07/01/19

Experiment:

conducting or observing something happen resulting in certain outcomes.

Deterministic experiment:

If the conditions of the experiment is fixed, the outcome is known and it is called deterministic experiment.

Eg: Boiling of water to form vapour.

Non-Deterministic Experiment:

If all the conditions are fixed even then the outcome cannot be predicted in advance, then, it is said to be random experiment.

Eg: Tossing a coin.

A random experiment can have several possible outcomes.

The set of all possible outcomes of a random experiment is called as sample space. It is usually denoted by Ω or S.

Examples:

1. Tossing a coin.

$$\Omega = \{H, T\}$$

2. Tossing of a Dice

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

3. Drawing of a card:

$\Omega = \{4 \times 1\text{-}10, \ 4 \times K, Q, J\}$

$\Omega_1 = \{R, B\}$

$\Omega_2 = \{\text{club}, \text{diamond}, \text{spade}, \text{heart}\}$

$$\Omega_3 = \{c_1, c_2, \dots, c_{13}, \\ d_1, d_2, \dots, d_{13}, \\ h_1, h_2, \dots, h_{13}, \\ s_1, s_2, \dots, s_{13}\}$$

4. Birth of a child:

$\Omega_1 = \{\text{Boy}, \text{Girl}\}$, [redacted]

$\Omega_2 = \{S, D\} = \{\text{Survival}, \text{Death}\}$

$\Omega_3 = (0, 100) \rightarrow$ Age in years

$\Omega_4 = \{\text{Language}_1, \dots\}$

5. King of Tollywood:

$\Omega = \{\text{Mahesh Babu}\}$

<u>Event:</u> An Event is a subset of the sample.

Tossing of a Dice: $E = \{\text{Number is prime}\}$
$= \{2, 3, 5\}$

$\phi \rightarrow$ Impossible Event
(LLR winning GC)

$\Omega \Rightarrow$ Sure Event
(RK, RP domination in Illu)

Union of Events: $A \cup B$

occurance of A or B or Both.

$$A_1 \cup A_2 \cup A_3 \dots = \bigcup_{i=1}^{n} A_i$$

→ occurance of atleast one of A_i.

$\bigcup_{i=1}^{\infty} A_i$ → occurance of atleast one of A_i.

Intersection of Events: $A \cap B$

simultaneous occurance of A and B.

$$A_1 \cap A_2 \cap A_3 \dots = \bigcap_{i=1}^{n} A_i$$

If $\bigcup_{i=1}^{\infty} A_i = \Omega$, A_i are said to be exhaustive event.

If $A \cap B = \emptyset$, They are called disjoint sets.

Mutually exclusive events.

A^c → Events-not Happening of A.

$A-B$ → Happening of A But not of B.
$(A \cap B^c)$

Mathematical or classical Definition of probability:-

Suppose a random expt has N possible outcomes. suppose these outcomes are equally likely & mutually exclusive. For an even E if there are m favourable outcomes, then the probability of E is defined as

$$P(E) = \frac{m}{N}.$$

Limitations of classical Def^n

1. N need not to be finite

2. The use of term "equally likely" means that it has equal chance of happenning which makes the definition circular. Moreover there must be prior knowledge of chances of outcomes.

Relative Frequency of statistical Definition of probability:

suppose a random experiment is conducted a large number of times independently under identical conditions. suppose n is the number of trails of the experiment event E occurs a_n times. Then probability of E is defined by,

$$P(E) = \lim_{n\to\infty} \frac{a_n}{n}$$

Example: HHT HHT HHT. . .

$$\frac{a_n}{n} \rightarrow \frac{1}{1}, \frac{2}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{4}{6}, \frac{5}{7}, \ldots$$

$$\frac{a_n}{n} = \frac{2k}{3k} ;\ n=3k \Big/ \frac{2k}{3k-1} ;\ n=3k-1 \Big/ \frac{2k-1}{3k-2},\ n=3k-2$$

$$P(H) = \frac{2}{3}.$$

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Limitations:

1. Data should be available to us. Observations may not be available all the time.

2. Outcome is difficult to observe.

Eg: currently engine of Airbus, has failed midair probability of failure of engine 15-20 per million.

Is probability that low? observing the exact outcomes is very difficult.

Probability of rare events, $\theta(a_n) < n$ e.g., $\frac{\sqrt{n}}{n} \to 0$

This doesn't give proper picture of risk.

e.g., $\lim_{n\to\infty} \frac{n^{5/6}}{n} = 0$

But take $n = 100$, $\frac{1}{3} < \frac{1}{100^{1/6}} < \frac{1}{2}$ This is high than predicted.

Axiomatic Definition of probability:

suppose, we have a random experiment resulting in a sample space (Ω).

A σ-algebra (σ-field) of subsets of Ω is a set $\mathcal{B}$ satisfying the following properties:

(i). If, $A_1, A_2, \ldots, A_n \in B$; then $\bigcup_{i=1}^{\infty} A_i \in B$

(ii). If, $A_i \in B$; $A_i^c \in B$.

(Ω, B) is called measurable space.

Def^n: (Ω, B) is a probability space if we can define a function $P : B \rightarrow R$ satisfying the following three axioms

A_1 : $P(E) \geq 0 \quad \forall E \in B$

A_2: $P(\Omega) = 1$

A_3 : If $E_1, E_2, \ldots$ are pairwise disjoint events in B, then

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

$(\Omega, B, P) \longrightarrow$ Probability Space

Some properties of Probability Function:

P_1: $P(\emptyset)=0$

proof: Let $E_1=\Omega$ and $E_2=E_3=\dots=\emptyset$.

using Axiom 3,

$\Rightarrow P(E_1 \cup E_2 \dots \cup E_n) = P(E_1)+P(E_2)+\dots$

$\Rightarrow 1 = P(\Omega)+P(\emptyset)(\dots)$

$\therefore \boxed{P(\emptyset)=0}$

P_2: If $A_1,\dots,A_n$ are pairwise disjoint events in $\mathcal{B}$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

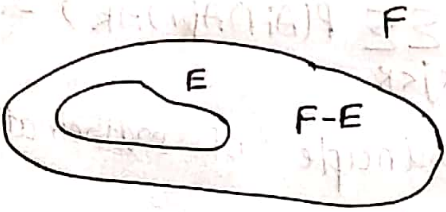
proof: In Axiom A_3

$E_{n+1}=E_{n+2}=\dots=\emptyset$

Then $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$

P_3: Probability Function is monotone.

if $E \subset F$ then $P(E) \le P(F)$



$F = E \cup (F-E)$

$P(F) = P(E)+P(F-E) \ge P(E)$

Also, $P(F)-P(E) = P(F-E)$ if $E \subset F$

$\Rightarrow P(\emptyset) \le P(E) \le P(\Omega) \Rightarrow \boxed{0 \le P(E) \le 1}$

P4: $P(E^c) = 1 - P(E)$

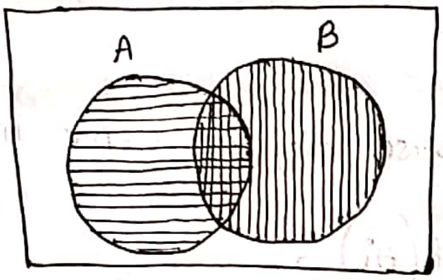
proof: E and E^c are disjoint.

$$P(E^c \cup E) = P(E) + P(E^c)$$

$$\Rightarrow \boxed{P(E^c) = 1 - P(E)}$$

Addition Rule: For Any two events A and B $\in Q$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$$A \cup B = A + B - A \cap B = A \cup (B - (A \cap B))$$

$$P(A \cup B) = P(A) + P(B - (A \cap B))$$

$$= P(A) + P(B) - P(A \cap B)$$

General Addition Rule: If $A_1, A_2, \dots, A_n \in B$ then,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}\sum P(A_i \cap A_j)$$

$$+ \sum_{i<j<k}\sum\sum P(A_i \cap A_j \cap A_k) - \dots$$

proof: We will use the principle of mathematical induction.

obviously $P(1)$ is True (trivially).

Let $P(n)$ be true for $n = k$.

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right)$$

$$= P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\bigcup_{i=1}^{k} A_i \cap A_{k+1}\right)$$

Applying the previous addition rule we can write,

$$= \sum_{i=1}^{K} P(A_i) - \sum_{i<j}^{K}\sum^{K} P(A_i \cap A_j) + \dots + (-1)^{K+1} P\left(\bigcap_{i=1}^{K} A_i\right)$$

$$+ P(A_{K+1}) - P\left(\bigcup_{i=1}^{K}(A_i \cap A_{K+1})\right)$$

$$= \sum_{i=1}^{K+1} P(A_i) - \sum_{i<j}^{K}\sum^{K} P(A_i \cap A_j) + \dots + (-1)^{K+1} P\left(\bigcap_{i=1}^{K} A_i\right)$$

$$- \sum_{i=1}^{K} P(A_i \cap A_{K+1}) + \sum_{i<j}^{K}\sum^{K} P((A_i \cap A_{K+1}) \cap (A_j \cap A_{K+1})) - \dots$$

$$+ (-1)^{K+1} P\left(\bigcap_{i=1}^{K} P(A_i \cap A_{K+1})\right)$$

$$= \sum_{i=1}^{K+1} P(A_i) - \sum_{i<j}^{K+1}\sum^{K+1} P(A_i \cap A_j) + \dots + (-1)^{K+2} P\left(\bigcap_{i=1}^{K+1} P(A_i)\right)$$

Theorem: (Subadditivity property of probability)

Let $A_1, A_2, A_3, \dots \in \mathbb{B}$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{m} P(A_i)$$

Conditional Probability:

Let A, B be any two events. Then the conditional probability of event A given that B has already occured is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

We prove that the conditional probability function is a valid probability function.

A_1: $P(A/B) \geq 0$

A_2: $P(\Omega/B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

$$A_3 = \frac{P(A_1 \cup A_2 \cup \dots A_n \cap B)}{P(B)}$$

$$= \frac{P(\cup_i (A_i \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} + \dots$$

$$= \sum_{i=1}^{n} P(A_i / B)$$

Multiplication Rule:

$$P(A \cap B) = P(A/B)\,P(B)$$
$$= P(B/A)\,P(A)$$

General Multiplication Rule:

Let $A_1, A_2, \dots A_n$ be events in B with $P(\bigcap_{i=1}^{n} A_i) > 0$

Then $P(\bigcap_{i=1}^{n} A_i) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2) \dots P(A_n)|\bigcap_{i=1}^{n-1} P(A_i))$

proof (by induction):

1. Base Case: P(1) is trivially true

2. $P(\bigcap_{i=1}^{k} A_i) = \text{True} = P(A_1)\,P(A_2|A_1) \dots P(A_k|\bigcap_{i=1}^{k-1} P(A_i))$

$$P\left(\left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}\right) = P\left(A_{k+1} / \bigcap_{i=1}^{k} A_i\right) P\left(\bigcap_{i=1}^{k} A_i\right)$$

$$= P(A_1)\,P(A_2/A_1) \dots, P\left(A_{k+1} / \bigcap_{i=1}^{k+1} A_i\right)$$

Theorem of Total Probability:

Let $A \subset \Omega$ and $B_1, B_2, \dots, B_n \in B$. Assume B_i's are mutually exhaustive events and exclusive. with $P(B_i) > 0$ Then

$$P(A) = \sum_{i=1}^{n} P(A/B_i)\,P(B_i)$$

proof: $\bigcup_{i=1}^{n} B_i = \Omega$

$$P(A) = P(A \cap \Omega) = P\left(A \cap \bigcup_{i=1}^{n} B_i\right)$$

$$= P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right)$$

$$= \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} \underline{P(A/B_i)P(B_i)}$$

Baye's Theorem:

Let $A \in B_0$ with $P(A) > 0$ and $B_1, B_2, \dots, B_n \in B$. Assume $B_1, B_2, \dots, B_n$ are mutually exclusive and exhaustive events. Then,

$$P(B_j|A) = \frac{P(A|B_j)\,P(B_j)}{\sum_{j=1}^{n} P(B_j)\,P(A/B_j)}$$

proof:

$$P(B_j/A) = \frac{P(B_j \cap A)}{P(A)}$$

$$= \frac{P(A/B_j)\,P(B_j)}{\sum_{j=\text{[illegible]}}^{n} P(A/B_j)\,P(B_j)}$$

← Theorem of Total probability.

Independence of events:

$$P(A/B) = P(A)$$

$$\Rightarrow \frac{P(A\cap B)}{P(B)} = P(A)$$

$$\Rightarrow P(A\cap B) = P(A)P(B)$$

so we define A and B be independent events of

$P(A\cap B) = P(A)P(B)$

If A,B,C are mutualty indepent,

$$P(B\cap C) = P(B)P(C)$$
$$P(A\cap C) = P(A)P(C)$$
$$P(A\cap B\cap C) = P(A)P(B)P(C)$$

- Total number of conditions $= 2^n - n - 1.$

Problem:

Birthday Problem: Suppose there are n-persons in a hall and none has birthday on 29th feb ($n \le 365$).

P (atleast 2 persons share same birthday)

Sol: $$P = 1 - \left(\frac{^{365}P_n}{365^n}\right)$$

n	$P(A^c)$	$P(A)$
10	0.871	0.129
20	0.589	0.411
23	0.493	0.507
30	0.294	0.706
50	0.030	0.970
60	0.006	0.994

Ex: six cards are drawn with replacement from an ordinary deck, what is prob that each of spades, etc, will appear atleast among the six cards?

Sol: $P(A^c) = \underbrace{\frac{4\times 39^6}{52^6}}_{\text{3 types}} + \underbrace{\frac{6\times 26^6}{52^6}}_{\text{2 types}} + \underbrace{4\times\frac{13^6}{52^6}}_{\text{1 type}}$

$= 4\times\left(\frac{3}{4}\right)^6 + 6\times\left(\frac{1}{2}\right)^6 + 4\times\left(\frac{1}{4}\right)^6$

$= \frac{4\times 3^6 + 6\times 2^6 + 4}{4^6}$

$= \frac{2916+384+4}{4096} = \frac{\cancel{2804}\ \cancel{1402}\ 701}{\cancel{4096}\ \cancel{2048}\ 1024} = \frac{701}{1024}$

$\frac{\cancel{2804}\ 701}{\cancel{4096}\ \cancel{1024}}$

$\frac{\cancel{3204}\ 801}{\cancel{4096}\ 1024}$

$P(A) = \frac{223}{1024}$ ⊗ [wrong]

sir method:

A → All suits appear once

A^c → All suits do not appear once

B_1 → spades do not appear, ...

$P(B_i) = \left(\frac{3}{4}\right)^6 \quad i=1,2,3,4$

$P(B_i \cap B_j) = \left(\frac{2}{4}\right)^6 \quad i=1,2,3,4;\ j>i$

$P(B_i \cap B_j \cap B_k) = \left(\frac{1}{4}\right)^6$

$P(B_1\cap B_2\cap B_3\cap B_4) = \underline{0}$

$P(B) = P(B_1 \cup B_2 \cup B_3 \cup B_4)$

$= \sum_{i=1}^{n} P(B_i) - \sum\sum P(B_i\cap B_j) + \sum\sum\sum P(B_i\cap B_j\cap B_k) - P(B_1\cap B_2\cap B_3\cap B_4)$

$= 4\times\left(\frac{3}{4}\right)^6 - 6\times\left(\frac{2}{4}\right)^6 + 4\times\left(\frac{1}{4}\right)^6$

$= \frac{317}{512} \qquad P(A) = \frac{195}{512} \simeq \underline{0.38}$

Q. 4 married people sit in a row, probability that no husband sits next to his wife.

Sol: $B_1 \quad B_2 \quad B_3 \quad B_4$

$$P(B_1 \cup B_2 \cup B_3 \cup B_4) = \sum P(B_i) - \sum\sum P(B_i \cap B_j) + \sum\sum\sum P(B_i \cap B_j \cap B_k)$$

$=$

$$P(B_i) = \frac{7!\,2!}{8!} = \frac{1}{4}$$

$$P(B_i \cap B_j) = \frac{6!\,2!\,2!}{8!} = \frac{4}{7\times 8} = \frac{1}{14}$$

$$P(B_i \cap B_j \cap B_k) = \frac{5!\,2!\,2!\,2!}{8!} = \frac{1}{42} \qquad P(B_1 \cap B_2 \cap B_3 \cap B_4) = \frac{4!\times 16}{8!}$$

$$\therefore P(B) = 1 - \frac{6}{14} + \frac{4}{42} = 1 - \frac{3}{7} + \frac{2}{21} - \frac{16}{5\times 6\times 7\times 8}$$

$$= \frac{21-9+2}{21} \overset{P(4)}{+} = \frac{23}{35}$$

$$P(A) = \frac{12}{35} \approx \underline{\underline{0.34}}$$

Q. 3 player A, B, C throw dic in order ABC, ABC, ...

(i). A is second player to get 6 in first time

(ii). A is last player to get 6 in first time.

Sol:

(i).

$$P = \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\times 2$$

$$+ \left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\times 2 + \cdots$$

~~$= \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2\cdot 2 + \left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right)^2\cdot 2 + \cdots$~~

(i). $P_1 = 2\times \sum_{r=1}^{\infty}\left(\frac{5}{6}\right)^{2r}\left(1-\left(\frac{5}{6}\right)^{r}\right)\frac{1}{6} = \frac{300}{1001}$ ✓

(ii). $P_2 = \frac{395}{1001}$

Example:

Events pairwise independent But ~~but~~ not mutually independent.

$\Omega = \{(0,0,0), (1,0,1), (0,1,1), (1,1,0)\}$

Let A_i, first digit in i, $i = 0, 1$.
B_i, second digit in i, $i = 0, 1$.
C_i third digit in i, $i = 0, 1$.

$P(A_0) = \frac{1}{2}$ $\quad P(A_1) = \frac{1}{2}$

$P(B_0) = \frac{1}{2}$ $\quad P(B_1) = \frac{1}{2}$

$P(C_0) = \frac{1}{2}$ $\quad P(C_1) = \frac{1}{2}$

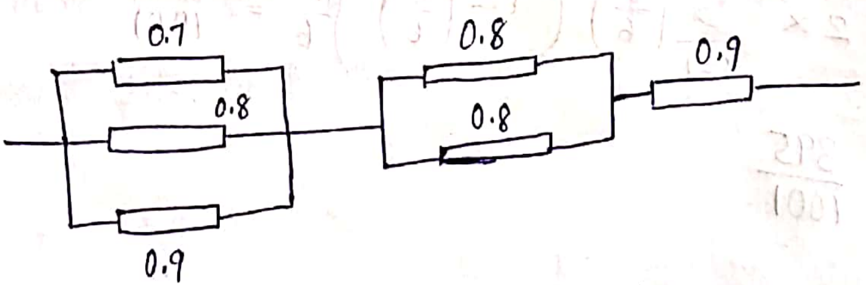
$$\left.\begin{array}{l} P(A_i \cap B_j) = \frac{1}{4} \\ P(B_i \cap C_j) = \frac{1}{4} \\ P(A_i \cap C_j) = \frac{1}{4} \end{array}\right\}$$ They are thus pairwise independent

$$P(A_i \cap B_j \cap C_k) = \begin{cases} (0,0,0) \rightarrow 1/4 \\ (0,1,1) \rightarrow 1/4 \\ (0,0,1) \rightarrow 1/4 \\ (1,1,0) \rightarrow 1/4 \end{cases}$$

$P(A_0 \cap B_0 \cap C_0) = \frac{1}{4} \neq \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

∴ They are not mutually independent.

Q.



What is the probability of it working.

$P(A) = 1-(0.2)(0.3)(0.1) = 1-0.006 = 0.994$

$P(B) = 1-(0.2)(0.2) = 1-0.04 = 0.96$

$P(C) = 0.9 = 0.90$

$P(A\cap B\cap C) = P(A)P(B)P(C) = (0.994)(0.96)(0.90)$

$= \underline{0.859}$

Ex: A laboratory blood test is 95% effective in detecting a certain disease when 'in fact' presents. However, the test also yields a "false positive" result for 1% of the healthy person tested. If 0.5% of population actually have the disease, what is the probability a person has the disease given that he is positive?

Sol: $P\left(\frac{D}{P}\right) = \frac{P(D\cap P)}{P(P)}$

$= \frac{P(P/D)\,P(D)}{P(P/D)P(D) + P(P/\bar{D})\,P(\bar{D})}$

$= \frac{(0.95)(0.005)}{(0.95)(0.005) + (0.01)(0.995)}$

$= \frac{95\times5}{95\times5+995} = \underline{0.323}$

Ex: There are 6 boxes of which 2 are round and 4 are square.

Round Box: 2 green + 3 blue

square Box: 1 green + 3 blue

A Box is chosen at random and a marble is chosen at random what is the probability that chosen marble is blue.

Sol: $P = P(R)P\left(\frac{B}{R}\right) + P(S)P\left(\frac{B}{S}\right)$

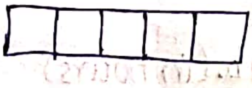
$= \left(\frac{1}{3}\right)\left(\frac{3}{5}\right) + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)$

$= \frac{1}{5} + \frac{1}{2} = \underline{0.70}$

(Total probability Theorem)

Ex: Consider a professor who sign 5 letter of references for 5 of the students, and then puts then into 5 preadressed envelopes at random. What is the probability that none of the letters is put in the correct envelope?

Sol:



$P(5) = 5! - 5\times4! + 10\times3! - \overset{10}{\cancel{5}}\times2! + 1$

$= 60 - 10 + 1 = \underline{51}$ ×

$P(\text{None goes to correct}) = 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)$

$P(5) = 120 - 5P(4) + 10P(3) - 10P(2) -$

$P(2) = 1$ $\quad P(3) = 2$ $\quad P(4) = 24 - 4P(3) + 6P(2) - 4P_1$

$= 24 - 8 + 6 = 22 - 4 = 18$

$P(3) = 3P(2) - P(1)$

$= 100 - 90 + 20 - 10 + 5 = \underline{45} - 1 = \underline{40}$

$P(5) = 120 - 5P(4) + 10P(3) - 10P(2) + 5P(1)$

$P(1) = $ [illegible]

$P(2) = 1$

$P(3) = 6 - 3P(2) + 3P(1) = 6$

$P(4) = 24 - 4P(3) - 6P(2) + 4P(1) = 8$

$P(5) = 120 - 90 + 30 - 10 + 5 = \underline{\underline{45}}$

Random Variables

Quite often in a random experiment, we assign real values to the outcomes. Say we assign a real valued function,

$X : \Omega \to \mathbb{R}$

i.e., X is said to be a random variable if it assigns real numbers to the elements of the sample space.

1. **Examples:** Suppose n coins are tossed.

$X \to$ the number of heads.

Then X is a random variables. It takes values $X = 0$ to n,

Let us take $n = 2$, $\Omega = \{HH, HT, TH, TT\}$

$X(HH) = 2$

$X(HT) = 1$

$X(TH) = 1$

$X(TT) = 0$

2. Let us consider the life of a bulb. (in hours)

$X \to$ the life

$X \in [0, \infty)$ $\qquad \Omega = [0, \infty)$

3. Tossing of Two dice

$X_1 \leftarrow$ sum $\qquad X_2 \leftarrow$ product

$X_1 = \{2, 3, \ldots, 12\}$ $\qquad \Omega = \{11, 22, \ldots, 66\}$

$X_2 = \{1, 2, \ldots, 36\}$

" IIT Kharagpur Biotechnologists discovered that below southern plate had bacterial start with high probability"

1. Discrete Random Variables: If a random variable takes finite or countably infinite number of variables, it is called discrete random variable.

2. continuos Random variables: If the range of random variable is an interval, it is called continuos random variable.

Probability Distribution:

Let X be a discrete random variable taking values from the set $X = \langle x_1, x_2, \ldots, \ldots \}$

The probability distribution of X is defined is defined by a function $P_X(x_i)$ (Probability Mass Function) satisfying the following properties:

1. $P_X(x_i) \geq 0$
2. $\sum_{i=0}^{\infty} P_X(x_i) = 1$
3. $P(X = x_i) = P_X(x_i)$

In the experiment of tossing of a coin n times,

$p(\text{Head in a single toss}) = P$

$P(X=0) = (1-p)^n$

$P(X=1) = n.p(1-p)^{n-1}$

$P(X=r) = {}^nC_r \, p^r (1-p)^{n-r} \quad r = 0,1,2,\ldots,n$

$\therefore \boxed{P_X(r) = {}^nC_r \, p^r (1-p)^{n-r}}$

Tossing of two Dice and $X_1 =$ sum. (Fair Dice)

$P_{X_1}(2) = P(X_1=2) = \frac{1}{36}$

$x_1+x_2=r$

$^{r-1}C_{2-1} = r-1$

$P(X_1=3) = \frac{2}{36}$

$P(X_1=4) = \frac{3}{36}$ .....

$$P(X_1=r) = \begin{cases} \frac{r-1}{36} & (r<8) \\ \frac{13-r}{36} & (8 \le r \le 12) \end{cases}$$

Probability density function:

" All of you have paid 2500 ₹, suppose you are shifted to Kolkata and upto 1 lakh is covered, How many students in Kgp every year will be sick? All students combined paid 2.5cr but Insurance company spent 1.5cr. They got a profit of 1cr, LIC people trap It is not that every will fall sick will die. upper class people. We want to know how many people die before 50 yrs etc.,"

Let X be a continuos random variable on $R = (-\infty, \infty)$

The probability distribution is defined by function $f_X(x)$ called as probability Density function (PDF), satisfying the following properties:

1. $f_X(x) \geq 0 \quad \forall x \in R$
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
3. $P(a \leqslant x \leqslant b) = \int_a^b f_X(x)\,dx$

Ex: Let X be life of a bulb, it has been found that,

[redacted] $f_X(x) = \begin{cases} 3e^{-3x} & ; x \geq 0 \\ 0 & ; x \leq 0 \end{cases}$

$p(1 < x < 2) = e^{-3} - e^{-6} = 0.0473$

$p(x > 7) = e^{-21}$

"Nowadays Hostels rooms are crowded, strength increases every year. In the mess queue to take chapathis, we need to stand 2-5 minutes (10min - 15min) Sometimes 0 minutes too. Out of 30 days, on 3 days no waiting time? we need to modify the probability function."

Mixed Random Variable:

Sometimes a r.v. X may have positive probabilities for some points and pdf over some intervels. In such a case we have partly pmf and partly pdf.

Ex: Waiting time for chapati. (Time)

$P(X=0) = 1/10$

& $f_X(x) = \begin{cases} \frac{1}{20} \cdot \frac{9}{10} & 0 < x < 20 \\ 0 & ; o/w \end{cases}$

Cummulative distribution function:

Let X be a r.v., we define

$F_X(x) = P(X \leq x), x \in R.$

In case X is a discrete r.v. with values in $\mathcal{X} = \{x_1, x_2, \ldots, \}$, then

$F_X(x) = \sum_{x_i \leq x} P_X(x_i)$

conversely, $P_X(x_i) = F_X(x_i) - F_x(x_{i-1})$

In case X is a continuos r.v.,

$$F_X(x) = \int_{-\infty}^{x} f_x(t)\,dt$$

conversely, $\frac{d}{dx} F_X(x) = f_X(x)$

characterization properties of CDF:

If $F_X(x)$ is cdf of a r.v. Then,

1. $\lim_{x \to -\infty} F_X(x) = 0$
2. $\lim_{x \to +\infty} F_X(x) = 1$
3. If $x_1 < x_2$ $F_X(x_1) \le F_X(x_2)$
4. F_X is continuos from rigid at every point
   $\lim_{h \to 0^+} F_X(x+h) = F_X(x) \ \forall x \in R$

conservely, if a function F satisfies the above properties, then F is a cdf of some r.v. X.

Example: $X \to$ no. of heads in two tosses of a coin where $P(H) = 2/3$

$X \to 0, 1, 2$

$P_X(0) = \frac{1}{9}$

$P_X(1) = \frac{4}{9}$

$P_X(2) = \frac{4}{9}$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{9}, & x < 0 \\ \frac{1}{9}, & x = 1/2 \\ \frac{5}{9}, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

For chapati problem:

$$F_X(x) = 0, \quad x<0$$
$$= \frac{1}{10}, \quad x=0$$
$$= \frac{1}{10} + \int_0^x \frac{9}{100}\,dt, \quad 0<x<20$$
$$= 1, \quad x \geq 20$$

$$F_X(x) = \begin{cases} 0; & x<0 \\ \frac{1}{10} + \frac{9x}{200}; & 0 \leq x < 20 \\ 1; & x \geq 20 \end{cases}$$

Mathematical Expectation of a R.V.:

Let X be a discrete r.v. with prob, $P_X(x_i)$, $x_i \in \{x_1, x_2 \dots\}$

Expected value of X,

$$E(X) = \sum_{i=1}^{\infty} x_i P_X(x_i) \leftarrow$$ absolute convergence is required

$$E(X) = 0\left(\frac{1}{9}\right) + 1\left(\frac{4}{9}\right) + 2\left(\frac{4}{9}\right) = \underline{\frac{4}{3}}$$

If X is continuos r.v. with pdf $f_X(x)$

we define,
$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

provided the integral is absolutely convergent.

$$E(X) = \int_0^{\infty} x \cdot 3e^{-3x}\,dx = \frac{1}{3}\int_0^{\infty} te^{-t}\,dt$$
$$= \frac{1}{3}\Gamma(0) = \underline{\frac{1}{3}}$$

$$E(\text{time of getting chapati}) = 0 \cdot \frac{1}{10} + \frac{9}{200}\int_0^{20} x\,dx$$

$$= \underline{\underline{9\text{ min}}}$$

Ex: suppose a box has 4 bulbs out of which one is defective, Bulbs are tested one by one without replacement to identify the defective.

$X \rightarrow$ no. of testing required.

$$P(X=1) = \frac{1}{4}$$

$$P(X=2) = \frac{3}{4}\left(\frac{1}{3}\right) = \frac{1}{4}$$

$$\left.\begin{aligned} P(X=3) &= \frac{3}{4}\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \\ P(X=4) &= \frac{3}{4}\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1}\right) = \frac{1}{4} \end{aligned}\right\} \text{Logically } P(X=3) = 1/2$$

$$E(X) = \frac{1}{4}(1) + \frac{1}{4}(2) + \frac{1}{2}(3)$$

$$= \frac{3}{4} + \frac{3}{2} = \underline{\underline{\frac{9}{4}}}$$

"Parato distribution"

Ex: $F(x) = \begin{cases} 0 \; ; & x \leq 1 \\ 1 - \frac{1}{x} \; ; & x > 1 \end{cases}$ is cdf?

[redacted]

$\lim_{x \to -\infty} F(x) = 0$ ✓

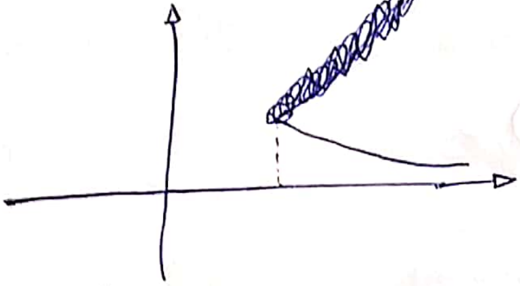
$\lim_{x \to \infty} F(x) = 1$ ✓

$\lim_{x \to 1^-} = \lim_{x \to 1^+}$ ✓

$F(x_1) \leq F(x_2)$ for $x_1 < x_2$.

$\therefore$ cdf of continuos random variable x

$$\text{pdf} = \begin{cases} \frac{1}{x^2} \; ; & x > 1 \\ 0 \; ; & o/w \end{cases} ✓$$



$p(x>3) =$ [redacted] $F(3) =$ [redacted] $\frac{1}{3}$ [redacted]

$p(x>n) =$ [redacted] $\frac{1}{n}$

Ex: $F(X) = \frac{1}{\pi}\tan^{-1} x$

$\lim_{x\to-\infty} F(x) = \frac{-1}{2}$ ×

$\lim_{x\to\infty} F(x) = \frac{1}{2}$ ×

(convert it to a cdf)

Ex: $F(x) = \begin{cases} 0 & ; x<0 \\ x & ; 0\le x\le 1/2 \\ 1 & ; x>1/2 \end{cases}$

Not continuos at $x = 1/2$

Generalization of concept of expectation:

Let X be a r.v. with pmf/pdf and g be real valued function $g: \mathbb{R}\to\mathbb{R}$

Then $y=g(x)$ is a r.v.

$E(Y) = \sum_{i=1}^{\infty} g(x_i)p(x_i)$ if X is discrete

$=$ [redacted] $\int_{-\infty}^{\infty} g(x_i) f(x_i)\,dx$ if x is continuos e.g., $g(X) = aX + b$

$$E(g(x)) = \sum_{x_i \in X} (ax_i + b)\, p_X(x_i)$$

$$= a\,E(x) + b$$

Expectation is a linear function.

Moments: $\mu'_K = E(X^K)$, $K = 1, 2, 3, \ldots$

These are called non-central moments of X

In particular $\mu'_1 = E(X)$ is mean value of X.

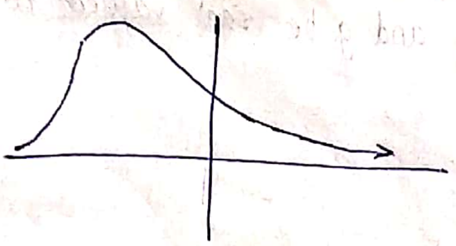
$$\mu_K = E(X - \mu'_1)^K$$

are called central moments about mean of X.

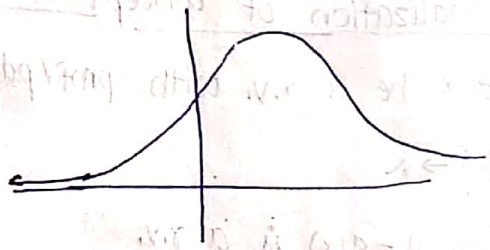
$$\mu_1 = E(x - \mu'_1) = E(X) - \mu'_1 = 0$$

$\mu_2 = E(x - \mu'_1)^2$ is called variance of X.

$\sqrt{\mu_2}$ = standard deviation of X (s.d. of X)



positively skewed



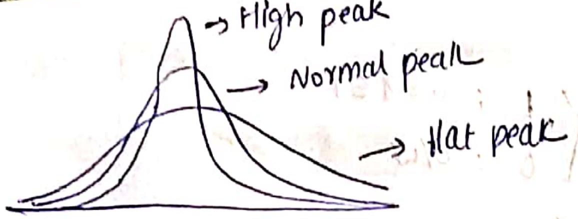
negatively skewed

μ_3 = sign / indicator of skewness of the distribution.

$\beta_1 = \dfrac{\mu_3}{\mu_2^{3/2}} \rightarrow$ Measure of skewness

$\beta_1 = 0$ symmetric

$\beta_1 > 0$; positive skewed; $\beta_1 < 0$: negatively skewed.



$\beta_2 = 0 \rightarrow$ normal
$\beta_2 > 0 \rightarrow$ High
$\beta_2 < 0 \rightarrow$ platykurtic peak.

$$\beta_2 = \left(\frac{\mu_4}{\mu_2^2} - 3\right)$$ → measure of peakedness

change of variable formula:

Find cdf of $Y = X^2$.

$$G(y) = P(Y \le y)$$
$$= P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$g(y) = G'(y)$$
$$= \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

Eg., $f_X(x) = \begin{cases} \frac{2x}{R^2} \; ; & 0 \le x \le R \\ 0 \; ; & o/w \end{cases}$

$$g(y) = \frac{1}{2\sqrt{y}}\left(\frac{2\sqrt{y}}{R^2} + 0\right) = \underline{\underline{\frac{1}{R^2}}}$$

continuous uniform random variable;

ex: $f_X(x) = \begin{cases} \frac{1}{b-a} \; ; & a \le x \le b \\ 0 \; ; & o/w \end{cases}$

Then x is said to be following uniform density with parameter a & b $X \sim U(a,b)$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \frac{x-a}{b-a} \; ; \; a < x < b$$

$$F_X(x) = \begin{cases} 0 \; ; & x < a \\ (x-a)/(b-a) \; ; & a < x < b \\ 1 \; ; & x > b \end{cases}$$

Ex: Let $X \sim U(0,1)$

pdf, $f_X(x) = \begin{cases} 1 ; & 0 \le x \le 1 \\ 0 ; & o/w \end{cases}$

$$F_X(x) = \begin{cases} 0 ; & x < 0 \\ x ; & 0 < x < 1 \\ 1 ; & x > 1 \end{cases}$$

consider the transformation, $Y = \frac{-1}{\lambda} \log(1-x) \quad \lambda > 0$

$G_Y(y) =$

$$F_X(y) = F_X(Y \le y)$$

$$= P_X\left(\frac{-1}{\lambda} \log(1-x) \le y\right)$$

$$= P_X\left(\log(1-x) \ge -\lambda y\right)$$

$$= F_X\left(1 - x \ge e^{-\lambda y}\right)$$

$$= F_X\left(X \le 1 - e^{-\lambda y}\right)$$

$$= \begin{cases} 0 ; & e^{-\lambda y} \le 0 \\ 1 - e^{-\lambda y} ; & 0 \le e^{-\lambda y} \le 1 \\ 0 ; & e^{-\lambda y} \ge 1 \end{cases}$$

$$= \begin{cases} 1 - e^{-\lambda y} ; & 0 \le y \le \infty \\ 0 ; & o/w \end{cases}$$

$$f_X(y) = \begin{cases} \lambda e^{-\lambda y} ; & 0 \le y \le \infty \\ 0 ; & o/w \end{cases}$$

$P(X > x+y) = e^{-\lambda(x+y)}$

$P(X > x+y \mid X > y) = ?$

$P(X > x+y) = e^{-\lambda x} . e^{-\lambda y}$

$= P(X > x+y) = P(X > x) . P(X > y)$

$$\frac{P(X > x+y)}{P(X > y)} = P(X > x)$$

$\therefore \; P(X > x+y \mid X > y) = \underline{P(X > x)}$

Theorem: Let Q be a differentiable function which is strictly increasing (or) strictly decreasing on an interval I.

Let $\varphi(I)$ denote the range of Q.

and φ^{-1} be inverse of φ on I.

Let X be a continuos random variable having density $f(x)$ such that $f(x) \neq 0$ on I. Then $Y = Q(x)$ whose density is given by $g(y) = f(\varphi^{-1}(y)) \left| \frac{d}{dy} \varphi^{-1}(y) \right|$ $y \in Q(I)$.

$$= P(Y \leq y)$$

$$G_y(y) = P(\varphi(x) \leq y)$$

S.I

$$= P(x \leq \varphi^{-1}(y))$$

$$= F(\varphi^{-1}(y))$$

$$g(y) = \frac{d}{dy} F(\varphi^{-1}(y))$$

$$= F'(\varphi^{-1}(y)) \left| \frac{d}{dy} \varphi^{-1}(y) \right|$$

$$= f(\varphi^{-1}(y)) \left[\frac{d}{dy} \varphi^{-1}(y) \right]$$

S.D

$$G_y(y) = P(Y \leq y)$$

$$= P(\varphi(x) \leq y)$$

$$= P(x \geq \varphi^{-1}(y))$$

$$= 1 - P(x \leq \varphi^{-1}(y)) = 1 - F(\varphi^{-1}(y))$$

$$g(y) = -f(\varphi^{-1}(y)) \left(\frac{d}{dy} \varphi^{-1}(y) \right)$$

Ex: Let X be a continioos r.v. with density f. Let $a,b \in \mathbb{R}$ and $b \neq 0$. Then define $Y = a + bX$. Then what is the density of Y.

Sol:

$$
\begin{aligned}
G_Y(y) &= P(Y \le y) \\
&= P(a + bX \le y) \\
&= P\left(X \le \frac{y-a}{b}\right) \\
&= F_X\left(\frac{x-a}{b}\right)
\end{aligned}
$$

$$\underline{g(y) = f\left(\frac{x-a}{b}\right)\left(\frac{1}{|b|}\right)}$$

Ex:

$$f_X(x) = \begin{cases} 2x/R^2 & ; 0 \le x < R \\ 0 & ; o/w \end{cases}$$

$$Y = \frac{X}{R} \qquad 0 < y < 1$$

$$
\begin{aligned}
G_Y(y) &= P(Y \le y) \\
&= P(X \le yR) \\
&= F(yR)
\end{aligned}
$$

$$g_Y(y) = f(yR)(R) = \frac{2yR}{R} = 2y$$

$$g(y) = \begin{cases} 2y ; & 0 \le y \le 1 \\ 0 ; & o/w \end{cases}$$

symmetric density:-

f is symmetric density if $f(x) = f(-x) \quad \forall\, x \in \mathbb{R}$

A random variable is symmetric if its $f_X(x)$ is symmetric density.

Ex: If X is a symmetric random variable with CDF $F_X(x)$ Then $F_X(0) = 1/2$.

$$F_X(-x) = \int_{-\infty}^{-x} f(y)\,dy = \int_{x}^{\infty} f(-y)\,dy = \int_{x}^{\infty} f(y)\,dy$$

$$\therefore \int_{-\infty}^{0} f(y)\,dy = \int_{0}^{\infty} f(y)\,dy = \frac{1}{2}$$

$$\therefore F_X(0) = \int_{-\infty}^{0} f(y)\,dy = \underline{\underline{0}}.$$

04/02/19

For any random variable X, the moment generating function defined as $M_X(t) = E[e^{tX}]$ provided it exists and $t \in \mathbb{R}$.

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \dots$$

$$\therefore M_X(t) = E[1] + tE[X] + \dots$$
$$= E[1] + t\mu_1' + \dots$$

That is in the expansion of $M_X(t)$, the coefficient of $\frac{t^r}{r!}$ is μ_r' → rth non central moment.

We can also consider, $g(t) = M_X(t)$

Then $g(0) = 1$

$g'(0) = \mu_1' \quad g''(0) = \mu_2' \quad \dots$

$$\left.\frac{d^r}{dt^r} M_X(tX)\right|_{t=0} = \mu_r' \ ; \ r = 0,1,2,3,\dots$$

If mgf exists then all moments exists.

$$\mu_K = E(X-\mu_1')^K$$
$$= E\left[X^K - \binom{K}{1} X^{K-1}\mu_1' + \dots\right]$$
$$= \mu_K' - \binom{K}{1}\mu_{K-1}'\mu_1' + \dots$$

In particular, $\mu_2 = \mu_2' - (\mu_1')^2$

$$Var(X) = E(X^2) - (E(X))^2$$

Quantiles of population:

A number satisfying, (Q_p)

$P(X \le Q_p) \ge p$ and

$P(x \ge Q_p) \ge 1-p$

is called pth quantile of distn of X.

if F is a absolute continuos cdf, then

$$F(Q_p) = P(X \le Q_p) = p$$

For $p = 1/2$, Q_p is called median is denoted by m.

For $p = 1/4, 1/2, 3/4$, $Q_{\frac{1}{4}} = Q_{\frac{2}{4}} = M, Q_{\frac{3}{4}}$ are called quartiles.

$Q_{1/100}, \dots, Q_{99/100}$ are called percentiles.

let $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ $\quad -\infty < x < \infty.$

we want median.

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi}\left(\tan^{-1}(x) + \frac{\pi}{2}\right)$$

$$= \frac{\tan^{-1}(x)}{\pi} + \frac{1}{2}$$

$F(x) = \frac{1}{2} \Rightarrow \boxed{x = 0}$

$F(Q_{\frac{1}{4}}) = \frac{1}{4} \Rightarrow \frac{-1}{4} = \frac{\tan^{-1}(x)}{\pi} \Rightarrow \boxed{x = -1}$

$F(Q_{\frac{3}{4}}) \Rightarrow \boxed{x = 1}$

-1, 0, 1 are the quartile distances of this distribution.

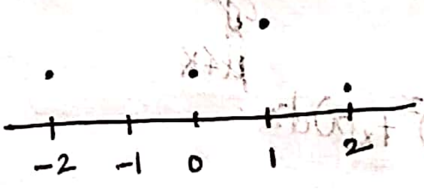
$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi(1+x^2)} dx = \frac{1}{2\pi}\left(\ln|1+x^2|\right)_{-\infty}^{\infty} = \underline{0}$$

Ex: $P(X=-2) = \frac{1}{4}$

$P(X=0) = \frac{1}{4}$

$P(X=1) = \frac{1}{3}$

$P(X=2) = \frac{1}{6}$



Any point between 0 and 1 is the median.

Q. $f_X(x) = \begin{cases} \frac{e^{-x/2}}{2} & ; x > 0 \\ 0 & ; x = 0 \end{cases}$

$$M_X(t) = \frac{1}{2}\int_{0}^{\infty} e^{xt} e^{-x/2}\, dx = \frac{1}{2}\left(\frac{e^{x(t-1/2)}}{t-1/2}\right)_0^{\infty}$$

$$= \left(\frac{-1}{2t-1}\right); \quad t < \frac{1}{2}$$

Ex: Find $\mu_1', \mu_2', \mu_3', \mu_4', \mu_2, \mu_3$ and μ_4. β_1, β_2 also.

Chebyshev Inequality:

Let X be a random variable with mean μ and variable σ^2. Then for any $K > 0$,

$$P(|X-\mu| \geq K) \leq \frac{\sigma^2}{K^2}$$

proof: (Let us take continuos case)

$$P(|X-\mu| \geq K) = P(X-\mu \geq K \;\&\; X-\mu \leq -K)$$

$$= P(X-\mu \geq K \;\cup\; X-\mu \leq -K)$$

$$= P(X-\mu \geq K) + P(X-\mu \leq -K)$$

$$= P(X \geq \mu+K) + P(X \leq \mu-K)$$

$$= \cancel{\int_{\mu+K}^{\infty}}$$

(do ulta)

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$$

$$= \int_{|x-\mu| \geq K} (x-\mu)^2 f_X(x)\, dx + \int_{|x-\mu| < K} (x-\mu)^2 f_X(x)\, dx$$

$$\geq \int_{|x-\mu| \geq K} (x-\mu)^2 f_X(x)\, dx = K^2 P(|x-\mu| \geq K)$$

other alternative forms;

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Ex: Numbers of cstumers in a shop has $\sigma = 2.5$ and $\mu = 18$. what is minimum probability that 8 to 28 costumers appear.

Sol: $P(8 \leq X \leq 28) = P(|X-18| \leq 10)$

$$\geq 1 - \frac{\sigma^2}{100} \geq 1 - \frac{1}{16} = \frac{15}{16}$$

Ex: Show that for 40,000 flips of a fair coin, $\overset{S.T.}{P} \geq 0.99$ that $19,000 \leq N(\text{Heads}) \leq 21,000$.

Sol: $P(19000 \leq X \leq 21,000) = P(|X-20,000| \leq 1,000)$

$$\mu = \sum \left(\tfrac{1}{2}\right)(1) = 20,000$$

$\mu = np = 20000$

$\sigma = \sqrt{npq} = 100$

Ex: Mean $= \mu$

variance $= 1$

How many observations are needed in order that prob ≥ 0.9 that mean of observations differs μ by not more than 1.

Sol: $E(\bar{X}) = \mu$

$$V(\bar{X}) = V\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) = \frac{\sum V(X_i)}{n^2} = \frac{1}{n}$$

$$P(|\bar{X} - \mu| < 1) \geq 1 - \frac{1}{n} > 0.9$$

$\Rightarrow n > 10$

Ex: $f_X(x) = \frac{1}{\beta}\left\{1 - \frac{|x-\alpha|}{\beta}\right\}$ $\qquad \alpha-\beta<x<\alpha+\beta$

Find $\mu_1', \mu_2, \mu_3, \mu_4, Q_1, Q, Q_3$.

Special Distributions: (Discrete)

Degenerated Distribution:

$P(X=a)=1$ for some point a.

$E(X)=a$

$V(X)=0$

Discrete uniform distribution:

X can take 1, 2, 3, ..., N

$P(X=k)=\frac{1}{N} \qquad k=1,2,\dots,N$

$E(X)=\frac{N+1}{2}$

$$V(X)=\sum_{k=1}^{N}\left(\frac{N+1}{2}-k\right)^2\left(\frac{1}{N}\right)$$

$$=\left(\frac{1}{N}\right)\left(\left(\frac{N+1}{2}\right)^2\cdot N + \frac{N(N+1)(2N+1)}{6} - \frac{(N+1)N(N+1)}{2}\right)$$

$$=\frac{(N+1)(2N+1)}{6}-\frac{(N+1)(N+1)}{4}$$

$$=\frac{N^2-1}{12}$$

$$M_X(t)=E(e^{tX})=\sum_{j=1}^{N} e^{tj}\cdot\frac{1}{N}$$

$$=\begin{cases}\frac{1}{N}\cdot\left(\frac{e^{tN}-1}{e^t-1}\right) & ;\ t\neq 0\\ 1 & ;\ t=0\end{cases}$$

Bernoullian Trail:

If trials results in two cases 'success' and 'failure' it is called Bernoullian trial.

$X(\text{success}) = 1$; $X(\text{failure}) = 0$:

$P(X=1) = p$ $\quad$ $P(X=0) = 1-p = q$ $\quad$ $0 \le p \le 1$

$E(X) = p$

$E(X^2) = p$

$\mu_k = p$

$\mu_2 = var(X) = p - p^2 = pq$

$M_X(t) = q + pe^t$

Binomial Distribution:

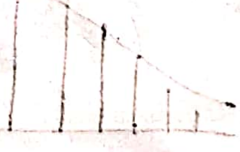
consider n independent trials of Bernoulli under identical conditions with probability of success in each trial as p.

$X \rightarrow$ numbers of successes in n trials.

$P(X=K) = {}^nC_K \, p^K (1-p)^{n-K}$

$\mu_1' = \sum_{K=1}^{n} K \cdot {}^nC_K \cdot p^K (1-p)^{n-K}$

$= n \cdot \sum_{K=1}^{n} \frac{(n-1)!}{(K-1)!(n-K)!} \, p^{K-1} (1-p)^{n-1-K} \cancel{(1-p)} \, p$



$= n$ [illegible] $p) \left[(q+p)^{n-1} \right] = \underline{np}$

$$E(x(x-1)) = \sum j(j-1)\frac{n!}{j!\,(n-j)!}\; p^j q^{n-j}$$

$$= \underline{\underline{n(n-1)p^2}}$$

called as second factorial moment.

$$E(X^2) = n(n-1)p^2 + E(X)$$

$$= n(n-1)p^2 + np$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= n(n-1)p^2 + np - n^2p^2 = \underline{npq}$$

<u>Ex:</u> Find $E(X^3)$, $E(X^4)$, μ_3, μ_4

$$\mu_3 = np(1-p)(1-2p)$$

$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{np(1-p)(1-2p)}{(npq)^{3/2}} = \frac{1-2p}{\sqrt{npq}}$$

$p < 1/2$; positively skewed.

$p > 1/2$; negatively skewed

$p = 1/2$; symmetric.

$p = 1/2$

$p < 1/2$

$p > 1/2$

$$\mu_4 = 3(npq)^2 + npq(1-6pq)$$

$$\beta_2 = \frac{\mu_4}{\mu_3^2} - 3 = \frac{1-6pq}{npq}$$

$$M_X(t) = (q + pe^t)^n$$

suppose independent and identical Bernoulli trials are performed till weget the success. Let X be no. of trials required

$$P_X(j) = q^{j-1}p \quad ; \; j = 1, 2, 3, \ldots$$

This is called as geometric distribution.

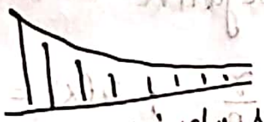
$$E(X) = \sum_{j=1}^{\infty} x \cdot q^{x-1} p = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \underline{\underline{\frac{1}{p}}}$$

<u>Ex:</u> calculating higher order factorial moments this distribution we can use the following formula

$$\frac{1}{(1-r)^{k+1}} = \sum_{j=k}^{\infty} \binom{j}{k} r^{j-k}$$

$$= \sum_{i=0}^{\infty} \binom{k+i}{k} r^{i} \qquad 0<r<1$$

$$\mu_2' = E(X^2) = \frac{q+1}{p^2}$$



$$\mathrm{Var}(X) = \mu_2 = \mu_2' - \mu_1'^2 = \underline{\underline{\frac{q}{p^2}}}$$

positively skewed.

<u>Ex:</u> Find $\mu_3, \mu_4, \beta_1, \beta_2$.

$$\mu_3 = E(X-\mu)^3 = \sum_{j=1}^{\infty} (j-\mu)^3 p^1 q^{j-1} \longleftarrow \underline{\mu = 1/p}$$

$$M_X(t) = \sum_{j=1}^{\infty} e^{tj} q^{j-1} p = pe^t \sum_{j=1}^{\infty} (qe^t)^{j-1}$$

$$= pe^t \left(\frac{1}{1-qe^t} \right)$$

$$qe^t < 1 \rightarrow \boxed{t < -\ln q} \qquad 1 + tj + \frac{t^2 j^2}{2!}$$

$$M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots$$

$$P(X>m) = \sum_{j=m+1}^{\infty} p\, q^{j-1} = q^m p(1+q+q^2+\ldots) = \underline{\underline{q^m}}$$

$$P(X>n+m \mid X>n) = P(X>m)$$
$$= q^m$$
↳ Memoryless property

Ex: $P(\text{success}) = 1/3$

$$P(X \geq 5) = \left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) + \dots$$
$$= \frac{\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)} = \left(\frac{2}{3}\right)^4$$

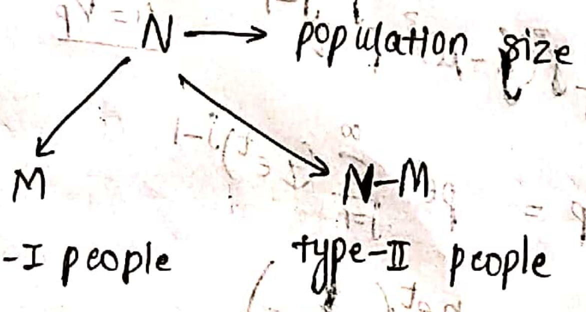
Independent bernoullian trials are performed under identical conditions until we get r^{th} success:

$$P_X(X=K) = {}^{K-1}C_{r-1}\, p^r q^{K-r} \; ; \; K = r, r+1, r+2, \dots$$

Negative binomial distribution / Inverse binomial

$$E(X) = \frac{r}{p} \; ; \; V(X) = \frac{rq}{p^2}$$

$$M_X(t) = \left(\frac{pe^t}{1-qe^t}\right)^r \quad / \; t < -\ln_e q$$



A random sample of size n is selected probability that there are x items of type-I.

$$P_X(x) = P(X=x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \; ; \; x = 0, 1, 2, \dots, n$$
$$x \leq M$$
$$n-x \leq N-M$$

Hypergeometric distribution

$$\mu_1' = E(x) = \frac{Mn}{N}$$

$$E(X(X-1)) = \frac{M(M-1)\, n(n-1)}{N(N-1)}$$

$$E(X^2) = \frac{Mn\,(Mn - M - n + N)}{N(N-1)}$$

$$V(X) = \left(\frac{N-n}{N-1}\right)\frac{Mn}{N}\left(1-\frac{M}{N}\right)$$

suppose we want to estimate the number of tigers in a reserived forest.

$$X \to \frac{Mn}{N}$$

$$\Rightarrow N \approx \frac{nM}{X}$$

Capture-Recapture Technique:

Theorem: Let $X \sim$ Hypergeometric(M, N, n). As $M \to \infty$, $N \to \infty$,

$$\frac{M}{N} \to p \qquad P(X=x) \longrightarrow \binom{n}{x} p^x q^{n-x}$$

Poisson process

The events occuring/being observed over time/area/space etc., are refered as being a poisson process if they satisfying these assumptions:

1. The number of occurances during disjoint time intervals are independent.
2. The probability of a single occurance during a small time interval is proportional to the length of the interval.
3. The probability of more than one occurance in a small time interval is negligible.

$X(t)$ = No. of occurances on an interval of time '$\underline{t}$'.

$P(X(t)=n)$: $\underline{n}$ events occur in the time $\underline{t}$.

$= P_n(t)$

Assumptions:

1.

2. $P_1(h) = \lambda h$

3. $P(X(t)>1) = P_2(h) + P_3(h) + \dots = \frac{O(h)}{h} \to 0$ as $h \to 0$

under assumptions 1-3;

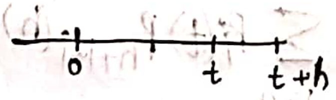
$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0,1,2,\dots$$

proof:

We will use principle of mathematical induction.

To prove $P_0(t) = e^{-\lambda t}$ we consider

$P_0(t+h) = e^{-\lambda(t+h)}$



p(no occurance in interval (0 to t+h])

$P_0(t+h) = p($ no occurance in $(0,t] \cap$ no occurance in $(t, t+h])$

$= P_0(t) \cdot P_0(t \text{ to } t+h)$

$= P_0(t) P_0(h) = P_0(t)(1 - \lambda h - o(h))$

$\Rightarrow \dfrac{P_0(t+h) - P_0(t)}{h} = -\dfrac{\lambda h P_0(t)}{h} - \dfrac{o(h) P_0(t)}{h}$

take limit as $h \to 0$

$P_0'(t) = -\lambda P_0(t)$

$P_0(0) = e^{-\lambda t}$

$P_1(t+h) = P_1(0 \le t \le t) \cdot P_0(t \le t < t+h) +$
$P_0(0 < t \le t) P_1(t \le t < t+h)$

$= P_1(t) P_0(h) + P_0(t) P_1(h)$

P_1 ~~$(t+h)$~~ P

$\Rightarrow P_1(t+h) = P_1(t)(1 - \lambda h - o(h)) + e^{-\lambda t}(\lambda h + o(h))$

$\dfrac{P_1(t+h) - P_1(t)}{h} = -\lambda P_1(t) - \dfrac{o(h)}{h} P_1(t) + \lambda e^{-\lambda t} + \dfrac{o(h)}{h} e^{-\lambda t}$

Take limit as $h \to 0$

$P_1'(t) = \lambda(e^{-\lambda t} - P_1(t))$

$P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}$

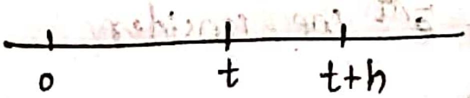
$P_1(t) = \lambda t e^{-\lambda t} + c$

$P_1(0) = 0 \Rightarrow c = 0$

$\therefore P_1(t) = \lambda t (e^{-\lambda t})$

Let us assume $P_n(t)$ Holds for all $n \le k$.

Now for $k=n+1$:



$$P_{k+1}(t+h) = \sum_{i=0}^{k+1} P_i(t)\, P_{k+1-i}(h)$$

$$= P_{k+1}(t)(1-\lambda h - o(h)) + \frac{e^{-\lambda t}(\lambda t)^k}{k!}(\lambda h + o(h))$$

$$+ \sum_{j=0}^{k-1} \frac{e^{-\lambda t}(\lambda t)^{k+1-j}}{(k+1-j)!}\, o(h)$$

$$\frac{P_{k+1}(t+h) - P_{k+1}(t)}{h} = -\lambda P_{k+1}(t) - \frac{o(h)}{h} P_{k+1}(t)$$

$$+ \frac{\lambda e^{-\lambda t}(\lambda t)^k}{k!}\left(\lambda + \frac{o(h)}{h}\right)$$

Taking limit as $h \to 0$,

$$P'_{k+1}(t) = -\lambda P_{k+1} + \frac{\lambda^2 e^{-\lambda t}(\lambda t)^k}{k!}$$

$$\frac{d}{dt}\left(e^{\lambda t} P_{k+1}\right) = \frac{\lambda^2 (\lambda t)^{k+1}}{\lambda\, k!}$$

$$e^{\lambda t} P_{k+1} = \frac{\lambda^2 t (\lambda t)^{k+1}}{\lambda\, k!\,(k+1)}$$

$$P_{k+1} = \frac{e^{-\lambda t}(\lambda t)^{k+1}}{(k+1)!}$$

Ex: Customers arraive at a shopping mall at 5 per minute. Probabibility that no customer came in 1 minute period.

$$P_0(1) = P(X(1)=0) = e^{-5}$$

$$P_2(3) = \frac{e^{-15}(15)^2}{2!} \qquad \boxed{\lambda = 5}$$

Ex: Average of delivery 2 months.

$\lambda = 1$ 2 months

$\lambda = 1/2$ 1 month

$$P_0(1\text{ month}) = e^{-(1/2)} = \frac{1}{\sqrt{e}} \simeq 0.6065$$

Ex: Total = 3000 positions.

$$\lambda = \frac{1}{6000}$$

$$P_n(3000) = \frac{e^{-1/2}\left(\frac{1}{6000} \times 3000\right)^n}{n!} = \frac{1}{\sqrt{e}\, 2^n . n!} \checkmark$$

$$P_0(3000) = \frac{1}{\sqrt{e}} \checkmark$$

$$P_0(3000 \times 16) = \frac{e^{-8}\left(\frac{1}{\cancel{6000}} \times 8\right)^{n!}}{n!} = \frac{8^n}{e^8 . n!} = \frac{1}{e^8} \checkmark$$

Putting $\lambda t = \mu$, we call it a poisson distribution.

$$P_X(K) = P(X=K) = \frac{e^{-\mu}\mu^K}{K!} \; ; K = 0,1,2,3,\dots$$

$$\sum_{K=0}^{\infty} P_X(K) = e^{-\mu}e^{\mu} = 1$$

$$E(X) = \sum_{K=0}^{\infty} K P_X(K) = \sum_{K=1}^{\infty} \frac{K}{K!} e^{-\mu}\mu^K$$

$$= e^{-\mu}\cdot\mu \sum_{K=0}^{\infty} \frac{\mu^K}{K!}$$

$$= e^{-\mu}\mu e^{\mu} = \underline{\mu}$$

$$E(X(X-1) = \sum_{K=2}^{\infty} \frac{}{(K-2)!} e^{-\mu}\mu^{K-2} \; \mu^2$$

$$= e^{-\mu}\cdot e^{\mu} = \mu^2$$

$$E(X^2) = \mu^2 + \mu$$

$$Var(X) = -(E(X))^2 + E(X^2)$$

$$= \underline{\underline{\mu}}$$

Ex: Find $\mu_3', \mu_4', \mu_3, \mu_4, \beta_1, \beta_2$.

$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda$ ✓

$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$?

$\mu_3 = \lambda$

$\mu_4 = \lambda + 3\lambda^2$

$\beta_1 = \frac{1}{\sqrt{\lambda}} > 0$ positively skewed.

$\beta_2 = \frac{1}{\lambda} > 0$ Leptokusitic

$$M_X(t) = E(e^{tX}) = \sum_{K=0}^{\infty} \frac{e^{tK} e^{-\mu} \mu^K}{K!}$$

$$= e^{-\mu} \sum_{K=0}^{\infty} \frac{(\mu e^t)^K}{K!} = e^{-\mu} e^{\mu e^t}$$

$$= \underline{\underline{e^{\mu(e^t - 1)}}}$$

Theorem: Let $X \sim Bin(n,p)$

If $n \to \infty$ and $p \to 0 \to np \to \lambda$ then

$$P(X=K) \longrightarrow \frac{e^{-\lambda} \lambda^K}{K!}$$

$$M_X(t) = (q + pe^t)^n \qquad np \simeq \lambda$$

$$\Rightarrow p \simeq \frac{\lambda}{n}$$

$$= (1 - p + pe^t)^n$$

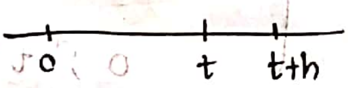
$$= (1 + p(e^t - 1))^n \cong \left(1 + \frac{\lambda}{n}(e^t - 1)\right)^n \to e^{\lambda(e^t - 1)}$$

which is mgf of $\underline{\underline{P(\lambda) \text{ dist}^n}}$

Ex: ~~Givex~~ Give a direct proof.

poisson process with rate $\underline{\lambda}$. Y will be time of first occurance. what is distribution of Y.

Sol: $$P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$



$$P(Y > t) = P(X(t) = 0) = P_0(t)$$

$$F_y(t) = 1 - e^{-\lambda t};\ t > 0 \quad ;\ 0 \quad t \le 0$$

$$f_y(t) = \underline{\underline{\lambda e^{-\lambda t}}}$$

$$E(Y^k) = \int_0^{\infty} t^k \lambda e^{-\lambda t} dt = \frac{\lambda \Gamma{K+1}}{\lambda^{K+1}} = \underline{\underline{\frac{K!}{\lambda^K}}}$$

Memoryless property:

$$P(Y>t) = e^{-\lambda t}$$

$$P(Y>s+t \mid Y>s) = \frac{P(Y>s+t)}{P(Y>s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = P(Y>t)$$

Let Y_r be time of r^{th} occurance.

$$P(Y_r > t) = \sum_{j=0}^{r-1} P(X(t) \le j)$$

$$= \begin{cases} \sum \frac{e^{-\lambda t}(\lambda t)^j}{j!} & t>0 \\ 1 ; & t \le 0 \end{cases}$$

$$CDF = 1 - P(Y_r > t)$$

$$= \begin{cases} 0 ; & t \le 0 \\ 1 - \sum \frac{e^{-\lambda t}(\lambda t)^j}{j!} ; & t>0 \end{cases}$$

$$f_{Y_r}(t) = \begin{cases} \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!} ; & t>0 \\ 0 ; & t \le 0 \end{cases}$$

This is Erlang or gamma distribution.

General form,

$$f_X(x) = \frac{\lambda^r}{\Gamma r} e^{-\lambda x} x^{r-1} \qquad \begin{matrix} x>0 \\ \lambda>0 \\ r>0 \end{matrix}$$

$$\mu'_k = \int_0^\infty \frac{\lambda^r}{\Gamma r} e^{-\lambda x} x^{r+k-1} dx$$

$$= \frac{\Gamma k+r}{\Gamma r} \cdot \frac{1}{\lambda^k} \qquad k = 1, 2, 3 \dots$$

$\mu_1' = E(X) = \frac{r}{\lambda}$

$\mu_2 = var(X) = \frac{r}{\lambda^2}$

$MGE = E(e^{Y_r(t)} = \left(\frac{\lambda}{\lambda - t}\right)^r$ +ve skewed.

poisson → gamma functions methods

<u>continucos uniform distribution:</u>

Let x be a continucos random variable with uniform density over an interval $[a,b]$. Then,

$$f(x) = \begin{cases} c & ; a < x < b \\ 0 & ; o/w \end{cases}$$

$$\int_a^b f(x)\,dx = c(b-a) = 1 \Rightarrow c = \frac{1}{b-a}$$

The geometric shape is that of a rectangle.

$$\mu_1' = E(X) = \int_a^b x \frac{1}{b-a}\,dx = \frac{b+a}{2}$$

$$\mu_2' = E(X^2) = \frac{a^2 + ab + b^2}{3}$$

$$\mu_2 = E(X^2) - (E(X))^2 = \frac{(b-a)^2}{12}$$

<u>MGF:</u> $E(e^{tx}) = \int_a^b \frac{e^{tx}}{b-a}\,dx = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} & ; t \neq 0 \\ [illegible] & ; t = 0. \end{cases}$

Notations:

Bernoulli (p) $\quad$ Bin (n,p) $\quad$ Geometric (p)

$X \sim NB(r,p)$ $\quad$ $X \sim HG((M,N;n))$

$X \sim P(\lambda t)$ $\quad$ $X \sim P(\lambda)$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{e^{-\lambda t}(\lambda t)^j}{j!} \qquad \frac{e^{-\lambda}\lambda^j}{j!} \quad / j = 0,1,2,3,\ldots$$

$$\text{Exp}(\lambda) \longrightarrow f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x > 0 \\ 0 & ; x \le 0 \end{cases}$$

$X \sim$ Gamma(r,λ)

$$f_X(x) = \frac{\lambda^r}{\Gamma r} e^{-\lambda x} x^{r-1}$$

$X \sim U(a,b)$: uniform density on the interval a to b.

Q. □□□ $\quad P(sp) = 0.9$

Number of trials of rth success

$$P(X=n) = {}^{n-1}C_{r-1} (0.9)^r (0.1)^{n-r}$$

$$= {}^{n-1}C_2 (0.9)^3 (0.1)^{n-3}$$

$$\text{mean} = \frac{r}{p}$$

$$V(X) = \frac{rq}{p^2}$$

□□□

$$(P(X=3))^3 \cdot {}^{10}C_3 (1 - P(X=3))^7 \; ✓$$

$n=11$

---

4. DDD

$P(D=4) = 0.30$

$P(D=1) = 0.70$

D	9
4	6

~~P(R) =~~

$$P(A) = (0.30)\left(\frac{{}^{6}C_3}{{}^{10}C_3}\right) + (0.70)\left(\frac{{}^{9}C_3}{{}^{10}C_3}\right) \checkmark = \frac{54}{100}$$

---

5. $N(D) = 300$

$\lambda = 300 \times 0.02 = 6$

$$P(x=n) = \sum_{n=0}^{4} \frac{e^{-6}\, 6^{n}}{n!}$$

---

8. 9–5pm ⇒ 8 hrs

$\lambda = 500/\text{day}$

$$P_x(x=t) = \frac{e^{-\lambda t} . \lambda (\lambda t)^{r-1}}{(r-1)!} \quad x$$

T→ waiting time for first call after 9:00 am

$$f_T(t) = \frac{50}{8} e^{-50t/8}$$