III RELATIONSHIP BETWEEN CONTINUITY AND THE EXISTENCE OF PARTIAL DERIVATIVES

A function can have partial derivatives with respect to both x and y at a point without being continuous there. On the other hand a continuous function may not have partial derivatives.

EXAMPLE: Show that the function

$$f(x,y) = \begin{cases} (x+y)\,\mathrm{Sin}\left(\frac{1}{x+y}\right), & (x,y) \neq (0, \\ 0, & (x,y) = (0, \end{cases}$$

is continuous at $(0,0)$ but its partial derivatives do not exist at $(0,0)$.

SOLUTION:

$$|f(x,y)-f(0,0)|$$

$$= \left|(x+y)\sin\left(\frac{1}{x+y}\right)\right|$$

$$\leq |x+y|$$

$$\left.\begin{aligned} &\leq |x|+|y| \\ &\leq \sqrt{2}\cdot\sqrt{x^2+y^2} \end{aligned}\right\}^{*}$$

$$< \epsilon$$

Choose $\delta < \frac{\epsilon}{\sqrt{2}}$, then

$|f(x,y)-f(0,0)| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} <$

$$\Rightarrow \boxed{\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)}$$

Hence the function is continuous at $(0,0)$

(*)

$$\left(|x| - |y|\right)^2 \geq 0$$

$$\Rightarrow \quad x^2 + y^2 \geq 2\,|x|\,|y|$$

$$\Rightarrow \quad 2\left(x^2+y^2\right) \geq x^2 + y^2 + 2\,|x|\,|y|$$

$$\Rightarrow \quad 2\left(x^2+y^2\right) \geq \left(|x| + |y|\right)^2$$

$$\Rightarrow \quad \left(|x| + |y|\right) \leq \sqrt{2}\sqrt{\left(x^2+y^2\right)}$$

<u>ALTERNATIVE</u> : 1

$$\lim_{(x,y)\to(0,0)} (x+y)\,\sin\left(\frac{1}{x+y}\right)$$

put $x + y = t$

Then $\displaystyle\lim_{t\to 0} t\,\sin\left(\frac{1}{t}\right)$

$$= 0$$

Now consider

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \ \sin\left(\frac{1}{\Delta x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \sin\left(\frac{1}{\Delta x}\right)$$

$\Rightarrow \ f_x(0, 0)$ does not exist.

Similarly $f_y(0, 0)$ does not exist.

EXAMPLE:

Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0, 0)$.

SOLUTION:

Choose the path $y = mx$

The limit

$$\lim_{x \to 0} \frac{x \cdot mx}{x^2 + 2m^2x^2}$$

$$= \frac{m}{1+m^2}$$

depends on the path.

Hence the function is not continuou at $(0,0)$.

Now consider

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x}$$

$$= 0 \quad = f_x(0,0)$$

$$\lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y}$$

$$= 0 = f_y(0,0)$$

$\Rightarrow$ The partial derivatives f_x and f_y exist at $(0,0)$

THEOREM: [SUFFICIENT CONDITION FOR CONTINUITY AT (0,0)]

One of the first order partial derivative exist and is bounded in the neighbourhood of (x_0, y_0) and the other exist at (x_0, y_0).

PARTIAL DERIVATIVES OF HIGHER ORD

$$\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}_{f_{xx}}, \quad \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}_{f_{yx}}, \quad \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}_{f_{xy}}$$

are called second order partial derivatives of f.

The derivatives f_{xy} and f_{yx} are called mixed derivatives.

If the mixed derivatives f_{yx} and f_{xy} are continuous in an open domain D, then at any point $(x, y) \in D$

$$f_{xy} = f_{yx}.$$

EXAMPLE: Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function

$$f(x, y) = \begin{cases} \dfrac{xy^3}{x + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

SOLUTION:

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y}$$

$$= 0$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \cdot y^3}{\Delta x + y^2} \cdot \frac{1}{\Delta x}$$

$$= y$$

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{x \cdot \Delta \cancel{y^3}^{\,2}}{x + \Delta y^2} \cdot \frac{1}{\cancel{\Delta y}}$$

$$= 0$$

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$

$$= 0$$

$$\therefore \boxed{f_{xy}(0,0) = 0}$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(\Delta y - 0)}{\Delta y}$$

$$= 1$$

$$\therefore \boxed{f_{yx}(0,0) = 1}$$

Since $f_{xy}(0,0) \neq f_{yx}(0,0)$, f_{xy} and f_{yx} are not continuous at $(0,0)$.

III <u>CHECK:</u> For $(x,y) \neq (0,0)$

$$f_{yx}(x,y) = \frac{y^6 + 5xy^4}{(x+y^2)^3} = f_{xy}(x,y)$$

Along the path $x = my^2$ the limit $\lim_{(x,y)\to(0,0)} f_{yx}(x,y)$ depends on the path.

This implies, f_{yx} is not continuous at $(0,0)$.