NC 41

INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR
CLASS TEST / LABORATORY TEST

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Signature of the Invigilator

EXAMINATION (Mid Semester / End Semester)	SEMESTER (Autumn / Spring)
Roll Number	Section	Name: Koeli Ghoshal
Subject Number: MA20103	Subject Name: Partial Diff. Equations

Lecture-1 17.7.2017

Power series solution of 2nd order ODE

Some basic definitions

Power series

An infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = c_0 + c_1 (x-x_0) + c_2 (x-x_0)^2 + \cdots \quad (1)$$

is called power series in $(x-x_0)$. In particular, a power series in x is an infinite series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

For example, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

The power series (1) converges for $|x| < R$, where

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$$ → provided the limit exists.

Analytic function

A f^n. $f(x)$ on an interval containing the point $x=x_0$ is called analytic at x_0 if its Taylor's series

$$\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n$$

exists and converges to $f(x)$ $\forall$ x in the interval of convergence.

Ordinary and singular point

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{——(1)}$$

A point $x=x_0$ is called an ordinary point of the eqn. (1) if both $P(x)$ and $Q(x)$ are analytic at $x=x_0$. If $x=x_0$ is not an ordinary point of (1), then it is called a singular point of (1).

There are two types of singular points

(i) regular singular point

(ii) irregular " "

A singular point $x=x_0$ of (1) is called a regular singular pt. of (1) if $(x-x_0)P(x)$ & $(x-x_0)^2 Q(x)$ are analytic at $x=x_0$. A singular pt. which is not regular is called an irregular singular point.

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Ex Determine whether $x=0$ is an ordinary point or a regular singular point of

$$2x^2 \frac{d^2y}{dx^2} + 7x(x+1)\frac{dy}{dx} - 3y = 0$$

Soln

$$\frac{d^2y}{dx^2} + \frac{7(x+1)}{2x}\frac{dy}{dx} - \frac{3}{2x^2}y = 0$$

$\because$ $P(x)$ & $Q(x)$ are undefined at $x=0$, so both of $P(x)$ & $Q(x)$ are not analytic at $x=0$.

Thus $x=0$ is not an ordinary point and so $x=0$ is a singular point.

$$x P(x) = \frac{7(x+1)}{2} \qquad x^2 Q(x) = -\frac{3}{2}$$

Both are analytic at $x=0$.

$\therefore$ $x=0$ is a regular singular pt.

Power series sol^n. about an ordinary point $x = x_0$

$$y'' + P(x)\, y' + Q(x)\, y = 0 \quad \text{---}(1)$$

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n \quad \text{---}(2)$$

$$y' = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1} \quad \text{---}(3)$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n (x - x_0)^{n-2} \quad \text{---}(4)$$

Putting the values of y, y', y'' in (1),

$$A_0 + A_1 (x - x_0) + A_2 (x - x_0)^2 + \cdots + A_n (x - x_0)^n + \cdots = 0 \quad \text{---}(5)$$

where the coefficients A_0, A_1, A_2 etc are some f^n- of $c_0, c_1, c_2 \cdots$ etc. Since (5) is an identity, all of $A_0, A_1, A_2 \cdots$ must be zero.

i.e. $A_0 = 0 \quad A_1 = 0 \quad \cdots \quad A_n = 0, \quad \cdots$ ---(6)

Solving (6), we obtain the coefficients in (2) in terms of c_0 & c_1. Substituting in (2), we obtain the reqd. series sol^n. of (1) in powers of $x - x_0$.

Ex Solve by power series method $y' - y = 0$

$y' - y = 0 \quad -(1)$

Solve We assume that a sol^n. of (1) is given by

$$y = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{n=0}^{\infty} c_n x^n \quad -(2)$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow (c_1 + 2c_2 x + 3c_3 x^2 + \cdots) - (c_0 + c_1 x + c_2 x^2 + \cdots) = 0$$

$$\Rightarrow (c_1 - c_0) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 + \cdots = 0$$

$c_1 - c_0 = 0 \longrightarrow c_1 = c_0$

$2c_2 - c_1 = 0 \rightarrow c_2 = \frac{c_1}{2} = \frac{c_0}{2}$

$3c_3 - c_2 = 0$

$c_3 = \frac{c_2}{3} = \frac{c_0}{3!}$

$$y = c_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= c_0 e^x$$

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Lecture - 2 & 3

Ex Find the power series sol^n ~~of the~~ eqn.

$(x^2+1)y'' + xy' - xy = 0$ in powers of x

i.e. about $x=0$.

Sol^n

$$y'' + \frac{x}{x^2+1}y' - \frac{x}{x^2+1}y = 0$$

$$P(x) = \frac{x}{x^2+1} \qquad q(x) = -\frac{x}{x^2+1}$$

$P(x)$ & $q(x)$ are both analytic at $x=0$. So $x=0$ is an ordinary point. Therefore, to solve the eqn we take the power series

$$y = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

$$\therefore y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substituting in the diff. eqn.

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$$

$$\Rightarrow 2c_2 + (6c_3 + c_1 - c_0)x + \sum_{n=2}^{\infty}\left[n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_{n-1}\right]x^n$$

$\therefore 2c_2 = 0 \qquad \therefore c_2 = 0$

$6c_3 + c_1 - c_0 = 0 \qquad \therefore c_3 = \dfrac{c_0 - c_1}{6}$

$n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_{n-1} = 0 \quad \forall\, n \geq 2$

$$\therefore c_{n+2} = \frac{c_{n-1} - n^2 c_n}{(n+1)(n+2)} \quad \forall\, n \geq 2$$

This is called recurrence relation.

Putting $n=2$ in the rec. relation,

$$c_4 = \frac{c_1 - 4c_2}{12} = \frac{c_1}{12} \quad \text{as } c_2 = 0$$

Putting $n=3$, $c_5 = -\dfrac{9c_3}{20} = -\dfrac{9}{20}\left(\dfrac{c_0 - c_1}{6}\right)$

$$= -\frac{3}{40}(c_0 - c_1)$$

$$y = c_0 + c_1 x + \left(\frac{c_0 - c_1}{6}\right)x^3 + \frac{1}{12}c_1 x^4 - \frac{3}{40}(c_0 - c_1)x^5 + \cdots$$

$$= c_0\left(1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + \cdots\right)$$
$$+ c_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \cdots\right)$$

which is the reqd. solⁿ. near $x=0$ where c_0 & c_1 are arbitrary constants.

Series solution about a regular singular point

Frobenius method

$$x^2 \frac{d^2y}{dx^2} + y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y'' =$$

$$\sum c_n n(n-1) x^n + \sum c_n x^n = 0$$

For values of $n = 0, 1, 2, \dots$ $c_n = 0$. So, there is no series of the form $y = \sum_{n=0}^{\infty} c_n x^n$ which may be the solution of the above eqn.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y = 0 \quad \text{— (1)}$$

which can be written in the form

$$x^2 \frac{d^2y}{dx^2} + xP(x)\, x\frac{dy}{dx} + x^2 Q(x)\, y = 0 \quad \text{— (2)}$$

$xP(x)$ and $x^2 Q(x)$ are regular at $x = 0$. So we can write

$$xP(x) = p_0 + p_1 x + \dots + p_s x^s + \dots$$

$$x^2 Q(x) = q_0 + q_1 x + \dots + q_s x^s + \dots \quad \text{— (3)}$$

For the solⁿ. of (2),

$$y = x^k (c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots), \; c_0 \neq 0$$

$$= c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \dots + c_n x^{k+n} + \dots \quad \text{— (4)}$$

where $c_0, c_1, c_2, \ldots$ are constants and $c_0 \neq 0$.

From (4),

$$\frac{dy}{dx} = c_0 k x^{k-1} + c_1(k+1)x^k + \cdots + c_n(k+n)x^{k+n-1} + \cdots$$

$$\frac{d^2y}{dx^2} = c_0 k(k-1)x^{k-2} + c_1(k+1)kx^{k-1} + \cdots$$
$$+ c_n(k+n)(k+n-1)x^{k+n-2} + \cdots$$

Substituting these, the LHS of (2) becomes

$$c_0 k(k-1)x^k + c_1(k+1)kx^{k+1} + \cdots + c_n(k+n)(k+n-1)x^{k+n} + \cdots$$
$$+ (p_0 + p_1 x + \cdots + p_s x^s + \cdots) \times$$
$$\left\{ c_0 k x^k + c_1(k+1)x^{k+1} + \cdots + c_n(k+n)x^{k+n} + \cdots \right\}$$
$$+ (q_0 + q_1 x + \cdots + q_s x^s + \cdots) \times$$
$$(c_0 x^k + c_1 x^{k+1} + \cdots + c_n x^{k+n} + \cdots) = 0 \quad \text{--- (5)}$$

From (1), (2) & (5)

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y$$
$$= c_0\left\{k(k-1) + kp_0 + q_0\right\}x^{k-2}$$
$$+ \left[c_1\left\{(k+1)k + (k+1)p_0 + q_0\right\}\right.$$
$$\left. + c_0(kp_1 + q_1)\right]x^{k-1} + \cdots$$

$$\text{--- (6)}$$

Equating the coefficients of x^{k-1} to zero,

$$c_1\{(k+1)k + (k+1)p_0 + q_0\} + c_0(kp_1 + q_1) = 0$$

$$\Rightarrow \frac{c_1}{c_0} = \text{a } f^n \text{ of } k = f_1(k) \text{ say.}$$

Similarly, equating the coeff. of x^k to zero, we shall get $\frac{c_2}{c_0} = f_2(k)$ and so on.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = c_0\{k(k-1) + kp_0 + q_0\} x^{k-2} \quad \text{---(7)}$$

y is obtained as

$$y = c_0 x^k \left(1 + \frac{c_1}{c_0}x + \frac{c_2}{c_0}x^2 + \cdots + \frac{c_n}{c_0}x^n + \cdots\right)$$

$$= c_0 x^k \left(1 + x f_1(k) + x^2 f_2(k) + \cdots + x^n f_n(k) + \cdots\right) \quad \text{---(8)}$$

From (7), $k^2 + (p_0 - 1)k + q_0 = 0 \quad \because c_0 \neq 0$

This eqn. is called indicial equation.

The roots of the indicial eqn.

(i) may have distinct roots, not differing by an integer

(ii) may have equal roots

(iii) are distinct but differ by an integer.

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Case (i) Here $k_1 \neq k_2$ and $k_1 - k_2$ is not an integer.

we put k_1 and k_2 for k in (8) and drop the arb. const. c_0 and get the two series sol^n, namely

$$y_1 = x^{k_1}\{1 + x f_1(k_1) + x^2 f_2(k_1) + \cdots + x^n f_n(k_1) + \cdots\}$$

and $$y_2 = x^{k_2}\{1 + x f_1(k_2) + x^2 f_2(k_2) + \cdots + x^n f_n(k_2) + \cdots\}$$

The G.S. is $y = Ay_1 + By_2$

where A, B are arb constants.

Case ii) Here $k_1 = k_2$

We put in (8), $y = c_0 W$

where $$W = x^k\{1 + x f_1(k) + x^2 f_2(k) + \cdots + x^n f_n(k) + \cdots\} \quad -(10)$$

From (7), we have

$$\frac{d^2W}{dx^2} + P(x)\frac{dW}{dx} + Q(x)W = \{k^2 + k(p_0 - 1) + q_0\}x^{k-2}$$

$$= (k-k_1)^2 x^{k-2} \quad \text{since } k_1 = k_2 \quad -(11)$$

Differentiating both sides of (11) w.r.t. k, we get

$$\frac{d^2}{dx^2}\left(\frac{dW}{dk}\right) + P(x)\frac{d}{dx}\left(\frac{dW}{dk}\right) + Q(x)\left(\frac{dW}{dk}\right)$$
$$= (k-k_1)^2 \frac{d}{dk}x^{k-2} + 2(k-k_1)x^{k-2} \quad -(12)$$

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1. Advanced Engineering Mathematics — Kreyszig
2. An elementary course in PDE — T. Amaranath
3. Elements of PDE — Ian N Sneddon
4. Introduction to PDE — K Sankara Rao

From (12), we see that $\frac{dW}{dk}$ is a sol^n. of the eqn. if $k=k_1$. First sol^n. is

$$[W]_{k=k_1} = x^{k_1}\{1 + x f_1(k_1) + x^2 f_2(k_1) + \cdots + \cdots$$

$$= W_1 \text{ (say)}$$

Second sol^n. is

$$\left[\frac{dW}{dk}\right]_{k=k_1} = W_2 \text{ (say)}$$

G.S. is $\quad y = AW_1 + BW_2$

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Roll Number	Section	Name: Koeli Ghoshal
Subject Number: MA20103	Subject Name: Partial Diff. Equations

Lecture 2 & 3 (continued) 19.7.2017

Example of Type I of Frobenious method (Roots of indicial eqn. are unequal & not differing by an integer)

Ex Solve in series $9x(1-x)y'' - 12y' + 4y = 0$

Sol^n Putting in standard form

$$\frac{d^2y}{dx^2} - \frac{4}{3x(1-x)}\frac{dy}{dx} + \frac{4}{9x(1-x)}y = 0 \quad \text{---(1)}$$

$$P(x) = -\frac{4}{3x(1-x)} \qquad Q(x) = \frac{4}{9x(1-x)}$$

$\because$ $P(x)$ & $Q(x)$ are not both analytic at $x=0$, so $x=0$ is not an ordinary pt. of the diff-eqn. $xP(x)$ & $x^2Q(x)$ are analytic at $x=0$. So $x=0$ is a regular singular point of (1).

We take the sol^n. in the form

$$y = \sum_{m=0}^{\infty} c_m x^{k+m} \quad \text{where } c_0 \neq 0.$$

$$\therefore y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} \quad \text{and} \quad \text{---(2)}$$

$$y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} \quad \text{---(3)}$$

Lecture 4 24.7.2017

Substituting these values in (1)

$$9x(1-x)\sum_{m=0}^{\infty} c_m(k+m)(k+m-1)x^{k+m-2}$$

$$-12\sum_{m=0}^{\infty} c_m(k+m)x^{k+m-1} + 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\Rightarrow 9x\sum_{m=0}^{\infty} c_m(k+m)(k+m-1)x^{k+m-2}$$

$$-9x^2\sum_{m=0}^{\infty} c_m(k+m)(k+m-1)x^{k+m-2}$$

$$-12\sum_{m=0}^{\infty} c_m(k+m)x^{k+m-1} + 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m\left\{9(k+m)(k+m-1) - 12(k+m)\right\}x^{k+m-1}$$

$$+\sum_{m=0}^{\infty} c_m\left\{4 - 9(k+m)(k+m-1)\right\}x^{k+m} = 0 \quad \text{--- (4)}$$

$$9(k+m)(k+m-1) - 12(k+m) = 3(k+m)(3k+3m-7) \quad \text{--- (5)}$$

$$4 - 9(k+m)(k+m-1) = -(3k+3m-4)(3k+3m+1) \quad \text{--- (6)}$$

Using (5) & (6), (4) can be rewritten as

$$3\sum_{m=0}^{\infty} c_m(k+m)(3k+3m-7)x^{k+m-1}$$

$$-\sum_{m=0}^{\infty} c_m(3k+3m-4)(3k+3m+1)x^{k+m} = 0$$

$$\text{--- (7)}$$

$$\Rightarrow 3\sum_{m=0}^{\infty} c_m (k+m)(3k+3m-7)\, x^{k+m-1}$$
$$-\sum_{m=1}^{\infty} c_{m-1}\left[3k+3(m-1)-4\right]\cdot\left[3k+3(m-1)+1\right] \times x^{k+m-1} = 0 \quad \text{—(7)}$$

Equating to zero the coefficient of the smallest power of x i.e. x^{k-1}, (7) gives the indicial eqn.

$$3c_0 k(3k-7) = 0$$

or, $k(3k-7) = 0$, $\because c_0 \neq 0$

$$k = 0 \text{ and } \frac{7}{3} \quad \text{—} (8)$$

which are not equal and not differing by an integer

For recurrence relation, we equate to zero the coeff. of x^{k+m-1} and obtain

$$3c_m (k+m)(3k+3m-7) - c_{m-1}\left[3k+3(m-1)-4\right] \times \left[3k+3(m-1)+1\right] = 0$$

$$\Rightarrow c_m = \frac{3k+3m-2}{3(k+m)}\, c_{m-1} \quad \text{—} (9)$$

Taking $m=1$ in (9), $\quad c_1 = \frac{c_0}{3} \times \frac{3k+1}{k+1} \quad$ —(10)

" $\quad m=2$ in (9), $\quad c_2 = \frac{3k+4}{3(k+2)}\, c_1$

$$= \frac{c_0}{3^2} \times \frac{(3k+1)(3k+4)}{(k+1)(k+2)} \quad \text{—(11)}$$

and so on.

Putting these values in

$$y = x^k (c_0 + c_1 x + c_2 x^2 + \cdots) \text{ gives}$$

$$y = c_0 x^k \left[1 + \frac{1}{3} \frac{3k+1}{k+1} x + \frac{1}{3^2} \frac{(3k+1)(3k+4)}{(k+1)(k+2)} x^2 + \cdots \right] \quad -(12)$$

Putting $k=0$ and replacing c_0 by a in (12),

$$y = a \left(1 + \frac{1}{3} x + \frac{1.4}{3.6} x^2 + \cdots \right) = au \text{ (say)}$$

Putting $k = \frac{7}{3}$ and replacing c_0 by b in (12)

$$y = b x^{\frac{7}{3}} \left(1 + \frac{8}{10} x + \frac{8.11}{10.13} x^2 + \cdots \right) = bv \text{ (say)}$$

Reqd solⁿ. is $y = au + bv$.

$$y = x^k (c_0 + c_1 x + c_2 x^2 + \cdots) = \sum_{m=0}^{\infty} c_m x^{k+m}$$

$$= c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \cdots$$

$$\frac{dy}{dx} = c_0 k x^{k-1} + c_1 (k+1) x^k + c_2 (k+2) x^{k+1} + \cdots$$

$$= \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$$

Example on Type 2 Frobenious method

Roots of indicial eqn. are equal.

Ex1 Solve in series $(x-x^2)y'' + (1-5x)y' - 4y = 0$ — (1)

about $x = 0$.

Solⁿ. Let the series solⁿ. of (1) be

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0 \quad \text{— (2)}$$

$$\left.\begin{aligned} y' &= \sum_{m=0}^{\infty} c_m (k+m)\, x^{k+m-1} \\ y'' &= \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m-2} \end{aligned}\right\} \text{— (3)}$$

Putting these values in (1),

$$(x-x^2)\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m-2} + (1-5x)\sum_{m=0}^{\infty} c_m (k+m)\, x^{k+m-1}$$

$$- 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m \left\{ (k+m)(k+m-1) + (k+m) \right\} x^{k+m-1}$$
$$- \sum_{m=0}^{\infty} c_m \left\{ (k+m)(k+m-1) + 5(k+m) + 4 \right\} x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m (k+m)^2 x^{k+m-1} - \sum_{m=0}^{\infty} c_m (k+m+2)^2 x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m (k+m)^2 x^{k+m-1} - \sum_{m=1}^{\infty} c_{m-1} (k+m+1)^2 x^{k+m-1} = 0$$

— (4)

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Equating to zero, the coeff. of smallest powers of x i.e. x^{k-1}, (4) gives the indicial eqn.

$$c_0 k^2 = 0 \quad \therefore k^2 = 0 \quad \because c_0 \neq 0$$

$$\therefore k = 0, 0 \rightarrow \text{equal roots}$$

Finally, equating to zero the coefficient of x^{k+m-1} in (4)

$$c_m (k+m)^2 - c_{m-1}(k+m+1)^2 = 0$$

$$\Rightarrow \quad c_m = \frac{(k+m+1)^2}{(k+m)^2} c_{m-1} \quad \text{—(5)}$$

Putting $m = 1, 2, 3, \ldots$ in (5),

$$c_1 = \left[\frac{(k+2)^2}{(k+1)^2}\right] c_0$$

$$c_2 = \frac{(k+3)^2}{(k+2)^2} c_1 = \frac{(k+3)^2}{(k+2)^2} \frac{(k+2)^2}{(k+1)^2} c_0 = \frac{(k+3)^2}{(k+1)^2} c_0$$

$$c_3 = \frac{(k+4)^2}{(k+1)^2} c_0 \quad \text{and so on.}$$

Putting these values in (2)

$$y = x^k c_0 \left\{ 1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \cdots \right\} \quad \text{—(6)}$$

Putting $k = 0$ and substituting c_0 by a in (6),

$$y = a(1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots) = au \text{ (say)}$$

$$\frac{\partial y}{\partial k} = c_0 x^k \ln x \left\{ 1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \cdots \right\}$$

$$+ c_0 x^k \left\{ \frac{(k+2)^2}{(k+1)^2} x \left(\frac{2}{k+2} - \frac{2}{k+1} \right) + \frac{(k+3)^2}{(k+1)^2} x^2 \left(\frac{2}{k+3} - \frac{2}{k+1} \right) + \cdots \right\}$$

Putting $k = 0$ and replacing c_0 by b gives

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = b \ln x (1 + 2^2 x + 3^2 x^2 + \cdots) + b\left\{2^2 x (1-2) + 3^2 x^2 \left(\frac{2}{3} - 2\right) + \cdots\right\}$$

$$= b\left[u \ln x - 2(1.2x + 2.3x^2 + \cdots)\right] = bv \text{ as say}$$

$$y = au + bv$$

Lecture – 5 & 6

26.7.2017

Examples of Type 3 on Frobenious method :

Roots of indicial eqn. are unequal and differing by an integer.

Ex Find the series solution near $x=0$ of the eqn.

$$x^2 y'' + (x+x^2) y' + (x-9) y = 0 \quad -(1)$$

Sol^n Let $y = x^k (c_0 + c_1 x + c_2 x^2 + \cdots) = \sum_{m=0}^{\infty} c_m x^{k+m}, \; c_0 \neq 0$ — (2)

$$y' = \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1}, \quad y'' = \sum_{m=0}^{\infty} (k+m)(k+m-1) c_m x^{k+m-2} \quad -(3)$$

$$x^2 \sum_{m=0}^{\infty} (k+m)(k+m-1) c_m x^{k+m-2} + (x+x^2) \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1} + (x-9) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \{ (k+m)(k+m-1) + (k+m) - 9 \} c_m x^{k+m} + \sum_{m=0}^{\infty} \{ (k+m) + 1 \} c_m x^{k+m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (k+m+3)(k+m-3) c_m x^{k+m} + \sum_{m=0}^{\infty} (k+m+1) c_m x^{k+m+1} = 0 \quad -(4)$$

Equating to zero the coeff. of smallest power of x i.e. x^k

Indicial eqn. is $(k+3)(k-3) c_0 = 0$

$\Rightarrow k = 3, -3$ as $c_0 \neq 0$

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Equating to zero the coeff. of x^{k+m} in (4)

$$(k+m+3)(k+m-3)c_m + (k+m)c_{m-1} = 0$$

$$\Rightarrow c_m = -\frac{(k+m)}{(k+m+3)(k+m-3)}c_{m-1} \quad \text{--- (5)}$$

Putting $m = 1, 2, 3, \ldots$ in (5)

$$c_1 = -\frac{k+1}{(k+4)(k-2)}c_0$$

$$c_2 = +\frac{(k+1)(k+2)}{(k-1)(k-2)(k+4)(k+5)}c_0$$

$$c_3 = -\frac{(k+1)(k+2)(k+3)}{k(k-1)(k-2)(k+4)(k+5)(k+6)}c_0$$

and so on. Putting these values in (2),

$$y = c_0 x^k\left[1 - \frac{k+1}{(k-2)(k+4)}x + \frac{(k+1)(k+2)}{(k-1)(k-2)(k+4)(k+5)}x^2 - \frac{(k+1)(k+2)(k+3)}{k(k-1)(k-2)(k+4)(k+5)(k+6)}x^3 + \cdots\right] \quad \text{--- (6)}$$

Putting $k=3$ and replacing c_0 by a

$$y = ax^3\left[1 - \frac{4}{2.7}x + \frac{4.5}{2.1.7.8}x^2 + \frac{4.5.6}{3.2.1.7.8.9}x^3 - \cdots\right]$$

$= au$ (say)

Putting $k=-3$ and replacing c_0 by (b) in (6)

$$y = bx^{-3}\left\{1 - \frac{2}{5}x + \frac{1}{20}x^2 + 0 + 0 + 0 + \cdots\right\} = bv$$

$y = au + bv$ is the reqd. soln.

Legendre Equation and Legendre function

The diff. eqn. of the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{—(1)}$$

is called Legendre's eqn. where n is a non-ve integer (n is a real const)

The eqn. can be written in the form

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

The solution in descending powers of x is more used than the ascending powers.

Let us assume $y = \sum_{m=0}^{\infty} c_m x^{k-m}$ —(2) where $c_0 \neq 0$

$y' =$ $\quad y'' =$ —~~(2)~~

$$(1-x^2)\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - 2x\sum_{m=0}^{\infty} c_m(k-m)x^{k-m-1}$$

$$+ n(n+1)\sum_{m=0}^{\infty} c_m x^{k-m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m\left\{(k-m)(k-m-1) + 2(k-m) - n(n+1)\right\}x^{k-m} = 0$$

—(3)

$$(k-m)(k-m-1) - 2(k-m) - n(n+1)$$

$$= \cancel{(k-m)} \quad (k-m-n)(k-m+n+1)$$

$$\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m(k-m-n)(k-m+n+1)x^{k-m} = 0 \quad \text{--- (4)}$$

$$\Rightarrow \sum c_{m-2}(k-m+2)(k-m+1)x^{k-m} - \sum_{m=0}^{\infty} c_m(k-m-n)(k-m+n+1)x^{k-m} = 0$$

To get the indicial eqn., we equate to zero the coeff. of highest power of x i.e. x^k in (4) and obtain

$$c_0(k-n)(k+n+1) = 0 \quad \text{--- (5)}$$

$$\Rightarrow (k-n)(k+n+1) = 0 \quad \because c_0 \neq 0$$

$$k = n, \; -(n+1)$$

& We equate to zero the coeff. of x^{k-1} in (4) and obtain $c_1(k-1-n)(k+n) = 0$ --- (6)

For $k = n$ & $-(n+1)$, neither $(k-1-n)$ nor $(k+n)$ is zero. So from (6), $c_1 = 0$. Finally equating to zero the coeff. of x^{k-m}, we have

$$c_{m-2}(k-m+2)(k-m+1) - c_m(k-m-n)(k-m+n+1) = 0$$

$$\Rightarrow c_m = \frac{(k-m+2)(k-m+1)}{(k-m-n)(k-m+n+1)} c_{m-2} \quad \text{--- (7)}$$

Putting $m = 3, 5, 7 \ldots$ in (7) and noting that $c_1 = 0$

$$c_1 = c_3 = c_5 = c_7 = \cdots = 0 \quad \text{--- (8)}$$

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Case I When $k = n$. Then (7) becomes

$$C_m = -\frac{(n-m+2)(n-m+1)}{m(2n-m+1)} C_{m-2} \quad \text{---(9)}$$

Putting $m = 2, 4, 6, \ldots$ in (9),

$$c_2 = -\frac{n(n-1)}{2(2n-1)} c_0, \quad c_4 = -\frac{(n-2)(n-3)}{4\cdot(2n-3)} c_2$$

$$= \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)} c_0$$

Rewriting (2), for $k = n$

$$y = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + c_3 x^{n-3} \ldots \quad (10)$$

Using (8) and the above values of $c_2, c_4, \ldots$ (10) becomes after replacing c_0 by a,

$$y = a\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\,(2n-1)(2n-3)} x^{n-4} \ldots\right]$$

(11)

Case II When $k = -(n+1)$; $C_m = \frac{(n+m-1)(n+m)}{m(2n+m+1)} C_{m-2}$ (12)

$$c_2 = \frac{(n+1)(n+2)}{2(2n+3)} c_0, \quad c_4 = \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)} c_0$$

and so on,

$$y = c_0 x^{-n-1} + c_1 x^{-n-2} + c_2 x^{-n-3} + c_3 x^{-n-4}$$

(13)

$$y = b\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\,(2n+3)(2n+5)} x^{-n-5} \ldots\right] \quad (14)$$

If we take $a = [1.3.5...(2n-1)]/n!$, the solⁿ. (11) is denoted by $P_n(x)$ and is called Legendre function of the 1st kind (Legendre polynomial). Thus $P_n(x)$ is a solⁿ. of (1). Again if we take

$b = n!/[1.3.5...(2n+1)]$, the solⁿ. (14) is denoted by $Q_n(x)$ and is called Legendre fⁿ. of the 2nd kind. Thus $P_n(x)$ & $Q_n(x)$ are two L.I. solⁿs. of (1). Hence the general solⁿ. of (1) is

$$y = A\,P_n(x) + B\,Q_n(x)$$ where

A, B are arbitrary constants.

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Roll Number									Section		Name	Koeli Ghoshal
Subject Number	M	A	2	0	1	0	3		Subject Name			PDE

Lecture-7

31.7.2017

§ $P_n(x)$ is the coefficient of h^n in the expansion in ascending powers of $(1-2hx+h^2)^{-1/2}$, $|x|<1$, $|h|<1$

Proof $(1-2hx+h^2)^{-1/2} = \{1-h(2x-h)\}^{-1/2}$

$$= 1+\frac{1}{2}h(2x-h)+\frac{1.3}{2.4}h^2(2x-h)^2+\cdots$$

$$+\frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}h^{n-1}(2x-h)^{n-1}$$

$$+\frac{1.3\ldots(2n-1)}{2.4\ldots2n}h^n(2x-h)^n+\cdots$$

Coefficient of h^n

$$=\frac{1.3\ldots(2n-1)}{2.4\ldots2n}(2x)^n-\frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}{}^{n-1}c_1(2x)^{n-2}$$

$$+\frac{1.3\ldots(2n-5)}{2.4\ldots(2n-4)}{}^{n-2}c_2(2x)^{n-4}+\cdots$$

$$=\frac{1.3\ldots(2n-1)}{n!}\left[x^n-\frac{2n}{2n-1}(n-1)\frac{x^{n-2}}{2^2}\right.$$

$$\left.+\frac{2n(2n-2)}{(2n-1)(2n-3)}\frac{(n-2)(n-3)}{2!}\frac{x^{n-4}}{2^4}-\cdots\right]$$

$$=\frac{1.3\ldots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}-\cdots\right]$$

$= P_n(x)$

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$$

$(1-2xh+h^2)^{-1/2}$ is called generating function of the Legendre polynomials.

Orthogonal properties of Legendre's polynomial

(i) $\int_{-1}^{+1} P_m(x) P_n(x)\, dx = 0$ if $m \neq n$

(ii) $\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}$ if $m=n$.

Proof (i) $\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$

$$\therefore \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0 \quad -(1)$$

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)P_m = 0 \quad -(2)$$

Multiplying (i) by P_m & (ii) by P_n and then subtracting,

$$P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} - P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}$$
$$+ \left\{n(n+1) - m(m+1)\right\} P_n P_m = 0$$

Integrating between -1 and $+1$,

$$\left[P_m(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1}\frac{dP_m}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx$$

$$-\left[P_n(1-x^2)\frac{dP_m}{dx}\right]_{-1}^{+1} + \int_{-1}^{+1}\frac{dP_n}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx$$

$$+\left[n(n+1)-m(m+1)\right]\int_{-1}^{1} P_m P_n\,dx = 0$$

$$\int_{-1}^{1} P_m P_n\,dx = 0 \quad \because m \neq n.$$

(ii) We have $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Squaring both sides

$$(1-2xh+h^2)^{-1} = \sum_{n=0}^{\infty} h^{2n}\{P_n(x)\}^2 + 2\sum_{\substack{m,n=0\\ m\neq n}}^{\infty} h^{m+n} P_m(x)P_n(x)$$

Integrating between -1 and $+1$,

$$\sum_{n=0}^{\infty}\int_{-1}^{+1} h^{2n}\left[P_n(x)\right]^2 dx + 2\sum_{\substack{m,n=0\\ m\neq n}}^{\infty}\int_{-1}^{+1} h^{m+n} P_m(x)P_n(x)\,dx$$

$$= \int_{-1}^{+1}\frac{dx}{(1-2xh+h^2)}$$

or, $$\sum_{n=0}^{\infty}\int_{-1}^{+1} h^{2n}\left[P_n(x)\right]^2 dx = \int_{-1}^{+1}\frac{dx}{(1-2xh+h^2)}$$

$$= -\frac{1}{2h}\left\{\ln(1-2xh+h^2)\right\}_{-1}^{+1}$$

$$= -\frac{1}{2h}\left\{\ln(1-h)^2 - \ln(1+h)^2\right\}$$

$$= \frac{1}{2h}\left[\ln\left(\frac{1+h}{1-h}\right)^2\right] = \frac{1}{h}\ln\left\{\frac{1+h}{1-h}\right\}$$

$$= \frac{2}{h}\left\{ h + \frac{h^3}{3} + \frac{h^5}{5} + \cdots \right\}$$

$$= 2\left\{ 1 + \frac{h^2}{3} + \frac{h^4}{5} + \cdots \right\} = \sum_{n=0}^{\infty} \frac{2h^{2n}}{2n+1}$$

Equating the coeff. of h^{2n},

$$\int_{-1}^{+1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}$$

<u>Result</u> : $\int_{-1}^{+1} P_m(x) P_n(x) \, dx = \frac{2}{2n+1} \delta_{m,n}$

where $\delta_{m,n}$ is the Kronecker delta defined as

$$\delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Lecture-8 1.8.2017

$$P_n(x) = \sum_{r=0}^{n/2} (-1)^r \frac{(2n-2r)!}{2^n r! (n-2r)! (n-r)!} x^{n-2r}$$

$$\text{where } \frac{n}{2} = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$$a = \frac{1.3.5\ldots(2n-1)}{n!} = \frac{(2n)!}{2^n (n!)^2}$$

$$= \frac{1.2.3.5.6\cdots \cancel{2n}(2n-1)2n}{2.4.6\cdots 2n \; n!}$$

$$= \frac{(2n)!}{2^n n! \, n!}$$

Recurrence formulae

I $\quad (2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$

Proof $\quad (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Differentiating both sides w.r.t. h,

$$-\frac{1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h) = \sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-1/2} = (1-2xh+h^2)\sum n h^{n-1} P_n(x)$$

$$\Rightarrow (x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

$$\Rightarrow (x-h)\left[P_0(x) + hP_1(x) + \cdots + h^{n-1}P_{n-1}(x) + h^n P_n(x) + \cdots\right]$$

$$= (1-2xh+h^2)\left[P_1(x) + 2hP_2(x) + \cdots + (n-1)h^{n-2}P_{n-1}(x) + nh^{n-1}P_n(x) + (n+1)h^n P_{n+1}(x) + \cdots\right]$$

Equating coeff. of h^n from both the sides,

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

In short, $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$

Equating the coeff. of h^{n-1} from both sides,

$$nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$$

II $\quad nP_n = xP_n' - P'_{n-1}$ dashes denote differentials w.r.t. x

Proof $\quad (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad -(1)$

Differentiating (1) w.r.t. 'h',

$$(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} n h^{n-1} P_n(x) \quad -(2)$$

Again differentiating (1) w.r.t. 'x',

$$h(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x)$$

$$\Rightarrow h(x-h)(1-2xh+h^2)^{-3/2} = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) \quad -(3)$$

From (2) and (3),

$$h\sum_{n=0}^{\infty} n h^{n-1} P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x)$$

$$\Rightarrow h\left[h^0 P_1(x) + 2hP_2(x) + \cdots + nh^{n-1}P_n(x) + \cdots\right]$$

$$= (x-h)\left[P_0'(x) + hP_1'(x) + \cdots + h^{n-1}P'_{n-1}(x) + h^n P_n'(x) + \cdots\right]$$

Equating coeff. of h^n on both the sides,

$$nP_n(x) = xP_n'(x) - P'_{n-1}(x)$$

i.e. $nP_n = xP_n' - P'_{n-1}$

$\underline{\text{III}}$ $(2n+1) P_n = P'_{n+1} - P'_{n-1}$

<u>Proof</u> From rec. for. I

$$(2n+1)\, x P_n = (n+1) P_{n+1} + n P_{n-1}$$

Differentiating w.r.t. x,

$$(2n+1)\, x P_n' + (2n+1) P_n = (n+1) P'_{n+1} + n P'_{n-1} \quad \text{---(1)}$$

From rec. formula II

$$x P_n' = n P_n + P'_{n-1} \quad \cancel{\text{---(2)}}$$

$$\Rightarrow (2n+1)\, x P_n' = (2n+1)(n P_n + P'_{n-1}) \quad \text{---(2)}$$

Eliminating $x P_n'$ from (1) & (2),

$$(2n+1)(n P_n + P'_{n-1}) + (2n+1) P_n = (n+1) P'_{n+1} + n P'_{n-1}$$

$$\Rightarrow (2n+1)(n+1) P_n = (n+1) P'_{n+1} + n P'_{n-1} - (2n+1) P'_{n-1}$$

$$\Rightarrow (2n+1)(n+1) P_n = (n+1) P'_{n+1} - (n+1) P'_{n-1}$$

$$\Rightarrow (2n+1) P_n = P'_{n+1} - P'_{n-1}$$

$\underline{\text{IV}}$ $(n+1) P_n = P'_{n+1} - x P_n'$

<u>Proof</u> From rec. formular II & III

$$\Rightarrow n P_n = x P_n' - P'_{n-1} \quad \text{---(1)}$$

$$(2n+1) P_n = P'_{n+1} - P'_{n-1} \quad \text{---(2)}$$

Subtracting (1) from (2),

$$(n+1) P_n = P'_{n+1} - x P_n'$$

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V $\quad (1-x^2) P_n' = n(P_{n-1} - xP_n)$

Proof Replacing n by $n-1$ in rec. for. IV,

$$nP_{n-1} = P_n' - xP'_{n-1} \quad \text{—(1)}$$

II rec. for $\rightarrow$ $\quad nP_n = xP_n' - P'_{n-1} \quad$ —(2)

Multiplying (2) by x and then subtracting from (1),

$$n(P_{n-1} - xP_n) = (1-x^2) P_n'$$

$$\text{i.e. } (1-x^2) P_n' = n(P_{n-1} - xP_n)$$

VI $\quad (1-x^2) P_n' = (n+1)(xP_n - P_{n+1})$

Proof From rec. for. I

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

which can be written as

$$(n+1)xP_n + nxP_n = (n+1)P_{n+1} + nP_{n-1} \quad \text{—(1)}$$

$$\Rightarrow (n+1)(xP_n - P_{n+1}) = n(P_{n-1} - xP_n)$$

Rec formula V $\rightarrow$ $(1-x^2) P_n' = n(P_{n-1} - xP_n)$ —(2)

From (1) & (2),

$$(1-x^2) P_n' = (n+1)(xP_n - P_{n+1})$$

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Rodrigue's Formula

$$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n}(x^2-1)^n$$

Ex Show that $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$

Soln

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$$

$$= \left\{1 - h(2x-h)\right\}^{-1/2}$$

$$= 1 + \frac{h}{2}(2x-h) + \frac{1.3}{2.4} h^2 (2x-h^2) + \cdots$$

$$\Rightarrow P_0(x) + hP_1(x) + h^2 P_2(x) + \cdots$$

$$= 1 + xh + \frac{1}{2}(3x^2-1)h^2 + \cdots$$

Equating the coeff. of likes powers of h,

$P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$

Bessel eqn. and Bessel function

The differential eqn. of the form

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

$$\Rightarrow \quad y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0 \quad \text{--- (1)}$$

is called Bessel eqn. of order n, n being a constant. We solve (1) by Frobenius method.

Let the series solution be

$$\left.\begin{aligned} y &= \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0 \\ y' &= \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} \\ y'' &= \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} \end{aligned}\right\} \text{--- (2)}$$

Substituting in (1),

$$x^2 \sum c_m (k+m)(k+m-1) x^{k+m-2} + x \sum c_m (k+m) x^{k+m-1} + (x^2 - n^2) \sum c_m x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m \left\{ (k+m)(k+m-1) + (k+m) - n^2 \right\} x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m (k+m+n)(k+m-n) x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0 \quad \text{--- (3)}$$

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Equating to zero the smallest power of x, namely x^k, (3) gives the indicial eqn.

$$c_0 (k+n)(k-n) = 0 \quad \text{i.e. } (k+n)(k-n) = 0 \quad , c_0 \neq 0$$

Roots are $k = n, -n$

Next equating to zero the coeff. of x^{k+1} in (3),

$$c_1 (k+1+n)(k+1-n) = 0$$

so that $c_1 = 0$ for $k = n$ & $k = -n$.

Finally equating to zero the coeff. of x^{k+m} in (3),

$$c_m (k+m+n)(k+m-n) + c_{m-2} = 0$$

$$\Rightarrow c_m = + \frac{1}{(k+m+n)\underset{(n-k-m)}{(\cancel{k+m-n})}} c_{m-2} \quad \text{--- (4)}$$

Putting $m = 3, 5, 7 \ldots$ in (4) & using $c_1 = 0$,

$$c_1 = c_3 = c_5 = c_7 = \ldots = 0 \quad \text{--- (5)}$$

Putting $m = 2, 4, 6, \ldots$ in (4) gives

$$c_2 = \frac{1}{(k+2+n)(n-k-2)} c_0$$

$$c_4 = \frac{1}{(k+4+n)(n-k-4)} c_2$$

$$= \frac{1}{(k+4+n)(n-k-4)(k+2+n)(n-k-2)} c_0$$

and so on.

Putting these values in (2),

$$y = c_0 x^k \left[1 + \frac{x^2}{(n+k+2)(n-k-2)} + \frac{x^4}{(n+k+2)(n-k-2)(n+k+4)(n-k-4)} + \cdots \right]$$

Replacing k by n and $-n$ and also replacing c_0 by a and b in the above eqn

$$y = ax^n \left\{ 1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4 \cdot 8 \cdot (1+n)(2+n)} - \cdots \right\} \text{—(6)}$$

$$\text{and } y = bx^{-n} \left\{ 1 - \frac{x^2}{4(1-n)} + \frac{x^4}{4 \cdot 8 (1-n)(2-n)} - \cdots \right\} \text{—(7)}$$

If $a = 1 / \{ 2^n \, \Gamma(n+1) \}$, then (6) is called Bessel f^n of the 1st kind of order n. It is denoted by $J_n(x)$. Thus,

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8 \cdot (n+1)(n+2)} - \cdots \right] \text{—(8)}$$

$$\text{or, } J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2} \right)^{2r+n} \text{—(9)}$$

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EXAMINATION (Mid Semester / End Semester)		SEMESTER (Autumn / Spring)	
Roll Number		Section	Name: Koeli Ghoshal
Subject Number	MA20103	Subject Name	Partial Diff. Equations

Lecture-1 17.7.2017

Power series solution of 2nd order ODE

Some basic definitions

Power series

An infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = c_0 + c_1 (x-x_0) + c_2 (x-x_0)^2 + \cdots \quad (1)$$

is called power series in $(x-x_0)$. In particular, a power series in x is an infinite series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

For example, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

The power series (1) converges for $|x| < R$, where

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \longrightarrow \text{provided the limit exists.}$$

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Analytic function

A f^n. $f(x)$ on an interval containing the point $x=x_0$ is called analytic at x_0 if its Taylor's series

$$\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n$$

exists and converges to $f(x)$ $\forall x$ in the interval of convergence.

Ordinary and singular point

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{---(1)}$$

A point $x=x_0$ is called an ordinary point of the eqn. (1) if both $P(x)$ and $Q(x)$ are analytic at $x=x_0$. If $x=x_0$ is not an ordinary point of (1), then it is called a singular point of (1).

There are two types of singular points

(i) regular singular point

(ii) irregular " "

A singular point $x=x_0$ of (1) is called a regular singular pt. of (1) if $(x-x_0)P(x)$ & $(x-x_0)^2 Q(x)$ are analytic at $x=x_0$. A singular pt. which is not regular is called an irregular singular point.

Ex Determine whether $x=0$ is an ordinary point or a regular singular point of

$$2x^2 \frac{d^2y}{dx^2} + 7x(x+1)\frac{dy}{dx} - 3y = 0$$

Sol^n

$$\frac{d^2y}{dx^2} + \frac{7(x+1)}{2x}\frac{dy}{dx} - \frac{3}{2x^2}y = 0$$

$\because$ $P(x)$ & $Q(x)$ are undefined at $x=0$, so both of $P(x)$ & $Q(x)$ are not analytic at $x=0$.

Thus $x=0$ is not an ordinary point and so $x=0$ is a singular point.

$$x P(x) = \frac{7(x+1)}{2} \qquad x^2 Q(x) = -\frac{3}{2}$$

Both are analytic at $x=0$.

$\therefore$ $x=0$ is a regular singular pt.

Power series solⁿ. about an ordinary point $x=x_0$

$$y'' + P(x)\, y' + q(x)\, y = 0 \quad \text{---(1)}$$

$$y = \sum_{n=0}^{\infty} c_n (x-x_0)^n \quad \text{---(2)}$$

$$y' = \sum_{n=1}^{\infty} n c_n (x-x_0)^{n-1} \quad \text{---(3)}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n (x-x_0)^{n-2} \quad \text{---(4)}$$

Putting the values of y, y', y'' in (1),

$$A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots + A_n(x-x_0)^n + \cdots = 0 \quad \text{---(5)}$$

where the coefficients A_0, A_1, A_2 etc are some fⁿ. of $c_0, c_1, c_2 \ldots$ etc. Since (5) is an identity, all of $A_0, A_1, A_2 \ldots$ must be zero.

i.e. $A_0 = 0 \quad A_1 = 0 \quad \ldots \quad A_n = 0, \quad \ldots$ ---(6)

Solving (6), we obtain the coefficients in (2) in terms of c_0 & c_1. Substituting in (2), we obtain the reqd. series solⁿ. of (1) in powers of $x-x_0$.

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E1 Solve by power series method $y' - y = 0$

Solve: $y' - y = 0$ — (1)

We assume that a sol^n. of (1) is given by

$y = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{n=0}^{\infty} c_n x^n$ — (2)

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$\Rightarrow (c_1 + 2c_2 x + 3c_3 x^2 + \cdots) - (c_0 + c_1 x + c_2 x^2 + \cdots) = 0$

$\Rightarrow (c_1 - c_0) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 + \cdots = 0$

$c_1 - c_0 = 0 \longrightarrow c_1 = c_0$

$2c_2 - c_1 = 0 \rightarrow c_2 = \frac{c_1}{2} = \frac{c_0}{2}$

$3c_3 - c_2 = 0$ $\quad c_3 = \frac{c_2}{3} = \frac{c_0}{3!}$

$$y = c_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= c_0 e^x$$

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19.7.2017

Lecture - 28 & 3

Ex Find the power series solⁿ of the eqn.

$(x^2+1)y'' + xy' - xy = 0$ in powers of x i.e. about $x=0$.

Solⁿ

$$y'' + \frac{x}{x^2+1}y' - \frac{x}{x^2+1}y = 0$$

$$P(x) = \frac{x}{x^2+1} \qquad Q(x) = -\frac{x}{x^2+1}$$

$P(x)$ & $Q(x)$ are both analytic at $x=0$. So $x=0$ is an ordinary point. Therefore, to solve the eqn we take the power series

$$y = c_0 + c_1x + c_2x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

$$\therefore y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

Substituting in the diff. eqn.

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$$

$$\Rightarrow 2c_2 + (6c_3 + c_1 - c_0)x + \sum_{n=2}^{\infty}\left[n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_{n-1}\right]x^n = 0$$

$\therefore 2c_2 = 0 \qquad \therefore c_2 = 0$

$6c_3 + c_1 - c_0 = 0 \qquad \therefore c_3 = \dfrac{c_0 - c_1}{6}$

$n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_{n-1} = 0 \quad \forall\, n \geq 2$

$$\therefore c_{n+2} = \frac{c_{n-1} - n^2 c_n}{(n+1)(n+2)} \quad \forall\, n \geq 2$$

This is called recurrence relation.

Putting $n=2$ in the rec. relation,

$$c_4 = \frac{c_1 - 4c_2}{12} = \frac{c_1}{12} \quad \text{as } c_2 = 0$$

Putting $n=3$, $c_5 = -\dfrac{9c_3}{20} = -\dfrac{9}{20}\left(\dfrac{c_0 - c_1}{6}\right)$

$$= -\frac{3}{40}(c_0 - c_1)$$

$$y = c_0 + c_1 x + \left(\frac{c_0 - c_1}{6}\right)x^3 + \frac{1}{12}c_1 x^4 - \frac{3}{40}(c_0 - c_1)x^5 + \cdots$$

$$= c_0\left(1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + \cdots\right)$$
$$+ c_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \cdots\right)$$

which is the reqd. solⁿ. near $x=0$ where c_0 & c_1 are arbitrary constants.

Frobenius method

$$x^2 \frac{d^2y}{dx^2} + y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y'' =$$

$$\sum c_n n(n-1) x^n + \sum c_n x^n = 0$$

For values of $n = 0, 1, 2, \ldots$ $c_n = 0$. So, there is no series of the form $y = \sum_{n=0}^{\infty} c_n x^n$ which may be the solution of the above eqn.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \quad \text{--- (1)}$$

which can be written in the form

$$x^2 \frac{d^2y}{dx^2} + xP(x)\, x\frac{dy}{dx} + x^2 Q(x)\, y = 0 \quad \text{--- (2)}$$

$xP(x)$ and $x^2 Q(x)$ are regular at $x=0$. So we can write

$$xP(x) = p_0 + p_1 x + \cdots + p_s x^s + \cdots$$

$$x^2 Q(x) = q_0 + q_1 x + \cdots + q_s x^s + \cdots \quad \text{--- (3)}$$

For the soln. of (2),

$$y = x^k (c_0 + c_1 x + c_2 x^2 + \cdot + c_n x^n + \cdots),\ c_0 \neq$$

$$= c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \cdots + c_n x^{k+n} + \text{---}$$

--- (4)

where $c_0, c_1, c_2 - -$ are constants and $c_0 \neq 0$.

From (4),

$$\frac{dy}{dx} = c_0 k x^{k-1} + c_1(k+1)x^k + \cdots + c_n(k+n)x^{k+n-1} + \cdots$$

$$\frac{d^2y}{dx^2} = c_0 k(k-1)x^{k-2} + c_1(k+1)kx^{k-1} + \cdots$$
$$+ c_n(k+n)(k+n-1)x^{k+n-2} + \cdots$$

Substituting these, the LHS of (2) becomes

$$c_0 k(k-1)x^k + c_1(k+1)kx^{k+1} + \cdots + c_n(k+n)(k+n-1)x^{k+n} + \cdots$$
$$+ (p_0 + p_1 x + \cdots + p_s x^s + \cdots) \times$$
$$\{c_0 k x^k + c_1(k+1)x^{k+1} + \cdots + c_n(k+n)x^{k+n} + \cdots\}$$
$$+ (q_0 + q_1 x + \cdots + q_s x^s + \cdots) \times$$
$$(c_0 x^k + c_1 x^{k+1} + \cdots + c_n x^{k+n} + \cdots) = 0 \quad \text{--- (5)}$$

From (1), (2) & (5)

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y$$
$$= c_0\{k(k-1) + kp_0 + q_0\}x^{k-2}$$
$$+ [c_1\{(k+1)k + (k+1)p_0 + q_0\}$$
$$+ c_0(kp_1 + q_1)]x^{k-1} + \cdots$$

$$\text{--- (6)}$$

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Equating the coefficients of x^{k-1} to zero,

$$c_1\left\{(k+1)k + (k+1)p_0 + q_0\right\} + c_0(kp_1 + q_1) = 0$$

$\Rightarrow \frac{c_1}{c_0}$ = a f^n. of $k = f_1(k)$ say.

Similarly, equating the coeff. of x^k to zero,

we shall get $\frac{c_2}{c_0} = f_2(k)$ and so on.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = c_0\left\{k(k-1) + kp_0 + q_0\right\} x^{k-2} \quad \text{---(7)}$$

y is obtained as

$$y = c_0 x^k\left(1 + \frac{c_1}{c_0}x + \frac{c_2}{c_0}x^2 + \cdots + \frac{c_n}{c_0}x^n + \cdots\right)$$

$$= c_0 x^k\left(1 + x f_1(k) + x^2 f_2(k) + \cdots + x^n f_n(k) + \cdots\right) \quad \text{---(8)}$$

From (7), $k^2 + (p_0 - 1)k + q_0 = 0 \quad \because c_0 \neq 0$

This eqn. is called indicial equation.

The roots of the indicial eqn.

(i) may have distinct roots, not differing by an integer

(ii) may have equal roots

(iii) are distinct but differ by an integer.

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Case (i) Here $k_1 \neq k_2$ and $k_1 - k_2$ is not an integer. We put k_1 and k_2 for k in (8) and drop the arb. const. c_0 and get the two series sol^n, namely

$$y_1 = x^{k_1}\left\{1 + x f_1(k_1) + x^2 f_2(k_1) + \cdots + x^n f_n(k_1) + \cdots\right\}$$

and $$y_2 = x^{k_2}\left\{1 + x f_1(k_2) + x^2 f_2(k_2) + \cdots + x^n f_n(k_2) + \cdots\right\}$$

The G.S. is $y = Ay_1 + By_2$

where A, B are arb constants.

Case ii) Here $k_1 = k_2$

We put in (8), $y = c_0 W$

where $$W = x^k\left\{1 + x f_1(k) + x^2 f_2(k) + \cdots + x^n f_n(k) + \cdots\right\} \quad \text{—(10)}$$

From (7), we have

$$\frac{d^2W}{dx^2} + P(x)\frac{dW}{dx} + Q(x)W = \left\{k^2 + k(p_0 - 1) + q_0\right\}x^{k-2}$$
$$= (k - k_1)^2 x^{k-2} \quad \text{since } k_1 = k_2 \quad \text{—(11)}$$

Differentiating both sides of (11) w.r.t. k, we get

$$\frac{d^2}{dx^2}\left(\frac{dW}{dk}\right) + P(x)\frac{d}{dx}\left(\frac{dW}{dk}\right) + Q(x)\left(\frac{dW}{dk}\right)$$
$$= (k - k_1)^2 \frac{d}{dk} x^{k-2} + 2(k - k_1)x^{k-2} \quad \text{—(12)}$$

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1. Advanced Engineering Mathematics — Kreyszig
2. An elementary course in PDE — T. Amaranath
3. Elements of PDE — Ian N Sneddon
4. Introduction to PDE — K Sankara Rao

From (12), we see that $\frac{dW}{dk}$ is a solⁿ. of the eqn. if $k=k_1$. First solⁿ. is

$$[W]_{k=k_1} = x^{k_1}\left\{1 + x f_1(k_1) + x^2 f_2(k_1) + \cdots + \cdots\right\}$$
$$= W_1 \text{ (say)}$$

Second solⁿ. is

$$\left[\frac{dW}{dk}\right]_{k=k_1} = W_2 \text{ (say)}$$

G.S. is $y = AW_1 + BW_2$

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EXAMINATION (Mid Semester / End Semester) SEMESTER (Autumn / Spring)

Roll Number		Section		Name	Koeli Ghoshal
Subject Number	MA20103	Subject Name			Partial Diff. Equations

Lecture 28 3 (continued) 19.7.2017

Example of Type I of Frobenious method (Roots of indicial eqn. are unequal & not differing by an integer)

Ex Solve in series $9x(1-x)y'' - 12y' + 4y = 0$

Sol^n Putting in standard form

$$\frac{d^2y}{dx^2} - \frac{4}{3x(1-x)}\frac{dy}{dx} + \frac{4}{9x(1-x)}y = 0 \quad \text{—(1)}$$

$$P(x) = -\frac{4}{3x(1-x)} \qquad Q(x) = \frac{4}{9x(1-x)}$$

$\because$ $P(x)$ & $Q(x)$ are not both analytic at $x=0$, so $x=0$ is not an ordinary pt. of the diff-eqn. $xP(x)$ & $x^2Q(x)$ are analytic at $x=0$. So $x=0$ is a regular singular point of (1).

We take the soln. in the form

$$y = \sum_{m=0}^{\infty} c_m x^{k+m} \quad \text{where } c_0 \neq 0$$

$$\therefore y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} \text{ and } \quad \text{—(2)}$$

$$y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} \quad \text{—(3)}$$

Lecture 4

24.7.2017

Substituting these values in (1)

$$9x(1-x)\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2}$$
$$-12\sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} + 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\Rightarrow 9x\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2}$$
$$-9x^2\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2}$$
$$-12\sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} + 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m \left\{9(k+m)(k+m-1) - 12(k+m)\right\}x^{k+m-1}$$
$$+\sum_{m=0}^{\infty} c_m \left\{4 - 9(k+m)(k+m-1)\right\}x^{k+m} = 0 \quad \text{---(4)}$$

$$9(k+m)(k+m-1) - 12(k+m) = 3(k+m)(3k+3m-7) \quad \text{---(5)}$$

$$4 - 9(k+m)(k+m-1) = -(3k+3m-4)(3k+3m+1) \quad \text{---(6)}$$

Using (5) & (6), (4) can be rewritten as

$$3\sum_{m=0}^{\infty} c_m (k+m)(3k+3m-7)x^{k+m-1}$$
$$-\sum_{m=0}^{\infty} c_m (3k+3m-4)(3k+3m+1)x^{k+m} = 0$$
$$\text{---(7)}$$

$$\Rightarrow 3\sum_{m=0}^{\infty} c_m (k+m)(3k+3m-7)\, x^{k+m-1} - \sum_{m=1}^{\infty} c_{m-1}\left[3k+3(m-1)-4\right]\left[3k+3(m-1)+1\right] x^{k+m-1} = 0 \quad -(7)$$

Equating to zero the coefficient of the smallest power of x i.e. x^{k-1}, (7) gives the indicial eqn.

$$3c_0 k(3k-7) = 0$$

$$\text{or,} \quad k(3k-7) = 0, \quad \because c_0 \neq 0$$

$$k = 0 \text{ and } \frac{7}{3} \quad -(8)$$

which are not equal and not differing by an integer.

For recurrence relation, we equate to zero the coeff. of x^{k+m-1} and obtain

$$3c_m (k+m)(3k+3m-7) - c_{m-1}\left[3k+3(m-1)-4\right] \times \left[3k+3(m-1)+1\right] = 0$$

$$\Rightarrow c_m = \frac{3k+3m-2}{3(k+m)} c_{m-1} \quad -(9)$$

Taking $m=1$ in (9), $c_1 = \frac{c_0}{3} \times \frac{3k+1}{k+1} \quad -(10)$

" $m=2$ in (9), $c_2 = \frac{3k+4}{3(k+2)} c_1$

$$= \frac{c_0}{3^2} \times \frac{(3k+1)(3k+4)}{(k+1)(k+2)} \quad -(11)$$

and so on.

Putting these values in

$y = x^k(c_0 + c_1 x + c_2 x^2 + \dots)$ gives

$$y = c_0 x^k \left[1 + \frac{1}{3}\frac{3k+1}{k+1}x + \frac{1}{3^2}\frac{(3k+1)(3k+4)}{(k+1)(k+2)}x^2 + \dots \right] \quad -(12)$$

Putting $k=0$ and replacing c_0 by a in (12),

$$y = a\left(1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \dots\right) = au \text{ (say)}$$

Putting $k = \frac{7}{3}$ and replacing c_0 by b in (12)

$$y = bx^{\frac{7}{3}}\left(1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \dots\right) = bv \text{ (say)}$$

Reqd soln. is $y = au + bv$.

$$y = x^k(c_0 + c_1 x + c_2 x^2 + \dots) = \sum_{m=0}^{\infty} c_m x^{k+m}$$

$$= c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \dots$$

$$\frac{dy}{dx} = c_0 k x^{k-1} + c_1(k+1)x^k + c_2(k+2)x^{k+1} + \dots$$

$$= \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$$

Example on Type 2 Frobenious method

Roots of indicial eqn. are equal.

Ex Solve in series $(x-x^2)y'' + (1-5x)y' - 4y = 0$ — (1)
about $x=0$.

Soln. Let the series soln. of (1) be

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0 \quad \text{— (2)}$$

$$\left.\begin{aligned} y' &= \sum_{m=0}^{\infty} c_m(k+m)\, x^{k+m-1} \\ y'' &= \sum_{m=0}^{\infty} c_m(k+m)(k+m-1)\, x^{k+m-2} \end{aligned}\right\} \text{— (3)}$$

Putting these values in (1),

$$(x-x^2)\sum_{m=0}^{\infty} c_m(k+m)(k+m-1)x^{k+m-2} + (1-5x)\sum_{m=0}^{\infty} c_m(k+m)x^{k+m-1}$$
$$- 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m\{(k+m)(k+m-1) + (k+m)\}x^{k+m-1}$$
$$- \sum_{m=0}^{\infty} c_m\{(k+m)(k+m-1) + 5(k+m) + 4\}x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m(k+m)^2 x^{k+m-1} - \sum_{m=0}^{\infty} c_m(k+m+2)^2 x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m(k+m)^2 x^{k+m-1} - \sum_{m=1}^{\infty} c_{m-1}(k+m+1)^2 x^{k+m-1} = 0$$

— (4)

Equating to zero, the coeff. of smallest powers of x i.e. x^{k-1}, (4) gives the indicial eqn.

$$c_0 k^2 = 0 \quad \therefore k^2 = 0 \quad \because c_0 \neq 0$$

$$\therefore k = 0, 0 \rightarrow \text{equal roots}$$

Finally, equating to zero the coefficient of x^{k+m-1} in (4)

$$c_m (k+m)^2 - c_{m-1}(k+m+1)^2 = 0$$

$$\Rightarrow c_m = \frac{(k+m+1)^2}{(k+m)^2} c_{m-1} \quad \text{— (5)}$$

Putting $m = 1, 2, 3, \ldots$ in (5),

$$c_1 = \left[\frac{(k+2)^2}{(k+1)^2}\right] c_0$$

$$c_2 = \frac{(k+3)^2}{(k+2)^2} c_1 = \frac{(k+3)^2}{(k+2)^2} \frac{(k+2)^2}{(k+1)^2} c_0 = \frac{(k+3)^2}{(k+1)^2} c_0$$

$$c_3 = \frac{(k+4)^2}{(k+1)^2} c_0 \quad \text{and so on.}$$

Putting these values in (2)

$$y = x^k c_0 \left\{ 1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \cdots \right\} \quad \text{— (6)}$$

Putting $k=0$ and substituting c_0 by a in (6),

$$y = a(1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots) = au \text{ (say)}$$

$$\frac{\partial y}{\partial k} = c_0 x^k \ln x \left\{ 1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \cdots \right\}$$
$$+ c_0 x^k \left\{ \frac{(k+2)^2}{(k+1)^2} x \left(\frac{2}{k+2} - \frac{2}{k+1} \right) + \frac{(k+3)^2}{(k+1)^2} x^2 \left(\frac{2}{k+3} - \frac{2}{k+1} \right) + \cdots \right\}$$

Putting $k=0$ and replacing c_0 by b gives

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = b \ln x (1 + 2^2 x + 3^2 x^2 + \cdots) + b\left\{ 2^2 x(1-2) + 3^2 x^2 \left(\frac{2}{3} - 2\right) + \cdots \right\}$$

$$= b\left[u \ln x - 2(1.2x + 2.3x^2 + \cdots)\right] = bv \text{ as say}$$

$$y = au + bv$$

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Lecture - 5 & 6 26.7.2017

Examples of Type 3 on Frobenious method:

Roots of indicial eqn. are unequal and differing by an integer

Ex Find the series solution near $x=0$ of the eqn.

$$x^2 y'' + (x+x^2) y' + (x-9) y = 0 \quad -(1)$$

Solⁿ Let $y = x^k (c_0 + c_1 x + c_2 x^2 + \cdots) = \sum_{m=0}^{\infty} c_m x^{k+m}, \; c_0 \neq 0$ — (2)

$$y' = \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1}, \quad y'' = \sum_{m=0}^{\infty} (k+m)(k+m-1) c_m x^{k+m-2} \quad -(3)$$

$$x^2 \sum_{m=0}^{\infty} (k+m)(k+m-1) c_m x^{k+m-2} + (x+x^2) \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1} + (x-9) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left\{ (k+m)(k+m-1) + (k+m) - 9 \right\} c_m x^{k+m} + \sum_{m=0}^{\infty} \left\{ (k+m) + 1 \right\} c_m x^{k+m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (k+m+3)(k+m-3) c_m x^{k+m} + \sum_{m=0}^{\infty} (k+m+1) c_m x^{k+m+1} = 0 \quad -(4)$$

Equating to zero the coeff. of smallest power of x i.e. x^k

Indicial eqn. is $(k+3)(k-3) c_0 = 0$

$$\Rightarrow k = 3, -3 \quad \text{as } c_0 \neq 0$$

Equating to zero the coeff. of x^{k+m} in (4)

$$(k+m+3)(k+m-3)c_m + (k+m)c_{m-1} = 0$$

$$\Rightarrow c_m = -\frac{(k+m)}{(k+m+3)(k+m-3)} c_{m-1} \quad \text{--- (5)}$$

Putting $m = 1, 2, 3, \ldots$ in (5)

$$c_1 = -\frac{k+1}{(k+4)(k-2)} c_0$$

$$c_2 = +\frac{(k+1)(k+2)}{(k-1)(k-2)(k+4)(k+5)} c_0$$

$$c_3 = -\frac{(k+1)(k+2)(k+3)}{k(k-1)(k-2)(k+4)(k+5)(k+6)} c_0$$

and so on. Putting these values in (2),

$$y = c_0 x^k \left[1 - \frac{k+1}{(k-2)(k+4)} x + \frac{(k+1)(k+2)}{(k-1)(k-2)(k+4)(k+5)} x^2 - \frac{(k+1)(k+2)(k+3)}{k(k-1)(k-2)(k+4)(k+5)(k+6)} x^3 + \ldots\right] \quad \text{--- (6)}$$

Putting $k = 3$ and replacing c_0 by a

$$y = ax^3\left[1 - \frac{4}{2.7}x + \frac{4.5}{2.1.7.8}x^2 + \frac{4.5.6}{3.2.1.7.8.9}x^3 - \ldots\right]$$

$= au$ (say)

Putting $k = -3$ and replacing c_0 by (b) in (6)

$$y = bx^{-3}\left\{1 - \frac{2}{5}x + \frac{1}{20}x^2 + 0 + 0 + 0 + \ldots\right\} = bv$$

$y = au + bv$ is the reqd. soln.

Legendre Equation and Legendre function

The diff. eqn. of the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

is called Legendre's eqn. where n is a non-ve integer. (n is a real const)

The eqn. can be written in the form

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

The solution in descending powers of x is more used than the ascending powers.

Let us assume $y = \sum_{m=0}^{\infty} c_m x^{k-m}$ --- (2) where $c_0 \neq 0$

$y' =$ $\qquad$ $y'' =$

$$(1-x^2)\sum_{m=0}^{\infty} c_m (k-m)(k-m-1)x^{k-m-2} - 2x\sum_{m=0}^{\infty} c_m (k-m)x^{k-m-1}$$

$$+ n(n+1)\sum_{m=0}^{\infty} c_m x^{k-m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m (k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m\left\{(k-m)(k-m-1)\right.$$

$$\left.+ 2(k-m) - n(n+1)\right\}x^{k-m} = 0$$

--- (3)

$$(k-m)(k-m-1) - 2(k-m) - n(n+1)$$

$$= \cancel{(k-m)}\ (k-m-n)(k-m+n+1)$$

$$\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m(k-m-n)(k-m+n+1)x^{k-m} = 0 \quad \text{--- (4)}$$

$$\Rightarrow \sum c_{m-2}(k-m+2)(k-m+1)x^{k-m} - \sum_{m=0}^{\infty} c_m(k-m-n)(k-m+n+1)x^{k-m} = 0$$

To get the indicial eqn., we equate to zero the coeff. of highest power of x i.e. x^k in (4) and obtain

$$c_0(k-n)(k+n+1) = 0 \quad \text{--- (5)}$$

$$\Rightarrow (k-n)(k+n+1) = 0 \quad \because c_0 \neq 0$$

$$k = n, \ -(n+1)$$

& We equate to zero the coeff. of x^{k-1} in (4) and obtain $c_1(k-1-n)(k+n) = 0$ — (6)

For $k = n$ & $-(n+1)$, neither $(k-1-n)$ nor $(k+n)$ is zero. So from (6), $c_1 = 0$. Finally equating to zero the coeff. of x^{k-m}, we have

$$c_{m-2}(k-m+2)(k-m+1) - c_m(k-m-n)(k-m+n+1) = 0$$

$$\Rightarrow c_m = \frac{(k-m+2)(k-m+1)}{(k-m-n)(k-m+n+1)} c_{m-2} \quad \text{--- (7)}$$

Putting $m = 3, 5, 7 \ldots$ in (7) and noting that $c_1 = 0$

$$c_1 = c_3 = c_5 = c_7 = \cdots = 0 \quad \text{--- (8)}$$

Case I When $k = n$. Then (7) becomes

$$C_m = -\frac{(n-m+2)(n-m+1)}{m(2n-m+1)} C_{m-2} \quad \cdots (9)$$

Putting $m = 2, 4, 6, \cdots$ in (9),

$$C_2 = -\frac{n(n-1)}{2(2n-1)} C_0, \quad C_4 = -\frac{(n-2)(n-3)}{4\cdot(2n-3)} C_2$$

$$= \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot \cancel{\ } (2n-1)(2n-3)} C_0$$

Rewriting (2), for $k = n$

$$y = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + C_3 x^{n-3} \cdots (10)$$

Using (8) and the above values of $C_2, C_4, \cdots$

(10) becomes after replacing C_0 by a,

$$y = a\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\,(2n-1)(2n-3)} x^{n-4} - \cdots\right] \quad (11)$$

Case II When $k = -(n+1)$; $C_m = \frac{(n+m-1)(n+m)}{m(2n+m+1)} C_{m-2}$ (12)

$$C_2 = \frac{(n+1)(n+2)}{2(2n+3)} C_0, \quad C_4 = \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot (2n+3)(2n+5)} C_0$$

and so on,

$$y = C_0 x^{-n-1} + C_1 x^{-n-2} + C_2 x^{-n-3} + C_3 x^{-n-4} \quad (13)$$

$$y = b\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\,(2n+3)(2n+5)} x^{-n-5} - - \right]$$

If we take $a = [1.3.5 \ldots (2n-1)] / n!$, the solⁿ. (11) is denoted by $P_n(x)$ and is called Legendre's function of the 1st kind (Legendre polynomial). Thus $P_n(x)$ is a solⁿ. of (1). Again if we take

$b = n! / [1.3.5 \ldots (2n+1)]$, the solⁿ. (14)

is denoted by $Q_n(x)$ and is called Legendre fⁿ. of the 2nd kind. Thus $P_n(x)$ & $Q_n(x)$ are two L.I. solⁿs. of (1). Hence the general solⁿ. of (1) is

$$y = A\,P_n(x) + B\,Q_n(x)$$ where

A, B are arbitrary constants.

NC141

INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR
CLASS TEST / LABORATORY TEST

3

Signature of the Invigilator

EXAMINATION (Mid Semester / End Semester)	SEMESTER (Autumn / Spring)
Roll Number	Section	Name: Koeli Ghoshal
Subject Number: MA20103	Subject Name: PDE

Lecture-7 31.7.2017

§ $P_n(x)$ is the coefficient of h^n in the expansion in ascending powers of $(1-2hx+h^2)^{-1/2}$, $|x|<1$, $|h|<1$.

Proof $(1-2hx+h^2)^{-1/2} = \{1-h(2x-h)\}^{-1/2}$

$$= 1+\frac{1}{2}h(2x-h)+\frac{1.3}{2.4}h^2(2x-h)^2+\cdots$$

$$+\frac{1.3\cdots(2n-3)}{2.4\cdots(2n-2)}h^{n-1}(2x-h)^{n-1}$$

$$+\frac{1.3\cdots(2n-1)}{2.4\cdots 2n}h^n(2x-h)^n+\cdots$$

Coefficient of h^n

$$=\frac{1.3\cdots(2n-1)}{2.4\cdots 2n}(2x)^n-\frac{1.3\cdots(2n-3)}{2.4\cdots(2n-2)}\,{}^{n-1}C_1(2x)^{n-2}$$

$$+\frac{1.3\cdots(2n-5)}{2.4\cdots(2n-4)}\,{}^{n-2}C_2(2x)^{n-4}+\cdots$$

$$=\frac{1.3\cdots(2n-1)}{n!}\left[x^n-\frac{2n}{2n-1}(n-1)\frac{x^{n-2}}{2^2}\right.$$

$$\left.+\frac{2n(2n-2)}{(2n-1)(2n-3)}\frac{(n-2)(n-3)}{2!}\frac{x^{n-4}}{2^4}-\cdots\right]$$

$$=\frac{1.3\cdots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}-\cdots\right.$$

$= P_n(x)$

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$$

$(1-2xh+h^2)^{-1/2}$ is called generating function of the Legendre polynomials.

Orthogonal properties of Legendre's polynomial

(i) $\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = 0$ if $m \neq n$

(ii) $\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$ if $m = n$.

Proof (i) $\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$

$$\therefore \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0 \quad -(1)$$

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)P_m = 0 \quad -(2)$$

Multiplying (i) by P_m & (ii) by P_n and then subtracting,

$$P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} - P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + \{n(n+1) - m(m+1)\} P_n P_m = 0$$

Integrating between -1 and $+1$,

$$\left[P_m(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1}\frac{dP_m}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx$$

$$-\left[P_n(1-x^2)\frac{dP_m}{dx}\right]_{-1}^{+1} + \int_{-1}^{+1}\frac{dP_n}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx$$

$$+\left[n(n+1)-m(m+1)\right]\int_{-1}^{1}P_mP_n\,dx=0$$

$$\int_{-1}^{1}P_mP_n\,dx=0 \quad \because m\neq n.$$

(ii) We have $(1-2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty}h^nP_n(x)$

Squaring both sides

$$(1-2xh+h^2)^{-1}=\sum_{n=0}^{\infty}h^{2n}\{P_n(x)\}^2+2\sum_{\substack{m,n=0\\m\neq n}}^{\infty}h^{m+n}P_m(x)P_n(x)$$

Integrating between -1 and $+1$,

$$\sum_{n=0}^{\infty}\int_{-1}^{+1}h^{2n}[P_n(x)]^2dx+2\sum_{\substack{m,n=0\\m\neq n}}^{\infty}\int_{-1}^{+1}h^{m+n}P_m(x)P_n(x)dx$$

$$=\int_{-1}^{+1}\frac{dx}{(1-2xh+h^2)}$$

or, $$\sum_{n=0}^{\infty}\int_{-1}^{+1}h^{2n}[P_n(x)]^2dx=\int_{-1}^{+1}\frac{dx}{(1-2xh+h^2)}$$

$$=-\frac{1}{2h}\left\{\ln(1-2xh+h^2)\right\}_{-1}^{+1}$$

$$=-\frac{1}{2h}\left\{\ln(1-h)^2-\ln(1+h)^2\right\}$$

$$=\frac{1}{2h}\left[\ln\left(\frac{1+h}{1-h}\right)^2\right]=\frac{1}{h}\ln\left\{\frac{1+h}{1-h}\right\}$$

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$$= \frac{2}{h}\left\{ h + \frac{h^3}{3} + \frac{h^5}{5} + \cdots \right\}$$

$$= 2\left\{ 1 + \frac{h^2}{3} + \frac{h^4}{5} + \cdots \right\} = \sum_{n=0}^{\infty} \frac{2h^{2n}}{2n+1}$$

Equating the coeff. of h^{2n},

$$\int_{-1}^{+1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}$$

Result $\int_{-1}^{+1} P_m(x) P_n(x) \, dx = \frac{2}{2n+1} \delta_{m,n}$

where $\delta_{m,n}$ is the Kronecker delta defined as

$$\delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Lecture-8 1.8.2017

$$P_n(x) = \sum_{r=0}^{n/2} (-1)^r \frac{(2n-2r)!}{2^n r! \, (n-2r)! \, (n-r)!} x^{n-2r}$$

$$\text{where } \frac{n}{2} = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$$a = \frac{1.3.5 \ldots (2n-1)}{n!} = \frac{(2n)!}{2^n (n!)^2}$$

$$= \frac{1.2.3.5.6 \ldots \cancel{2n}(2n-1)\,2n}{2.4.6 \ldots 2n \quad n!}$$

$$= \frac{(2n)!}{2^n \, n! \, n!}$$

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Recurrence formulae

I $\quad (2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$

Proof $\quad (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Differentiating both sides w.r.t. h,

$$-\frac{1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-1/2} = (1-2xh+h^2)\sum nh^{n-1}P_n(x)$$

$$\Rightarrow (x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

$$\Rightarrow (x-h)\left[P_0(x) + hP_1(x) + \cdots + h^{n-1}P_{n-1}(x) + h^nP_n(x) + \cdots\right]$$

$$= (1-2xh+h^2)\left[P_1(x) + 2hP_2(x) + \cdots + (n-1)h^{n-2}P_{n-1}(x) + nh^{n-1}P_n(x) + (n+1)h^nP_{n+1}(x) + \cdots\right]$$

Equating coeff. of h^n from both the sides,

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

In short, $\underline{(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}}$

Equating the coeff. of h^{n-1} from both sides,

$$nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$$

II $nP_n = xP_n' - P'_{n-1}$ dashes denote differentials w.r.t. x

Proof $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$ —(1)

Differentiating (1) w.r.t. 'h',

$$(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} n h^{n-1} P_n(x) \quad \text{—(2)}$$

Again differentiating (1) w.r.t. 'x',

$$h(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x)$$

$$\Rightarrow h(x-h)(1-2xh+h^2)^{-3/2} = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x) \quad \text{—(3)}$$

From (2) and (3),

$$h\sum_{n=0}^{\infty} n h^{n-1} P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x)$$

$$\Rightarrow h\left[h^0 P_1(x) + 2hP_2(x) + \cdots + nh^{n-1}P_n(x) + \cdots\right]$$

$$= (x-h)\left[P_0'(x) + hP_1'(x) + \cdots + h^{n-1}P'_{n-1}(x) + h^n P_n'(x) \cdots\right]$$

Equating coeff. of h^n on both the sides,

$$nP_n(x) = xP_n'(x) - P'_{n-1}(x)$$

i.e. $nP_n = xP_n' - P'_{n-1}$

III $\quad (2n+1)P_n = P'_{n+1} - P'_{n-1}$

Proof From rec. for. I

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

Differentiating w.r.t. x,

$$(2n+1)xP_n' + (2n+1)P_n = (n+1)P'_{n+1} + nP'_{n-1} \quad \text{---(1)}$$

From rec. formula II

$$xP_n' = nP_n + P'_{n-1} \quad \cancel{\text{---(2)}}$$

$$\Rightarrow (2n+1)xP_n' = (2n+1)(nP_n + P'_{n-1}) \quad \text{---(2)}$$

Eliminating xP_n' from (1) & (2),

$$(2n+1)(nP_n + P'_{n-1}) + (2n+1)P_n = (n+1)P'_{n+1} + nP'_{n-1}$$

$$\Rightarrow (2n+1)(n+1)P_n = (n+1)P'_{n+1} + nP'_{n-1} - (2n+1)P'_{n-1}$$

$$\Rightarrow (2n+1)(n+1)P_n = (n+1)P'_{n+1} - (n+1)P'_{n-1}$$

$$\Rightarrow (2n+1)P_n = P'_{n+1} - P'_{n-1}$$

IV $\quad (n+1)P_n = P'_{n+1} - xP_n'$

Proof From rec. formular II & III

$$\Rightarrow nP_n = xP_n' - P'_{n-1} \quad \text{---(1)}$$

$$(2n+1)P_n = P'_{n+1} - P'_{n-1} \quad \text{---(2)}$$

Subtracting (1) from (2),

$$(n+1)P_n = P'_{n+1} - xP_n'$$

V $\quad (1-x^2) P_n' = n(P_{n-1} - xP_n)$

Proof $\quad$ Replacing n by $n-1$ in rec. for. IV,

$$nP_{n-1} = P_n' - xP'_{n-1} \quad \text{---}(1)$$

II rec. for $\rightarrow \quad nP_n = xP_n' - P'_{n-1} \quad \text{---}(2)$

Multiplying (2) by x and then subtracting from (1),

$$n(P_{n-1} - xP_n) = (1-x^2) P_n'$$

i.e. $(1-x^2) P_n' = n(P_{n-1} - xP_n)$

VI $\quad (1-x^2) P_n' = (n+1)(xP_n - P_{n+1})$.

Proof $\quad$ From rec. for. I

$$(2n+1) xP_n = (n+1) P_{n+1} + nP_{n-1}$$

which can be written as

$$(n+1) xP_n + nxP_n = (n+1) P_{n+1} + nP_{n-1} \quad \text{---}(1)$$

$$\Rightarrow (n+1)(xP_n - P_{n+1}) = n(P_{n-1} - xP_n)$$

Rec ^{n} formula V $\rightarrow \quad (1-x^2) P_n' = n(P_{n-1} - xP_n) \text{---}(2)$

From (1) & (2),

$$(1-x^2) P_n' = (n+1)(xP_n - P_{n+1})$$

$$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n}(x^2-1)^n$$

<u>Ex</u> Show that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

<u>Soln</u>
$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$$
$$= \{1 - h(2x-h)\}^{-1/2}$$
$$= 1 + \frac{h}{2}(2x-h) + \frac{1.3}{2.4} h^2 (2x-h^2) + \cdots$$

$\Rightarrow$ $P_0(x) + hP_1(x) + h^2 P_2(x) + \cdots$
$$= 1 + xh + \frac{1}{2}(3x^2-1)h^2 + \cdots$$

Equating the coeff. of likes powers of h,
$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

Bessel eqn. and Bessel function

The differential eqn. of the form

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

$$\Rightarrow \quad y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0 \quad \text{—(1)}$$

is called Bessel eqn. of order n, n being a constant. We solve (1) by Frobenius method.

Let the series solution be

$$\left.\begin{aligned} y &= \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0 \\ y' &= \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} \\ y'' &= \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} \end{aligned}\right\} \text{—(2)}$$

Substituting in (1),

$$x^2 \sum c_m (k+m)(k+m-1) x^{k+m-2} + x \sum c_m (k+m) x^{k+m-1}$$
$$+ (x^2 - n^2) \sum c_m x^{k+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m \left\{ (k+m)(k+m-1) + (k+m) - n^2 \right\} x^{k+m}$$
$$+ \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m (k+m+n)(k+m-n) x^{k+m}$$
$$+ \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0 \quad \text{—(3)}$$

Equating to zero the smallest power of x, namely x^k, (3) gives the indicial eqn.

$$c_0 (k+n)(k-n) = 0 \quad \text{i.e.} \quad (k+n)(k-n) = 0 \quad , c_0 \neq 0$$

Roots are $k = n, -n$

Next equating to zero the coeff. of x^{k+1} in (3),

$$c_1 (k+1+n)(k+1-n) = 0$$

so that $c_1 = 0$ for $k = n$ & $k = -n$.

Finally equating to zero the coeff. of x^{k+m} in (3)

$$c_m (k+m+n)(k+m-n) + c_{m-2} = 0$$

$$\Rightarrow c_m = + \frac{1}{(k+m+n)(\underset{(n-k-m)}{\cancel{k+m-n}})} c_{m-2} \quad \text{— (4)}$$

Putting $m = 3, 5, 7 \ldots$ in (4) & using $c_1 = 0$,

$$c_1 = c_3 = c_5 = c_7 = \cdots = 0 \quad \text{— (5)}$$

Putting $m = 2, 4, 6, \ldots$ in (4) gives

$$c_2 = \frac{1}{(k+2+n)(n-k-2)} c_0$$

$$c_4 = \frac{1}{(k+4+n)(n-k-4)} c_2$$

$$= \frac{1}{(k+4+n)(n-k-4)(k+2+n)(n-k-2)} c_0$$

and so on.

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Putting these values in (2),

$$y = c_0 x^k \left[1 + \frac{x^2}{(n+k+2)(n-k-2)} + \frac{x^4}{(n+k+2)(n-k-2)(n+k+4)(n-k-4)} + \cdots \right]$$

Replacing k by n and $-n$ and also replacing c_0 by a and b in the above eq.

$$y = a x^n \left\{ 1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4 \cdot 8 .\,(1+n)(2+n)} - \cdots \right\} \quad (6)$$

$$\text{and } y = b x^{-n} \left\{ 1 - \frac{x^2}{4(1-n)} + \frac{x^4}{4 \cdot 8\,(1-n)(2-n)} - \cdots \right\} \quad (7)$$

If $a = 1 / \left\{ 2^n \Gamma(n+1) \right\}$, then (6) is called Bessel f^n. of the 1st kind of order n. It is denoted by $J_n(x)$. Thus,

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8 .\,(n+1)(n+2)} - \cdots \right] \quad (8)$$

or, $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n} \quad (9)$$

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Replacing b by $1/\{2^n \Gamma(n+1)\}$ in (7) and proceeding as above gives,

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \quad \text{--- (10)}$$

When n is not integer, $J_{-n}(x)$ is distinct from $J_n(x)$. The G.S. of Bessel eqn. (1) when n is not an integer is

$$y = A J_n(x) + B J_{-n}(x)$$

When n is an integer, $J_{-n}(x) = (-1)^n J_n(x)$

Bessel eqn. for $n=0$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

Solⁿ. $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

where $J_0(x)$ is called Bessel fⁿ. of zeroth order

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Relation between $J_n(x)$ and $J_{-n}(x)$, n being an integer

When n is an integer, $J_{-n}(x) = (-1)^n J_n(x)$

Case (i) Let n be a +ve integer.

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \quad -(1)$$

Since $n>0$, for $r = 0, 1, 2, \ldots (n-1)$, $\Gamma(-n+r+1)$ is infinite and so $\frac{1}{\Gamma(-n+r+1)}$ is zero. Keeping this in mind we see that the sum over r in (1) must be taken from n to ∞. Thus

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \quad -(2)$$

Put $m = r-n \quad \therefore r = m+n$

$\therefore m = 0$ when $r = n$

$m = \infty$ when $r = \infty$

$$J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^{m+n} \frac{1}{(m+n)!\,\Gamma(m+1)} \left(\frac{x}{2}\right)^{2(m+n)-n}$$

$$= \sum_{m=0}^{\infty} (-1)^m (-1)^n \frac{1}{\Gamma(m+n+1)\, m!} \left(\frac{x}{2}\right)^{2m+n}$$

$$= (-1)^n \sum_{m=0}^{\infty} (-1)^m \frac{1}{\Gamma(m+n+1)\, m!} \left(\frac{x}{2}\right)^{2m+n}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma(r+n+1)\, r!} \left(\frac{x}{2}\right)^{2r+n}$$

$\therefore J_{-n}(x) = (-1)^n J_n(x)$

Case (ii) Let $n < 0$

Let p be a +ve integer such that $n = -p$

$\because p > 0$, for case I

$$J_{-p}(x) = (-1)^p J_p(x)$$

so that $J_p(x) = (-1)^{-p} J_{-p}(x)$

But $p = -n$ $\therefore$ $J_{-n}(x) = (-1)^n J_n(x)$

$\therefore$ The result holds for any integer.

Generating function for $J_n(x)$

$$\exp\left\{\frac{1}{2}x\left(z - \frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

Prove that when n is a +ve integer, $J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ in ascending and descending powers of z. Also prove that $J_n(x)$ is the coeff. of z^{-n} multiplied by $(-1)^n$ in the expansion of the above expression.

Proof

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{\frac{xz}{2}} \cdot e^{-\frac{x}{2z}}$$

$$= \left[1+\left(\frac{x}{2}\right)z+\left(\frac{x}{2}\right)^2\frac{z^2}{2!}+\cdots+\left(\frac{x}{2}\right)^n\frac{z^n}{n!}+\left(\frac{x}{2}\right)^{n+1}\frac{z^{n+1}}{(n+1)!}+\cdots\right]$$

$$\times\left[1-\left(\frac{x}{2}\right)z^{-1}+\left(\frac{x}{2}\right)^2\frac{z^{-2}}{2!}+\cdots+\left(\frac{x}{2}\right)^n\frac{(-1)^n z^{-n}}{n!}+\left(\frac{x}{2}\right)^{n+1}\frac{(-1)^{n+1} z^{-(n+1)}}{(n+1)!}+\cdots\right] \quad \text{--- (1)}$$

The coeff. of z^n in the product (1) is obtained by multiplying the coeff. of $z^n, z^{n+1}, z^{n+2}\ldots$ in the 1st bracket with the coeff. of $z^0, z^{-1}, z^{-2}\ldots$ in the 2nd " respectively.

$\therefore$ Coeff. of z^n in the product (1),

$$= \left(\frac{x}{2}\right)^n\frac{1}{n!}-\left(\frac{x}{2}\right)^{n+2}\frac{1}{(n+1)!}+\left(\frac{x}{2}\right)^{n+4}\frac{1}{(n+2)!\,2!}-\cdots$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} = J_n(x)$$

The coeff. of z^{-n} in the product (1) is obtained by multiplying the coeff. of $z^{-n}, z^{-n-1}, z^{-n-2}$ ... of the 2nd bracket with the coeff. of z^0, z^1, z^2 .. in the 1st " respectively.

$\therefore$ coeff. of z^{-n} in the product (1),

$$= \left(\frac{x}{2}\right)^n \frac{(-1)^n}{n!} + \left(\frac{x}{2}\right)^{n+1} \frac{(-1)^{n+1}}{(n+1)!} \frac{x}{2} + \cdots$$

$$= (-1)^n \left[\left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+1} \frac{1}{(n+1)!} \cdots \right]$$

$$= (-1)^n J_n(x) \quad \text{as before.}$$

Coeff. of $z^{-n} = (-1)^n J_n(x)$

$$\Rightarrow J_n(x) = (-1)^n \times \text{coeff. of } z^{-n}$$

Finally, in the product (1), the coeff. of z^0 is obtained by multiplying the coeff. of $z^0, z^1, z^2, \ldots$ in the 1st bracket with the " " $z^0, z^{-1}, z^{-2}, \ldots$ in " 2nd " and is thus

$$= 1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 \left(\frac{1}{2!}\right)^2 - \left(\frac{x}{2}\right)^6 \left(\frac{1}{3!}\right)^2 + \cdots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \cdots = J_0(x)$$

We observe that the coeff. of z^0, $(z - z^{-1})$, $(z^2 + z^{-2})$, ... are $J_0(x)$, $J_1(x)$, $J_2(x)$ ....

Thus (1) gives,

$$\exp\left\{\frac{x}{2} \times \frac{z-1}{z}\right\} = J_0(x) + (z - z^{-1}) J_1(x) + (z^2 + z^{-2}) J_2(x) + \cdots + [z^n + (-1)^n z^{-n}] J_n(x) + \cdots$$

$$= \sum_{n=-\infty}^{\infty} z^n J_n(x) \quad \text{as } J_{-n}(x) = (-1)^n J_n(x)$$

Recurrence formulae

I. $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

Proof $\quad J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

Differentiating w.r.t. x

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)}{r!\,\Gamma(n+r+1)} \frac{1}{2} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$\Rightarrow x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{n+2r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{n}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$+ \sum_{r=0}^{\infty} (-1)^r \frac{2r}{r!\,\Gamma(n+r+1)} \frac{x}{2} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n(x) + x \sum_{r=1}^{\infty} (-1)^r \frac{1}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$r - 1 = s$

$$= n J_n(x) - x \sum_{s=0}^{\infty} (-1)^s \frac{1}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$= n J_n(x) - x J_{n+1}(x)$$

$$\therefore \quad x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

which can also be written as

$$\frac{d}{dx}\left(x^{-n} J_n\right) = - x^{-n} J_{n+1}$$

<u>II</u> $\quad x J_n'(x) = - n J_n(x) + x J_{n-1}(x)$

<u>Proof</u> By (I),

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{n+2r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{2n+2r-n}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -n \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$+ \sum_{r=0}^{\infty} (-1)^r \frac{2n+2r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -n J_n(x) + \sum_{r=0}^{\infty} (-1)^r \frac{2(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{x}{2}\right)$$

$$= -n J_n(x) + x \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -n J_n(x) + x \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{n-1+2r}$$

$$= -n J_n(x) + x J_{n-1}(x)$$

$$\frac{d}{dx}\left[x^n J_n\right] = x^n J_{n-1}$$

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III $\quad 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

Proof $\quad$ Rec. I & II

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

& $\quad x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

Adding $\quad 2x J_n'(x) = x\left[J_{n-1}(x) - J_{n+1}(x)\right]$

$$\therefore 2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

IV $\quad 2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

Proof $\quad$ Rec formula I & II

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

& $\quad x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

Subtracting,

$$0 = 2n J_n(x) - x\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

$$\Rightarrow 2n J_n(x) = x\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

(V) $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

Proof $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -n x^{-n-1} J_n(x) + x^{-n} J_n'(x)$$

$$= x^{-n-1}\left[-n J_n(x) + x J_n'(x)\right]$$

$$= x^{-n-1}\left[-n J_n(x) + \left\{n J_n(x) - x J_{n+1}(x)\right\}\right]$$

by rec I

$$= x^{-n-1}\left[-x J_{n+1}(x)\right]$$

$$= -x^{-n} J_{n+1}(x)$$

VI $$\frac{d}{dx}\left[x^{n} J_n(x)\right] = x^{n} J_{n-1}(x)$$

Proof: $$\frac{d}{dx}\left[x^{n} J_n(x)\right] = n x^{n-1} J_n(x) + x^{n} J_n'(x)$$

$$= x^{n-1}\left[n J_n(x) + x J_n'(x)\right]$$

$$= x^{n-1}\left[n J_n(x) + \left\{-n J_n(x) + x J_{n-1}(x)\right\}\right]$$

by rec II

$$= x^{n-1}\left[x J_{n-1}(x)\right]$$

$$= x^{n} J_{n-1}(x)$$

Lecture – 11 $\qquad$ 7~~6~~.8.2017

Partial Differential Eqn

A partial diff. eqn. (in short pde) is an eqn. that contains the independent variables $x_1, x_2, \ldots x_n$ the dependent variable or the unknown function z and its partial derivatives upto some order. It has the form

$$F(x_1, x_2, \ldots x_n, z, z_{x_1}, z_{x_2}, \ldots z_{x_n}, \ldots; z_{x_1 x_1}, \ldots, z_{x_i x_j}) = 0$$

where F is a given f^n., $z_{x_j} = \frac{\partial z}{\partial x_j}$

$z_{x_i x_j} = \frac{\partial^2 z}{\partial x_i \partial x_j}$, $i, j = 1, 2, \ldots n$ are the partial derivatives of z.

Order and degree

$$\frac{\partial^2 z}{\partial y^2} = k\left(\frac{\partial^3 y}{\partial z^3}\right)^2 \rightarrow \text{3rd order, 2nd degree}$$

Derivation of PDE

(a) By the elimination of arbitrary constants:—

$$f(x, y, z, a, b) = 0 \quad \text{—— (1)}$$

a, b are arb. constants. Differentiating (1) partially w.r.t. x and y, we have,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0 \quad \text{—— (2)}$$

and $\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0$ —— (3)

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q$$

Eliminating a and b from (1), (2) & (3), we have diff. eqn. of the form

$$F(x, y, z, p, q) = 0$$

which is a pde of first order.

(b) By the elimination of arbitrary functions:—

Let u and v be two f^ns. of x, y and z which are connected by

$$f(u, v) = 0 \quad \text{—— (1)}$$

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Differentiating (1) w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0 \quad \text{--- (2)}$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0 \quad \text{--- (3)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) & (3),

$$\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)$$

which can be written as

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p+\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q$$
$$=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$\Rightarrow Pp+Qq=R \quad \text{--- (4)}$$

where $$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}=\frac{\partial(u,v)}{\partial(y,z)}$$

$$Q=\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial(u,v)}{\partial(z,x)}$$

and $$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial(u,v)}{\partial(x,y)}$$

(4) is the reqd. pde.

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