Linear Algebra

Lecture 3

Definition of vector space.

→ Examples

<u>Thm</u>: (Cancelation theorem)

Given a vector space V over a field $\mathbb{F}$, $x, y, z \in V$.

If $x + z = y + z$

$\Rightarrow x = y$

<u>Corollary</u>: In a vector space V over the field $\mathbb{F}$, additive identity is unique.

<u>Pf</u>: On contrary assume that additive identity is <u>NOT</u> unique.

Let 0 and $0'$ be two additive identities.

Since 0 is additive identity, $x + 0 = x$ $\forall x \in V$. In particular, $0' + 0 = 0'$ —(1)

Since $0'$ is also an additive identity,

$0 + 0' = 0$ —(2), From (1) & (2) and Cancelation $0 = 0'$.

Corollary: Additive inverse is unique.

Thm: In vector space V over a field $\mathbb{F}$, the following statements are true.

1) $0x = 0 \quad \forall\ x \in V$

2) $(-a)x = a(-x) = -(ax)$
$\forall\ a \in \mathbb{F},\ x \in V$

3) $a0 = 0 \quad \forall\ a \in \mathbb{F}$

Proof: (1) $0x + 0x = (0+0)x$
$= 0x$

$\cancel{0x} + 0x = \cancel{0x} + 0$

$\Rightarrow \quad 0x = 0$

3) $a(0+0) = a0 + 0$

$\cancel{a0} + a0 = \cancel{a0} + 0$

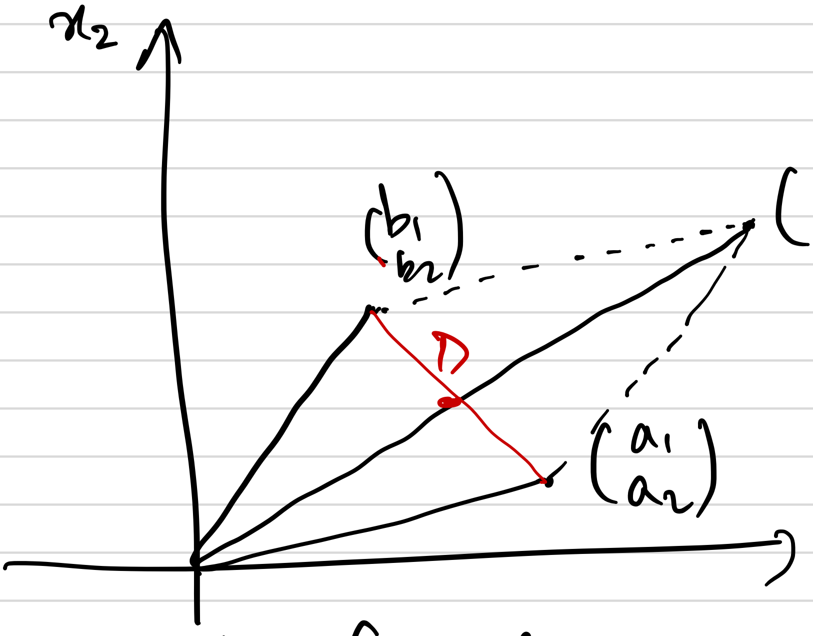
$\Rightarrow \quad a0 = 0$

Example: It is easy to check that $\mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x \in \mathbb{R} \right\}$ is an $\mathbb{R}$-vector space with addition defined as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cancel{x_1 + x_2} \; x_1+y_1 \\ \cancel{y_1 + y_2} \; x_2+y_2 \end{pmatrix}$$

and the scalar multiplication defined as $c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix}$.

Why the addition is component-wise??



$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

By mid-point formula, Co-ordinates are:

$$\frac{1}{2}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \dfrac{a_1+b_1}{2} \\[2ex] \dfrac{a_2+b_2}{2} \end{pmatrix}$$

__Example:__ Recall that $\mathbb{C}(\mathbb{R})$ denotes set of all real valued functions on $\mathbb{R}$. Then it is easy to verify that $C[a,b]$ is a $\underbrace{\text{real vector space}}_{\text{vectorspace over } \mathbb{R}}$.

Addition $(f+g)(t) = f(t) + g(t) \quad \forall t \in [a,b]$

$(\alpha f)(t) = \alpha f(t) \quad \forall t \in [a,b]$

Further consider $C_E(\mathbb{R}) \subseteq C(\mathbb{R})$ to be set of all even functions on $\mathbb{R}$.

Is $C_E(\mathbb{R})$ a vector space over $\mathbb{R}$.

An $f \in C_E(\mathbb{R})$ if

$$f(-t) = f(t) \quad \forall \; t \in \mathbb{R}.$$

Example:

Let S be the set of all real convergent sequences. It is easy to verify that S is a real vector space.

$x, y \in S$

$x = (x_n)_{n=1}^{\infty}$; $y = (y_n)_{n=1}^{\infty}$

$x+y = (x_n+y_n)_{n=1}^{\infty}$

$\alpha x = (\alpha x_n)_{n=1}^{\infty}$ for $\alpha \in \mathbb{R}$.

Let $S_0 \subseteq S$ be the set of sequences converging to 0.

Is S_0 a real vector space??

Let $x, y \in S_0$, $x+y \in S_0$

for $\alpha \in \mathbb{R}$, $\alpha x \in S_0$

Definition: Let V be a vector space over the field $\mathbb{F}$. Let $W \subseteq V$. Then W is called a vector subspace (or simply subspace) of V if W is a vector space over $\mathbb{F}$.

Trivial cases:

1) $V \subseteq V$
2) $\{0\} \subseteq V$

V and $\{0\}$ are called as trivial subspaces of V.

For a subspace $W \subsetneq V$, W is called a proper subspace of V.

We have already seen two examples of proper subspaces.

Q: How to check if a subset W of an $\mathbb{F}$-vector space V is a subspace or not??

We know that V is a vector space over $\mathbb{F}$ and hence all the axioms in the definition of a vector space are satisfied.

In particular for any subset $W \subseteq V$ the following properties hold true.

i) $\forall\, x, y \in W$, $x+y = y+x$

ii) $\forall\, x, y, z \in W$, $x+(y+z) = (x+y)+z$

Similarly, (v), (vi), (vii) and (viii) are also satisfied.

Therefore, it is enough to check whether W is closed under vector addition & scalar multiplication and ensure that $0 \in W$ in order to conclude that W is a subspace of V.

Theorem: Let V be a vector space over $\mathbb{F}$ and W be a subset of V. Then W is a subspace of V if and only if the following conditions hold.

a) $0 \in W$.

b) $x+y \in W$ for every $x, y \in W$.

c) $\alpha x \in W$ for $\alpha \in \mathbb{F}$, $x \in W$.

Theorem: Let V be a vector space over $\mathbb{F}$ and $W \subseteq V$. Then W is a subspace of V if and only if

$$\alpha x + \beta y \in W \quad \forall \alpha, \beta \in \mathbb{F}, \; x, y \in W.$$

Examples: Let $M_{m \times n}(\mathbb{R})$ denote the set of all matrices of size $m \times n$ with real entries. Is $M_{m \times n}(\mathbb{R})$ a real vector space??

Addition: Usual matrix addition.

Scalar multiplication: Usual scalar multiplication.

Consider $M_{n\times n}(\mathbb{R})$.

Let $M^{0}_{n\times n}(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$ be a subset of all matrices whose trace is zero.

Is $M^{0}_{n\times n}(\mathbb{R})$ a subspace of $M_{n\times n}(\mathbb{R})$??

Let $X, Y \in M^{0}_{n\times n}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$.

Then to check whethere

$$\alpha X + \beta Y \in M^{0}_{n\times n}(\mathbb{R}).$$

Note: $\text{trace}(\alpha X + \beta Y) = \alpha\,\text{trace}(X) + \beta\,\text{trace}(Y)$

$= \alpha \cdot 0 + \beta \cdot 0$

$= 0$ $\boxtimes$

Let $M^{S}_{n\times n}(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$ be a subset of all symmetric matrices.

(For any symmetric matrices X, Y and scalar α, β

$$(\alpha X + \beta Y)^T = \alpha X^T + \beta Y^T = \alpha X + \beta Y$$

Let $M^{d}_{n\times n}(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$ be a set of all diagonal matrices.

Trivial to say $M^{d}_{n\times n}(\mathbb{R})$ is a subspace.

Let $M^{ss}_{n\times n}(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$ be a set of all skew-symmetric matrices.

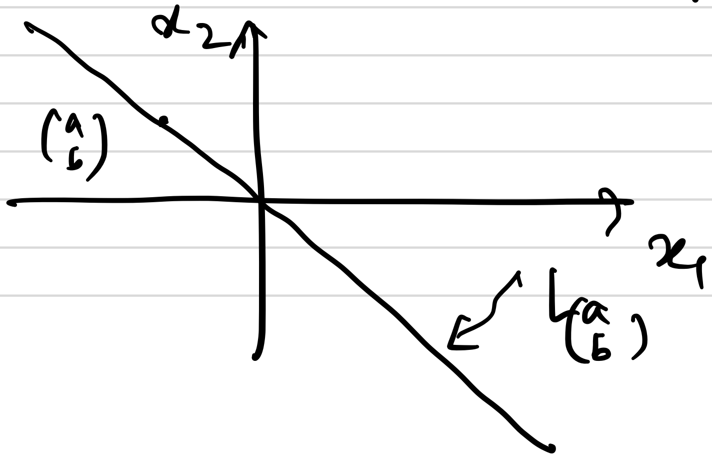
Example: We know $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

For a vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$, define

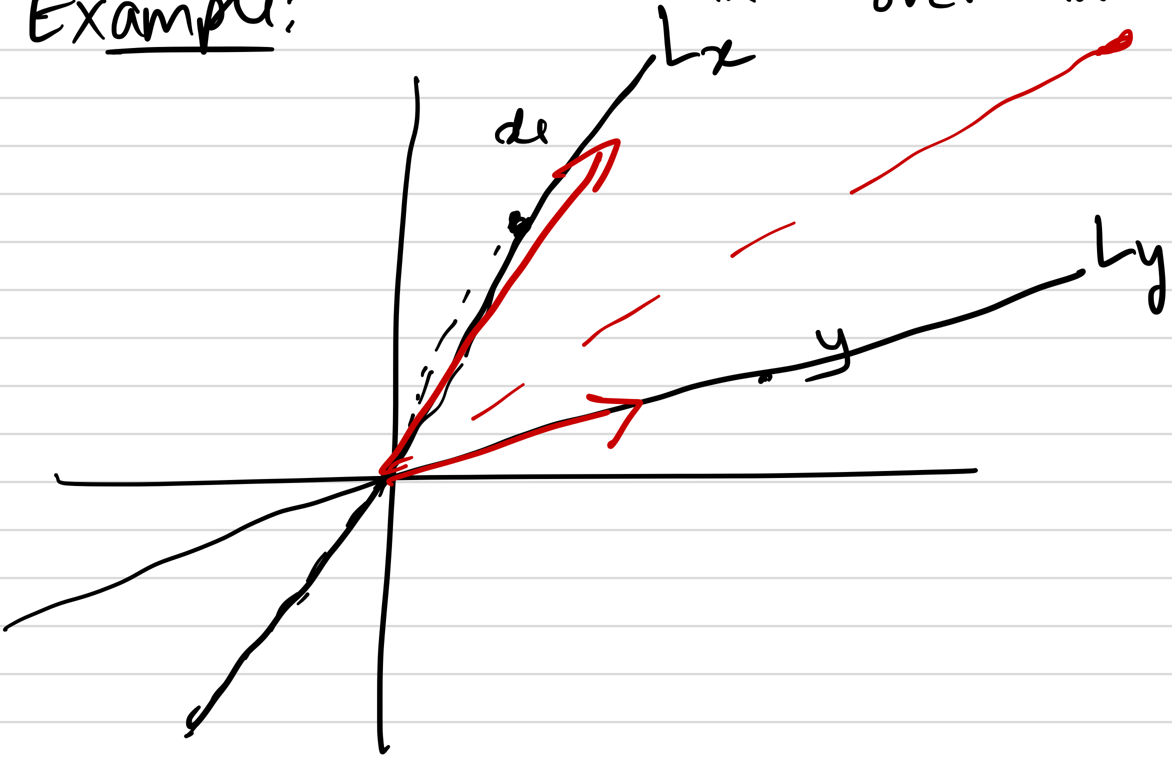
$$L_{\begin{pmatrix} a \\ b \end{pmatrix}} = \left\{ t\begin{pmatrix} a \\ b \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Is $L_{\begin{pmatrix} a \\ b \end{pmatrix}} \subseteq \mathbb{R}^2$ a subspace??



Example:

$\mathbb{R}^2$ over $\mathbb{R}$



For $x, y \in \mathbb{R}^2$ and define L_x and L_y.

$L_x = \{tx \mid t \in \mathbb{R}\}$, $L_y = \{ty \mid t \in \mathbb{R}\}$

Is $L_x \cup L_y$ a subspace of $\mathbb{R}^2$??

No. (refer to the figure)

Is $L_x \cap L_y$ a subspace of $\mathbb{R}^2$??

$$\underset{\{(\begin{smallmatrix}0\\0\end{smallmatrix})\}}{\overset{\shortparallel}{L_x \cap L_y}}$$

trivial subspace of $\mathbb{R}^2$.

Example: We know that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

Let x & y be two vectors in $\mathbb{R}^3$.

$$x, y \in \mathbb{R}^3$$

$$P_{x,y} = \{ \alpha x + \beta y \mid \alpha, \beta \in \mathbb{R} \}$$

$$P_{x,y} \subseteq \mathbb{R}^3.$$

Is $P_{x,y}$ a subspace of $\mathbb{R}^3$??

For any $u, v \in P_{x,y}$ and $a, b \in \mathbb{R}$,

$$\begin{aligned} au + bv &= a(\alpha_1 x + \beta_1 y) + b(\alpha_2 x + \beta_2 y) \\ &= (a\alpha_1 + b\alpha_2) x + (a\beta_1 + b\beta_2) y \\ &\in P_{x,y} \end{aligned}$$

$\Rightarrow P_{x,y}$ is a subspace of $\mathbb{R}^3$.

Example:

Let $x_1, y_1, x_2, y_2 \in \mathbb{R}^3$.

Let $W_1 = P_{x_1, x_2}$

$W_2 = P_{y_1, y_2}$

Is $W_1 \cup W_2$ a subspace??

Is $W_1 \cap W_2$ a subspace??