A_n is Simple for $n \geq 5$

Q1 (c) $\phi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]$.

$f(x) \in \mathbb{Z}[x]$.

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$.

$$\phi(f(x)) = \frac{(f(x) - f(0))}{x}.$$

$$\phi(f(x)) = a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_1$$

ϕ is surjective.

$\ker \phi = \mathbb{Z}$.

$\therefore$ By 1st isomorphism Thm,

$$\mathbb{Z}[x] / \mathbb{Z} \cong \mathbb{Z}[x].$$

$\boxed{\text{Group homomorphism}}$

Q. $R = \mathbb{Z}[\sqrt{2}] = \{a + \sqrt{2}\, b \mid a, b \in \mathbb{Z}\}$.

If $R/M_1 \cong R/M_2$

is $M_1 = M_2$?

Ans. $\varphi_1 : \mathbb{Z}[\sqrt{2}] \longrightarrow \mathbb{Z}/7\mathbb{Z}$.

$\varphi_2 : \mathbb{Z}[\sqrt{2}] \longrightarrow \mathbb{Z}/7\mathbb{Z}$.

In $\mathbb{Z}/7\mathbb{Z}$, $\bar{2} = \bar{3}^2$.

also $\bar{2} = \bar{4}^2$

$\varphi_1(a + b\sqrt{2}) = \bar{a} + \bar{3}\bar{b}$

$\varphi_2(a + b\sqrt{2}) = \bar{a} + \bar{4}\bar{b}$.

Check φ_1 & φ_2 are surjective ring homomorphism.

$R/\ker\varphi_1 \cong R/\ker\varphi_2 \cong \mathbb{Z}/7\mathbb{Z}$.

Note that $-3+\sqrt{2} \in \ker \varphi_1$

$$\varphi_1(-3+\sqrt{2}) = -\bar{3} + \bar{3} = \bar{0}.$$

But $\varphi_2(-3+\sqrt{2}) = -\bar{3} + \bar{4} = \bar{1} \neq \bar{0}$

$$-3+\sqrt{2} \notin \ker \varphi_2.$$

Thus $\ker \varphi_1 \neq \ker \varphi_2$.

$$\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]/\mathbb{Z}.$$

$$\phi(f(x)) = xf(x) + \mathbb{Z}.$$

Fact: A_n is a normal subgp of S_n.

Q Does A_n has a non-trivial normal subgp?

Thm. A_n is simple for $n \geq 5$.

Claim 1: A_n is gen by 3-cycles.

Any $\sigma \in A_n$ is a product of even number of trasposetions.

The transpotions will be of the form either

$(a\ b)(c\ d)$ or $(a\ b)(a\ c)$

Since $(a\ b)(c\ d) = (a\ c\ b)(a\ c\ d)$

and $(a\ b)(a\ c) = (a\ c\ b)$.

$\therefore A_n$ is gen by 3-cycles.

Claim 2: Let N be a normal subgp of A_n.

WTS N contain a 3-cycle.

Suppose $\rho \neq (1)$ and $\rho \in N$.

Pick $\rho \in N$ s.t ρ has maximum number of fixed pts. [i.e $i \in [n]$ is called a fixed pt. of $\sigma \in S_n$ if $\sigma(i) = i$].

S_7

$(1\ 2\ 3\ 4) \to$ 3 fixed pts $(5, 6, 7)$.

$(1\ 2\ 3)(4\ 5\ 6) \to$ 1 fixed pt (7).

Write f as product of disjoint cycles

$$f = (a_1 \ldots a_m)(b_1 \ldots b_l) \ldots$$

<u>WTS</u> In cycle decomposition of f all cycles have equal length.

Suppose $m < l$. Then $f^m \in N$, and f^m has more fixed pts than f which is a contradiction as f has maximum no. of fixed pts.

<u>Therefore</u> all cycles have same length in the cycle decomposition of f.

First assume that $m \geq 4$ then

$$\tau = (a_1\, a_2)(a_3\, a_4)\, \rho \left[(a_1\, a_2)(a_3\, a_4)\right]^{-1}$$

$$= (a_2\, a_1\, a_4\, a_3 \ldots a_m)(b_1\, b_2 \ldots b_m) \ldots$$

$\in N$.

$$\tau \rho = (a_2\, a_1\, a_4\, a_3\, a_5 \ldots a_m)(b_1\, b_2 \ldots b_m)$$
$$(a_1\, a_2 \ldots a_m)(b_1\, b_2 \ldots b_m)$$

$\tau \rho (a_1) = a_1$

$\Rightarrow \tau \rho$ has more fixed pt than ρ which is a contradiction.

Now consider $m = 3$ and ρ has at least 2 disjoint cycles

say $\rho = (a_1\, a_2\, a_3)(b_1\, b_2\, b_3)$

$$\tau = (a_1 a_2)(a_3 b_1)\,\rho\,[(a_1 a_2)(a_3 b_1)]^{-1}$$

$$= (a_2\ a_1\ b_1)(a_3\ b_2\ b_3) \in N.$$

$\tau\rho(a_1) = a_1$

$\Rightarrow \tau\rho \in N$ has more fixed pts than ρ, which is a contradiction.

Now let $m = 2$ then

$$\rho = (a_1 a_2)(a_3 a_4) \cdots (a_{2r-1}, a_{2r}),$$

$r \geq 2.$

$$\tau = (a_1\ a_2\ a_3)\,\rho\,(a_1\ a_2\ a_3)^{-1}$$

$$= (a_2\ a_3)(a_1\ a_4)(a_5\ a_6) \cdots (a_{2r-1}, a_{2r})$$

$\tau\rho = (a_1\ a_3)(a_2\ a_4)$ has more fixed pts if $r \geq 3$.

Let $r = 2$, $\rho = (a_1\ a_2)(a_3\ a_4)$

$$\tau = (a_1\ a_2\ a_5)(a_1\ a_2)(a_3\ a_4)(a_1\ a_2\ a_5)^{-1}$$

$$= (a_2\ a_5)(a_3\ a_4) \in N.$$

$$\tau\rho = \text{—} \quad \text{—} \quad \text{—}$$

$$= (a_1\ a_5\ a_2) \in N.$$

Claim 3: If a 3-cycle $\in N$ then all 3 cycles $\in N$.

Suppose $(1\ 2\ 3) \in N$ WTS $(i\ j\ k) \in N$.

Consider the permutation $\beta = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$

Then $\beta (1\ 2\ 3) \beta^{-1} = (i\ j\ k) \in N$.

$\therefore\ N = A_n$.