Linear Algebra

Lecture 7

Subspaces: (Intersection / Union / sums)

Let V be a vector space over a field $\mathbb{F}$ and let U & W be two subspaces of V.

<u>Definition</u>: The sum of subspaces U & W denoted as $U+W$ is defined as

$$U+W = \{v \in V \mid v = u+w \text{ for } u \in U \text{ \& } w \in W\}$$

<u>Theorem</u>: Let V be vector space with U & W its subspaces. Then

i) $U+W$ and $U \cap W$ are subspaces of V.

ii) If V is finite dimensional, and let $l = \dim(U \cap W)$, $p = \dim(U)$, $q = \dim(W)$, then there exists a basis $\{v_1, \ldots, v_m\}$ of $U+W$ such that $\{v_1, \ldots, v_l\}$ is a basis of $U \cap W$, $\{v_1, \ldots, v_p\}$ is a basis in U & $\{v_1, \ldots, v_l, v_{p+1}, \ldots, v_{p+q-l}\}$ is basis in W.

Proof:

i) Let $u_1, u_2 \in U$ and $w_1, w_2 \in W$.

Then for scalars a & b

$$a(u_1+w_1) + b(u_2+w_2)$$
$$= (au_1+bu_2) + (aw_1+bw_2)$$
$$\in U+W$$

$\Rightarrow$ $U+W$ is a subspace.

ii) Let $\{v_1, \cdots, v_e\}$ be a basis in $U \cap W$.

Complete this linearly independent to a basis in U & W respectively.

Let $\{v_1, \cdots, v_e, v_{e+1}, \cdots, v_p\}$ be a completed basis in U.

Let $\{v_1, \cdots, v_e, v_{p+1}, \cdots, v_{p+q-e}\}$ be a completed basis in W.

Observe for any $u \in U$, $w \in W$

$$u+w \in \text{span}\{v_1, v_2, \cdots, v_{p+q-e}\}$$

$$\Rightarrow \quad U+W \subseteq \text{Span}\{v_1, \cdots, v_{p+q-e}\}$$ ①

$\divideontimes$

To show that $v_1, v_2, \dots, v_{p+q-l}$ are linearly independent.

Consider

$$a_1 v_1 + a_2 v_2 + \dots + a_{p+q-l} v_{p+q-l} = 0$$

So let $u = a_1 v_1 + \dots + a_p v_p$

$$= -a_{p+1} v_{p+1} \dots - a_{p+q-l} v_{p+q-l}$$

$$= w$$

$\Rightarrow \quad w \in U \cap W$

$$w = b_1 v_1 + \dots + b_l v_l$$

You can check $a_1, \dots, a_{p+q-l} = 0$

This means $\{v_1, \dots, v_{\underline{p+q-l}}\}$ is a basis of $U+W$. $\blacksquare$

$\divideontimes$

$$\Downarrow$$

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Definition: The subspace $X = U+W$ of V is called a direct sum of U & W if any vector $v \in U+W$ has a unique representation of the form $v = u+w$ for $u \in U$, $w \in W$.

Equivalently, $v = u_1 + w_1 = u_2 + w_2$

$$\Rightarrow u_1 = u_2 \ \& \ w_1 = w_2$$

A direct sum of U & W is denoted $U \oplus W$.

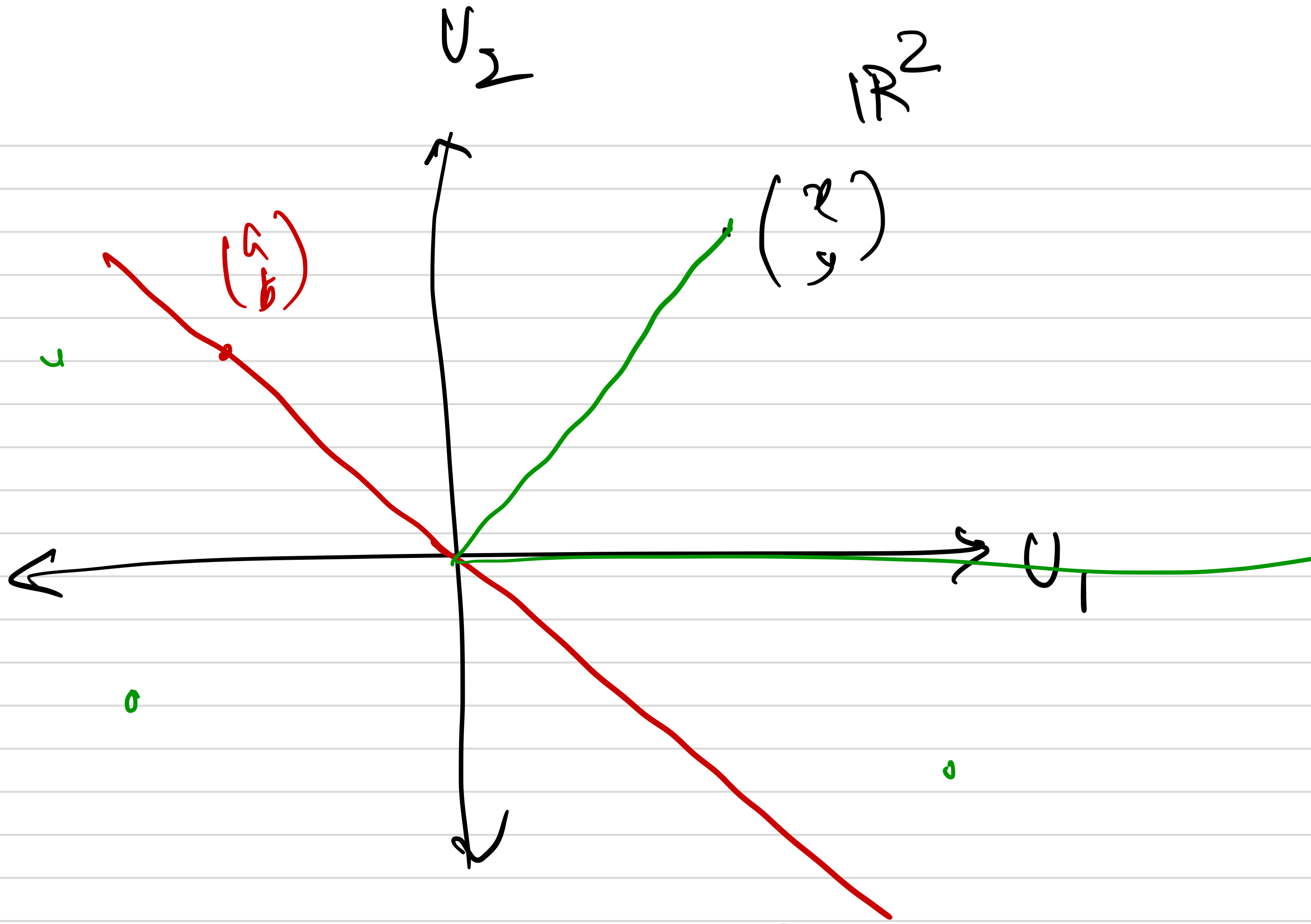
Here U is said to be a complement of W in $U \oplus W$.

Example: Can we say

$$\mathbb{R}^2 = U_1 \oplus U_2$$

where $U_1 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$

$U_2 = \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} : x \in \mathbb{R} \right\}$



$$\underbrace{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}}_{\in U_1} + \underbrace{\begin{pmatrix} 0 \\ y \end{pmatrix}}_{\in U_2} \quad : \text{unique}$$

$$\Rightarrow \mathbb{R}^2 = U_1 \oplus U_2$$

U_2 is a complement of U_1 in $\mathbb{R}^2$.

Note: $U_3 = \left\{ t \begin{pmatrix} a \\ b \end{pmatrix} : \text{for } t \in \mathbb{R} \right\}$ $\quad \begin{pmatrix} a \\ b \end{pmatrix} \notin U_1$

$$\boxed{\mathbb{R}^2 = U_1 \oplus U_2}$$

Lemma: For two finite dimensional subspaces U & $W \subseteq V$, the following statements are equivalent.

1) $U+W = U \oplus W$

2) $U \cap W = \{0\}$

3) $\dim(U \cap W) = 0$

4) $\dim(U+W) = \dim(U) + \dim(W)$

5) For any bases $\{u_1, \ldots, u_p\}$ of U & $\{w_1, \ldots, w_q\}$ of W, the set $\{u_1, \ldots, u_p, w_1, \ldots, w_q\}$ is a basis of $U+W$.

Linear Transformations.

Definition: Let V and W be vector spaces over a field $\mathbb{F}$. We call a function $T: V \to W$ a linear transformation from V to W if for all $x, y \in V$ and $\alpha \in \mathbb{F}$, we have

(a) $T(x+y) = T(x) + T(y)$

(b) $T(\alpha x) = \alpha T(x)$

Examples: Let $\mathbb{R}$ be a vector space over $\mathbb{R}$.

Then $T(x) = cx$ for any $c \in \mathbb{R}$ is a linear transformation.

However $T(x) = x^2$ is not a linear transformation

Is $T(x) = c_1 x + c_2$ for $c_1, c_2 \in \mathbb{R}$??

Is $T(x) = |x|$??

Proposition: Let V and W be vector spaces over a field $\mathbb{F}$ and let $T: V \to W$ be a linear transformation.

Then

1) $T(0) = 0$
2) $T(cx + y) = cT(x) + T(y) \quad \forall c \in \mathbb{F}$ and $x, y \in V$
3) $T(x - y) = T(x) - T(y) \quad \forall x, y \in V.$
4) For $x_1, x_2, \ldots, x_n \in V$ and $c_1, c_2, \ldots, c_n \in \mathbb{F}$

$$T\left(\sum_{i=1}^{n} c_i x_i\right) = \sum_{i=1}^{n} c_i T(x_i)$$

Example: $V = W = \mathbb{R}^2$, over $\mathbb{R}$.

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + 3 \\ 3x_1 + 4x_2 \end{pmatrix}$$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$

$c \in \mathbb{R}$

$$cx + y = \begin{pmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{pmatrix}$$

To check whether

$$T(cx+y) = cT(x) + T(y)$$

<u>Example:</u> $V = W = \mathbb{R}^2$ over $\mathbb{R}$

$T : V \to W$

i) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ projection.

ii) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$

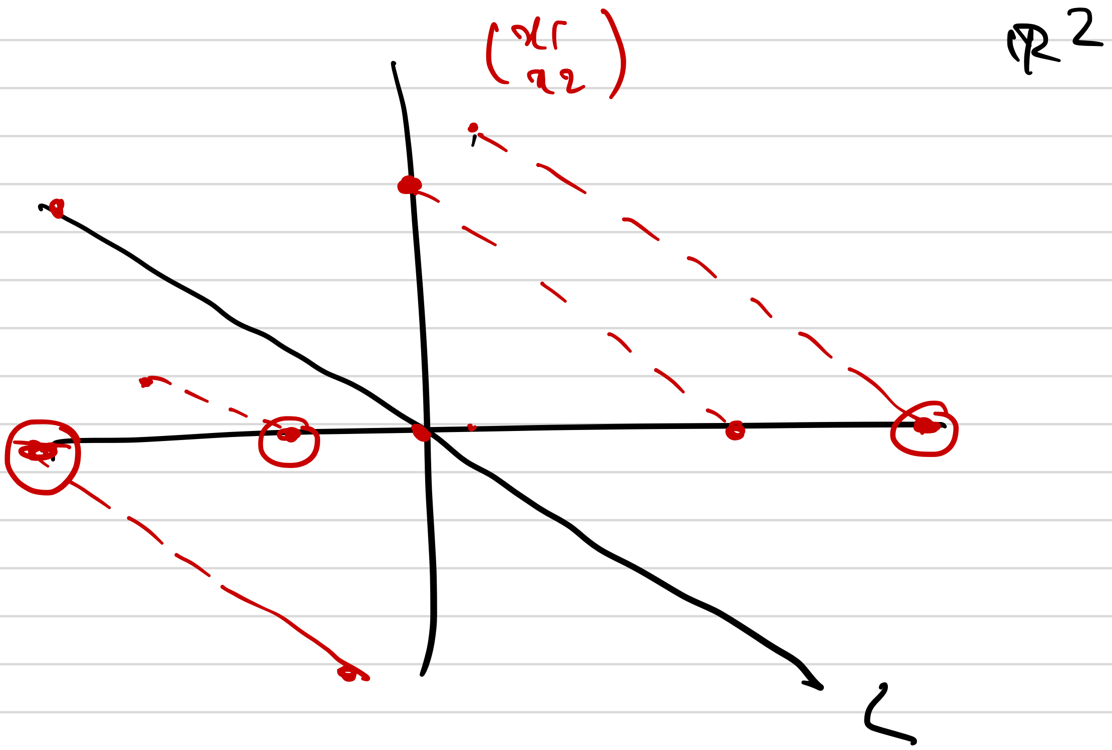
iii) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$ for $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

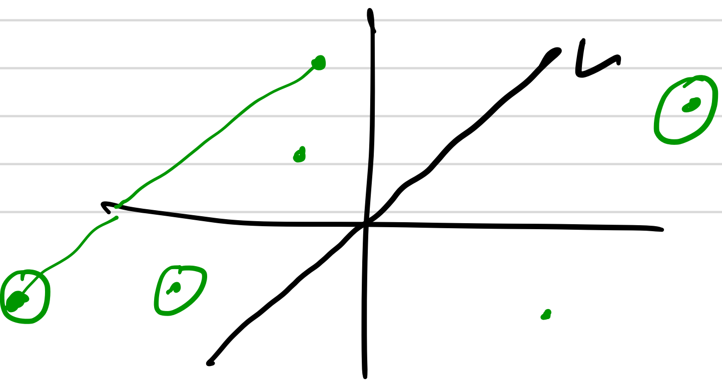
Let $x, y \in \mathbb{R}^2$

$$T(cx+y) = A(cx+y)$$
$$= cAx + Ay$$
$$= cT(x) + T(y)$$

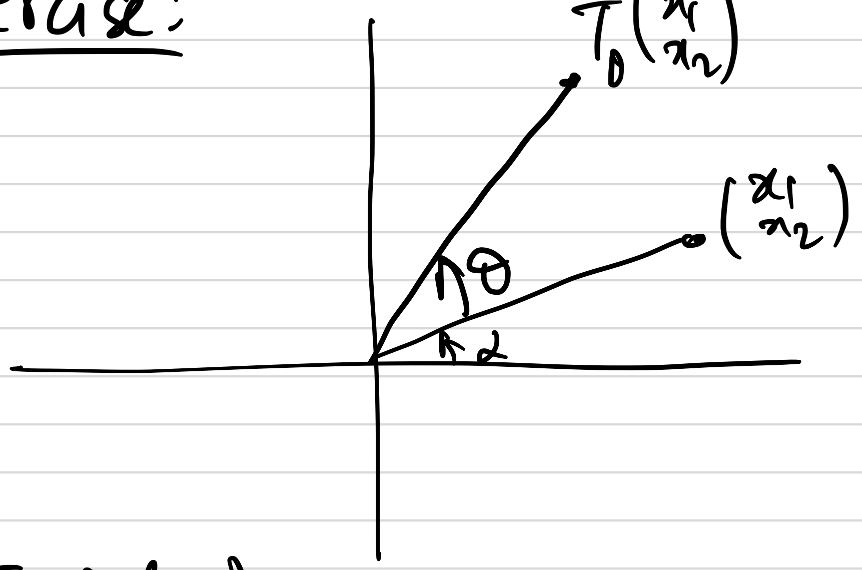
Exercise: Projection along the Line L.



Exercise: Reflection of every vector thru' X-axis along L.



Exercise:



$T_\theta\begin{pmatrix}x_1\\x_2\end{pmatrix} \longrightarrow$ rotates vector $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ by angle θ in anti-clockwise direction.

Fix a point $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ in $\mathbb{R}^2$.

$$r = \sqrt{x_1^2+x_2^2}$$

$$x_1 = r\cos\alpha$$
$$x_2 = r\sin\alpha$$

$T_\theta\begin{pmatrix}x_1\\x_2\end{pmatrix}$ has same magnitude r and makes an angle $(\alpha+\theta)$ with positive X-axis.

$$T_\theta\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix}$$

$$= \begin{pmatrix} r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_2\cos\theta + x_1\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

<u>Example:</u> Define $T: M_{n\times n}(\mathbb{R}) \to \mathbb{R}$

i) $T(A) = \operatorname{trace}(A)$ ✓

ii) $T(A) = \det(A)$ ✗✗

<u>Example:</u> Let $V = W = P_n(\mathbb{R})$ over $\mathbb{R}$

$$T(P(x)) = \frac{d}{dx} P(x)$$