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Lecture 6

Generating function for the Bessel's function $J_n(x)$

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Prove that

$$\exp\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

or, show that when n is a positive integer, $J_n(x)$ is the co-efficient of z^n in the expansion of $\exp\left\{\left(\frac{x}{2}\right)\times\left(z-\frac{1}{z}\right)\right\}$

i.e., $e^{\left(x/2\right)\left(z-1/z\right)}$ in ascending & descending powers of z.

Also, show that J_n is the coefficient of z^{-n} multiplied by $(-1)^n$ in the expansion of the above expression.

[Note :- $\exp\left\{\left(\frac{x}{2}\right)\times\left(z-\frac{1}{z}\right)\right\}$ is called the generating function for $J_n(x)$.]

Proof :- We know,

$$\exp\left\{\left(\frac{x}{2}\right)\times\left(z-\frac{1}{z}\right)\right\} = e^{\left(\frac{xz}{2}-\frac{x}{2z}\right)}$$

$$= e^{\frac{xz}{2}} \cdot e^{-\frac{x}{2z}}.$$

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$$= \left[1 + \left(\frac{x}{2}\right)z + \frac{\left(\frac{x}{2}\right)^2 z^2}{2!} + \cdots + \left(\frac{x}{2}\right)^n \frac{z^n}{n!} + \frac{\left(\frac{x}{2}\right)^{n+1} z^{n+1}}{(n+1)!} + \cdots\right].$$

$$\times \left[1 - \left(\frac{x}{2}\right)z^{-1} + \frac{\left(\frac{x}{2}\right)^2 z^{-2}}{2!} + \cdots + \frac{\left(\frac{x}{2}\right)^n (-1)^n z^{-n}}{n!} + \frac{\left(\frac{x}{2}\right)^{n+1} (-1)^{n+1} z^{-(n+1)}}{(n+1)!} + \cdots\right] \longrightarrow ①$$

The co-efficient of z^n in the product ① is obtained by multiplying the coefficients of $z^n, z^{n+1}, z^{n+2}, \ldots$ in the first bracket with the coefficient of $z^0, z^{-1}, z^{-2}, \ldots$ in the second resply.

$\therefore$ Coefficient of z^n in product (i)

$$= \left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{(n+2)!} \frac{1}{2!} - \cdots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}.$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}$$

$\left[\because (n+m)! = \Gamma(n+m+1),\ (n+m)\right.$ being positive integer$\left.\right]$

$$= J_n(x)$$

The coefficient of z^{-n} in the product (1) is obtained by multiplying the coefficients of $z^{-n}, z^{-(n+1)}, z^{-(n+2)}, \ldots$ of the second bracket with the coefficients of $z^0, z^1, z^2, \ldots$ in the first bracket only.

$\therefore$ Coefficient of z^{-n} in product (1)

$$= \left(\frac{x}{2}\right)^n \frac{(-1)^n}{n!} + \left(\frac{x}{2}\right)^{n+2} \frac{(-1)^{n+1}}{(n+1)!}$$

$$+ \left(\frac{x}{2}\right)^{n+4} \frac{(-1)^{n+2}}{(n+2)!\,2!} + \cdots$$

$$= (-1)^n \left[\left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \cdot \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \cdot \frac{1}{(n+2)!\,2!} - \cdots\right]$$

$$= (-1)^n J_n(x), \text{ as before}$$

Thus, the coefficient of z^{-n}

$$= (-1)^n J_n(x)$$

$$\Rightarrow J_n(x) = (-1)^n \times (\text{coeff. of } z^{-n})$$

Finally, in the product (1) the co-efficient of z^0 is obtained by multiplying the coefficient of $z^0, z^1, z^2, \ldots$ in the first bracket with the coefficient of $z^0, z^{-1}, z^{-2}, \ldots$ in the second bracket & is thus

$$= 1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 \cdot \left(\frac{1}{2!}\right)^2$$

$$- \left(\frac{x}{2}\right)^6 \cdot \left(\frac{1}{3!}\right)^2 + \cdots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$= J_0(x).$$

we observe that the coefficients of

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$z^0, (z - z^{-1}), (z^2 + z^{-2}), \ldots, \{z^n + (-1)^n z^{-n}\}, \ldots$

are $J_0(x), J_1(x), J_2(x), \ldots, J_n(x), \ldots$ resp.

Thus, (1) gives

$$\exp\left\{\left(\frac{x}{2}\right)\times(z - 1/z)\right\}$$

$$= J_0(x) + (z - z^{-1})J_1(x) + (z^2 + z^{-2})J_2(x) + (z^3 - z^{-3})J_3(x) + \cdots + \{z^n + (-1)^n z^{-n}\}J_n(x)$$

$$= \sum_{n=-\infty}^{\infty} z^n J_n(x), \text{ as } J_{-n}(x) = (-1)^n J_n(x)$$

§/ Trigonometric expansions involving Bessel's functions

Show that

(i) $\cos(x\sin\phi) = J_0 + 2\cos(2\phi)\cdot J_2 + 2\cos(4\phi)\cdot J_4 + \cdots$

(ii) $\sin(x\sin\phi) = 2\sin\phi\cdot J_1 + 2\sin(3\phi)J_3 + \cdots$

(iii) $\cos(x\cos\phi) = J_0 - 2\cos(2\phi)J_2 + 2\cos(4\phi)J_4 - \cdots$

(iv) $\sin(x\cos\phi) = 2\cos\phi J_1 - 2\cos(3\phi)J_3 + 2\cos(5\phi)J_5 - \cdots$

(11)

(v) $\cos x = J_0 - 2J_2 + 2J_4 - \cdots$

$$= J_0(x) + 2\sum_{n=1}^{\infty} (-1)^n J_{2n}(x)$$

(vi) $\sin x = 2J_1 - 2J_3 + 2J_5 - \cdots$

$$= 2\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x).$$

prm 2 :- We know that

$$e^{\frac{x}{2}(z-1/z)} = J_0 + (z - z^{-1})J_1 + (z^2 - z^{-2})J_2 + (z^3 - z^{-3})J_3 + \cdots \rightarrow (1)$$

$\textcircled{2}$

Let $z = e^{i\phi}$

so that $z^{n} = e^{in\phi}$

$z\, z^{-n} = e^{-in\phi}$.

Thus $\textcircled{1}$ gives

$$e^{(x/2)}\left(e^{i\phi} - e^{-i\phi}\right) = J_0 + \left(e^{i\phi} - e^{-i\phi}\right) J_1$$

$$+ \left(e^{2i\phi} + e^{-2i\phi}\right) J_2$$

$$+ \left(e^{3i\phi} - e^{-3i\phi}\right) J_3$$

$$+ \ldots$$

Also

$$\cos(n\phi) = \left(e^{ni\phi} + e^{-ni\phi}\right)/2$$

$$\sin(n\phi) = \left(e^{ni\phi} - e^{-ni\phi}\right)/2i$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$e^{xi\sin\phi} = J_0 + 2i\sin\phi\, J_1$$

$$+ 2\cos(2\phi)\, J_2$$

$$+ 2i\sin(3\phi)\, J_3 + \cdots$$

$$\Rightarrow \cos(x\sin\phi) + i\sin(x\sin\phi)$$

$$= \left(J_0 + 2\cos(2\phi)J_2 + 2\cos(4\phi)J_4 + \cdots\right) + i\left(2\sin\phi\, J_1 + 2\sin(3\phi)J_3 + \cdots\right)$$

Equating real & imaginary parts, we have

$$\cos(x\sin\phi) = J_0 + 2\cos(2\phi)J_2 + 2\cos(4\phi)J_4 + \cdots \rightarrow (1)$$

$$\& \sin(x\sin\phi) = 2\sin\phi\, J_1 + 2\sin(3\phi)J_3 + 2\sin(5\phi)J_5 + \cdots \rightarrow (2)$$

iii) Hint :-

Replace ϕ by $(\pi/2 - \phi)$ in eq^n (1), we obtain

$$\cos(x\cos\phi) = J_0 - 2\cos(2\phi)J_2 + 2\cos(4\phi)J_4 - \dots$$

HW iv), v), vi)
