CONVERGENCE OF IMPROPER INTEGRALS:

PROPER INTEGRALS:

$\int_a^b f(x)\,dx$ Range of integration is finite and integrand $f(x)$ is bounded.

IMPROPER INTEGRALS:

INTEGRAL $\int_a^b f(x)\,dx$ is called improper

(i) $a = -\infty$ or $b = \infty$ or both and $f(x)$ is bounded (first kind)

(ii) $f(x)$ is unbounded at one or more points of $a \le x \le b$ (second kind)

(iii) Both conditions (i) & (ii) (third kind or mixed kind)

Examples: $\int_0^\infty \cos x \, dx$ first kind

$\int_0^1 \frac{dx}{x-1}$ second kind

$\int_0^\infty \frac{dx}{(1-x)^2}$ third kind

Evaluation of integrals of first kind:

(i) $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$

(ii) $\int_{-\infty}^b f(x)\,dx = \lim_{R\to\infty} \int_{-R}^b f(x)\,dx$

(iii) $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1\to\infty}\int_{-R_1}^{c} f(x)\,dx + \lim_{R_2\to\infty}\int_{c}^{R_2} f(x)\,dx$$

OR

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{R_1\to\infty \\ R_2\to\infty}}\int_{-R_1}^{R_2} f(x)\,dx.$$

<u>Examples:</u> (i) $$\int_0^{\infty} \sin x\,dx = \lim_{R\to\infty}\int_0^{R} \sin x\,dx$$

$$= \lim_{R\to\infty}(1-\cos R)$$

does not exist.

(ii) $$\int_2^{\infty} \frac{2x^2}{x^4-1}\,dx = \lim_{R\to\infty}\int_2^{R} \frac{2x^2}{x^4-1}\,dx$$

$$= \lim_{R\to\infty}\int_2^{R}\left(\frac{1}{x^2+1}+\frac{1}{x^2-1}\right)dx$$

$$= \lim_{R\to\infty}\left[\int_2^{R}\frac{1}{x^2+1}\,dx + \frac{1}{2}\int_2^{R}\frac{1}{x-1}\,dx - \frac{1}{2}\int_2^{R}\frac{1}{x+1}\,dx\right]$$

$$= \lim_{R\to\infty}\left[\tan^{-1}R - \tan^{-1}(2) + \frac{1}{2}\ln\left(\frac{R-1}{R+1}\right) + \frac{1}{2}\ln 3\right]$$

$$= \frac{\pi}{2} - \tan^{-1}(2) + \frac{1}{2}\ln(3).$$

Evaluation of improper integrals of the second kind:

(i) if $f(x)\to\infty$ as $x\to b$ then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+}\int_a^{b-\varepsilon} f(x)\,dx$$

(ii) if $f(x)\to\infty$ as $x\to a$ then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+}\int_{a+\varepsilon}^{b} f(x)\,dx$$

(iii) if $f(x) \to \infty$ as $x \to c$ only. Here $a < c < b$.

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon \to 0+} \int_{c+\varepsilon}^b f(x)\,dx$$

(iv) if $f(x) \to \infty$ as $x \to a$ and $x \to b$.

$$\int_a^b f(x)\,dx = \lim_{\substack{\varepsilon_1 \to 0+ \\ \varepsilon_2 \to 0+}} \int_{a+\varepsilon_1}^{b-\varepsilon_2} f(x)\,dx .$$

Examples: $\int_0^1 \frac{dx}{\sqrt{1-x}}$

$$= \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{\varepsilon \to 0+} \left[-2\sqrt{1-x} \right]_0^{1-\varepsilon}$$

$$= - \lim_{\varepsilon \to 0+} 2\left(\sqrt{\varepsilon} - 1\right)$$

$$= 2.$$

(ii) $$\int_0^2 \frac{dx}{2x-x^2} = \lim_{\varepsilon_1 \to 0+} \int_{\varepsilon_1}^1 \frac{dx}{2x-x^2} + \lim_{\varepsilon_2 \to 0+} \int_1^{2-\varepsilon} \frac{dx}{2x-x^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \frac{1}{2}\left[\ln \frac{x}{2-x}\right]_{\varepsilon_1}^1 + \lim_{\varepsilon_2 \to 0+} \frac{1}{2}\left[\ln \frac{x}{2-x}\right]_1^{2-\varepsilon}$$

$$= -\frac{1}{2} \lim_{\varepsilon_1 \to 0+} \ln\left(\frac{\varepsilon_1}{2-\varepsilon_1}\right) + \frac{1}{2} \lim_{\varepsilon_2 \to 0} \ln\left(\frac{2-\varepsilon_2}{\varepsilon_2}\right)$$

$$= \infty$$

$\Rightarrow$ integral diverges.

CONVERGENCE TEST FOR IMPROPER INTEGRALS: (TYPE I)

COMPARISON TEST – I :

If f and g are positive or non-negative, $f \geq 0$, $g \geq 0$

and $f(x) \leq g(x)$ for all x in $[a, R]$

then

(i) $\int_a^\infty f\,dx$ converges if $\int_a^\infty g\,dx$ converges

(ii) $\int_a^\infty g\,dx$ diverges if $\int_a^\infty f\,dx$ diverges

COMPARISON TEST – II :

Suppose $f(x) \geq 0$ & $g(x) > 0$ $\forall\ x > a$.

If $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = k\ (\neq 0)$

then both the two integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together.

In case : $k = 0$ and $\int_a^\infty g\,dx$ converges then

$\int_a^\infty f\,dx$ converges

In case : $k = \infty$ and $\int_a^\infty g\,dx$ diverges then

$\int_a^\infty f\,dx$ diverges.

A useful Comparison test:

We know: for $a>0$.

$$\int_a^R \frac{C}{x^n}\,dx = \begin{cases} C\log\left(\frac{R}{a}\right) & n=1. \\ \frac{1}{1-n}\left[\frac{1}{R^{n-1}} - \frac{1}{a^{n-1}}\right], & n\neq 1 \end{cases}$$

Therefore,

$$\int_a^\infty \frac{C}{x^n}\,dx = \lim_{R\to\infty}\int_a^R \frac{C}{x^n}\,dx = \begin{cases} +\infty & n\leq 1 \\ \frac{C}{(n-1)a^{n-1}} & n>1. \end{cases}$$

The μ-test: (Comparison test + above result)

Let $f(x)$ be bounded and integrable, $f(x)\geq 0$ in the interval $[a,R]$, $a>0$.
or $f(x)\in C$ $\quad a\leq x<\infty$.

a) If $\exists$ $\mu>1$ such that $\lim_{x\to\infty} x^\mu f(x)$ exists then

$$\int_a^\infty f(x)\,dx \text{ is convergent}$$

b) If $\exists$ $\mu\leq 1$ such that $\lim_{x\to\infty} x^\mu f(x)$ exists and $\neq 0$

then the integral $\int_a^\infty f(x)\,dx$ is divergent. and the same is true if

$$\lim_{x\to\infty} x^\mu f(x) \text{ is } +\infty \text{ or } -\infty.$$

OR (alternate for b))

$$\begin{cases} \text{test} & \lim_{x\to\infty} x f(x) = A\neq 0 \quad (\text{or} = \pm\infty) \\ \Rightarrow & \int_a^\infty f(x)\,dx \text{ diverges.} \end{cases}$$

test fails if $A=0$.

Examples:

(i) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

Sol: Let $f(x) = \dfrac{1}{x\sqrt{x^2+1}}$ $\left(\sim \dfrac{1}{x^2}\right)$

$g(x) = \dfrac{1}{x^2}$ $\boxed{\text{OR } \mu = 2}$

Note that $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1.$ (non zero finite)

$\Rightarrow \displaystyle\int_1^\infty f(x)\,dx$ & $\displaystyle\int_1^\infty g(x)\,dx$ converge or diverge together.

As $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges $\Rightarrow \displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ converges.

(ii) $\displaystyle\int_0^\infty \frac{x^2\,dx}{\sqrt{x^5+1}}$

Let $f(x) = \dfrac{x^2}{\sqrt{x^5+1}}$ $\left(\sim \dfrac{1}{\sqrt{x}}\right)$

and $g(x) = \dfrac{1}{\sqrt{x}}$. $\boxed{\mu = \frac{1}{2}}$

$\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{\sqrt{x^5+1}} \cdot \sqrt{x} = 1.$

As $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\,dx$ diverges

by Comparison test II $\displaystyle\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$ diverges.

(iii) $\int_{0}^{\infty} e^{-x^2}\,dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\text{proper}} + \int_1^{\infty} e^{-x^2}dx$

We know. $e^{x^2} = 1+x^2+\cdots > x^2 \quad \forall x.$

$$\Rightarrow e^{-x^2} < \frac{1}{x^2}$$

Since $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges, the integral $\int_0^{\infty} e^{-x^2}dx$ converges.

OR: $\boxed{\mu = 2}$ & $\lim_{x\to\infty} x^2 e^{-x^2} = 0 \Rightarrow \int_0^{\infty} e^{-x^2}dx$ converges.

(iv) $\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx$

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \underbrace{\int_0^1 \frac{\sin^2 x}{x^2}\,dx}_{\text{proper}} + \int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx.$$

Also. $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ and $\int_1^{\infty} \frac{dx}{x^2}$ converges.

$\Rightarrow \int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx$ converges.

(v) $\int_1^{\infty} \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx.$

$$f(x) = \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} = \frac{\tan^{-1}x}{x^{1/3}(1+x^{-4})^{1/3}} \quad (\sim x^{-1/3} \text{ at } \infty)$$

$g(x) = \frac{1}{x^{1/3}}$. $\boxed{\text{OR: } \mu = 1/3}$

$$\frac{f(x)}{g(x)} = \frac{\tan^{-1}x}{(1+x^{-4})^{1/3}} \to \pi/2 \text{ as } x\to\infty.$$

$\Rightarrow \int_1^{\infty} f(x)\,dx$ diverges.

Absolute Convergence:

Def: The integral $\int_a^\infty f(x)\,dx$ converges absolutely $\Leftrightarrow$

$\int_a^\infty |f(x)|\,dx$ converges.

Def: The integral $\int_a^\infty f(x)\,dx$ converges conditionally $\Leftrightarrow$

it converges but not absolutely.

Example: $\int_1^\infty \frac{\sin x}{x^2}\,dx$ converges absolutely

$$\frac{|\sin x|}{x^2} \le \frac{1}{x^2}$$

By comparison test I, the integral.

$$\int_1^\infty \frac{|\sin x|}{x^2}\,dx < \infty. \quad \text{(converges)}$$

Theorem: $\int_a^\infty f\,dx$ exists if $\int_a^\infty |f|\,dx$ exists.

The converse is not true.

Example: $\int_0^\infty \frac{\sin x}{x}\,dx$ converges conditionally.

Test for absolute Convergence:

(I):
1. $f(x), g(x) \in C$ $a \le x < \infty$
2. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = A$
3. $\int_a^\infty |g(x)|\,dx < \infty$

$\Rightarrow \int_a^\infty f(x)\,dx$ converges absolutely

(II):
1. $f(x) \in C$ $0 < a \le x < \infty$
2. $\lim_{x\to\infty} x^{\mu} f(x) = A$ $\mu > 1$.

$\Rightarrow \int_a^\infty f(x)\,dx$ converges absolutely.

Abel's test:

(1) the improper integral $\int_a^\infty f(x)\,dx$ is convergent

(2) g is monotone and bounded on $[a,\infty)$

Then the improper integral $\int_a^\infty f(x)\,g(x)\,dx$ converges.

Dirichlet's Test:

1) f is integrable on each interval $[a,b]$, $b>a$, and the integrals $\int_a^b f(x)dx$ are uniformly bounded, that is $\exists$ a constant $C>0$ such that

$$\left|\int_a^b f(x)\,dx\right| \le C \quad \text{for all } b>a.$$

2) g is monotone and bounded on $[a,\infty)$ and $\lim_{x\to\infty} g(x)=0$

Then the improper integral

$$\int_a^\infty f(x)\,g(x)\,dx$$

Converges.

Examples: $\int_1^\infty \frac{\sin x}{x^p}\,dx$ is convergent for $p>0$.

Let $f(x) = \sin x$ and $g(x) = \frac{1}{x^p}$.

Here: $\left|\int_1^b \sin x\,dx\right| = |\cos(1) - \cos(b)| \le |\cos(1)| + |\cos(b)|$

≤ 2

for $1 \le b < \infty$.

(Abel's test is not possible because $\int_1^\infty \sin x\,dx$ is not convergent)

Also $g(x) = \frac{1}{x^p}$ is montone decreasing function tending to 0 as $x\to\infty$, $p>0$.

Using Dirichlet test $\int_1^\infty f(x)\,g(x)$ Converges for $p>0$.

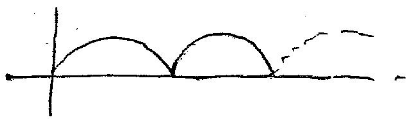
Note that $\int_1^\infty \frac{\sin x}{x^p}$, $p>1$ converges absolutely.

Question: Prove that $\int_0^\infty \frac{\sin x}{x}\,dx$ converges conditionally.

We know that $\int_0^\infty \frac{\sin x}{x}\,dx$ converges.

We show now that this integral does not converge absolutely.

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx$$



Substitute $x = n\pi + y$. Then

$$\sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin y}{n\pi + y} \cdot dy \geqslant \sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin y}{n\pi + \pi} \cdot dy$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{1}{(n+1)\pi} \cdot 2.}_{\text{divergent series.}}$$

By comparison test: The series

$$\sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin y}{n\pi + y}\,dy \text{ diverges}$$

and hence the improper integral

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx \text{ diverges.}$$

Ex: $$\int_a^{\infty} (1-e^{-x}) \frac{\cos x}{x^2}\, dx. \qquad a>0.$$

Let $f(x) = \frac{\cos x}{x^2}$ and $g(x) = 1-e^{-x}$.

We know. $$\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$$

& $\int_a^{\infty} \frac{1}{x^2}\, dx$ converges.

$\Rightarrow$ By comparison test the improper integral

$$\int_a^{\infty} \frac{\cos x}{x^2}\, dx \quad \text{converges.}$$

Also, $g(x)$ is montonic & bounded function for $x \in [a, \infty[$

Hence by Abel's lemma the improper integral

$$\int_a^{\infty} (1-e^{-x}) \frac{\cos x}{x^2}\, dx \quad \text{Converges.}$$

Ex: Test the convergence of $\int_0^{\infty} \frac{\sin x}{x} \cdot e^{-x}\, dx$.

Sol:
$$\int_0^{\infty} e^{-x} \cdot \frac{\sin x}{x}\, dx = \underbrace{\int_0^{1} e^{-x} \frac{\sin x}{x}\, dx}_{\text{proper}} + \int_1^{\infty} e^{-x} \frac{\sin x}{x}\, dx.$$

We know that $\int_1^{\infty} \frac{\sin x}{x}\, dx$ is convergent

and the function e^{-x} is monotone and bounded.

This implies using Abel's test that the improper integral converges.

§ Integral of the type $\int_{-\infty}^{b} f(x)\, dx$

Substitution $x=-t$: $\int_{-b}^{\infty} f(-t)\, dt$

$\square$

Review of all test for $\int_a^\infty f(x)\,dx$.

§. Comparison test: I

a) 1. $0 \le f(x) \le g(x)$
2. $\int_a^\infty g(x)\,dx < \infty$

$\Rightarrow \int_a^\infty f(x)\,dx < \infty$

b) 1. $0 \le g(x) \le f(x)$
2. $\int_a^\infty g(x)\,dx = \infty$

$\Rightarrow \int_a^\infty f(x)\,dx = \infty$

(II):

$0 \le f(x)$ and $g(x) \ge 0$ find $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = k$.

1) $k \ne 0$ then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ behave the same.

2) $k=0$ and $\int_a^\infty g(x)\,dx$ converges then $\int_a^\infty f(x)\,dx$ converges.

3) $k=\infty$ and $\int_a^\infty g(x)\,dx$ diverges then $\int_a^\infty f(x)\,dx$ diverges.

§. Test Integral: $\int_a^\infty \dfrac{dx}{x^p}$. where p is a constant and $a>0$.

Converges for $p>1$ & diverges if $p \le 1$.

§ Comparison test II with $g(x) = \dfrac{1}{x^\mu}$ + Test integral.

$\Rightarrow \mu$ test.

§ Absolute convergence $\Rightarrow$ convergence.

§ Abel's test: 1) $\int_a^\infty f(x)\,dx < \infty$ 2) g is monoton & bounded

$\Rightarrow \int_a^\infty f(x)\,g(x)\,dx < \infty$.

§ Dirichlet test: 1) $\left|\int_a^b f(x)\,dx\right| \le C \quad \forall\, b>a$.

2) g is monotone, bounded and $\lim\limits_{x\to\infty} g(x) = 0$

$\Rightarrow \int_a^\infty f(x)\,g(x)\,dx < \infty$.

Convergence of improper integrals of the second kind:

Test integral: $\int_a^b \frac{dx}{(x-a)^n}$.

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon\to 0+} \frac{1}{1-n}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right] \quad \text{if } n \neq 1$$

$$= \begin{cases} \frac{1}{(1-n)(b-a)^{n-1}} & \text{if } n<1 \\ \infty & \text{if } n>1 \end{cases}$$

for $n=1$:

$$\int_a^b \frac{dx}{(x-a)} = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b \frac{dx}{x-a} = \lim_{\varepsilon\to 0} \log(x-a)\Big|_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon\to 0} \left[\log(b-a) - \log\varepsilon\right]$$

$$= \infty$$

Hence:

$$\boxed{\int_a^b \frac{dx}{(x-a)^n} \text{ converges if } n<1 \text{ and diverges if } n\geq 1.}$$

We consider the convergence of the integral $\int_a^b f(x)\,dx$ when $f(x)$ becomes unbounded at $x=a$. We can denote such integral by $\int_{a+}^b f(x)\,dx$.

For the case $\int_a^{b-} f(x)\,dx$ we can set $x=b-t$ and get

$$\int_{0+}^{b-a} f(b-t)\,dt.$$

Convergence test for $\int_{a+}^{b} f(x)\,dx$:

§ Comparison test (I):

a) 1) $0 \le f(x) \le g(x)$, $a<x\le b$; 2) $\int_{a+}^{b} g(x)\,dx$ converges $\Rightarrow \int_{a+}^{b} f(x)\,dx$ converges

b) 1) $0 \le g(x) \le f(x)$; 2) $\int_{a+}^{b} g(x)\,dx$ diverges $\Rightarrow \int_{a+}^{b} f(x)\,dx$ diverges.

(II) Let $f(x) \ge 0$ & $g(x) \geq 0$ for $a<x\le b$

and $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = k$. Then

1) if $k \ne 0$ then $\int_{a+}^{b} f(x)\,dx$ & $\int_{a+}^{b} g(x)\,dx$ behave the same.

2) if $k=0$ & $\int_{a+}^{b} g(x)\,dx$ converges, then $\int_{a+}^{b} f(x)\,dx$ converges.

3) if $k=\infty$ & $\int_{a+}^{b} g(x)\,dx$ diverges, then $\int_{a+}^{b} f(x)\,dx$ diverges.

Ex:

1. Test the convergence of $\int_0^3 \dfrac{dx}{(3-x)\sqrt{x^2+1}}$. Unbounded at upper end.

set $3-x=t \Rightarrow dx=-dt$

$$\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}} = \int_0^3 \frac{dt}{t\sqrt{(3-t)^2+1}}$$

Let $g(t) = \frac{1}{t}$ and find $\lim\limits_{t\to 0}\dfrac{f(t)}{g(t)} = \lim\limits_{t\to 0} \dfrac{1}{\sqrt{(3-t)^2+1}} = \dfrac{1}{\sqrt{10}} \ne 0.$

$\Rightarrow \int_0^3 \dfrac{dx}{(3-x)\sqrt{x^2+1}}$ diverges, since $\int_0^3 g(t)\,dt$ diverges.

2. $\int_{\pi}^{4\pi} \dfrac{\sin x}{\sqrt[3]{x-\pi}}\,dx$ converges since $\left|\dfrac{\sin x}{\sqrt[3]{x-\pi}}\right| \le \dfrac{1}{\sqrt[3]{(x-\pi)}}$ and $\int_{\pi}^{4\pi} \dfrac{1}{\sqrt[3]{x-\pi}}$ converges.

In fact integral $\int_{\pi}^{4\pi} \dfrac{\sin x}{\sqrt[3]{x-\pi}}$ converges absolutely.

Note: Improper integrals of the third kind can be expressed in terms of improper integrals of the first and second kind.

μ-test: a) if $\exists$ $0<\mu<1$ such that

$$\lim_{x\to a+} (x-a)^{\mu} f(x) \text{ exists}$$

then $\int_{a+}^{b} f(x)\,dx$ converges (absolutely)

b) If $\exists$ $\mu \geq 1$ such that

$$\lim_{x\to a+} (x-a)^{\mu} f(x) \text{ exists } (\neq 0) \quad (\text{it may be } \pm\infty)$$

then $\int_{a+}^{b} f(x)\,dx$ diverges.

Abel's test: $\int_{a+}^{b} f(x)\,dx$ converges + $\varphi(x)$ is bounded & monotonic

then $\int_{a+}^{b} f(x)\,\varphi(x)\,dx$ converges.

Dirichlet test: $\int_{a+\varepsilon}^{b} f(x)\,dx$ is bounded & $\varphi(x)$ is bounded, monotonic and converging to zero as $x\to a$

then $\int_{a+}^{b} f(x)\,\varphi(x)\,dx$ is convergent.

Example: Discuss the convergence or divergence of the integral

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\cdot dx \;; \qquad \alpha>0.$$

Solution: Let $\alpha \geq 1$. (first kind)

$$\lim_{x\to\infty} x^{\mu}\,\frac{x^{\alpha-1}}{x+1} = \lim_{x\to\infty} \frac{x^{\mu+\alpha-1}}{1+x}.$$

$$= 1 \quad \text{if} \quad \mu+\alpha-1 = 1 \quad \text{or} \quad \mu = 2-\alpha.$$

Since $\alpha \geq 1$, $\mu \leq 1$

$\Rightarrow$ The given integral is divergent for $\alpha \geq 1$.

Let $0<\alpha<1$, The integrand becomes unbounded at $x=0$.

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \underbrace{\int_0^a \frac{x^{\alpha-1}}{1+x}\,dx}_{I} + \underbrace{\int_a^\infty \frac{x^{\alpha-1}}{x+1}\,dx}_{II}.$$

For the integral I:

$$\lim_{x\to 0} x^{\mu}\frac{x^{\alpha-1}}{1+x}$$

$$= \lim_{x\to 0} \frac{x^{\mu+\alpha-1}}{1+x} = 1 \quad \text{if } \mu+\alpha-1=0$$

$$\mu = 1-\alpha.$$

Since $0<\alpha<1$, μ lies between 0 & 1.

This implies that $\int_0^a \frac{x^{\alpha-1}}{1+x}\,dx$ converges

For the integral II:

$$\lim_{x\to \infty} x^{\mu}\frac{x^{\alpha-1}}{x+1}$$

$$= \lim_{x\to \infty} \frac{x^{\mu+\alpha-1}}{x+1} = 1 \quad \text{if } \mu+\alpha-1=1$$

$$\mu = 2-\alpha$$

Since $0<\alpha<1$, μ lies between 1 & 2.

$\Rightarrow \int_a^\infty \frac{x^{\alpha-1}}{x+1}\,dx$ will converge.

$\Rightarrow \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx$ converges for $0<\alpha<1$.