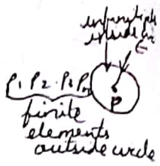
SEQUENCE AND SERIES

→ The set of values in $\{p_n\}$ is called the range of a sequence.

→ $f: \mathbb{N} \to X$,
$f(1), f(2) \ldots \quad f(n) = x_n$.
$\{x_n\}$ is a sequence.
if $x_i \in A \Rightarrow \{x_n\}$ is a sequence in A.

Eg. $f(n) = 1/n$ gives an infinite sequence.

→ A sequence $\{p_n\}$ converges in X (metric space) if $\forall \epsilon > 0, \exists N$ s.t. $\forall n \geq N \; d(p_n, p) < \epsilon$.
i.e. $\lim_{n\to\infty} p_n = p$. Eg. $\{\frac{1}{n}\}$ converges in $\mathbb{R}$.



→ Convergency of sequence depends on
1) elements of sequence
2) the metric space.

Eg $\{1/n\}$ converges in $\mathbb{R}$ but not in $\mathbb{R}^+$.

→ So if $\{p_n\}$ converges in X & $p_n \to p \Rightarrow p \in X$.

→ $\{i^n\}$ is a finite range, bounded sequence but doesn't converge.
$\{i, -1, -i, 1\}$

→ $\{p_n\}$; $p_n = n^2$ is unbounded, divergent, infinite range.

→ $S_n = 1 + \left(\frac{(-1)^n}{n}\right)$; is bounded, infinite range but converges

→ $S_n = 1$; $n = 1, 2 \ldots$ is bounded & finite range.
constant sequence always converges. but has NO limit pt.

<u>Theorem</u>: Let $\{p_n\}$ be a sequence in X.
(a). $\{p_n\}$ converges to $p \in X$ iff every nbhd of p contains all but finitely many (meaning all except finite) of the terms $\{p_n\}$
(b) If $p \in X$, $p' \in X$ & p_n converges to p & p' then $p = p'$
i.e. Limit of sequence is unique
(c) If $\{p_n\}$ converges then $\{p_n\}$ bounded [but reverse may not because Eg $\{\frac{1}{n}\}$ bounded but not converges]
(d) If $E \subset X$ & p is a limit pt. of E then $\exists$ a sequence $\{p_n\}$ in E s.t. $\lim_{n\to\infty} p_n = p$.

proof (a). Suppose $p_n \to p$. For some fix $\epsilon > 0$, $V = d(p, q) < \epsilon$, $q \in X$.

$V = \{ q \mid d(q, p) < \epsilon \}$

Now let W be a nbhd of p, of some $\epsilon > 0$,

$d(q, p) < \epsilon,\ q \in X \Rightarrow q \in W$.

Corresponding to this ϵ, $\exists\ N \in \mathbb{Z}$ s.t

$n \geq N \Rightarrow d(p_n, p) < \epsilon \Rightarrow p_n \in W$

conversely, suppose every nbhd of p contains all but finitely many p_n. Then prove $p_n \to p$.

~~Since~~ Fix $\epsilon > 0$, let $V = \{ q \in X \mid d(p, q) < \epsilon \}$

$\Rightarrow V$ is nbhd of p.

By assumption $\exists\ N$ corresponding to this V.

s.t. $p_n \in V$ if $n \geq N$.

$d(p, p_n) < \epsilon$ for $n \geq N \Rightarrow \lim\limits_{n \to \infty} p_n = p$.

(b) $p_n \to p \Rightarrow$ for $\epsilon > 0$ $\exists\ N_1 \in \mathbb{Z}$ st $d(p_n, p) < \frac{\epsilon}{2}$, $n \geq N_1$

$p_n \to p' \Rightarrow$ for $\epsilon > 0$ $\exists\ N_2 \in \mathbb{Z}$ st $d(p_n, p') < \frac{\epsilon}{2}$, $n \geq N_2$.

$\therefore d(p, p') \leq d(p, p_n) + d(p_n, p')$

let $N = \max(N_1, N_2)$.

$\Rightarrow d(p, p') \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ \forall\ n \geq N$.

$\Rightarrow d(p, p') = 0 \Rightarrow p, p'$.

(c). Given $p_n \to p$, to show that $\{p_n\}$ bdd.

choose $\epsilon = 1$, so $\exists\ N$ (integer) st $n \geq N$.

$d(p_n, p) < 1$.

$r = \max(d(p_1, p), d(p_2, p) \cdots d(p_n, p), 1)$

$\Rightarrow d(p_n, p) \leq r$; $n = 1, 2 \cdots$

(d) As $p \in X$ is limit pt of E,

for each +ve integer n, there is a pt $p_n \in E$ st $d(p_n, p) < \frac{1}{n}$.

By archimedian principle, given $\epsilon > 0$, choose N st $\epsilon N > 1$. If $N > n \Rightarrow \frac{1}{n} < \frac{1}{N} < \epsilon \Rightarrow d(p_n, p) < \epsilon$

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Theorem: Suppose $\{S_n\}$, $\{t_n\}$ are complex sequences and $\lim_{n\to\infty} S_n = S$, $\lim_{n\to\infty} t_n = t$, then.

a). $\lim_{n\to\infty} (S_n + t_n) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} t_n = S + t$.

b) $\lim_{n\to\infty} c.S_n = cS$

$\lim_{n\to\infty} (c + S_n) = c + s$

c) $\lim_{n\to\infty} S_n t_n = st$.

d) $\lim_{n\to\infty} 1/S_n = 1/S$. provided $S_n \neq 0$ $(n = 1, 2, \cdots)$ & $S \neq 0$.

Proof:

a) $\epsilon > 0$, $\exists$ integers N_1, N_2 s.t. $n \geq N_1 \Rightarrow |S - S_n| < \epsilon/2$ for $n \geq N_1$ and $n \geq N_2 \Rightarrow |t_n - t| < \epsilon/2$,

Let $N = \max(N_1, N_2)$

Then for $n \geq N$ $|S - S_n| + |t - t_n| \geq |S + t - (S_n + t_n)|$.

$\therefore |S + t - S_n + t_n| < \epsilon \quad \forall n \geq N$.

$\Rightarrow$ $\{S_n + t_n\}$ converges to $S + t$.

b). $|cS_n - cS| = |c(S_n - S)| = |c||S_n - S|$

$\forall \epsilon > 0 \ \exists N$ s.t $n \geq N \Rightarrow |S_n - S| < \epsilon/|c|$

$\Rightarrow |c||S_n - S| < |c| \epsilon/|c| = \epsilon$.

so $|cS_n - cS| < \epsilon \quad \forall n \geq N$.

$|||^{ly}$ ~~$|(c+S_n)$~~ $|c + S_n - (c + S)| < \epsilon \quad \forall n \geq N$. for some $N \in \mathbb{N}$.

c). $|S_n t_n - St| = |(S_n - S)(t_n - t) + S(t_n - t) + t(S_n - S)|$

$< |S_n - S||t_n - t| + |S(t_n - t)| + |t(S_n - S)|$

given $\epsilon > 0$, $\exists N_1, N_2$ st: $n \geq N_1 \Rightarrow |S_n - S| \leq \sqrt{3}$.

also $\exists n \geq N_2$ s.t $|t_n - t| < \sqrt{3}$ let $N = \max(N_1, N_2)$.

$\Rightarrow n \geq N$ then $|S_n - S||t_n - t| < \epsilon$.

$|S(t_n - t)| < \frac{|S|\epsilon}{|S|}$, i.e $|S(t_n - t)| \to 0$.

d) $\lim_{n\to\infty} 1/S_n = 1/s$. $s \neq 0$, $s_n \neq 0$ $(n = 1, 2, 3, \cdots)$

$$\left|1/S_n - 1/S\right| = \frac{|S_n - S|}{|S_n S|} = \frac{0}{S^2} \to 0 \quad (\text{if } S \neq 0)$$

$S_n \to S$, then for $\epsilon = |S|/2$, $\exists K_1$ st $|S_n - S| < |S|/2$

$\Rightarrow \big||S_n| - |S|\big| \le |S_n - S| < |S|/2 \Rightarrow |S| - |S_n| < |S|/2 \Rightarrow |S_n| > |S|/2$

Now $\epsilon > 0$, $\exists K_2 \in N$ st $|S_n - S| < \frac{\epsilon}{2}|S|^2$ $\forall n \geq K_2$

let $K = \max(K_1, K_2)$

$$\left|\frac{1}{S_n} - \frac{1}{S}\right| = \frac{|S_n - S|}{|S_n||S|} \le \frac{\epsilon/2\,|S|^2}{\frac{|S|}{2}.|S|} = \epsilon \quad \forall\ n \geq K$$

Theorem a) Suppose $x_n \in R^k$ ← finite $(n = 1, 2, \cdots)$ and $x_n = (\alpha_{1,n}, \alpha_{2,n}, \cdots \alpha_{k,n})$. Then $\{x_n\}$ converges to $x = (\alpha_1, \alpha_2 \cdots \alpha_k)$ if & only if

$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \quad (1 \le j \le k)$ [convergence implies componentwise converg. & vice versa]

b) Suppose $\{x_n\}$, $\{y_n\}$ are sequences in R^k, $\{\beta_n\}$ is a sequence of real nos & $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$, then $\lim_{n\to\infty}(x_n + y_n) = x + y$, $\lim_{n\to\infty}(x_n y_n) = xy$

$\lim_{n\to\infty} \beta_n x_n = \beta x$.

NOTE: If $x \in R^k$, then $|x| = \sqrt{\sum_{j=1}^{k} |\alpha_j|^2}$ (for safe side $|\alpha_j|^2 \neq |\alpha_j^2|$ (if $\alpha_j \in C$?))

Proof. a) If $x_n \to x$, given $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t. $n \geq N$,

$|x_n - x| < \epsilon$

$\therefore \left|(\alpha_{1,n} - \alpha_1, \alpha_{2,n} - \alpha_2, \cdots, \alpha_{k,n} - \alpha_k)\right| < \epsilon$

$\therefore \left[\sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2\right]^{1/2} < \epsilon$

Absolute value of a component always less than norm of whole vector $\Rightarrow$ If $x = (\alpha_1 \cdots \alpha_k)$ then $|\alpha_j| < |x|$.

$\therefore |\alpha_{j,n} - \alpha_j| \le |x_n - x| < \epsilon$ for $n \geq N$.

i.e. $|\alpha_{j,n} - \alpha_j| \le \epsilon$ for $n \geq N \Rightarrow \lim_{n\to\infty} \alpha_{j,n} = \alpha_j$ $(1 \le j \le k)$

Proof: (b) Suppose K contains more than one point.
$\Rightarrow$ diam $K > 0$. But for each n, K_n contains K. i.e $K_n \supset K$.
$\Rightarrow$ diam $K_n \geq$ diam $K >$ which contradicts $\lim_{n\to\infty}$ diam $K_n = 0$
why? let diam $K = \delta$.
so $\exists\, 0 < \epsilon < \delta$ s.t. diam $K_n > \epsilon \quad \forall n \in \mathbb{N}$.
so. $\lim_{n\to\infty}$ diam $K_n = 0$ is not possible.

Theorem: b) If X is a compact metric space and if $\{p_n\}$ is a cauchy sequence in X. Then $\{p_n\}$ converges to some point of
a) Every convergent sequence is a Cauchy sequence in any metric space X.
c) In $\mathbb{R}^k$, every cauchy sequence converges. (converse of (a))

Proof. a) Suppose $p_n \mapsto p$ in X. i.e. given $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $n \geq N$ $d(p_n, p) < \epsilon$ & $m \geq N, d(p_m, p) < \epsilon$
$d(p_n, p_m) \leq d(p_n, p) + d(p_m, p) < 2\epsilon$ for $m, n \geq N$.
$\Rightarrow \{p_n\}$ is a cauchy sequence.

b) Let $\{p_n\}$ be a cauchy sequence in the compact set X.
For $N = 1, 2, 3, \ldots$, let E_N be the set consisting of $p_N, p_{N+1}, \ldots$
i.e. $E_N = \{p_N, p_{N+1}, \ldots\}$
$\lim_{n\to\infty}$ diam $E_N = \lim_{n\to\infty} \overline{E_N} = 0$.
$\overline{E_N}$ is a closed subset of compact set X, also $E_N \supset E_{N+1}$.
$\Rightarrow \overline{E_N} \supset \overline{E_{N+1}}$. so from previous theorem,
$\bigcap_{N=1}^{\infty} \overline{E_N}$ consists of exactly one point. i.e $\exists$ unique. $p \in X$
such that $p \in \overline{E_N}\ \forall N$. also $\lim_{N\to\infty}$ diam $\overline{E_N} = 0$. meaning
$\forall \epsilon > 0$, $\exists\, N_0$ s.t diam $\overline{E_N} < \epsilon\ \forall N \geq N_0$
since $p \in \overline{E_N}$, $d(p, q) < \epsilon\ \forall q \in \overline{E}$. Hence $\exists\, p_n \mapsto q$. ; $p_n \in E$
~~i.e $d(p, p_n) < \epsilon\ \forall n \geq N_0$.~~

c) Let $\{p_n\}$ be a cauchy sequence in $\mathbb{R}^k$. For $N = 1, 2, 3, \ldots$
let $E_N = \{p_{N+1}, p_{N+2}, \ldots\}$.
As $\lim_{N\to\infty}$ diam $E_N = 0$, for $\epsilon = 1$ (fix), $\exists$ some N s.t diam $E_N < 1$
Range of $\{p_n\}$ is the union of E_N & the finite set $\{p_1, p_2, \ldots p_{N-1}\}$
Hence $\{p_n\}$ is bounded. Since every bounded subset of $\mathbb{R}^k$, its closure is compact in $\mathbb{R}^k$. So $\{p_n\}$ has a limit in its closure. So $\{p_n\}$ converges.
so, $\mathbb{R}^k$ is a complete metric space.

→ A real sequence is said to be monotonically increasing if $s_n \leq s_{n+1}$ $(n = 1, 2 \ldots)$
→ A real sequence is said to be monotonically decreasing if $s_n \geq s_{n+1}$ $(n = 1, 2 \ldots)$
→ A real sequence is said to be monotone if it is either monotonically increasing or monotonically decreasing.

Theorem: A monotonic sequence $\{S_n\}$ converges if & only if it is bdd.

Proof: Suppose $\{S_n\}$ is monotonically increasing.

Let $E = \{S_1, S_2, \cdots\}$ be the range of $\{S_n\}$.

If $\{S_n\}$ is bounded, then E has a lub.

Let $S = \text{lub} E$ i.e. $S_n \leq S$ $(n = 1, 2, 3, \cdots)$

$\Rightarrow$ For every $\epsilon > 0$, $\exists N$ s.t. $S - \epsilon < S_N \leq S$

since $\{S_n\}$ is increasing $n \geq N \Rightarrow S_n \geq S_N$.

so $S - \epsilon < S_n \leq S$; $n \geq N$ i.e. $S_n \longrightarrow S$.

Q Application: Let $\{y_n\}$ be a sequence defined as $y_1 = 1$, $y_{n+1} = \frac{1}{4}(2y_n + 3)$, $n \geq 1$.

show that $\lim_{n \to \infty} y_n = 3/2$.

$\rightarrow y_1 = 1$, $y_2 = \frac{1}{4}(2+3) = 5/4$. $y_1 < y_2 < 2$.

To show that $y_n < 2$ $\forall n$.

Suppose $y_K < 2$ for some K.

then $y_{K+1} = \frac{1}{4}(2y_K + 3) < \frac{1}{4}(2 \times 2 + 3) = 7/4 < 2$

So $y_n < 2$ $\forall n$ by method of induction.

To show $y_n < y_{n+1}$ $\forall n$.

$\lim_{n \to \infty} y_{n+1} = \frac{1}{4}(2 \lim_{n \to \infty} y_n + 3)$

let $\lim_{n \to \infty} y_n = y$

$\Rightarrow y = \frac{1}{4}(2y + 3) \Rightarrow y = 3/2$

Q. If $S_1 = \sqrt{2}$, $S_{n+1} = \sqrt{2 + \sqrt{S_n}}$. show that $\{S_n\}$ converges &.

$S_n < 2$ for $n = 1, 2, 3 \ldots$

. $S_1 = \sqrt{2} < 2$ (Base hypothesis)

Let $S_K < 2$ for some K (inductive hypothesis).

$\therefore S_{K+1} = \sqrt{2 + \sqrt{S_n}} < \sqrt{2+2} < 2$

$\Rightarrow S_N < 2$ $\forall n \in \mathbb{N}$.

~~To show $S_N \leq S_{N+1}$~~.

let $S = \lim_{N \to \infty} S_N$

$\therefore \lim_{N \to \infty} S_{N+1} = \lim_{N \to \infty} S_N$.

$\sqrt{2 + \sqrt{S}} = S$

$\therefore 2 + \sqrt{S} = S^2$

$\therefore S \approxeq 1.8311$ (by Newton Raphson)

* LIMIT SUPREMUM / LIMI[illegible]

→ Let $\{S_n\}$ be a bounded sequence and $M_n = \sup\{S_n, S_{n+1}, \ldots\}$

~~$\Rightarrow M_{n+1} \subset M_n$ & $\sup M_{n+1} \leq \sup M_n$~~ ~~(show check)~~

$\{S_n, S_{n+1} \ldots\} \supset \{S_{n+1}, \ldots\} \Rightarrow M_{n+1} < M_n$ (check proof)

$\Rightarrow \{M_n\}$ is a decreasing sequence.

→ $\lim_{n\to\infty} \sup\{S_n\} = \lim_{n\to\infty} M_n$. is called the limit supremum

Eg. Let $S_n = \{(-1)^n\}$, so $M_n = \sup\{(-1)^n, (-1)^{n+1}, \ldots\} = 1$.

→ $m_n = \inf\{S_n, S_{n+1}, \ldots\}$.

so $m_{n+1} > m_n$ (check proof)

$\{m_n\}$ is an increasing sequence.

$\lim_{n\to\infty} \inf\{S_n\} = \lim_{n\to\infty} m_n$ is called the limit infimum

Eg. Let $S_n = \{(-1)^n\}$, so $m_n = \inf\{(-1)^n, (-1)^{n+1}, \ldots\} = -1$

→ Let $\{P_n\}$ be an infinite sequence. Suppose $\exists$ real no. S satisfy.

1) for every $\epsilon > 0$, $\exists$ N s.t $P_n < S + \epsilon$.

2) for every $\epsilon > 0$ & $M > 0$ $\exists$ $n \geq M$, st $P_n > S - \epsilon$.

Then $S = \lim_{n\to\infty} \sup\{P_n\} = \overline{\lim_{n\to\infty}} P_n$.

* SOME SPECIAL SEQUENCES.

→ 1) If $0 \leq x_n \leq S_n$ for $n \geq N$ where N is some fixed no. & if $S_n \to 0$ then $x_n \to 0$. (squeezing lemma)

2) If $p > 0$, then $\lim_{n\to\infty} 1/n^p = 0$

3) If $p > 0$, then $\lim_{n\to\infty} p^{1/n} = 1$.

4) $\lim_{n\to\infty} n^{1/n} = 1$

5) If $p > 0$ and α is a real no then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

6) If $|x| < 1$ then $\lim_{n\to\infty} |x|^n = 0$.

→ A series of the form $\sum_{n=1}^{\infty} a_n$ is infinite series.

$S_n = a_1 + a_2 + \cdots + a_n$ is n^{th} partial sum

→ $\sum_{n=1}^{\infty} a_n$ converges if & only if $\{S_n\}$ converges.

→ $\sum a_n$ converges iff for every $\epsilon > 0$, $\exists$ N st.

$\sum_{k=n}^{m} |a_k| < \epsilon$ if $m \geq n \geq N$

(Cauchy criterion for convergence)

Theorem: If $\sum_{n=1}^{\infty} a_n < \infty$, then $\lim_{n\to\infty} a_n = 0$. (Converse not true)

Proof: $S_n = a_1 + \cdots + a_n$

$S_{n+1} = a_1 + \cdots + a_{n+1}$

$S_{n+1} - S_n = a_{n+1}$

$\lim_{n\to\infty} S_n = S < \infty$

$\therefore \lim_{n\to\infty}(S_{n+1} - S_n) = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = S - S = 0 = \lim_{n\to\infty} a_n$

- Comparison Test:

If $|a_n| \le C_n$ for $n \geq N_0$ where N_0 is fixed +ve integer

and if $\sum_{n=1}^{\infty} C_n < \infty \Rightarrow \sum a_n < \infty$.

If $|a_n| \geq b_n$ for $n \geq N_0$ & $\sum b_n = \infty$ then $\sum a_n = \infty$.

- Ratio Test:

The series $\sum a_n$.

a) converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for $n \geq N_0$.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ further test is required.

- Root Test:

Given $\sum_{n=1}^{\infty} a_n$, put $\alpha = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

Then

a) if $\alpha > 1$, $\sum a_n < \infty$

b) if $\alpha > 1$, $\sum a_n = \infty$

c) $\alpha = 1$, the test gives no information.

- Alternating Series

$\sum_{n=1}^{\infty} (-1)^n a_n \qquad a_n > 0$.

- Alternating Series Test

Suppose the sequence $\{a_n\}$ of real nos satisfies

(a) alternating + & − (b) $|a_n| \le |a_{n-1}|$ $\forall n$ (c) $\lim_{n\to\infty} a_n = 0$.

Then the series $\sum a_n$ is convergent

A series Σa_n is called absolutely convergent if $\Sigma |a_n|$ converges.

→ $f: E \to Y$, X & Y are metric spaces & $E \subset X$.
Let p be a limit pt of E. We write $f(x) \to q$ as $x \to p$ or $\lim_{x \to p} f(x) = q$ if there is a point q in Y with the following property (p may not $\in E$). For every $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $d_Y(f(x), q) < \epsilon$ whenever $0 < d_X(x, p) < \delta$.

→ f is continuous at $x = p$ if $\forall\, \epsilon > 0$ $\exists\, \delta > 0$, s.t. $d_Y(f(x), f(p)) < \epsilon$ whenever $0 < d_X(x, p) < \delta$

→ $\lim_{x \to p} f(x) = q$ & $\lim_{x \to p} g(x) = q_1$; $\lim_{x \to p} (f(x) + g(x)) = q + q_1$
$= \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$

→ $\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{q}{q_1} = \frac{\lim_{x \to p} f(x)}{\lim_{x \to p} g(x)}$

→ $\lim_{x \to p} [\alpha f(x)] = \alpha \lim_{x \to p} f(x) = \alpha q$.

→ $f: E \to R^k$; $E \subset X$ is called a vector valued function
$f(x) \in R^k$; $f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$; $f_i: E \to R$, $i = 1, \ldots k$

→ f is continuous iff f_i is continuous

Theorem: Let X, Y, E, f and p be as in definition. Then $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E s.t. $p_n \neq p$ $\lim_{n \to \infty} p_n = p$

Proof: Suppose $\lim_{x \to p} f(x) = q$. Choose a sequence $\{p_n\}$ satisfying $p_n \neq p$. $\lim_{n \to \infty} p_n = p$. Given $\epsilon > 0$. $\exists\, \delta > 0$ s.t. $d_Y(f(x), q) < \epsilon$ if $x \in E$ & $0 < d_X(x, p) < \delta$.
As $p_n \to p$, $\exists\, N$ s.t. $d_X(p_n, p) < \delta$, $n \geq N$
Thus for $n \geq N$, we have $d_Y(f(p_n), q) < \epsilon$
i.e $\lim_{n \to \infty} f(p_n) = q$ for $\{p_n\}$ in E with $\lim_{n \to \infty} p_n = p$.

Conversely, given $\lim_{n\to\infty} f(p_n) = q$ for $\{p_n\}$ in E with $p_n \neq p$ & $\lim_{n\to\infty} p_n = p$. To show that $\lim_{x\to p} f(x) = q$. Suppose $\lim_{x\to p} f(x) \neq q$; Then $\exists\ \epsilon > 0$ s.t. for every $\delta > 0$. $\exists\ p \in E$ (depending on δ) for which $d_Y(f(x), q) > \epsilon$ but $d_X(x, p) < \delta$.

Taking $\delta_n = \frac{1}{n}$ $(n = 1, 2, 3, \dots)$ $\quad 0 < d_X(p_n, p) < \frac{1}{n}$ & $d_Y(f(p_n), q) \geq \epsilon$ which contradicts the assumption.

→ f is continuous on E if f is continuous at every pt of E.

→ f is not continuous at a point p if $\exists\ \epsilon > 0\ \forall\ \delta > 0$ & $\exists\ x \in E$ with $d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) \geq \epsilon$.

→ To show $f(x) = \begin{cases} 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ is discontinuous.

$|x| < \delta \Rightarrow \frac{1}{|x|} > \frac{1}{\delta} > \epsilon$.

→ If $f : E \to Y$. where E is isolated set, then f is continuous on E.

Proof: Let $p \in E$. Now for any $\epsilon > 0$, we choose a $\delta > 0$ st.

$d_X(x, p) < \delta \Rightarrow x = p$.

$d_Y(f(x), f(p)) = 0 < \epsilon$

Eg. Any function from $\mathbb{N} \to \mathbb{R}$ or $\mathbb{Z} \to \mathbb{R}$ is continuous as $\mathbb{N}$ & $\mathbb{Z}$ are isolated set.

Theorem

$f : X \to Y$; X & Y are two topological spaces. f is continuous on X iff $\forall$ open subset $V \subset Y$, $f^{-1}(V)$ is open in X

Here: $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$, $f(E) = \{f(x) \mid x \in E\}$.

Proof: Suppose f is continuous in X and V is open in Y.

We have to show $f^{-1}(V)$ is open in X i.e every pt of $f^{-1}(V)$ is an interior pt. So, suppose $p \in X$ and $f(p) \in V$. Since V is open, $\exists\ \epsilon > 0$ s.t. if $y \in V$ if $d_Y(f(p), y) < \epsilon$ & since f is continuous at p, $\exists\ \delta$ s.t. if $d_X(x, p) < \delta$, then $d_Y(f(x), f(p)) < \epsilon \Rightarrow f(x) \in V$ or $x \in f^{-1}(V)$ ($\because V$ is open)

Conversely, suppose $f^{-1}(V)$ is open in X for all open subsets V in Y. To show that f is continuous on X.
Fix $p \in X$ & $\epsilon > 0$, Let V be set of all $y \in Y$ s.t $d_Y(y, f(p)) < \epsilon$
i.e. V is open subset of Y. $f^{-1}(V)$ is open in X.
and hence $\exists\ \delta > 0$ s.t $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$
But $f(x) \in V$, $d_Y(f(x), f(p)) < \epsilon$

<u>Corollary</u>: $f: X \to Y$ is continuous iff $f^{-1}(E)$ is closed in X $\forall$ closed E in Y.

<u>Proof</u>: E^c is open.

$$f^{-1}(E^c) = (f^{-1}(E))^c.$$

$\left[\begin{array}{l}\text{If } f \text{ is continuous} \\ f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \\ f^{-1}(E^c) = (f^{-1}(E))^c\end{array}\right]$

$\Rightarrow f^{-1}(E)$ is closed.

→ A <u>mapping</u> $f: E \to \mathbb{R}^k$ is said ~~to be~~ <u>bounded</u> if $\exists\ M$ s.t $\underset{\text{(norm)}}{\|f(x)\|} \leq M\ \forall\ x \in E$

Theorem: Let f be a continuous map from a compact metric space X into a metric space Y. Then $f(X)$ is compact.

Proof: Let $\{V_\alpha\}$ be an open covering of $f(X)$. To show that it has a finite subcovering.
V_α is open in Y, As f is continuous, $f^{-1}(V_\alpha)$ is open in X. As X is compact.

$$X \subset f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cdots f^{-1}(V_{\alpha_n}).$$

$$f(X) \subset f(f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cdots f^{-1}(V_{\alpha_n}))$$

$$= f(f^{-1}(V_{\alpha_1} \cup V_{\alpha_2} \cdots V_{\alpha_n})) \subset V_{\alpha_1} \cup V_{\alpha_2} \cdots V_{\alpha_n}$$

$$f(X) \subset V_{\alpha_1} \cup V_{\alpha_2} \cdots V_{\alpha_n}.$$

$\Rightarrow X$ has a a finite subcover.

$\boxed{\text{NOTE: } f(f^{-1}(A)) \subset A \quad f^{-1}(f(A)) \supset A}$

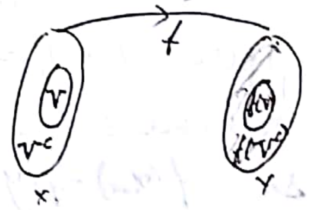
→ Open Mapping: A f is open map if the image is open for every open set in domain.

→ If $f: X \longrightarrow \mathbb{R}^k$ then,
↑ compact
↑ continuous

f is bounded. as $f(X)$ is compact subset of $\mathbb{R}^k$
$\Rightarrow f(X)$ is closed & bounded } proof
$\Rightarrow f(X)$ is bounded

Theorem: Suppose f is continuous 1-1 mapping of a compact set X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x \ \forall x \in X$ is continuous mapping of Y onto X.

Proof: $f: X \to Y$. As f is bijective $f^{-1}: Y \to X$.
Let V be open in X. To show that $(f^{-1})^{-1}(V)$ is open.
(or) To show $f(V)$ is open
V^c is closed. $\Rightarrow V^c$ is compact
$\Rightarrow f(V^c)$ is compact subset of Y. (from prev. theorem)
$\Rightarrow f(V^c)$ is closed
$\Rightarrow f(V^c)^c$ is open.
$\Rightarrow f(V)$ is open.



The converse is not true. (counterexample).
Eg. $f: [0, 2\pi) \to \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$
$f(t) = (\cos t, \sin t) \quad 0 \le t < 2\pi$

$[0, 2\pi)$ is not closed & hence not compact

here f is bijective & continuous.
However f^{-1} is not continuous.
$f^{-1}: \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\} \to [0, 2\pi)$.

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UNIFORMLY CONTINUOUS

→ $f: X \to Y$ is said to be uniformly continuous on X. (not at a point)
if given $\epsilon > 0$ $\exists$ $\delta > 0$ s.t
$d_Y(f(p), f(q)) < \epsilon$ when $\forall p, q \in X$, $d_X(p,q) < \delta$.

→ δ depends on ϵ and independent of any $p \in X$.

→ uniform continuity at a point is meaningless

→ $\delta(\epsilon)$ is one which will do the job for all points in X. but it is not the case in only continuous case.

Eg. $f(x) = \frac{1}{x}$; $x \in (0,1)$. is continuous.

→ To show f is not uniformly continuous. ($\because (0,1)$ not compact)
i.e $\exists \epsilon$ s.t $\forall \delta > 0$ $\exists$ $x, y \in (0,1)$ with $|x-y| < \delta$
and $|f(x) - f(y)| \geq \epsilon$
choose 'n' s.t $\frac{1}{n} < \delta$ (archimedian principle)
choose $\epsilon < 1$.
· let $x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$; $x_n, y_n \in (0,1)$. (you can choose any two points)
$|x_n - y_n| = \frac{1}{n(n+1)} < \frac{1}{n} < \delta$
$|f(x_n) - f(y_n)| = 1$
Since we choose $\epsilon < 1$ (we have freedom).
so $|f(x_n) - f(y_n)| \geq \epsilon$ for $|x_n - y_n| < \delta$ $\forall \delta$
so f is not uniformly continuous.

Theorem: Let f be a continuous map from a compact metric space X to a metric space Y. Then f is uniformly continuous.

Proof: Let $\epsilon > 0$ be given. Now, $\because f$ is continuous, we can associate to each $p \in X$, a +ve no. $\phi(p)$ s.t $q \in X$ $d_X(p,q) < \phi(p)$ ~~$d_X(p,q) < \phi(p)$~~ $\Rightarrow d_Y(f(p), f(q)) < \frac{\epsilon}{2}$
Let $J(p) = \{q \in X \mid d_X(p,q) < \frac{1}{2}\phi(p)\}$
$J(p)$ is non empty as $p \in J(p)$ as $d_X(p,q) < \frac{1}{2}\phi(p)$ → non negative
The collection $\{J(p) \mid p \in X\}$ is an open covering of X.
$\because X$ is compact there is a finite set of points $\{p_1 \cdots p_n\}$ s.t $X \subset \bigcup_{i=1}^{n} J(p_i)$
~~choose~~ $\delta = \frac{1}{n}\min\{\phi(p_1), \cdots \phi(p_n)\}$. Now take $p, q \in X$ s.t $d_X(p,q) \leq \delta$. By 2. $\exists$ m $1 \leq m \leq n$ st $p \in J(p_m)$.
Hence $d_X(p, p_m) < \frac{1}{2}\phi(p_m)$. $d_X(q, p_m) \leq d_X(q,p) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m)$ $\leq \phi(p_m)$

$$d_Y(f(p), f(q)) \le d_Y(f(p), f(p_n)) + d_Y(f(q), f(p_n))$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

→ Connectedness & Compactness are topological property.

Theorem: If E is a connected subset of compact metric space X & f is a continuous map from X to metric space Y. Then $f(E)$ is connected.

Proof: Suppose $f(E)$ is not connected.

$f(E) = A \cup B$, $A, B \neq \phi$ $\bar{A} \cap B = \phi$, $A \cap \bar{B} = \phi$.

$G = f^{-1}(A) \cap E$, $H = f^{-1}(B) \cap E$

$$G \cup H = E \cap f^{-1}(A) \cup E \cap f^{-1}(B)$$
$$= E \cap (f^{-1}(A) \cup f^{-1}(B)) = E \cap f^{-1}(A \cup B)$$
$$= E$$

Note: $A \subset \bar{A} \Rightarrow f^{-1}(A) \subset f^{-1}(\bar{A})$

$\because G \subset f^{-1}(A)$

$\Rightarrow G \subset f^{-1}(\bar{A})$

$\bar{A}$ is closed $\Rightarrow \bar{G} \subset f^{-1}(\bar{A})$

|||ly $H \subset f^{-1}(B)$

$$\bar{G} \cap H \subset f^{-1}(\bar{A}) \cap f^{-1}(B)$$
$$= f^{-1}(\bar{A} \cap B) = f^{-1}(\phi) = \phi.$$

$\bar{G} \cap H = \phi. \Rightarrow E$ is separated.

Theorem: Let f be a continuous function on $[a,b]$. If $f(a) < f(b)$ and λ is a number s.t $f(a) < \lambda < f(b)$, then $\exists$ $x \in (a,b)$ s.t $f(x) = \lambda$.

Proof: As $[a,b]$ is connected (as it is an interval in $\mathbb{R}$).
$\Rightarrow f([a,b])$ is a connected subset of $\mathbb{R}$ as f is continuous. Now $f(a) \in E$ & $f(b) \in E$.
~~so~~ ~~since~~ $f(a) < \lambda < f(b)$ and E is connected then $\lambda \in E$, so $\exists$ $p \in [a,b]$ s.t. $f(p) = \lambda$.
$p \neq a$ since $f(a) < \lambda$ ~~or~~ i.e. $f(a) = f(p) = \lambda$
$p \neq b$ since $f(b) > \lambda$ i.e. $f(b) = f(p) = \lambda$

→ A type of discontinuity
If $f : (a,b) \to \mathbb{R}$.
if $a \leq x \leq b$ $f(x+) \to$ right limit
$f(x-) \to$ left limit

$f(x^+) \neq f(x^-)$ (i)
$f(x^+) = f(x^-) \neq f(x)$ (ii)

$x^- \quad x \quad x^+$

1. Type one discontinuity : Both left & right limit exist
2. Type two discontinuity :
   Eg. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$

→ A function is said to be monotone if it is either monotonically increasing or decreasing.
i.e $a < x < y < b$ then $f(x) \leq f(y)$ in inc, $f(x) \geq f(y)$ in dec.

Theorem: Let f be monotonically increasing on (a,b). Then $f(x^+)$ & $f(x^-)$ exist at every pt of x of (a,b).

Corollary: Monotonic functions have no discontinuities of 2^{nd} type (as $f(x+)$, $f(x-)$ exist).

Theorem: Let f be monotonic on (a,b), then set of points of (a,b) at which f is discontinuous is atmost countable.

NOTE: f is differentiable at x if $\phi(t) = f'(x)$.

$$\phi(t) = \frac{f(x)-f(t)}{x-t}$$

if $x \in \mathbb{R}^n$, division is not define so we divide by say eg $\|x-t\|$.

Theorem: Suppose f is differentiable in (a,b).

a) If $f'(x) \geq 0 \ \forall x \in (a,b)$ then f is monotonically increasing

b) If $f'(x) \leq 0 \ \forall x \in (a,b)$, then f is monotonically decreasing

c) If $f'(x) = 0 \ \forall x \in (a,b)$, then f is constant.

→ Let $f: X \to \mathbb{R}$ be a given vector. We say f has a local maximum at pt $p \in X$ if $\exists \ \delta > 0$ s.t. $f(q) \leq f(p) \ \forall q$ with $d(p,q) < \delta$

We say f has a local minimium at $p \in X$ if $\exists \ \delta > 0$ s.t $f(q) \geq f(p) \ \forall q$ with $d(p,q) < \delta$

Theorem: Let f be defined on $[a,b]$: If f has local (minimum/maximum) at a pt $x \in (a,b)$ and $f'(x)$ exists, then $f'(x) = 0$.

Theorem: Suppose f is a real differentiable function on $[a,b]$ and suppose and $f'(a) < \lambda < f'(b)$, then $\exists \ x \in (a,b)$ s.t. $f'(x) = \lambda$.

Proof: Let $g(t) = f(t) - \lambda t$

$g'(t) = f'(t) - \lambda \Rightarrow g'(a) = f'(a) - \lambda < 0$,

so $g(t_1) < g(a)$ for $t_1 \in (a,b)$, $t_2 \in (a,b)$

also $g'(b) > 0$ so that $g(t_2) < g(b)$ for

$\Rightarrow g$ has a minumum on $[a,b]$ i.e at some $a<x<b$

$g'(x) = 0 \Rightarrow f'(x) - \lambda = 0 \Rightarrow f'(x) = \lambda$

Q. Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b). Let g be the inverse of f. Prove that g is differentiable and that $g'(f(x)) = \frac{1}{f'(x)}$ $(a<x<b)$.

→ Let $a < x_1 < x_2 < b$. f is differentiable on $[x_1, x_2]$

$f(x_2) - f(x_1) = (x_2 - x_1) \cdot f'(x^*)$; $x^* \in (x_1, x_2)$

$\because f'(x^*) > 0$

$\Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1)$ if $x_2 > x_1$

→ f is strictly increasing

As g is the inverse of f,

$g(f(x)) = x$ & $f(g(y)) = y$.

$\Rightarrow g'(f(x)) \cdot f'(x) = 1$. → $g'(f(x)) = \frac{1}{f'(x)}$

→ MVT and L'Hôpital's Rule are not ~~defined~~ valid for complex valued functions.

Eg. $f(x) = e^{ix}$

$f : [0, 2\pi] \longrightarrow \mathbb{C}$

$f(2\pi) - f(0) = e^{i(2\pi)} - e^{i0}$

$= 1 - 1 = 0$

$\Rightarrow$ by mvt $\exists\, c \in (0, 2\pi)$ s.t $f'(c) \cdot (2\pi - 0) = 0$

$\Rightarrow f'(c) = 0$ for some $c \in (0, 2\pi)$. — (1)

$f'(x) = i \cdot e^{ix}$

$|f'(x)| = 1 \;\; \forall x \in (0, 2\pi)$

$\Rightarrow f'(x) \neq 0 \;\; \forall x \in (0, 2\pi)$ — (2)

which is a contradiction from (1) & (2)

$\Rightarrow$ MVT not valid.

Eg. $f : (0,1) \to \mathbb{C}$

$f(x) = x$, $g(x) = x + x^2 e^{(i/x^2)}$

$$\lim_{x\to 0} \frac{f}{g} = \lim_{x \to 0} \frac{x}{x + x^2 e^{(i/x^2)}} = \frac{x}{x(1 + x e^{i/x^2})} = 1$$

But if

lim

$x \to 0$

* RIEMA

→ Let $f : [a,$

Let $P = \{$

s.t $a = x$

in $[a, b$

Let M

m

Let U(

$\int_a^b$

$\int_a^b$

But if we apply L'Hôpital

$$g'(x) = 1 + 2xe^{i/x^2} + ie^{(i/x^2)}\left(-\frac{2}{x^3}\right)$$

$$= 1 + \left(2x - \frac{2i}{x}\right)e^{i/x^2}$$

$$|g'(x)| \geq \left|2x - \frac{2i}{2x}\right| = \frac{2}{x}\sqrt{x^4+1} - 1$$

$$\Rightarrow |g'(x)| \geq \frac{2}{x} - 1.$$

$$\left|\frac{f'(x)}{g'(x)}\right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x}.$$

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0 \neq 1$$

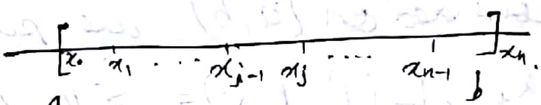
$\Rightarrow$ L'Hôpital not valid

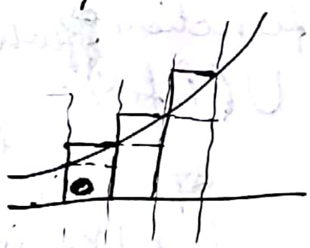
* RIEMANN INTEGRATION

→ Let $f:[a,b] \to R$ be a bounded function.

Let $P = \{x_0, \dots x_n\}$ with $\Delta x_j = x_j - x_{j-1}$

s.t $a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = b$ be a partition in $[a,b]$.





Let $M_j = \sup f(x)$, ; $x \in [x_{j-1}, x_j]$.

$m_j = \inf f(x)$; $x \in [x_{j-1}, x_j]$.

Let $U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$ & $L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i$

upper Reimam sum   lower Reimann sum

$\overline{\int_a^b} f dx = \inf U(f,P)$ $\forall$ partition P = upper reimann integral

$\underline{\int_a^b} f dx = \sup L(f,P)$ $\forall$ partition P = lower reimann integral

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$\overline{\int_a^b} f\,dx = \inf\limits_{\forall P} U(P,f)$. partition in $[a,b]$

$\underline{\int_a^b} f\,dx = \sup\limits_{\forall P} L(P,f)$

$f \in \mathcal{R}$ (Riemann Integrable)

$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$.

$\int_a^b f\,dx$ (common)

since f is bounded on $[a,b]$ $\exists$ bounds.

M & m s.t $m \le f(x) \le M$ $(a \le x \le b)$

$\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

$m_i \le M \;\forall i$

$m \le M_i \;\forall i$

Proof: $U = \sum M_i \Delta x_i \le \sum M \Delta x_i < M \sum \Delta x_i \le M(b-a)$

$L = \sum m_i \Delta x_i \geq \sum m \Delta x_i \geq m \sum \Delta x_i \geq m(b-a)$

$\int f\,d\alpha(x)$. α is mon

$\rightarrow$ Let α be a monotonically increasing function on $[a,b]$. Since $\alpha(a), \alpha(b)$ are finite, α is bounded. corresponding to each partition P of $[a,b]$, we write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$, $\Delta\alpha_j \geq 0$ as α is monotonically increasing. Now for any real function which is bounded on $[a,b]$ we put.

$U(P,f,\alpha) = \sum\limits_{i=1}^{n} M_i \Delta\alpha_i$, $L(P,f,\alpha) = \sum\limits_{i=1}^{n} m_i \Delta\alpha_i$

$\overline{\int_a^b} f\,d\alpha = \inf\limits_{\forall P} U(P,f,\alpha)$ —— (3) Upper R-S Integr

$\underline{\int_a^b} f\,d\alpha = \sup\limits_{\forall P} L(P,f,\alpha)$ —— (4). Lower R-S Integral

If (3) & (4) are equal, we denote their common value by $\int_a^b f\,d\alpha$ or $\int_a^b f(x)\,d\alpha(x)$ =

~~when $\alpha(x)=x$, it reduces to~~ :

This is Riemann Streitjes Integral

when $\alpha(x) = x$, it reducdes to Riemann Integral.

→ If P_1 & P_2 are two partition in $[a,b]$.
If $P_1 \supset P_2$ then we say P_1 is refinement of P_2.

→ Given two partition P_1 & P_2, we say $P_0 = P_1 \cup P_2$ is their common refinement

Theorem: If P^* is a refinement of P, then.
a) $L(P, f, \alpha) \leq L(P^*, f, \alpha)$ and
b) $U(P^*, f, \alpha) \leq U(P, f, \alpha)$.

Proof: $S_0 \subset S$. $S_0 \neq \phi$. S is bounded.
$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$

a) Suppose P^* contains just one pt more than P.
Let extra point be x^* & suppose $x_{i-1} < x^* < x_i$
where x_{i-1} & x_i are two consecutive pts of P.
Let $w_1 = \inf\limits_{x \in [x_{i-1}, x^*]} f(x)$
$w_2 = \inf\limits_{x \in [x^*, x_i]} f(x)$

we have $m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$.

$\Rightarrow w_1 \geq m_i$ & $w_2 \geq m_i$

Hence $L(P^*, f, \alpha) - L(P, f, \alpha)$

$$= \Big(w_1[\alpha(x^*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x^*)]\Big)$$
$$- \Big(m_i[\alpha(x_i) - \alpha(x_{i-1})]\Big).$$

$$= w_1[\alpha(x^*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x_{i-1})].$$
$$- m_1[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{j-1})]$$
$$= (w_1 - m_1)[\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_1)[\alpha(x_i) - \alpha(x_{i-1})]$$

$\geq 0 \Rightarrow L(P^*, f, \alpha) \geq L(P, f, \alpha)$

$|||^{ly}$ prove. b)

Theorem: $f \in \mathcal{R}(\alpha)$ on $[a,b]$ iff $\forall \epsilon > 0\ \exists$ a partition P s.t $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

Proof: suppose $\forall \epsilon > 0\ \exists P$ st $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

we know $L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha)$

subtracting L from all sides

$$\Rightarrow 0 \le \underline{\int_a^b} f\,d\alpha - L \le \overline{\int_a^b} f\,d\alpha - L \le U - L < \epsilon$$

$$\Rightarrow 0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \epsilon$$

If $0 \le a < \epsilon \quad \forall \epsilon \Rightarrow a = 0$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha = 0$$

$\Rightarrow f$ is R-S integral

$\Rightarrow f \in \mathcal{R}(\alpha)$.

Conversely, let $f \in \mathcal{R}(\alpha)$., to show that $\exists P$ s.t $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \quad \forall \epsilon > 0$ ~~for some P~~.

$\because f \in \mathcal{R}(\alpha)$. ~~$U(P,f,\alpha) = L(P,f,\alpha)$~~

$$\overline{\int_a^b} f\,d\alpha = \inf_P U(P,f,\alpha)$$

$$\underline{\int_a^b} f\,d\alpha = \sup_P L(P,f,\alpha).$$

$\forall \epsilon > 0,\ \exists P_1$ s.t.

$$\frac{\epsilon}{2} + \int_a^b f\,d\alpha > U(P_1,f,\alpha)$$

$$\Rightarrow U(P_1,f,\alpha) - \int_a^b f\,d\alpha < \frac{\epsilon}{2} \quad \text{—— (1)}$$

Ily $\forall \epsilon > 0\ \exists P_2$ s.t.

$$\int_a^b f\,d\alpha - L(P_2,f,\alpha) < \frac{\epsilon}{2} \quad \text{—— (2)}$$

adding ① & ②.

~~$U(P_1, f, \alpha) - L(P_2, f, \alpha) < \epsilon$~~

Let $P = P_1 \cup P_2$

$$U(P_1, f, \alpha) \le U(P_2, f, \alpha) < \int f\,d\alpha + \frac{\epsilon}{2} < L(P, f, \alpha) + \frac{\epsilon}{2}$$
$$\le L(P_1, f, \alpha) + \epsilon.$$

$$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Q. $f(x) = \begin{cases} 1 & x \text{ is rational in } [a,b] \\ 0 & \text{otherwise} \end{cases}$

Is $f(x)$ rieman integrable?

$\rightarrow$ $M_i = \sup(f(x)) = 1$. $x \in [x_{i-1}, x_i]$.

$m_i = \inf(f(x)) = 0$. $x \in [x_{i-1}, x_i]$.

$\Rightarrow \overline{\int_a^b} f\,dx = b - a$ $\quad \because U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum \Delta x_i$

$\underline{\int_a^b} f\,dx = 0.$ $\quad \because L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum 0 \Delta x_i$

so $\overline{\int_a^b} \ne \underline{\int_a^b}$ $\quad \therefore f$ is not Riemann Integrable.

Theorem If f is continuous on $[a,b]$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$

Proof: As α is monotonically increasing

(For riemann integrable st $\alpha(x) = x$)

$\Rightarrow$ ~~$\alpha(b) \le \alpha$~~ $\alpha(a) \le \alpha(b)$ i.e. $\alpha(b) - \alpha(a) \ge 0$.

Let $\epsilon > 0$ be given. Choose $\eta > 0$ s.t.

$[\alpha(b) - \alpha(a)]\eta < \epsilon$ $\quad \left(\eta < \frac{\epsilon}{\alpha(b) - \alpha(a)} \;\; ; \alpha(b) - \alpha(a) \ne 0\right)$

Since f is uniformly continuous on $[a,b]$, $\exists\, \delta > 0$ s.t.

$|f(x) - f(t)| < \eta$ if $x \in [a,b]$, $t \in [a,b]$.
$|x - t| < \delta$.

If P is any partition of $[a,b]$ s.t. $\Delta x_i < \delta \;\; \forall i$

then $|f(x) - f(t)| < \eta \Rightarrow |M_i - m_i| \le \eta \quad i = 1, \dots n.$

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$U(P, f, \alpha) - L(P, f, \alpha)$

$= \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i \leq \eta \sum_{i=1}^{n} \Delta\alpha_i = \eta[\alpha(b) - \alpha(a)] < \epsilon$

Theorem: Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$ and α is continuous at every pt at which f is discontinuous, then $f \in \mathcal{R}(\alpha)$.

→ f is integrable on $[a, b]$ iff f has countable pts of discontinuity

⇒ PROPERTIES OF INTEGRAL

→ If $f_1 \in \mathcal{R}(\alpha)$, $f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$ then $f_1 + f_2 \in \mathcal{R}(\alpha)$

→ $\int_a^b c f \, d\alpha = c \int f \, d\alpha$

→ $\int_a^b (f_1 + f_2) \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$

→ $f_1(x) \leq f_2(x)$ on $[a, b]$.

⇒ $\int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha$.

→ $\left| \int_a^b f \, d\alpha \right| \leq M[f(b) - f(a)]$ if $|f(x)| \leq M$ ~~on~~ $[a, b]$

→ If $f \in \mathcal{R}(\alpha_1)$ & $f \in \mathcal{R}(\alpha_2)$ then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$

$$\int_a^b f \, d(\alpha_1 + \alpha_2) = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2.$$

→ If $f \in \mathcal{R}(\alpha)$ & c is some const. then

$$\int_a^b f \, d(c\alpha) = c \int_a^b f \, d\alpha.$$

Theorem: If $f \in R(\alpha)$ and $g \in R(\alpha)$, on $[a,b]$, then,

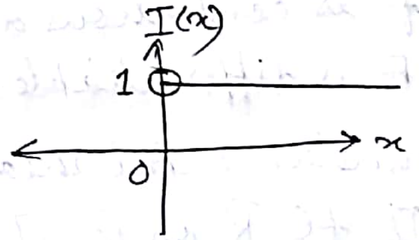
a) $fg \in R(\alpha)$

b) $|f| \in R(\alpha)$

c) $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$.

***UNIT STEP FUNCTION:**

→ $I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$



Theorem: If $a < s < b$, f is founded on $[a,b]$, f is continuous at s (maybe f is discontinuous everywhere except s). and $\alpha(x) = I(x-s)$, then.

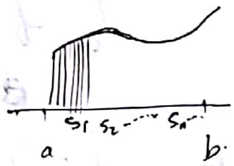
$$\int_a^b f\,d\alpha = f(s)$$

Theorem: Suppose $C_n \geq 0$ for $n = 1, 2, \cdots$ & $\sum C_n < \infty$,

$\{S_n\}$ is a sequence of distinct points in (a,b) and.

$$\alpha(x) = \sum_{n=0}^{\infty} C_n I(x - S_n).$$

Let f be continuous on $[a,b]$, then.



$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} C_n f(S_n)$$

Theorem: Assume α increases monotonically and $\alpha' \in R$ on $[a,b]$

Let f be a bounded real function on $[a,b]$,

then $f \in R(\alpha)$ iff $f\alpha' \in R$.

In that case

$$\int_a^b f\,d\alpha = \int_a^b f(x) \cdot \alpha'(x)\,dx.$$

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*INTEGRATION AND DIFFERENTIATION

<u>Theorem</u>: Let $f \in \mathcal{R}$ on $[a,b]$. For $a \le x \le b$, put

$$F(x) = \int_a^x f(t)\,dt$$

then F is continuous on $[a,b]$. Furthermore, if
(f may not be continuous).
f is continuous at a point x_0 of $[a,b]$, then
F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

<u>Theorem</u>: (Fundamental Theorem of Calculus)
If $f \in \mathcal{R}$ on $[a,b]$, and if there is a differentiable function F on $[a,b]$ s.t $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{(F is primitive of f)}$$

→ <u>Vector Integration</u>
Let $\vec{f}$ be a vector valued function

$$\vec{f} = (f_1, f_2, f_3, \cdots f_n)$$

$$\int_a^b \vec{f}\,d\alpha = \left(\int_a^b f_1\,d\alpha, \int_a^b f_2\,d\alpha, \cdots, \int_a^b f_n\,d\alpha\right) \in \mathbb{R}^n$$

α is monotonically increasing scalar.

Eg. $\vec{f}(x) = (\cos x, \sin(x))$
$= (f_1(x), f_2(x))$

$$\int_a^b \vec{f}(x)\,dx = \left(\int_a^b \cos x\,dx, \int_a^b \sin x\,dx\right).$$

→ $\|^{ly}$ we define Matrix Integration.

* SEQUENCE AND SERIES OF FUNCTIONS.

→ $\{f_n\}$ is a sequence of continuous functions

$f_n : E \to \mathbb{R}$; $E \subset X$.

→ (f_n) converges to $f(x)$ pointwise if given $\epsilon > 0$, $\exists N$ s.t.

$|f_n(x) - f(x)| < \epsilon$; $n \geq N$

→ In short $\{f_n\}$ converges pointwise if $\{f_n(x)\}$ converges $\forall x \in E$.

Eg. $f_n(x) = \dfrac{2n}{1+x^2}$, ~~then f_n = fix n and the~~ fix x.

$f_n(0) = (2n)$ divergent

Here N depends on ϵ & x i.e $N(\epsilon, x)$

Q. If $f_n \to f$ & f_n is continuous, then is f continuous?

→ NOT always

Q. If $f_n \to f$, then does $f_n' \to f'$?

→ NOT always

Q. If $f_n \to f$, then does $\int f_n \to \int f$?

→ NOT always.

→ <u>Double sequence</u>

$$S_{m,n} = \frac{m}{m+n} \qquad m, n = 1, 2, 3, \ldots$$

Fix n and vary m.

$$\lim_{m\to\infty} S_{m,n} = \lim_{m\to\infty} \frac{m}{m+n} = 1$$

$$\lim_{n\to\infty} \lim_{m\to\infty} S_{m,n} = \lim_{n\to\infty} 1 = 1$$

Now, Fix m & vary n.

$$\lim_{n\to\infty} S_{m,n} = 0.$$

$$\lim_{m\to\infty} \lim_{n\to\infty} S_{m,n} = \lim_{m\to\infty} 0 = 0.$$

$1 \neq 0. \Rightarrow \lim_{m\to\infty} \lim_{n\to\infty} S_{m,n} \overset{\text{not always}}{\neq} \lim_{n\to\infty} \lim_{m\to\infty} S_{m,n}.$