Theorem:

If S is ∞ set, then $\aleph_0 \le |S|$.

theorem: let S be any set then

$$|S| < |P(S)|$$

In case of ∞ sets.

need to map injective mapping & prove there cannot be any bijecⁿ.

01/02/18

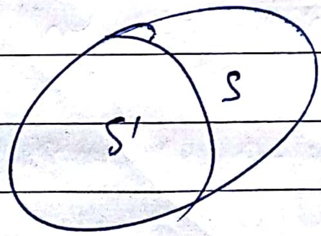
$\aleph_0 = N$

$c = |[0,1]| = |R| = |R^2| = \dots |R^n|$

S is a finite set then $|S| < \aleph_0 < c$

theorem: If S is an ∞ set then $\aleph_0 < |S|$.

proof: If S is ∞, then S contains a countable ∞ subset S'.



define a Mapping

$f : S' \to S$

$f(s') = s'$.

$|f(S')| = |S'| \le |S$

I manage to construct an injective mapping from S' to S.

Also S' can be put on a one-one [illegible] with set of Natural No.

Theorem: Let S be any set. Then $|P(S)| > |S|$!

Proof: Consider injective mapping $f : S \to P(S)$ defined as $f(a) = \{a\}$

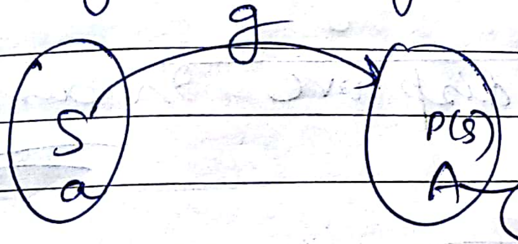
$\Rightarrow |S| \leq |P(S)|$.

Claim its strictly less $= |P(S)| \neq |S|$ never be equal

i.e no bijec^n from S to $P(S)$

proof by contradiction.

Let 'g' be any arbitrary func^n from S to $P(S)$



Idea is to make sure there always exist a $A \in P(S)$ such that $\nexists$ no 'a' $\in S$ satisfying $g(a) = A$

For any $x \in S$, $g(x) \in P(S) \Rightarrow g(x) \subseteq S$

x — may / may not be in $g(x)$

Define a subset A of S follows:— construct a set A such that it has no preimage $a \in S$ $g(a) = A$ under $g(x)$

$$A = \{x \in S \mid x \notin g(x)\}$$

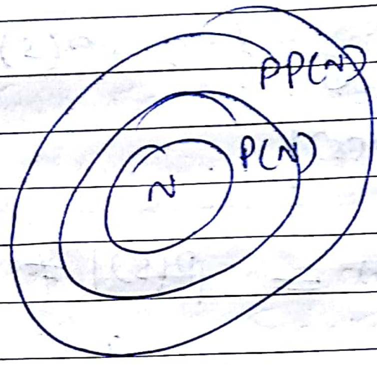
which imply for no $a \in A$ $\quad a \notin g(a) = A$.

Date ___ / ___ / ___

(•) $C: |R| < |P(R)| < |P(P(R))| \ldots$

$\aleph_0: |N| < |\underbrace{P(N)}_{\aleph_1}| < |\underbrace{P(P(N))}_{\aleph_2}| \ldots$

uncountable ??



Continum hypothesis

$\nexists$ no set 'S' s.t $\aleph_0 < |S| < C$.

cannot be proved / disprove in our system of logic

Algebraic No:

A real no. root of poly. eqn with rational coeff.

Transcedental : SEE

Ex: Show that ~~set~~ of all algebraic No = countable

$\mathbb{Z}$ Countable

Theorem A: let P_n : set of All polynomial fun^n of deg 'n' with $\mathbb{Z}$ co-eff. defined By

$f(n) = a_0 x^n + a_1 x^{n-1} \ldots + a_n$ $a_0 \neq 0$

$a_i \in \mathbb{Z}$.

now if we can show P_n — countable, we are done since roots are $n \times (\ \ )$ at max.

As Q_n - countable, we have enumeration of Q_n
$\langle f_{n_1}, f_{n_2} \ldots \rangle$

let A_{nk}: set of roots of $f_{nk} = 0$ — Countable.

How to prove Theorem A now?

proof By Induction:

- Base $(n=0)$ — a_n — all set of Natural No.

- Induc^n step. let p_k — countable

For each $\mathbb{Z}$ m, let

$$S_m = \begin{cases} f : f = m x^{k+1} + g & g \in P_k \} \\ f : f = -m x^{k+1} + g & g \in P_k \} \end{cases}$$

S_{-m}

Page No.

Date ___/___/___

$\{S_m \cup S_{-m}\} = T_m$.

$P_{R+1} = \left(\bigcup_{m=1}^{\infty} T_m\right)$

S_m & S_{-m} Countable → one - one correspondence possible.

Well ordering Principle:
every non-empty subset of Natural no has a least element.

~~ex: N~~

Archimedian prop. of S.

For any $x, y \in S$ with $x > 0$ ∃ a (+)ve Z 'n' s.t $nx > y$.

ex: N is archemedian

i.e to prove that $\forall x, y \in N$, ∃ $n \in N$ st $nx > y$

soln: If possible N not be Archemedian,

∃ some x, y for which $nx \leq y$ $\forall n \in N$.

$S = \{ y - nx \mid n\ (+)ve\ Z \}$

so, $S \neq \phi$.

Date ___ / ___ / ____

- S is a non-empty subset of N.
- By Well-ordering principle, S must have a smallest element, say $y = mx$, $m \in N$ //

Consider $y-(m+1)x < y-mx$.

then $y-(m+1)x \in S$, but it is smaller than the smallest element of S.

now, we get smaller than smallest.
hence ($\longrightarrow$ $\longleftarrow$).

Example :

Q : countable $\Rightarrow$ ~~R : Uncountable~~

Q = dense = R $\Leftarrow$ B/w any 2 Q/R $\exists$ ∞ Q/R.

Q = Archimedian = R //

Lets. Prove.

Q = Archimedian.

ie

to proove that if $a>0$, $a \in Q$, $b \in Q$ any rational no then $\exists$ a (+ve) Z n s.t $na > b$.

soln:

Trivial Case : If $b < 0$ Since $a \in Q$ $a > b$ always. $n=1$

$0 < b < a$ for $n=1$ holds

Date ___ / ___ / ___

Case 2: $b>0$ $b>a$

~~lets use well ordering principle to proove this;~~

claim : $\exists$ a (+)ve $\mathbb{Z}$ 'n' s.t $na>b$

(by contradiction)

If possible, let claim is false

i.e $na \leq b$ $\forall n \in \mathbb{N}$.

$(na)b^{-1} \leq 1 <$ any (+)ve int. m other than 1

$(na)b^{-1} \leq 1 \leq m$ $m \in \mathbb{N}$

$n m^{-1} \leq b a^{-1}$

any arbitary Rational no $\leq$ fixed rational no.

not possible. ($\rightarrow \leftarrow$)

$\underline{na > b}$ //.

hence $\mathbb{Q}$ Archimedian

Date __/__/__

prove that $n < 2^n \ \forall n \geq 1$ using well ordering principle.

sol^n say A: set of natural no for which the inequality does not hold.

$= \{ n \in N \mid n \geq 2^n \}$

∴ claim $A = \phi$ // If we prove it we are done //.

say if not $A \neq \phi$

So, $A \neq \phi \quad A \subseteq N$.

- By well ordering principle, A must have smallest element, say 'm'.

$m > 2^m$

then $m \neq 1$ $\quad m \cdot 1 \ngtr 2^{(m-1)} \quad \frac{m}{2} > 2^{m-1}$

m must be greater than 1.

$m > 1 \ \| \ m \geq 2$. $\quad m \geq 2^m$

m - least element of A

Also

$(m-1) \geq \frac{m}{2} \geq 2^{m-1}$. we get $(m-1)$ //

$\frac{m}{2}$

hence now $m-1$ be the smallest

hence contradicted.

Date ___ / ___ / ___

Theorem: First Principle of finite Induction:

let S be a set of (+)ve Z with following properties:
(a) the Z 1 ∈ S
(b) whenever Z K ∈ S, the next Z K+1 ∈ S (Induction step)
hence S = N

essentially the set S = set of all Natural Numbers.

2nd Principle (stronger
- upto K belong to S.

Theorem: (2nd principle of finite Induction)

b - (b') ⇒ If K is a (+)ve Z such that 1, 2, ... K ∈ S, then K+1 ∈ S.

Proving Mathematical Induction by well ordering principle
Theorem 1: by contradiction.

let $T \neq \phi$ be set of all (+)ve Z not in S

i.e. $T = N - S$

T - non empty set, so by well-ordering principle T - has min. element say a.

Page No.

Scanned by CamScanner

Date

Since $1 \in S$, Also $a \neq 2$,

$a \neq 1$

$a-1 \geq 0$.

$a-1 < a$

'a' least element of T

a-1 smaller than a.

S, a-1 ∉ T.

a-1 ∈ S.

then by cond n 'b' a ∈ S.

contradiction //

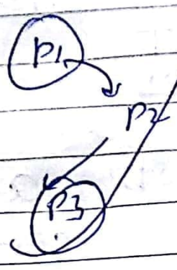
Ex: In a Round Robbin tournament every player plays every player plays every other player exactly once and each match has a winner & a looser. We say that the player $p_1, p_2 \ldots p_m$ form a cycle if p_1 beats p_2, p_2 beats p_3 ... p_m beats. Use Well Ordering principle to show if there is a cycle of length 'm' ($m \geq 3$) among the players in a round-robin tournament then there must be cycle of 3 players.

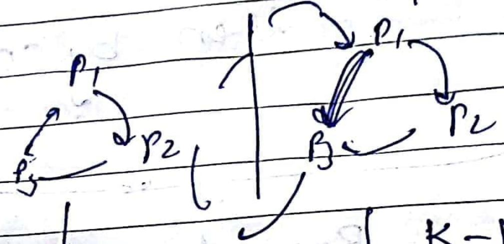
so by well - ordering principle S must have smallest element, say k.

let p_i $i = 1$ to k be a cycle.

$k \geq 3$.

Since every player plays either p_1 wins p_3
p_1 loss p_3





cycle 3

k-1 cycle length contradiction hence

Cycle of 3 exists.

feb 18

Theorem : If x and y are 2 real no's s.t $x<y$, then $\exists$ a rational no. r where $x<r<y$

Proof :

I. Suppose $x>0$, $0<x<y$
then by archedian property of real no.
$x<y$ $\exists$ $nx>y$

$\exists$ a (+)ve int 'n' s.t

$$n(y-x) > 1$$

$$\boxed{y-x > \frac{1}{n}} \text{ — (1).}$$

Let $A = \{m \in N \mid \frac{m}{n} > x\} \subseteq N$

$A \neq \phi$ as $\frac{1}{n}, x \in R$ $\frac{1}{n} > 0$ & by archimedean property $\exists$ a (+)ve z m. s.t

$$m\left(\frac{1}{n}\right) > x$$

so, by well ordering principle, A must have a smallest element, say $p > 1$

As $p \in A$, we must have $\frac{p}{n} > x$.

$p-1 \notin A$ $\because$ p - smallest.

$$\Rightarrow \frac{(p-1)}{n} \leq x \leq p/n < x + 1/n \quad y$$

$$\frac{p}{n} \leq x + \frac{1}{n} \leq y.$$

$$\frac{p}{n}$$

$$x < \frac{p}{n} < y$$

Case III ✓

$x < 0 < y$,

$y > 0$ $1 \in R$

$\underline{n \cdot y > 1}$. $y > \frac{1}{n} > 0$

<u>Excercise</u> :

For any (+)ve real no. x, ∃ a (+)ve Z n st

$\boxed{n-1 \le x < n}$

<u>Theorem</u> :

Date ___/___/___

Saathi

$\Rightarrow A \subseteq R$, bounded above by y.

A must contain Sup. A — least upper Bound.

let $M = \text{Sup} A \in A$.

$x > 0 \Rightarrow M - x < M$
$\Rightarrow M - x$ is not an upper bound of A.
$\Rightarrow M - x > y$.

$\exists$ some element say

$mx \in A$ s.t

$mx \leq M - x$

$mx > M - x$

$(m+1)x > M$ (— e —).

Example. (# of subset of a finite set).
Use M.I.

S: finite : 'n' elements :
S — 2^n subsets.

Soln :

let $P(n)$ — proposn that a set of 'n element' has 2^n subsets

Date

Base : P(0) true as a set of 0 elements, the empty set has exactly $2^0 = 1$ subset, namely empty set itself.

Induc^n step :

let P(k) be true

claim : P(k+1) - true

let T - set of cardinality k+1

$T = S \cup \{a\}$ $\quad S = T \setminus \{a\}$

$|S| = k$.

For each subset X of S, there are exactly 2 subsets of T.

namely

X & $X \cup \{a\}$

$S - 2^k$

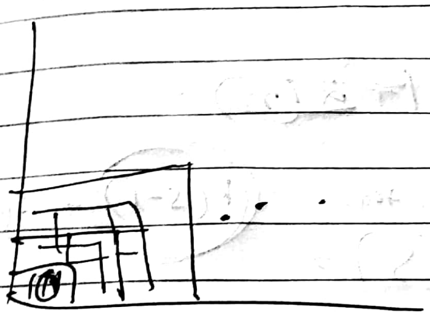
$T = 2 \cdot 2^k = 2^{k+1}$ subsets

Scanned by CamScanner

$1+3+4+5 \ldots \; 4^2$

$\vdots$

$1+\ldots \; (2n+1) = n^2$



geometric proof

Date / /

Example :
Frame a formula for the following.

(Nicomachus theorem) $1^3 = 1$

$2^3 = 3+5$

$3^3 = 7+9+11$

$4^3 = 13$

1

8 3 5 $2(n-1)$

27 7 9 11

13 15 17 19

$n^3 =$ F $+ 2(n) = S$

n^{th} — n term — n^3

$(n-1)^3$

n

$1+2(n)$

$\sum (n-1)3 + 2$

n^{th}

$F(S) = F(S-1) + 2(n-1)$

Proof

$1^3 + 2^3 + \dots \; n^3 = (1+2+3+\dots n)^3$

geometrical interpretn of formulae

Take a sq. having side of length $[n(n+1)]$

break it into $(n+1)$ subsequence, each with side length 'n'.

$n+1$ sq.

n – length

Date ___ / ___ / ___

Saathi

ex:
$9 \times 1 + 2 = 11$
$9 \times 12 + 3 = 111$
$9 \times 123 + 4 = 1111$
$9 \times 1234 + 5 = 11111$

$$= \frac{1}{9}(10^{n+1} - 1)$$

$9 \times (\cancel{1 + 2 \times 10} + \cancel{1 \times 10}$

$(n+1) + 9 \times (n \times 10^n + (n-1) \times 10^{n-1} \ldots + 1)$

$$= \frac{1}{9}(10^{n+1} - 1).$$

euler's poly

Ex: $f(n) = n^2 + n + 41.$

$f(n)$ generates prime $\forall n \in N,$
$0 \le n \le 100$

Correct or not?

$0 \le n \le 10^6$

$g(n) = n^2 + n + 27941.$

$0 \le n \le 10^6 \longrightarrow$ $g(n)$ produces 286129 prime.