e every point of $f^{-1}(V)$ is an interior pt. So, suppose $p \in X$ and $f(p) \in V$. Since V is open, $\exists\ \varepsilon > 0$ s.t $y \in V$ if $d_Y(f(p), y) < \varepsilon$ and since f is continuous at p, $\exists\ \delta > 0$ s.t if $d_X(x, p) < \delta$ then $d_Y(f(x), f(p)) < \varepsilon$.

$\Rightarrow f(x) \in V$ i.e $x \in f^{-1}(V)$ as soon as $d(x, p) < \delta$

24/10/18

Converse: Suppose $f^{-1}(V)$ is open in X for all open subset V in Y. To show that f is continuous in X. Fix $p \in X$ and $\varepsilon > 0$. Let V be set of all $y \in Y$ s.t $d_Y(y, f(p)) < \varepsilon$. i.e V is open subset of Y.
$f^{-1}(V)$ is open in X, and hence $\exists\ \delta > 0$ s.t
$f^{-1}(V)$ as soon as $d_X(x, p) < \delta$. But $f(x) \in V$
i.e $d_Y(f(x), f(p)) < \varepsilon$.

Corollary: $f: X \to Y$ is ~~open~~ continuous iff $f^{-1}(E)$ is closed in X $\forall$ E is closed in Y.

$$f^{-1}(E^c) = (f^{-1}(E))^c$$

$\Rightarrow E^c$ is open $\Rightarrow f^{-1}(E^c)$ is open

$(f^{-1}(E^c))^c = f^{-1}(E)$ is closed.

A mapping $f: E \to R^k$ is said to be bounded if $\exists$ a number M s.t $|f(x)| \leq M$ $\forall$ $x \in E$.

Thm: Let f be a continuous map from a compact metric space X into a metric space Y then $f(X)$ is compact.

proof: Let $\{V_\alpha\}$ be an open covering of $f(X)$.
To show that it has a finite subcovering.
$\{V_\alpha\}$ is open in Y. As f is continuous, $f^{-1}(V_\alpha)$ is open in X.

As X is compact $X \subset f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \dots \cup f^{-1}(V_{\alpha_n})$

$f(X) \subset f(f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \dots f^{-1}(V_{\alpha_n}))$

$f(X) \subset f(f^{-1}(V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}))$

$f(f^{-1}(A)) \subset A$

$f^{-1}(f(A)) \supset A$

$f(X) \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}$.

Hence $f(X)$ is compact.

Scanned by CamScanner

* If $f : X \longrightarrow R^k$ $\Rightarrow$ f is bounded.

$\uparrow$ Continuous $\quad$ $\uparrow$ Compact.

Proof: As $f(X)$ is compact subset of R^k

$\Rightarrow$ $f(X)$ is closed and bounded.

Hence $f(X)$ is bounded.

Thm: Suppose f is continuous 1-1 mapping of a compact space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x$ $\forall$ $x \in X$ is continuous mapping of Y onto X.

$$f : X \xrightarrow{\text{bijective}} Y \Rightarrow f^{-1} \text{ is continuous.}$$

$\uparrow$ Compact

Proof:

$f : X \rightarrow Y$

As f is bijective $f^{-1} : Y \rightarrow X$.

Let V be open in X

To show that $(f^{-1})^{-1}(V)$ is open.

i.e $f(V)$ is open.

V^c is closed. $\Rightarrow$ V^c is compact. (closed subset of a compact set)

$\Rightarrow$ $f(V^c)$ is compact.

$\Rightarrow$ $f(V^c)$ is closed.

$\Rightarrow$ $f(V^c)^c$ is open.

$\Rightarrow$ $f(V)$ is open

25-10-18

eg: $f : [0, 2\pi) \to S'$, $S' = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$

$f(t) = (\cos t, \sin t)$ $t \in [0, 2\pi)$

f^{-1} is not continuous.

→ Any bijective ~~map~~ continuous mapping, its inverse mapping may not be continuous.

☆ Uniformly continuous function:

$f : X \to Y$. f is said to be uniformly continuous on X if given $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t $d_Y(f(p), f(q)) < \varepsilon$ whenever $d_X(p, q) < \delta$ $\forall$ $p, q \in X$.

Here δ is independent of the point $x \in X$.

eg: $f(x) = \frac{1}{x}$ on $(0,1)$.

$\exists$ ε s.t $\forall$ $\delta > 0$ $\exists$ $x, y \in (0,1)$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

Choose a positive integer n s.t $\frac{1}{n} < \delta$.

Let $x_0 = \frac{1}{n}$, $y_0 = \frac{1}{n+1}$

Now $x_0, y_0 \in (0, 1)$, $|x_0 - y_0| = \left|\frac{1}{n} - \frac{1}{n+1}\right|$

$= \frac{1}{n(n+1)} < \frac{1}{n} < \delta$

$|f(x_0) - f(y_0)| = |n - (n+1)| = 1 > \varepsilon_0$.

Thm: Let f be a continuous map from a compact metric space to a metric space Y. Then f is uniformly continuous.

Proof: Let ε_0 is given. Since f is continuous we can associate to each point $p \in X$ a positive number $\phi(p)$ s.t $q \in X$, $d_X(q, p) < \phi(p) \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$. —(*)

Scanned by CamScanner

Let $J(p) = \{ q \in X \mid d_X(p,q) < \frac{1}{2}\phi(p) \}$

$J(p) \neq \phi$ as $p \in J(p)$ $\quad$ $\left| d_X(p,p) = 0 < \frac{1}{2}\phi(p) \right.$ $\downarrow$ non-ve

and the collection of all sets $J(p)$ is an open covering of X;

and since X is compact, there is a finite set of points $p_1, p_2, \ldots, p_n$ in X s.t $X \subset J(p_1) \cup J(p_2) \cup \ldots \cup J(p_n)$ $\;**$

Put $\delta = \frac{1}{2}\min(\phi(p_1), \phi(p_2), \ldots, \phi(p_n))$, $\delta > 0$.

Now, take p, q in X s.t $d_X(p,q) < \delta$.

By $(**)$ there is an integer $1 \leq m \leq n$ s.t $p \in J(p_m)$

hence $\quad d_X(p, p_m) < \frac{1}{2}\phi(p_m)$

$$d_X(q, p_m) \leq d_X(q,p) + d_X(p, p_m)$$
$$< \delta + \frac{1}{2}\phi(p_m)$$
$$\leq \phi(p_m)$$

$$d_Y(f(p), f(q)) \leq d_X(f(p), f(p_m)) + d_X(f(q), f(p_m))$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$f: X \to Y$
$\;\downarrow$
Compact

1. f is continuous $\Rightarrow$ $f(X)$ is compact subset of Y.

2. ~~for~~ $f: X \xrightarrow{\text{continuous}} Y$ $\quad$ f is uniformly continuous function.
$\;\downarrow$
compact

3. If E is a connected subset of X then $f(E)$ is connected.

Scanned by CamScanner

31-10-18

Proof :- Suppose $f(E)$ is not connected.

$f(E) = A \cup B$, $\quad A, B \neq \phi$, $\bar{A} \cap B = \phi$, $A \cap \bar{B} = \phi$.

$G = f^{-1}(A) \cap E$, $\quad H = f^{-1}(B) \cap E$.

$G \cup H = (E \cap f^{-1}(A)) \cup (f^{-1}(B) \cap E)$

$= E \cap (f^{-1}(A) \cup f^{-1}(B))$

$= E \cap f^{-1}(A \cup B)$

$= E$ $\quad \longrightarrow G \subset f^{-1}(A)$, $G \subset E$.

As $A \subset \bar{A} \Rightarrow f^{-1}(A) \subset f^{-1}(\bar{A})$

$G \subset f^{-1}(A) \subset f^{-1}(\bar{A})$

$\Rightarrow G \subset f^{-1}(\bar{A})$ $\uparrow$ closed.

$\Rightarrow \bar{G} \subset f^{-1}(\bar{A})$

~~Similarly~~ $H \subset f^{-1}(B)$

~~$\bar{G} \cup H \subset$~~ $\quad \bar{G} \cap H = \phi$.

Similarly $G \cap \bar{H} = \phi$.

$\bar{G} \cap H \subset f^{-1}(\bar{A} \cap B)$

$\bar{G} \cap H = \phi$.

$\therefore f(E)$ is connected

$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Thm: Let f be a continuous function in $[a, b]$. If $f(a) < f(b)$ and λ is a number s.t $f(a) < \lambda < f(b)$, then $\exists\ x \in (a, b)$ s.t $f(x) = \lambda$.

Proof: As $[a, b]$ is connected. (as it is an interval in $\mathbb{R}$.) $E = f([a, b])$ is a connected subset of $\mathbb{R}$ (as f is continuous). Now $f(a) \in E$, $f(b) \in E$. Since $f(a) < \lambda < f(b)$ and

E is connected $\lambda \in E$. So $\exists\, p \in [a,b]$, $f(p) = \lambda$.

~~[illegible]~~, if $p = a$, $f(a) = f(p) = \lambda$
$f(a) < \lambda$ ✗.

if $p = b$ $f(b) = f(p) = \lambda$
$f(b) > \lambda$ ✗.

$\therefore p \in (a,b)$.

Type of discontinuity.

$f(x+) \to$ right limit $f(x^+) \to$ left limit

(i) $f(x+) \neq f(x)$ $f : (a,b) \to \mathbb{R}$

(ii) $f(x+) = f(x-) \neq f(x)$ $a \leq x < b$
$a < x \leq b$.

Type 1 (simple) limit : both left and right limit exist.

Type 2 : both don't exist

$f : (a,b) \to \mathbb{R}$ $a < x < y < b$

if $f(x) \leq f(y)$ (monotonically increasing

$f(x) \geq f(y)$ monotonocally decreasing.

A function is said to be monotonic if either it is monotonical. increasing or monotonocally decreasing.

<u>Thm</u>: Let ~~A~~ f be monotonically increasing on (a,b). Then $f(x+)$ and $f(x-)$ exist ~~[illegible]~~ at every pt. ~~of~~ at x of (a,b). Same for monotonic decreasing function.

31-10-18

Corollary: Monotonic functions have no discontinuities of 2nd type. (as $f(x+)$, $f(x-)$ exist).

Thm: Let f be monotonic on (a,b). Then the set of points of (a,b) at which f is discontinuous is atmost countable.

$f:(a,b) \longrightarrow R$

f is diff. at x $\qquad \phi(t) = \dfrac{f(x)-f(t)}{x-t}$

$\lim_{t \to x} \phi(t) = f'(x)$

Thm: Suppose f is diff. in (a,b).

(a) If $f'(x) \geq 0$ $\forall x \in (a,b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$, $\forall x \in (a,b)$, then f is constant.

(c) If $f'(x) < 0$, $\forall x \in (a,b)$, then f is monotonically decreasing.

Let $f: X \to R$ be a given function. We say f has a local maximum at a point $p \in X$ if $\exists$ $\delta > 0$ s.t $f(q) \leq f(p)$ $\forall q$ with $d(p,q) < \delta$.

if $f(q) \geq f(p)$ $\rightarrow$ then f has a local minimum.

Thm: let f be defined on $[a,b]$. If f has a local maximum at a pt. $x \in (a,b)$, and $f'(x)$ exists, then $f'(x) = 0$.

Thm: Suppose f is a real differentiable fn. on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a pt. $x \in (a,b)$ s.t $f'(x) = \lambda$.

Proof:

$g(t) = f(t) - \lambda t$

~~$g(a) = f(a)$~~

$g'(t) = f'(t) - \lambda$

$g'(a) = f'(a) - \lambda < 0$

so $g(t_1) < g(a)$ for $t_1 \in (a,b)$ and $g'(b) > 0$

~~that~~ so that $g(t_2) < g(b)$ for some $t_2 \in (a,b)$

g has a minimum on $[a,b]$. i.e at some pt. x.

$a < x < b$ $\quad g'(x) = 0$.

$g'(x) = f'(x) - \lambda = 0$

$\Rightarrow f'(x) = \lambda$.

Rudin

Ex: Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b) and g be the inverse of f. Prove that g is differentiable and

$$g'(f(x)) = \frac{1}{f'(x)}$$

Let $a < x_1 < x_2 < b$, on $[x_1, x_2]$. f is differentiable

$f(x_2) - f(x_1) = (x_2 - x_1) f'(x^*)$, $x^* \in (x_1, x_2)$

$f(x_2) - f(x_1) > 0 \qquad f'(x^*) > 0$

$f(x_1) < f(x_2)$

As g is the inverse of f, $g(f(x)) = x$ and $x \in$

$f(g(y)) = y$, $y \in Y$.

$g'(f(x)) \times f'(x) = 1$

01-11-18

Mean value thm. and L' Hospital rule are not defined for complex valued function.

$$f(x) = e^{ix}$$

$$f: [0, 2\pi] \to \mathbb{C}$$

$f(x) = e^{ix}$.

$$f(2\pi) - f(0) = e^{i2\pi} - e^{i\cdot 0}$$

$$= 1 - 1 = 0$$

$$f'(x) = ie^{ix}$$

$$|f'(x)| = 1$$

So derivative never vanishes.

So. $f'(x) \neq 0 \quad \forall x$.

$$f: (0,1) \to \mathbb{R}$$

$$g: (0,1) \to \mathbb{C}$$

$f(x) = x \qquad g(x) = x + x^2 e^{i/x^2}$

$$\lim_{x\to 0} \frac{x}{x + x^2 e^{i/x^2}} = \frac{x}{x(1 + x e^{i/x^2})} = 1$$

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 0.$$

$$g'(x) = 1 + \left(2x - \frac{2i}{x}\right) e^{i/x^2} \qquad 0<x<1$$

$$|g'(x)| \geq \left|2x - \frac{2i}{x}\right| - 1$$

$$= \frac{2}{x}\sqrt{x^4+1} - 1$$

$$|g'(x)| \geq \frac{2}{x} - 1$$

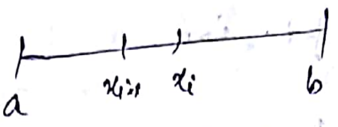
$$\left|\frac{f'(x)}{g'(x)}\right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x}$$

Riemann Integration.

Let ~~f be~~ a $f : [a,b] \to \mathbb{R}$ be a bounded fn..

Let $P = \{x_0, x_1, x_2, \ldots x_n\}$ with $a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n$

be a partition in $[a,b]$.



$$\Delta x_i = x_i - x_{i-1}$$

$$M_i = \sup f(x) \qquad x \in [x_{i-1}, x_i]$$

$$m_i = \inf(f(x)) \qquad x = [x_{i-1}, x_i]$$

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$\overline{\int_a^b} f\,dx = \inf U(f,P) \quad \forall \text{ partition } P. \qquad \text{upper Riemann integral}$$

$$\underline{\int_a^b} f\,dx = \sup L(f,P) \quad \forall P. \qquad \text{Lower Riemann integral.}$$

$f \in \mathcal{R}$ (Riemann integrable) if $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$

$$= \int_a^b f\,dx \quad \text{(Common)}.$$

Since f is bounded on $[a,b]$, $\exists$ bounds M, and s.t

$$m \leq f(x) \leq M. \qquad (a \leq x \leq b)$$

$$\Rightarrow m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

02-11-18

$M_i \leq M \quad \forall i \qquad . \qquad m \leq m_i \quad \forall i$

$$\sum_{i=1}^{n} M \Delta x_i = M \sum_{i=1}^{n} \Delta x_i = M(b-a)$$

$$\sum_{i=1}^{n} m \Delta x_i \leq \sum_{i=1}^{n} m_i \Delta x$$

Def: Let α be a monotonically ~~decreasing~~ increasing function on $[a,b]$. Since $\alpha(a), \alpha(b)$ are finite, α is bounded. Corresponding to each partition P ~~of~~ of $[a,b]$, we write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$, $\Delta\alpha_i \geq 0$, as α is monotonically increasing. Now for any real function f which is bounded on $[a,b]$, we put

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \; ; \; L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

(3.) $\overline{\int_a^b} f\,d\alpha = \inf_{\forall P} U(P,f,\alpha)$.

(4.) $\underline{\int_a^b} f\,d\alpha = \sup_{\forall P} L(P,f,\alpha)$

If L.H.S of (3) and (4) are equal, we denote their Common value by $\int_a^b f\,d\alpha$ or $\int_a^b f(x)\,d\alpha(x)$

→ Riemann - Stieltjes Integral

when $\alpha(x) = x$, it removes to Riemann integral.

$f \in R(\alpha)$. f is Riemann-Stieltjes integral.

⇒ P_1 and P_2 are two partition in $[a,b]$.

$P_1 \supset P_2$ →

refinement of P_2

We say P_1 is refinement of P_2. Given two partition P_1 and P_2, we say $P^* = P_1 \cup P_2$ is their common refinement.

<u>Thm</u>: If P^* is a refinement of P, then

(a) $L(P, f, \alpha) \leq L(P^*, f, \alpha)$ and

(b) $U(P^*, f, \alpha) \leq U(P, f, \alpha)$

So. $S_0 \subset S$, $S_0 \neq \phi$, S is bounded.

$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$.

To prove (a) suppose P^* contains just one pt. more than P. Let the extra point be x^* and suppose $x_{i-1} < x^* < x_i$, x_{i-1}, x_i are two consecutive points of P.

Put $w_1 = \inf(f(x))$
$x \in [x_{i-1}, x^*]$

$w_2 = \inf(f(x))$
$x \in [x^*, x_i]$

$m_i = \inf(f(x)) \quad x \in [x_{i-1}, x_i]$

$w_1 \geq m_i$

$w_2 \geq m_i$

Hence $L(P^*, f, \alpha) - L(P, f, \alpha)$

$= w_1[\alpha(x^*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x^*)]$
$- m_i[\alpha(x_i) - \alpha(x_{i-1})]$

$= w_1[\alpha(x^*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x_{i-1})]$
$- m_i[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})]$

02-11-18

$= (\omega_1 - m_i)\{\alpha(x^*) - \alpha(x_{i-1})\} + (\omega_2 - m_i)[\alpha(x_i) - \alpha(x^*)]$

$$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$

Thm: $f \in R(\alpha)$ on $[a,b]$ iff $\forall\ \varepsilon > 0\ \exists$ a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$. (*)

Proof: Suppose (*) holds

we know, $L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha)$

$$L(P,f,\alpha) - L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha - L \leq \overline{\int_a^b} f\,d\alpha - L \leq U(P,f,\alpha) - L(P,f,\alpha)$$

$$\Rightarrow 0 \leq \underline{\int_a^b} f - \overline{\int_a^b} f < \varepsilon$$

$$\Rightarrow \overline{\int_a^b} f - \underline{\int_a^b} f = 0 \quad \text{i.e. } f \in R(\alpha).$$

$0 \leq a < \varepsilon \Rightarrow a = 0$

Converse

$f \in R(\alpha)$

to show that $\forall\ \varepsilon > 0\ \exists$ a partition P s.t $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

$\varepsilon > 0$, $\exists$ a partition P_1 s.t

$$\frac{\varepsilon}{2} + \int_a^b f\,d\alpha > U(P_1,f,\alpha)$$

$$\Rightarrow U(P_1,f,\alpha) - \int_a^b f\,d\alpha < \frac{\varepsilon}{2} \quad \text{— (I)}$$

$$\int f\,d\alpha - L(P_2,f,\alpha) < \frac{\varepsilon}{2} \quad \text{— (II)}$$

$$P = P_1 \cup P_2$$

adding (I) and (II)

$U(P_1,f,\alpha) \leq U(P_2,f,\alpha) < \int f\,d\alpha + \frac{\varepsilon}{2} < L(P_1,f,\alpha) + \frac{\varepsilon}{2} \leq L(P,f,\alpha) + \varepsilon$

$\Rightarrow \boxed{U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon.}$

eg.

$$f(x) = \begin{cases} 1 & x \text{ is rational in } [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$M_j = \sup f(x) = 1. \quad x \in [x_{i-1}, x_i]$

$0 = m_j = \inf(f(x))$

$x \in [x_{i-1}, x_i]$

$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b-a$

$L(P,f) = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0.$

$\int_a^b f\,d\alpha = b-a$ and $\int_a^b f\,d\alpha = 0$

so not a Riemann integral.

<u>Thm</u>: If f is continuous on $[a,b]$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$

Proof: As α is monotonically increasing

$a < b \Rightarrow \alpha(a) < \alpha(b)$ i.e $\alpha(b) - \alpha(a) \geq 0$.

Let $\varepsilon > 0$ be given. Choose $\eta > 0$ s.t

$[\alpha(b) - \alpha(a)]\eta < \varepsilon$ $\quad \left(\eta < \frac{\varepsilon}{\alpha(b)-\alpha(a)}\right)$

if $\alpha(b) - \alpha(a) = 0$ then $0 \cdot \eta < \varepsilon$ $\quad$ for $\alpha(b) - \alpha(a) > 0$.

Since f is uniformly continuous on $[a,b]$, $\exists\ \delta > 0$ s.t

$|f(x) - f(t)| < \eta$ if $x \in [a,b]$

$t \in [a,b]$

$|x - t| < \delta$

If P is any partition of $[a,b]$ such that $\Delta x_i < \delta \ \forall i$, then $|f(x) - f(t)| < \eta$

$\Rightarrow |M_i - m_i| \leq \eta \quad (i = 1, 2, \dots n)$

and therefore $U(P,f,\alpha) - L(P,f,\alpha)$

$= \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i \leq \eta \sum_{i=1}^{n} \Delta\alpha_i = \eta[\alpha(b) - \alpha(a)] < \varepsilon$

Thm: Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$ and α is Continuous at every points at which f is discontinuous then $f \in R(\alpha)$.

f is integrable on $[a,b]$ iff f has countable number discountable points.

Properties of Integral

1. If $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$ on $[a,b]$ then

$f_1 + f_2 \in R(\alpha)$, $cf \in R(\alpha)$ for every constant.

$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$, $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$

2. $f_1(x) \leq f_2(x)$ in $[a,b]$

$\Rightarrow \int_a^b f_1(x)\, d\alpha \leq \int_a^b f_2\, d\alpha$

3. $\int_a^b f(x)\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha$. $\quad a \leq c \leq b$

4. $|f(x)| \leq M$ then $\left|\int_a^b f\,d\alpha\right| \leq M[\alpha(b) - \alpha(a)]$

5. If $f \in R(\alpha_1)$ $f \in R(\alpha_2)$ then

$f \in R(\alpha_1 + \alpha_2)$. $\int_a^b f\,d(\alpha_1 + \alpha_2)$.