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NAME: AKASH MANDAL STD.: Sem 6 SEC.: _____ ROLL NO.: 16MA2037 SUB.: MODERN ALGEBRA

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3.12.18

Evaluation

Midsem 30%
End Sem 50%
Class Test 10%
Surprise Test 10%

Textbook (not just notes)

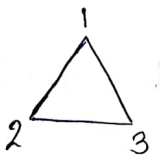
1. Algebra – Artin ✱
2. Abstract Algebra – by Dummit and Foote
3. Algebra – by Herstein
4. Abstract Algebra – Galion

→ Group Theory
  → Subgroups
  → Cyclic
  ⋮
  → Sylow's Theorems
→ Ring Theory.

Intro

Started to study symmetry of objects. (Motivation)

Consider



$\circlearrowleft 120^\circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

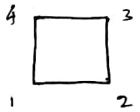
followed by

ϕ

gives

ϕ^2

Also for square



4R
4F

Certain operations.

Dihedral group

∴ Operations on elements give back same elements of the set. 3 rotations and 3 reflections.

Also for any transformation ∃ another transformation which composed with first returs 0° or identity transformation

Symmetries → Transformations to preserve invariance.

Eg

$GL_n(\mathbb{R})$ ≡ The set of all Invertible matrices.

Operation: Matrix multiplication

- Inverse exists.
- AB invertible
- $AI = A$

Eg $SL_n(\mathbb{R})$ ≡ The set of all $n \times n$ matrices such that determinant is 1.

- $|AB| = |A||B| = 1$
- $AI = A$
- A^{-1} exists. $|A| \neq 0$

Eg $O_n(\mathbb{R})$ ≡ The set of all $n \times n$ orthogonal matrices

- AB orthogonal
- I exists
- A^{-1} exists.

Eg $SO_n(\mathbb{R}) = \{ A \in O_n(\mathbb{R}) \mid \det(A) = 1 \}$

Eg $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ under addition

Eg $\mathbb{R}^*, \mathbb{Q}^*, \mathbb{C}^*$: non zero sets under multiplication.

Eg $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$

Definition of Group :-

A group is a set on which a law of composition is defined with certain properties. A law of composition on a set S is a map $F : S \times S \to S$

$f(a,b)$ often denoted by $a+b$ or ab

Examples:

(1) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as $f(a,b) = a + b$

(2) $GL_n(\mathbb{R}) = A$ $f : A \times A \to A$ defined as $f(B,C) = BC$

(3) S = Set of all maps from $T \to T$, T is a set

$f(f_1, f_2) = f_1 \circ f_2$ $\quad f_1, f_2 \in S$

Group if f has only bijective maps
- Identity map
- Invertible

This group : Permutation / Symmetric group

(1) A law of composition on a set S is said to be associative if

$(ab)c = a(bc)$ $\quad \forall a, b, c \in S$

(2) A law of composition is said to be commutative if

$ab = ba$ $\quad \forall a, b \in S$.

(3) An identity for a law of composition is an element $e \in S$

s.t. $ea = ae = a$ $\quad \forall a \in S$

(4) Suppose we have an identity element then an element a is said to be invertible if $\exists\, b \in S$ s.t. $ab = ba = e$

A group is a non empty set G with a law of composition that is associative, and has identity and every element has inverse. (1) (3) (4)

EOC

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EOC

7.1.19

Dihedral groups.

$D_4 = \{ I_d, (1\,2\,3\,4), (1\,3)(2\,4), (1\,4\,3\,2), (2\,4), (1\,2)(3\,4), (1\,3), (1\,4)(2\,3) \}$

Rotations (1 2 3 4); Flips (2 4), (1 2)(3 4), (1 3), (1 4)(2 3)

Consecutive mapping

Study symmetry of regular n-gon

$D_{2n} \rightarrow 2n$ elements

Dihedral group: The group of symmetries of a regular polygon including both rotations and reflections.

A regular polygon with n sides has n different rotations and n different reflections with ~~total~~ $2n$ different elements and denoted as D_n (not D_{2n} as is another convention)

A group G where commutativity holds ie $ab = ba \ \forall a, b \in G$ is called an abelian group. Eg. $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$ abelian over +

$GL_n(\mathbb{R})$ not abelian over x.

Eg D_n: not abelian

Ex. Identity element is unique in a group

Let e, e' be unique identity elements in $G(*)$

~~$e * e' = e$~~ $a * e = a \quad \forall a \in G$

$\therefore e' * e = e' \quad \ldots (1)$

$e' * a = a \quad \forall a \in G$

$\therefore e'' * e = e \quad \ldots (2)$

$\therefore (1), (2) \rightarrow e' = e \quad \rightarrow\leftarrow$

Ex ☆ The inverse of an element is unique in a group.

Similar

Remark (i) If the law of composition is written additively then

$$na = \underbrace{a + a + a + \cdots + a}_{n \text{ times}}$$

Then the inverse of a denoted as

$$-a$$

(ii) If law of composition is written multiplicatively

$$a^n = \underbrace{a * a * a * \cdots * a}_{n \text{ times}}$$

$a^{-1} \rightarrow$ Inverse.

Ex ☆

Show that in a group.

(i) $(a^{-1})^{-1} = a$ (ii) $(ab)^{-1} = b^{-1}a^{-1}$

Finite Group: A group G with a finite number of elements in the associated set.

Note In a finite group $\forall a \in G$, $\exists n \in \mathbb{N}$ s.t. $a^n = e = Id$.

Def Let G be a group and $g \in G$. If there exists no $n \in \mathbb{N}$ s.t. $g^n = Id$ we say g has an infinite order.

If $\exists n$ s.t. $g^n = 1$ then

$$O(g) = \text{order of } g = \min \{i : g^i = 1\}$$

Eg. In a finite group, every element is of finite order
& Group of all roots of unity is an infinite group with elements, each having finite order.

Subgroup (Trivial → $G, \{e\}$)

A subset H of a group G is called a subgroup if you have the following properties:

i) $a, b \in H \Rightarrow ab \in H$

ii) $\exists\ 1 \in H$

iii) If $a \in H$ then $a^{-1} \in H$

Equivalently if $a, b \in H$ then $ab^{-1} \in H$ is satisfied. $H \neq \phi$.

Eg. ① $\mathbb{Q}^* \subseteq \mathbb{R}^* \subseteq \mathbb{C}^*$

② $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ subgroup of $\mathbb{C}^*$

③ $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$

④ $i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, Id

Consider $H = \{\pm I_d, \pm i, \pm j, \pm k\} \subseteq GL_2(\mathbb{C})$

Ex. Check $i^2 = j^2 = k^2$ $\quad i^4 = I_d$ $\quad ji = i^3 j$

$H \rightarrow$ Quaternian group.

Ex Identify the subgroups of Integers. → Integer multiples of $\mathbb{N}$ /

EOC

8.1.19

Permutation Group

Let Ω be a non empty set. A permutation of Ω is simply a bijection $f: \Omega \to \Omega$. The set of all permutations on Ω forms a group and is denoted by S_Ω (aka the symmetric group of Ω)

$$S_\Omega = \{f : \Omega \to \Omega \mid f \text{ is bijective}\}$$

is a group with respect to composition operation

When we study for natural number set (not any Ω), ~~the~~ we have $[n] = \{1, 2, 3 \ldots, n\}$. The symmetric group on $[n]$ is denoted by S_n and the elements are termed permutations.

We can relate S_3 with D_3 but S_4 has $4! = 24$ elements while D_4 has 8 elements. Thus D_4 is a subgroup of S_4. This shows not every bijection corresponds to a transformation.

Notation :

Consider $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 1 & 2 \end{pmatrix} \in S_6$

We can write it compactly as

$\sigma = (1\ 5)(2\ 4\ 6)$ [Can also write (15)(246)(3) (not used)]

Known as cycle decomposition of permutation

The permutation $(a_1\ a_2\ a_3 \dots a_k)$ where $a_i \in [n]$ are distinct is called a k cycle.

A two cycle is called a transposition. The k cycle $\sigma = (a_1 a_2 \dots a_k)$ is a permutation described as

$\sigma(i) = i$ if $i \notin \{a_1, \dots, a_k\}$

$\sigma(a_i) = a_{i+1}$ for $i = 1, 2, \dots, k-1$

$\sigma(a_k) = a_1$

Ex Every permutation is a product of disjoint cycles. Disjoint cycles commutes with each other.

The product of disjoint cycles is defined as follows :-

$(1\ 2\ 3)(1\ 2)(3\ 4)$

done later — done first

$(1\ 2\ 3) \cdot (1\ 2)(3\ 4) = (1\ 3\ 4)$

How to find inverse of a permutation

Let $\sigma = (1\ 5)(2\ 4\ 6)$ then

$\sigma^{-1} = (5\ 1)(6\ 4\ 2)$

Check: $\sigma \cdot \sigma^{-1}$

$(1\ 5)(2\ 4\ 6) \cdot (5\ 1)(6\ 4\ 2) = Id$

For any $\sigma \in S_n$, the cycle decomposition of σ^{-1} is obtained by the numbers of each cycle of the cycle decompositions of σ in reverse order.

To write every permutation as product of transposition :-

$$(a_1 a_2 a_3 \dots a_k) = (a_1 a_k)(a_1 a_{k-1}) \dots (a_1 a_3)(a_1 a_2)$$

Ex ☆

① $S_n = \langle (1,2), (1,3), \dots, (1\ n) \rangle$ [S_n can be generated by transposition of the form : $\langle \dots \rangle$]

② $S_n = \langle (1,2), (2,3), (3\ 4), \dots, (n-1\ n) \rangle$

Even and odd permutation

If permutation can be written as product of even k cycles

$\Rightarrow$ even permutation

else $\Rightarrow$ odd permutation.

Thus definition is not well defined since many transposition decomposition are possible.

Alt. Definitions.

Consider polynomial: $P(x_1, x_2, \dots, x_n) = \prod (x_i - x_j) \quad 1 \le j < i \le n$

Let $\sigma \in S_n$ we define

$$\sigma P(x_1, x_2, \dots, x_n) = \prod_{1 \le j < i \le n} (x_{\sigma(i)} - x_{\sigma(j)})$$

Eg $n = 3 \quad \sigma = (1\ 2\ 3) \in S_n.$

$$P(x_1, x_2, x_3) = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

$$\sigma P(x_1, x_2, x_3) = (x_{\sigma(3)} - x_{\sigma(2)})(x_{\sigma(3)} - x_{\sigma(1)})(x_{\sigma(2)} - x_{\sigma(1)})$$

$= (x_1 - x_3)(x_1 - x_2)(x_3 - x_2)$

Obviously $\sigma P = P$ or $-P$

Define

~~if $\sigma(P) = +$~~

if $\sigma P = P \Rightarrow$ even permutation ~~transposition~~

if $\sigma P = -P \Rightarrow$ odd permutation

$\therefore$ transposition is always an odd permutation *

This definition being well defined and together with (*) makes the original ~~permutation~~ definition well defined

Therefore, formalising

If σ is a transposition $\sigma P = -P$

If σ is a product of an even number of transposition then $\sigma P = P$

Definition σ is said to be an even permutation if $\sigma P = P$

and σ is said to be an odd permutation if $\sigma P = -P$

Let σ be a k cycle. If k is even then σ is odd permutation and if k is odd then σ is even permutation.

Thm: The set of all even permutations forms a subgroup of the set of permutations.

Proof. $|S_n|$ (order of) $= n!$ (Try to establish bijection)

Let X = set of all even permutations
and Y = set of all odd permutations.
define $f: X \to Y$ as $f(\sigma) = (1\ 2)\sigma$

Injective: $(1\ 2)\sigma_1 = (1\ 2)\sigma_2$

$\Rightarrow (1\ 2)(1\ 2)\sigma_1 = (1\ 2)(1\ 2)\sigma_2$

$\Rightarrow \sigma_1 = \sigma_2$

Surjective:

If τ is an odd permutation,

then $(1\ 2)\tau$ is an even permutation

and $f((1\ 2)\tau) = \tau$

$\therefore f$ is a surjective map.

$\therefore |X| = |Y| = \frac{n!}{2}$ $\left[\because |S_n| = n! \text{ and } S_n = X \sqcup Y\right]$

Ex If σ is a k cycle then $\text{ord}(\sigma) = k$

Let Π be a permutation which is a product of disjoint cycles $k_1, k_2, \ldots, k_r$ then $\text{ord}(\Pi) = \text{L.C.M.}(k_1, k_2, \ldots, k_r)$

For each permutation in S_n, let

$$\varepsilon(\sigma) = \begin{cases} 1 & \text{if } \sigma P = P \\ -1 & \text{if } \sigma P = -P \end{cases}$$

$\varepsilon(\sigma)$ is called the sign of σ

X ——— X

Prev Ex

4. Identify all subgroups of $\mathbb{Z}$

Let H be a subgroup of $\mathbb{Z}$, Let b be the smallest positive integer $\in H$. ~~(H is defined as smallest subgroup containing b)~~

Want to show (WTS) $H = b\mathbb{Z}$

Obviously $b\mathbb{Z} \subseteq H$

WTS $H \subseteq b\mathbb{Z}$,

Let $n \in H$ be any (+) integer

Then by division algorithm $n = qb + r \quad 0 \leq r < b$

$\therefore r = \underset{\in H}{n} - q\underset{\in H}{b} \in H$

but b is the smallest positive integer in H.

Hence $r = 0 \quad \therefore H = b\mathbb{Z}$

$\therefore$ Every subgroup of $\mathbb{Z}$ is of the form $b\mathbb{Z}$ for some integer b. (under addition)

Cyclic groups

Let G be a group. $x \in G$. Consider $H = \{\ldots x^{-3}, x^{-2}, x^{-1}, 1, x, x^2, \ldots\}$

H is obviously a group $\therefore$ H is a subgroup of G.

Also H is a cyclic group, since H generated by single element x.

Also H is the smallest subgroup of G containing x.

Ex ☆ How many subgroups are there in S_3?

(1) Consider the cyclic group generated by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R})$$

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \forall n$$

$\langle A \rangle$: This group is an infinite cyclic group of $GL_2(\mathbb{R})$

(2) $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$

generates a cyclic group of order 6.

$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

(3) $\mathbb{Z}/p\mathbb{Z}$

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$

$\langle \bar{5} \rangle = \{\bar{0}, \bar{5}, \overline{10}, \bar{3}, \bar{8}, \bar{1}, \bar{6}, \overline{11}, \bar{4}, \bar{9}, \bar{2}, \bar{7}\} = \langle T \rangle$

$= \langle \bar{7} \rangle = \langle \overline{11} \rangle$

Thus every number coprime to 12 generates the group.

$\bar{2} = 12k+2$

$x, y \in \bar{2}$

$\Rightarrow x \equiv 2 \bmod n$

$y \equiv 2 \bmod 12$

$6(12k+12)$

$12\mathbb{Q}$.

10.1.19

Proposition. Let $H = \langle x \rangle$. Then

(i) If $|H| = \infty$, then $x^n \neq 1$ for any $n = 0$

(ii) If $|H| = n$ then $x^n = 1$ and $H = \{1, x, x^2, \ldots, x^{n-1}\}$

Pf (i) Suppose $x^n = 1$ for some $n \neq 0$ and assume $n > 0$.

Let $m \geq n$. Then

$$m = qn + r \quad , \quad 0 \leq r < n$$

$$\therefore x^m = x^{qn+r} = x^{qn} \cdot x^r = 1 \cdot x^r = x^r$$

Choose smallest n s.t. $x^n = 1$, $n > 0$ Then

$$x^n = 1 \Rightarrow x^{-1} = x^{n-1}$$

Then $H = \{1, x, x^2, \ldots, x^{n-1}\}$

But order of H is infinite, ie $|H| = \infty$.

Thus $x^n \neq 1$ for any $n \neq 0$

(ii) Enough to prove, order of $x = n$

Since $H = \langle x \rangle$ and H is of finite order.

$\exists a, b \in \mathbb{Z}$ s.t. $x^a = x^b$

$$\Rightarrow x^{a-b} = 1$$

Pick smallest int $m > 0$ s.t $x^m = 1$ $\quad |x| = m$

Then $H = \{1, x, x^2, \ldots, x^{m-1}\}$ ie $|H| = m$

But given $|H| = n \Rightarrow m = n$

$\therefore |x| = n$

Thm: Let H be a cyclic group generated by x

~~$H = \langle x \rangle$~~ b

(i) Every subgroup K of H is either $\{1\}$ or $\langle x^d \rangle$ where d is the smallest power of x in K

(ii) If $|H| = n$ then for every divisor d of n $\exists$ a subgroup K of order d. This subgroup is cyclic and $\langle x^{n/d} \rangle$

Proof (i) $K \neq \{1\}$ K is a subroup of H.

Consider $P = \{ n \geq 1 \mid x^n \in K \}$

By well ordering principle, P has a smallest positive integer, say d.

To show $K = \langle x^d \rangle$

It is obvious $\langle x^d \rangle \subseteq K$

Choose any element $b \in P$ s.t $x^b \in K$ & $b > d$

Then, $b = qd + r$, $0 \leq r < d$

$\Rightarrow r = b - qd$

$\therefore x^r = x^{b-qd} \in K$ but $r < d$ $\therefore r = 0$

$x^b \in \langle x^d \rangle$

$\therefore K \subseteq \langle x^d \rangle$

$\Rightarrow K = \langle x^d \rangle$

(ii) $\therefore |H| = n \Rightarrow |x| = n$. Let $n = dd'$. Now consider $\langle x^{d'} \rangle$

To show $|x^{d'}| = d$

Assum $|x^{d'}| = m$

$\Rightarrow x^{d'm} = 1$

But $|x| = n$

$\Rightarrow n \mid d'm$ or $d \mid m$

Also $(x^{d'})^d = 1$

$\therefore |x^{d'}| = d$

Ex ★ Try to prove there exists unique subgroup of order d for every divisor of n

Ex ★ Let G be a group and $x \in G$

If $|x| = n$ the $|x^a| = \frac{n}{\gcd(n,a)}$

14.1.19

Group homomorphism.

Let G and G' be groups. Then a map $f: G \to G'$ is called a group homomorphism if.

$$f(gh) = f(g)\, f(h) \qquad \forall\ g, h \in G$$

Example of Group homomorphism.

(1) Consider map det : $GL_n(\mathbb{R}) \to \mathbb{R}^\times$

$$\det(AB) = (\det A)(\det B).$$

det is a group homomorphism.

(2) $f: (\mathbb{R}, +) \longrightarrow (\mathbb{R}^\times, \cdot)$

$f(a) = e^a$ is a group homomorphism.

(3) $f: S_n \to \{\pm 1\}$

$f(\sigma) = \text{sign}(\sigma)$ is a group homomorphism.

(4) $\phi: \mathbb{Z} \to G$, G is a group with op^n multiplicatively

$n \longrightarrow a^n$ when a is fixed element in G.

$\phi(m+n) = a^{m+n} = a^m a^n = \phi(m)\,\phi(n)$

$\therefore$ This is a group homomorphism.

(5) Let H be a subgroup of G. Then

$i: H \to G$ defined as $i(x) = x$ is a group hmo—

(6) Let G_1, G_2 be 2 groups. Define their product

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$$

For $(a,b), (c,d) \in G_1 \times G_2$ define a law of composition on $G_1 \times G_2$ as

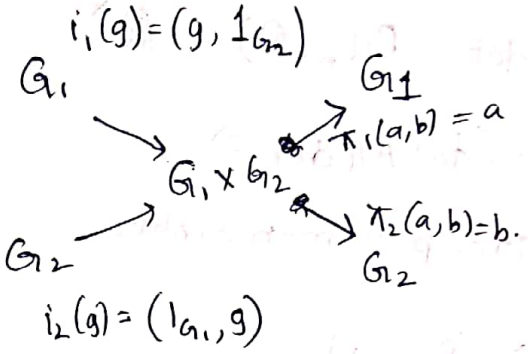
$$(a,b)\cdot(c,d) = (ac, bd)$$

Identity : $(1_{G_1}, 1_{G_2})$

Inverse of (a,b) is (a^{-1}, b^{-1})

1_{g_1}
$i(g_1, g_2)$
$= i(g_1)\ i(g_2)$
$= i($

Consider



i_1, i_2, π_1, π_2 are gp. homomorphism.

Properties of group homomorphisms.

(1) If $f: G \to G'$ be a gp homomorphism then

$f(1_G) = 1_{G'}$ and $f(a^{-1}) = (f(a))^{-1}$

(2) Let $G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3$ where $f * g$ are group homo. Then $gof : G_1 \to G_3$ is also a gp. homo.

(3) Every group homo. defines two impodant ~~sug~~ subgroups its image and kernel.

The image of a gp. homo is defined as

$$\text{im}\,\phi = \{ x \in G' \mid x \in \phi(a) \text{ for some } a \in G \}$$

Check : $\text{im}\,\phi$ is a subgrp.

The kernel of a gp. homo is defined as

$$\ker \phi = \{ x \in G \mid \phi(x) = 1_{G'} \}$$

Note that $1 \in \ker \phi$

$\ker \phi$ is a subgroup of G.

Let $g \in \ker \phi$ and $h \in G$.

$$\phi(hgh^{-1}) = \phi(h)\, g\, \phi(g)\, \phi(h^{-1})$$
$$= \phi(h)(\phi(h))^{-1}$$
$$= 1_{G'}$$
$$hgh^{-1} \in \ker \phi$$

Defⁿ A subgroup N of G is called a normal subgp. of G $(N \triangle G)$ if $ghg^{-1} \in N$ for all $h \in N$ and $g \in G$.

Ex

① $SL_n(\mathbb{R}) \triangle GL_n(\mathbb{R})$ since $\ker(\det) = SL_n(\mathbb{R})$ so it is a normal subgroup of $GL_n(\mathbb{R})$

② If $f : G_1 \longrightarrow G_2$ is a gp. homo. then

$$\ker f \triangleleft G_1$$

③ A_n is a normal subgroup of S_n.

↳ Set of all even permutation of S_n since

$$\ker(\text{sign}) = A_n .$$

(4) Every subgroup of an abelian group is a normal subgroup.

(5) Consider sub.grp. U of invertible Upper triangular matrices of $GL_2(\mathbb{R})$. Ex. Is U a normal subgroup of $GL_2(\mathbb{R})$ [Ans: No, show counter eg]

Center of a Group:

Def: Let G be any group. Then its ~~Center~~ centre $Z(G) = \{g \in G \mid gh = hg \ \forall h \in G\}$

Ex Is $Z(G)$ is a subgp. of G?

Ex. Let $G = S_n$ then $Z_G = \begin{cases} S_n & \text{if } n = 1, 2 \\ 1 & \text{if } n \geq 3 \end{cases}$

Show that $Z(GL_n(\mathbb{R})) = \{c I_n \mid c \in \mathbb{R}^\times\}$

EoC

15.1.19

Q. Let $f: G \to G'$ be a group homomorphism. When is it injective?

Let $\phi: S \to T$ be a mapping of sets. This map defines an equivalence relation on the domain S by the rule $a \sim b$ if $\phi(a) = \phi(b)$. For every element $t \in T$ the inverse image of t is defined as $\phi^{-1}(t) = \{s \in S \mid \phi(s) = t\}$

The inverse images are called the fibers of ϕ

Consider the group homomorphism.

$\phi : \mathbb{C}^* \longrightarrow \mathbb{R}^*_{>0}$ (modulus of complex no)

defined by $\phi(z) = |z|$

Then fibers of ϕ are concentric circles about the origin and $\mathbb{C}^* = \bigcup \phi^{-1}(r)$

Prop^{n} Let $\phi: G \to G'$ be a group homomorphism with $\ker(\phi) = N$

Let $a, b \in G$ then $\phi(a) = \phi(b)$ iff $b = an$ for some $n \in N$ or equivalently, if $a^{-1}b \in N$

Proof. Let $\phi(a) = \phi(b)$

$\phi(a)^{-1}\phi(b) = 1_{G'}$

$\phi(a^{-1}b) = 1_{G'}$

$a^{-1}b \in N$

$a^{-1}b = n$ for some $n \in N$

$\therefore \quad b = an$

Conversely if $b = an$

$$\begin{aligned}\phi(b) &= \phi(an) \\ &= \phi(a)\,\phi(n) \\ &= \phi(a)\,1_{G'} \\ &= \phi(a)\end{aligned}$$

Corollary: A grp. homo $\phi: G \to G'$ is injective if $\ker\phi = \{1_G\}$

Defn: A gp homomorphism $\phi: G \to G'$ is called an isomorphism if ϕ is 1-1 and onto. If $G' = G$ then an isomorphism is called an automorphism.

Propn: Let $\phi: G \to G'$ be a gp. homo. of finite groups.

Then

$$|G| = |\ker\phi| \cdot |\mathrm{Im}\,\phi|$$

There is a bijective map b/w elements of the image and the equivalence classes.

E.T.F the number of eqv. classes given $|G|$ on $|\ker\phi|$

* Let G be a grp and H be a subgroup of G.

A Left coset of H is defined as the subset

$$aH = \{ ah \mid h \in H \}$$

Let us define an equivalence relation on G &

Let $a, b \in G$ then $a \sim b$ if $a = bh$ for some $h \in H$

ies $b^{-1}a \in H$. This is an equiv. relation & eq. classes are of the form gH.

Therefore either $g_1H = g_2H$ or $g_1H \cap g_2H = \phi$ and G is the disjoint union of distinct left cosets of H

Def^n Number of left cosets of a subgroup H in a group G is known as the index of H in G $= [G:H]$

Is a left coset aH a subgrp of G.

→ It is not a subgroup of G.

Eg $G = \mathbb{R}^2$ and $H = \langle (1,0) \rangle$.

Then the left coset are of the form

$$v + H = \{ v + \alpha(1,0) \mid \alpha \in \mathbb{R} \}$$

This is the line parallel to x-axis that passes through pt v.

Two left cosets are either parallel or ~~disjoint~~ same.

Ex $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$

$$\left.\begin{aligned} 1+H &= \{\ldots -9, -4, 1, 6, 11, 16, \ldots\} \\ 2+H &= \{\ldots -8, -3, 2, 7, 12, \ldots\} \\ 3+H &= \{\ldots -7, -2, 3, 8, \ldots\} \\ 4+H &= \{\ldots -6, -1, 4, 9, \ldots\} \end{aligned}\right\} \text{Covers } \mathbb{Z}$$

$\mathbb{Z}$ is disjoint union of these left cosets

Let $H = n\mathbb{Z}$. Then G & H are of infinite order but there are only finitely many cosets of H namely $n\mathbb{Z}, 1+n\mathbb{Z}, 2+n\mathbb{Z}, \ldots n-1+n\mathbb{Z}$

$\therefore [\mathbb{Z} : n\mathbb{Z}] = n$.

Let G be a finite group and H is a subgroup of G. Then $|H| = |aH|$ for some $a \in G$

Define a map $f : H \to aH$ by $f(h) = ah$.

Let $f(h_1) = f(h_2) \Rightarrow ah_1 = ah_2 \Rightarrow h_1 = h_2$

$\therefore f$ is 1-1 map

f is clearly a surjective map.

$\therefore f$ is a bijection. Then $|H| = |aH|$

Lagranges Thm:

Let G be a finite gp. and H is a subgp. of G. Then $|H|$ divides $|G|$ and $|G|/|H|$ = no. of left cosets of H $= [G : H]$

Pf Let $G = H \cup a_1H \cup a_2H \cup \ldots \cup a_{r-1}H$

$$|G| = |H| r \quad \ldots (*)$$

Note that. Left cosets are equivalence classes and G can be written as disjoint union of these left cosets. Let there be r distinct left cosets say $H, a_1H, \ldots a_{r-1}H$. Then $* \Rightarrow |H|/|G|$

Examples

Let $O_n(\mathbb{R})$ be the group of all $n \times n$ orthogonal matrices. $\det : O_n(\mathbb{R}) \to \{1, -1\}$. $\ker(\det) = SO_n(\mathbb{R})$

$A \in O_n(\mathbb{R})$ now consider the left coset $A SO_n(\mathbb{R})$

$$[O_n(\mathbb{R}) : SO_n(\mathbb{R})] = 2$$

Cor Let G be a group of prime order. Then G is cyclic group.

Pf: Let $1 \neq a \in G$. Consider the cyclic subgroup $H = \langle a \rangle$ of G Then by Lagranges thm. $|H| \mid |G|$ but $|G|$ is a prime no. say p. Then $|H| = p$. $\therefore H = G$.

Warning Converse is not true. ie, if $n \mid |G|$ for a group G then G need not have a subgroup of order n

Cor: Let G be a finite gp. and $a \in G$ then $|a| \mid |G|$ and also $a^{|G|} = 1$

Proof Let $H = \langle a \rangle$ generated by a.

$|H| = |a|$.

but $|a| \mid |G|$

Let $|G| = |a| r$

$\therefore a^{|G|} = a^{|a| r} = 1$.

Prop^n Let $\phi : G \to G'$ be a gp. homo. of finite groups. Then $|G| = |\ker \phi| \cdot |\mathrm{Im}\, \phi|$

We have seen that the left cosets of $\ker \phi$ are the fibers of the map ϕ. There is a bijection b/w the elements of image set and no. of left cosets of $\ker \phi$.

$$\therefore [G : \ker \phi] = [\mathrm{Im}\, \phi]$$

$$\frac{|G|}{|\ker \phi|} = |\mathrm{Im}\, \phi|$$

$$|G| = |\ker \phi| \, |\mathrm{Im}\, \phi|$$

Eg. $\mathrm{sign} : S_n \longrightarrow \{1, -1\}$

$$[S_n : A_n] = 2$$

[EOC]

17.1.19

Example (1)

Q $K \subset H \subset G$ Let $K \triangleleft H$ and $H \triangleleft G$. Is $K \triangleleft G$?

A: Not always

Consider $W = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\} \triangleleft SL_2(\mathbb{R})$

and $SL_2(\mathbb{R}) \triangleleft GL_2(\mathbb{R})$

Ex: Show that W is not a normal subgroup of $GL_2(\mathbb{R})$

But isomorphic to $\mathbb{R}$

Define: $\phi : W \to \mathbb{R}$

$$\phi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = x$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$$

We have $\phi(AB) = \phi(A)\,\phi(B)$

$\therefore \phi$ is a group homomorphism.

ϕ is obviously 1-1 and onto. $\Rightarrow \phi$ is an isomorphism.

Example (2) Let G be an infinite cyclic group. Show that G & $\mathbb{Z}$ are isomorphic.

Define: $\phi : \mathbb{Z} \to G = \langle a \rangle$

$n \longmapsto a^n$

Check: ϕ is an isomorphism.

Exer (3) Any finite cyclic group of order n is isomorphic to $(\mathbb{Z}/n\mathbb{Z}, +)$

Hint: ϕ is same as in (2)

Rem: Let G be a group.

$$Aut(G) = \{ \phi : G \to G \mid \phi \text{ is an automorphism} \}$$

automorphism: isomorphism from $G \to G$

$Aut(G)$ is a group w.r.t. fn. composition operation

Ex: Let $G = (\mathbb{Z}/3\mathbb{Z}, +)$. Then find $Aut(G)$

Eg. Inner automorphism.

Let G be a group. Fix $g \in G$

Define $i_g : G \to G$

$i_g(x) = gxg^{-1}$

$i_g(xy) = gxyg^{-1}$

$= gxIyg^{-1}$

$= gxg^{-1}gyg^{-1}$

$= i_g(x)\, i_g(y)$

$\therefore$ Homomorphism.

Let $i_g(x) = i_g(y)$

$\Rightarrow gxg^{-1} = gyg^{-1}$

$\Rightarrow x = y$

i_g is 1-1.

Note, $i_g(g^{-1}xg) = x$ i_g is onto.

$\therefore i_g$ is an automorphism.

The elts gxg^{-1} are called conjugates of x $\forall g \in G$

You can talk about the right coset of a subgroup H in a gp. G as well, which is denoted by

$$Hg = \{hg \mid h \in H\}$$

In general left and right cosets need not be equal.

Ex Let $H = \langle (1,2) \rangle$ be the subgroup of S_3
Find all left and right cosets of H.

Left Cosets:

$$gH = \{gh \mid h \in H\}$$

Proposition

A subgroup H of G is normal iff every left coset is also a right coset

Pf Let H be a normal subgroup of G.

Let aH be a left coset.

$$\underbrace{ah}_{\in aH} = \underbrace{aha^{-1}}_{\in H}a = \underbrace{h'a}_{\in Ha} \Rightarrow aH \subseteq Ha$$

Silly starting with elt in Ha, we can reach element aH

$$\Rightarrow Ha \subseteq aH$$

$$\Rightarrow aH = Ha.$$

Conversely, let every left coset of H be a right coset

Let $aH = Hb$

Then $a \in Ha$ (identity in H)

$a \in aH = Hb$ $\Rightarrow a \in Ha \cap Hb$

Since two left cosets are either disjoint or equal

$\Rightarrow Ha = Hb \Rightarrow aH = Ha \Rightarrow aHa^{-1} = H$

$\therefore H$ is normal sgp.

Cor If $[G:H] = 2$ Then $H \triangle G$ $a \in Ha$

Let $a \notin H$ Then $G = H \cup aH$ (aH)
$= H \cup Ha$

$\Rightarrow aH = Ha \Rightarrow H \triangleleft G.$
$\forall a \in G.$

$aH = bH \overset{?}{\Longrightarrow} abH.$

[EOC]

22.1.19

Product of Groups

Eg Let G be a group and H, K are subgroups of G.

$H \times K$. $HK = \{hk \mid h \in H, k \in K\}$

$(h_1k)(h_2k_2) \in HK$?

Let H be a normal subgroup.

$h_1k_1h_2k_2 = h_1h^*k_1k_2 \in HK$.

Ex If H or K is a normal subgroup of G then $HK = KH$ and HK is a subgroup of G.

$k^{-1}h^{-1}$

h^*k^{-1}

Thm Let H, K are subgroups of G.

① if $H \cap K = \{1\}$. then $p: H \times K \to G$ defined by $p(h,k) = hk$ is injective and its image is HK.

Pf Let (h_1, k_1) and (h_2, k_2)

$p(h_1, k_1) = p(h_2, k_2)$

$\Rightarrow h_1k_1 = h_2k_2$

$\Rightarrow h_2^{-1}h_1 = k_2k_1^{-1} \in H \cap K = \{1\}$

$h_2^{-1}h_1 = 1 \Rightarrow h_1 = h_2$

$k_2k_1^{-1} = 1 \Rightarrow k_1 = k_2$

② If either H or K is normal. Then $HK = KH$ and HK is a sgp. of G

Pf : Ex -

(3) If both H and K are normal, and $H \cap K = \{1\}$ identity and $G_1 = HK$. Then G_1 is isomorphic to $H \times K$

$$p((h',k')\cdot(h,k)) = p(h',k')\,p(h,k)$$

$$p(h'h, k'k) = \quad = h'k'hk$$

$$\| \\ h'hk'k$$

ETS $hk' = k'h$

$\Leftrightarrow k'^{-1} h k' = h$

$\Rightarrow h^{-1}k'^{-1}hk' = 1$

$h^{-1}k'^{-1}hk' \in H$ as $k'^{-1}hk^{-1} \in H$

$h^{-1}k'^{-1}hk' \in K$ as $h^{-1}k'^{-1}h \in K$

$h^{-1}k'^{-1}hk' \in H \cap K = \{1\}$

Ex Show that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \simeq \mathbb{Z}/6\mathbb{Z}$

Ex $\mathbb{Z}_r \times \mathbb{Z}_s \simeq \mathbb{Z}_{rs}$ if $\gcd(r,s) = 1$

Quotient Group : Let G be a group and H is a s'gp of G.

$G/H = \{aH \mid a \in G \text{ where } H \triangle G\}$

$=$ set of all left cosets of H.

Lemma:

The product of two left cosets, aH and bH is the left coset abH. ie $aH \cdot bH = abH$

Pf $aH \cdot bH = \{ah_1 bh_2 : h_1, h_2 \in H\}$

$ah_1 bh_2 = abh'h_2 \in abH$

Note that this is well defined.

Let $aH = a'H$ and $bH = b'H$

$a'^{-1}a \in H$ $\quad b'^{-1}b \in H$

W.T.S. $abH = a'b'H$

ie $b'^{-1}a'^{-1}ab \in H$

$b'^{-1}a'^{-1}ab = b'^{-1}hb \in H$

Prop^n G/H is a group w.r.t the operation defined as $aH \cdot bH = abH$, and identity is H, and inverse of $(gH)^{-1} = g^{-1}H$. Therefore G/H is a group.

First isomorphism Thm:

Let $\phi : G \to G'$ be a surjective gp homo. Then $G/\ker(\phi) \cong G'$

Pf Let $N = \ker \phi$

Define a map $\bar{\Phi} : G/N \to G'$

$$\bar{\Phi}(gN) = \phi(g)$$

E: Check $\bar{\phi}$ is well defined, $\bar{\phi}$ is a gp homo

Since ϕ is surjective so is $\bar{\phi}$.

$$\bar{\Phi}(g_1N) = \bar{\Phi}(g_2N)$$

$$\Rightarrow \phi(g_1) = \phi(g_2)$$

$$\Rightarrow [\phi(g_2)]^{-1}\phi(g_1) = 1.$$

$$\Rightarrow \phi(g_2^{-1}g_1) = 1$$

$$\Rightarrow g_2N = g_1N$$

$\bar{\phi}$ is injective. $\therefore \bar{\phi}$ is an isomorphism.

Examples

(1) $\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$

This is a subjective gp homo with ker as $SL_n(\mathbb{R})$. Therefore by 1st isomorphism thm $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^\times$

(2) $\det : O_n(\mathbb{R}) \longrightarrow \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$

This is a surjective gp. homo

$\therefore O_n(\mathbb{R})/SO_n(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

(3) $\text{sign} : S_n \longrightarrow \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$

This is a surj. gp. homo.

$\therefore S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$

(4) $\phi : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times_{>0}$

$z \longmapsto |z|$

u is
You are

ϕ is a surj. gp homo.

$\mathbb{C}^\times/U \cong \mathbb{R}^\times_{>0}$

where $U = \ker \phi = \{ z \in \mathbb{C}^\times \mid |z| = 1 \}$

2nd Isomorphism Thm

For a non empty subset A of G define the normaliser as $N_G(A) = \{ g \in G \mid gAg^{-1} = A \}$

Thm Let G be a gp. Let A, B be subgroups of G and assume that $A \subseteq N_G(B)$ Then $A \cap B$ is a normal subgroup of A and $AB/B \cong A/A \cap B$

AB
$a_1b_1 a_2b_2$

Use $gB^{-1}g = B$
$\forall g \in A$

Ex. Show AB is a subgroup of G

WTS: $A \cap B \trianglelefteq A$

Let $x \in A \cap B$. Then $gxg^{-1} \in A$ $\forall g \in A$ i.e. $gxg^{-1} \in B$

$[\because A \subseteq N_G(B)]$

$\therefore A \cap B \trianglelefteq A$.

Now define $\phi : AB \longrightarrow A/A \cap B$.

$$\psi(ab) = a(A \cap B)$$

Swap as A in $N_G(B)$

$$\psi(ab \cdot a_1 b_1) = \psi(ab\, a_1 b_1) = \phi(a\, a_1 a_1 b_2 b_1) = aa_1(A \cap B)$$

$$\psi(ab)\psi(a_1 b_1) = a(A \cap B)\, a_1(A \cap B)$$

$$= aa_1(A \cap B)$$

ϕ is surjective as well

$$\ker \phi = \{ab \mid \phi(ab) = A \cap B\}$$

$$= \{ab \mid a(A \cap B) = A \cap B\}$$

$$= \{ab \mid a \in A \cap B\}$$

check $\ker \phi = B$.

Then by 1st isomorphism thm.

$$AB/B \cong A/A \cap B.$$

Third isomorphism theorem.

Let G be a gp and H and K are normal subgroups of G with H is a subgroup of K

Then $K/H \triangle G/H$

and $\dfrac{G/H}{K/H} \cong G/K$

Pf: $\phi : G/H \to G/K$.

$\phi(gH) = gK$

Show gp homo; find K.

—

22.1.19

$Z(GL_n(\mathbb{R})) = \{cI_n \mid c \in \mathbb{R}^*\}$ also holds for $\mathbb{C}$

$Z(SL_n(\mathbb{C})) = \{cI_n \mid c \in \mathbb{C}^*, c^n = 1\}$

$GL_n(\mathbb{C}_n) / Z(GL_n(\mathbb{C})) = PGL_n(\mathbb{C})$

$SL_n(\mathbb{C}) / Z(SL_n(\mathbb{C}))$

Ch. 2

M. 67 Pg 10

$a = (a_1 \dots a_k) \in \mathbb{R}^*$

$b = (b_1 \dots b_k) \in \mathbb{R}$

$X : [0,1] \longrightarrow \mathbb{R}^n$

$t \longrightarrow X(t) = (x_1(t) \dots x_t(t))$

~~24.1.19~~

24.1.19

Let G be a gp. and H, K are normal subgroups of G with H/K, Then $K/H \trianglelefteq G/H$.

and $\dfrac{G/H}{K/H} \cong G/K$

Pf $K/H = \{kH \mid k \in K\}$ is a subgroup of G/H.

$gH\ kH\ g^{-1}H = (gkg^{-1})H \in K/H$

$\therefore$ K/H is a normal subgroup of G/H

maybe not needed since $\ker(\phi)$ is always normal ??

Let $\varphi : G/H \to G/K$

$\varphi(gH) = gK$

Check φ is well defined

φ is a gp homomorphism.

φ is surjective.

$\ker \varphi = \{gH \mid gK = K\} = \{gH \mid g \in K\} = K/H$

$\therefore$ By first isomorphism thm

$\dfrac{G/H}{K/H} \cong G/K$

Prop Let $f: G \to \tilde{G}$ be a surj gp. homa and $K = \ker(f)$. Then $\exists$ bijection:

$$\{\text{subgroups of } G \text{ containing } K\} \longleftrightarrow \{\text{Subgroups of } \tilde{G}\}$$

$$H \longmapsto f(H)$$

verify it is a sgp of $\tilde{G}$

$$f^{-1}(\tilde{H}) \longmapsto \tilde{H}$$

And the bijection restricts to

$$\{\text{normal subgroups of } G \text{ cont. } K\} \longleftrightarrow \{\text{normal subgrps of } \tilde{G}\}$$

Pf Let H be a subgroup of G containing K.

① Then $f(H)$ is a subgroup of $\tilde{G}$ (Ex)

$f^{-1}(f(H)) \supseteq H$ is trivial.

To show $f^{-1}(f(H)) \subseteq H$

Choose $g \in f^{-1}(f(H))$

$f(g) \in f(H)$

$f(g) = f(h)$ for some $h \in H$

$f(h^{-1}g) = 1$

$h^{-1}g \in K$

$g = hk$ for some $k \in K \subset H$

$\in H$

$\therefore f^{-1}(f(H)) \subseteq H$

$\therefore H = f^{-1}(f(H))$

~~To show~~ ~~$f(f^{-1}(\tilde{H}) =$~~

Let $\tilde{H}$ be a subgroup of $\tilde{G}$. Consider ~~$f^{-1}(\tilde{G})$~~

$f^{-1}(\tilde{H})$ is a subgroup of G (Ex) containing K.

To show $f(f^{-1}(\tilde{H})) \subseteq \tilde{H}$ is clear.

Using surjectivity Prove the other containment [Ex]

The bijection restricts to normal subgroups [Ex].

Ex $G = \mathbb{Z}$ $K = 2\mathbb{Z}$ $H = 6\mathbb{Z}$

$G/H = \{\mathbb{Z}/6\mathbb{Z}$

$K/H = \{m + 6\mathbb{Z} \mid m$ is a multiple of $2\}$

~~$= \{6\mathbb{Z}, 6$~~

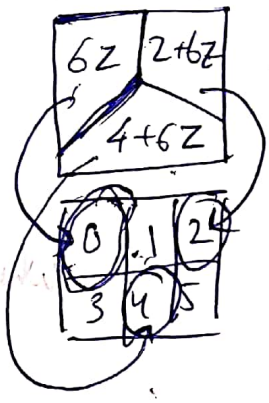
$\frac{G/H}{K/H} \cong G/K$

(2)

$6\mathbb{Z}$

$2+6\mathbb{Z}$

$4+6\mathbb{Z}$



$\{[0][2][4]\}$,

$\{[1],[3][5]$

Converse of Lagranga Thm is not true!

Eg $|S_4| = 24$ $G = A_4$ $|A_4| = 12$

W.T.S There doesnt exist any subgroup of order G in A_4.

Let H be a subgroup of G of order 6.

$[G:H] = |G|/|H| = 2$. Since index is 2, so H is a normal subgroup of G.

$V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$

Klien 4-grp $\subseteq A_4$

$|H \cap V_4| \geq 2$

$(1\,2\,3)(12)(34)(1\,2\,3)^{-1} \in H$

$[\because H \trianglelefteq A_4]$

$= (14)(23)$

which implies $V_4 \subseteq H$ (contradiction)

$|H \cap V_4) = 1$

$H = \{Id, 5\ 3\text{-cycles}\}$

Note: 1st Class Test

5th Feb

29.1.19

Group Action.

29.1.19

A gp action of a gp. G on a set A is a map

from $G \times A \to A$

$$(g, s) \longmapsto gs$$

satisfying the following properties.

(1) $g_1(g_2 s) = (g_1 g_2)s \quad \forall g_1, g_2 \in G$ & $s \in A$

(2) $1 \cdot s = s \quad \forall s \in A$ and $1 \in G$

Example (1) $G = \{1, \tau\}$. where τ is reflection w.r.t x axis.

$$G \times \mathbb{C} \longrightarrow \mathbb{C}$$

$$(g, z) \longmapsto gz$$

$$gz = \begin{cases} z & , g = 1 \\ \bar{z} & , g = \tau \end{cases}$$ is a gp. action

(2) $G \times G \longrightarrow G$

$(g, x) \longmapsto gxg^{-1}$ is a gp. action

(3) G is a gp. and H a subgroup of G.

$$G \times G/H \longrightarrow G/H$$

$(g, aH) \longmapsto gaH$ is a gp. action

(4) $G \times G \longrightarrow G$

$(g, x) \longmapsto gx$.

Another representation of group action :-

Let G be a gp, A be a set. Let $g \in G$, be a fixed elt, we get a map.

$$\sigma_g : A \longrightarrow A$$
$$\sigma_g(a) = ga$$

W.T.S

σ_g is an inj. map.

$a_1, a_2 \in A$ s.t

$$\sigma_g(a_1) = \sigma_g(a_2)$$
$$\Rightarrow g a_1 = g a_2$$
$$\Rightarrow g^{-1}(g a_1) = g^{-1}(g a_2)$$
$$\Rightarrow (g^{-1}g) a_1 = (g^{-1}g) a_2 \quad \text{(Prop 1)}$$
$$\Rightarrow 1.a_1 = 1.a_2$$
$$\Rightarrow a_1 = a_2 \quad \text{(Prop 2)}$$

Surjective.

Since $\sigma_g(g^{-1}a) = a$. $\therefore$ Thus σ_g is a bijective map.

Define $\phi : G \longrightarrow S_A$ (permutation)

$$g \longrightarrow \sigma_g$$

$$[\phi(g_1)\,\phi(g_2)](a)$$
$$= \sigma_{g_1}(\sigma_{g_2}(a)) = \sigma_{g_1}(g_2 a) = g_1 g_2 a$$
$$= \sigma_{g_1 g_2}(a) = \phi(g_1 g_2)(a)$$

$\Rightarrow \phi(g_1)\phi(g_2) = \phi(g_1 g_2)$

ϕ is a gp. homomorphism.

The above homomorphism ϕ is called the permutation representation of a given group action.

Defn The kernel of a group action is defined as

$$\{g \in G \mid gs = s \quad \forall\, s \in A\}$$

Defn For each $s \in A$, the stabilizer of s is denoted by

$$stb(s) = \{g \in S \mid gs = s\}$$

$\therefore$ Is $stb(s)$ is a subgroup?

(i) $g_1, g_2 \in stb(s)$

$\Rightarrow g_1 s = s \quad g_2 s = s.$

$g_1 g_2 s = g_1 s = s \quad \Rightarrow g_1 g_2 \in S.$

(ii) $g_1 \in stb(s)$

$g_1 s = s.$

$g_1^{-1} g_1 s = g_1^{-1} s.$

$g_1^{-1} s = s. \quad \Rightarrow g_1^{-1} \in S.$

$\therefore$ $stb(s)$ is a subgp.

Def^n

The orbit of an elt $s \in S$ denoted by $O(s) = \{gs \mid g \in G\}$

[$s \sim t$ if $\exists\, g \in G$ st. $t = gs$. $s, t \in A$]

$\sim$ is an equiv. relation on A

The orbits, therefore are equivalence classes and hence, $O(s)$ partition A

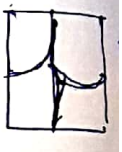
* Relation b/w $O(s)$ and $Stb(s)$

Set $O(s) = \{gs \mid g \in G\}$

$G_s = Stb(s) = \{g \in G \mid gs = s\}$

$G/G_s \xrightarrow{\phi} O(s)$

$gG_s \longmapsto gs$



Check: ϕ is well defined

W.T.S ϕ is injective.

Let $\phi(g_1 G_s) = \phi(g_2 G_s)$

$\Rightarrow g_1 s = g_2 s$

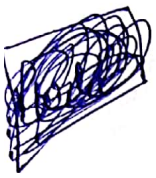
$\Rightarrow g_2^{-1} g_1 s = s$.

$g_2^{-1} g_1 \in G_s \Rightarrow g_1 G_s = g_2 G_s$

$\therefore \phi$ is injective.

$|O(s)| = [G : G_s]$

Prop^n Let G be a gp. acting on A.
For each $s \in A$ $|O(s)| = [G : G_s]$



Second hour

Def^n A gp. action of G on a set A is called transitive if there is only one orbit.
i'e given any two elements ~~from~~ $x, y \in A$

$\exists\, g \in G$ st. $x = gy$

Ex Find example of transitive group action

$\phi : G \longrightarrow S_A$

Calay's Thm

Every finite gp. G is isomorphic to a subgroup of a permutation gp. If $|G| = n$ then G is isomorphic to a subgroup of S_n.

Pf: $G \times G \longrightarrow G$

$(g, x) \longmapsto gx$

$\phi : G \longrightarrow S_n$

$\phi(g) = \sigma_g$ where $\sigma_g(x) = gx$

$\ker \phi = \{g \in G \mid \sigma_g = 1_G\} = \{g \in G \mid gx = x \ \forall x \in G\}$

$= \{1_G\}$

$\therefore$ Injective gp. homo. from $G \to S_n$.

$\therefore$ G is isomorphic to sgp of S_n.

~~Thm (Gau~~

Thm (Cauchy):

Let G be a finite group and p be a prime number s.t. $p \mid |G|$. Then G has a subgroup of ord p.

Pf Consider the set.

$$S = \{(x_1, \ldots, x_p) \mid (x_1, \ldots, x_p) \in G \times \cdots \times G \text{ and } x_1 \circ x_2 \circ \cdots x_p = 1\}$$

Let $\sigma = (1, 2, \ldots, p) \in S_p$.

$H = \langle \sigma \rangle$ & $|H| = p$

Define a gp. action.

$$H \times S \longrightarrow S$$

$$(\sigma^i, (x_1, \ldots, x_p)) \mapsto (x_{\sigma^i(1)}, x_{\sigma^i(2)}, \ldots x_{\sigma^i(p)})$$

$$|O(s)| = [H : Hs] = \frac{|H|}{|Hs|}.$$

$\therefore |O(s)|$ is either 1 or p

Suppose that $|O(s)| = 1$.

$$O((x_1, \ldots x_p)) = (x_1, \ldots, x_p)$$

$(\sigma, (x_1, \ldots, x_p)) = (x_2, x_3, \ldots, x_1)$

$\Rightarrow$ But $O(s)$ has 1 elt.

$\therefore x_1 = x_2 = \ldots = x_p$.

Observe $\sigma((1,1,1,\ldots,1)) = (1,1,\ldots,1)$

Note that $S = O(s_1) \cup \ldots \cup O(s_k)$

$|O(s)| = 1$.

$|S| = np^{-1}$; if $|G| = n$.

Since $p \mid n$ $\quad p \mid |S|$

If rest of Orbits have cardinality p.

then $1 + (k-1)p = |S|$.

But $p \mid |S|$

$\therefore \exists$ element with cardinality 1 other than identity

Infact $p-1$ elts. exist with $x^p = 1$.

Group action by Conjugation

$G \times G \longrightarrow G$

$(g, x) \longrightarrow gxg^{-1}$

$\text{Stb}(x) = \{g \in G \mid gx = xg\}$

$=: G(x)$ known as centraliser

$O(x) = \{gxg^{-1} \mid g \in G\}$

↳ Conjugacy class of $x := Gx$

$|O(x)| = 1$

$\Rightarrow gxg^{-1} = x \quad \forall g \in G$

$\Rightarrow x \in Z(G)$

$x \in Z(G)$ iff $|O(x)| = 1$

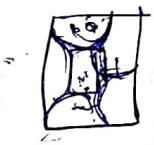
$$|G| = \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} |G_i|$$

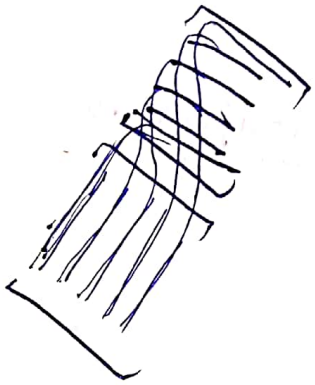
This is class eqn:

$$|G| = |Z(G)| + \sum_{i=1}^{m} |G_i|$$

5th Feb. Class Test

$aH = \{ah \mid h \in H\}$





aH

G/H

ab

9

[7]

aH

bH

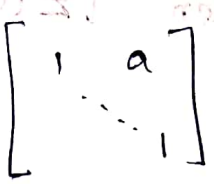
abH

29.1.19

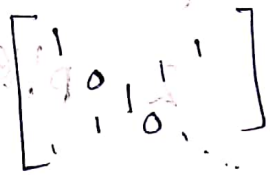
Tutorial.

Elementary matrices generate $GL_n(\mathbb{R})$.

Type 1



Type 2



Type 3

31.1.19

Defⁿ A gp. of order p^n, p is a prime number and $n \in \mathbb{N}$ is called a p group.

Propⁿ Let G be a p-group. Then $|Z(G)| \geq p$.

Let $|Z(G)| = 1$

Then $|G| = 1 + \sum_{i=1}^{n} |G_{g_i}|$

Now $p \mid |C_{g_i}| \ \forall i$ & $p \mid |G|$

$\Rightarrow p \mid 1$ which is a contradiction

$|Z(G)| \geq p$.

Corr

An group of order p^2 is abelian.

By prev. propⁿ. we have $|Z(G)| \geq p$

WTS $|Z(G)| = p^2$

Let $|Z(G)| = p$ & $x \in G \setminus Z(G)$

& $Z(G) \subsetneq C(x)$ as $x \in C(x) \setminus Z(G)$

$|C(x)| = p^2$ i.e $C(x) = G$

which is a contradiction

~~$\therefore |C(x)| =$~~

$\therefore Z(G) = G$.

$\therefore G$ is abelian gp.

Corr Any group of order p^2 is ismorphic to

$\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

↳ check if G has ~~order~~ elt. of order p^2
if yes we are done
else go for 2nd case.

Pf Let $1 \neq x \in G$. Then $|x| = p$ or $|x| = p^2$.

If $|x| = p^2$ then $G \cong \mathbb{Z}/p^2\mathbb{Z}$

O/w Assume that $\nexists$ any elt. of order p^2 in G.
∴ All elts. of order p.
Consider $H = \langle x \rangle$
Let $1 \neq y \in G \setminus H$
Then $|y| = p$. Let $K = \langle y \rangle$
$H \cap K = \{1\}$. [As its order divides p. But cannot be p.]

Consider HK [normal subgrps H, K as G abelian]
↳ is a subgroup.

$|HK| = p^2$
∴ $G = HK$

Consider $\phi : H \times K \longrightarrow HK \equiv G$
$(h, k) \longrightarrow hk$

is an isomorphism.
∴ $G \cong H \times K$, where $H \cong \mathbb{Z}/p\mathbb{Z}$ and $K \cong \mathbb{Z}/p\mathbb{Z}$

∴ $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

Sylow's Th...

Defn Let G be a group of order $p^\alpha m$ where $p \mid m$ then the subgroup of order p^α is called a Sylow p subgroup of G.

n_p denotes no. of Sylow p subgroup of ~~order~~ G.

Thm Let G be a gp. of order $p^\alpha m$ where $p \nmid m$ and p is a prime number. Then

1. Sylow p subgroup exist. ie a sbgp of order p^α in G
2. If P is a Sylow p-subgp of G and Q is any p-subgroup of G then $\exists\ g \in G$ s.t. $Q \subseteq gPg^{-1}$ ie Q is contained in some conjugate of P

In particular any two Sylow p subgroups are conjugate in G.

Remark If there is only 1 Sylow p-subgp of G then it will be a normal subgroup of G

3. $n_p \equiv 1 \pmod p$ and $n_p \mid m$

Group of order 15

$15 = 3 \cdot 5$

$n_3 \equiv 1 \pmod 3 \qquad n_3 \mid 5 \Rightarrow n_3 = 1$

$n_5 \equiv 1 \pmod 5 \qquad n_5 \mid 3 \Rightarrow n_5 = 1$

Let H and K be subgroups of G of order 3 and 5 respectively

Let $H = \langle x \rangle$ and $K = \langle y \rangle$

$|x| = 3$ and $|y| = 5$

$H \cap K = \{1\}$

H, K are normal subgroups.

$|HK| = 15 \Rightarrow HK = G$

$G \cong H \times K$

$\cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/15\mathbb{Z}$

$\therefore \exists$ only one gp. of ord. 15 which is $\mathbb{Z}/15\mathbb{Z}$ upto isomorphism.

Ex Characterize the group of order 21