Date 24/10/2017

Lecture 24

Properties of Fourier Transform

Th-1 / (Linearity property)

If $F(\alpha)$ & $G(\alpha)$ be the Fourier transforms of $f(x)$ & $g(x)$ resly. Then

$$\mathcal{F}\{c_1 f(x) + c_2 g(x)\} = c_1 \mathcal{F}\{f(x)\} + c_2 \mathcal{F}\{g(x)\}.$$

&

$$\mathcal{F}^{-1}\{c_1 F(\alpha) + c_2 G(\alpha)\} = c_1 f(x) + c_2 g(x);$$

where c_1 & c_2 are constants.

Eg; (1) $\mathcal{F}\{2e^{-t}u(t) + 3e^{-2t}u(t)\} = \mathcal{F}\{2e^{-t}u(t) + 3e^{-2t}u(t)\}$

$= 2.\,\mathcal{F}\{e^{-t}u(t)\} + 3\,\mathcal{F}\{e^{-2t}u(t)\} = \frac{2}{(1+i\omega)} + \frac{3}{(2+i\omega)}$

H.W

(2) If $f(t) = \begin{cases} 4, & -3 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$, find $\hat{f}(\omega)$?

Hint. - $\hat{f}(\omega) = 8/3 \sin(3\omega)$

~~Th-2~~ HW (change of scale)

If $F(\alpha)$ is the Fourier transform of $f(x)$. Then the Fourier transform of $f(ax)$ is

$$\frac{1}{|a|} F(\alpha/a), \quad a \neq 0$$

EX

Also, $\mathcal{F}_s\{f(ax)\} = \frac{1}{a} F_s(\alpha/a), \; a>0$

$\mathcal{F}_c\{f(ax)\} = \frac{1}{a} F_c(\alpha/a), \; a>0$

H.W. EX e.g. ? $\mathcal{F}(e^{-x^2}) = ?$

We know that $\mathcal{F}\{f(x)\} = \mathcal{F}(e^{-x^2/2}) = e^{-\omega^2/2} = \hat{f}(\omega)$

$\boxed{\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a} \text{ were } a=1/2}$

Now $f(x) = e^{-x^2} = f(\sqrt{2}x)$.

$\therefore \mathcal{F}(e^{-x^2}) = \mathcal{F}\{f(x)\} = \mathcal{F}\{f(\sqrt{2}x)\} = \frac{1}{\sqrt{2}} \mathcal{F}(\omega/\sqrt{2})$

$= \frac{1}{\sqrt{2}} \cdot e^{-\frac{1}{2}(\omega/\sqrt{2})^2} = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$. //

3/

Shifting on t - axis

(First shifting theorem)

If $\mathcal{F}\{f(t)\} = \hat{f}(\omega)$, &

t_0 is any real no. (constant), then

$$\mathcal{F}\{f(t-t_0)\} = e^{-i\omega t_0}\hat{f}(\omega)$$

i.e. Shifting (or, translating) a function in one domain corresponds to a multiplication by a complex exponential fn in the other domain.

Soln:- From the defn, we get

$\mathcal{F}\{f(t-t_0)\}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-t_0)e^{-i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}e^{-i\omega t_0}\int_{-\infty}^{\infty} f(t-t_0)e^{-i\omega(t-t_0)}\,dt.$$

We have,

$$\hat{f}(\omega) = \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

&

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}\,d\omega$$

Let $t - t_0 = \tau$, as $t \to \infty$, $\tau \to \infty$

then $dt = d\tau$, $t \to -\infty$, $\tau \to -\infty$

$$= e^{-i\omega t_0} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega\tau}\, d\tau}_{\tau=}$$

$$= e^{-i\omega t_0} \hat{f}(\omega)$$

$$\therefore \mathcal{F}\{f(t-t_0)\} = e^{-i\omega t_0} \hat{f}(\omega)$$

$$\Rightarrow \mathcal{F}^{-1}\{e^{-i\omega t_0} \hat{f}(\omega)\} = f(t-t_0).$$

***Ex/** Find $\mathcal{F}^{-1}\left\{\dfrac{e^{4i\omega}}{3+i\omega}\right\}$

Sol^n:- Now, $\mathcal{F}\{e^{-3t} u_0(t)\}$

Let $f(t) = \begin{cases} 0, & t < 0 \\ e^{-\alpha t}, & t \geq 0, \ \alpha > 0 \end{cases}$

$= e^{-\alpha t} u_0(t)$, where $u_0(t)$ is the unit step f^n.

~ f ^ $f(t)$ has a jump discontinuity at $t=0$ & is of magnitude 1.

Also, $\int_{-\infty}^{\infty} |f(t)|\, dt = \int_0^{\infty} e^{-\alpha t}\, dt$

$$= \frac{1}{\alpha},$$

∴ $f(t)$ is absolutely integrable.

∴ the Fourier Transform of $f(t)$ exist.

We have, $\mathcal{F}\{f(t)\} = \hat{f}(\omega) \overset{or}{=} \boxed{F(\omega)}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \mathop{lt}_{p\to\infty} \int_0^p e^{-\alpha t - i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \mathop{lt}_{p\to\infty} \left[-\frac{e^{-(\alpha+i\omega)t}}{\alpha+i\omega} \right]_0^p = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(\alpha+i\omega)}$$

$$\therefore \mathcal{F}\left[e^{-\alpha t} u_0(t)\right] = \frac{1}{\sqrt{2\pi}} \frac{1}{(\alpha + i\omega)}$$

$$\Rightarrow \boxed{\mathcal{F}^{-1}\left[\frac{1}{\alpha + i\omega}\right] = \sqrt{2\pi} \cdot e^{-\alpha t} u_0(t)}$$

We have,

$$\mathcal{F}\left\{e^{-3t} u_0(t)\right\} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(3 + i\omega)}$$

or, $$\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(3 + i\omega)}\right] = e^{-3t} u_0(t)$$

$$= f(t) \quad \text{(a)}$$

Using the first shifting theorem (or, time shifting theorem) we get

$$\mathcal{F}^{-1}\left[\frac{e^{-(-4i\omega)}}{\sqrt{2\pi}\,(3 + i\omega)}\right] = f(t - (-4))$$

$$= f(t+4)$$

$$= e^{-3(t+4)} u_{-4}(t). \quad \text{(from (a))}$$

$$\mathcal{F}^{-1}\left\{\frac{e^{4i\omega}}{\sqrt{2\pi}\,(3+i\omega)}\right\}$$

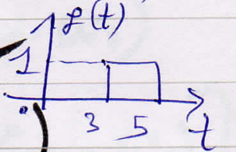
$$= \begin{cases} 0, & t < -4 \\ e^{-3(t+4)}, & t \geq -4. \end{cases}$$

H.W

Ex2/ Use the time-shifting property to find the Fourier transform of the function $f(t) = \begin{cases} 1, & 3 \leq t \leq 5 \\ 0, & \text{otherwise.} \end{cases}$

* * *

Hint: $\hat{f}(\omega) = e^{-4i\omega}\,\frac{2}{\omega}\sin\omega$



Th-4/ (Frequency shifting)

or, Second shifting theorem for Fourier Transform

If $\mathcal{F}\{f(t)\} = \hat{f}(\omega)$, ω_0 is any real no., then

$$\mathcal{F}\left[e^{i\omega_0 t} f(t)\right] = \hat{f}(\omega - \omega_0)$$

Proof:- From the defn, we get

proof :- From the defⁿ, we get

$$\hat{\mathcal{F}}\left[\underbrace{e^{i\omega_0 t} f(t)}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega_0 t} \cdot e^{-i\omega t} f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{e^{-i(\omega-\omega_0) t}} f(t)\, dt$$

$$= \hat{f}(\omega - \omega_0)$$

Q) Use Frequency shift property to obtain the F.T of the modulated wave

$f(t) = f(t) \cos \omega_0 t$,

where $f(t)$ is an arbitrary signal whose F.T is $\hat{f}(\omega)$.

Solⁿ:- $\mathcal{F}\{f(t)\} = \frac{1}{2}\mathcal{F}\{f(t)\, e^{i\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{f(t)\, e^{-i\omega_0 t}\}$

$A(\omega) = \hat{f}(\omega) = \frac{1}{2}\tilde{F}(\omega-\omega_0) + \frac{1}{2}\tilde{F}(\omega+\omega_0)$ [by 2nd shifting]

This implies,

H.W

EX Find the inverse F.T

$$L(\omega) = 2\,\frac{\sin\{3(\omega-2\pi)\}}{(\omega-2\pi)}$$

Here, $\omega_0 = 2\pi$ (2nd shift)

$l(t) = \mathcal{F}^{-1}\{L(\omega)\} = e^{i2\pi t} P_3(t)$, $P_3(t) = \begin{cases} 1, & -3 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$

$$\mathcal{F}^{-1}\left[\hat{f}(\omega - \omega_0)\right] = e^{i\omega_0 t} f(t).$$

$\hat{f}(\omega) = F(\omega)$

$\hat{f}(\omega) = A(\omega)$

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Th-5/ (Modulation Theorem)

If $\mathcal{F}\{f(t)\} = \hat{f}(\omega)$ & ω_0 is any real no., then

$$\mathcal{F}\left[f(t)\cos(\omega_0 t)\right] = \frac{1}{2}\left[\hat{f}(\omega+\omega_0) + \hat{f}(\omega-\omega_0)\right]$$

& $$\mathcal{F}\left[f(t)\sin(\omega_0 t)\right] = \frac{i}{2}\left[\hat{f}(\omega+\omega_0) - \hat{f}(\omega-\omega_0)\right]$$

[EX/~~Do~~ it by 2nd shifting theorem]

Sol^n:- L.H.S $= \mathcal{F}\left[f(t)\cos(\omega_0 t)\right]$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,\underbrace{\cos(\omega_0 t)}\, e^{-i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left(\frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2}\right)e^{-i\omega t}\,dt$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i(\omega+\omega_0)}\, dt\right.$$

$$\left. + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i(\omega-\omega_0)}\, dt\right]$$

$$= \frac{1}{2}\left[\hat{f}(\omega+\omega_0) + \hat{f}(\omega-\omega_0)\right]$$

EX (the other one).

(or, Time differentiation Property)

Th-6/ Fourier transform of derivatives.

Let $f(t)$ be continuous & $f^{(k)}(t)$, $k = 1, 2, \ldots, n$ be piece-wise continuous on every interval $[-l, l]$ & $\int_{-\infty}^{\infty} |f^{(k-1)}(t)|\, dt$, $k = 1, 2, \ldots, n$ converge.

If $\mathcal{F}\{f(t)\} = \hat{f}(\omega)$, then

$$\mathcal{F}\{f^{(n)}(t)\} = (i\omega)^n \hat{f}(\omega)$$

proof :- Done

[ie, $\mathcal{F}\{f'(t)\} = i\omega \hat{f}(\omega)$

$\mathcal{F}\{f''(t)\} = -\omega^2 \hat{f}(\omega)$

(ie, Differentiating a function is said to amplify the higher frequency components because of the additional multiplying factor ω)

Q) Find the solution of the d. eqn

$$y' - 2y = H(t)\, e^{-2t}, \quad -\infty < t < \infty,$$

using Fourier transforms, where $H(t) = u_0(t)$ is the unit step fn.

Soln :- Applying Fourier transform to the d. eqn, we get

$$\mathcal{F}[y'] - 2\mathcal{F}[y] = \mathcal{F}[H(t)\, e^{-2t}]$$

$$\Rightarrow i\omega\, \hat{y}(\omega) - 2\,\hat{y}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(2+i\omega)}$$

$$\Rightarrow \hat{y}(\omega) = \frac{-1}{\sqrt{2\pi}\,(2+i\omega)(2-i\omega)}$$

$$\left[\mathcal{F}\left\{e^{-\alpha t} H(t)\right\} = \frac{1}{\sqrt{2\pi}} \frac{1}{(\alpha+i\omega)}\right]$$

$$= -\frac{1}{\sqrt{2\pi}\,(4+\omega^2)},$$

where $\mathcal{F}[y(t)] = \hat{y}(\omega)$.

$$\Rightarrow \quad y(t) = \mathcal{F}^{-1}\left\{\hat{y}(\omega)\right\}$$

$$= \mathcal{F}^{-1}\left[-\frac{1}{\sqrt{2\pi}\,(4+\omega^2)}\right]$$

$$= -\frac{1}{4} e^{-2|t|}$$

$$= \begin{cases} -\frac{1}{4} e^{2t}, & t<0 \\ -\frac{1}{4} e^{-2t}, & t>0. \end{cases}$$

~~E/~~ we know, $\mathcal{F}\left\{e^{-a|t|}\right\} = \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{(a^2+\omega^2)}$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{0} e^{at} \cdot e^{-i\omega t}\, dt + \frac{1}{\sqrt{2}} \int_{0}^{\infty} e^{-at} e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a-i\omega} + \frac{1}{a+i\omega}\right)$$

where, $f(t) = e^{-a|t|} = \begin{cases} e^{at}, & t<0 \\ e^{-at}, & t>0 \end{cases}$

Fourier Transform has an interesting property

(Symmetry property)

(or, Duality property)

Let $\mathcal{F}\{f(t)\} = \hat{f}(\omega)$. Then

$$\mathcal{F}\{\hat{f}(t)\} = f(-\omega).$$

[i.e., it states that we can interchange the time & frequency domains provided we put $(-\omega)$ rather than ω in the second term, this corresponding to a reflection in the vertical axis. If $f(t)$ is even fn then this latter is irrelevant.]

Sol^n:- We have,

$$f(t) = \mathcal{F}^{-1}[\hat{f}(\omega)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega.$$

$$\Rightarrow f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\, e^{ist}\, ds$$

(changing ω by s)

Since ω is a dummy variable of integration. Hence, Let $\boxed{t = -\omega}$, we get

$$\sqrt{2\pi}\, f(-\omega) = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{-i\omega s}\, ds$$

$$\Rightarrow f(-\omega) = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\, e^{-i\omega s}\, ds}$$

$$\Rightarrow f(-\omega) = \mathcal{F}\left[\hat{f}(t)\right]$$

$$\Rightarrow \mathcal{F}\left[\hat{f}(t)\right] = f(-\omega).$$

Ex/ Find the Fourier transform of $f(t) = \dfrac{1}{5+it}$

Sol^n: We shall use the symmetry property to solve it

We know, $\dfrac{1}{\sqrt{2\pi}} \dfrac{1}{(5+i\omega)} = \mathcal{F}\left\{\underline{H(t)\, e^{-5t}}\right\}$

Let $b(t) = H(t)\, e^{-5t} \to (1)$

Then $\mathcal{F}\{b(t)\} = \mathcal{F}\{H(t)\, e^{-5t}\}$

$$= \frac{1}{\sqrt{2\pi}\,(5+i\omega)} = \hat{b}(\omega).$$

Using the symmetry result we get

$$\mathcal{F}\{\hat{b}(t)\} = b(-\omega).$$

or, $\mathcal{F}\{\hat{b}(t)\} = \mathcal{F}\left\{\frac{1}{\sqrt{2\pi}\,(5+it)}\right\}$

$$= b(-\omega) \quad (\text{from } (1))$$

$$= H(-\omega)\, e^{+5\omega}.$$

$$= \sqrt{2\pi}\, H(-\omega)\, e^{+5\omega}$$

$\Rightarrow \mathcal{F}\{f(t)\} = \hat{f}(\omega) = \sqrt{2\pi}\, H(-\omega)\, e^{5\omega}$

$$= \begin{cases} \sqrt{2\pi}\, e^{+5\omega}, & \omega \le 0 \\ 0, & \omega > 0. \end{cases} \quad //$$

Q) If the Fourier sine transform of $f(x)$ is $\frac{\alpha}{1+\alpha^2}$, find $f(x)$. , $\alpha > 0$

Soln:- We have,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{\alpha}{1+\alpha^2}\right) \sin \alpha x \, d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{[(\alpha^2+1)-1]}{\alpha(1+\alpha^2)} \sin \alpha x \, d\alpha$$

$\alpha > 0$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \alpha x}{\alpha} d\alpha - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \alpha x}{\alpha(1+\alpha^2)} d\alpha.$$

Recall,

$$\mathcal{F}_S\{f(x)\} = \frac{\alpha}{1+\alpha^2} = \hat{f}_S(\omega)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_S(\omega) . \sin(\omega x) d\omega$$

($\omega = \alpha$) here

But, $\int_0^\infty \frac{\sin \alpha x}{\alpha} d\alpha = \pi/2$ (how?)

Hence, $$f(x) = \sqrt{\pi/2} - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin(\alpha x)}{\alpha(1+\alpha^2)}\, d\alpha$$

Differentiating under the integral sign. $\rightarrow (1)$

$$\frac{df(x)}{dx} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cancel{\alpha} \cos(\alpha x)}{\cancel{\alpha}(1+\alpha^2)}\, d\alpha$$

$$\Rightarrow \frac{df}{dx} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos(\alpha x)}{1+\alpha^2}\, d\alpha \rightarrow (2)$$

$$\Rightarrow \frac{d^2 f}{dx^2} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\alpha \sin(\alpha x)}{1+\alpha^2}\, d\alpha \rightarrow (3)$$

From (1), (2) & (3), it follows that

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\alpha \sin(\alpha x)}{1+\alpha^2}\, d\alpha - \sqrt{\pi/2} + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin(\alpha x)}{\alpha(1+\alpha^2)}\, d\alpha$$

$$\frac{d^2 f(x)}{dx^2} - f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha - \sqrt{\pi/2}$$

$$= \sqrt{\pi/2} - \sqrt{\pi/2} = 0$$

$$\Rightarrow \frac{d^2 f}{dx^2} - f = 0$$

$$\Rightarrow f(x) = c_1 e^{x} + c_2 e^{-x}$$

A.E

$\longrightarrow (4)$ $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

$$\left[\frac{\alpha}{1+\alpha^2} + \frac{1}{\alpha(1+\alpha^2)}\right] = \frac{(\alpha^2+1)}{\alpha(1+\alpha^2)} = \frac{1}{\alpha}$$

$$\Rightarrow \frac{df(x)}{dx} = c_1 e^{x} - c_2 e^{-x}$$

When $x = 0$, $c_1, c_2 = ?$

[we use eqn (1) only, as our d. eqn is obtained from that $f(x)$]

$$f(0) = \sqrt{\pi/2} \quad [\text{by } (1)]$$

$$\& \ \frac{df(0)}{dx} = -\sqrt{\frac{2}{\pi}} \underbrace{\int_0^\infty \frac{d\alpha}{1+\alpha^2}}_{} \ (\text{by } (2))$$

$$= \sqrt{\pi/2}. \quad (\text{how?})$$

Using (4), we get

$$c_1 + c_2 = \sqrt{\pi/2}$$

$$c_1 - c_2 = -\sqrt{\pi/2}$$

$$\Rightarrow c_1 = 0, \quad c_2 = \sqrt{\pi/2}.$$

$$\therefore f(x) = \boxed{\sqrt{\pi/2}\, e^{-x}}$$

H.W.

~~Pr~~ **Differentiation w.r.t frequency**

(or, Frequency differentiation property)

Let $f(t)$ be piecewise continuous on every interval $[-l, l]$. Let $\int_{-\infty}^{\infty} |t^n f(t)|\, dt$ converge. Then

$$\mathcal{F}[t^n f(t)] = i^n \hat{f}^{(n)}(\omega).$$

$$\Rightarrow \mathcal{F}^{-1}\left[\hat{f}^{(n)}(\omega)\right] = (-i)^n t^n f$$

Hint:-

$$\hat{f}'(\omega) = \frac{d}{d\omega}\left[\hat{f}(\omega)\right]$$

$$= \frac{d}{d\omega}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d}{d\omega}\left[f(t)e^{-i\omega t}\right]dt = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[t f(t)\right]e^{-i\omega t}\,dt$$

$$= -i\,\mathcal{F}\left[t f(t)\right] \quad \text{etc}$$

Th:-/ **Fourier Transform of an integral**

Th:-/ Let $f(t)$ be piece-wise continuous on every ~~fini~~ interval $[-l, l]$, & $\int_{-\infty}^{\infty}|f(t)|\,dt$ converge

Let $\mathcal{F}\{f(t)\} = \hat{f}(\omega)$ & $\hat{f}(\omega)$ satisfy $\hat{f}(0) = 0$

Then

$$\tilde{F}\left[\int_{-\infty}^{t} f(\tau)\,d\tau\right] = \frac{1}{i\omega}\hat{f}(\omega).$$

Hint:- $\int_{-\infty}^{t} f(\tau)\,d\tau = \int_{-\infty}^{\infty} f(\tau)\,u(t-\tau)\,d\tau$
$= f(t) * u(t)$

H.W

EX/ Find the inverse Fourier transform of

$$\left(\sqrt{\pi}\,\omega\, e^{-\omega^2/8}\right) \Big/ \left(4\sqrt{2}\, i\right)$$

Hint:- we have

$$\frac{\sqrt{\pi}\,\omega}{4\sqrt{2}\,i} e^{-\omega^2/8} = -\frac{\sqrt{\pi}}{\sqrt{2}\,i}\frac{d}{d\omega}\left(e^{-\omega^2/8}\right)$$

Apply (Diff. w.r.t frequency) formula.

$$\therefore \tilde{F}^{-1}\left[\frac{\sqrt{\pi}\,\omega}{4\sqrt{2}\,i} e^{-\omega^2/8}\right] = -\frac{\sqrt{\pi}}{i\sqrt{2}}\left\{\tilde{F}^{-1}\left[\hat{f}'(\omega)\right]\right\}, \text{ where } \hat{f}(\omega) = e^{-\omega^2/8}$$

Ans:- $\tilde{F}^{-1}\left(\frac{\sqrt{\pi}\,\omega}{4\sqrt{2}\,i} e^{-\omega^2/8}\right) \boxed{= t e^{-2t^2}}$

Ex/ Using convolution theo. for Fourier transform find

$$\mathcal{F}^{-1}\left[\frac{1}{(12+7i\omega-\omega^2)}\right]$$

Sol:- We have

$$12+7i\omega-\omega^2 = (4+i\omega)\cdot(3+i\omega)$$

$$\therefore \mathcal{F}^{-1}\left[\frac{1}{(4+i\omega)(3+i\omega)}\right] = \mathcal{F}^{-1}\left[\frac{1}{(4+i\omega)}\cdot\frac{1}{(3+i\omega)}\right]$$

But $\mathcal{F}^{-1}\left(\frac{1}{4+i\omega}\right) = \sqrt{2\pi}\, e^{-4t} H(t)$

& $\mathcal{F}^{-1}\left(\frac{1}{3+i\omega}\right) = \sqrt{2\pi}\, e^{-3t} H(t)$, Using the convolution theorem we get

$$F^{-1}\left[\frac{1}{(4+i\omega)(3+i\omega)}\right]$$

$$= \frac{1}{\cancel{\sqrt{2\pi}}}\left[\cancel{\sqrt{2\pi}} \cdot e^{-4t} H(t)\right] * \left[\sqrt{2\pi}\, e^{-3t} H(t)\right]$$

Recall,

$$f * g = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp$$

$$\frac{1}{\sqrt{2\pi}}(f * g) = \hat{F}^{-1}\left[\hat{f}(\omega) \cdot \hat{g}(\omega)\right]$$

$$= \sqrt{2\pi} \int_{-\infty}^{\infty} e^{-4\tau} H(\tau)\, e^{-3(t-\tau)} H(t-\tau)\, d\tau$$

$$= \sqrt{2\pi}\, e^{-3t} \int_{-\infty}^{\infty} e^{-\tau}\, \underbrace{H(\tau)\, H(t-\tau)}\, d\tau$$

But $H(\tau)\,H(t-\tau) = \begin{cases} 0, & \text{for } \tau < 0 \ \& \ \tau > t \\ 1, & \text{for } 0 < \tau < t \end{cases}$

$$\therefore \ = \sqrt{2\pi}\, e^{-3t} \int_0^t e^{-\tau}\, d\tau$$

$$= \sqrt{2\pi}\, e^{-3t} \left[-e^{-\tau}\right]_0^t = \sqrt{2\pi}\, e^{-3t}\left[1 - e^{-t}\right]_{t \geq 0}$$

$$\therefore \hat{\mathcal{F}}^{-1}\left[\frac{1}{(4+i\omega)(3+i\omega)}\right]$$

$$\underline{\underline{H.W}} = \sqrt{2\pi}\, e^{-3t}\left[1-e^{-t}\right],\ t \geq 0$$

Ex Conjugation

$$\hat{\mathcal{F}}\left[\overline{f}(x)\right] = \overline{\hat{f}(-\omega)}$$

where $\overline{f}(x)$ & $\overline{\hat{f}(-\omega)}$

are the complex conjugate of $f(x)$ & $\hat{f}(-\omega)$.

Time - Reversal property:

$$\mathcal{F}\{f(-t)\}(\omega) = \mathcal{F}\{f(t)\}(-\omega).$$

Fourier transform of Dirac-delta fn

We know that the Laplace transform of the Dirac-Delta fn is

$$\mathcal{L}\left[\delta(t)\right] = 1$$

We also proved the filtering property of Dirac-Delta fn as

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

Slly, we can prove that having defined the delta fn as

$$\delta(t) = \lim_{k\to 0} \frac{1}{k}\left[H(t) - H(t-k)\right]$$

where $H(t)$ is the unit step fn.

$$\mathcal{F}\{\sqrt{2\pi}\,\delta(t)\} = 1.$$

i.e.,
$$H(t) - H(t-k) = \begin{cases} 0, & t<0 \\ 1, & 0 \le t < k \\ 0, & t \ge k \end{cases}$$

$\Updownarrow$

i.e., $\mathcal{F}\left[\delta(t)\right] = \lim_{k\to 0}\left\{\frac{1}{k}\,\mathcal{F}\left[H(t) - H(t-k)\right]\right\}$

$= \lim_{k\to 0} \frac{1}{k\sqrt{2\pi}} \int_0^k 1\cdot e^{-i\omega t}\,dt \quad \cancel{= \lim_{k\to 0} \frac{1}{\sqrt{2\pi}\,k}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_0^k}$

$= \lim_{k\to 0} \frac{1}{\sqrt{2\pi}\,k}\left[\frac{1 - e^{-i\omega k}}{i\omega}\right] = \frac{1}{\sqrt{2\pi}} \Rightarrow \mathcal{F}\left[\sqrt{2\pi}\,\delta(t)\right] = 1$

Duality property :–

we know that if $f(t) = p_1(t) = \begin{cases} 1, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

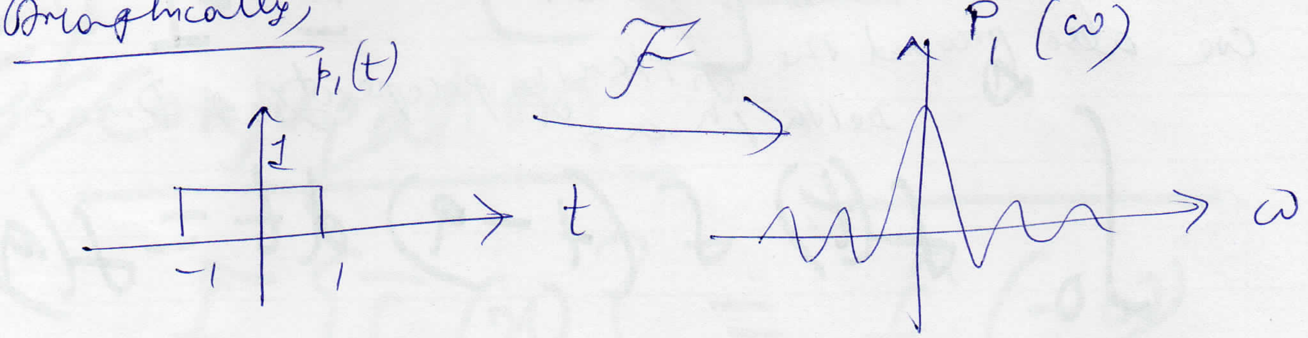
then $\hat{f}(\omega) = \dfrac{2\sin\omega}{\omega}$

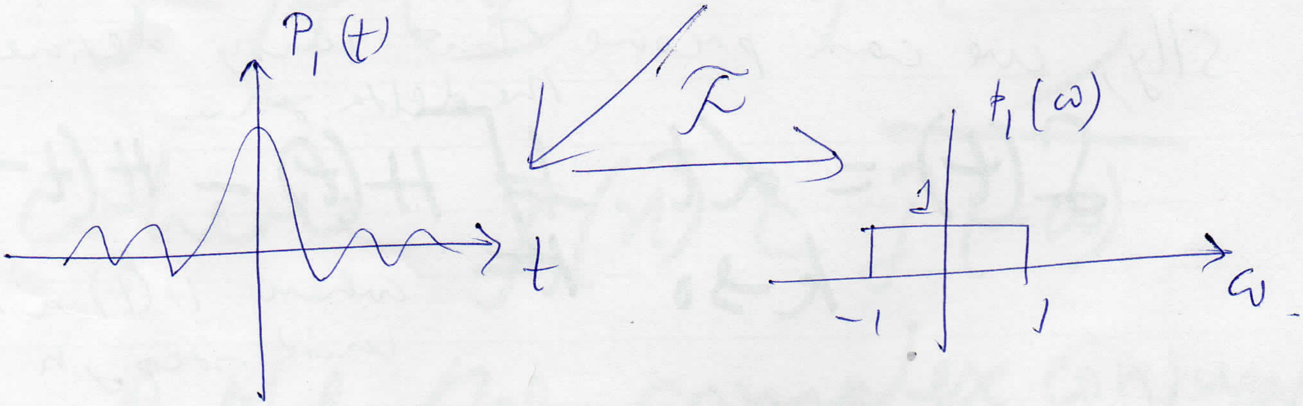
Then, by the duality property

$$\mathcal{F}\left\{2\,\frac{\sin t}{t}\right\} = f(-\omega) = p_1(-\omega)$$

$$= p_1(\omega) \quad [\text{since } p_1(\omega) \text{ is even}]$$

Graphically,





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cont:– $\therefore\ \mathcal{F}^{-1}(1) = \sqrt{2\pi}\,\delta(t)$ $\left[\text{Using } \int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a) \text{ \& first shifting theorem}\right]$

Also, $\mathcal{F}[\delta(t-a)] = e^{-ia\omega}\hat{f}(\omega) = \dfrac{e^{-ia\omega}}{\sqrt{2\pi}}$ $\left[\text{ie, } \mathcal{F}[\hat{f}(t)] = f(-\omega)\right]$

Using duality property, we get

$$\mathcal{F}(1) = \sqrt{2\pi}\,\delta(-\omega) = \sqrt{2\pi}\,\delta(\omega)$$

[as $\delta(\omega)$ is even f^n]

—X—

To prove (Scaling property of F.T)

$$\mathcal{F}\{f(at)\}(\omega) = \frac{1}{|a|}\mathcal{F}\{f(t)\}(\omega/a).$$

By Time-Reversal property.

$$\mathcal{F}\{f(-t)\}(\omega) = \mathcal{F}\{f(t)\}(-\omega)$$

Now, $\mathcal{F}\{f(-t)\}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-t)e^{-i\omega t}\,dt$

Let $u = -t$, when $t \to \infty$, $u \to \infty$
$\Rightarrow dt = -du$, $t \to \infty$, $u \to -\infty$.

$$= \frac{1}{\sqrt{2\pi}}\int_{u=\infty}^{-\infty} f(u)e^{+i\omega u}(-du)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{i\omega u}\,du$$

$$= \hat{f}(-\omega) = \mathcal{F}\{f(t)\}(-\omega).$$

Then, if $a<0$, then $a = -|a|$, so that

$$\mathcal{F}\{f(at)\} = \mathcal{F}\, f(-\underline{|a|}\, t)(\omega)$$

$$= \mathcal{F}\{f(|a|t)\}(-\omega)$$

$$= \frac{1}{|a|}\mathcal{F}\{f(t)\}(-\omega/|a|)$$

$$= \frac{1}{|a|}\mathcal{F}\{f(t)\}(\omega/a).$$