VECTOR CALCULUS

VECTORS and SCALARS:

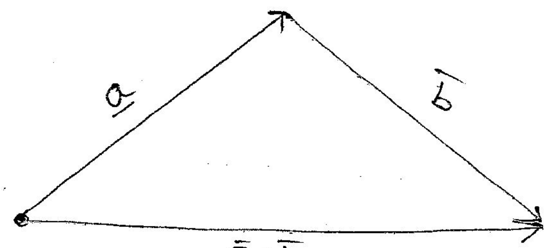
* A vector is a physical quantity which has both magnitude and direction.

  Examples: velocity, force, electric field.

* A scalar is a physical quantity which has magnitude only.

  Examples: mass, temperature and pressure.

Addition of vectors: (Geometric definition)



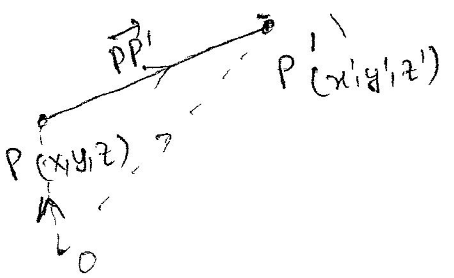
Addition of vectors: (Algebraic definition)

$\underline{a} = (a_1, a_2, a_3)$ $\quad b = (b_1, b_2, b_3)$ OR $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$

$\underline{b} = b_1\underline{i} + b_2\underline{j} + c_3\underline{k}$

$\underline{a} + \underline{b} = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3).$ $\quad \bar{a} + \bar{b} = (a_1 + b_1)\underline{i} + (a_2 + b_2)\underline{j} + (a_3 + c_3)\underline{k}.$

The vector joining two points:

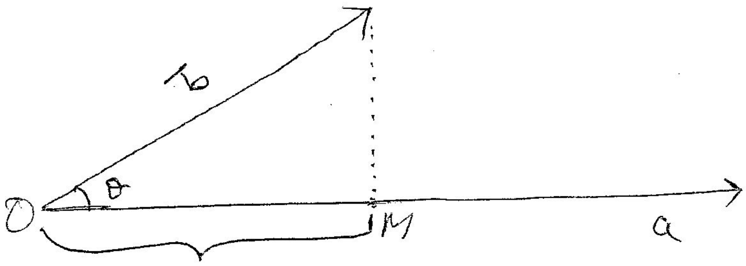


$\overline{PP'} = (x' - x,\ y' - y,\ z' - z).$

$\overline{OP} + \overline{PP'} = OP' \Rightarrow \overline{PP'} = \overline{OP'} - \overline{OP}.$

Dot product: (scalar product) $\rightarrow$ scalar quantity.

$$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$



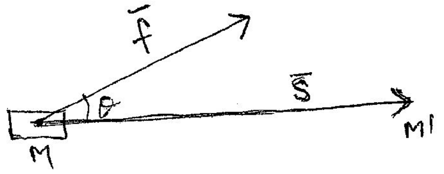
$|b| \cos\theta$

$\uparrow$

projection of $\bar{b}$ on $\bar{a}$.

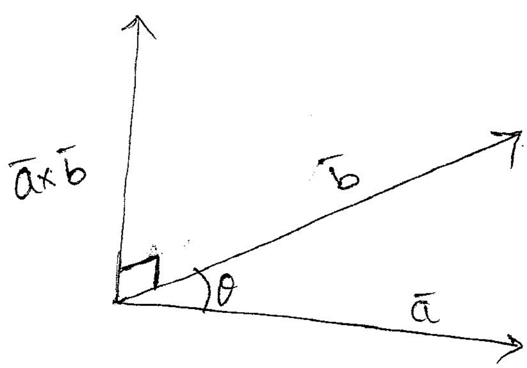
$$\boxed{\text{Projection of } \bar{b} \text{ on } \bar{a} = \frac{\bar{a} \cdot \bar{b}}{|a|}}$$

Application: Evaluation of work done:



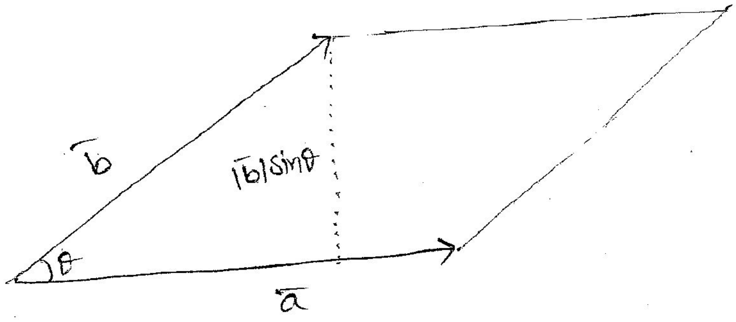
$$\boxed{W = \bar{f} \cdot \bar{S}} = |\bar{f}||\bar{S}| \cos\theta$$

Cross product: (vector product) $\rightarrow$ vector quantity



$$|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}| \sin\theta$$

Direction: perpendicular to both $\bar{a}$ and $\bar{b}$ in a right handed sense.



Area of the parallelogram $= |\bar{a}|\,|\bar{b}|\sin\theta$.

$= |\bar{a}\times\bar{b}|$.

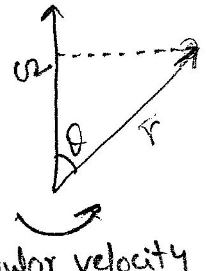
Evaluation of cross product:

$\bar{a} = (a_1, a_2, a_3)$ $\qquad$ $\bar{b} = (b_1, b_2, b_3)$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \bar{i}\left(a_2 b_3 - b_2 a_3\right) + \bar{j}\left(a_3 b_1 - b_3 a_1\right) + \bar{k}\left(a_1 b_2 - b_1 a_2\right).$$

Application:



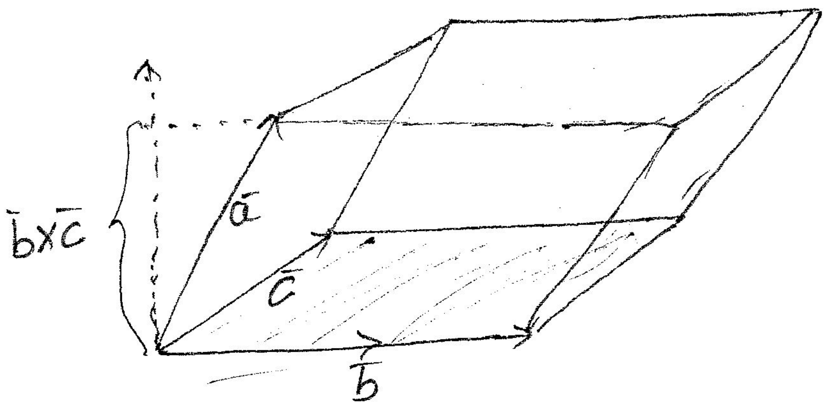
Velocity vector of the point r is

$$v = \Omega \times r$$

Scalar triple product:

$$a\cdot b\times c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \cancel{(a\,b\,c)}$$

$$= (a \;\; b \;\; c)$$



height of the parallelopiped $= \dfrac{|\bar{a}\cdot\bar{b}\times\bar{c}|}{|\bar{b}\times\bar{c}|}$

Area of the base $= |\bar{b}\times\bar{c}|$

Volume of the parallelopiped $= |a\cdot\bar{b}\times\bar{c}|$

Volume of the tetrahedron made by the vectors $\underline{a}, \underline{b}, \underline{c}$ is

$$\frac{1}{6}\,|a\cdot b\times c|.$$

Scalar fields and vector fields:

A scalar or vector quantity is said to be field if it is a function of position.

Examples: Scalar field: Temperature inside a room.

Vector field: Velocity of the air within a room.

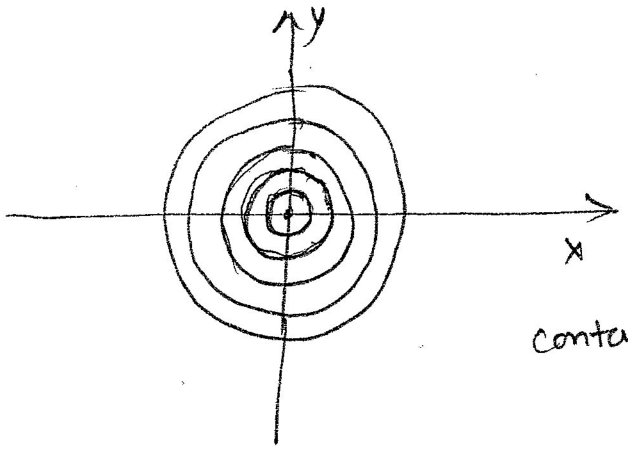
In general scalar or vector fields depends ~~is~~ three dimensional.

– difficult to visualise.

If, for example, a scalar field $T = T(x,y)$ depends on two coordinates then it can be visualised by sketching a contour plot ie the line $T(x,y) = \text{constant}$ is ploted for different values of the constant.

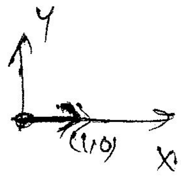
Consider $T(x,y) = x^2+y^2$

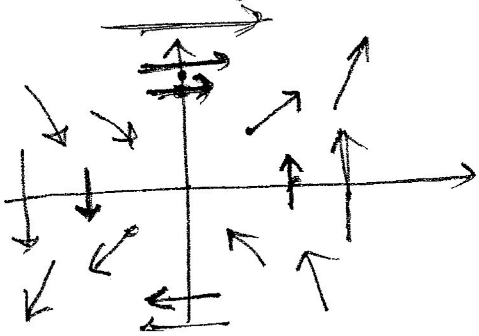
contour lines $x^2+y^2 = \text{constant}$.



contours of the scalar field $T(x,y) = x^2+y^2$

Consider vector field: $\underline{u}(x,y) = (y,x)$.





Vector field
$\underline{u}(x,y) = (y,x)$

Level surfaces: Let $f(x,y,z)$ be a single valued continuous scalar function defined at every point in D.

Then the equation of surface

$$f(x,y,z) = c$$

is called a level surface.

Examples:

$$f(x,y,z) = z - \sqrt{x^2+y^2}$$

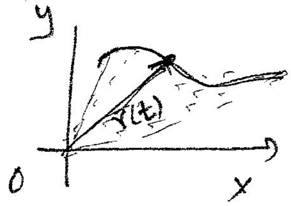
$$\Rightarrow \quad z - \sqrt{x^2+y^2} = c$$

$$\Rightarrow \quad (z-c)^2 = (x^2+y^2)$$

The level surfaces are cones.

Parametric representation of curves:

Curve in two dimension: $x = x(t) \quad y = y(t) \quad a \le t \le b$.

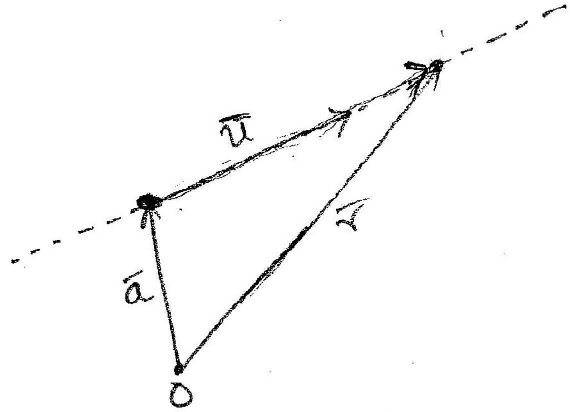


Position vector $\boxed{\bar{r}(t) = x(t)i + y(t)j}$

Similarly in 3D:

$$\bar{r}(t) = x(t)\,\bar{i} + y(t)\,\bar{j} + z(t)\,\bar{k}, \qquad a \le t \le b.$$

Equation of a straight line:



$\bar{a}$: position vector of a particular fixed point on the line

$\bar{u}$: vector pointing along the line.

equation of the straight line:

$$\bar{r} = \bar{a} + \lambda \bar{u}$$

$$= (a_1 + \lambda u_1)\bar{i} + (a_2 + \lambda u_2)\bar{j} + (a_3 + \lambda u_3)\bar{k}.$$

Equation of a circle in a plane in 3-dimensions:

Consider the circle as $x^2 + y^2 = a^2$, $z = d$.

$$\bar{r}(t) = a\cos\theta\, \bar{i} + a\sin\theta\, \bar{j} + d\bar{k}.$$

$$0 \le \theta \le 2\pi.$$

Parametric representation of surfaces:

Let $f(x,y,z) = c$. be the equation of a surface.

(i) write $z = f(x,y)$ then.

$$r(u,v) = u\bar{i} + v\bar{j} + f(u,v)\bar{k}.$$

OR

(ii) $r(u,v) = x(u,v)\bar{i} + y(u,v)\bar{j} + z(u,v)\bar{k}$. ; $(u,v) \in D$.

Parametric representation of a sphere: $x^2+y^2+z^2=a^2$.

$$r(u,v) = a\cos u \sin v\,\bar{i} + a\sin u \sin v\,\bar{j} + a\cos v\,\bar{k}$$

$$0 \le u \le 2\pi \qquad 0 \le v \le \pi.$$

Limit and Continuity of vector functions:

Limit:

$$\lim_{t\to a} |\bar{v}(t) - \bar{l}| = 0$$

Continuity: $\bar{v}(t)$ is said to be continuous at $t=a$ if

i) $\bar{v}(t)$ is defined in some neighbourhood of a

ii) $\lim_{t\to a} v(t)$ exists and

iii) $\lim_{t\to a} \bar{v}(t) = \bar{v}(a)$

Differentiability:

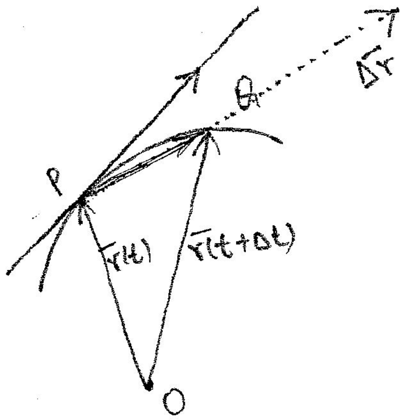
$\bar{v}(t)$ is said to be differentiable if

$$\lim_{\Delta t\to 0} \frac{\bar{v}(t+\Delta t) - \bar{v}(t)}{\Delta t} \quad \text{exists}$$

Let $V(t) = r(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ be the parametric representation of a curve C. Then

$$\frac{d\bar{r}}{dt} = r'(t) = \frac{dx(t)}{dt}\bar{i} + \frac{dy(t)}{dt}\bar{j} + \frac{dz(t)}{dt}\bar{k}$$

Geometric representation of $r'(t)$: (tangent to a curve)



Let the parametric equation of the curve be $\bar{r}(t)$.

Note that the Direction of $\Delta\bar{r} = \bar{r}(t+\Delta t) - \bar{r}(t)$ and

$\dfrac{\Delta\bar{r}}{\Delta t}$ is the same.

Then the limiting position ~~vector~~ of the vector $\dfrac{\Delta\bar{r}}{\Delta t}$, that is,

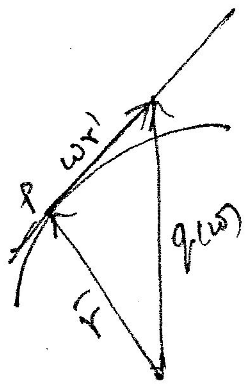
$\lim\limits_{\Delta t\to 0} \dfrac{\Delta\bar{r}}{\Delta t}$ is the tangent to the curve at P.

$$\boxed{\text{tangent vector} = \bar{r}'(t) = \lim_{\Delta t\to 0} \frac{\Delta\bar{r}}{\Delta t}}$$

$$\boxed{\text{Unit tangent vector } \bar{u} = \frac{\bar{r}'(t)}{|r'(t)|}}.$$

Equation of tangent to C at P:

$$\bar{q}(w) = \bar{r} + w r'$$



Rules of Differentiation:

$$(\bar{u}\cdot\bar{v})' = \bar{u}'\cdot\bar{v} + \bar{u}\cdot\bar{v}'$$

$$(\bar{u}\times\bar{v})' = u'\times v + u\times v'$$

$$(\bar{u}\ \bar{v}\ \bar{w})' = (u'\ v\ w) + (u\ v'\ w) + (u\ v\ w')$$

Partial derivatives of a vector function:

Let $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ & v_1, v_2, v_3 are differentiable functions of n variables $t_1, t_2, \cdots t_n$.

Then the partial derivative of v with respect to t_j is given by.

$$\frac{\partial\bar{v}}{\partial t_j} = \frac{\partial v_1}{\partial t_j}\bar{i} + \frac{\partial v_2}{\partial t_j}\bar{j} + \frac{\partial v_3}{\partial t_j}\bar{k}.$$

Examples. i) find $v'(t)$ for $v(t) = (\cos t + t^2)(t\bar{i} + \bar{j} + 2\bar{k})$

$$v'(t) = [(-\sin t + 2t)t + (\cos t + t^2)]\bar{i}$$
$$+ [-\sin t + 2t](\bar{j} + 2\bar{k})$$
$$= (3t^2 - t\sin t + \cos t)\bar{i} + (2t - \sin t)(\bar{j} + 2\bar{k})$$

ii) Partial derivatives: $r(t_1, t_2) = a\cos t_1\, i + a\sin t_1\, j + t_2 k.$

$$\frac{\partial\bar{r}}{\partial t_1} = -a\sin t_1 i + a\cos t_1 j$$

$$\frac{\partial\bar{r}}{\partial t_2} = \bar{k}.$$

D.

Gradient and Directional Derivative

* Gradient of a scalar function $f(x,y,z)$ is a vector given by:

$$\text{grad } f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

* Nabla or Del operator:

$$\nabla \equiv \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k} \quad \text{or} \quad \nabla \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

So, $\text{grad } f = \nabla f$

• If a surface is given by $f(x,y,z) = c$. Then $\nabla f(P)$ is the vector normal to the surface $f(x,y,z) = c$ at the point P.

Consider a smooth curve C on the surface passing through a point P on the surface. Let $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ be the position vector of P.

Since the curve lies on the surface we have

$$f(x(t), y(t), z(t)) = c$$

Then, $$\frac{d}{dt} f(x(t), y(t), z(t)) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = 0$$ by chain rule

$$\Rightarrow \left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}\right)\cdot\left(\frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}\right) = 0$$

$$\Rightarrow \nabla f \cdot \bar{r}'(t) = 0$$

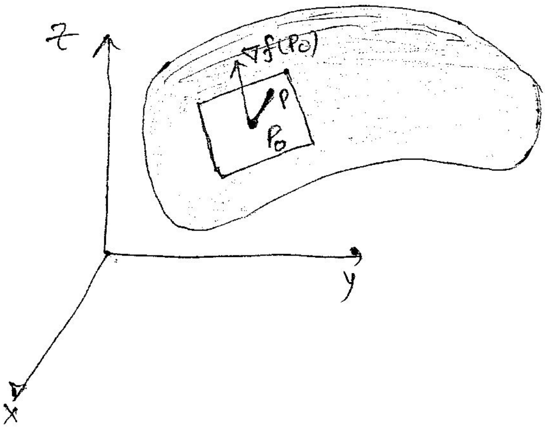
Note that $\bar{r}'(t)$ is a tangent vector at P and lies in the tangent plane at P

$\Rightarrow \nabla f(P)$ is the vector normal to the surface $f(x,y,z) = c$ at P.

- Unit normal vector to a surface $f(x,y,z) = c$:

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

- Equation of the tangent plane:



$P_0\,(x_0, y_0, z_0)$
$P\,(x, y, z)$

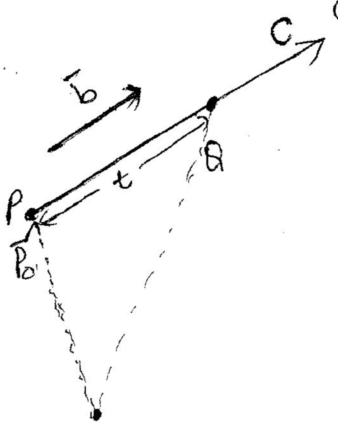
$$\overline{P_0P} \cdot \nabla f(P_0) = 0 \qquad \text{since } \overline{P_0P} \text{ lies in the tangent plane.}$$

$$\Rightarrow \left[(x-x_0)\hat{i} + (y-y_0)\hat{k} + (z-z_0)\bar{k}\right] \cdot \left[\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\bar{k}\right] = 0$$

$$\Rightarrow \boxed{(x-x_0)\frac{\partial f}{\partial x}(P_0) + (y-y_0)\frac{\partial f}{\partial y}(P_0) + (z-z_0)\frac{\partial f}{\partial z}(P_0) = 0}$$

Directional Derivative: Generalization of the notion of partial derivatives.

Partial derivative: direction is parallel to one of the coordinate axes.



$$D_b f = \frac{df}{dt} = \lim_{t \to 0} \frac{f(Q) - f(P)}{t}$$

If the limit exists, it is called the directional derivative of f

at the point P in the direction of b. The position vector of the line C is: $r(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k} = P_0 + t\bar{b}$.

Using chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right)\cdot\left(\frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k\right)$$

$$= \nabla f \cdot \frac{d\bar{r}}{dt}$$

$$= \nabla f \cdot \bar{b}$$

$$\boxed{\text{directional derivative } D_b f = \nabla f \cdot \bar{b}}$$

At any point P, the directional derivative of f represents the rate of change in f along $\bar{b}$ at the point P.

Remark: $\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$

Note that component of ∇f in the i direction $= \frac{\nabla f \cdot i}{|i|}$

rate of change of f in x direction $= \frac{\partial f}{\partial x}$.

In general, the component of ∇f in any direction is the rate of change of f in that direction. This leads to the formula $\nabla f \cdot \bar{b}$: $\bar{b}$ unit vector.

Maximum rate of change of a scalar field:

Note that

Rate of change of f in the direction of $\bar{b}$

$$= D_b f = \nabla f \cdot \bar{b} = |\nabla f|\,|\bar{b}| \cos\theta$$

$$= |\nabla f| \cos\theta .$$

$\Rightarrow -|\nabla f| \leq D_b f \leq |\nabla f|$ since $-1 \leq \cos\theta \leq 1$

$\Rightarrow$ Rate of change is maximum when θ is 0, that is, in the direction of ∇f.

$\Rightarrow$ Rate of change is minimum when θ is π, that is, in the opposite direction of ∇f.

$\Rightarrow$ Gradient vector ∇f points in the direction in which f increases most rapidly and $-\nabla f$ points in the direction in which f decreases most rapidly.

Example: Find the unit normal to the surface $x^2+y^2-z=0$ at the point $(1,1,2)$.

Sol: Define $f = x^2+y^2-z = 0 \Rightarrow \nabla f = (2x, 2y, -1)$

$$\nabla f(1,1,2) = (2, 2, -1).$$

Unit normal vector $\hat{n} = \frac{1}{\sqrt{4+4+1}}(2, 2, -1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$. Ans

The other one is $-\hat{n} = (-2/3, -2/3, +1/3)$.

Example: Find the directional derivative of the scalar field $f = 2x + y + z^2$ in the direction of the vector $(1,1,1)$ and evaluate this at the origin.

Sol:
$$\nabla f = (2, 1, 2z)$$

$$D_{(1,1,1)} f = \nabla f \cdot (1, 1, 1) \frac{1}{\sqrt{3}}$$

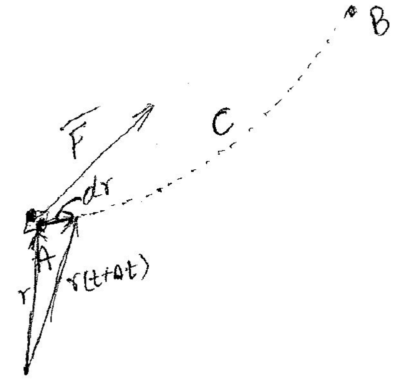
$$= (2, 1, 2z) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2z}{\sqrt{3}}\right)$$

At the origin: $Df = \sqrt{3}$. Ans.

Conservative vector field: A vector field $\bar{V}$ is said to be conservative if the vector function can be written as the gradient of a scalar function f, that is, $\bar{V} = \nabla f$. The function f is called a potential function or a potential of $\bar{V}$.

In such a vector field the work done in moving a particle from a point P to a point Q depends on the points P and Q and is independent of path along which the particle is displaced from P to Q.



Suppose $F = \nabla f$

Total work done displacing the particle from A to B...

$$W = \int_C \bar{F} \cdot d\bar{r} = \int_C \nabla f \cdot d\bar{r}$$

Recall from the Directional derivative $df = \nabla f \cdot d\bar{r}$

$$\Rightarrow W = \int_C df = f\Big|_a^b = f(b) - f(a).$$

$\Rightarrow$ F is conservative.

Example: Show that the vector field $\bar{F} = (2x+y, x, 2z)$ is conservative.

Sol: $\bar{F}$ is conservative if it can be written as $\bar{F} = \nabla\varphi$.

$$\Rightarrow \frac{\partial \varphi}{\partial x} = 2x+y \quad (1) \qquad \frac{\partial \varphi}{\partial y} = x \quad (2) \qquad \frac{\partial \varphi}{\partial z} = 2z \quad (3)$$

(1) $\Rightarrow$ $\varphi = x^2 + xy + h(y,z)$.

(2) $\Rightarrow$ $x + \frac{\partial h}{\partial y} = x \Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow$ h is independent of y.

$h = h(z)$

(3) $0 + \frac{dh}{dz} = 2z \Rightarrow h = z^2 + c$

$$\Rightarrow \varphi = x^2 + xy + z^2 + c.$$

Ans

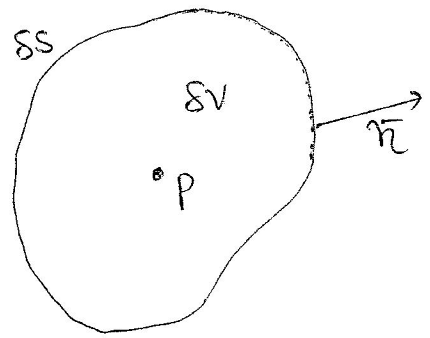
Divergence of a vector field:

The divergence of a vector field is defined as

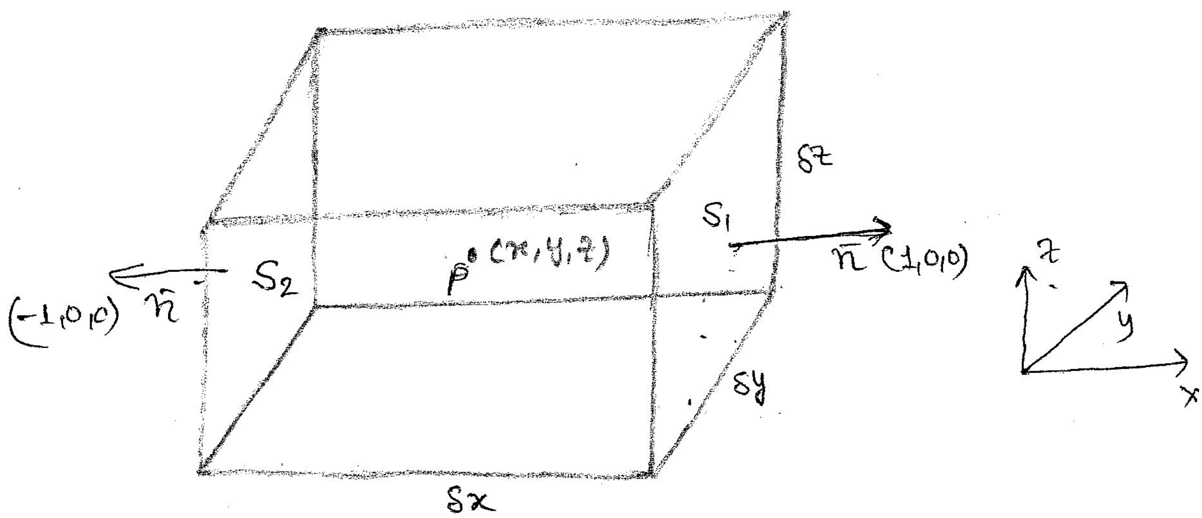
$$\operatorname{div} \bar{V} = \lim_{\delta V \to 0} \frac{1}{\delta V} \iint_{\delta S} \bar{V} \cdot \bar{n} \, ds .$$

Flux of the vector field out of a small closed surface

where δV is a small volume enclosing P with surface δS and $\bar{n}$ is the outward pointing normal to δS.



Divergence interms of components:



$$\iint_{S_1} \bar{u} \cdot \bar{n} \, ds \simeq u_1\left(x + \frac{\delta x}{2}, y, z\right) \delta y \, \delta z .$$

$$\iint_{S_2} \bar{u} \cdot \bar{n} \, ds \simeq -u_1\left(x - \frac{\delta x}{2}, y, z\right) \delta y \, \delta z .$$

Adding the contribution from these two sides we get

$$\iint_{S_1+S_2} \bar{u}\cdot\bar{n}\, ds \simeq \left(u_1(x+\tfrac{\delta x}{2}, y, z) - u_1(x-\tfrac{\delta x}{2}, y, z)\right)\delta y\, \delta z$$

$$\simeq \frac{\partial u_1}{\partial x}\,\delta x\,\delta y\,\delta z$$

$$\simeq \frac{\partial u_1}{\partial x}\,\delta V$$

Similarly from others sides.

$$\iint_{S_3+S_4} \bar{u}\cdot\bar{n}\, ds \simeq \frac{\partial u_2}{\partial y}\,\delta V$$

& $$\iint_{S_5+S_6} \bar{u}\cdot\bar{n}\, ds \simeq \frac{\partial u_3}{\partial z}\,\delta V.$$

Therefore

$$\boxed{\operatorname{div} \bar{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}}$$

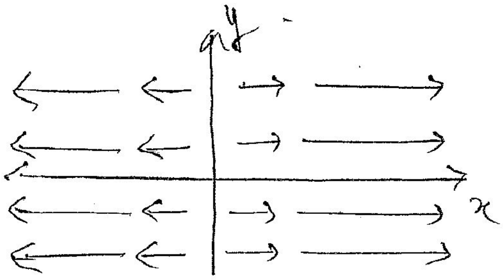
It can also be written as:

$$\boxed{\operatorname{div} \bar{u} = \bar{\nabla}\cdot\bar{u}}$$

Physical Interpretation :

Divergence can be interpreted as the rate of expansion or compression of the vector field.

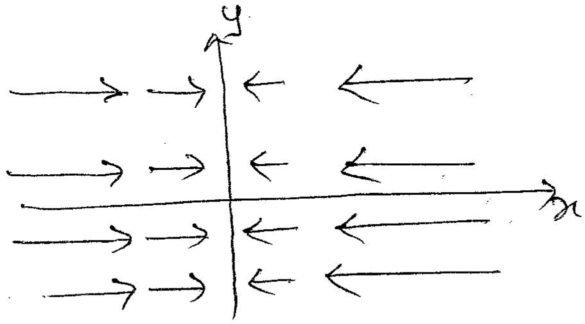
Examples: (1) $\bar{u} = (x, 0, 0)$



Tendency of the fluid is expansion.

$\operatorname{div} \bar{u} = \nabla \cdot \bar{u} = 1.$ (positive)

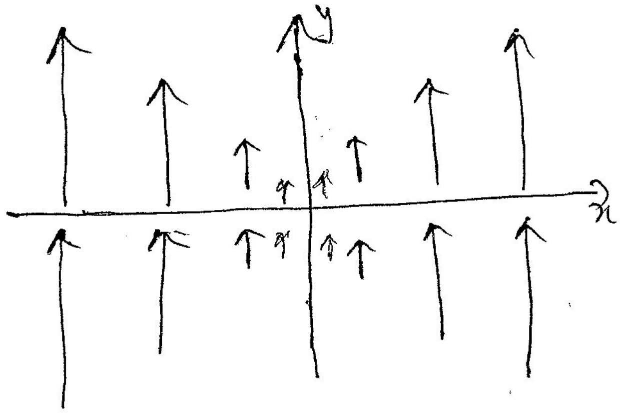
(2) $\bar{u} = (-x, 0, 0)$



Tendency of the fluid is compression.

$\operatorname{div} \bar{u} = -1.$ (negative)

(3) $\bar{u} = (0, x, 0)$



neither expanding nor contracting.

$\operatorname{div} \bar{u} = 0.$

<u>Note:</u> A vector field $\bar{V}$ for which $\nabla \cdot \bar{V} = 0$ everywhere is said to be <u>solenoidal</u>. The relation $\operatorname{div} \bar{V} = 0$ is also known as the condition of incompressibility.

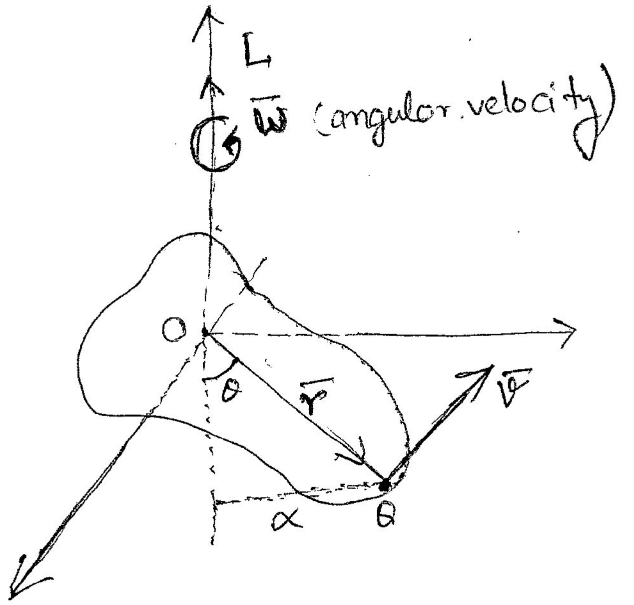
Curl of a Vector field:

Curl of a differentiable vector function $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ is given by

$$\text{Curl } \bar{v} = \nabla \times \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\bar{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\bar{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}.$$

Physical Interpretation:



$$\alpha = |\bar{r}| \sin\theta$$

$$|\bar{v}| = |\bar{\omega}|\alpha = |\bar{\omega}||\bar{r}| \sin\theta = |\bar{\omega}||\bar{r}|\sin\theta$$

$$\Rightarrow \quad \bar{v} = \bar{\omega} \times \bar{r}$$

Also note that $\bar{\omega} = |\bar{\omega}|\bar{k}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Thus: $\bar{v} = \bar{\omega} \times \bar{r} = -|\bar{\omega}|y\,\bar{i} + |\omega|x\,\bar{j}$

Also $\text{Curl } \bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -|\omega|y & |\omega|x & 0 \end{vmatrix} = 2|\omega|\bar{k} = 2\bar{\omega}$

Curl is directed along the axis of rotation with magnitude twice the angular speed.

Example: Given in the discussion of divergence:

(i) $\bar{u} = (x, 0, 0)$

$$\nabla \times \bar{u} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} = i(0) + j(0) + k(0) = 0.$$

No sense of rotation.

(ii) $\bar{u} = (-x, 0, 0)$ Again there is no sense of rotation.

(iii) $\bar{u} = (0, x, 0)$

$$\nabla \times \bar{u} = \begin{vmatrix} i & j & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = i(0) + j(0) + \bar{k}(1)$$

$$= \bar{k}$$

Since any point the velocity component in the y direction to the right of the particle is greater than that of the left, the rotation is about an axis in the z direction.

Note: A vector field $\bar{u}$ for which $\nabla \times \bar{u} = 0$ everywhere is said to be irrotational.

Curl and Conservative vector field: Suppose $\bar{u}$ is conservative.

that is, $\bar{u} = \nabla \varphi. = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right)$

$$\nabla \times \bar{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} = \bar{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial y}\right)\right] + \bar{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial z}\right)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right)\right] = \bar{0}.$$

Any vector field that can be written as the gradient of a scalar field is irrotational.