$$\frac{dx}{1} = \frac{dy}{-2q} = \frac{dz}{p - 2q^2} = \frac{dp}{f(6x)} = \frac{dq}{-1}$$

$x = q + c$

$\underline{dy}$

$q = -x + a$

$p = 3x^2 - y + (x-a)^2$

$z = p\,dx + q\,dy$ Solve.

---

linear Eq^{ns} with constant coeff.

$$u_{xx} - 2u_{xy} + u_{yy} = x^2 + y$$

$$D_x = \frac{\partial}{\partial x} \qquad D_y = \frac{\partial}{\partial y}$$

$$D = \frac{\partial}{\partial x} \qquad D' = \frac{\partial}{\partial y}$$

Theorem:-

If $(a_i D + b_i D' + C_i)$ is a factor of $F(D, D')$, then a sol^n of $F(D,D')u(x,y) = 0$ is of form

$$\boxed{u_i = e^{-C_i/a_i\, x} f(b_i x - a_i y)}$$

Note:- $F(D,D')u = (D+D')(D-D')u = 0$

then any 'u' such that $(D - D')u = 0$ or $(D + D')u = 0$ will be a sol^n of $F(D,D')u = 0$.

aim to find soln of

$(a_iD + b_iD' + c_i)u = 0.$ $a_iu_x + b_iu_y + c_iu = 0.$

$$\frac{dx}{a_i} = \frac{dy}{b_i} = -\frac{du}{c_iu}$$

$\underline{b_ix - a_iy = c}$

$$\frac{du}{u} = -\frac{c_i}{b_i}dy$$

$$\frac{du}{u} = -\frac{c_i}{a_i}dx$$

$$u = \left(e^{-c_i/b_iy}\right).c_2$$

$$u = \left(e^{-c_i/a_ix}\right).c_1$$

$$u_i = e^{-c_i/a_ix} f(b_ix - a_iy)$$

Eg: $(2D - 5D' + 7)u = 0$ $a_i = 2$ $b_i = -5$ $c_i = 7.$

$$u_1 = e^{-7/2x} f(-5x - 2y) = e^{-7/2x} g(5x + 2y).$$

$$u_2 = e^{+7/5y} f(-5x - 2y).$$

Eg: $(2D^2D' - 3DD'^2 + D'^3)u = 0.$

$F(D, D')u = 0.$

factorising: $D'(2D^2 - 3DD' + D'^2)$

$D'(2D^2 - 2DD' - DD' + D'^2).$

$D'(D - D')(2D - D')u = 0.$

$D'u = 0$; $Du - D'u = 0$; $(2D - D')u = 0$

$(c_i = 0).$

$u_1 = f(x)$; $u_2 = g(x+y)$ $u_3 = h(x+2y).$

eg

$$(D^2 - D'^2 + 2D + 1)u = 0.$$

$$(D + D' + 1)(D - D' + 1)u = 0.$$

complementary function.

$$= e^{-x} f_1(x-y) + e^{-x} f_2(x+y).$$

Theorem 2). If $(a_i D + b_i D' + c_i)^2$, $(a_i \neq 0)$ is a factor of $F(D, D')$ then the soln of $F(D,D')u=0$ is of the form $u_i = e^{-c_i/a_i\, x}\left[g(b_i x - a_i y) + \frac{x}{a_i} f(b_i x - a_i y)\right]$

proof consider $(a_i D + b_i D' + c_i)^2 u = 0$ —— $(*)$ let $(a_i D + b_i D' + c_i) u = z$ —— $(+)$

$(*) \Rightarrow (a_i D + b_i D' + c_i) z = 0$

$$\therefore z = e^{-c_i/a_i\, x} f(b_i x - a_i y).$$

$(+) \Rightarrow (a_i D + b_i D' + c_i)u = e^{-c_i/a_i\, x} f(b_i x - a_i y)$

$$\Rightarrow a_i u_x + b_i u_y = -c_i u + e^{-c_i/a_i\, x} f(b_i x - a_i y).$$

$$\Rightarrow \frac{dx}{a_i} = \frac{dy}{b_i} = \frac{du}{-c_i u + e^{-c_i/a_i\, x} f(b_i x - a_i y)}.$$

$b_i x - a_i y = \text{constant}$

$$\frac{dx}{a_i} = \frac{du}{-c_i u + e^{-c_i/a_i\, x} f(d_1)}$$

$$\Rightarrow \frac{du}{dx} + \frac{c_i}{a_i} u = \frac{1}{a_i} e^{-c_i/a_i\, x} f(d_1).$$

$$\Rightarrow \frac{du}{dx} e^{c_i/a_i\, x} + \frac{c_i}{a_i} u e^{c_i/a_i\, x} = \frac{1}{a_i} f(d_1).$$

$$\Rightarrow \frac{d}{dx}\left(u e^{c_i/a_i\, x}\right) = \frac{1}{a_i} f(d_1)$$

$$\Rightarrow u e^{c_i/a_i\, x} = \frac{x}{a_i} f(d_1) + d_2$$

$\Rightarrow$ general soln.

$$u = e^{-\frac{c_i}{a_i} x}\left[g(b_i x - a_i y) + \frac{x}{a_i} f(b_i x - a_i y)\right]$$

Corollary

If $(a_i D + b_i D' + c_i)^n$, $a_i \neq 0$ is a factor, then

$u = e^{-c_i/a_i \, x}\left(\sum_{j=1}^{n} x^{j-1} f_{ij}(b_i x - a_i y)\right)$ will be soln of $F(D, D')u = 0$

Corollary 2) If $a_i = 0$, $b_i \neq 0$ and $(b_i D' + c_i)^n$ is a factor then $e^{-c_i/b_i \, y} \sum_{j=1}^{n} (x)^{j-1} f_{ij}(b_i x)$

will be the Soln of $F(D, D')u = 0$

Eg) ① $(D^3 + 3D^2 D' - 4D'^3)u = 0.$

② $(D^3 - 3D^2 D' + 3DD'^2 - D'^3)u = 0$

③ $(2D + D' - 1)^2 (D - 2D' + 2)^3 u = 0$

Soln

① $(D^3 + 3D^2 D' - 4D'^3)u$

$(D^3 - D'^3 + 3D^2 D' - 3D'^3)u = 0$

$(D - D')(D^2 + D'^2 + DD' + 3D'(D + D'))u = 0.$

$(D - D')(D^2 + D'^2 + DD' + 3DD' + 3D'^2)u = 0$

$(D - D')(D^2 + 4DD' + 4D'^2)u = 0$

$(D - D')(D + 2D')^2 \; u = 0$

$u_1 = \underline{f(x + y)}$ $\quad u_2 = \underline{x f_1(2x - y) + g_1(2x - y)}$

Soln. $u = u_1 + u_2$.

② $(D^3 - 3D^2 D' + 3DD'^2 - D'^3)u = 0$

$(D - D')^3 \cdot u = 0.$

$u = x^2 f(x + y) + x g(x + y) + h(x + y).$

③ $(D'+2D-1)^2 \quad (D-2D'+2)^3 u = 0.$

$$u_1 = x e^{x} f_{11}(2x-y) + e^{-x} f_{12}(2x-y)$$

$$u_2 = x^2 e^{-2x} f_{21}(2x+y) + x e^{-2x} f_{22}(2x+y) + e^{-2x} f_{23}(2x+y)$$

$$u = Au_1 + Bu_2$$

(no need to write A, B. it will be absorbed in u_1, u_2).

Irreducible Equations.

① $(2D^2 - D'^2 + D)u = 0$

Consider $F(D,D')u=0$ where

$$F(D,D') = \sum_{\alpha}\sum_{\beta} C_{\alpha\beta} D^{\alpha} D'^{\beta}$$

$\Rightarrow u = \sum_{j=1}^{\infty} d_j e^{(a_j x + b_j y)}$ is a solⁿ.

provided $F(a_j, b_j) = 0$.

$$u = \sum_{j=1}^{\infty} d_j e^{a_j x + b_j y}$$

where a_j, b_j are $2a_j^2 - b_j^2 + a_j = 0$

$$\Rightarrow b_j = \pm\sqrt{2a_j^2 + a_j}$$

$$\therefore u = \sum_{j=1}^{\infty} d_j e^{a_j x \pm \sqrt{2a_j^2 + a_j}\, y}$$

Eg. $(D-2D'+5)(D^2+D'+3)u = 0.$

CF of $(D-2D'+5)v = 0$ is $v = e^{-5x} f_1(2x+y)$

CF of $(D^2+D'+3)u_1 = 0$ is $u_1 = \sum_{j=1}^{\infty} d_j e^{a_j x + b_j y}$

$a_j^2 + b_j + 3 = 0. \quad \Rightarrow b_j = -3 - a_j^2$

$$\therefore u_1 = \sum_{j=1}^{\infty} d_j e^{a_j x - (3+a_j^2) y}.$$

$$\therefore u = e^{-5x} f_1(2x+y) + \sum_{j=1}^{\infty} d_j e^{(a_j x - (3+a_j^2) y)}.$$

Finding particular integral of non-homogenous eqn.

① $(D^2+2DD'+D'^2)u = 12xy$.

$u_c = f(x-y) + x\,g(x-y)$.

(complementary funcn)

$$u_p = \frac{1}{D^2+2DD'+D'^2}\,12xy.$$

$$= \frac{1}{(D+D')^2}\,12xy = \frac{12}{D^2}\left(1+\frac{D'}{D}\right)^{-2} xy.$$

$$= \frac{12}{D^2}\left(1-\frac{2D'}{D}+\frac{3D'^2}{D^2}-\cdots\right)(xy)$$

(both differentiation and integration in one equation)

$$= \frac{12}{D^2}(xy - x^2).$$

(further terms are 0).

$$= 12\left(\frac{x^3}{6}y - \frac{x^4}{12}\right) = 2x^3y - x^4.$$

② $(D+D'-1)(D+2D'-3)u = 4+3x+6y$.

$u_c = e^{x} f_1(x-y) + e^{3x} f_2(2x-y)$

$$u_p = \frac{1}{(D+D'-1)(D+2D'-3)}(4+3x+6y).$$

$$= \left[1-(D+D')\right]^{-1}\frac{1}{3}\left[1-\left(\frac{D+2D'}{3}\right)\right]^{-1}(4+3x+6y).$$

$$= \frac{1}{3}\left[1+(D+D')+(D+D')^2+\cdots\right]\left[1+\frac{D+2D'}{3}+\cdots\right](4+3x+6y)$$

$$= \frac{1}{3}\left[1+(D+D')+(D+D')^2+\cdots\right]\left(\underbrace{4+3x+6y+\frac{3+12}{3}}_{9}\right)$$

$$= \frac{1}{3}(9+3x+6y+3+6) = x+2y+6.$$

Theorem. If $f(x,y)$ is of the form e^{ax+by}, then P.I. of $F(D,D') = f(x,y)$

is $\frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}$.

provided $F(a,b) \neq 0$.

Eg: $(D^2 - 3DD' + 2D'^2)u = e^{2x-y}$ $\quad \underline{a=2 \quad b=-1.}$

$$u_p = \frac{1}{D^2 - 3DD' + 2D'^2} e^{2x-y}, \quad \cancel{a=2}$$

$$F(a,b) = (2^2 - 3.2.(-1) + 2.(-1)^2) \neq 0.$$
$$= 12.$$

$$\therefore u_p = \frac{1}{12} e^{2x-\cancel{b}y}$$

Eg: $(D^2 - D')u = e^{x+y}$. $\quad a=1, b=+1.$

$F(a,b) = 0$

$$u_p = \frac{e^{x+y}}{(D^2 - D')} = \frac{e^{x+y}}{0} \; ?$$

Theorem: If $f(x,y) = e^{ax+by}\phi(x,y)$ then $\frac{1}{F(D,D')}(e^{ax+by}\phi(x,y))$ $\cancel{e^{ax+by}}$

$$= e^{ax+by} \cdot \frac{1}{F(D+a, D'+b)} \phi(x,y)$$

$$u_p = \frac{1}{(D^2 - D')} e^{x+y} \cdot 1 = e^{x+y} \cdot \frac{1}{\{(D+1)^2 - (D'+1)\}} \cdot 1$$

$$= e^{x+y} \cdot \frac{1}{(D^2 + 2D - D')} \cdot 1$$

$$= e^{x+y} \cdot \frac{1}{(-D')\left(1 - \frac{2D + D^2}{D'}\right)} \cdot 1 = e^{x+y}\left(-\frac{1}{D'}\right)\left(1 + \frac{2D+D^2}{D'} + \dots\right)$$

$$= e^{x+y}\left(-\frac{1}{D'}\right)(1)$$

$$= \underline{-y\, e^{x+y}}.$$

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Q $(D^2 + DD' + D + D' + 1)u = e^{-2x}(x^2+y^2)$

$$u_p = e^{-2x} \cdot \frac{1}{(D-2)^2 + (D-2)(D') + D-2 + D' + 1}(x^2+y^2),$$

$$= e^{-2x} \cdot \frac{1}{D^2+DD'-3D-D'+3}(x^2+y^2)$$

$$= e^{-2x} \cdot \frac{1}{3}\left[1 + \frac{(D^2+DD'-3D-D')}{3} + \frac{(D^2+DD'-3D-D')^2}{9} + \cdots\right](x^2+y^2).$$

$$= \frac{e^{-2x}}{3}\left[x^2+y^2 - \frac{1}{3}(2-6x-2y) + \frac{1}{9}(18+2)\right].$$

$$= \frac{e^{-2x}}{27}(9x^2 + 9y^2 + 18x + 6y + 14).$$

Periodic functions-

$$(D^2 + 2DD' + D'^2)u = \cos(x+2y).$$

Let $u_p = c_1 \cos(x+2y) + c_2 \sin(x+2y)$.

$$D^2u = -c_1\cos(x+2y) - c_2\sin(x+2y)$$

$$D'^2 u = -4c_1\cos(x+2y) - 4c_2\sin(x+2y)$$

$$DD'u = -2c_1\cos(x+2y) - 2c_2\sin(x+2y).$$

Substitute in the eq^n and equate coeff. of independent func^n.

$$-9c_1\cos(x+2y) - 9c_2\sin(x+2y) = \cos(x+2y)$$

$$C_2 = 0, \quad c_1 = -1/9.$$

$$\therefore u_p = -\frac{1}{9}\cos(x+2y)$$

(1) $(D^2 - D')\, u = x\sin y$.

(2) $(D^2 - D')\, u = e^{y} \cos(2x + 3y)$

D 1

u_c: $a_j^2 - b_j = 0$ $\quad b_j = a_j^2$.

$$u = \sum_{j=1}^{\infty} d_j\, e^{a_j x + a_j^2 y}$$

$$u_p = \frac{1}{(D^2 - D')}\, x\sin y = \frac{1}{(1 + (D^2 - D' - 1))} = \frac{1}{(1 - (D' + 1 - D^2))}\, x\sin y$$

$$\cancel{= \left[1 + (D' + 1 - D^2) + (D' + 1 - D^2)^2 + \cdots\right] x\sin y}$$

$$\cancel{= (1 + x\cos y + x\sin y +}$$

$$= \operatorname{Im}\left(\frac{1}{(D^2 - D')}\, x e^{iy}\right)$$

$$= \operatorname{Im}\left[e^{iy} \cdot \frac{1}{(D^2 - D' - i)}\, x\right] \quad \cancel{= \operatorname{Im}\left[e^{iy} \cdot \frac{1}{(1 - (D' + i + 1 - D^2))}\, x\right]}$$

$$\cancel{= \operatorname{Im}\left[e^{iy}\, (1 + (D' + i + 1 - D^2)^2 +\right.}$$

$$= \operatorname{Im}\left[e^{iy} \cdot \frac{1}{i\left(\frac{D^2 - D'}{i} - 1\right)}\, x\right].$$

$$= \operatorname{Im}\left[\frac{(-1)\, e^{iy}}{i}\, \frac{x}{\left(1 - \left(\frac{D^2 - D'}{i}\right)\right)}\right].$$

$$= \operatorname{Im}\left[(-1)\, \frac{e^{iy}}{i}\, (x)\right]$$

$$u_p = \underline{x\cos y} \qquad u_c + u_p$$

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Classification of 2nd order PDE and Integration

→ Some motivation

1) Conservation of vector field

$$\nabla \cdot \overline{V} = 0 \quad \text{—} (1)$$

if irrotational $\vec{\nabla} \times \vec{V} = 0$ —(2), $\overline{V} = \nabla \phi$.

(1) or (2) → $\nabla^2 \phi = 0$.

2D $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0.$

when water drop falls in steady state water

flux $J = -D \nabla c$

total flux $\vec{\nabla} \cdot J = \nabla(-D \nabla c)$.

$$\frac{\partial c}{\partial A} + D \nabla^2 c = 0$$

diffusion / unsteady condition

Consider a general linear 2nd order PDE with variable coefficients.

$$a(x,y)\, u_{xx} + b(x,y)\, u_{xy} + c(x,y) u_{yy} + d(x,y)\, u_x + e(x,y) u_y + f(x,y) u(x,y) = g(x,y) \quad \text{—} (*)$$

Eg 1) $x^2 u_{xx} + 2xy\, u_{xy} + y u_{yy} + x u_x - y u_y + u = \ln x$.

2) $x u_{xx} - u_{xy} + y^2 u_{xy} = 0$

aim:- to integrate (*)

$u_{xx} + u = 0$.

if (*) can be to one of the following forms.

$u_{\xi y} = F(x, y, u, u_\xi, u_y)$

$u_{\xi\xi} =$ ______

$u_{yy} =$ ______

i.e. we need to transform (x,y) coordinate to (ξ, Y) coordinates, we reduce the 3 2nd order derivatives terms in (*) to above form (having one 2nd order

Motivation

$a u_x + b u_y = 0$; $a, b \in \mathbb{R}$. ; $a^2+b^2 \neq 0$.

$(a,b)\cdot(u_x, u_y) = 0$.

$\bar{v}\cdot\nabla u = 0$

we need to transform to another (ξ, Y)

such that only u_ξ or u_Y remains in the PDE.

$$(\xi, Y) = (ax+by,\ bx-ay)$$

because

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{0} \Rightarrow u = c_1,\ bx-ay = c_2.$$

$$\Rightarrow u = f(bx-ay)$$

$\Rightarrow$ $\xi = ax+by$ $Y = bx-ay$ $u = f(\xi, Y)$ (my assumption)

$$u_x = u_\xi \xi_x + u_Y Y_x = u_\xi \cdot a + u_Y \cdot b$$

$$u_y = u_\xi \xi_y + u_Y Y_y = u_\xi b - u_Y \cdot a$$

$\therefore a u_x + b u_y = 0 \Rightarrow a(a u_\xi + b u_Y) + b(b u_\xi - a u_Y) = 0.$

$\Rightarrow (a^2+b^2) u_\xi = 0 \Rightarrow u_\xi = 0$ $\because a^2+b^2 \neq 0$.

$$\therefore u = f(Y)$$

$$u = f(bx-ay).$$

We use this motivation for eq^n having multiple 2nd order terms to having only one by choosing appropriate (ξ, η) coordinate axis.

$(x,y) \longrightarrow (\xi, \eta)$. $J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq \{0, \infty\}$

So after transforming, we get a PDE of below form

$u_{\xi\eta} = \ldots$

$u_{\xi\xi} = \ldots$

$u_{\eta\eta} = \ldots$

consider $\xi = \xi(x,y)$, $\eta = \eta(x,y)$

$$u_x = u_\xi \xi_x + u_\eta \eta_x \quad , \quad u_y = u_\xi \xi_y + u_\eta \eta_y .$$

$$u_{xx} = u_\xi (\xi_{xx}) + \xi_x (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) + u_\eta \eta_{xx} + \eta_x (u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x).$$

$$u_{xy} = \quad '' y$$

$$u_{yy} = \quad '' y .$$

eq^n ⊛

$$a u_{xx} + b u_{xy} + c u_{yy} + \dots = g(x,y).$$

reduces to

$$(a \xi_x^2 + b \xi_x \xi_y + c \xi_y^2) u_{\xi\xi} + (2a(\xi_x \eta_x) + b(\xi_x \eta_y + \xi_y \eta_x) + 2c(\xi_y \eta_y)) u_{\xi\eta}$$

$$+ (a \eta_x^2 + b \eta_x \eta_y + c \eta_y^2) u_{\eta\eta} + F(u, \xi, \eta, u_\xi, u_\eta) = 0. \quad \text{(**)}$$

(**) can be written as

$$\tilde{A} u_{\xi\xi} + \tilde{B} u_{\xi\eta} + \tilde{C} u_{\eta\eta} + F = 0$$

$$A = a \xi_x^2 + b \xi_x \xi_y + c \xi_y^2$$

$C =$

$B =$

note that $(\tilde{B}^2 - 4\tilde{A}\tilde{C}) = (b^2 - 4ac)(\xi_x \eta_y - \xi_y \eta_x)^2$

$$= (b^2 - 4ac) \cdot J^2$$

If $J \neq 0$, discriminant is invariant.

Let $\chi = \xi$ or η.

consider $a \chi_x^2 + b \chi_x \chi_y + c \chi_y^2 = 0$

i.e. $a \left(\frac{\chi_x}{\chi_y}\right)^2 + b \left(\frac{\chi_x}{\chi_y}\right) + c = 0. \quad$ (3)

aim $\Rightarrow \chi = \text{const}$

$$d\chi = \chi_x dx + \chi_y dy = 0 \quad \text{(3)}$$

$$\frac{dy}{dx} = -\frac{\chi_x}{\chi_y} \quad \text{—} (e_3)$$

(e_1) and (e_3) $\quad a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c = 0.$

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a}.$$

Soln if the above PDE is curve

$\xi = \text{const.}, \; \eta = \text{const.}$

We are killing $\tilde{A}$ and $\tilde{C}$ provided $\tilde{B}$ is non zero.

$\xi = \text{const.}$ & $\eta = \text{const}$ are called "characteristic curves"!

$\xi = \xi(x,y) = \phi(x,y).$

$\eta = \eta(x,y) = \psi(x,y)$ reduce given PDE to

$u_{\xi\eta} = F$ provided $\tilde{B} \neq 0.$

→ PDE of form $\boxed{u_{\xi\eta} = F}$ are called canonical forms.

Q) $u_{xx} + 2u_{xy} + u_{yy} + u_x = 0.$

$a = 1 \quad b = 2 \quad c = 1$

Q) $y^2 u_{xx} - x^2 u_{yy} = 0$

$a = y^2 \quad b = 0 \quad c = -x^2$

$$\frac{dy}{dx} = \pm \frac{\sqrt{4x^2y^2}}{2y^2} = \pm \frac{2xy}{2y^2} = \pm x/y.$$

$\exists \; \xi = \xi(x,y) = y^2 - x^2 = c_1$

$\eta = \eta(x,y) = y^2 + x^2 = c_2.$

$u_x = u_\xi(-2x) + u_\eta(2x) = -2x u_\xi + 2x u_\eta.$

$u_{xx} = -2\left(u_\xi + x\left(u_{\xi\xi}(-2x) + u_{\xi\eta}(2x)\right)\right) + 2\left(u_\eta + x\left(u_{\eta\xi}(-2x) + u_{\eta\eta}(2x)\right)\right)$

$u_y = u_\xi(2y) + u_\eta(2y).$

Characteristic curve : A smooth curve $\bar{x}(k) \in \mathbb{R}^2$ is called a characteristic curve for the 2nd order PDE. (*) if its tangent vector

$$\dot{\bar{x}} = (\dot{x}, \dot{y})^T \text{ satisfy the quadratic } a\dot{y}^2 - b\dot{y} + cy = 0$$

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + fu = g.$$

$$\Delta = b^2 - 4ac.$$

$$Q(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2 + \cdots + fu = 0.$$

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

- if $b^2 - 4ac > 0$ + 2 real sol^ns

$\xi = \xi(x,y)$, $\eta = \eta(x,y)$.

characteristic curve.

PDE reduces to $u_{\xi\eta} = F$

and PDE of hyperbolic type.

- if $b^2 - 4ac = 0$ then one real root parabolic type.
- if $b^2 - 4ac < 0$, 2 imaginary roots, ~~imaginary~~ elliptic type.

(I). Choice 1) if we choose $(b^2 - 4ac > 0)$

$$\xi_x + \frac{b + \sqrt{b^2 - 4ac}}{2a} \xi_y = 0.$$

then $\tilde{A} = 0$

$$\eta_x + \frac{b - \sqrt{b^2 - 4ac}}{2a} \eta_y = 0.$$

then $\tilde{C} = 0$.

PDE is hyperbolic type.

$$\tilde{B} = -\frac{\Delta \xi_x \eta_y}{a} \neq 0.$$

(**) reduces to $\boxed{u_{\xi\eta} = F}$ Canonical form.

I) $u_{xx} - 2u_{xy} - 3u_{yy} = 0$

$a = 1$, $b = -2$, $c = -3$.

$b^2 - 4ac = 4 + 12 = 16 > 0$

∴ the given PDE is hyperbolic.

& we ~~except~~ pect two real characteristics

∴ consider

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm 4}{2} = 1, -3$$

$$\Rightarrow \underline{\underline{x - y = c_1}}$$

$$\underline{\underline{3x + y = c_2}}$$

∴ $\xi = x - y$, $\eta = 3x + y$

$u_x = u_\xi \cdot 1 + u_\eta \cdot 3$

$u_y = u_\xi(-1) + u_\eta(1) \qquad = -u_\xi + u_\eta.$

$u_{xx} = u_{\xi\xi} \cdot 1 + u_{\xi\eta}(3) + 3(u_{\eta\xi} + u_{\eta\eta} \cdot 3)$

$u_{yy} =$

$u_{xy} =$

$\Rightarrow \quad 16 u_{\xi\eta} = 0 \qquad \Rightarrow u_{\xi\eta} = 0$

$u_\xi = f(\xi)$

$u = g(\xi) + h(\eta)$

$u = \underline{\underline{g(x-y) + h(3x+y)}}$

II) Choice 2: $\Delta = b^2 - 4ac = 0$ (parabolic)

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{b}{2a}.$$

$\tilde{A} = a\xi_x^2 + b\xi_x \xi_y + c\xi_y^2.$

$= a\left(\xi_x + \frac{b}{2a}\xi_y\right)^2 \qquad \Rightarrow \frac{\xi_x}{\xi_y} = \frac{-b}{2a}. \qquad \tilde{A} = 0.$

$u_y = u_\xi(2y) + u_\eta(2y).$

choose η such that $\frac{\eta_x}{\eta_y} \neq \frac{-b}{2a} = \tilde{c} \neq 0.$

(a) $u_{xx} + 2u_{xy} + u_{yy} + u_x = 0.$

$a = 1,\ b = 2,\ c = 1$

$b^2 - 4ac = 0 \quad , \quad \frac{dy}{dx} = \frac{b}{2a} = 1.$

$x - y = c_1. \quad \xi(x,y) = x - y$

along which $\tilde{A} = 0.$

Choose η such that (ξ, η) are independent

let $\eta = y$ (we can take $x+y$ also, take y to make calculation easy)

$u_x = u_\xi + u_\eta(0) = u_\xi \quad , \quad u_{xx} = u_{\xi\xi}.$

$u_y = -u_\xi + u_\eta \quad ; \quad u_{xy} = u_{\xi\xi}(-1) + u_{\xi\eta}(1)$

$u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

$\therefore u_{\xi\xi} - 2u_{\xi\xi} + 2u_{\xi\eta} + u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} + u_\xi = 0$

$\boxed{u_{\eta\eta} + u_\xi = 0}$ Cannonical form.

$u_\eta = -\int u_\xi \, d\eta + g(\xi)$

$u = -\iint u_\xi \, d\eta \, d\eta + \eta g(\xi) + h(\xi)$

Choice 3) $\Delta = b^2 - 4ac < 0$ (elliptic)

$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$ gives two linearly independent soln.

$\bar{\xi}, \bar{\eta}$ but are imaginary

$\xi = \bar{\xi} + \bar{\eta} \quad , \quad \eta = (\bar{\xi} - \bar{\eta})i$ are real

(real)

a) $u_{xx} + 2u_{xy} + u_{yy} = 0$

$a = 1, \quad b = 2, \quad c = 2$

$b^2 - 4ac = -4$

$\frac{dy}{dx} = \frac{2 \pm 2i}{2} = 1 \pm i$

$\bar{\xi} = y - (1+i)x \qquad \bar{\eta} = y - (1-i)x.$

$\xi = 2(y-x), \quad \eta = 2x$

$\xi = 2y - 2x, \quad \eta = 2x.$

$u_x = u_\xi(-2) + u_\eta(2).$ $\qquad u_{xx} = \ldots$

$u_y = u_\xi(2) + u_\eta(0)$

$u_{xy} = -2 \ldots$

$u_{\xi\xi} + u_{\eta\eta} = 0$

Parabolic- diffusion process

$C = c(\bar{x}; A)$ chemical | nutrient conduction process.

$T = T(\bar{x}, A)$ temperature.

Any transformation say from (x,y) to (u,v) we also need to come back from (u,v) to (x,y) so $dx\,dy = J\,du\,dv$ such that $J \neq 0$

$\hat{J} = -k\nabla T$

$\hat{J} = -\nabla(k\nabla T) = -k\nabla^2 T$ (k const)

Rate of change of temp. = total flux accumulated

$\frac{\partial T}{\partial A} = k\nabla^2 T$

In 1D: $\frac{\partial T}{\partial A} = k\frac{\partial^2 T}{\partial x^2}$ (gives Palabola)

Steady conduction $\nabla^2 T = 0$ (elliptic)

$\nabla \times \bar{v} = 0$ & $\bar{v} = \nabla \phi$ $\nabla \cdot \bar{v} = 0$

& $\nabla^2 \phi = 0$.

wave eq^n. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Sol^n of 2nd order PDE

- Motivation to seperation of variables.

$2u_{xx} - u_y = 0$ (linear, homogenous)

$u = u(x,y)$ let u be $X(x)Y(y)$

$u_x = X'Y$; $u_y = XY'$

$\therefore u_{xx} = X''Y$; $u_{yy} = XY''$

$u_x = \partial u/\partial x$

$u_y = \partial u/\partial y$.

$2X''Y - XY' = 0$. ——— ①

looking for a non trivial sol^n & $XY \neq 0$

hence divide by XY.

& $2\frac{X''}{X} - \frac{Y'}{Y} = 0$. & $2\frac{X''}{X} = \frac{Y'}{Y}$.

f(x) only f(y) only.

$\therefore 2\frac{X''}{X} = \frac{Y'}{Y} = \lambda$.

& $X'' - \frac{\lambda}{2} X = 0$, $Y'' - \lambda Y = 0$ (eigen value problems)

why eigen value

in general

$AX = \lambda X$.

transform.

here,

$LX = \lambda X$

operator. eigen value

a). $u_{xx} + 2u_{xy} + u_{yy} = 0$

$\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = c_1 x + c_2$

$\lambda > 0 \Rightarrow X'' - \frac{\lambda}{2} x = 0 \Rightarrow X(x) = c_1 e^{\sqrt{\lambda/2}\,x} + c_2 e^{-\sqrt{\lambda/2}\,x}$.

$\lambda < 0 \Rightarrow \lambda = -m^2$

$$X'' + \frac{m^2}{2} X = 0$$

$$X(x) = c_1 \cos\left(\frac{m}{\sqrt{2}} x\right) + c_2 \sin\left(\frac{m}{\sqrt{2}} x\right)$$

$\lambda = 0 \Rightarrow Y' = 0 \Rightarrow Y(y) = d_1$

~~$\lambda > 0 \Rightarrow Y' = \lambda Y$.~~

$\Rightarrow$ for $\lambda = 0$

$$u(x,y) = (c_1 x + c_2) d_1 = a_1 x + a_2.$$

$\lambda > 0 \Rightarrow Y' = \lambda Y$ $\qquad$ $\lambda < 0 \quad Y = d_2 e^{-m^2 y}$.

$\Rightarrow Y = d_1 e^{\lambda y}$.

$$u(x,y) = (d_1 e^{\lambda y}) (c_1 e^{\sqrt{\lambda/2}\,x} + c_2 e^{-\sqrt{\lambda/2}\,x})$$

on eigenvalues

$u_{xx} - u_y = 0 \quad , \quad 0 < x < L.$

$u(x,0) = f(x) \quad ; \quad u(0,y) = 0 \quad ; \quad u(L,y) = 0.$

$u = XY \qquad \Rightarrow \frac{X''}{X} = \frac{Y'}{Y} = \lambda.$

$\lambda = 0 \qquad X(x) = (c_1 x + c_2)$

$c_2 = 0 \quad \because u(0,y) = 0$

$\Rightarrow X(0) \cdot Y(y) = 0$

$\Rightarrow X(0) = 0 \qquad \because Y(y) \neq 0.$

$u(L,Y) = 0 \; X(L) \cdot Y(y).$

$= c_1 L = 0$

$\Rightarrow c_1 = 0.$

$\therefore$ trivial solⁿ for $\lambda = 0$.

$\underline{\lambda > 0}$

$$X(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$$

$$X(0) = 0 \quad \Rightarrow C_1 + C_2 = 0$$

$$X(L) = 0 \Rightarrow C_1 e^{\sqrt{\lambda} L} + C_2 e^{-\sqrt{\lambda} L} = 0$$

$$C_1 = 0 \;;\; C_2 = 0$$

$\underline{\lambda < 0}$

$\underline{\lambda = -m^2}$

$$X(x) = C_1 \cos mx + C_2 \sin mx$$

$$X(0) = 0 \Rightarrow \underline{C_1 = 0}$$

$$\Rightarrow X(x) = C_2 \sin mx$$

$$X(L) = 0 \qquad \Rightarrow C_2 \sin mL = 0$$

we are looking for non trivial soln.

$\therefore C_2 \neq 0$, $mL = n\pi$

$$m = \frac{n\pi}{L} \quad , n = 1, 2 \ldots$$

$$\lambda = -m^2 = -\frac{n^2 \pi^2}{L^2}$$

($\lambda = 0$ is excluded so n starts from 1, 2, ...)

$$\therefore \lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

($\because$ there are multiple eigen values).

eigen functions corresponding to λ_n

$$\sin\left(\frac{n\pi}{L} x\right).$$

there's no sense to compute $Y(y)$ for $\lambda \geq 0$

($\because X(x)$ is zero & $u = XY$)

$\lambda < 0$, $\lambda = -m^2$

$$Y' - \lambda Y = 0 \cdot \qquad \Rightarrow Y = C_1 \cos mx + C_2 \sin mx$$

$$\therefore \underline{\underline{Y = d_1 e^{-m^2 y}}}$$

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Note: if $u(0,y) = 1$

$\Rightarrow X(0) \cdot Y(y) = 1$

doesnt mean $X(0) = 1$.

instead we have infinite sol^n.

$u = XY$.

$= C_2 \sin\left(\frac{n\pi x}{L}\right) d_1 e^{-\left(\frac{n\pi}{L}\right)^2 y}$.

$$u_n(x,y) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 y}.$$

because each eigen value yields a sol^n u.

$\therefore$ the general sol^n is

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 y}$$

$$u(x,0) = f(x)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$ (fourier series).

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Recall $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, then

$$a_n = \frac{f^{(n)}(a)}{n!} \quad \text{Taylor's series.}$$

along those lines, we wish to expand a function $f(x)$ as $[-\pi, \pi]$ which is periodic (i.e. $f(x+\pi) = f(x)$) as $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ — ⊛

& need to determine a_0, a_n, b_n $(n \geq 1)$ assume that ⊛ can be integrated term by term.

Simply integrate ⊛ from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \sum_{n\geq 1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx.$$

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx = a_0 \cdot 2\pi.$$

$$\underline{\underline{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx.}}$$

$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ — ⊛

multiply ⊛ with $\cos mx$ and $\int_{-\pi}^{\pi}$

$$\int_{-\pi}^{\pi} f(x) \cos mx\,dx = a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx\,dx = 0. \quad \forall\, m, n.$$

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$$\int_{-\pi}^{\pi} f(x)\cos mx \, dx = a_m \pi.$$

$$\therefore \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos nx \, dx.$$

$\|{}^{ly}$ $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

f piecewise continuous on $(-\pi, \pi]$ and periodic

Eg. $$f(x) = \begin{cases} 0 & , \ -\pi \leq x < 0 \\ 1 & , \ 0 \leq x < \pi \end{cases}$$

$$f(x+2\pi) = f(x).$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \left[\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} dx \right].$$

$$= \frac{\pi}{2\pi} = \underline{\underline{1/2}}.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{0}^{\pi} 1 \cdot \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \cdot \left. \frac{\sin nx}{x} \right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_{0}^{\pi} \sin nx \, dx$$

$$= \frac{-1}{\pi} \left. \frac{\cos nx}{x} \right]_0^{\pi} = \frac{-1}{n\pi} \left(\cos n\pi - \cos 0 \right)$$

$$= \frac{-1}{n\pi} (\cos n\pi - 1) = \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n\pi}, & n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x \ldots$$

Heat conduction in a rod (wire) of finite length $0 \le x < L$.

assumptions: homogenous material thin so that heat distribution is uniform.

$x=0$ $\qquad\qquad$ $x=L$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0<x<L$$

$u=u(x,t)$

defined as an Initial Boundary value Problem.

init. condition: $u(x,0)=f(x)$.

fixed ends (fixed temperature)

homogenous b.c.s: $u(0,t)=0$, $u(L,t)=0$ $\quad \forall\ t>0$.

insulated ends

$$\frac{\partial u}{\partial x}(0,t)=0, \quad \frac{\partial u}{\partial x}(L,t)=0 \quad \forall\ t>0.$$

IBVP.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0<x<L.$$

i.c. $u(x,0)=f(x)$.

b.c. $u(0,t)=0$, $u(L,t)=0$ $\quad \forall\ t>0$.

let $u(x,t)=X(x)\,T(t)$

$$XT' = kTX''$$

$$\frac{X}{X''} = \frac{1}{k}\frac{T'}{T} = \text{const} := d.$$

$k=1$.

$X'' - \lambda X = 0.$

$u(0,t) = X(0).\,T(t) = 0$

$\Rightarrow X(0) = 0$

$u(L,t) = X(L)T(t) = 0$

$\Rightarrow X(L) = 0$

$\lambda = 0$ $\quad X(x) = ax + b$

$X(0) = 0, \quad X(L) = 0 \quad \Rightarrow a = 0$
$\qquad b = 0.$

trivial solution.

$\lambda > 0$ $\quad X(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$

$X(0) = 0 = c_1 + c_2$

$X(L) = 0 = c_1 e^{\sqrt{\lambda} L} + c_2 e^{-\sqrt{\lambda} L}.$

$\Rightarrow c_1 = 0, \; c_2 = 0$

trivial solⁿ.

$\lambda < 0$, $\lambda = -m^2$ (say)

$X(x) = c_1 \cos mx + c_2 \sin mx.$

$X(0) = 0 \;\Rightarrow c_1 = 0$

$X(x) = c_2 \sin mx$ $\qquad x(L) = 0 = c_2 \sin mL.$

non-trivial solⁿ

$c_2 \neq 0 \;\Rightarrow \sin mL = 0$

$\Rightarrow m = \frac{n\pi}{2}, \; n = 1, 2 \cdots$

for $\lambda < 0$, $= -m^2$, $\quad \frac{T'}{T} = \lambda = -m^2$

$\Rightarrow T = d_1 e^{-m^2 t} = \cancel{ded}\, d_1 e^{-(n\pi/L)^2 t}$

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$d \quad u_n(x,t) = X_n(t)\, T_n(t)$

$$= C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

$\cancel{C_n \sin(\frac{n\pi x}{L})}$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 t}.$$

need to determine C_n from i.c. $u(x,0) = f(x)$.

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right).$$

$$\int_0^L f(x) \sin\frac{m x\pi}{L}\,dx = \sum_{n=1}^{\infty} C_n \int_0^L \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\,dx$$

$$C_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

suppose $f(x) = (1-x)$

$$C_n = \frac{2}{L}\int_0^L (1-x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$$= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx - \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx.$$

$$\int_0^L x \sin\left(\frac{n\pi x}{L}\right) = \left[-\frac{Lx\cos}{n\pi}\left(\frac{n\pi x}{L}\right)\right]_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right) dx$$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx = \frac{L}{n\pi}\left[-\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$\frac{L}{n\pi}\left(-\cos n\pi + 1\right)$$

$$= \frac{-L^2}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\left[\cancel{\frac{L}{n\pi}\left(\sin(n\pi - 0\right)}\right]$$

$$= \frac{-L^2}{n\pi}\cos(n\pi)$$

$$\frac{2}{L} + \frac{-2}{L}\cdot\frac{L}{n\pi}\cos n\pi + \frac{2}{L}\cdot\frac{L^2}{n\pi}\cos n\pi \quad \ldots \text{solve}$$

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Heat conduction equation. - insulated ends.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad , \quad 0 < x < L$$

$$\boxed{u(x,0) = f(x).}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad ; \quad \frac{\partial u}{\partial x}(L,t) = 0 \qquad \forall\, t > 0.$$

look for $u(x,t) = X(x)\,T(t)$.

$$XT' = X''T \quad \Rightarrow \boxed{\begin{array}{l} X'' - \lambda X = 0 \\ T' - k\lambda T = 0. \end{array}}$$

b.c.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(X, T) = X'T$$

$$\frac{\partial u}{\partial x}(0,t) = X'(0)\,T(t) = 0$$

$$\Rightarrow \boxed{\begin{array}{l} X'(0) = 0 \\ X'(L) = 0 \end{array}}$$

$$\frac{\partial u}{\partial x}(L,t) = 0 \Rightarrow$$

$\lambda = 0$

$$X'' - \lambda X = 0 \quad \Rightarrow X(x) = ax + b.$$

$X'(0) = 0 \Rightarrow a$

$X'(L) = 0$ (satisfied).

$\Rightarrow b$ is arbitrary.

$$\therefore \boxed{X(x) = b.}$$

non-trivial Sol^n.

$\lambda > 0$.

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}.$$

$$\Rightarrow X' = \sqrt{\lambda}\left(c_1 e^{\sqrt{\lambda}x} - c_2 e^{-\sqrt{\lambda}x}\right)$$

$$X'(0) = 0 = \sqrt{\lambda}(c_1 - c_2).$$

$X'(L) = 0 = \sqrt{\lambda}\,(c_1 e^{\sqrt{\lambda}L} - c_2 e^{-\sqrt{\lambda}L})$.

$\Rightarrow c_1 = 0,\ c_2 = 0$.

$\therefore$ trivial solⁿ

$\lambda < 0$ $\quad X(x)$, $\lambda = -m^2$.

$X(x) = c_1 \cos mx + c_2 \sin mx$.

$X' = m(-c_1 \sin mx + c_2 \cos mx)$ $\qquad X' = -mc_1 \sin mx + mc_2 \cos mx$.

$X'(0) = 0 = c_2$

$X'(L) = 0 = mc_1 \sin mL$

$\Rightarrow m = \dfrac{n\pi}{L}$, $n = 1, 2 \ldots$

$\therefore \lambda_n = -\left(\dfrac{n\pi}{L}\right)^2$, $n = 1, 2 \ldots$ $(n \neq 0)$.

$X_n = \cos \dfrac{n\pi}{L} x$.

$\therefore u_n(x,t) = X_n T_n$.

$\lambda = 0$. $\quad T' - \lambda T = 0 \Rightarrow T = d_1$

$\lambda < 0$. $\quad T' + m^2 T = 0 \Rightarrow T = d_2 e^{-m^2 t}$.

$u_n(x,t) = \underset{(\lambda = 0)}{X_n T_n} + \underset{(\lambda < 0)}{X_n T_n}$

$\therefore \tilde{u}_n(x,t) = bd_1 + c_n \cos\left(\dfrac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$.

$u(x,t) = A_0 + \displaystyle\sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$.

$= \displaystyle\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$

I.C. $\quad u(x,0) = f(x)$

$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$

$\therefore A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

Let $f(x) = x$

$= \frac{2}{L} \int_0^L x \cos\left(\frac{n\pi x}{L}\right) dx.$

1). $u(0,t) = 0.$

$\frac{\partial u}{\partial x}(L,t) = 0$

$u(x,0) = \begin{cases} 1, & 0 < x < L/2 \\ 2, & L/2 < x < L \end{cases}$

2). $\frac{\partial u}{\partial x}(0,t) = 0$

$\frac{\partial u}{\partial x}(L,t) = 0$

$u(x,0) = \begin{cases} x, & 0 < x < L/2 \\ 1-x, & L/2 < x < L \end{cases}$

2) $X'' - \lambda X = 0$

$T' - \lambda T = 0.$

$X'(0,t) = 0$

$X'(L,t) = 0.$ } init.

Solving for X

$\lambda = 0.$ $X = ax + b.$

b is arbitrary

$X(x) = b.$

$\lambda > 0$

no sol^n.

$\lambda < 0.$

$X_n = \cos\left(\frac{n\pi x}{L}\right)$

||^{ly} $\lambda = 0$ $T = d_1$

$\lambda < 0$ $T = d_2 e^{-m^2 t}$

$u_n(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$

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$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = \begin{cases} x & , 0 < x < L/2 \\ 1-x & , L/2 < x < L . \end{cases}$$

$$A_n = \frac{2}{L} \int_0^{L/2} x \cos\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L} \int_{L/2}^{L} (1-x) \cos\left(\frac{n\pi x}{L}\right) dx .$$

$$= \frac{2}{L} \left[\int_0^{L/2} x \cos\left(\frac{n\pi x}{L}\right) dx + \int_{L/2}^{L} (1-x) \cos\left(\frac{n\pi x}{L}\right) dx \right].$$

$$\int_0^{L/2} x \cos\left(\frac{n\pi x}{L}\right) dx = \left[\frac{L}{n\pi} . x \sin\left(\frac{n\pi x}{L}\right) \right]_0^{L/2} - \int_0^{L/2} \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{L^2}{2n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{(n\pi)^2} \left[\cos\left(\frac{n\pi x}{L}\right) \right]_0^{L/2} .$$

1). b.c. $X(0) = 0$ $X'(L) = 0$

$X'' - \lambda X = 0$

$\underline{\lambda = 0}$ $X = ax + b$.

$X'(L) = 0$ $\Rightarrow a = 0$

$X(x) = b$

$\underline{\lambda > 0}$ $C_1 + C_2 = 0$

$C_1 e^{\sqrt{\lambda} L} - C_2 e^{-\sqrt{\lambda} L} = 0$

$\underline{C_1 = C_2 = 0}$.

$\underline{\lambda < 0}$ $X(x) = C_1 \cos mx + C_2 \sin mx$.

$\underline{C_1 = 0}$; $m C_2 \sin mL = 0$

$\Rightarrow m$ $\underline{\text{arbitrary}}$.

$X(x) = \underline{C_2 \sin mx}$.

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(I). Determine the region in which the given eq^n is hyperbolic, parabolic, elliptic. Identify the transformation in respective regions that transforms the given eq^n to cannonical form. Reduce it to corresponding cannonical form.

(a). $x u_{xx} + u_{yy} = x^2$

(b). $u_{xx} + y^2 u_{yy} = y$.

(c). $x^2 u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} = e^x$.

(d). $u_{xx} - \sqrt{y}\, u_{xy} + x u_{yy} = \cos(x^2 - 2y)$, $y \geq 0$

Exercises on classification.

The type of PDE is a local property.

An equation can change its behaviour in different regions of a plane. Can change its behaviour depending on a parameter.

Eg 1) Tricomi's eq^n

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$a = y$
$b = 0$
$c = 1$

$b^2 - 4ac = -4y$

- $y = 0$ parabolic
- $y < 0$ hyperbolic
- $y > 0$ elliptic

Eg 2) $$\frac{1}{1-m^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$b^2 - 4ac = \dfrac{-1}{1-m^2}$

- $m^2 > 1$ hyperbolic
- $m^2 < 1$ elliptic

Eg 3) Reduce to cannonical form $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} = \frac{1}{xy}\left(y^3 u_x + x^3 u_y\right)$

$a = y^2 \quad b = -2xy \quad c = x^2$

$b^2 - 4ac = 4x^2y^2 - 4x^2y^2 = 0$. parabolic.

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2xy}{2y^2} = -\frac{x}{y}.$$

$\rightarrow \xi = x^2 + y^2$ only one characteristic (real).

Let $\eta = y$

$$u_x = u_\xi \cdot 2x + u_\eta \cdot 0 = 2x \cdot u_\xi.$$

$$u_{xx} = 2(u_\xi + x \cdot 2x\, u_{\xi\xi})$$

$$u_{xy} = 2x(u_{\xi\xi}\, 2y + u_{\xi\eta} \cdot 1).$$

$$u_y = u_\xi \cdot 2y + u_\eta \cdot 1$$

$$u_{yy} = \cancel{u_{\xi\xi}\, 2y + u_\xi \, 2 + u_{\xi\eta} 2y}.$$

$$2(u_\xi + y\, u_{\xi\xi} \cdot 2y + y\, u_{\xi\eta} + u_{\eta\xi}\, 2y + u_{\eta\eta})$$

$$u_{\eta\eta} - \frac{1}{\eta} u_\eta = 0 \qquad \text{Cannonical / standard form.}$$

$$\cancel{\eta\, u_{\eta\eta} - u_\eta = 0.}$$

$$\cancel{\frac{\partial}{\partial \eta}(\eta u_\eta - 2u) = 0.}$$

$$\cancel{\eta u_\eta - 2u = f(\xi)}$$

~~rather do integration by parts.~~

$$\cancel{u_\eta} \quad \frac{1}{\eta} u_{\eta\eta} - \frac{1}{\eta} u_\eta = 0.$$

$$\frac{\partial}{\partial \eta}\left(\frac{1}{\eta} u_\eta\right) = 0.$$

$$u_\eta = \eta f(\xi)$$

~~[illegible]~~

$$u = \eta^2 f(\xi) + g(\xi)$$

$$= y^2 f(x^2+y^2) + g(x^2+y^2).$$

Eg 4) $u_{xx} + x^2 u_{yy} = 0.$

$a=1 \quad b=0 \quad c = x^2$

$b^2 - 4ac = -4x^2$ — $x = 0$ parabolic.

— $x \neq 0$ elliptic.

$$\frac{dy}{dx} = \frac{\pm\sqrt{-4x^2}}{2} = \pm ix.$$

$$y \pm \frac{ix^2}{2} = c_{1,2}.$$

$\xi(x,y) = y + \frac{ix^2}{2}$ $\quad \eta(x,y) = y - \frac{ix^2}{2}$ imaginary

(this is a characteristic of ~~[illegible]~~ elliptical form)

Need to switch to a plane where the given PDE admits a cannonical form

$$\alpha = \frac{\xi+\eta}{2} \quad ; \quad \beta = \frac{\xi-\eta}{2i}$$

$$= y \qquad\qquad = x^2/2.$$

So $\alpha = y, \ \beta = x^2$

↳ remove 2 as it can be absorbed in the family no point working with the factor.

$u_x = u_\alpha \cdot 0 + u_\beta \cdot 2x = 2u_\beta x$ $\qquad u_{xx} = 2u_\beta + (2x)^2 u_{\beta\beta}$

$u_{xy} = 2u_{\beta\alpha} \cdot xy$

$u_y = u_\alpha \cdot 1 + u_\beta \cdot 0 = u_\alpha.$ $\qquad u_{yy} = u_{\alpha\alpha}$

$u_{xx} + x^2 u_{yy} = 0.$

$\Rightarrow 2u_\beta + 4x^2 u_{\beta\beta} + x^2 u_{\alpha\alpha} = 0.$

$\Rightarrow \quad u_{\alpha\alpha} + 4u u_{\beta\beta} = \frac{-1}{2x^2} u_\beta \quad = \frac{-1}{2\beta} u_\beta.$

$\Rightarrow \boxed{u_{\alpha\alpha} + u_{\beta\beta} = \frac{-1}{2\beta} u_\beta.}$ cannonical

Eg 5) Reduce $u_{xx} + u_{xy} - 2u_{yy} + 1 = 0.$

in $0 \leq x \leq 1$, $y > 0.$

to cannonical form and solve with $u = \frac{\partial u}{\partial y} = x$ on $y = 0.$

$a = 1,\ b = 1,\ c = -2$

$b^2 - 4ac = 9 > 0.$ **hyperbolic**

$\therefore \frac{dy}{dx} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2} = 2, -1.$

$\Rightarrow \xi = y - 2x \quad \eta = y + x$

$\Rightarrow x =$

$y =$

$u_x = u_\xi(-2) + u_\eta(1) \quad = -2u_\xi + u_\eta.$

$u_y = u_\xi + u_\eta.$

$u_{xy} = -2(u_{\xi\xi} + u_{\xi\eta}) + u_{\eta\xi} + u_{\eta\eta}.$

$u_{yy} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}$

$u_{xx} = -2u_{\xi\xi}\cdot(-2)$
$-2u_{\xi\eta} + u_{\eta\eta} + u_{\eta\xi}(-2)$
$= 4u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - 2u_{\eta\xi}.$

$u_{xx} + u_{xy} - 2u_{yy} + 1 = 0$

$$u_{\xi\eta} = 1/9.$$

Cannonical form

$u_{\xi} = \frac{1}{9}\eta + f(\xi).$

$u = \frac{1}{9}\eta\xi + g(\xi) + h(\eta).$

$u = \frac{1}{9}(y-2x)(y+x) + g(y-2x) + h(y+x).$

$x = \frac{-2}{9}x^2 + g(-2x) + h(x)$

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① Solve $u_t = 9u_{xx}$, $0 < x < 2$
$t > 0$

$u(0,t) = 0$ $\frac{\partial u}{\partial x}(2,t) = 0$

$u(x,0) = x$ ~~boundary conditions~~ → $u(0,t)=0$
$u(x,0) = x$
$x'(2,t) = 0.$

$X'' - \lambda X = 0$

for $\lambda = 0$

$\Rightarrow X = ax + b.$

$X'(0) = 0 = a$

$X'(2) = 0 = a$

$\Rightarrow a = 0$

$b = \cancel{\text{arbitrary}}\ 0$ as $x(0) = 0.$

for $\lambda > 0$

$X(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$

$X'(x) = C_1\sqrt{\lambda}e^{\sqrt{\lambda}x} - C_2\sqrt{\lambda}e^{-\sqrt{\lambda}x}$

$X'(2) = C_1\sqrt{\lambda}e^{2\sqrt{\lambda}} - C_2\sqrt{\lambda}e^{-2\sqrt{\lambda}}$

~~$X'(0) = C_1 - C_2 = 0$~~ $(X(0)) = 0.$

$\Rightarrow C_1 = C_2 = 0.$

for $\lambda < 0$

$\lambda = -m^2$

$X(x) = C_1\cos mx + C_2\sin mx.$

$X'(x) = -C_1 m\sin mx + C_2 m\cos mx$

$X'(0) = \cancel{C_2 m = 0}$

$\Rightarrow C_2 = 0.$

$X'(2)=0$

$-\cancel{C_1 m \sin 2m}$ $C_2 m \cos(2m)=0$.

$2m = (2n+1)\frac{\pi}{2}$

$m = \frac{(2n+1)\pi}{4}$; $n = 0,1,2 \ldots$

$T' - \lambda T = 0$

for $\lambda > 0$ $\quad Y(x) = 0$

so no T needed

$\lambda < 0$ $\quad T' + m^2 T = 0$

$T = d_2 e^{-m^2 t}$

2nd explanation

let $u(x,t) = \underbrace{X(x)}_{\text{func}^n \text{ of } x} \cdot \underbrace{T(t)}_{\text{func}^n \text{ of } t} \quad \times$

PDE = $\frac{\partial u}{\partial t} = \frac{9\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(X(x)\ T(t)) = X(x)\frac{dT}{dt} = X(x)\ T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(X(x)\,T(t)) = \frac{\partial^2 X}{\partial x^2}(T(t))$$

$$= X''(x)\ T(t)$$

$\Rightarrow \quad X(x)\ T'(t) = 9\,X''(x)\,T(t)$

$\Rightarrow \quad f_n \cdot f_x = f_n \cdot f_t$.

$\Rightarrow \quad \frac{T'}{9T} = \frac{X''}{X} = \lambda$ const.

$$\Rightarrow \boxed{X'' - \lambda X = 0 \quad ; \quad T' - 9\lambda T = 0.}$$

b.c

$u(0,t) = 0 \Rightarrow X(0)\ T(t) = 0.$

for non trivial sol^n $\quad T(t) \neq 0 \quad \Rightarrow \boxed{X(0) = 0}$

$\frac{\partial u}{\partial x}(2,t) = 0 \Rightarrow \quad X'(2)\ T(t) = 0$

$\Rightarrow \boxed{X'(2) = 0}$

to solve.

$X'' - \lambda X = 0$, $X(0) = 0$, $X'(2) = 0$.

$\lambda = 0$ $\Rightarrow$ $X'' = 0$ $\Rightarrow$ $X(x) = ax + b$

$X(0) = 0 \Rightarrow b = 0$

$X'(2) = 0 \Rightarrow a = 0$.

$\Rightarrow$ trivial solution.

$\lambda > 0$: $\Rightarrow X(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$.

$X(0) = 0 = c_1 + c_2$

$X'(2) = 0 \Rightarrow \lambda (c_1 e^{2\sqrt{\lambda}} - c_2 e^{-2\sqrt{\lambda}})$

$\Rightarrow c_1 = 0, c_2 = 0$ $\Rightarrow$ trivial solution.

$\lambda < 0$: let $\lambda = -m^2$.

$X(x) = c_1 \cos mx + c_2 \sin mx$.

$X(0) = 0 \Rightarrow c_1 = 0$

$X(x) = c_2 \sin mx$

$X'(x) = m c_2 \cos mx$

$X'(2) = 0 = m c_2 \cos 2m$.

non trivial soln's $\Rightarrow$ $c_2 \neq 0$

$2m = (2n+1)\pi/2$ $m = (2n+1)\pi/4$.

$\therefore$ eigenvalues : $\lambda_n = -[(2n+1)\pi/4]^2$

eigen functions : X_n: $\cancel{\cos\left(\frac{(2n+1)\pi}{2}x\right)}$

$\sin\left(\frac{(2n+1)\pi}{4}x\right)$

$T' - g\lambda T = 0.$

$$T = Be^{g\lambda T} = Be^{-g\left[\frac{(2n+1)\pi}{4}\right]^2 t}$$

$u(x,t) = \sum u_n$

$$= \sum_{n=0}^{\infty} A_n \sin\frac{(2n+1)\pi}{4}x \cdot e^{-g\left[\frac{(2n+1)\pi}{4}\right]^2 t}$$

$u(x,0) = x$

$$x = \sum_{n=0}^{\infty} A_n \sin\left[\frac{(2n+1)\pi}{4}x\right].$$

$\cancel{A_n = \frac{2}{2}\int_0}$ $\quad A_n = \frac{2}{2}\int_0^2 x \sin\frac{(2n+1)\pi}{4}x \, dx.$

This was one ~~case~~ end insulated and other end flux inflow case

This can be extended to more dimensions

$$\frac{\partial u}{\partial t} = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Wave Equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$, x \in \mathbb{R}$

$-\infty < x < \infty$

$t > 0$

$u = u(x,t)$

↳ displacement

initial data

initial displacement $u(x,0) = f(x)$

initial velocity $\frac{\partial u}{\partial t}(x,0) = g(x)$

(infinite string) $x \in \mathbb{R}$.

↓

this is one of the ~~two~~ 3 kinds of possible cases

infinite string ∃ vanishing disturbances at far field at $x \to \infty$.

hyperbolic eq $c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$

$A = c^2 \quad B = 0 \quad C = -1.$

$b^2 - 4ac = 4c^2 > 0.$

$$\frac{dt}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{2c}{2c^2} = \pm 1/c.$$

$dt = \pm \frac{dx}{c}$

$$x - ct = c_1 = \xi(x,t)$$
$$x + ct = c_2 = \eta(x,t)$$

two real characteristics.

$u_x = u_\xi + u_\eta \qquad u_t = -cu_\xi + cu_\eta$

$u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta} \qquad u_{tt} = c^2 u_{\xi\xi} - c^2 u_{\xi\eta} + c^2 u_{\eta\eta} - c^2 u_{\xi\eta}$

$= c^2 u_{\xi\xi} + c^2 u_{\eta\eta} - 2c^2 u_{\xi\eta}$

$c^2 u_{xx} - u_{tt} = 0 \Rightarrow \quad c^2 u_{\xi\xi} + c^2 u_{\eta\eta} + 2c^2 u_{\xi\eta} - c^2 u_{\xi\xi} - c^2 u_{\eta\eta} + 2c^2 u_{\xi\eta} = 0.$

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$\Rightarrow 4c^2 u_{\xi\eta} = 0.$ (Cannonical form)

$\Rightarrow \boxed{u_{\xi\eta} = 0}$

$\Rightarrow u_{\xi} = f(\xi)$

$\Rightarrow u = g(\xi) + h(\eta)$

$\therefore u(x,t) = f(x-ct) + g(x+ct)$ $\quad u(x,t) = f(x+ct) + g(x-ct)$

$u(x,t) = \phi(x+ct) + \psi(x-ct)$

Cond^{n}s. 1). $u(x,0) = f(x) = \phi(x) + \psi(x)$ —— (A) $\quad f(x)$ = initial displacement

2). $u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x)$ —— (B) $\quad g(x)$ = initial velocity

integrate (B) from x_0 to x.

$$\int_{x_0}^{x} g(J)\,dJ = c\int_{x_0}^{x} \phi'(J)\,dJ - c\int_{x_0}^{x} \psi'(J)\,dJ$$

$$= c(\phi(x) - \psi(x)) - c(\phi(x_0) - \psi(x_0))$$

$\cancel{-c\phi(x) + c\phi(x)}$

$$= 2c\phi'(x) - c(f(x)) - c(\phi(x_0) - \psi(x_0))$$

$$= 2c\phi(x) - cf(x) - \tilde{k}$$

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(J)\,dJ + \frac{\tilde{k}}{2c}$$

$\tilde{k} = c(\phi(x_0) - \psi(x_0))$

lly

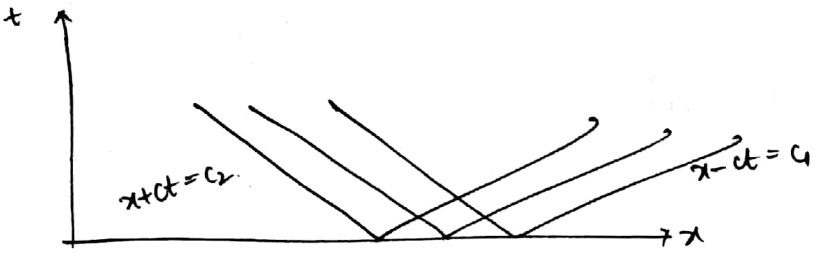
$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(J)\,dJ - \frac{\tilde{k}}{2c}.$$

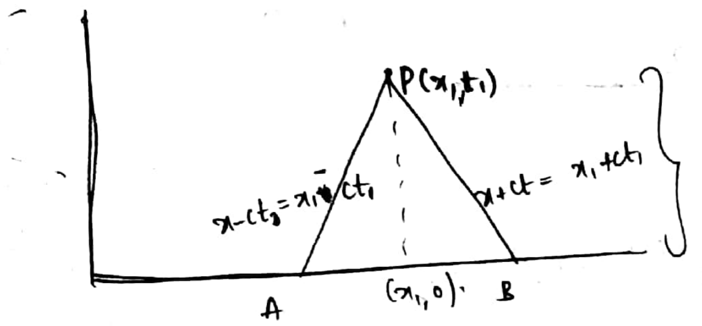
$$\therefore u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(J)\,dJ + \cancel{\frac{\tilde{k}}{2c}} + \frac{1}{2} f(x-ct) - \frac{1}{2}\int_{x_0}^{x-ct} g(J)\,dJ - \cancel{\frac{\tilde{k}}{2c}}$$

$$\therefore u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(J)\,dJ.$$

(in exams, derive cannonical form and solve)

d'Alembert's solution.



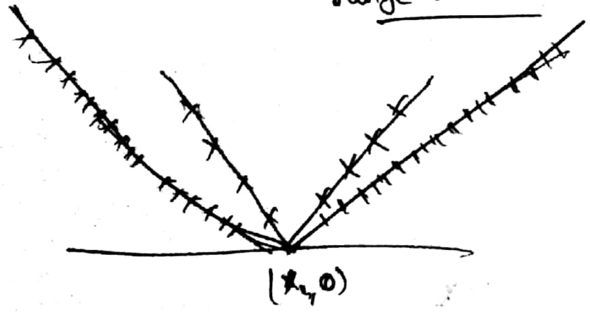


this is the complete scope for getting to point P.

$$u(x_1,t_1) = u_P = \frac{1}{2}\left[f(A) + g(B)\right] + \frac{1}{2c}\int_{A}^{B} g(J)\,dJ$$

AB: domain of dependence.

range of influence.



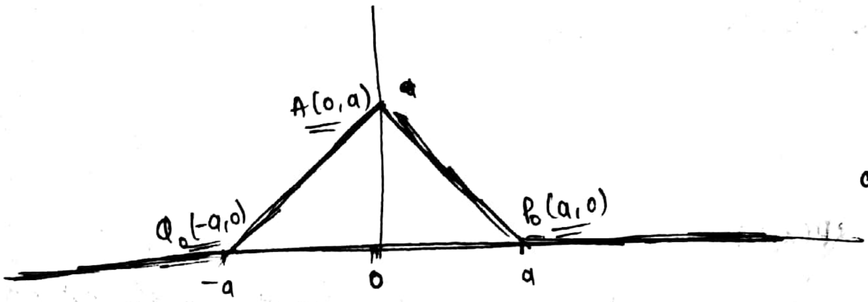
d'Alembert's Sol^n

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi.$$

Case-1) $g(x) = 0$. initial velocity $= 0$.

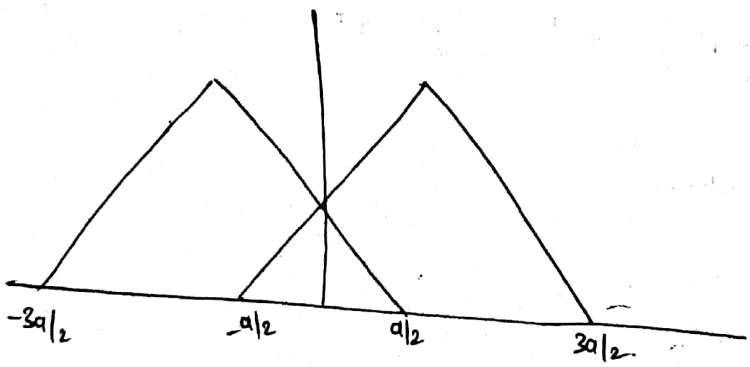
let
$$f(x) = \begin{cases} 0 & , -\infty < x \le -a \\ x+a & , -a \le x \le 0 \\ a-x & , 0 \le x \le a \\ 0 & , a \le x < \infty. \end{cases}$$

$$\therefore u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right].$$



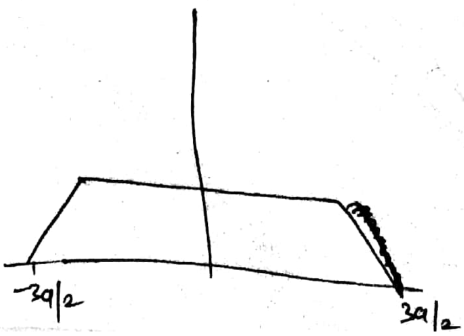
at $t=0$

$$u(x, t=a/2c) = \frac{1}{2}\left[f(x+a/2) + f(x-a/2)\right]$$

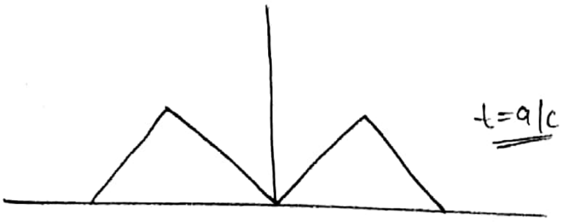


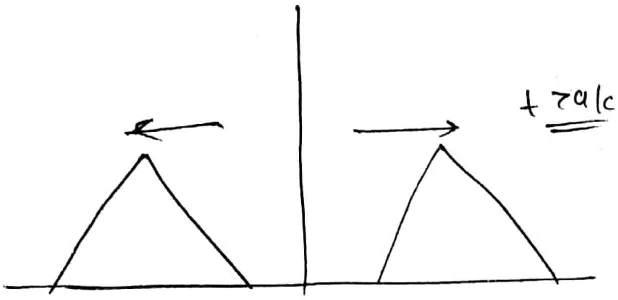
$t = a/2c$

$=$



$u(x, t = a/c) = \frac{1}{2}\left[f(x+a/c) + f(x-a/c)\right]$





Case-2 $\quad f(x) = 0$

$g(x) \neq 0$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$

let $g(x) = x$ $\quad$ then $u(x,t) = xt$.

problem.

$u_{tt} = 9u_{xx}$, $x \in \mathbb{R}$, $t > 0$.

$u(x,0) = \sin x = f(x)$.

$c = 3$

$u_t(x,0) = \cos x = g(x)$.

$$u(x,t) = \frac{1}{2}\left[\sin(3t+x) + \sin(x-3t)\right] + \frac{1}{6}\int_{x-3t}^{x+3t} \cos\xi\, d\xi.$$

$\left[\sin\xi\right]_{x-3t}^{x+3t}$

$= \frac{1}{2}\left[\cancel{2\sin}\cancel{A+B}\ 2\sin(x)\cos(3t)\right]$

$+ \frac{1}{6}\left[\sin(x+3t) - \sin(x-3t)\right]$

$= \sin(x)\cos(3t)$

$+ \frac{1}{6}\left\{\sin x\cos 3t + \cos x \sin 3t - \sin x \cancel{\cos 3t} + \cos x \sin 3t\right\}$

$= \frac{1}{3}(\cos x \sin 3t).$

$= \sin x \cos 3t + \frac{1}{3} \cos x \sin 3t$

Wave eq^n finite string.

$$u_{tt} = c^2 u_{xx} \quad \text{— (1)} \qquad x \in [0, L].$$

BC: $u(0,t) = 0$, $u(L,t) = 0$ — fixed ends.

IC: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

Look for seperable sol^n $u = X(x)\,T(t)$. — (*)

(*) in (1).

$$X T'' = c^2 X'' T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda \text{ (say)}.$$

pair of 2nd order ODEs:

$$X'' - \lambda X = 0 \quad ; \quad T'' - c^2 \lambda T = 0$$

BC: $X(0) = 0$ $X(L) = 0$.

$\lambda = 0, \lambda > 0$ → trivial sol^n.

$\lambda < 0$. $X(x) = C_1 \cos mx + C_2 \sin mx.$

$X(0) = C_1 = 0$

$X(L) = C_2 \sin(mL)$ $mL = n\pi$, $n \geq 1$

$m = \frac{n\pi}{L}.$

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$\therefore$ eigen values. $\lambda_n = -m^2 = -\left(\frac{n\pi}{L}\right)^2$

eigen functions $X_n : \sin\left(\frac{n\pi x}{L}\right)$.

correspondingly, ($\lambda<0$)

$T_n(t) = T'' + c^2\left(\frac{n\pi}{L}\right)^2 T = 0.$

$\Rightarrow T_n(t) = d_n \cos\left(\frac{n\pi c}{L} t\right) + e_n \sin\left(\frac{n\pi c}{L} t\right).$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t\right) \sin \frac{n\pi x}{L}.$

IC: $u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$

$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$

$\Rightarrow \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx}_{L/2 \text{ (by orthogonality property)}}.$

$\Rightarrow A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$

^{11}ly $B_n = \frac{2}{L} \cdot \frac{L}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$

let $f(x) = 1$ $g(x) = x$

$B_n = \frac{2}{n\pi c} \int_0^L x \sin \frac{n\pi x}{L}\, dx.$

$A_n = \frac{2}{\cancel{L}} \left[\frac{-\cancel{L} \cos\left(\frac{n\pi x}{L}\right)}{n\pi}\right]_0^L$

$\Rightarrow \frac{2}{n\pi}\left[-\cos(n\pi) + 1\right] = \frac{2}{n\pi}\left[1 - (-1)^n\right].$

get the answer

Remark.

in case of f, g algebraic terms only, we proceed through the previous process as only algebraic terms can be expressed as fourier series.

Suppose the initial data is

$$u(x,0) = \frac{1}{3}\sin 5\pi x$$

$$u_t(x,0) = 6\sin 10\pi x - 100 \sin 3\pi x$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi c t}{L}\right)\sin n\pi x.}$$

general solution

$$u(x,0) = \frac{1}{3}\sin 5\pi x = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}.$$

$$\Rightarrow A_5 = \frac{1}{3} ; A_n = 0 \ \forall\, n \neq 5.$$

$$u_t(x,0) = 6\sin 10\pi x - 100\sin 3\pi x$$

$$= \sum_{n=1}^{\infty}\left(B_n \cdot \frac{n\pi c}{L}\right)\sin n\pi x.$$

for $n = 3$

$$-100 = B_3 \cdot \frac{3\pi c}{L} \qquad B_3 = -\frac{L \cdot 100}{3\pi c}.$$

for $n = 10$

$$B_{10} \cdot \frac{10 \times \pi c}{L} = 6.$$

$$B_{10} = \frac{6L}{10\pi c}.$$

and all other zero.

#

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \alpha_n \cos\frac{n\pi c}{L}(t+\delta_n)\sin\frac{n\pi x}{L}.}$$

harmonic motion with amplitude $\alpha_n \sin\frac{n\pi x}{L}$.

each motion is a standing wave. with nodes at $\sin \frac{n\pi x}{L} = 0$.

max. amplitude at $\sin \frac{n\pi x}{L} = \pm 1$.

for a fixed t,

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi x}{L}.$$

$$C_n(t) = \alpha_n \cos \omega_n (t + \delta_n)$$

$$\omega_n = \frac{n\pi c}{L}.$$

free ends.

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0.$$

one end fixed other free

$u(x,0) = x$

$$u(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0.$$

$u_t(x,0) = x^2$.

$$X'' - \lambda X = 0 \qquad T'' - c^2 \lambda T = 0.$$

$\lambda = 0$

$X = ax + b$

$X(0) = 0 \quad b = 0.$

$X'(L) = \quad a = 0.$

trivial

$\lambda > 0$

$X(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}.$

$X(0) = C_1 + C_2 = 0.$

$X'(L) = C_1 \sqrt{\lambda} e^{\sqrt{\lambda} x} - C_2 \sqrt{\lambda} e^{-\sqrt{\lambda} x}$

$C_1 e^{\sqrt{\lambda} x}$

$C_1 = C_2 = 0$

$\lambda < 0$

$X(x) = C_1 \cos mx + C_2 \sin(mx)$

$X(0) = C_1 = 0$

$X'(L) = \cancel{-mC_1 \sin mx} + mC_2 \cos mx$

$\cos mL = 0$

$mL = (2n+1)\pi/2 \qquad m = \frac{(2n+1)\pi}{2L}$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{(2n+1)\pi c t}{2L} \right) + B_n \sin\left(\frac{(2n+1)\pi c t}{2L} \right) \right) \sin\left(\frac{(2n+1)\pi}{2L} x \right).$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n+1)\pi}{2L} x \right). = x.$$

$$A_n = \frac{2}{L} \int_0^L x \sin\left(\frac{(2n+1)\pi}{2L} x \right) dx.$$

$$= \frac{2}{L} \left\{ 2L \left[\frac{x \cos\left(\frac{(2n+1)\pi}{2L} x \right)}{(2n+1)\pi} \right]_0^L - \int_0^L \frac{2L \cdot \cos\left(\frac{(2n+1)\pi}{2L} x \right)}{(2n+1)\pi} dx \right].$$

$$= \frac{2}{\cancel{L}} \left[\cancel{\frac{2L}{(2n+1)\pi} \cdot L \cos} \quad 0 \quad + \quad \frac{2\cancel{L}}{(2n+1)\pi} \cdot \frac{2L}{(2n+1)\pi} \sin \frac{(2n+1)\pi}{2\cancel{L}} \cdot L \right]$$

$$A_n = \frac{8L}{((2n+1)\pi)^2} (-1)^n$$

$$B_n \quad u_t(x,0) = \sum \frac{B_n \cdot 2L}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi}{2L} x \right) = x^2.$$

$$B_n = \frac{2}{L} \int_0^L \frac{(2n+1)\pi}{2L} x^2 \sin\left(\frac{(2n+1)\pi}{2L} x \right) dx$$

$$B_n = \frac{(2n+1)\pi}{L^2} \underline{\int_0^L x^2 \sin\left(\frac{(2n+1)\pi}{2L} x\right) dx.}$$

$$\int_0^L x^2 \sin\left(\frac{(2n+1)\pi}{2L} x\right) = \cancel{\left[- 2L \frac{x^2 \cos \frac{(2n+1)\pi x}{2L}}{(2n+1)\pi} \right]_0^L} + 2\int_0^L x \frac{2L}{(2n+1)\pi} \cos \frac{(2n+1)\pi x}{2L} dx$$

$$= \frac{4L}{(2n+1)\pi} \int_0^L x \cos \frac{(2n+1)\pi x}{2L} dx.$$

$$\int_0^L x \cos \frac{(2n+1)\pi x}{2L} dx = \left[2L \frac{x \cdot \sin \frac{(2n+1)\pi x}{2L}}{(2n+1)\pi} \right]_0^L - \int_0^L \frac{2L}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi x}{2L}\right) dx$$

$$= \frac{2L \cdot L \cdot \sin\left(\frac{(2n+1)\pi}{2}\right)}{(2n+1)\pi} - \frac{2L}{(2n+1)\pi} \cdot \frac{2L}{(2n+1)\pi}$$

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