Department of Mathematics
I I T Kharagpur
MA 20101 Transform Calculus -

Mid - Autumn 2011 Max. Marks : 30 Time : 2hrs.

No. of Students : 600

Instructions :(i) Answer ALL the questions. Provide answers to all parts of each question together, otherwise it will be ignored.
(ii) L and L^{-1} denote the Laplace and inverse Laplace transforms, respectively.

1.a). Use method of partial fractions to evaluate [3+2 marks]

$$a).\ L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} \text{ and } b).\ L^{-1}\left\{\frac{6s^2+22s+18}{s^3+6s^2+11s+6}\right\}.$$

c). Use Convolution theorem to evaluate $L^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$. [2 marks]

2.a). Evaluate $L\left\{(t-\frac{\pi}{2})\sin(t)H(t-\frac{\pi}{2})\right\}$, where [3 marks]

$$H(t-a) = \begin{cases} 1, & \text{if } t > a, \\ 0, & \text{if } t < a. \end{cases}$$

b). Evaluate $L\left\{\dfrac{2\sin(t)\sinh(t)}{t}\right\}$. [2 marks]

c). Using Laplace transform, evaluate the following integral,

$$\int_{-\infty}^{\infty} \frac{x\sin(tx)}{x^2+a^2}dx.$$ [2 marks]

3.a). Evaluate $L\{|\cos(t)|\}$. [2 marks]

b). Evaluate $L\{\tan^{-1}(s+2)\}$. [2 marks]

c). Find $L\{F(t)\}$, if $F(t) = \cosh(t)\displaystyle\int_0^t e^x\cosh(x)dx$. [2 marks]

d). Find $L\{\text{erf}(\sqrt(t)\}$ and deduce that $\int_0^\infty e^{-t}\text{erf}(\sqrt(t)dt = \frac{\sqrt{2}}{2}$. [2 marks]

4). Solve the following differential equation using Laplace transform

$a).\ y''+2y'+25y = 150,\ \ y(0) = 0,\ \ y'(0) = 0,$ [3 marks]

$b).\ ty''+(2t+3)y'+(t+3)y = 3e^{-t},\ y(0) = 0.$ [5 marks]

Q:1: Use method of partial fraction to evaluate

a) $\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$ b) $\mathcal{L}^{-1}\left\{\frac{6s^2+22s+18}{s^3+6s^2+11s+6}\right\}$

Sol: a) $s^4+4a^4 = (s^2+2a^2)^2-(2sa)^2$

$$= (s^2+2a^2+2sa)(s^2+2a^2-2sa)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+2a^2+2sa)(s^2+2a^2-2sa)}\right\}$$

$$= \frac{1}{4a}\mathcal{L}^{-1}\left\{\frac{1}{s^2-2as+2a^2}-\frac{1}{s^2+2as+2a^2}\right\}$$

$$= \frac{1}{4a}\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+a^2}-\frac{1}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{4a}e^{at}\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}e^{-at}\cdot\frac{1}{4a}$$

$$= \frac{1}{4a}e^{at}\frac{\sin at}{a}-\frac{1}{4a}\cdot e^{-at}\cdot\frac{\sin at}{a}$$

$$= \frac{1}{2a^2}\sin at\,\frac{\{e^{at}-e^{-at}\}}{2}$$

$$= \frac{1}{2a^2}\sinh at\,\sin at.$$

$\square$.

Q.1 b): $\mathcal{L}^{-1}\left\{\dfrac{6s^2+22s+18}{s^3+6s^2+11s+6}\right\}$

Sol: $\mathcal{L}^{-1}\left\{\dfrac{6s^2+22s+18}{(s+1)(s+2)(s+3)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{2}{s+2}+\frac{3}{s+3}\right\}$$

$$= e^{-t}+2e^{-2t}+3e^{-3t}$$

1c) Use convolution theorem to evaluate

$\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$

Sol: $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\} = \sin t * e^{-t}$

$$= \int_0^t \sin u \; e^{-(t-u)}\,du$$

$$= e^{-t}\int_0^t \sin u \; e^{u}\,du$$

$$= \frac{1}{2}\left(\sin t - \cos t + e^{-t}\right)$$

Q:2a: Evaluate $\mathcal{L}\{(t-\pi/2)\sin t\, H(t-\pi/2)\}$

Sol: $\mathcal{L}\{(t-\pi/2)\sin t\, H(t-\pi/2)\}$

Using second shifting th. $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}\{t\sin(t+\pi/2)\}$$

$$= e^{-\pi/2\, s}\mathcal{L}\{t\cos t\}$$

$$= e^{-\pi/2\, s}\left\{-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right\}$$

$$= e^{-\pi/2\, s}\left[\frac{s^2-1}{(s^2+1)^2}\right]$$

2b) evaluate $\mathcal{L}\left\{\frac{2\sin t\sinh t}{t}\right\}$

$$= \int_s^\infty \mathcal{L}\{\sin t\,(e^t-e^{-t})\}\,ds$$

$$= \int_s^\infty \left\{\frac{1}{1+(s-1)^2} - \frac{1}{1+(s+1)^2}\right\} ds$$

$$= \tan^{-1}(s+1) - \tan^{-1}(s-1)$$

$$= \tan^{-1}\left(\frac{2}{s^2}\right)$$

$$\boxed{\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)}$$

2c) Using L.T. evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{x \sin(tx)}{x^2+a^2} \, dx$$

Sol :

$$f(t) = \int_{-\infty}^{\infty} \frac{x \sin(xt)}{x^2+a^2} \, dx$$

Taking laplace transform

$$F(s) = \int_{-\infty}^{\infty} \frac{x}{x^2+a^2} \cdot \frac{x}{x^2+s^2} \, dx$$

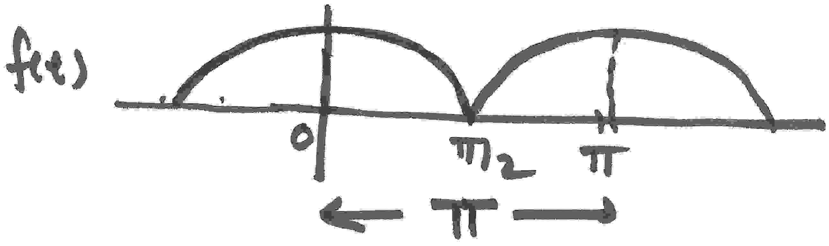
$$= \int_{-\infty}^{\infty} \left\{ \frac{1}{(x^2+s^2)} - \frac{a^2}{(x^2+a^2)(x^2+s^2)} \right\} dx$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{(x^2+s^2)} - \frac{a^2}{s^2-a^2} \cdot \left\{ \frac{1}{x^2+a^2} - \frac{1}{x^2+s^2} \right\} \right] dx$$

$$= \pi \frac{1}{s+a}$$

$$\Rightarrow \boxed{f(t) = \pi e^{-at}}$$

3a) Evaluate $\mathcal{L}\{|\cos(t)|\}$.



$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-\pi s}}\left[\int_0^{\pi/2} e^{-st}\cos t\,dt - \int_{\pi/2}^{\pi} e^{-st}\cos t\,dt\right]$$

$$= \frac{1}{1-e^{-\pi s}}\left[\frac{s+e^{-\frac{\pi s}{2}}}{1+s^2} + \frac{-se^{-\pi s}+e^{-\frac{\pi s}{2}}}{1+s^2}\right]$$

$$= \frac{1}{1-e^{-\pi s}}\left[\frac{s+2e^{-\frac{\pi s}{2}}-se^{-\pi s}}{1+s^2}\right]$$

3b) Evaluate $\mathcal{L}^{-1}\{\tan^{-1}(s+2)\}$

Sol: Let $F(s) = \tan^{-1}(s+2)$

$$\Rightarrow F'(s) = \frac{1}{1+(s+2)^2}$$

$$\Rightarrow \mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-2t}\sin t$$

$$\Rightarrow -t\,\mathcal{L}^{-1}\{F(s)\} = e^{-2t}\sin t$$

$$\boxed{\mathcal{L}\{t f(t)\} = -\frac{d}{ds}\mathcal{L}(f(t))}$$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\{F(s)\} = -\frac{e^{-2t}\sin t}{t}}$$

Q3c: Find $\mathcal{L}\{F(t)\}$ if $F(t) = \cosh t \int_0^t e^x \cosh x \, dx$

Sol: $\mathcal{L}\{\cosh t\} = \dfrac{s}{s^2-1}$

$$\mathcal{L}\{e^t \cosh t\} = \frac{s-1}{(s-1)^2-1} = \frac{s-1}{s^2-2s}$$

$$\mathcal{L}\left\{\int_0^t e^x \cosh x \, dx\right\} = \frac{s-1}{s(s^2-2s)} = \frac{s-1}{s^2(s-2)}$$

$$\therefore \mathcal{L}\left\{\cosh t \int_0^t e^x \cosh x \, dx\right\} = \frac{1}{2}\,\frac{s-2}{(s-1)^2(s-3)} + \frac{1}{2}\,\frac{s}{(s+1)^2(s-1)}$$

Q: 3d): $\mathcal{L}\{\operatorname{erf}(\sqrt{t})\}$ and deduce $\int_0^\infty e^{-t} \operatorname{erf}(\sqrt{t}) \, dt = \dfrac{\sqrt{2}}{2}$.

Sol: $\mathcal{L}\{\operatorname{erf}(\sqrt{t})\} = \dfrac{1}{s(s+1)^{1/2}}$

$$\Rightarrow \int_0^\infty e^{-st} \operatorname{erf}(\sqrt{t}) \, dt = \frac{1}{s\sqrt{s+1}}$$

putting $s = 1$:

$$\int_0^\infty e^{-t} \operatorname{erf}(\sqrt{t}) \, dt = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Q:4 : Solve the following D.E. using L.T.

a) $y'' + 2y' + 25y = 150$; $y(0) = 0$, $y'(0) = 0$.

b) $t y'' + (2t+3)y' + (t+3)y = 3e^{-t}$; $y(0) = 0$.

Sol: a): $s^2 Y(s) + 2s Y(s) + 25 Y(s) = 150/s$

$$Y(s) = \frac{6}{s} + \frac{6s+12}{s^2+2s+25}$$

$$= \frac{6}{s} - 6\frac{(s+1)}{(s+1)^2+(\sqrt{24})^2} - \frac{6}{\sqrt{24}}\frac{\sqrt{24}}{[(s+1)^2+(\sqrt{24})^2]}$$

$$y(t) = 6 - 6e^{-t}\cos\sqrt{24}\,t - \frac{6}{\sqrt{24}}e^{-t}\sin\sqrt{24}\,t$$

b) taking L.T. we get

$$\frac{dY}{ds} - \frac{1}{s+1}Y(s) = \frac{-3}{(s+1)^3}$$

$$\text{I.F.} = e^{\ln\left(\frac{1}{s+1}\right)} = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)^2} + c(s+1)$$

Since $\lim_{s\to\infty} Y(s) = 0 \Rightarrow c = 0$

$$Y(s) = \frac{1}{(s+1)^2}$$

$$\Rightarrow \boxed{y(t) = te^{-t}}$$