# ADVANCED NUMERICAL TECHNIQUES

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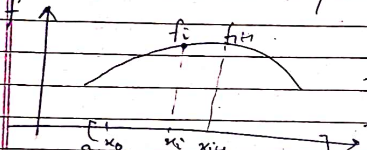
we have $(x_0, x_1, \ldots x_n) \to$ points in $[a,b]$ at which $f_i = f(x_i)$ for $i = 0, 1, 2, \ldots, n$.




$f'(x)$ at $x = x_i$ ??

$f(\xi_i) = ?$ $x_i < \xi_i < x_{i+1}$

$\int_{a_i}^{b_i} f(x)\,dx = ?$

$[a_i, b_i] \subseteq [a,b]$

**Now**

$f(x) \sim p(x)$ in $[a,b]$, $p(x) \to$ Polynomial

$p(x) = \sum_{i=0}^{m} a_i x^i$, which is degree m
$m \to$ integer.

$\downarrow$

Interpolation Polynomial.

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^m \\ 1 & x_1 & x_1^2 & \cdots & x_1^m \\ & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_m \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}}_{b}$$

$a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m = P_m(x)$

$P_m(x)$ is an interpolation polynomial iff

$P_m(x_i) = f_i$, $i = 0, 1, 2, \ldots n$.

So, we get,

$$\left.\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 + \cdots &+ a_m x_0^m = f_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 + \cdots &+ a_m x_1^m = f_1 \\ \cdots\cdots &= \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots &+ a_m x_n^m = f_n \end{aligned}\right\} \circledast \ (n+1) \text{ equations}$$

& $(m+1)$ unknowns - $(a_0, a_1, a_2, \ldots a_m)$

$AX = b$,

$X^T = [a_0 \; a_1 \ldots \; a_m]$

$A \to (m+1) \times (n+1)$. $b \to 1 \times (n+1)$.

$\text{Rank}[A] = \text{Rank}[A{:}b] = n$.

$m = n \to$

$$A = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ - & - & - & - & - \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

$|A| = \prod_{\substack{i,j=0 \\ (i \neq j)}}^{n} (x_i - x_j) \neq 0$ iff $x_i \neq x_j$

All the discrete points $x_i$s are distinct.
If the interpolation points or node points are distinct, then a unique polynomial $P_n(x)$ exists, s.t. $f(x) \sim P_n(x)$ in $[a,b]$

where $[a, b] = I[x_0, x_1, \ldots, x_n]$
smallest interval containing
$x_0, x_1, \ldots, x_n$.

$P_n(x) = ?$

Let. $f(x_i) = f_i$ be s.T. $f_j = 0 \;\forall j \neq i$
$\qquad\qquad\neq 0 \;\; j = i$

$$P_n(x_i) = 0 \;, \; i \neq j$$
$$= f_j \;, \; i = j \;, \; i = 0, 1, \ldots, n$$

i.e. $P_n(x)$ has roots at $x_0, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$. i.e. $n$ distinct points.

$$P_n(x) = C_j (x - x_0)(x - x_1) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_n)$$

Since $P_n(x_j) = f_j$, $\boxed{C_j = \dfrac{f_j}{(x_j - x_0)(x_j - x_1) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)}}$

let $L(x) = (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)$
$$= \prod_{i=0}^{n} (x - x_i)$$

$$L'(x_j) = (x_j - x_0)(x_j - x_1) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)$$

$$\therefore \quad C_j = \frac{f_j}{L'(x_j)}$$

$$\boxed{P_n^j(x) = \frac{L(x) f_j}{(x - x_j) L'(x_j)}}$$

when
$f(x_i) = f_i$, $i = 0, 1, \ldots, n$
$f_i \neq 0 \;\forall\; i$

Now, let's Generalise



$$P_n(x) = \sum_{i=0}^{n} \frac{L(x) f_i}{L'(x_i)(x - x_i)} = \sum_{j=0}^{n} P_n^j(x).$$

$$= \frac{(x - x_1) \cdots (x - x_n)}{(x_0 - x_1) \cdots (x_0 - x_n)} f_0 + \frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} f_1$$

$$+ \cdots + \frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})} f_n.$$

Lagrange's Interpolating Polynomial.

$[x_0, x_1, \ldots, x_n, x_{n+1}] \rightarrow n+2 \rightarrow P_{n+1}$

$P_{n+1}(x) \rightarrow P_n(x) \rightarrow P_i(x) = P_{i-1}(x) + Q_i(x)$

$f(x) \sim P_n(x)$ in $[a, b]$

$$\int_a^b f(x)\,dx \simeq \int_a^b P_n(x)\,dx$$

$L(x) = \prod_{i=0}^{n} (x - x_i) \rightarrow$ Lagrange's Polynomial

Error - ??

$E(x) = f(x) - P_n(x)$ in $[a, b]$.
$[a, b] = I[x_0, x_1, \ldots, x_n]$.
$\rightarrow$ smallest interval having $x_0, x_1, \ldots x_n$

$E(x_i) = 0$ for $i = 0, 1, \ldots, n$.
i.e. $E(x)$ have $(n+1)$ distinct zeroes.
$x_0, x_1, \ldots, x_n$ as $f(x_i) = P_n(x_i)$
$i = 0, 1, \ldots, n$.

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$E'(x)$ will have $n$ zeroes

$E''(x)$ " " $n-1$ "

- ——— .

$E^{(n+1)}(x)$ is a non-zero at least in one point in $[a,b]$.

We apply Rolle's Theorem $E(x_i) = E(x_{i+1}) = 0$.

$E'(\xi_i) = 0$ in $x_i < \xi_i < x_{i+1}$

$E(x) = f(x) - P_n(x) = g(x)(x-x_0) \cdots (x-x_n)$.

$g(x)$ is unknown, to determine.

$$\cancel{E^{(n+1)}}\ g(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!\,(x-x_0)\cdots(x-x_n)}$$

Error

$$E(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!\,(x-x_0)\cdots(x-x_n)}.$$

where $\xi$ is an arbitrary pt. in $[a,b]$.

Error of interpolation

$$E(x) = \frac{f^{(n+1)}(\xi)}{\lfloor(n+1)\ (x-x_0)\cdots(x-x_n)}, \quad \xi \in (a,b)$$

$E(x)$ is small when $(n+1)$ is large

This implies that the number of interpolation points reduces the error.

Preliminary Discussions

Mathematical modeling of Physical situations involve IVP (initial value Problem) or BVP (boundary " " ).

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IVP, BVP involves functions and its derivatives

IVP → where all conditions are prescribed at a single point. eg.

$F(x, y', y'', \ldots, y^{(n)}) = 0, \quad y^{(n)} = \frac{d^n y}{dx^n}$

which is $n^{th}$ order ODE and all the conditions are prescribed at a single pt. $x=0$.

i.e, $y(0) = y_0,\ y'(0) = y'_0, \ldots, y^{(n-1)}(0) = y_0^{n-1}$.

which $n$ conditions prescribed at $x=0$.

This problem is equivalent

$w = y',\ z = w' = y'', \ldots, X = y^{(n-1)}$.

$$\frac{dX}{dx} = f(x, y, w, z, \ldots X)$$

which are $n$ first order ODE with conditions

$y(0) = y_0,\ w(0) = y'_0, \ldots, X(0) = y_0^{n-1}$.

Theorem: Any $n^{th}$ order IVP is equivalent to $n$-first order IVP.

$$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + \sinh y = \cos x$$

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**Error in interpolation:-**

$f(x) \sim P_n(x)$ in $I\,[x_0, x_1, x_2, \ldots, x_n]$ or $[a,b]$

$$E(x) = f(x) - P_n(x).$$

Let $G(x) = E(x) - \dfrac{l(x)}{l(t)} E(t)$, $t$ is a parameter distinct from $(x_0, x_1, x_2, \ldots x_n)$

$E(x_i) = 0$ for $i = 0, 1, 2, \ldots, n \Rightarrow G(x_i) = 0$ and $G(t) = 0$.

$$l(x) = \prod_{i=0}^{n} (x - x_i).$$

So, $G(x)$ is $0$ at $(n+2)$ distinct points $x_0, x_1, \ldots, x_n$, $t$ in $I\,\{t, x_0, \ldots x_n\}$.

$G(x)$ is $(n+1)$ times continuously differentiable in $I_t$ and has $(n+2)$ distinct zeros.

$G'(x)$ has $(n+1)$ " " in $I_t$.

by Rolle's theorem

$G''(x)$ has $(n)$ " $0$'s.

$G^{(j)}(x)$ has $(n+2-j)$ distinct zeros.

& $G^{(n+1)}(\xi) = 0$, $\xi$ be any arbitrary pt in $I_t$.

$$G(x) = E(x) - \frac{l(x)}{l(t)} E(t)$$

$$E^{(n+1)}(x) = f^{(n+1)}(x), \quad l^{(n+1)}(x) = \lfloor n+1$$

So,

$$G^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{\lfloor n+1}{l(t)} E(t) = 0$$

So $E(t) = \dfrac{l(t) f^{(n+1)}(\xi)}{\lfloor n+1}$, $\xi \in I_t$.

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The error of interpolation at any pt. $x$ other than the node point is

$$E(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{\lfloor n+1} f^{(n+1)}(\xi), \quad \xi \in I_x$$

$\{x_0, x_1, \ldots, x_n\} \longrightarrow (n+1) \longrightarrow \boxed{P_n(x)}$

Piecewise interpolation.

$f(x) \sim p^{(i)}(x)$ in $x_i \le x \le x_{i+1}$

$$S(x) = \{ p^{(i)}(x) \mid x_i \le x \le x_{i+1} \quad i = 0, 1, \ldots n-1 \}$$

$f(x) \sim S(x)$ $\quad$ $p^{(i)}(x) \to$ cubic.

$\Rightarrow S(x) \to$ cubic spline.

IVP $\to$ Initial Value Problem

BVP $\to$ Boundary Value Problem

Any $n^{th}$ order IVP is equivalent to $n$-first order IVP.

A 1st order IVP is $\left[ \dfrac{dy}{dx} = f(x, y), \; y(0) = y_0 \right]$

linear IVP if $f(x,y)$ is linear function of $y$.

nonlinear " " " " nonlinear "

eg. $\dfrac{dy}{dx} = x \sinh y \to$ non linear. $\quad \dfrac{dy}{dx} = x^3 y$

Euler method / Predictor corrector / linear

Runge-Kutta method.

BVP. in which the conditions are prescribed at the boundary points.

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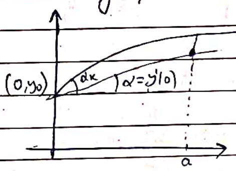
$$\left\langle \frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \quad 0 < x < a \right\rangle$$

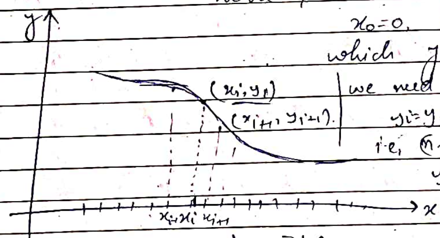
$y(0) = y_0$, $y(a) = y_a$ are prescribed
which can not be reduced to 1st order IVP

$y'(0) = \alpha \rightarrow$ shooting Method




Solving a BVP, i.e. find $y(x)$ in $0<x<a$
when $\frac{d^2y}{dx^2} = \left(f(x, y), \frac{dy}{dx}\right)$, $y(0) = y_0$, $y(a) = y_a$

Find $y_i = y(x_i)$, $i = 0, 1, \dots, n$
where $x_0, x_1, \dots x_n$ are distinct node pts in $[0, a]$
$x_0 = 0$, $x_n = a$, in which $y$ is prescribed.




we need to find $y_i = y(x_i)$ $i=1, \dots n-1$
i.e. $(n-1)$ pt. at which $y_i$ need to obtain

$y_i = y(x_i)$, $i=1, 2, \dots, n-1$

at $(n-1)$ distinct points.

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Finite difference method:-
The derivatives are replaced by difference of function values at finite number of points

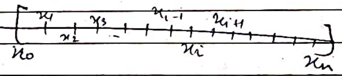
Satisfy the ODE at $x = x_i$.

$$y_i'' - f(x_i, y_i, y_i') = 0. \quad 0 < x < a.$$
$$i = 1, 2, \dots, n-1$$

$y_i = ?$

$$y_i' = \frac{(y_{i+1} - y_{i-1})}{2h} \; ; \; y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$y_{i+1} = y(x_i + h)$ where $x_i = x_0 + ih$, $i = 0, 1, \dots n$
$h \rightarrow$ step size. $\quad h = \frac{(x_n - x_0)}{n}$ $\neq 0$,
$h$ is constant for equi-spaced points.




$$y_{i+1} = y(x_{i+1}) = y(x_i + h) = y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{3!}y_i''' + \dots$$

$$y_{i-1} = y(x_i - h) = y_i - hy_i' + \frac{h^2}{2}y_i'' - \frac{h^3}{3!}y_i''' + \dots$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + \frac{h^2}{3!}y_i''' \mp \dots \longrightarrow$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2).$$

if $h<1$ $\Rightarrow$ $h^2 << 1$, the contribution of the terms become negligible as we increase no. of terms

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If we approximate

$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$, then the error is the infinite series which involves $h^2$.

An infinite series is truncated upto a finite no. of terms. The error is referred as truncation error .. TE. Most significant term which is truncated is the least order term of $h$, ie, $O(h^2)$.

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2) \rightarrow \text{central diff approximation.}$$

$$y_i' = \frac{y_{i+1} - y_i}{h} + O(h) \rightarrow \text{forward diff.}$$

$$y_i' = \frac{y_i - y_{i-1}}{h} + O(h) \rightarrow \text{backward diff.}$$

$$y_{i+1} + y_{i-1} = 2y_i + 2\frac{h^2 y_i''}{2!} + 2\frac{h^4}{4!} y^{(iv)} + \cdots$$

$$y_i'' = \frac{(y_{i+1} - 2y_i + y_{i-1})}{h^2} + \underbrace{\left(\frac{h^2}{4!} y_i^{(iv)} + h^4\right)}_{TE}$$

$$\boxed{y_i'' = \frac{(y_{i+1} - 2y_i + y_{i-1})}{h^2} + O(h^2)} \rightarrow \text{central diff. approximation}$$

finite differences of $y_i', y_i'', \ldots$

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(0) = y_0, \quad y(a) = y_a$$

$0 < x < a$

at $x = x_i$

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$$y_i'' - f(x_i, y_i, y_i') = 0, \quad i = 1, 2, \ldots, n-1.$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) = 0 \quad \text{—} (*)$$

$i = 1, 2, \ldots$

with $y_0 = y_0$, $y_n = y_a$.

which are $(n-1)$ algebraic equations in $(n-1)$ variables

Solve (*) to get $y_1, y_2, \ldots, y_{n-1}$.

Ex: $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 1$, $y(0) = 0$, $y(1) = 1$.

let $h = 0.25$ $\quad y_0 = 0$ $\quad$ ~~$x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$.~~

$x_0 = 0$
$x_1 = 0.25$
$x_2 = 0.5$
$x_3 = 0.75$
$x_4 = 1.$

$y'' - f(x_i, y_i, y_i') = 0 \quad y_4 = 1.$

$$\Rightarrow \frac{(y_{i+1} - 2y_i + y_{i-1})}{h^2} + x_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + x_i^2 y_i = 1$$

$, i = 1, 2, 3.$

$$\frac{(y_2 - 2y_1 + y_0)}{h^2} + \frac{x_1(y_2 - y_0)}{2h} + x_1^2 y_1 = 1.$$

~~$\frac{(y_3 - 2y_2 + y_1)}{h^2}$~~ $+ x_2$

$$\frac{(y_2 - 2y_1)}{(0.25)^2} + \frac{(0.25)(y_2)}{2 \times 0.25} + (0.25)^2 y_1 = 1$$

$$\frac{(y_3 - 2y_2 + y_1)}{(0.25)^2} + \frac{(0.5)(y_3 - y_1)}{2 \times 0.25} + (0.5)^2 y_2 = 1.$$

$$\frac{(1 - 2y_3 + y_2)}{(0.25)^2} + \frac{0.75(1 - y_2)}{2 \times 0.25} + (0.75)^2 y_3 = 1$$

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<u>Linear BVP:</u>

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = C(x), \quad 0 < x < a$$

Boundary conditions: $y(0) = y_0$, $y(a) = y_a$
where $A(x)$, $B(x)$, $C(x)$ are known functions of $x$ or constant.

Consider the grid pointer: $x_i = 0 + ih$, $i = 0, 1, \dots n$.

$y_0 = y_0$, $y_n = y_a$ are given. | $h \to$ step size/grid size

To obtain: $y_i$ at $i = 1, 2, 3, \dots, n-1$

Satisfy the ODE (1) at any $x_i$

$$y_i'' + A_i y_i' + B_i y_i = C_i, \quad i = 1, 2, \dots, n-1$$

Discretization: derivatives at $x_i$ are approximated by finite difference scheme.

Discretize (n) by central difference scheme

i.e. $y_i' = \left(\frac{y_{i+1} - y_{i-1}}{2h}\right)$, $y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$

we get

$$\frac{(y_{i+1} - 2y_i + y_{i-1})}{h^2} + \frac{A_i}{2h}(y_{i+1} - y_{i-1}) + B_i y_i = C_i \quad (*)$$

$i = 1, 2, \dots, n-1$

which leads to $(n-1)$ unknowns & $(n-1)$ equations

$(y_0, y_1, y_2, \dots, y_{n-1}, y_n)$ $\quad$ $\underline{y_1, y_2, y_3, \dots, y_{n-1}}$

Thus, system of eq$^n$ $(*)$ is $(n-1)$ linear algebraic equations involving $(n-1)$ unknowns, which can be expressed in a matrix eq$^n$ as.

$$Ax = d, \quad x = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$(n-1)$ unknowns

$1 \times (n-1)$

$A \to$ coeff. matrix $(n-1) \times (n-1)$

$d \to$ $1 \times (n-1)$ vector.

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Express $(*)$ as

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i, \quad i = 1, 2, \dots, n-1$$

$a_1 y_0 + b_1 y_1 + c_2 y_2 = d_1$

$\Rightarrow b_1 y_1 + c_2 y_2 = d_1 - a_1 y_0$

$a_2 y_1 + b_2 y_2 + c_2 y_3 = d_2$

$a_3 y_2 + b_3 y_3 + c_3 y_4 = d_3$

$- - -$

$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i$

$a_{n-1} y_{n-2} + b_{n-1} y_{n-1} = d_{n-1} - c_{n-1} y_n$

$Ax = d$

where $a_i = \frac{1}{h^2} - \frac{A_i}{2h}$, $b_i = \frac{-2}{h^2} + B_i$

$c_i = \frac{1}{h^2} + \frac{A_i}{2h}$, $d_i = c_i$

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & \cdots & 0 \\ \cdots & & & & & & \\ 0 & 0 & 0 & a_i & b_i & c_i & 0 \\ & & & a_{n-2} & b_{n-2} & c_{n-2} & \\ \cdots & & & & 0 & a_{n-1} & b_{n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - a_1 y_0 \\ d_2 \\ \vdots \\ \vdots \\ d_{n-1} \end{bmatrix}$$

$A$ is a $(n-1) \times (n-1)$ tri-diagonal matrix.

$X = A^{-1} d$, $A^{-1}$ exists as we are considering well posed problem.

Reduce the system as

$$\begin{bmatrix} 1 & c_1' & 0 & \cdots & & 0 \\ 0 & 1 & c_2' & \cdots & & 0 \\ & & & \ddots & & \\ 0 & & & & & \\ & \cdots & & & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} d_1' \\ d_2' \\ \vdots \\ d_{n-1}' \end{bmatrix}$$

Eliminate $a_i$s and reduce the diagonal element to 1. we get an upper triangular matrix

Soln is
$y_{n-1} = d'_{n-1}$
$y_i = d_i' - c_i' y_{i+1}$, $i = n-2, n-3, \dots 2, 1$

We need to obtain $c_i'$, $d_i$ $\forall i$

Thomas Algorithm:
$c_1' = c_1/b_1$, $d_1' = d_1/b_1$

verify
$c_i' = \dfrac{c_i}{b_i - a_i c_{i-1}'}$, $d_i' = \dfrac{d_i}{b_i - a_i c_{i-1}'}$

$i = 2, 3, \dots, n-1$

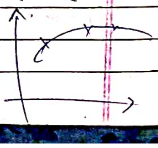
Working Rule: 1 Discretize the ODE, use b.c.s. to get the coeff. of the ensuing matrix

2 Use Thomas Algo to get the reduced coefficients (-) corresponding to the triangular matrix

3 Obtain the soln by back substitution

LabTask as ... wt

Q $x^2y'' + xy' = 1$ $y(1) = 0$, $y(1.4) = 0.0566$
$h = 0.1$. Solve also by Calculator
reduce $h$ to $0.05, 0.01$, & see the change



2 $y'' = x + y$, $y(0) = y(1) = 0$
$h = 0.1, 0.05, 0.01$

---

1 $x^2y'' + xy' = 1$ $y(1) = 0$, $y(1.4) = 0.0566$
$h = 0.1$

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$
$y_0 = 0$ $y_4 = 0.0566$

Unknowns $\rightarrow$ $y_1, y_2, y_3$

Now, let's discretize the ODE using B.C's.

$$x_i^2 y_i'' + x_i y_i' = 1$$

$$\Rightarrow x_i^2\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + x_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) = 1.$$

$i = 1, 2, 3$

$$\Rightarrow x_1^2\left(\frac{y_2 - 2y_1 + y_0}{h^2}\right) + x_1\left(\frac{y_2 - y_0}{2h}\right) = 1. \quad \text{(i)}$$

$$x_2^2\left(\frac{y_3 - 2y_2 + y_1}{h^2}\right) + x_2\left(\frac{y_3 - y_1}{2h}\right) = 1. \quad \text{(ii)}$$

$$x_3^2\left(\frac{y_4 - 2y_3 + y_2}{h^2}\right) + x_3\left(\frac{y_4 - y_2}{2h}\right) = 1 \quad \text{(iii)}$$

$$\underbrace{\left(\frac{-2x_1^2}{h^2}\right)}_{b_1} y_1 + \underbrace{\left(\frac{x_1^2}{h^2} + \frac{x_1}{2h}\right)}_{c_1} y_2 = 1 - \underbrace{\left(\frac{x_1^2}{h^2} - \frac{x_1}{2h}\right)}_{a_1} y_0$$

$$\underbrace{\left(\frac{x_2^2}{h^2} - \frac{x_2}{2h}\right)}_{a_2} y_1 + \underbrace{\left(\frac{-2x_2^2}{h^2}\right)}_{b_2} y_2 + \underbrace{\left(\frac{x_2^2}{h^2} + \frac{x_2}{2h}\right)}_{c_2}$$

$$a_i = \frac{x_i^2}{h^2} - \frac{x_i}{2h}, \quad b_i = \frac{-2x_i^2}{h^2}, \quad c_i = \frac{x_i^2}{h^2} + \frac{x_i}{2h}.$$

Now, we have,

$b_1 y_1 + c_1 y_2 = 1 - a_1 y_0$ (A)
$a_2 y_1 + b_2 y_2 + c_2 y_3 = 1$ (B)
$a_3 y_2 + b_3 y_3 = 1 - c_3 y_4$ (C)

$$Ax = d$$

$$A = \begin{bmatrix} b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1-a_1 y_0 \\ 1 \\ 1-c_3 y_4 \end{bmatrix}$$

Now, we can write (A) (B) & (C) as

$$y_1 + \frac{c_1}{b_1} y_2 = \frac{(1-a_1 y_0)}{b_1}$$

So, $c_1' = \frac{c_1}{b_1}$ $\qquad d_1' = \frac{(1-a_1 y_0)}{b_1}$

Now, $a_2 y_1 + b_2 y_2 + c_2 y_3 = 1.$

$$a_2 y_1 + \frac{a_2 \cdot c_1}{b_1} y_2 = \frac{a_2}{b_1}$$

$$\left(b_2 - \frac{a_2 c_1}{b_1}\right) y_2 + c_2 y_3 = \left(1 - \frac{a_2}{b_1}\right)$$

$$y_2 + \frac{c_2 y_3}{(b_2 - a_2 c_1')} = \frac{(1 - a_2 d_1')}{(b_2 - a_2 c_1')}$$

$$c_2' = \frac{c_2}{b_2 - a_2 c_1'}, \quad d_2' = \frac{1 - a_2 d_1'}{b_2 - a_2 c_1'}$$

$$y_2 + c_2' y_3 = d_2'$$

Now,

Now. $a_3 y_2 + b_3 y_3 = (1 - c_3 y_4)$

$$a_3 y_2 + a_3 c_2' y_3 = a_3 d_2'$$

$$y_3 (b_3 - a_3 c_2') = (1 - c_3 y_4) - a_3 d_2'$$

$$y_3 = \frac{\{(1 - c_3 y_4) - a_3 d_2'\}}{(b_3 - a_3 c_2')}$$

$\Rightarrow y_3 = 0.034437\ 45167$

Now, $y_2 = d_2' - c_2' y_3$

$= 0.01665574562$

Now,

$$y_1 = \frac{1}{b_1} - \frac{c_1 y_2}{b_1}$$

$= 4.574181079 \times 10^{-3}$

(2)

| | | |
|---|---|---|
| $a_1 = \frac{231}{2}$ | $b_1 = -242$ | $c_1 = \frac{253}{2}$ |
| $a_2 = 138$ | $b_2 = -288$ | $c_2 = 150$ |
| $a_3 = \frac{325}{2}$ | $b_3 = -338$ | $c_3 = \frac{351}{2}$ |

$y_0 = 0, \quad y_4 = 0.0566$

So, the equations are

$$\begin{bmatrix} -242 & \frac{253}{2} & 0 \\ 138 & -288 & 150 \\ 0 & \frac{325}{2} & -338 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -8.9333 \end{bmatrix}$$

$$\boxed{\begin{aligned} y_1 &= 4.574181079 \times 10^{-3} \\ y_2 &= 0.01665574562 \\ y_3 &= 0.0344374516 7 \end{aligned}} \rightarrow \text{ANS}$$

Thomas method

$c_1' = -\frac{23}{44}, \quad d_1' = \frac{(1-0)}{-242} = \frac{-1}{242}$

$c_2' = \frac{c_2}{b_2 - a_2 c_1'} = \frac{150}{-288 - 138 \times \left(-\frac{23}{44}\right)} = \frac{-1100}{1583}$

$d_2' = \frac{1 - a_2 d_1'}{b_2 - a_2 c_1'} = \frac{1 + 138 \times \frac{1}{242}}{-288 + 138 \times \frac{23}{44}} = \frac{-380}{52239}$

$$y_3 = \frac{\left(1 - \frac{357}{2}\times 0.0566 + \frac{325}{2}\times\frac{380}{52239}\right)}{-338 + \frac{325}{2}\times\frac{1100}{1583}}$$

$$= 0.03443745167$$

Now, $$y_2 = \frac{-380}{52239} + \frac{1100}{1583}\times y_3$$

$$= 0.01665574562$$

$$y_1 = \frac{1}{-242} + \frac{253}{2\times 242}\times y_2$$

$$= 4.574181079\times 10^{-3}$$

---

★ $A(x)y'' + B(x)y' + C(x)y = D(x)$

$$A(x_i)\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + B(x_i)\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + C(x_i)\,y_i = D(x_i)$$

$$\left(\frac{A(x_i)}{h^2} - \frac{B(x_i)}{2h}\right) y_{i-1} + \left(\frac{-2A(x_i)}{h^2} + C(x_i)\right) y_i + \left(\frac{A(x_i)}{h^2} + \frac{B(x_i)}{2h}\right) y_{i+1} = D(x_i)$$

$$a_i = \frac{A(x_i)}{h^2} - \frac{B(x_i)}{2h} \qquad b_i = \left(\frac{-2A(x_i)}{h^2} + C(x_i)\right)$$

$$c_i = \frac{A(x_i)}{h^2} + \frac{B(x_i)}{2h} \qquad d_i = D(x_i)$$

Now, we have.

$$a_1 y_0 + b_1 y_1 + c_1 y_2 = d_1$$
$$a_2 y_1 + b_2 y_2 + c_2 y_3 = d_2$$
$$a_3 y_2 + b_3 y_3 + c_3 y_4 = d_3$$
$$\cdots$$
$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i$$
$$\cdots$$

$$a_{n-1} y_{n-2} + b_{n-1} y_{n-1} + c_{n-1} y_n = d_n$$

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## Two point Linear BVP.

$$y'' + A(x)\,y' + B(x)\,y = C(x), \quad 0 < x < a. \quad \text{--}(**)$$

B.Cs. $y(0) = y_0, \quad y(a) = y_a$

FDM → $Ax = d$ TDMA

$x = A^{-1} d$ → Thomas Alg.

Central difference scheme

Discretized eqn at any grid pt. $x_i$ is

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i, \quad i = 1, 2, \dots, n-1 \quad \text{--}(*)$$

$a_i, b_i, c_i, d_i$ → are known coeff.

Let $y_i$ → numerical solⁿ of BVP. —(γ)

$Y_i$ → exact solⁿ " BVP —(γ *)

Error $= y_i - Y_i$; → two kinds of errors

Truncation error → when you choose a method.

Round-off Error → during computation.

Truncation error (T.E) is the residue by which the exact solⁿ of the differential eqⁿ (**) fails to satisfy the difference eqⁿ (*).

$$T.E \equiv a_i Y_{i-1} + b_i Y_i + c_i Y_{i+1} - d_i \text{ at a grid pt. } x_i, \quad i = 1, 2, \dots, n-1 \quad (***)$$

local T.E

which is the global T.E as the discretization is uniform.

Consistency: If T.E → 0 as $h \to 0$ then the numerical scheme / discretised eqⁿ (*) is said to be consistent with the ODE, i.e. (**).

$$\frac{dy}{dx} = f, \quad f_i = \frac{y_{i+1} - y_{i-1}}{2h} \quad \text{to } l'$$

To get an expression for the T.E (***), expand all the variables about $x_i$ by Taylor series.

$$Y_{i+1} = Y(x_i + h) = Y_i + hY_i' + \frac{h^2}{2} Y_i'' + \dots$$

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Note: $Y_i$'s satisfy the ODE, thus $Y_i'' + A_i Y_i' + B_i Y_i = C_i$

In the T.E, find the least order terms in $h$ which provides order of the T.E.

as $h \ll 1$, terms in least order of $h$ are the most significant terms of T.E.

Thus, we estimate the order of T.E.

> **Assignment:** Find the T.E and check for consistency for discretization of linear BVP by the central difference scheme.

$$T.E \equiv a_i Y_{i-1} + b_i Y_i + c_i Y_{i+1} - d_i$$

$$= a_i Y(x_i - h) + b_i Y_i + c_i Y(x_i + h) - d_i$$

$$= a_i \left[ Y_i - hY_i' + \frac{h^2}{2} Y_i'' + \dots \right] + b_i Y_i + c_i \left[ Y_i + hY_i' + \frac{h^2}{2} Y_i'' + \dots \right] - d_i$$

Apply $Y_i'' + A_i Y_i' + B_i Y_i - C_i = 0$

The sol of $y_i \approx Y(x = x_i) = Y_i$ $\quad Y_i \approx y_i$

A consistent & stable numerical scheme for a linear BVP converges to the exact solⁿ.

$y_i \to Y_i$, stability

$\bar{y}_0 = y_0 + \varepsilon_0$

$y = 1/3 = 0.333\dots$

if $|\varepsilon| \ll 1$

$|y_i - \bar{y}_i| \ll 1$

if A → diagonally dominant

$|b_i| \geq |a_i| + |c_i|$

## Derivative Boundary Condition:

B.C. $\alpha_0 y(0) + \beta_0 y'(0) = \gamma_0$
$\alpha_a y(a) + \beta_a y'(a) = \gamma_a$ } (I)

where not all $\alpha_0, \beta_0, \alpha_a, \beta_a$ are zeroes.

Discreticed eq

$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i, \; i=1,2,\dots,n-1$

we need to sym to reduce to a compact tri diagonal system. i.e. no. of unknowns and no. of eq^n are same.

$\alpha_0 y_0 + \beta_0 y_0' = \gamma_0$

$\alpha_a y_n + \beta_a y_n' = \gamma_a \rightarrow$

$[\;_{i=0}\; _{i=1} \quad \;]_{i=n}$ — $y_0'$ or $y_n'$ can not be replaced by central difference.

Consider two fictitious grid points $x_{-1}, x_{n+1}$ symmetrically in the system.

$x_i$ for $i = -1, 0, 1, \dots, n, n+1 \rightarrow (n+3)$

$\alpha_0 y_0 + \beta_0 (y_1 - y_{-1})/2h = \gamma_0$ —(i)

$\alpha_a y_n + \beta_a (y_{n+1} - y_{n-1})/2h = \gamma_n$ —(ii)

$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i$
$i = 0, 1, 2, \dots, n.$

for $i=0$, $a_0 y_{-1} + b_0 y_0 + c_0 y_1 = d_0 \rightarrow$ replace $y_{-1}$ by (i).

$\bar{b}_0 y_0 + \bar{c}_0 y_1 = \bar{d}_0$ —(1)

$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i, \; i = 1, 2, \dots, n-1$

$a_n y_{n-1} + b_n y_n + c_n y_{n+1} = d_n \rightarrow \bar{a}_n y_{n-1} + \bar{b}_n y_n = \bar{d}_n$

$AX = d$, $X^T = [y_0, y_1, \dots, y_n]$.

$A \rightarrow (n+1)\times(n+1)$. $\rightarrow$ closed tri-diagonal system i.e., same no. of eq^n & no. of variables.

(06 TSW) Q. $y'' - 2y = 0$, $y(0) = 1$, $y'(1) = 0$. $y_n'$
$h = 0.2$
$h = 0.2, 0.1, 0.05, 0.01$

0.2 0.4 0.6 0.8 (1.0) $n+1$
$[_{i=0}\; 1 \; 2 \; 3 \; 4 \;]_{i=n}$ 1
$n=5$

Ans

$y_i'' - 2y_i = 0$

$\Rightarrow \dfrac{(y_{i+1} - 2y_i + y_{i-1})}{h^2} - 2y_i = 0.$

$\Rightarrow \underbrace{\left(\frac{1}{h^2}\right)}_{a_i} y_{i-1} + \underbrace{\left(-\frac{2}{h^2} - 2\right)}_{b_i} y_i + \underbrace{\left(\frac{1}{h^2}\right)}_{c_i} y_{i+1} = \underbrace{0}_{d_i}.$

Now, $y_0 = 0$, $y_n' = 0$

$\Rightarrow \dfrac{(y_{n+1} - y_{n-1})}{2h} = 0.$

Now, we have

$\frac{1}{h^2} y_0 + \left(-2\left(\frac{1}{h^2}+1\right)\right) y_1 + \frac{1}{h^2} y_2 = 0.$

$\frac{1}{h^2} y_1 + \left(-2\left(\frac{1}{h^2}+1\right)\right) y_2 + \frac{1}{h^2} y_3 = 0.$

$\vdots$

$\frac{1}{h^2} y_3 + f^2\left(1 + \frac{1}{h^2}\right) y_4 + \frac{1}{h^2} y_5 = 0$

$\frac{y_4}{h^2}$

$x_{-1}, x_{n+1}$ → are not physically defined system becomes larger. for non linear it imparts instability.

$$y_0' = \frac{y_1 - y_0}{h} + O(h) \qquad [\quad ]_{0}^{n}$$

$$y_n' = \frac{y_n - y_{n-1}}{h} + O(h).$$

accuracy becomes lower

Looking for 2nd order scheme for 1st order derivatives.

$$y_i' = A_i y_i + B_i y_{i+1} + C_i y_{i+2} + O(h)$$

$$= \bar{A} y_i + \bar{B}_i y_{i-1} + \bar{C}_i y_{i-2} + O(h^2)$$

Find $A_i$'s $B_i$'s & $C_i$

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \ldots$$

$$y_{i-1} = y_i - h y_i' + \frac{h^2}{2} y_i''$$

$$y_{i+2} = y_{i+1} + h y_{i+1}' + \frac{h^2}{2} y_{i+1}''$$

$$y_{i+2} = y_{i+1} +$$

$$A_i + B_i + C_i = 0$$

$$B_i + 2C_i = \frac{1}{h}, \quad \frac{B_i h^2}{2} + \frac{C_i h^2}{2} = 0$$

$$B_i + 4C_i = 0$$

$$y_i' = \frac{-3y_i + 4y_{i+1} - y_{i+2}}{2h} + O(h^2)$$

Lab → means to be done with code & soln submitted to the Lab link.

H.T → means to write the solution manually.

2nd order backward difference.

$$y_i' = \frac{3y_i - 4y_{i-1} + y_{i-2}}{2h} + O(h^2)$$

Lab. Ex: $y'' - 2y = 0$, $y_0 = 1$, $y'(1) = 0$

use 2nd order backward diff. for $y'(1) = 0$ (H.T)

19/1/21 Lab Asgn

(1) $y'' - 2xy' - 2y = -4x$

$y(0) = y'(0)$, $2y(1) - y'(1) = 1$

$h = 0.25$ (H.T) → written

$h \leq 0.1$ → (Lab Asgn)

(2) $y'' + 2xy' + 2y = 4x$, $0 < x < 0.5$

$y(0) = 1$, $y(0.5) = 1.279$, $h = 0.1$

(Lab + H.T)

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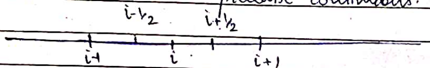
## Linear BVP :

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y = r(x). \qquad 0 < x < a \quad \text{--- (1)}$$

$$\left.\begin{aligned} \alpha_0 y(0) + \beta_0 y'(0) &= \gamma_0 \\ \alpha_1 y(a) + \beta_1 y'(a) &= \gamma_1 \end{aligned}\right\} \text{B.C.}$$

$p(x)$, $q(x)$ & $r(x)$ are piecewise continuous in $[0, a]$.




## Control Volume Method .

Integrate (1). between $x_{i-1/2}$ to $x_{i+1/2}$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]dx + \int_{x_{i-1/2}}^{x_{i+1/2}} q(x)\,y\,dx = \int_{x_{i-1/2}}^{x_{i+1/2}} r(x)\,dx$$

$$p(x)\frac{dy}{dx}\Big|_{x+1/2} - p(x)\frac{dy}{dx}\Big|_{x-1/2} + \int_{x_{i-1/2}}^{x_i} q(x)\,y\,dx + \int_{x_i}^{x_{i+1/2}} q(x)\,y\,dx$$

$$= \int_{x_{i-1/2}}^{x_i} r(x)\,dx + \int_{x_i}^{x_{i+1/2}} r(x)\,dx$$

We may have jump discontinuity of $q(x)$ & $r(x)$ at $x_i$ .

$$\int_{x_{i-1/2}}^{x_i} q(x)\,y\,dx \simeq \frac{\delta x_{i-1}}{2} q_{i-}\,y_i$$

$$\int_{x_i}^{x_{i+1/2}} q(x)\,y\,dx \simeq \frac{\delta x_{i}}{2} q_{i+}\,y_i$$

$$p_{i+1/2}\frac{y_{i+1}-y_i}{2\delta x_{i+1/2}} - p_{i-1/2}\frac{y_i - y_{i-1}}{2\delta x_{i-1/2}} + y_i\left[\frac{q_{i-}\,\delta x_{i-1} + q_{i+}\,\delta x_{i}}{2}\right]$$

$$= \frac{r_{i-}\,\delta x_{i-1} + r_{i+}\,\delta x_i}{2} \quad \text{--- } (*)$$

$r_{i-} = r_{i+} \rightarrow$ if continuous.

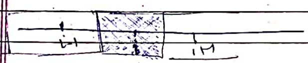
The discretized equation $(*)$ can be cast into a tri-diagonal form

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i \,, \qquad i = 1, 2, \ldots, n-1$$

$$p_{i\pm 1/2} = \frac{p_i + p_{i\pm 1}}{2} \,, \qquad \frac{dy}{dx}\Big|_{x+1/2} = \frac{y_{i+1} - y_i}{\delta x_i} \rightarrow \text{Central difference}$$

Variable step size is considered. We may have

$\delta x_{i+1} = \delta x_i = \delta x_{i-1}$ . H.T $\rightarrow$ Show that it reduces to central difference scheme if $p(x)$, $q(x)$ & $r(x)$ are continuous.




## Higher Order BVP .

$$y''' + A(x)\,y'' + B(x)\,y' + C(x)\,y = D(x) \,, \qquad 0 < x < a$$

3rd order BVP ; 3 b.c.s. need to be prescribed.

$y(0) = \alpha$, $y'(0) = \beta$, $y'(a) = \gamma$. $\longrightarrow$

$x_i = i\,\delta x$, $\quad i = 0, 1, 2, \ldots, n$

$x_0 = 0$, $\quad x_n = a$

$$y_i''' + A_i y_i'' + B_i y_i' + C_i y_i = D_i \,, \qquad i = 1, 2, \ldots, n-1.$$

$$y_i''' = (y_i'')' = \frac{y_{i+1}'' - y_{i-1}''}{2h} + O(h^2)$$

$$y_i''' = \frac{1}{2h^3}\left[y_{i+2} - 2y_{i+1} + y_i - (y_i - 2y_{i-1} + y_{i-2})\right]$$

$$= \frac{1}{2h^3}\left[y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right]$$

discretization of the ODE

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$$\frac{1}{2h^3}\left(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right) + \frac{A_i}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right]$$

$$+ \frac{B_i}{2h}\left[y_{i+1} - y_{i-1}\right] + C_i y_i = D_i \quad , \quad i = 1, 2, \ldots n-1$$

which are linear system of equations (not- tri-diagonal) unknowns $y_1, y_2, \ldots y_n$

Systems are $(n-1)$ eq$^n$ involving $\boxed{y_{-1}, y_1, y_n, y_{n+1}}$
$n+2$ unknowns -

discretize the B.C $\quad y'(0) = \beta \quad , \quad \dfrac{y_1 - y_{-1}}{2h} = \beta \quad$ (i)

$$y'(a) = \gamma, \qquad \frac{y_{n+1} - y_{n-1}}{2h} = \gamma \quad \text{(ii)}$$

ie, $(n-1) + 2 = n+1$. eq$^n$. involving $(n+2)$ unknowns.

Thus, it is a closed system of eq$^n$ & unique sol$^n$ canot be obtain

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# Higher Order BVP

$y''' + A(x)y'' + B(x)y' + C(x)y = D(x)$ , $0 < x < a$

$y(0) = y_0$ $y'(0) = y_0'$ , $y'(a) = y_a$. $x_0 = 0$, $x_n = a$

Discretize at $x_i$, $x_i = i\delta x$, $i = 0, 1, \ldots$

$i = 1, 2, \ldots, n-1$.

$i = 1$, $i = -1$

$$y_i''' = \frac{1}{2h^3}\left[y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right].$$

$i = n-1$, $i = n+1$.

$y_3 - 2y_2 + 2y_0 - y_{-1}$

$y_0 = y_0$ , $y'(0) = y_0' \Rightarrow y_{-1} = y_1 - y_0' \cdot 2h$

$y_{n+1} - y_{n-1} = 2h\, y_a$ ; $y_{n+1} = y_{n-1} + 2h y_a$.

Thus, we get $(n-1)$ linear Algebraic eq$^n$ involving $(n)$ unknowns $(y_1, y_2, \ldots, y_n)$

Thus unique sol$^n$ doesn't results — not a tridiagonal system

In order to overcome this we reduce the ODE as follows :-

Let $p = \frac{dy}{dx}$ —(i) $\frac{d^2p}{dx^2} + A(x)\frac{dp}{dx} + B(x)P + C(x)y = D(x)$ —(ii)

B.C.s $y(0) = y_0$ , $p(0) = y_0'$ , $p(a) = y'_a$

Discretize (i) & (ii) such that $y_n$ does not appear in the system. To discretize at the grid pt $x_i$, integrate (i) b/w $x_i$ & $x_{i-1}$.

$$\int_{x_{i-1}}^{x_i} dy = \int_{x_{i-1}}^{x_i} P\, dx \rightarrow \text{Trapezoidal Rule}$$

$$y_i - y_{i-1} = \frac{h}{2}\left[P_i + P_{i-1}\right] + O(h^3)$$

$$y_i - y_{i-1} - \frac{h}{2}(P_i + P_{i-1}) = 0 \quad \text{—(i)}$$

Apply central difference scheme in (ii)

$$\frac{P_{i+1} - 2P_i + P_{i-1}}{h^2} + A_i\frac{P_{i+1} - P_{i-1}}{2h} + B_i P_i + C_i y_i = D_i$$

$i = 1, 2, \ldots, n-1$

The system (i) & (ii) together $2(n-1)$ equations involving : $y_1, y_2, \dots, y_{n-1}$, $p_1, p_2, \dots, p_{n-1}$, unknowns as $y_0$, $p_0$ & $p_n$ are prescribed.

Thus, $2(n-1)$ equations involving $2(n-1)$ unknowns.

Thus, it is a closed system. A unique sol$^n$ results.

Note - (i) & (ii) are coupled.

An algorithm to be developed for solving (i) & (ii).

Let's combine (i) & (ii) to get single matrix eq$^n$ by introducing the variables. The variables

$$X_i = \begin{bmatrix} y_i \\ p_i \end{bmatrix} ; \quad A_i^* X_{i-1} + B_i^* X_i + C_i^* X_{i+1} = D_i^*$$

$A_i^*, B_i^*, C_i^* \to 2\times 2$ matrices.

$$y_i - y_{i-1} - \frac{h}{2}(p_i + p_{i-1}) = 0$$

$$a_i p_{i-1} + b_i p_i + c_i p_{i+1} + d_i y_i = e_i$$

$$\begin{bmatrix} -1 & -h/2 \\ 0 & a_i \end{bmatrix}\begin{bmatrix} y_{i-1} \\ p_{i-1} \end{bmatrix} + \begin{bmatrix} 1 & -h/2 \\ d_i & b_i \end{bmatrix}\begin{bmatrix} y_i \\ p_i \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & c_i \end{bmatrix}\begin{bmatrix} y_{i+1} \\ p_{i+1} \end{bmatrix} = \begin{bmatrix} 0 \\ e_i \end{bmatrix}$$

For $i = 1, \dots, n-1$

$$B_1^* X_1 + C_1^* X_2 = D_1^* - A_1^* X_0$$

$$A_2^* X_1 + B_2^* X_2 + C_2^* X_3 = D_2^*$$

$$A_{n-1}^* X_{n-2} + B_{n-1}^* X_{n-1} = D_{n-1} - C_{n-1}^* X_n$$

$X_0 \to$ known, $X_n = \begin{pmatrix} y_n \\ p_n \end{pmatrix}$

$$C_{n-1}^* X_n = \begin{pmatrix} 0 \\ c_{n-1} p_n \end{pmatrix} \text{ independent of } y_n$$

$$a_i = \left(\frac{1}{h^2} - \frac{A_i}{2h}\right)$$

$$b_i = \left(\frac{-2}{h^2} + B_i\right).$$

$$c_i = \left(\frac{1}{h^2} + \frac{A_i}{2h}\right).$$

$$d_i = (C_i)$$

$$e_i = D_i$$

Note: $y_n$ is not provided and it doesn't appear in the system.

B.C. ~~$y_0$~~ $p_0 = y_0'$, $p_n = y_a'$, $y_n = y_a$ i.e. $y(a) = y_a$ given

$$\int_{x_i}^{x_{i+1}} dy = \int_{x_i}^{x_{i+1}} p \, dx$$

$y_0$ doesn't appear in the system

$y''(0) = K$, $y'(a) = y'_a$, $p'(0) = K$

$y(0) = y_0$, $y(a) = y_n$, $p(0) = p_0$

We have a system of matrix equations as

$$A_i^* X_{i-1} + B_i^* X_i + C_i^* X_{i+1} = D_i^* \quad -(*)$$

Let $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \end{bmatrix}$, whose components $X_i$ are $1\times 2$ vectors. Then the system (*) can be expressed as

$$AX = D$$

where

$$A = \begin{bmatrix} B_1^* & C_1^* & 0 & 0 & \cdots & 0 \\ A_2^* & B_2^* & C_2^* & 0 & \cdots & 0 \\ 0 & A_3 & B_3 & \cdots & & \\ 0 & & & & & \\ 0 & 0 & 0 & & A_{n-1} & B_{n-1} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} D_1^* \\ D_2 \\ \vdots \\ D \end{bmatrix}$$

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The elements of the coeff. matrix are $2\times 2$ matrices.

H.T. → write $A_i$, $B_i$, $C_i$ in Base $f_n$

$A \to$ block tridiagonal matrix

we need to manipulate $AX = D$ to get $X$.

Q — Lab Task — H.T.

$$y''' + 4y'' + y' - 6y = 1, \quad 0 < x < 1$$

$$y(0) = 0, \quad y'(0) = 0, \quad y'(1) = 1.$$

Ans ↓ $h = 0.05$, $0.01$

~~$y_i''' + 4y_i''$~~ Let $p = \frac{dy}{dx}$ | $p_0 = 0$, $p_n = 1$.

$$\Rightarrow p_i'' + 4p_i' + p_i - 6y_i = 1.$$

$$\Rightarrow \frac{p_{i+1} - 2p_i + p_{i-1}}{h^2} + 4\left(\frac{p_{i+1} - p_{i-1}}{2h}\right) + p_i - 6y_i = 1$$

$$\Rightarrow p_{i-1}\underbrace{\left(\frac{1}{h^2} - \frac{2}{h}\right)}_{a_i} + \underbrace{\left(\frac{-2}{h^2} + 1\right)}_{b_i} p_i + \underbrace{\left(\frac{1}{h^2} + \frac{2}{h}\right)}_{c_i} p_{i+1} \underbrace{- 6}_{d_i} y_i = \underbrace{1}_{e_i}$$

&

$$y_i - y_{i-1} - \frac{h}{2}(p_i + p_{i-1}) = 0$$

$$\begin{bmatrix} -1 & -h/2 \\ 0 & a_i \end{bmatrix}\begin{bmatrix} y_{i-1} \\ p_{i-1} \end{bmatrix} + \begin{bmatrix} 1 & -h/2 \\ d_i & b_i \end{bmatrix}\begin{bmatrix} y_i \\ p_i \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & c_i \end{bmatrix}\begin{bmatrix} y_{i+1} \\ p_{i+1} \end{bmatrix} = \begin{bmatrix} 0 \\ e_i \end{bmatrix}$$

$$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i$$

$$D_1 = D_1 - A_1 X_0, \qquad \cancel{D_{n-1} = D_{n-1}} \quad D_{n-1} = D_{n-1} - C_{n-1} X_n$$

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$$\begin{bmatrix} B_1 & C_1 & 0 & 0 & \cdots & 0 \\ A_2 & B_2 & C_2 & \cdots & & 0 \\ 0 & A_3 & \cdots & & & \\ 0 & \cdots & & & A_{n-1} & B_{n-1} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_{n-1} \end{bmatrix}$$

$$X_{n-1} = D_{n-1}', \quad X_i = D_i' - C_i' X_{i+1}$$
$$i = n-2, n-3, \ldots, 2$$

$$\begin{bmatrix} I & C_1' & 0 & 0 & 0 & \cdots & 0 \\ 0 & I & C_2' & 0 & 0 & \cdots & \\ & \vdots & & & & I & C_{n-1}' \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} D_1' \\ D_2' \\ \vdots \\ D_{n-1}' \end{bmatrix}$$

Algorithm. $C_1' = (B_1)^{-1} C_1$,
$D_1' = (B_1)^{-1} D_1$

$B_i' = B_i - A_i C_{i-1}'$

$C_i' = (B_i')^{-1} C_i$; $\quad D_i' = (B_i')^{-1}(D_i - A_i D_{i-1}')$

$i = 2, 3, \ldots$

Through Block elimination method.
well posed problem

Q — H.T + Lab

$$y^{IV} = q/EI, \qquad y(0) = y''(0) = y(L) = y''(L).$$

$p = y''$, $\quad p'' = q/EI$
$y(0) = 0$, $y(L) = 0$, $p(0) = 0 = p(L) = 0$

Get the Block tridiagonal form for the reduced coupled set of BVP

$$y_i^{IV} = (y_i'')'' = \frac{1}{h^2}\left(y_{i+1}'' - 2y_i'' + y_{i-1}''\right)$$

$$= \frac{1}{h^4}\left(y_{i+2} - 2y_{i+1} + y_i - 2y_{i+1} + 4y_i - 2y_{i-1} + y_i - 2y_{i-1} + y_{i-2}\right)$$

$i = 1, 2, \ldots n-1$, $\quad y_{-1}, y_1, y_n, y_{n+1}$

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$y_0'' = 0 \Rightarrow y_{-1} - 2y_0 + y_1 = 0$

$y_n'' = 0 \Rightarrow y_{n+1} - 2y_n + y_{n-1} = 0.$

$\hookrightarrow y_1, y_2, y_3, \ldots, y_{n-1}$

---

Q (H.T + Lab)

$y^{iv} + 81y = 81x^2, \quad y(0) = y(1) = y''(0) = y''(1) = 0$

(find out in the same way as prev. one)

~~Do by Gauss elimination method.~~

$h = 0.25, \quad h < 0.1 \to$ Block-tridiagonal

---

$$\Rightarrow \begin{bmatrix} 0 & 1/h^2 \\ 1/h^2 & 0 \end{bmatrix} X_{i-1} + \begin{bmatrix} 81 & -2/h^2 \\ -2/h^2 & 1 \end{bmatrix} X_i + \begin{bmatrix} 0 & 1/h^2 \\ 1/h^2 & 0 \end{bmatrix} X_{i+1} = \begin{bmatrix} 0 \\ 81x_i^2 \end{bmatrix}$$

$i = 1, 2, 3, \ldots, n-1$

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## Higher order linear BVP

$Ax = d, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad X_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ \vdots \\ x_{3i} \end{bmatrix}$

$A \to$ Block matrix ; each element of $A$ is a square matrix.

Q $y^{iv} + 81y = 81x^2, \quad 0 < x < 1$

$y(0) = y(1) = y''(0) = y''(1) = 0$

Let $z = y''$, Reduced BVP

$$\left.\begin{aligned} z'' + 81y &= 81x^2, \\ y'' - z &= 0, \end{aligned}\right\} \quad 0 < x < 1$$

$y(0) = y(1) = 0; \quad z(0) = z(1) = 0$

$Ax = d \to$ Discretized through central difference scheme and get a block tri-diagonal form.

$z_i'' + 81y_i = 81x_i^2$

$\Rightarrow \dfrac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + 81y_i = 81x_i^2$

$\dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = z_i$

$X = \begin{bmatrix} y_i \\ z_i \end{bmatrix} \quad y_0 = y_n = 0, \quad z_0 = z_n = 0$

$i = 1, 2, \ldots, n-1.$

$\Rightarrow a_i z_{i-1} + b_i z_i + c_i z_{i+1} + 81y_i = 81x_i^2$

$d_i y_{i-1} + e_i y_i + f_i y_{i+1} - z_i = 0$

$2(n-1)$ eq.
$2(n-1)$ variable

$$\begin{bmatrix} 0 & a_i \\ d_i & 0 \end{bmatrix} \begin{bmatrix} y_{i-1} \\ z_{i-1} \end{bmatrix} + \begin{bmatrix} e_i & -1 \\ 81 & b_i \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix} + \begin{bmatrix} f_i & 0 \\ 0 & c_i \end{bmatrix} \begin{bmatrix} y_{i+1} \\ z_{i+1} \end{bmatrix} = \begin{bmatrix} 81x_i^2 \\ 0 \end{bmatrix}$$

$$X^T = [X_1\ X_2 \ldots X_{n-1}]$$
$$A \to (n-1)\times(n-1)$$
block matrix.

$\boxed{\text{Lab Task}}$ Solve for $h = 0.1, 0.05, 0.01$.

(*) $$y_i^{iv} = (y_i'')'' = \frac{1}{h^2}\left(y_{i+1}'' - 2y_i'' + y_{i-1}''\right)$$

$$= \frac{1}{h^4}\left[\underline{y_{i+2} - 2y_{i+1} + y_i} - 2y_{i+1} + 4y_i - 2y_{i-1} + y_i - 2y_{i-1} + y_{i-2}\right] + O(h^2)$$

$$= \frac{1}{h^4}\left[y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right] + O(h^2)$$

$i = 1, 2, \ldots, n-1$, $\quad y_{-1}, y_{n+1}$

$$\frac{1}{h^4}\left[y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right] + 81y_i = 81x_i^2$$
$i = 1, 2, \ldots, n-1$

System of eqn (*) consisting of $(n-1)$ eqn. involving $y_{-1}, y_1, y_2, \ldots, y_{n-1}, y_{n+1}$, ie $n-1+2 = n+1$ variables where ~~$y_{-1}$ & $y_{n+1}$ are fictitious~~ $x_{-1}$ & $x_{n+1}$ are fictitious points.

$\because\ y_0'' = 0$

$$\Rightarrow \frac{y_1 - 2y_0 + y_{-1}}{h^2} = 0$$

$$y_{-1} = -y_1 \quad (\because y_0 = 0) \quad \text{— (a)}$$

Similarly $y_n'' = 0$

$$\Rightarrow y_{n+1} - 2y_n + y_{n-1} = 0 \Rightarrow y_{n+1} = -y_{n-1} \quad \text{(b)}$$



$x_0 \quad x_1 \quad x_2 \quad x_3$
$0 \quad 1/3 \quad 2/3 \quad L$

Substitute (a), (b) in (*). we get $(n-1)$ eqn involving $(n-1)$ variables, which can be solved uniquely.

Note :- (*) is not a tri-diagonal system.

let $h = 1/3$. get the soln $\qquad y_3 = y_0 = 0$

$$\frac{\left[y_3 - 4y_2 + 6y_1 - 4y_0 + (-y_1)\right]}{h^4} + 81y_1 = 81x_1^2$$

$$\frac{\left[(-y_2) - 4y_3 + 6y_2 - 4y_1 + y_0\right]}{h^4} + 81y_2 = 81x_2^2$$

$$\Rightarrow \cancel{y_3} - 4y_2 + 6y_1 - \cancel{4y_0} - \cancel{y_1} + \cancel{y_1} = x_1^2$$

$$\Rightarrow 6y_1 - 4y_2 = x_1^2$$

$$-\cancel{y_2} - \cancel{4y_3} + 6y_2 - 4y_1 + 0 + \cancel{y_2} = x_2^2$$

$$\Rightarrow -4y_1 + 6y_2 = x_2^2$$

$$y_1 = \frac{11}{90}, \quad y_2 = \frac{7}{45}$$

Q $\boxed{\text{Lab Task}}$ $$y^{iv} + 2y = \frac{x^2}{9} + \frac{2}{3}x + 4. \qquad y(0) = y'(0) = y(3) = y'(3) = 0$$

$h = 0.1$ $\quad y_{-1}, y_{n+1}$. $\quad y_0, y_n \to$ given.

$y_0' = 0 \Rightarrow y_1 = y_{-1}$ $\qquad (y_1, y_2, \ldots, y_{n-1})$

$y_n' = 0 \Rightarrow y_{n+1} = y_{n-1}$

Not a tri-diag syst.

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(2) $a_i y_{i-2} + b_i y_{i-1} + c_i y_i + d_i y_{i+1} + e_i y_{i+2} = f_i$
$i = 1, 2, \ldots, n-1.$

(Q) $y^{iv} + 81y = 729x^2, \quad 0 < x < 1$

$y(0) = y'(0) = y''(1) = y'''(1) = 0$

$z = y''$ X $\qquad y'_0 = 0 \Rightarrow y_1 = y_{-1}$

$y''(1) = 0 \Rightarrow y''_n = 0$

$\Rightarrow y_{n+1} - 2y_n + y_{n-1} = 0$

$(y'''_n) = (y'')' = \left( \frac{y'_{n+1} - 2y'_n + y'_{n-1}}{h^2} \right) = 0$

$\Rightarrow \frac{y_{n+2} - y_n}{2h} - 2\frac{(y_{n+1} - y_{n-1})}{2h} + \frac{y_n - y_{n-2}}{2h} \Big) = 0$

$\Rightarrow y_{n+2} - \cancel{y_n} - 2y_{n+1} + 2y_{n-1} + \cancel{y_n} - y_{n-2} = 0$

$\Rightarrow y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2} = 0$

using backward diff

$(y''_n)' = \frac{1}{2h}\left(3y''_n - 4y''_{n-1} + y''_{n-2}\right) = 0$

$\Rightarrow 3(y_{n+1} - 2y_n + y_{n-1})$
$- 4(y_n - 2y_{n-1} + y_{n-2})$
$+ y_{n-1} - 2y_{n-2} + y_{n-3} = 0$

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Using Central diff. scheme for $i = 1, 2, \ldots, n-1$
$x_{-1}, x_{n+1}, x_{n}$

$\left.\begin{array}{l} y_n = \\ y_{n+1} = \end{array}\right\}$ interior pts $y_{n-1}, y_{n-2} \ldots$

$y_{n+1} - 2y_n + y_{n-1} = 0$ | $3y_{n+1} - 6y_n + 3y_{n-1}$
$- 4y_n + 8y_{n-1}$
$+ y_{n-1} - 2y_{n-2}$
$+ y_{n-3} = 0$

(Q) $y^{iv} - y''' + y = x^2$
$y(0) = y'(0) = 0 \quad y(1) = 2, y'(1) = 0$
$h = 1/3 \rightarrow$ with pen

Lab
$h = 0.1, 0.01$
$0.05$

## # Spline Interpolation Technique for linear two-point BVP

upto now.

$y_i = f(x_i), \quad i = 0, 1, \ldots, n$ are given.

Find $P_n(x) \rightarrow$ interpolation polynomial.

$y \sim P_n(x)$ in $I\ [x_0, x_1, \ldots x_n]$

$P_n(x_i) = y_i, \quad i = 0, 1, \ldots, n-1.$

### Spline Interpolation: $\quad S(x)$

let $P_k(x)$ be a polynomial which interpolates $y = f(x)$ in $[x_k, x_{k+1}]$

$y \sim P_k(x), \quad x_k \le x \le x_{k+1}$

$P_0(x)$ in $[x_0, x_1]$, $P_1(x)$ in $[x_1, x_2]$ ....

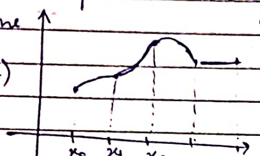
If $P_k(x)$ are cubic polynomials & $S(x)$ is the set of all $P_k(x)$, Then.

$y \sim S(x)$, a piecewise cubic polynomial interpolation where $S(x) = P_K(x)$, $x_K \le x \le x_{K+1}$

$K = 0, 1, 2, \ldots, n-1$

$S(x) = \{ P_K(x) \mid x_K \le x \le x_{K+1}, K = 0, 1, 2, \ldots, n-1 \}$

is called the piecewise cubic polynomial or spline




$y \sim S(x)$, $S(x) = P_K(x)$, $x_K \le x \le x_{K+1}$

Since $P_K(x)$ interpolates $y$ in $[x_K, x_{K+1}]$

Hence, $P_K(x_K) = y_K = f_K$, $P_K(x_{K+1}) = y_{K+1} = f_{K+1}$

$K = 0, 1, 2, \ldots, n-1$

$P_K(x)$ is a cubic polynomial

The polynomials $P_K(x)$ have the same slope & the same curvature at the knot points.

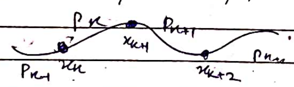
$P_K(x)$ are continuous (a) $P_K(x_K) = P_{K-1}(x_K)$

$K = 1, 2, \ldots, n-1$,

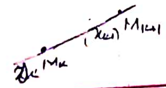
$P_{K-1}(x_K) = f_K = P_K(x_K)$

(b) $P_K'(x_K) = P_{K-1}'(x_K)$, $K = 1, 2, \ldots, n-1$

(c) $P_K''(x_K) = P_{K-1}''(x_K)$, $K = 1, 2, \ldots, n-1$




At the end points, $x_0$ & $x_n$, there is no continuity of slope (first derivative) or curvature ($2^{nd}$ derivative) can be impose




$\frac{M_{K+1} - M_K}{x_{K+1} - x_K} = \frac{y - M_K}{}$

$S(x)$ is piecewise cubic in $[x_0, x_n]$.

$S''(x)$ is piecewise linear in $[x_0, x_n]$.

ie, $P_K''(x)$ is a linear polynomial interpolates ~~in $[x_K, x_K]$~~ ~~$P_K''(x)$~~, ~~$[x_K, \ldots]$, $[x_{K+1}, \ldots]$~~

$[x_K, S''(x_K)]$, $[x_{K+1}, S''(x_{K+1})]$

$S''(x_K) = M_K$, $S''(x_{K+1}) = M_{K+1}$

ie, $P_K''(x)$ is a linear plynomial in $[x_K, x_{K+1}]$

with $P_K''(x_K) = M_K$, $P_K''(x_{K+1}) = M_{K+1}$

So, $P_K''(x)$ in $[x_K, x_{K+1}]$ is.

$$P_K''(x) = \frac{x - x_{K+1}}{x_K - x_{K+1}} M_K + \frac{x - x_K}{x_{K+1} - x_K} M_{K+1} \quad \Bigg| \quad y = \sum_{j=0}^{n} \frac{l(x)\, y_j}{l'(x_j)(x - x_j)}$$

$K = 0, 1, \ldots, n-1$.

$M_K \simeq f_K''$, $M_{K+1} \simeq f_{K+1}''$, which are not known

Let $h = x_{K+1} - x_K$, equi-spaced points. Integrate twice & get $P_K(x)$.

$$P_K(x) = \frac{M_K}{h} \frac{(x - x_{K+1})^3}{3!} + \frac{M_{K+1}}{h} \frac{(x - x_K)^3}{6}$$

$(x_{K+1} - x)^3$

$$+ C_K (x - x_K) + D_K (x_{K+1} - x)$$

$C_K, D_K \rightarrow$ two constants

use $P_K(x_K) = f_K$, $P'(x_K) = P'(x_{K+1})$

H.T.

To find $C_K$, $D_K$

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## Spline Interpolation:-

$y = f(x) \simeq s(x) = \{ P_k(x),\ x_k \le x \le x_{k+1} \mid k = 0, 1, 2, \ldots, n-1 \}$

$P_k(x)$ cubic polynomial.

$p_k(x_k) = f_k,\quad p_k(x_k) = p_{k+1}(x_k).$

$p_k'(x_k) = p'_{k+1}(x_k)$

$p_k''(x_k) = p''_{k+1}(x_k),\ k = 0, 1, \ldots, n-1$

$f_k'' \simeq M_k$, $p_k''(x)$ is a linear polynomial

$$p_k''(x) = \frac{M_k}{h}(x_{k+1} - x) + \frac{M_{k+1}}{h}(x - x_k) \qquad k = 0, 1, 2, \ldots, n-1$$

[Note: $M_k$ are unknown]

Integrate twice w.r.t. $x$ to get.

$$p_k(x) = \frac{M_k}{6h}(x_{k+1} - x)^3 + \frac{M_{k+1}}{6h}(x - x_k)^3 + C_k(x - x_k) + D_k(x_{k+1} - x) \qquad k = 0, 1, \ldots, n-1$$

$C_k, D_k$ are arbitrary const.

$p_k(x_k) = y_k = f_k,\quad p_k(x_{k+1}) = y_{k+1} = f_{k+1}$

$C_k = 1/h\left(y_{k+1} - \frac{M_{k+1}}{6}h^2\right)$

$D_k = 1/h\left(y_k - \frac{M_k}{6}h^2\right).$

Thus, $P_k(x) = \frac{M_k}{6}\left[\frac{(x_{k+1} - x)^3}{h} - h(x_{k+1} - x)\right] + \frac{y_k}{h}(x_{k+1} - x)$.

$+ \frac{M_{k+1}}{6}\left[(x - x_k)^3/h - h(x - x_k)\right] + \frac{y_{k+1}}{h}(x - x_k)$

we need to find $M_0, M_1, \ldots, M_n$. $\mid k = 0, 1, \ldots, n-1$

$p_k'(x) = p'_{k+1}(x_k),\ k = 1, 2, \ldots, n-1$

$$p_k'(x) = \frac{M_k}{6}\left[\frac{-3(x_{k+1} - x)^2}{h} + h\right] - \frac{y_k}{h} + M_{k+1}\left[\frac{3(x_{k+1} - x)^2}{h} - h\right] + \frac{y_{k+1}}{h}$$

$$p_k'(x_k) = \frac{M_k}{6}(-2h) + \frac{M_{k+1}}{6}(-h) + \frac{1}{h}(y_{k+1} - y_k)$$

$$p_k'(x_{k+1}) = \frac{M_k}{6}h + \frac{M_{k+1}}{6} \cdot 2h + \frac{1}{h}(y_{k+1} - y_k)$$

replace $k \to k-1$

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$$p'_{k-1}(x_k) = \frac{M_{k-1}}{6}h + \frac{M_k}{6}2h + \frac{1}{h}(y_k - y_{k-1}) \quad — (6)$$

let $\Delta y_k = y_{k+1} - y_k$ → forward difference

Since, $p_k'(x_k) = p'_{k-1}(x_k)$, so,

$$M_k(-2h) + M_{k+1}(-h) - M_{k-1}h - M_k \cdot 2h = \frac{6}{h}\Delta y_{k-1} - \frac{6}{h}\Delta y_k.$$

So,

$$hM_{k-1} + 4M_k h + hM_{k+1} = 6\left(\frac{\Delta y_k}{h} - \frac{\Delta y_{k-1}}{h}\right). \quad k = 1, 2, \ldots, n-1$$

$$hM_{k-1} + 4hM_k + hM_{k+1} = 6/h^2\,(y_{k+1} - 2y_k + y_{k-1}) — (*)$$

These are $(n-1)$ linear algebraic equations involving $(n+1)$ unknowns, ie, $M_0, M_1, \ldots, M_n$

We need to impose two end conditions for $M_0$ & $M_k$

(i) $M_0 = f_0'' = \delta_0$, $M_n = f_n'' = \delta_n$ → are prescribed.

(ii) $M_0 = M_1$, $M_n = M_{n-1}$, ie, $S''(x)$ approaches constant value at $x_0, x_n$ (Periodic condition).

(iii) $M_0 = M_n = 0$, free boundary condition, ie., $s(x)$ approaches a straightline at $x_0, x_n$.

Use $M_0, M_n$, solve $(*)$ as a tridiagonal system to get $M_1, M_2, \ldots, M_{n-1}$. (*)

Q. H.T

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 1.5 | 2.2 | 3.1 | 4.3 |
| $k=$ | 0 | 1 | 2 | 3 |

let free end condition.

(1) Find $y(1.2)$, $y'(1)$ by spline interpolation technique.

(2) Find $S(x) = \{p_k(x) \mid k = 0, 1, 2\}$

Ans. $p_k(x)$, $k = 0, 1, 2$, $M_0 = M_3 = 0$

$M_1, M_2 \to$ ??

$M_1 = 0.2$, $M_2 = 0.4 \to p_0(x)$, $1 \le x \le 2$

$y(1.2) = p_0(1.2)$ | $p_0(x) = \frac{M_0}{6h}(x_1 - x)^3 + \frac{M_1}{6h}(x - x_k)^3 +$

$y(1.2) = 1.6336$

$y'(1) \cong \underline{0.667}$

(#) B.V.P. by Spline interpolation :-

$y'' + b(x)\, y = c(x)$, $y'$ is absent.

$y_k, M_k \rightarrow$ two sets of unknowns. ($y_0, y_n$ are known).

$M_k = -b_k y_k - c_k$, $k = 0, 1, \dots, n$.

Substitute into the tri-diagonal relation

$$M_{k-1} + 4M_k + M_{k+1} = \frac{6}{h^2}\left(y_{k-1} - 2y_k + y_{k+1}\right)$$

$k = 1, 2, \dots, n-1$

$(-b_k y_{k-1} - c_k) + 4(\,--\,)$.

we get a tri-diagonal system

$a_k y_{k-1} + b_k y_k + c_k y_{k+1} = d_k$

$k = 1, 2, \dots, n-1$; $y_0, y_n \rightarrow$ known

which provides $y_k$ s.

HT (Q) $y'' - y = 0$, $y(0) = y(1) = 1$, $h = 1/2$ : Solve by spline interp.

Lab Task $h = 0.1, 0.05, 0.01$ & compare with finite differenece method.

Also compare with analytical soln.