Linear Algebra

Lecture 20

V : finite dimensional vector space over $\mathbb{F}$.

Let $T: V \to V$ be a linear map.

B: ordered basis for V.

$[T]_B$: matrix representation of T w.r.t. B.

Eigenvectors & eigenvalues of T.

$T(v_i) = \lambda_i v_i$ for some $\lambda_i \in F$

and $v_i \neq 0$.

Suppose $\{v_1, \dots, v_n\}$ forms a basis.

<u>Q</u>: Does there exist an ordered basis for V (Γ) such that

$[T]_\Gamma$: a diagonal matrix.

$$[T]_\Gamma = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

where $\Gamma = \{v_1, \dots, v_n\}$ ordered basis for V.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_B"'] & V \arrow[d, "\phi_B"] \\ \mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^n \end{tikzcd}$$

$v \in V$

$\downarrow$

$\phi(v) = [v]_B$

v is an eigenvector of T

$$\Updownarrow$$

$\phi_B(v)$ is an eigenvector of $[T]_B$. $\blacksquare$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\phi_B"'] & V \arrow[d, shift left, "\phi_B"] \\
\mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^n \arrow[u, shift left, "\phi_B^{-1}"]
\end{tikzcd}$$

ϕ_B is an isomorphism (one-one, onto, linear f$^{\underline{n}}$)

Suppose T is diagonalizable.

$\Rightarrow$ $\exists$ an ordered basis $B = \{v_1, \dots, v_n\}$ for V.

where $T(v_i) = \lambda_i v_i \quad i = 1, 2, \dots, n$
for some $\lambda_i \in \mathbb{F}$

$$[T]_B = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Consider the matrix $Q = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix}$
$\in M_{n\times n}(\mathbb{F})$

$$Q^{-1} A Q = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Ex:
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex:
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem: Let T be a linear transformation on a vector space V and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T. If $v_1, v_2, \dots, v_k$ are eigenvectors corresponding to $\lambda_1, \dots, \lambda_k$ resp., then $\{v_1, \dots, v_k\}$ is linearly independent.

Proof: By induction:

$k=1$; λ_1, v_1 clearly $\{v_1\}$ is linearly independent.

Assume $\{v_1, \ldots, v_{k-1}\}$ is a linearly ind. set.

To prove: $\{v_1, \ldots, v_{k-1}, v_k\}$ is linearly independent.

Let $a_1, \ldots, a_k \in \mathbb{F}$ s.t.

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0 \quad \text{—} (*)$$

Apply $(T - \lambda_k I)$ on both sides,

$$(T - \lambda_k I)(a_1 v_1) + \cdots + (T - \lambda_k I)(a_{k-1} v_{k-1})$$

$$+ \underbrace{(T - \lambda_k I)(a_k v_k)}_{0} = 0$$

$\Rightarrow a_1(\lambda_1 - \lambda_k) v_1 + a_2(\lambda_2 - \lambda_k) v_2 + \cdots$

$\cdots + a_{k-1}(\lambda_{k-1} - \lambda_k) v_{k-1} = 0$

$\because \{v_1, \ldots, v_{k-1}\}$ are independent

$$a_i(\lambda_i - \lambda_k) = 0 \quad \text{for } i = 1, 2, \ldots, k-1$$

$\Rightarrow a_i = 0 \quad$ for $i = 1, 2, \ldots, k-1$

since λ_i's are distinct.

$\Rightarrow a_k v_k = 0 \quad \Rightarrow a_k = 0$

$\Rightarrow \{v_1, \ldots, v_k\}$ is linearly indep. ∎

<u>Corollary</u>: Let T be a linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.

[Diagonalizability and factorization & characteristic polynomial.]

<u>Definition</u>: A polynomial $f(t) \in P(\mathbb{F})$ splits over $\mathbb{F}$ if there are scalars $c, a_1, a_2, \ldots, a_n$ (not necessarily distinct) in $\mathbb{F}$ s.t.

$$f(t) = c\,(t-a_1)(t-a_2)\cdots(t-a_n)$$

e.g. $t^2 - 1 = (t-1)(t+1)$

t^2+1 does not split over $\mathbb{R}$

$$\begin{array}{c} t^2+1 \\ \parallel \\ (t-i)(t+i) \end{array}$$

over $\mathbb{C}$.

<u>Theorem</u>: The characteristic polynomial of any diagonalizable linear operator splits.

$$(t^2+1)(t^2-1)$$

<u>Definition</u>: Let λ be an eigenvalue of a linear operator with characteristic polynomial $f(t)$. The algebraic multiplicity of λ is ~~the~~ the largest positive integer k s.t. $(t-\lambda)^k$ is a factor of $f(t)$.

$$(t-\lambda)^k \mid f(t)$$

Ex. 1 $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \longleftrightarrow$

Ex. 2 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

<u>Definition</u>: Let T be a linear operator on a vector space V and let λ be an eigenvalue of T. Define $E_\lambda = \{x \in V : T(x) = \lambda x\}$

$$E_\lambda = \{x \in V : T(x) = \lambda x\} = N(T - \lambda I)$$

The set E_λ is called the eigenspace of T corresponding to eigenvalue λ.

<u>Thm</u>: Let T be a linear operator on a finite dimensional vector space V and let λ be an eigenvalue of T with algebraic multiplicity m. Then

$$1 \le \dim(E_\lambda) \le m$$

<u>Proof</u>: Choose an ordered basis $\{v_1, v_2, \ldots, v_p\}$ of E_λ.

Extend this basis to the ordered basis $\beta = \{v_1, \ldots, v_p, v_{p+1}, \ldots, v_n\}$ of V.

Denote by $A = [T]_\beta$

How does this A look like?!

$$T(v_1) = \lambda v_1$$
$$= \lambda_1 v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$$

$\vdots$

$$T(v_p) = \lambda_p v_p$$
$$= 0 \cdot v_1 + \cdots + \lambda_p v_p + 0 \cdot v_{p+1} + \cdots + 0 \cdot v_n$$

$$A = \left[\begin{array}{ccc|c} \lambda & & 0 & \\ & \ddots & & B \\ 0 & & \lambda_p & \\ \hline & O & & C \end{array}\right]$$

The characteristic polynomial of T

$$f(t) = \det(A - tI_n)$$
$$= \det \left[\begin{array}{c|c} (\lambda - t) I_p & B \\ \hline O & C - tI_{n-p} \end{array}\right]$$
$$= \det\left[(\lambda - t) I_p\right] \times \det\left[C - tI_{n-p}\right]$$

$$f(t) = (\lambda - t)^p g(t)$$

$$\Rightarrow 1 \leq p \leq m$$

$\blacksquare$

$\dim(E_\lambda)$ = dimension of eigenspace corresponding to the eigenvalue λ

= geometric multiplicity of λ.

Lemma: Let T be a linear operator, and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. For each i, let $v_i \in E_{\lambda_i}$.
If $v_1 + v_2 + \cdots + v_k = 0$
then $v_i = 0$ for all i.

Proof: trivial.

Theorem: Let T be a linear operator on a vector space V. Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. For each $i = 1, \ldots, k$ Let S_i be a finite linearly independent subset of E_{λ_i}. Then
$S = S_1 \cup S_2 \cdots \cup S_k$ is linearly independent in V.

Proof: Suppose for each i

$$S_i = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$$

$$S = \{v_{ij} : j=1, \dots, n_i,\ i=1, \dots, k\}$$

Consider the scalars $\{a_{ij}\} \subseteq \mathbb{F}$

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = 0$$

For each i:

$$w_i = \sum_{j=1}^{n_i} a_{ij} v_{ij} \in E_{\lambda_i}$$

$$\Rightarrow \quad w_1 + w_2 + \dots + w_k = 0$$

Theorem:

Theorem: Let T be a linear operator on a finite dimensional vector space V. such that the characteristic polynomial splits. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigenvalues of T. Then

(a) T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$ for every $i = 1, 2, \cdots, k$.

(b) If T is diagonalizable and B_i is an ordered basis for E_{λ_i}, then $\beta = B_1 \cup B_2 \cup \cdots \cup B_k$ is an ordered basis for V consisting of eigenvectors of T.