Linear Algebra

Lecture 5

Example : $\mathbb{Q}$: field of rationals.

$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$

Then $\mathbb{Q}(\sqrt{2})$ is a vector space over $\mathbb{Q}$.
It can be shown that $1, \sqrt{2} \in \mathbb{Q}(\sqrt{2})$ are linearly independent vectors.
In fact, $\mathbb{Q}(\sqrt{2}) = \mathrm{span}\{1, \sqrt{2}\}$
Thus $\mathbb{Q}(\sqrt{2})$ is a finitely generated vector space over $\mathbb{Q}$.

Linear span :

Let V be a vector space over a field $\mathbb{F}$.

Lemma: Let $u_1, u_2, \dots, u_n \in V$. Then

$$\mathrm{span}\{u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_n\} \subseteq \mathrm{span}\{u_1, \dots, u_n\}$$

for any $i \in [n] = \{1, 2, \dots, n\}$

Pf: Without loss of generality, assume $i = n$.

$$\sum_{j=1}^{n-1} a_j u_j = 0 \cdot u_n + \sum_{j=1}^{n-1} a_j u_j \in \mathrm{span}\{u_1, \dots, u_n\}$$

Theorem: Let $n > 1$. Then there exists $i \in [n]$ such that

$\text{span}\{u_1, u_2, \dots, u_n\} = \text{span}\{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n\}$

if and only if $u_1, \dots, u_n$ are linearly dependent in V.

Proof: Let us assume

$\text{span}\{u_1, \dots, u_n\} = \text{span}\{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n\}$

Note that $u_i \in \text{span}\{u_1, \dots, u_n\}$

$$\therefore \quad u_i = b_1 u_1 + \dots + b_{i-1} u_{i-1} + b_{i+1} u_{i+1} + \dots + b_n u_n$$

for some $b_i \in \mathbb{F}$.

$$\Rightarrow b_1 u_1 + \dots + b_{i-1} u_{i-1} - u_i + b_{i+1} u_{i+1} + \dots + b_n u_n = 0$$

$\Rightarrow u_1, u_2, \dots, u_n$ are linearly dependent.

Conversely, assume that $u_1, \dots, u_n$ are linearly dependent.

Then to prove

$\text{span}\{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n\} = \text{span}\{u_1, \dots, u_n\}$

Since $u_1, \dots, u_n$ are linearly dependent, $\exists\, a_i \in \mathbb{F}$, not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$$

Let $a_i \neq 0$

$$u_i = -\frac{a_1}{a_i}u_1 - \frac{a_2}{a_i}u_2 - \cdots - \frac{a_{i-1}}{a_i}u_{i-1}$$

$$-\frac{a_{i+1}}{a_i}u_{i+1} - \cdots - \frac{a_n}{a_i}u_n \quad -(1)$$

Now take $u \in \operatorname{span}\{u_1, \dots, u_n\}$.

In order to prove the theorem, we want show $u \in \operatorname{span}\{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n\}$

Since $u \in \operatorname{span}\{u_1, \dots, u_n\}$

$$u = \sum_{j=1}^{n} b_j u_j$$

substitute (1)

$$= b_1u_1 + \cdots + b_{i-1}u_{i-1} + b_i u_i + b_{i+1}u_{i+1} + \cdots$$

$$\cdots + b_nu_n$$

$$u = b_1 u_1 + \cdots + b_{i-1} u_{i-1}$$

$$+ b_i \left[-\frac{a_1}{a_i} u_1 - \cdots - \frac{a_{i-1}}{a_i} u_{i-1} - \frac{a_{i+1}}{a_i} u_{i+1} - \cdots - \frac{a_n}{a_i} u_n \right]$$

$$+ b_{i+1} u_{i+1} + \cdots + b_n u_n$$

$$u = \left(b_1 - \frac{a_1 b_i}{a_i}\right) u_1 + \left(b_2 - \frac{a_2 b_i}{a_i}\right) u_2$$

$$+ \cdots + \left(b_{i-1} - \frac{a_{i-1} b_i}{a_i}\right) u_{i-1}$$

$$+ \left(b_{i+1} - \frac{a_{i+1} b_i}{a_i}\right) u_{i+1} + \cdots$$

$$\cdots + \left(b_n - \frac{a_n b_i}{a_i}\right) u_n$$

$$\Rightarrow \quad u \in \operatorname{span}\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n\} \quad \blacksquare$$

Example: $\mathbb{R}^3$ over $\mathbb{R}$.

$$\operatorname{span}\left\{ \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{u_2}, \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{u_3}, \underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{u_4}, \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{u_5} \right\}$$

$$u_1 + u_2 - u_3 + 0\cdot u_4 + 0\cdot u_5 = 0$$

Let $i=3$ and $a_3 = -1$

$$u_3 = u_1 + u_2 + 0\cdot u_4 + 0\cdot u_5$$

From the above theorem,

$$\operatorname{span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

$$= \operatorname{Span}\left\{ \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{v_3}, \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{v_4} \right\}$$

$$0\cdot v_1 + v_2 - v_3 + v_4 = 0$$

$i=2$, $a_i = 1$, $\quad v_2 = 0.v_1 + v_3 - v_4$

From the above theorem,

$$\operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$=\operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

This process in this example is summarized in the following corollary.

Corollary: Let $u_1, u_2, \cdots, u_n \in V$. Assume that not all u_i's are zero. Then there exists $d \geq 1$ and integers

$$1 \leq i_1 < i_2 < \cdots < i_d \leq n \quad \text{such that}$$

$u_{i_1}, u_{i_2}, \cdots, u_{i_d}$ are linearly independent and

$$\operatorname{span}\{u_1, \cdots, u_n\} = \operatorname{span}\{u_{i_1}, \cdots, u_{i_d}\}$$

Proof: If $u_1, \cdots, u_n$ are linearly independent vectors, then clearly $d = n$ and

$$i_k = k \quad \text{for } k = 1, \cdots, n.$$

Let $u_1, u_2, \dots, u_n$ be linearly dependent.
Then $\exists\, a_i \in \mathbb{F}$, not all zero, such that

$$\sum_{i=1}^{n} a_i u_i = 0$$

Assume that $a_i \neq 0$. Then from the previous theorem,

$$\operatorname{span}\{u_1, \dots, u_n\} = \operatorname{span}\{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n\}$$

Note that not all $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n$ are zero.

If $u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_n$ are linearly independent, then $d = n-1$

$$i_1 = 1, \dots, i_{i-1} = i-1, \quad i_i = i+1, \dots, i_{n-1} = n$$

and $\operatorname{span}\{u_1, \dots, u_n\} = \operatorname{span}\{u_{i_1}, \dots, u_{i_d}\}$

If $u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_n$ are linearly dependent,

$\vdots$

$\blacksquare$

Corollary: Let ~~be~~ V be a finitely generated non-trivial vector space over a field $\mathbb{F}$. Then there exists $d \in \mathbb{N}$ and 'd' linearly independent vectors $v_1, v_2, \ldots, v_d \in V$ such that

$$V = \text{span}\{v_1, \ldots, v_d\}$$

$\blacksquare$

Lemma: Let $n > m \geq 1$ be integers. Then any $w_1, w_2, \ldots, w_n \in \text{span}\{u_1, \ldots, u_m\}$ are linearly dependent.

Proof: Since $w_1, \ldots, w_n \in \text{span}\{u_1, \ldots, u_m\}$

$$w_i = a_{i1}u_1 + a_{i2}u_2 + \cdots + a_{im}u_m$$

$$i = 1, 2, \ldots, n$$

Let $x_i \in \mathbb{F}$ be such that

$$\sum_{i=1}^{n} x_i w_i = 0$$

$\blacksquare$

__Theorem__: Let $V = \text{Span}\{v_1, \dots, v_n\}$ and assume that $v_1, \dots, v_n$ are linearly independent. Then the following hold.

1. Any vector $u \in V$ can be expressed uniquely as a linear combination of $v_1, \dots, v_n$.

2. For any integer $N > n$, any N vectors in V are linearly dependent.

3. Assume that $u_1, u_2, \dots, u_n$ are linearly independent.

   Then $V = \text{span}\{u_1, \dots, u_n\}$

__Proof__:

1. Assume that u can be expressed in more than one way.

$$u = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i$$

$$\Rightarrow \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i = 0$$

$$\Rightarrow \alpha_i = \beta_i \text{ for } i = 1, 2, \dots, n$$ because v_i's are l.i.

2. Already proved in the lemma above

3. Let $u_1, u_2, \ldots, u_n$ be linearly independent. Consider any vector $v \in V$.

Then from part(2), $u_1, \ldots, u_n, v$ are linearly dependent.

Then $\exists$ scalars $a_1, \ldots, a_n$ and $b \in \mathbb{F}$, not all zero, such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n + bv = 0 \quad (*)$$

Let $b = 0$

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0$$

$$\Rightarrow \quad a_1 = \cdots = a_n = 0 \quad (\because u_1, \ldots, u_n \text{ are linearly independent.})$$

Contradicts $u_1, u_2, \ldots, u_n, v$ are linearly dependent.

$\therefore b \neq 0 \quad \Rightarrow \quad v = -\dfrac{a_1}{b} u_1 - \dfrac{a_2}{b} u_2 \cdots - \dfrac{a_n}{b} u_n$

$$\in \operatorname{span}\{u_1, \ldots, u_n\}$$

Dimension and Basis :

A vector space V over a field $\mathbb{F}$ is called finite dimensional vector space if there exists a finite set $\{x_1, x_2, \dots, x_n\} \subseteq V$ such that

$$V = \text{span}\{x_1, \dots, x_n\}.$$

Ex: $\mathbb{R}^3$ over $\mathbb{R}$.
$\mathbb{F}_2^n$ over $\mathbb{F}_2$

} Componentwise addition & scalar multiplication.

If a vector space V over $\mathbb{F}$ is not finite dimensional, then it is called as infinite dimensional vector space.

Ex: $P(\mathbb{R})$: set of all polynomials with real coefficients.

$$P(\mathbb{R}) = \text{span}\{1, x, x^2, \dots\}$$

Ex: $\mathbb{R}$ over $\mathbb{Q}$ $\Leftrightarrow$ Difficult to prove!!

The dimension of the vector space V is the number of vectors in any spanning linearly independent set.

(Note that if spanning linearly independent set has n vectors, then any collection of $n+1$ vectors from V is linearly dependent).

The dimension of V is generally denoted as

$$\dim_{\mathbb{F}} V \quad \text{or} \quad \dim V.$$

Assume $\dim_{\mathbb{F}} V = n$ and

$$V = \operatorname{span}\{v_1, v_2, \dots, v_n\}$$

Then $\{v_1, v_2, \dots, v_n\}$ is called a basis of V.