27/2/19

Book :- Shakarchi & Stein → Real Analysis

Lebesgue outer measure $\xrightarrow{\sigma\text{-additive}}$ Lebesgue measure.

Q.) Give a bijection between [0,1] and cantor set?

Q.) Give examples of two sets A and B s.t,

$$m(A) = m(B) = 0$$

but $m(A+B) > 0$ (strictly +ve)

cantor → uncountable

closed

Perfect

Bijection to [0,1]

has measure 0.

Riemann integrable,

$$U(f) = L(f)$$

{ ∵ U → upper sum
L → lower sum.

$$\overline{\int_a^b} f(x)\,dx = U(f) = \inf_P U(P, f) \quad \Big| \quad U(P,f) \Rightarrow \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

↳ upper Riemann integral

$$\underline{\int_a^b} f(x)\,dx = L(f) = \sup L(P, f) \quad \Big| \quad L(P,f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

↳ Lower Riemann integral

Another Def^n. of Riemann:-

$$S(P,f) = \sum_{i=1}^{n} f(x^*)(x_i - x_{i-1})$$

$$x_{i-1} < x^* < x_i$$

$$\lim_{\|P\| \to 0} S(P,f) = \int_a^b f(x)\,dx.$$

$$\text{Norm}(P) = x_i - x_{i-1}$$

$$\|P\| = x_i - x_{i-1}$$

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* A function is Riemann integrable if set of discontinous points has Lebesgue measure 0.

Q.) Give examples of sequences of Riemann integrable whose limit is not Riemann integrable.

i) $f_n(x) = \begin{cases} 1, & x \le q_n \\ 0, & \text{otherwise} \end{cases}$ $\to \{q_1, q_2, \dots, q_n\}$

$\{q_n\}$ is the enumeration of $\mathbb{Q}$.

$f = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases} \to$ Dirichlet f^n.

$f_n \xrightarrow{\infty} f$

ii) $f_n(x) = \begin{cases} n, & 0 < x < 1/n \\ 0, & 1/n \le x < 1 \end{cases}$

$$\int_0^1 f_n(x)\,dx = n \cdot \frac{1}{n} + 0 = 1.$$

$f_n \to f$. but $\int f_n \not\to \int f$.

Ex:- If f_n converges to f uniformly and each f_n is Riemann integrable then f is Riemann integrable.

Given :- $f_n \xrightarrow{\text{uniformly}} f$ & $f_n \in R$

proof :- i) $f \in R$
ii) $\int f_n \to \int f$

proof :- ii) $f \in C[a,b]$

$\Rightarrow \sup\limits_{x \in [a,b]} |f(x)| = \|f\|$.

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$f_n \rightrightarrows f$

$\Leftrightarrow \|f_n - f\| \to 0$

$\Rightarrow \sup\limits_{x \in [a,b]} |f_n(x) - f(x)| \to 0$

$$|\int f_n - \int f| \le |\int (f_n - f)|$$

$$\le \int_a^b |f_n - f|$$

$$\le (b-a)\|f_n - f\| \to 0 \text{ as } n \to \infty$$

* Fundamental Theorem of Integral Calculus:-

1.) $f \in C[a,b]$, continous function.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$..

continuity of 'f' implies existence of 'F'.

2.) $F(x) = \int_a^x f(x)\,dx$

→ if 'f' is Riemann integrable and bounded then F is continous.

→ if 'f' is continous then F is diff. and $\frac{d}{dx} F(x) = f(x)$.

Q.) $\frac{d}{dx}\left[\int_{x^2}^{x} e^{-t^2} dx\right] = ?$

$\cancel{F'(x) = e^{-x^2}}$

Sol:- $F(x) = \int_0^x e^{-t^2} dt$

$F'(x) = e^{-x^2}$

Now, $\frac{d}{dx}\left[\int_{x^2}^{0} e^{-t^2} dt + \int_0^x e^{-t^2} dt\right]$

$$= \frac{d}{dx}\left[-\int_0^{x^2} e^{-t^2}\,dt + F(x)\right]$$

$$= \frac{d}{dx}\left[-F(x^2) + F(x)\right]$$

$$= -F'(x^2)\cdot 2x + F'(x)$$

$$= -e^{-x^4}\cdot(2x) + e^{-x^2}.$$

Proof:- 1.) Use mean value theorem.

consider a partition of $[a,b]$.

$$F(x_k) - F(x_{k-1}) = (x_k - x_{k-1})\,F'(c)$$

$$= (x_k - x_{k-1})\,f(c) \qquad x_{k-1} < c < x_k$$

$$m_k \le f(c) \le \cancel{m_k}\ M_k$$

$$(x_k - x_{k-1})\,m_k \le (x_k - x_{k-1})\,f(c) \le (x_k - x_{k-1})\,M_k .$$

$$\sum m_k\,(x_k - x_{k-1}) \le \sum F(x_k) - F(x_{k-1})$$

$$\le \sum M_k\,(x_k - x_{k-1})$$

$$L(f) \le F(b) - F(a) \le U(f)$$

$$L(f) = U(f) = F(b) - F(a)$$

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2.) $\lim\limits_{h\to 0} \frac{1}{h} \int_a^{a+h} f(x)\,dx - f(a)$.

$= \lim\limits_{h\to 0} \frac{1}{h} \cdot \int_a^{a+h} \left(f(x) - f(a)\right) dx$

$\leq \frac{\varepsilon}{h} \cdot h$.

28/2/19

1.) Darboux integral $\leftarrow \int_a^b f(x)\,dx$

2.) Riemann integral $\leftarrow$

Fundamental theorem of integral calculus:-

i) $f \in R[a,b]$, if $\exists$ a differentiable function F such that,

$F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

ii) $\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$

f is $R[a,b]$, then $F(x) = \int_a^x f(x)\,dx$ is continous

Riemann integrable

Moreover if f is $C[a,b]$ $\longrightarrow$ continous.

then, $F(x) = \int_a^x f(t)\,dt$.

differentiable and $F'(x) = f(x)$

$m(E) = 0$.

when set can be expressed as union of intervals / which can be arbitrary small.

Q:) when is a function Riemann integrable?

Sol: E is the set of discontinuities for f

$f \in R[a,b] \Leftrightarrow |E| = 0$.

proof:- $W_\delta(x) = \sup\{ |f(y) - f(z)| ; y, z \in (x-\delta, x+\delta),\ \delta > 0 \}$

$E_n = \{ x \in E \mid W_\delta(x) \geq \frac{1}{n} \}$, then

$$E = \bigcup_{n \geq 1} E_n$$

Suppose f is Riemann-integrable.

T.S, $m(E) = 0$?

Case I: If $m(E_n) = 0 \ \forall n$ then $m(E) = 0$.

Let, $m(E_n) > 0$ for some N.

Take a partition, $x_0 = a < x_1 < x_2 < \cdots < x_n = b$.

$U(P,f) - L(P,f)$

$$= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

$$\geq \sum (M_i - m_i)(x_i - x_{i-1})$$

all the intervals that intersect with E_n

$(x_1, x_2) \cap E_n \neq \phi$

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$\geq \sum W_\delta(x) \; (x_i - x_{i-1})$

$(x_{i-1}, x_i) \cap E_n \neq \phi$

At least contains E_n so measure should be greater than $m(E_n)$

$\geq \frac{\alpha}{n} > 0$

$m(E_n) = \alpha$

Case II:- Suppose $m(E) = 0$

To show:- f is Riemann integrable.

If $m(E) = 0$ then, $m(E_n) = 0 \;\; \forall \; n.$

As we took a bounded f^n in the begining we know E_n is bounded. For now assume it is closed. Hence E_n is compact.

$\rightarrow$ E_n is compact $\Rightarrow E_n \subset \bigcup_{k=1}^{l} (a_k, b_k)$.

$$\sum_{k=1}^{l} (b_k - a_k) < \varepsilon.$$

As measure is '0' so we can write as arbitrarily small intervals.

$I = \bigcup_{k=1}^{l} (a_k, b_k)$, I^c is compact.

as complement is finite union of compact (as it is closed bounded)

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choose $(c_k, d_k)_{k=1}^{m}$

s.t, $w_\delta(x) < 1/n$

$U(P,f) - L(P,f)$

$$= \sum (b_k - a_k)(M_k - m_k)$$

$$+ \sum (d_k - c_k)(M_k - m_k)$$

$$\le 2M\varepsilon + \frac{b-a}{n} \to 0$$

as $\varepsilon \to 0$
$n \to \infty$.

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6/3/19

Recall. $f \in R[a,b] \Leftrightarrow m(E) = 0$,

E is the set of discontinuity of sets.

Measurable function :-

A function 'f' is said to be measurable if for every $a \in R$.

$\bigcup_{k=1}^{\infty} \{x : f(x) \leq a - \frac{1}{k}\} \Leftarrow \{x : f(x) < a\}$ is a measurable.

$\Downarrow$

$\{f < a\}$

$\Leftrightarrow \{x : f(x) \leq a\}$

$\Downarrow$

$\{f \leq a\} \Rightarrow \bigcap_{k=1}^{\infty} \{x : f(x) < a + 1/k\}$

* To show that :-

if f, g are measurable then $f+g$ is measurable.

Proof :- Let, $f+g$ is measurable,

$\{x : f+g < \alpha\}$

$\{x : f < \alpha - g\}$

$\rightarrow \{f+g > a\} = \bigcup_{r \in \mathbb{Q}} \{f > a - r\} \cap \{g > r\}$

these are measurable.

$\rightarrow$ $f \cdot g$

$\rightarrow f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ { $\because$ so this is measurable }

☆ If $\{f_n\}$ is a sequence of measurable f^n.

1) then $\sup\{f_n\}$ and $\inf\{f_n\}$ is measurable.

2) $\lim\sup(f_n)$ and $\lim\inf(f_n)$ are measurable.

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★ Littlewood's three principles:-

6/3/19 $(m(E)<\infty)$

① Every measurable set is almost finite union of closed intervals.

② Every measurable function is almost a conti function

③ Every point-wise convergent sequence of measurable function is almost uniformly convergence.

→ ② is called Lusin's theorem.

③ is called Egorov's theorem.

Proof:- ① Suppose E is a measurable set, then $\exists F$ (finite union of closed sets) such that $m(E \Delta F) < \varepsilon$ for any given $\varepsilon > 0$.

→ E is measurable. We can find a sequence of closed intervals $\{I_j\}$ s.t $\boxed{E \subset \bigcup_{j=1}^{\infty} I_j}$.

& $\sum_{j=1}^{\infty} m(I_j) < m(E) + \frac{\varepsilon}{2}$.

Now, $m(E) < \infty \Rightarrow \sum_{j=1}^{\infty} m(I_j)$ is a convergent series.

choose, N such that,

$$\sum_{j=N+1}^{\infty} m(I_j) < \frac{\varepsilon}{2}$$

Let, $F = \bigcup_{j=1}^{N} I_j$

$$\begin{aligned}
\to m(E \setminus F) &\leq m(E \setminus F) + m(F \setminus E) \\
&\leq \sum_{j=N+1}^{\infty} m(I_j) + m\left(\bigcup_{j=1}^{\infty} I_j \setminus E\right) \\
&\leq \frac{\varepsilon}{2} + \left(\sum_{j=1}^{\infty} m(I_j) - m(E)\right) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

$m(E \setminus F) \leq \varepsilon$

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Egorov's theorem (third principle):-

Let, $\{f_K\}$ be a sequence of measurable function. s.t, $f_K \to f$ (pointwise) on a measurable set E. ($m(E) < \infty$). then for $\varepsilon > 0$, we can find a closed set A_ε s.t $m(E \setminus A_\varepsilon) < \varepsilon$.

and $f_K \underset{\text{converges to}}{\longrightarrow} f$ uniformly on A_ε.

Proof:- $f_K \to f$ on E pointwise.

for fix pair of k and n define a set,

$E_K^n = \{x \in E \mid |f_j(x) - f(x)| < 1/n, \forall j > k\}$

it has fixed 'n' so it is an increasing sequence w.r.t. 'k'.

$E_K^n \subseteq E_{K+1}^n$ & $E_K^n \to E$,

we can find K_n s.t, $m(E - E_{K_n}^n) < \frac{1}{2^n}$

$\Rightarrow$ By construction of E_K^n

$\{x \in E \mid |f_j(x) - f(x)| < 1/n\}$, $\forall x \in E_K^n$. $j > k_n$

choose, N s.t, $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$ and

$$\boxed{\tilde{A}_\varepsilon = \bigcap_{n \geq N} E_{K_n}^n}$$

$$\to m(E \setminus \tilde{A}_\varepsilon) \leq \sum_{n \geq N}^{\infty} m(E \setminus E_{K_n}^n)$$

$$\leq \sum_{n \geq N}^{\infty} \frac{1}{2^n}$$

$$\leq \frac{\varepsilon}{2}$$

For, $\delta > 0$, choose $n > N$ s.t $\frac{1}{n} < \delta$

$x \in \tilde{A}_\varepsilon \Rightarrow x \in E_{K_n}^n$

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$\Rightarrow |f_n(x) - f(x)| < \frac{1}{n} < \delta, \ \forall \ x \in \tilde{A}_\varepsilon$

$\Rightarrow f_n \longrightarrow f$ uniformly on $\tilde{A}_\varepsilon$.

Find a closed set A_ε s.t $m(\tilde{A}_\varepsilon \setminus A_\varepsilon) < \frac{\varepsilon}{2}$

$$m(E \setminus A_\varepsilon) \leq m(E \setminus \tilde{A}_\varepsilon) + m(\tilde{A}_\varepsilon \setminus A_\varepsilon)$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$m(E \setminus A_\varepsilon) \leq \varepsilon.$$

★ Every non-negative measurable function can be approximated by an increasing sequence of step functions / simple functions.

$\exists \ \varphi_K \nearrow F$

sequence of simple functions

$-K \quad 0 \quad K$; $\varphi_K = [-K, K]$.

$$F_K(x) = \begin{cases} f(x), & x \in \varphi_K, \ f(x) \leq K \\ K, & x \in \varphi_K, \ f(x) > K \\ 0, & \text{otherwise} \end{cases}$$

as $K \to \infty$, $F_K(x) \to f(x)$ (pointwise)

range of $F_K(x)$ in $[0, K]$

For fixed K, $j \geq 1$ define

$$E_{l,j} = \left\{ x \in \varphi_K \mid \frac{l}{j} < F_K(x) \leq \frac{l+1}{j} \right\}$$

$$0 \leq l < Kj.$$

Define, $F_{K,j}(x) = \sum \frac{l}{j} \chi_{E_{l,j}}(x)$

Note:- $0 \leq F_K(x) - F_{K,j}(x) \leq \frac{1}{j}$; $\forall x$

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Let us construct, $\phi_K = F_{K,K}$.

at $K = \infty$, $F_{K,K}(x) = \phi_K(x) = F_K(x)$.

$\Rightarrow |F_K - \phi_K| \leq \frac{1}{K}$, as $K \to \infty$, $\phi_K \to f$.

<u>Lusin's theorem :-</u>

$f(x) = f^+(x) - f^-(x)$

where, $f^+(x) = \max\{f(x), 0\}$

$f^-(x) = \max\{-f(x), 0\}$

$f^+ \geq 0$; $f^- \geq 0$.

$\uparrow$ $\qquad\qquad \uparrow$

$\{\varphi_K^1\}$ $\qquad \{\varphi_K^{(2)}\}$

so, $|\varphi_K^{(1)} - \varphi_K^{(2)}| \to f$.

→ Suppose 'f' is a measurable function on a finite measurable set E. Then, $\exists$ a closed set, $F_\varepsilon \subset E$, s.t $F|_{F_\varepsilon}$ is continous and $m(E \setminus F_\varepsilon) < \varepsilon$ for any given $\varepsilon > 0$.

function restriction

<u>Proof:-</u> Let $\{f_n\}$ be a sequence of simple function converging pt-wise to 'f'. We can find E_n s.t $m(E_n) < \frac{1}{2^n}$ & 'f_n' is continous outside E_n.

By Egorov's theorem, we can find a closed set $A_{\varepsilon/3}$ s.t $m(E \setminus A_{\varepsilon/3}) < \varepsilon/3$ &

$f_n \to f$ uniformly on $A_{\varepsilon/3}$.

Define, $F' = A_{\varepsilon/3} \setminus \bigcup_{n \geq N} E_n$ $\left\{ \because N \text{ being closed s.t, } \sum_{n \geq N} \frac{1}{2^n} < \varepsilon/3 \right\}$

$\Rightarrow f|_{F'}$ is contn.

($\because f_n \to f$ uniformly on F')

(cont.)

Find closed set F_ε, s.t $m(F' \setminus F_\varepsilon) < \varepsilon/3$

$$m(E \setminus F_\varepsilon) \leq m(E \setminus A_{\varepsilon/3}) + m(A_{\varepsilon/3} \setminus F') + m(F' \setminus F_\varepsilon)$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$m(E \setminus F_\varepsilon) \leq \varepsilon$$

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Lebesgue Integral

Simple function:

$$\varphi(x) = \sum_{K=1}^{N} a_K \chi_{E_K}(x)$$

cannonical representation of φ is such that when; $a_K \neq 0$ and E_K's are disjoint.

→ Suppose φ takes finitely many distinct non-zero values $c_1, c_2, \ldots, c_M$.

$$E_K = \{x \mid \varphi(x) = c_K\}, \quad E_K \text{ is disjoint.}$$

★ **Lebesgue Integral :-**

$$\int_R \phi \, dx \Rightarrow \sum_{K=1}^{N} c_K \, m(E_K)$$

$$\int_E \phi \, dx = \int_R \phi \cdot \chi_E \, dx$$

★ **Properties of Lebesgue Integral:**

i) choice of independence of the representation for ϕ. If $\phi = \sum_{K=1}^{N} a_K \chi_{E_K}(x)$ is any representation of ϕ, then, $\int \phi \, dx = \sum_{K=1}^{N} a_K \, m(E_K)$.

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2.) <u>Linearity</u>,

$$\int (a\Psi + b\phi)\, dx = a\int\Psi + b\int\phi$$

where, Ψ and ϕ are simple f^n.

3.) <u>Additivity</u>.

E and F are disjoint measurable set, then:-

$$\int_{E\cup F}\phi = \int_E \phi + \int_F \phi \qquad \{\because \chi_{E\cup F} = \chi_E + \chi_F\}$$

4.) <u>Monotonicity</u>.

if $\phi \geq \Psi$

then, $\int\phi \geq \int\Psi$.

5.) <u>Triangle inequality</u>.

$$\left|\int \phi dx\right| \leq \int |\phi| dx$$

$$\phi = \Sigma a_K \chi_{E_K}$$

$$|\phi| = \Sigma |a_K| \chi_{E_K}$$

$$\left|\int\phi dx\right| = \left|\Sigma a_K m(E_K)\right| \leq \Sigma |a_K| m(E_K) = \int|\phi| dx$$

<u>Explanation</u>.

1.) E_K's are disjoint but a_K's are not distinct.

$$a_K = a$$

$$E_a' = \bigcup_{a_K = a} E_K \quad ; \quad m(E_a') = \sum_{a_K = a} m(E_K)$$

$$\int\phi dx = \sum_{a_K = a} a \cdot m(E_a') = \sum_{K=1}^{N} a_K m(E_K)$$

$\Rightarrow$ Suppose E_K's are not disjoint (but a_K's are distinct)

$$\bigcup_{K=1}^{N} E_K \Rightarrow \bigcup_{j=1}^{N} E_j^K \quad ; \quad E_i\text{'s are disjoint.}$$

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•) $\varphi = \sum_{k=1}^{M} c_k \chi_{E_k}$, $c_k \neq 0$

$E_k \cap E_j = \phi$

Lebesgue integral,

$$\int \varphi \, dx = \sum_{k=1}^{M} c_k \, m(E_k)$$

•) Integral of bounded measurable functions supported on a set of finite measure.

Def^n:- Support of $f := \{x \in E \mid f(x) \neq 0\}$

* If 'f' is measurable, then the supp(f) is also measurable.

* $f \geq 0$ measurable, $|f(x)| < M \;\forall x \in E$, $supp(f) \subseteq E$

we can find a sequence of simple functions φ_n, such that $supp(\varphi_n) \subseteq E \;\forall n$ & $\varphi_n \to F$ pointwise. ($|\varphi_n| < M$).

Lemma:-

⊙ Let 'f' be a non-negative bounded measurable function supported on a set E of finite measure. Let $\varphi_n \to f$ (φ_n's are as above).

Then;

i) $\lim_{n\to\infty} \int \varphi_n$ exists.

ii) If 'f' = 0 a.e, then $\lim_{n\to\infty} \int \varphi_n = 0$.

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Proof:- (i) By Egorov's theorem we can find a closed set $A_\epsilon \subset E$ s.t., $\varphi_n \to f$ uniformly on A_ϵ & $m(E \setminus A_\epsilon) < \epsilon$.

→ Define, $I_n = \int \varphi_n$.

$$|I_n - I_m| = \left| \int (\varphi_n - \varphi_m) \right|$$

$$\leq \int_{A_\epsilon} |\varphi_n - \varphi_m| + \int_{E\setminus A_\epsilon} |\varphi_n - \varphi_m|$$

$$\leq \int_{A_\epsilon} |\varphi_n - \varphi_m| + 2M\, m(E \setminus A_\epsilon)$$

$$\leq \int_{A_\epsilon} |\varphi_n - \varphi_m| + 2M\epsilon.$$

choose m, n large enough such that $|\varphi_n - \varphi_m| < \epsilon$ on A_ϵ. {Because of uniform convergence of $\{\varphi_n\}$}.

Thus, $|I_n - I_m| \leq \epsilon \cdot m(E) + 2M\epsilon$.

⇒ (I_n) is a cauchy sequence of real numbers.

⇒ I_n must converge, i.e., $\lim_{n\to\infty} \int \varphi_n$ exists.

(ii) $|I_n| \leq \epsilon \cdot m(E) + M \cdot \epsilon$. { this is the special case of (i) }.

⇒ $\int f = 0$.

Def^n:- If 'f' is a non-negative, bounded & measurable function supported on a set of finite measure ~~& $\{\varphi_n\}$ is a sequence of~~ then,

$$\int f = \lim_{n\to\infty} \int \varphi_n.$$

where, φ_n's are simple, $\text{supp}(\varphi_n) \subseteq \text{supp}(f)$ & $\varphi_n \longrightarrow f$ pointwise.

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Proposition:-

i) $\int (af + bg) = a\int f + b\int g$ (linearity)

ii) $f \le g \Rightarrow \int f \le \int g$. (monotonicity)

iii) $\int_{E \cup F} f = \int_E f + \int_F f$, $E \cap F = \phi$ (Additive)

iv) $\left| \int_E f \right| \le \int_E |f|$ (T.E).

Imp.

★ Bounded convergence theorem:-

Suppose $\{f_n\}$ is a sequence of measurable function supported on a set of finite measure and bounded by M. such that $f_n \to f$ pointwise. Then, 'f' is measurable bounded and supported on E, such that, $\int f_n \to \int f$.

$$\Rightarrow \int_E |f_n - f| \le \int_{A_\epsilon} |f_n - f| + \int_{E \setminus A_\epsilon} |f_n - f|$$

$$\le \epsilon \cdot m(E) + 2 \cdot M \cdot \epsilon$$

$$\Rightarrow \left| \int f_n - \int f \right| \le \int |f_n - f| \to 0 \text{ as } n \to \infty.$$

$$\Rightarrow \int f_n \to \int f.$$

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Exercise:- suppose $f \geq 0$, $supp(f) \subseteq E$, $m(E) < \infty$
If $\int f = 0$, then prove that $f = 0$ almost everywhere (a.e).

→ $E_K = \{x \in E \mid f(x) \geq \frac{1}{K}\}$, $K = 1, 2, 3, \ldots$

$$0 = \int_E f \geq \int_{E_K} f \geq \frac{1}{K} \cdot m(E_K). \Rightarrow m(E_K) = 0.$$

$$E = \{x \in R \mid f(x) > 0\}$$

$$E = \bigcup_{K=1}^{\infty} E_K \quad ; \quad m(E) = 0.$$

⊙ Prove that every Riemann integrable function on $[a,b]$ is measurable & Lebesgue integrable.

proof:- To show: $\int_{[a,b]}^{R} f(x)dx = \int_{[a,b]}^{L} f(x)dx$

→ Since 'f' is Riemann integrable. i.e, $f \in R[a,b]$ we get two sequence of simple functions 'namely' $\{\phi_K\}$ & $\{\psi_K\}$ such that,

$$\phi_1 \leq \phi_2 \leq \ldots \leq f \leq \ldots \leq \psi_2 \leq \psi_1.$$

with $\lim_{K \to \infty} \int_{[a,b]}^{R} \phi_K = \int_{[a,b]}^{R} f = \lim_{K \to \infty} \int_{[a,b]}^{R} \psi_K$

Observe:- $\int_{[a,b]}^{R} \phi_K = \int_{[a,b]}^{L} \phi_K$ and $\int_{[a,b]}^{R} \psi_K = \int_{[a,b]}^{L} \psi_K$

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Suppose, for $K \geq 1$, $\phi_K \longrightarrow \tilde{\phi}$ & $\psi_K \longrightarrow \tilde{\psi}$, Then,

$$\tilde{\phi} \leq f \leq \tilde{\psi}.$$

$$\& \quad \lim_{k\to\infty} \overset{L}{\int_{[a,b]}} \phi_k = \overset{L}{\int_{[a,b]}} \tilde{\phi}.$$

$$\lim_{k\to\infty} \overset{L}{\int_{[a,b]}} \psi_K = \overset{L}{\int_{[a,b]}} \tilde{\psi}.$$

{Here R → Riemann
L → Lebesgue}

{∵ By Bounded convergence theorem

$$\rightarrow \overset{L}{\int_{[a,b]}} \tilde{\phi} = \overset{L}{\int_{[a,b]}} \tilde{\psi} \Rightarrow \int(\tilde{\psi} - \tilde{\phi})$$

$$\Rightarrow \lim_{k\to\infty} \left[\overset{L}{\int_{[a,b]}} \psi_K - \overset{L}{\int_{[a,b]}} \phi_K \right]$$

$$\Rightarrow \lim_{k\to\infty} \left[\overset{R}{\int_{[a,b]}} \psi_K - \overset{R}{\int_{[a,b]}} \phi_K \right] = 0.$$

$$\rightarrow \tilde{\psi} - \tilde{\phi} \geq 0$$

$$\Rightarrow \tilde{\psi} = \tilde{\phi} \quad \text{a.e.,}$$

$$\Rightarrow \tilde{\psi} = \tilde{\phi} = f \Rightarrow \text{'}f\text{' is measurable.}$$

$$\rightarrow \overset{L}{\int_{[a,b]}} f = \lim_{k\to\infty} \overset{L}{\int_{[a,b]}} \phi_K = \lim_{k\to\infty} \overset{R}{\int_{[a,b]}} \phi_K = \overset{R}{\int_{[a,b]}} f.$$

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★ Lebesgue integral of non-negative measurable functions.

suppose $f \geq 0$; f is measurable.

$\int f = \sup\limits_{0 \leq g \leq f} \int g$ and g is a bounded measurable function supported on a set of finite measure.

→ we say 'f' is integrable if $\int f < \infty$.

$\Rightarrow \quad \int f + \int g \leq \int (f+g)$

$g_1 \leq f$

$g_2 \leq g$.

$\int \overset{\eta_1}{g_1} + \int \overset{\eta_2}{g_2} = \int g_1 + g_2 \leq \int (f+g)$

Take supremum in the LHS over all $g_1 \leq f$ & $g_2 \leq g$, then we have,

$$\int f + \int g \leq \int (f+g) \quad \text{——} \quad (1).$$

conversely, let $\eta \leq f+g$,

η is non-negative bounded measurable having finite support.

set, $\eta_1 = \min \{f, \eta\}$

$\eta_2 = \eta - \eta_1$

Then, $0 \leq \eta_1 \leq f$, $0 \leq \eta_2 \leq g$ $\quad \Big| \quad \begin{matrix} f \geq \eta \\ f < \eta \end{matrix}$

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$$\Rightarrow \quad f_n(x) = \begin{cases} n, & 0 < x \le 1/n \\ 0, & \frac{1}{n} \le x < 1 \end{cases}$$

$\int f_n = 1$ for each n. $\boxed{f \ge 0}$

$f_n \longrightarrow 0$ pointwise.

$1 = \int f_n \not\longrightarrow \int 0 = 0.$

$\int f \le \liminf \int f_n.$

★. <u>Fatou's Lemma</u> :-

Suppose $\{f_n\}$ is a sequence of non-negative measurable function s.t. $f_n \to f$ pointwise.

Then, $\int f \le \lim \inf \int f_n$

<u>proof</u>: Let 'g' be a bounded measurable function having support of finite measure s.t., $0 \le g \le f$. Let $g_n = \min\{g, f_n\}$.

g_n's are measurable function having finite measure.

$g_n \to g$ as $n \to \infty$ (pointwise)

$\Rightarrow \int g_n \to \int g.$

$\rightarrow$ ∵ $g_n \le f_n$

$\Rightarrow \int g_n \le \int f_n$

$\Rightarrow \int g \le \lim \inf \int f_n.$

Take supremum over all $g \le f$ in LHS.

$\Rightarrow \int f \le \lim \inf \int f_n.$

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$$f_n(x) = \begin{cases} -1/n, & x \in [n, 2n] \\ 0, & \text{otherwise.} \end{cases}$$

Does this lemma hold when $f < 0$?

Example, let $f_n = -1/n$, $x \in [n, 2n]$

0, otherwise.

$\rightarrow \int f_n(x) = -1 \quad \forall n$

$\liminf (f_n(x)) = -1$

$\rightarrow \int f = 0$

$\rightarrow \int f \geq \liminf (f_n)$

$0 \geq 1$

$\rightarrow$ Lemma doesn't hold for $f < 0$.

14/3/19.

Problem:- Suppose f is integrable i.e., $\int f < \infty$

Then prove that $f < \infty$ a.e.,

proof:- Take $E_K = \{x \mid f(x) \geq K\}$. Then $E_K \searrow E = \bigcap_{K=1}^{\infty} E_K$

converges.

$= \{x \mid f(x) = \infty\}$

$\rightarrow$ Now, $\underset{M}{\int f} \geq \int f \chi_{E_K} \geq K \cdot m(E_K) \quad \forall K \geq 1$

$\Rightarrow m(E_K) = 0$.

$m(E_K) \longrightarrow m(E)$.

$\Rightarrow \underbrace{\lim_{K \to \infty} m(E_K)}_{0} = m(E) = 0$

$\Rightarrow f < \infty$ a.e.,

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$\rightarrow \quad E_n = (n, \infty)$

$E_n \searrow E = (\infty, \infty) = \phi$

$$\begin{array}{ccc} m(E_n) & \not\rightarrow & m(\phi) \\ \Downarrow & & \Downarrow \\ \infty & & 0 \end{array}$$

★ Prove:- E_K is a decreasing sequence of measurable sets s.t $E_K \searrow E$. then $m(E_K) \rightarrow m(E)$ provided $\exists$ a 'K' s.t, $m(E_K) < \infty$.

proof:-

$$E = \bigcap_{K=1}^{\infty} E_K$$

$$E_1 = E \cup \left(\bigcup_{K=1}^{\infty} G_K \right)$$

where $G_K = E_K \setminus E_{K+1}$

$$m(E_1) = m(E) + \lim_{N \to \infty} \sum_{K=1}^{N-1} \left(m(E_K) \setminus m(E_{K+1}) \right).$$

$$\Rightarrow m(E) + \lim_{N \to \infty} \sum_{K=1}^{N-1} \left(m(E_K) - m(E_{K+1}) \right)$$

$$\Rightarrow \quad m(E) + \cancel{m(E_1)} - \lim_{N \to \infty} m(E_N)$$

$$\Rightarrow \lim_{N \to \infty} m(E_N) = m(E)$$

Corollary of Fatou's Lemma:-

Monotone Convergence Theorem:-

If $\{f_n\}$ be a sequence of non-negative measurable functions s.t, $f_n \nearrow f$, then, $\int f_n \rightarrow \int f$ or $\int |f_n - f| \rightarrow 0$

$\rightarrow \quad f_{n+1}(x) \geq f_n(x) \quad \forall \quad n = 1, 2, \ldots$

proof:- $f_n \le f$

$\Rightarrow \int f_n \le \int f$.

$\Rightarrow \limsup \int f_n \le \underbrace{\int f \le \liminf \int f_n}_{\text{By Fatou's}}$ —— (1)

We know, $\liminf \int f_n \le \limsup \int f_n$ —— (2)

(1) & (2):- $\limsup \int f_n = \liminf \int f_n = \int f$ —— (3).

* construct, $g_n = f_1 - f_n$

$\rightarrow g_n \nearrow f_1 - f$

$\rightarrow \lim \left(\int f_1 - f_n\right) \overset{\text{By MCT}}{=} \int f_1 - \int f$

$\therefore \lim \int f_n = \int f$.

⊙ Corollary:- suppose, $\sum_{k=1}^{\infty} a_k(x)$, $a_k(x) \ge 0$ is a series of measurable functions. Then $\int \sum_{k=1}^{\infty} a_k(x) = \sum_{k=1}^{\infty} \int a_k(x)$. Moreover,

if $\sum_{k=1}^{\infty} \int a_k(x) < \infty \Rightarrow \sum_{k=1}^{\infty} a_k(x) < \infty$ a.e.,

$\rightarrow f_n \Rightarrow \sum_{k=1}^{n} a_k(x)$, since $a_k(x) \ge 0$, $f_n \uparrow$ (incr)

$f = \sum_{k=1}^{\infty} a_k(x)$ $\quad f_n \uparrow f$.

Apply MCT, $\int f_n \to \int f$

$$\Rightarrow \lim_{n\to\infty} \int \sum_{k=1}^{n} a_k(x) = \sum_{k=1} a_k(x)$$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{\infty} \int a_k(x) = \sum_{k=1}^{\infty} \int a_k(x) = \int \sum_{k=1}^{\infty} a_k(x).$$

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★ **Borel - Cantelli Lemma :-**

$\{E_K\}$ is a countable family of measurable sets. s.t., $\sum_{K=1}^{\infty} m(E_K) < \infty$.

Define, $E = \limsup E_K$.

$$= \bigcap_{n=1}^{\infty} \bigcup_{K \geq n} E_K$$

$$= \{x \mid x \in E_K \text{ for infinitely many } K\}$$

→ Then, $m(E) = 0$.

proof: Define in previous corollary,

$a_K(x) = \chi_{E_K}(x)$. , $x \in E \Leftrightarrow x \in E_K$ for ∞ many K.

$\Leftrightarrow \sum_{K=1}^{\infty} a_K(x) = \infty$.

$\Rightarrow \sum_{K=1}^{\infty} \int a_K(x) = \sum_{K=1}^{\infty} m(E_K) < \infty$

$\Rightarrow \sum_{K=1}^{\infty} a_K(x) < \infty$ a.e.,

$\Rightarrow m(E) = 0$.

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18/3/19 TUTORIAL - 1

R. integral

4.) $f : [a,b] \longrightarrow R$ and cont at $c \in (a,b)$

$$\lim_{h\to 0^+} \frac{1}{2h} \int_{c-h}^{c+h} f(x)\,dx = f(c)$$

proof:- Let $\epsilon > 0$ arbitrary,

$\exists\ \delta > 0$ s.t $\Rightarrow |f(x) - f(c)| < \epsilon$, $x \in (c-\delta, c+\delta)$

$$\rightarrow \left|\frac{1}{2h}\int_{c-h}^{c+h} f(x)\,dx - f(c)\right| \leq \frac{1}{2h}\int_{c-h}^{c+h} |f(x) - f(c)|\,dx$$

$$\rightarrow \left|\frac{1}{2h}\int_{c-h}^{c+h} f(x) - \frac{1}{2h}\int_{c-h}^{c+h} f(c)\,dx\right| \leq \frac{1}{2h} \sup_{[c-h,\,c+h]} |f(x) - f(c)| \cdot 2h$$

$$\rightarrow \left|\frac{1}{2h}\int_{c-h}^{c+h} (f(x) - f(c))\,dx\right| \leq \frac{1}{2h} \cdot \epsilon \cdot 2h = \epsilon$$

(Hence proved)

In this q. we have supposed that $0 < h < \delta$

$\Rightarrow [c-h, c+h] \subseteq (c-\delta, c+\delta)$.

10) $\{f_n\}$ be a sequence of Riemann. Integral f^n.

$f_n \longrightarrow f$ pointwise in x.

Does 'f' is Riemann - integral?

If so, then $\lim_{n\to\infty} \int f_n = \int f$?

proof:- $f_n : [0,1] \longrightarrow R$

Let, $\{r_1, r_2, \dots\} = \{r_n : n \in N\}$ be an enumeration of ~~rection~~ rationals in $[0,1]$.

$$f_n(x) = \begin{cases} 1, & \text{if } x = \{r_1, r_2, \dots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

$f_n \to f$ pointwise.

$$f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

$f_n \to f$ pointwise, f_n & f are Riemann integrable

$\int f_n dx = 1 \neq 0 = \int f$.

$$\lim_{n\to\infty} \int f_n \neq \int f$$

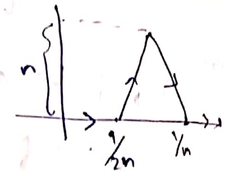
$f_n : R \to R$

$$f_n(x) = \begin{cases} n, & x \in (0, 1/n) \\ 0, & \text{otherwise.} \end{cases}$$

$f_n \to f$, $f \equiv 0$.

→ $f_n : [0,1] \to R$

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2n} \text{ or } x \ge 1/n \\ \text{linear } f^n \text{ as given in the diagram.} \end{cases}$$



$f \equiv 0$, $f_n \to f$ pointwise.

→ $\int f_n(x) dx$

$= \frac{1}{2} \cdot \left(\frac{1}{n} - \frac{1}{2n}\right) \cdot n$

$= \frac{1}{4}$.

→ $\lim \int f_n = \frac{1}{4} \neq 0 = \int f$.

TUTORIAL-2

Measurable f^n:

$f: E (\subseteq R) \longrightarrow R^* = [-\infty, \infty]$, where E is measurable set.

i) f is measurable $\forall\, \alpha \in R$, $\{x \in E, f(x) > \alpha\}$ is measurable $= f^{-1}(\alpha, \infty] \subseteq E$.

ii) $\{x \in E, f(x) \geq \alpha\}$ is measurable.

iii) $\{x \in E, f(x) < \alpha\}$ —— " ——

iv) $\{x \in E, f(x) \leq \alpha\}$ —— " ——

Proof:- (i) $\Rightarrow$ (ii) $f(x) > \alpha \Leftrightarrow f(x) \geq \alpha - \frac{1}{n}$ $\forall n$.

$\rightarrow \{x \in E : f(x) \geq \alpha\} \Rightarrow \bigcap_{n=1}^{\infty} \{x \in E \text{ s.t } f(x) \geq \alpha - \frac{1}{n}\}$

(ii) $\Rightarrow$ (iii) $\{x \in E : f(x) > \alpha$

Q.) $\alpha \in R^*$, show that $f^{-1}\{\alpha\}$ is a measurable set.

$\{x \in E : f(x) = \alpha\}$

i) $\alpha = -\infty$

ii) $\alpha = +\infty$

iii) $-\infty < \alpha < \infty$

i) $\{x \in E : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$

ii) $\{x \in E : f(x) = +\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) > n\}$

iii) $\{x \in E : f(x) = \alpha\} = \{x : f(x) \geq \alpha\} \cap \{x : f(x) \leq \alpha\}$

$f: E \to R^*$

$f(x) = c, \quad \forall x \in E$

$$\{x \in E : f(x) > \alpha\} = \begin{cases} E, & \alpha < c \\ \emptyset, & \alpha \geq c \end{cases}$$

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$$

$$\{x \in E : f(x) \geq \alpha\} = \begin{cases} A, & 0 \leq \alpha < 1 \\ E, & \alpha < 0 \\ \emptyset, & \alpha > 1 \end{cases}$$

Q.) $E \subseteq R, \quad m(E) = 0$

$f: E \to R^*$,

Is f measurable?

$\{x \in E : f(x) > \alpha\} \subseteq E$

$f_n \to f$ a.e pointwise. then prove that f is also Lebesgue measurable.

20/3/19

Integration for general functions:

'f' is a real valued function.

we say 'f' is Lebesgue integrable if $|f|$ is Lebesgue integrable

$\rightarrow f^+(x) = \max\{f(x), 0\}$

$f^-(x) = \min\{-f(x), 0\}$

Then, $f = f^+ - f^-$

$f^+ \le |f|$

$f^- \le |f|$

$\left\{ \because \int f = \sup_{g \le f} \int g \right.$

Define, $\int f = \int f^+ - \int f^-$

$= g_1 - g_2$

To show, $\int f_1 - \int f_2 = \int g_1 - \int g_2$.

$\rightarrow \int f_1 + g_2 = \int f_2 + g_1$

$\rightarrow \int f_1 + \int g_2 = \int f_2 + \int g_1$

$\Rightarrow g \in [f] \rightarrow$ equivalence class.

$g = f$ a.e.

Proposition:-

Let F be integrable on R^d. Then,

i) $\exists$ a ball 'B' of finite measure s.t,

$\int_{B^c} f < \epsilon$

ii) $\exists\ \delta$ s.t,

$\int_E f < \epsilon$ whenever $m(E) < \delta$.

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proof:- (i) consider $B_N = \{|x| < N\}$, define,

$f \geq 0$

By Defn

$f_N(x) = f \chi_{B_N}$.

$f_N \leq f_{N+1}$

$f_N \longrightarrow f$.

$\int |f - f_N| < \epsilon$ for large N.

$\Rightarrow \int (f - f \cdot \chi_{B_N}) < \epsilon$

$\Rightarrow \left| \int f (1 - \chi_{B_N}) \right| < \epsilon$

$\Rightarrow \left| \int f \chi_{B_N^c} \right| < \epsilon$

$\Rightarrow \left| \int_{B_N^c} f \right| < \epsilon$

$B_N \subseteq B_{N+1}$

$\chi_{B_N} \leq \chi_{B_{N+1}}$

$f \chi_{B_N} \leq f \chi_{B_{N+1}}$

$f_N \leq f_{N+1}$

(ii) Assume $f \geq 0$, $E_N = \{ x \mid f(x) \leq N \}$

$f_N = f \cdot \chi_{E_N}$

$f_N \leq f_{N+1}$, By MCT

$\exists$ large N s.t.,

$$\int_{E_N} |f - f_N| < \epsilon$$

→ Pick a 'δ' s.t, $N\delta < \epsilon$

$$\int_E f = \int_E (f - f_N) + \int_E f_N \leq N \cdot m(E) + \epsilon$$

$\leq 2\epsilon$

Let $\{f_n\}$ be a sequence of measurable f^n.

s.t., $f_n \to f$ a.e.,

suppose $|f_n(x)| \le g(x)$ & g is integrable.

Then, $\int |f_n - f| \to 0$ $\quad (\because \int f_n \to \int f)$.

Proof: For $N \geq 0$, let $E_N = \{x \mid |x| \le N, |g(x)| \le N\}$

Apply 1st part of the previous proposition to get a large 'N', s.t.,

$$\int_{E_N^c} g < \epsilon.$$

→ consider, $f_n \chi_{E_N}$ → bounded by 'N', easy to see that

$|f_n \cdot \chi_{E_N}| < N$.

Apply BCT to get,

$$\int_{E_N} |f - f_n| < \epsilon \quad \text{for large } n.$$

$$\rightarrow \left|\int (f_n - f)\right| = \left|\int_{E_N} (f_n - f)\right| + \left|\int_{E_N^c} (f_n - f)\right|$$

$$\le \epsilon + 2\left|\int_{E_N^c} g\right|$$

$$\le \epsilon + 2\epsilon$$

$$\le 3\epsilon.$$

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$\Rightarrow$ if, $f = u + iv$.

then, $\int f = \int u + i\int v$.

$u \le |f| \Rightarrow u$ is integrable.

$v \le |f| \Rightarrow v$ ——————.

so, f is integrable.

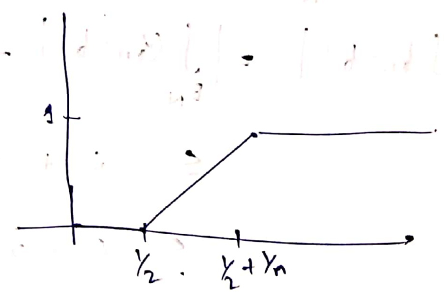
$\Rightarrow$ $C[a,b] \to$ continous

L^1 - norm.

Q.) whether $C[a,b]$ is complete ?

construct a function:-

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1/2 \\ n(x - 1/2) & \text{if, } 1/2 \le x \le 1/2 + 1/n \\ 1 & \text{if, } \frac{1}{2} + \frac{1}{n} \le x \le 1. \end{cases}$$



$\|f_n - f\|_1 \to 0$

Suppose there is a function, $f \in C[0,1]$

s.t, $d(f_n, f) \to 0$.

$$\int_0^{1/2} |f(x)| = \int_0^{1/2} |f_n - f| \le \int_0^1 |f_n - f| \longrightarrow 0$$

$$\Downarrow$$

$$\int_0^{1/2} |f(x)|\,dx = 0$$

$$\Downarrow$$

$$f(x) = 0 \ \forall \ x \in [0, 1/2).$$

→ If $\frac{1}{2} < x < 1$, then choose n_0 s.t, $\frac{1}{2} + \frac{1}{n_0} < x$; Then for all $n \geq n_0$,

$$\int_x^1 |1 - f(x)|\, dx$$

$$\Rightarrow \int_x^1 |f_n(x) - f(x)|\, dx \leq \int_0^1 |f_n(x) - f(x)|\, dx \to 0 \text{ as } n \to \infty.$$

★ $f(x) = 1, \ \forall \ \frac{1}{2} \leq x \leq 1$

$f \notin C[0,1]$.

So, $C[a,b]$ is not complete.

$L^1 \to$ Lebesgue measurable set.

★ <u>Riesz - Fischer Theorem :-</u>

$L^1(\mathbb{R})$ is complete ?

Proof :- consider $(f_n) \to$ a cauchy sequence.

$\|f_n - f_m\| < \epsilon$ whenever $m, n \geq N_0$.

If we can construct a subsequence (f_{n_k}) of (f_n) s.t.,

$\|f_{n_k} - f\| < \epsilon$, then the proof will be over.

$$\rightarrow \|f_n - f\| = \|f_n - f_{n_k} + f_{n_k} - f\|$$

$$\leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\|$$

$$\leq \epsilon + \epsilon$$

$$\leq 2\epsilon.$$

$\Rightarrow$ For each $\epsilon = \frac{1}{2^k}$, choose $N(\epsilon)$ s.t whenever $n_k \geq N(2^{-k})$ then, $\|f_n - f_{n_k}\|_1 < \frac{1}{2^k}$ $\forall$ $k \geq 1$.

Now, consider,

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x) \right)$$

Let, $g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} | \cancel{f_{n+1}} f_{n_{k+1}}(x) - f_{n_k}(x) |$.

$$\Rightarrow \int |f_{n_1}| + \cancel{\int} \sum_{k=1}^{\infty} \int | f_{n_{k+1}}(x) - f_{n_k}(x) |$$

$$\leq \|f_{n_1}\| + \sum_{k=1}^{\infty} \frac{1}{2^k} . < \infty \text{ (converges)}$$

converges a.e.

$\Rightarrow$ By MCT, f is integrable.

$f < \infty$ a.e.,

$\rightarrow f_{n_k} = f_{n_1}(x) + \sum_{j=1}^{k-1} \left(f_{n_{j+1}} - f_{n_j} \right)$.

for each k:-

$|f - f_{n_k}| \leq g$.

Apply DCT for:-

$\|f_{n_k} - f\| \to 0$ as $n_k \to \infty$.

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