Date
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Lecture 23

Fourier Transform & its Inverse

writing the exponential f^n in eq^n (4) as a product of exponential f^{ns}, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} \, dv \right]}_{= \hat{f}(\omega)} e^{i\omega x} \, d\omega \longrightarrow (5)$$

the expression in brackets is a f^n of ω, & is denoted by $\hat{f}(\omega)$, & is called the Fourier transform of f, writing $v = x$, we have

$$\boxed{\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \longrightarrow (6)}$$

With this, (5) becomes

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\hat{f}(\omega)}_{\hat{f}(\omega)} e^{i\omega x} \, d\omega \longrightarrow (7).}$$

& is called the Inverse Fourier transform of $\hat{f}(\omega)$.

Another notation is

$$\mathcal{F}(f) = \hat{f}(\omega) \text{ \&}$$

$\mathcal{F}^{-1}$ for the inverse.

the process of obtaining the Fourier transform

$$\mathcal{F}(f) = \hat{f}$$

from a given f is also called the Fourier ~~Transform~~ ~~method~~. or, the Fourier transform method.

Existence of the Fourier Transform (6).

Sufficient (condition) for the existence of the Fourier Transform (6) are the following two cond^ns:

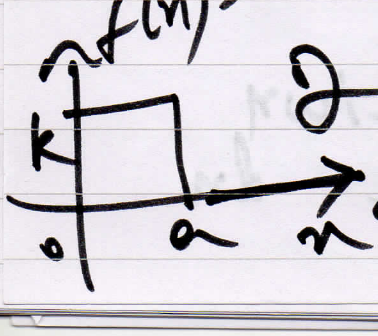
Find out counterexample!

1. $f(x)$ is piecewise-continuous on every finite interval.

2. $f(x)$ is absolutely integrable on the x-axis.

EX1 (Fourier transform)

Q) Find the Fourier transform of $f(x) = k$, if $0 < x < a$ & $f(x) = 0$ otherwise.



Sol^n:- We have,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{a} k\, e^{-i\omega x}\, dx$$

$$= \frac{k}{\sqrt{2\pi}} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_0^a$$

$$= \frac{k}{\sqrt{2\pi}} \left(\frac{e^{-i\omega a} - 1}{(-i\omega)} \right)$$

$$= \boxed{\frac{k\,(1 - e^{-i\omega a})}{i\omega \sqrt{2\pi}}}.$$

Ex/ Find the Fourier transform of $f(x) = 1$, if $|x| < 1$
& $f(x) = 0$, otherwise

Sol^n:- We have,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}.$$

(Find out something more).

Q) Find the Fourier transform of e^{-ax^2}, $a > 0$.

Sol^n:-

$$\mathcal{F}\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2 - i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\sqrt{a}\,x + \frac{i\omega}{2\sqrt{a}}\right)^2 + \left(\frac{i\omega}{2\sqrt{a}}\right)^2\right]}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{\left(-\omega^2/4a\right)} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}\,x + \frac{i\omega}{2\sqrt{a}}\right)^2} dx}_{I}. \quad \left[i^2 = -1.\right.$$

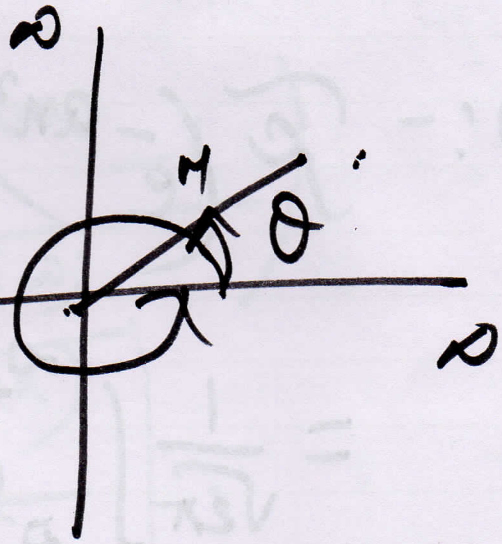
Let $\sqrt{a}\,x + \dfrac{i\omega}{2\sqrt{a}} = v$

$\Rightarrow dx = dv/\sqrt{a}$.

$$\therefore I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2}\, dv$$

(how?)

We now change to polar coordinates



$r = \sqrt{u^2 + v^2}$, θ

since $du\,dv = r\,dr\,d\theta$.

$$I^2 = \frac{1}{a} \int_{-\infty}^{\infty} e^{-u^2}\, du \int_{-\infty}^{\infty} e^{-v^2}\, dv$$

$$= \frac{1}{a} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}\, \underline{du\,dv}$$

$$= \frac{1}{a} \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} \cdot r \, dr \, d\theta$$

$$= \frac{1}{a} \underbrace{\left(\int_{\theta=0}^{2\pi} d\theta \right)}_{(=2\pi)} \cdot \left(\int_{r=0}^{\infty} r \, e^{-r^2} \right)$$

$$= \frac{2\pi}{\cancel{2} a} \cdot \int_0^{\infty} e^{-r^2} \, d(r^2)$$

$$= \frac{\pi}{a} \left[-e^{-r^2} \right]_0^{\infty}$$

$$= \pi/a \left[1 - 0 \right] = (\pi/a)$$

Hence, $I^2 = \pi/a \Rightarrow I = \sqrt{\pi/a}$.

$$\therefore \mathcal{F}\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2\pi}} e^{-(\omega^2/4a)} \cdot \sqrt{\pi/a}$$

$$= \boxed{\frac{1}{\sqrt{2a}} e^{-(\omega^2/4a)}} \; //.$$

Linearity Fourier transform / & Derivatives

EX/ Find the Fourier transform of $f(x) = e^{-ax}$, $x > 0$

& $f(x) = 0$, if $x < 0$.

Sol^n :- $\mathcal{F}(e^{-ax}) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \cdot e^{-i\omega x} \, dx$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right]_{x=0}^{\infty}$

Here, $a > 0$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{(a+i\omega)}$

Th-1/ The Fourier transform is a linear operation, that is, for any f^{ns} $f(x)$ & $g(x)$ whose Fourier transforms exist & any constant a & b,

$$\mathcal{F}(af+bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

This is true because integration is a linear operation, so that (8)

proof:- (6) gives

$$\mathcal{F}\left(a f(x) + b g(x)\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[a f(x) + b g(x)\right] e^{-i\omega x}\, dx.$$

$$= a\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \quad (\text{how?})$$

$$+ b\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\, e^{-i\omega x}\, dx$$

$$= a\cdot\mathcal{F}\{f(x)\} + b\,\mathcal{F}\{g(x)\}$$

Note:- In the application of the Fourier transform to d. eqns, the key property is that differentiation of fns corresponds to multiplication of transform by $i\omega$

Th-2 (Fourier transform of the derivative of $f(x)$)

Let $f(x)$ be continuous on the x-axis & $f(x) \to 0$ as $|x| \to \infty$.

Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\} = i\omega \mathcal{F}\{f(x)\} \quad (9)$$

proof:- From the defn of the Fourier transform, we have

$$\tilde{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{f'(x)}_{2nd\ f^n} \cdot \underbrace{e^{-i\omega x}}_{1^{st}\ f^n} \, dx$$

Integrating by parts, we obtain

$$\tilde{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \left[\underbrace{\left[f(x) \cdot e^{-i\omega x} \right]_{-\infty}^{\infty}}_{(=0)} - (-i\omega) \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} \, dx. \right.$$

$$= \frac{i\omega}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} \, dx}$$

$$= (i\omega) \cdot \tilde{F}\{f(x)\},$$

since $f(x) \to 0$ as $|x| \to \infty$.

Two successive applications of (9) given

$$\mathcal{F}\{f''(x)\} = i\omega\,\mathcal{F}\{f'(x)\}$$

$$= (i\omega)^2\,\mathcal{F}\{f(x)\}$$

$$= -\omega^2\,\mathcal{F}\{f(x)\} \quad (1')$$

s'lly, for higher derivative,

$$\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n\,\mathcal{F}\{f(x)\}$$

Ex/ Find the Fourier transform of $x e^{-x^2}$.

Sol:- Let $f(x) = \left(-\frac{1}{2} e^{-x^2}\right)$

$$f'(x) = x e^{-x^2}$$

$$\therefore \mathcal{F}\{x e^{-x^2}\} = \mathcal{F}\left\{-\frac{1}{2}\left(e^{-x^2}\right)'\right\}$$

$$= -\frac{1}{2}\mathcal{F}\left\{\left(e^{-x^2}\right)'\right\}$$

$$= -\frac{1}{2}\cdot(i\omega)\,\mathcal{F}\left(e^{-x^2}\right).$$

$$= -\frac{1}{2}(i\omega)\cdot\frac{1}{\sqrt{2}}\,e^{(-\omega^2/4)}$$

$$= \frac{-i\omega}{2\sqrt{2}}\,e^{-\omega^2/4}.$$

We know,

$$\mathcal{F}\left(e^{-x^2}\right) = \frac{1}{\sqrt{2}}\,e^{-\omega^2/4}$$

as

$$\mathcal{F}\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2a}}\left(e^{-\omega^2/4a}\right)$$

$(a=1)$

Note: The purpose is the same as in the case of Laplace transforms. The convolution of fns corresponds to the multiplication of their Fourier transforms.

Convolution of Fourier Transforms

The convolution $f*g$ of functions f & g is defined by

$$h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(p)\,g(x-p)\,dp$$

$$= \int_{-\infty}^{\infty} f(x-p)\,g(p)\,dp \longrightarrow (11)$$

Th-3 (Convolution theorem)

Suppose that $f(x)$ & $g(x)$ are piece-wise continuous, bounded & absolutely integrable on the X-axis.

then $$\mathcal{F}(f*g) = \sqrt{2\pi}\,\mathcal{F}(f)\cdot\mathcal{F}(g)$$

proof:- By the def^n $\longrightarrow (12)$

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)(x)\,e^{-i\omega x}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)\,g(x-p)\,e^{-i\omega x}\,dp\,dx$$

[An interchange of the order of integration gives (as f & g are absolutely integrable).

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x-p)\, e^{-i\omega x} \, \underbrace{dx\, dp}$$

Let $x - p = q$ be a new variable

Then $x = p + q$. [p — remains constant.]

$\therefore dx = dq$.

$$\therefore \mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(q)\, e^{-i\omega(p+q)}\, \underline{\underline{dq\, dp}}.$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, e^{-i\omega p}\, dp \right) \cdot \left(\int_{-\infty}^{\infty} g(q)\, e^{-i\omega q}\, dq \right)$$

$$\therefore \boxed{\mathcal{F}(f*g) = \sqrt{2\pi}\cdot\mathcal{F}(f)\cdot\mathcal{F}(g)}$$

We know that (from Advanced calculus) ✓

$$dx\,dp = |J|\,dq\,dp$$

$$= \left|\frac{\partial(x,p)}{\partial(q,p)}\right| dq\,dp$$

Now, $\dfrac{\partial(x,p)}{\partial(q,p)} = \begin{vmatrix} \dfrac{\partial x}{\partial q} & \dfrac{\partial x}{\partial p} \\ \dfrac{\partial p}{\partial q} & \dfrac{\partial p}{\partial p} \end{vmatrix} = \begin{vmatrix} x_q & x_p \\ p_q & p_p \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

$x_q = \dfrac{\partial x}{\partial q}$ etc.

By taking the inverse Fourier transform on both sides of (12), writing

$\hat{f} = \mathcal{F}(f)$ & $\hat{g} = \mathcal{F}(g)$,

as before & noting that

$\sqrt{2\pi}$ & $\frac{1}{\sqrt{2\pi}}$ in (12) & (7)

recall

$$\left[\text{i.e., } f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega \right] \rightarrow (7)$$

cancel each other, we obtain.

$$(f*g)(x) = \frac{1}{\cancel{\sqrt{2\pi}}} \int_{-\infty}^{\infty} \cancel{\sqrt{2\pi}} \cdot \left(\hat{f}\,\hat{g}\right) \cdot e^{i\omega x}\, d\omega$$

$$= \int_{-\infty}^{\infty} \hat{f}(\omega)\, \hat{g}(\omega)\, e^{i\omega x}\, d\omega \rightarrow (13)$$

which will help us in solving p-d-eqn.

Boundary value problems

Fourier series,
Fourier integrals
Fourier transform

In practice.
are probably the most important tools in solving boundary value problems.

Discrete Fourier transform

gives approximations to the Fourier transform of a f^n $f(x)$ from (equally spaced) recorded values of $f(x)$

An economical way of computing the discrete Fourier transform is the so-called Fast Fourier transform

Physical Interpretation :-

(Spectrum)

The nature of representation (7) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies. called a spectral representation

This name is given by optics, where light is such a superposition of colours (i.e. frequencies). In eqn (7), the spectral density $\hat{f}(\omega)$ measures the intensity of $f(x)$ in the frequency interval bet^n ω & $\omega+\Delta\omega$ ($\Delta\omega$ is small, fixed).

H-W
EX

Q. "Find out more about them"

Parseval's Identity for Fourier Transform.

It is used to find the energy of various signals.

Th- (Parseval's for Fourier Transform)

If $f(t)$ has a Fourier transform $\hat{f}(\omega)$ & $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$, then

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega.$$

ie, the integral of the power spectrum ie, $|\hat{f}(\omega)|^2$ equals the integral of the squared modulus of the function.

prm :- $\int_{-\infty}^{\infty} |f(t)|^2\, dt$

(ie, the energies in the frequency & time domains are equal).

In other words, the total energy of a signal can be calculated by summing power-per-sample across time or spectral power across frequency.

$$= \int_{-\infty}^{\infty} f(t)\, \underbrace{f^*(t)}\, dt$$

$$\left[z\bar{z} = |z|^2. \right]$$

where $f^*(t)$ is the complex conjugate of $f(t)$

$$= \int_{-\infty}^{\infty} f(t) \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}^*(\omega) e^{-i\omega t}\, d\omega \right\} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}^*(\omega) \underbrace{\left[\int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \right]} d\omega$$

$$= \int_{-\infty}^{\infty} \hat{f}^*(\omega) \cdot \hat{f}(\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

$\cancel{\hat{f}^*(t)} =$

$$f^*(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}^*(\omega) e^{-i\omega t}\, d\omega$$

-S(1)

$$\therefore \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega .$$

A.M

(Ex)

$$\int_{-\infty}^{\infty} f(t)\, g^{*}(t)\, dt$$

$$= \int_{-\infty}^{\infty} \hat{f}(\omega)\, \hat{g}(\omega)\, d\omega .$$

Let $f(t) = g(t)$

$\triangleright$ prove it

Scaling property:

Th- Let $f(x) \in G(\mathbb{R})$ & $a, b \in \mathbb{R}$, $a \neq 0$

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{-i\omega x} \, dx$$

Let $g(x) = f(ax+b)$. Then

$$\mathcal{F}(g) = \frac{1}{|a|} e^{i\omega(b/a)} \mathcal{F}(f)(\omega/a).$$

Here, $G(\mathbb{R})$ denotes the family of f^n defined on $\mathbb{R}$ with values in $\mathbb{C}$ which are piecewise & absolutely integrable.

Proof :- we have by defn

$$\mathcal{F}\{f(ax+b)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax+b)\, e^{-i\omega x} \cdot dx$$

Let $ax + b = t$

$$\Rightarrow x = (t-b)/a$$

$$\therefore a\,dx = dt$$

$$\Rightarrow dx = dt/a$$

As $x \to -\infty$, $t \to -\infty$,

$x \to \infty$, $t \to \infty$,

If $a > 0$, then

$$\mathcal{F}\{f(ax+b)\} = \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega (t-b)/a}\, dt$$

(Use Time Reversal property)

(how?) If $a < 0$, then

$$\mathcal{F}\{f(ax+b)\} = -\frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega (t-b)/a}\, dt$$

So, combining these results we get

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} e^{i\omega b/a} \underbrace{\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t/a} dt \right\}}$$

$$= \frac{1}{|a|} e^{i\omega b/a} \mathcal{F}\{f(\omega/a)\}$$

EY) — 2 ways of approaching the subject Fourier Transform.

— (1) ————

— (2) independently.

→ as an integral transform.

Defn:-

Let f be a function defined for all $x \in \mathbb{R}$ with values in $\mathbb{C}$. The Fourier transform is a mapping

$$\hat{f} : \mathbb{R} \to \mathbb{C} \text{ defined by}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx.$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega.$$

This is the inverse Fourier Transform.

— X —