Date
16/10/2017

Lecture 21

$H(x) \xrightarrow{\text{periodic}}$ F·S

Fourier Integral

Fourier series — periodic f^{ns}

non-periodic f^{ns}

Fourier series extend.

Q) What can be done to extend the method of Fourier series to non-periodic f^{ns}.

We begin with a special f^n

eg./ $f_L(x)$ of period $2L$ & see what happens to $f_L(x)$ as $L \to \infty$.

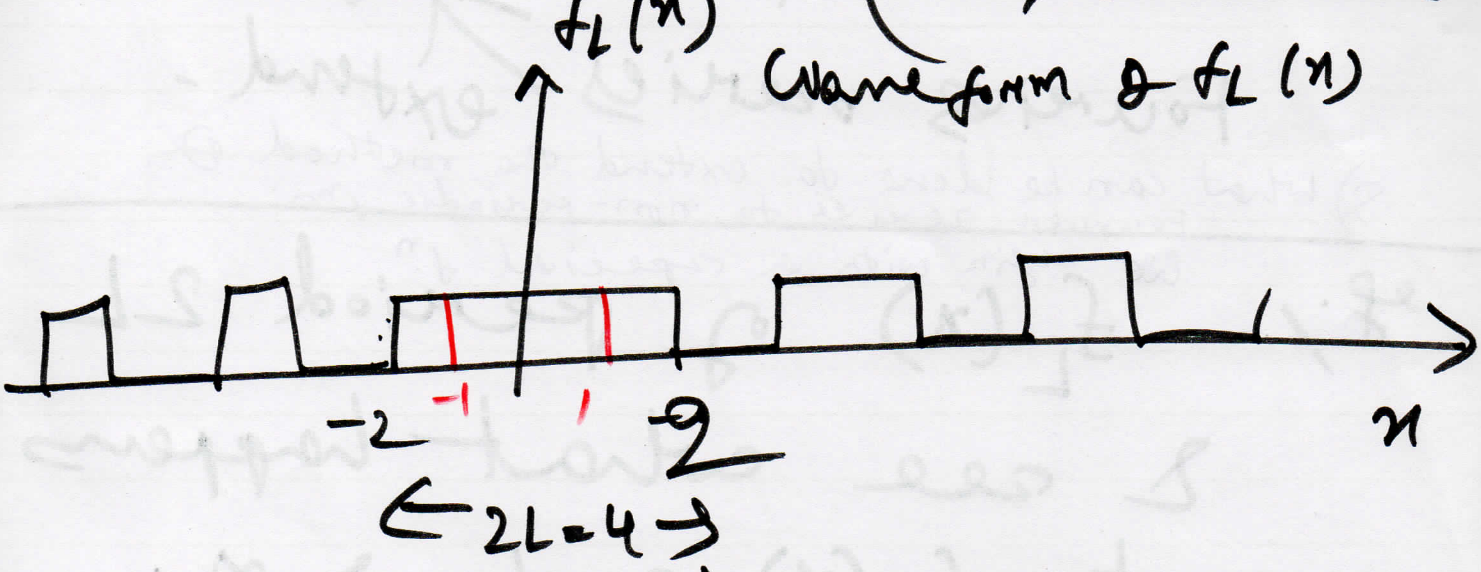
Ex1) Square wave

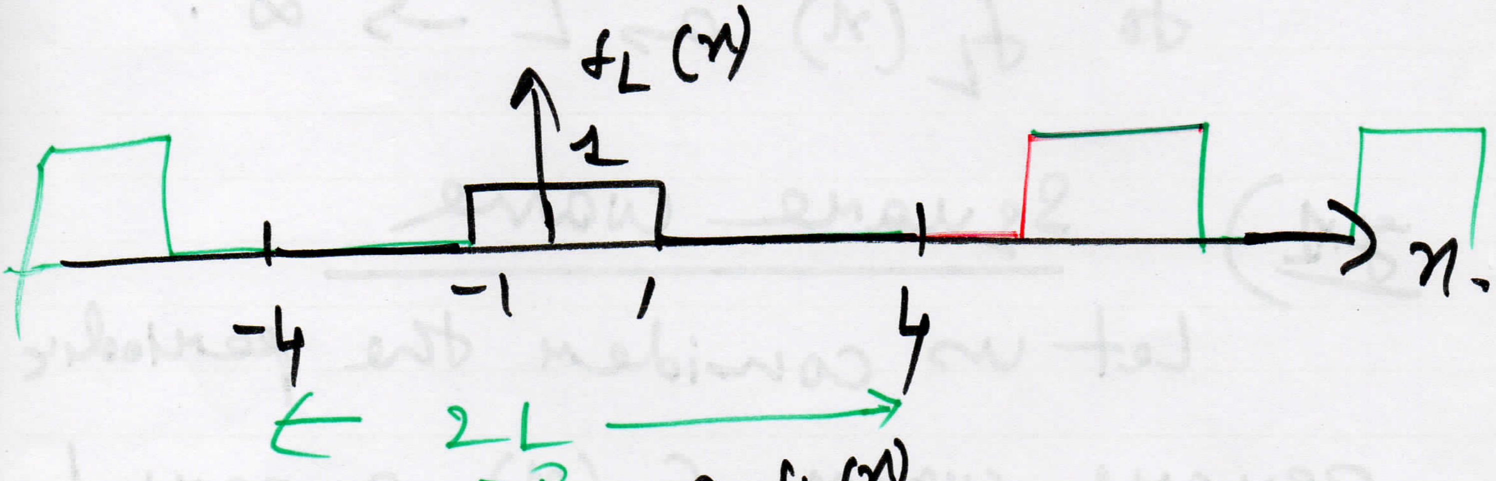
Let us consider the periodic square wave $f_L(x)$ of period $2L > 2$ given by

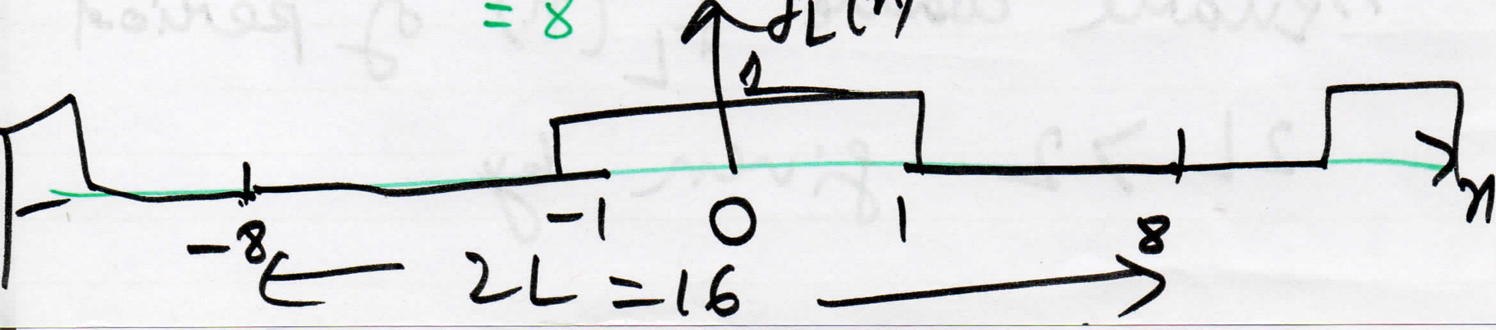
$$f_L(x) = \begin{cases} 0, & \text{if } -L < x < -1 \\ 1, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < L \end{cases}$$

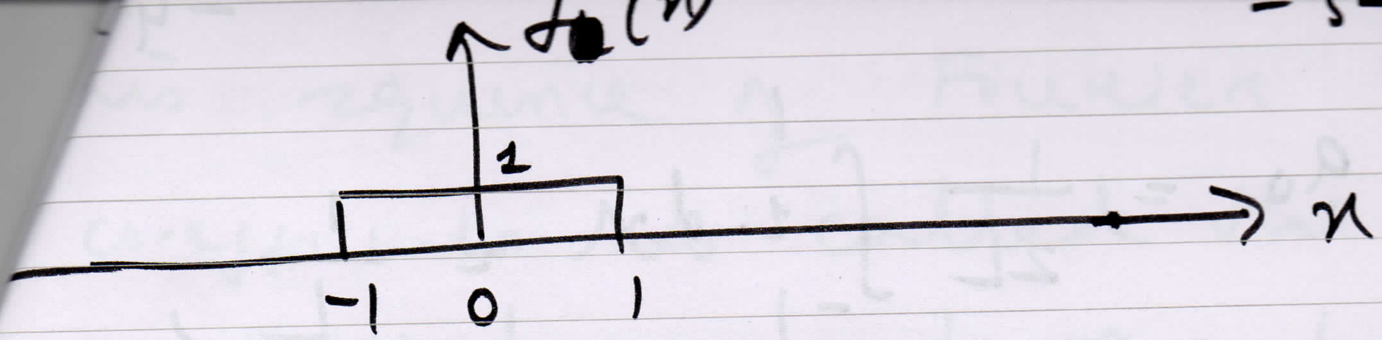
$$\& \lim_{L \to \infty} f_L(x) = \begin{cases} 1, & \text{if } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Waveform of $f_L(x)$









~~the~~ Fig 1 :- shows this f^n for $2L = 4, 8, 16$ as well as the non-periodic f^n, $f(x)$, which we obtain from $f_L(x)$ if we let $L \to \infty$.

We now explore what happens to the Fourier co-efficients of f_L as L increases.

Since f_L is even (why?),

$\boxed{b_n = 0}$ for all n.

For a_n, the Euler formulas give

$$a_0 = \frac{1}{2L}\int_{-1}^{1} 1.\,dx = \frac{1}{L},$$

$$a_n = \frac{1}{L}\int_{-1}^{1} 1.\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^1 \cos\left(\frac{n\pi x}{L}\right)dx \qquad \left[f_L(x)=1,\ -1<x<1.\right.$$

$$= \frac{2}{L}\,\frac{L}{n\pi}\cdot\left[\sin\frac{n\pi x}{L}\right]_{x=0}^{1}$$

$$= \frac{2}{L}\cdot\frac{\sin n\pi/L}{(n\pi/L)}$$

$\therefore \underset{L\to\infty}{Lt}\ a_n = 0$ (how?)

$\underline{\underline{EX}}$

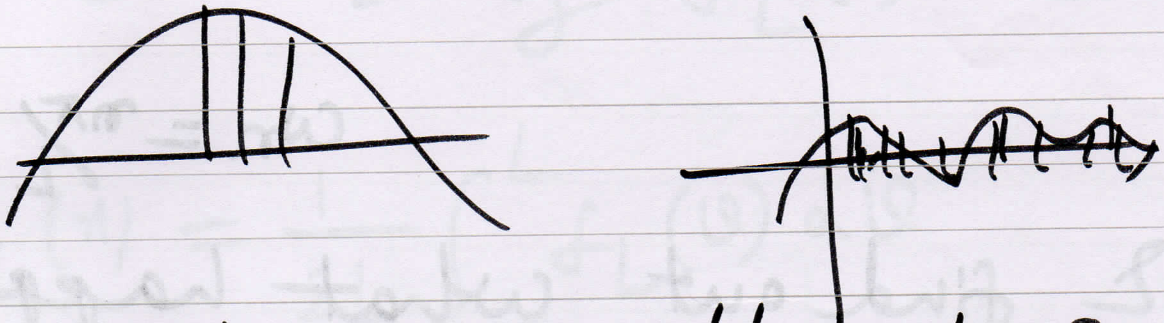
ie, $|a_n| \to 0$ as $L \to \infty$

this sequence of Fourier co-efficients is called the amplitude spectrum of f_L

because $|a_n|$ is the max^m amplitude of the wave

$a_n \cos(n\pi x/L)$

ie, these amplitudes an



will eventually be everywhere dense on the axis & will decrease to zero.

§ From Fourier series to the Fourier Integral

We now consider any periodic f^n $f_L(x)$ of period $2L$ that can be represented by a Fourier series.

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \omega_n x + b_n \sin \omega_n x\right)$$

$$\omega_n = n\pi / L.$$

& find out what happens if we let $L \to \infty$.

Here, $\cos \omega_n x$ & $\sin \omega_n x$ with ω_n is no longer restricted to integer multiples.

$$\omega = \omega_n = n\pi/L \text{ of } \pi/L$$

For all values of n.

We want to see, what form such an integral might have.

We now insert the values of a_0, a_n & b_n & denote the variable of integration by v, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv$$

$$+ \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos\omega_n x \int_{-L}^{L} f_L(v)\cos(\omega_n v)\,dv$$

$$+ \sin\omega_n x \int_{-L}^{L} f_L(v)\sin(\omega_n v)\,dv\Big]$$

We now let

$$\Delta\omega = \omega_{n+1} - \omega_n$$

$$= \frac{(n+1)\pi}{L} - \frac{n\pi}{L}$$

$$= \pi/L$$

Then $\frac{1}{L} = \frac{\Delta\omega}{\pi}$ & we may write the Fourier series in the form

$$f_L(x) = \underbrace{\frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv}_{(=0)}$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[(\cos\omega_n x)\,\Delta\omega\int_{-L}^{L} f_L(v)\cos(\omega_n v)\,dv$$

$$+ (\sin\omega_n x)\,\Delta\omega\int_{-L}^{L} f_L(v)\sin(\omega_n v)\,dv\Big] \longrightarrow (1)$$

This representation is valid for any fixed L arbitrarily large, but finite.

We now let $L \to \infty$ & assume that the resulting non-periodic

f^n $$f(x) = \lim_{L \to \infty} f_L(x)$$

is <u>absolutely integrable</u> on the x-axis

i.e., the following (finite!) limits exist!

$$\lim_{a \to -\infty} \int_a^0 |f(x)|\,dx + \lim_{b \to \infty} \int_0^b |f(x)|\,dx$$

(written as $\int_{-\infty}^{\infty} |f(x)|\,dx$)
exist.

$\longrightarrow$ 2

then $1/L \to 0$ & the value of the first term on the R.H.S of eqn (1) approaches zero. (ie, $\int_{-\infty}^{\infty} |f_L(u)|\,du$ converges ~~exists~~.)

Also, $\Delta\omega = \pi/L \to 0$ as $L \to \infty$

& it seems plausible
(how?)

that the infinite series in (1) becomes an integral from 0 to ∞ which represents $f(x)$ ~~namely~~.

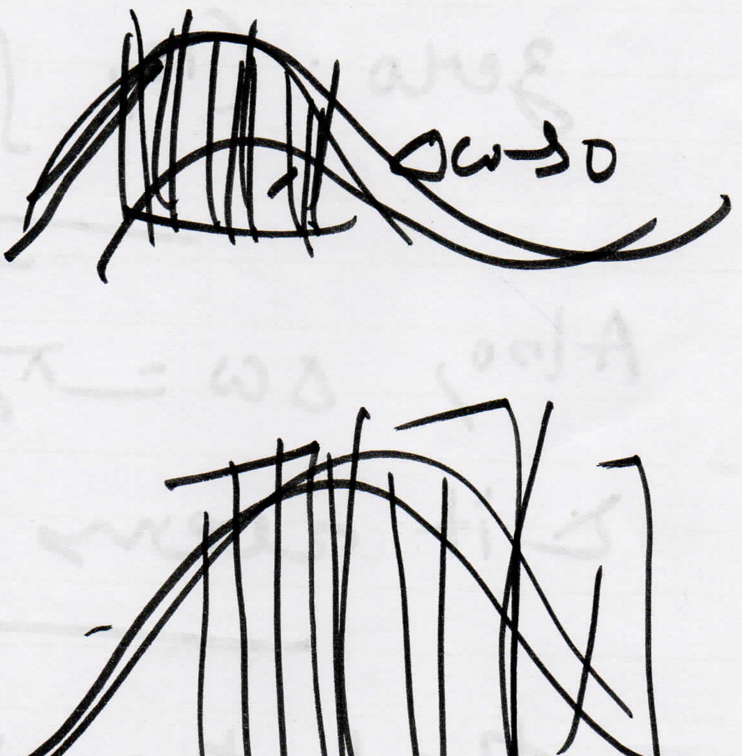
$$\text{ie, } \lim_{\Delta\omega \to 0} \left\{ \sum_{n=1}^{\infty} F(\omega_n)\, \Delta\omega \right\}$$

$$= \int_0^{\infty} F(\omega)\, d\omega$$

(Ex how?)

Which resembles a Riemann sum of a definite integral.

<u>Note</u> :- Periodic f^{ns} are represented by a Fourier series & non-periodic f^{ns} are represented in the form of an integral ie, by Fourier integral.



$$f(x) = \lim_{L \to \infty} f_L(x)$$

$$= \frac{1}{\pi} \int_0^\infty \Big[\cos(\omega x) \int_{-\infty}^{\infty} f(v) \cos(\omega v)\, dv$$

$$+ \sin(\omega x) \int_{-\infty}^{\infty} f(v) \sin(\omega v)\, dv \Big] d\omega \longrightarrow (3)$$

If we introduce the notation

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos(\omega v)\, dv$$

$$\& \; B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(\omega v)\, dv \longrightarrow (4)$$

we can write this in the form

$$f(x) = \int_0^{\infty} \Big[A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x) \Big] d\omega \quad \to (5)$$

This is called a representation of $f(x)$ by a Fourier integral.

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Th-1/ (Sufficient Condition)
(Fourier integral)

If $f(x)$ is piecewise continuous in every finite interval & has a

R. H. Derivative & a L. H. Derivative at every point & if the integral (2) exists i.e. ($\int_{-\infty}^{\infty} |f(x)|\,dx$ exists)

Then $f(x)$ can be represented by a Fourier integral (5) with A & B given by (4).

At a point where $f(x)$ is discontinuous, the value of the Fourier Integral equals the average of the left-hand & Right-hand limits of $f(x)$ at that point.

Equivalent forms of Fourier Integral Theorem.

Fourier's integral theorem can also be written in the forms

① $$f(x) = \frac{1}{2\pi}\int_{\alpha=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha$$

✓✗✗✓ (EX)

② (Complex form of Fourier integral)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(x-u)}\,du\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)\, e^{-i\alpha x}\,d\alpha, \text{ where } C(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\,dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\, d\alpha \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du$$

where it is understood that if $f(x)$ is not continuous at x, the L.H.S must be replaced by $\dfrac{f(x+0)+f(x-0)}{2}$.

These results can be simplified somewhat if $f(x)$ is either odd or even f^n.

(I) $f(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \sin(\alpha x)\, d\alpha \int_0^{\infty} f(u) \sin(\alpha u)\, du$ if $f(x)$ is odd

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos(\alpha x)\,d\alpha \int_0^\infty f(u)\cos(\alpha u)\,du$$

if $f(x)$ is even.

Note:- Fourier integrals are used mainly in solving O.D. eq^ns & P.D. eq^ns. However, they also help in evaluating integrals as well.

§/ Fourier Cosine & Sine Integrals

For an even or odd f^n, the Fourier integral becomes simpler, just as in the case of Fourier series.

Indeed, if $f(x)$ is an even f^n, then

// $B(\omega) = 0$ in eq^n (4)

& $$\boxed{A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos(\omega v)\,dv}$$

The Fourier integral (5) then reduces to the Fourier cosine integral

$$\boxed{f(x) = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega \quad (f \text{ even})}.$$

S'lly, if $f(x)$ is odd, then in (4), we have

$$A(\omega) = 0 \quad \&$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin(\omega v)\,dv$$

The Fourier integral (5) then reduces to the Fourier sine integral

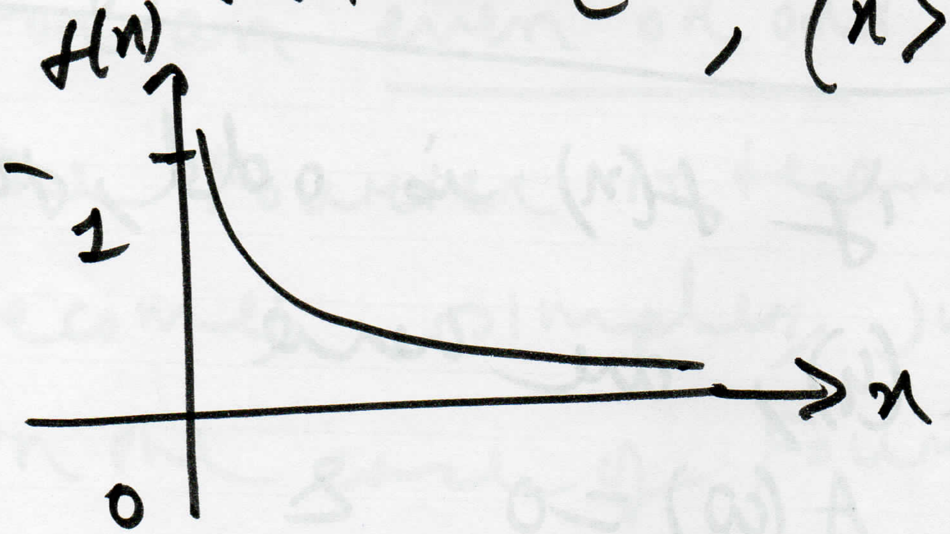
$$f(x) = \int_0^\infty B(\omega)\, \sin(\omega x)\, d\omega$$

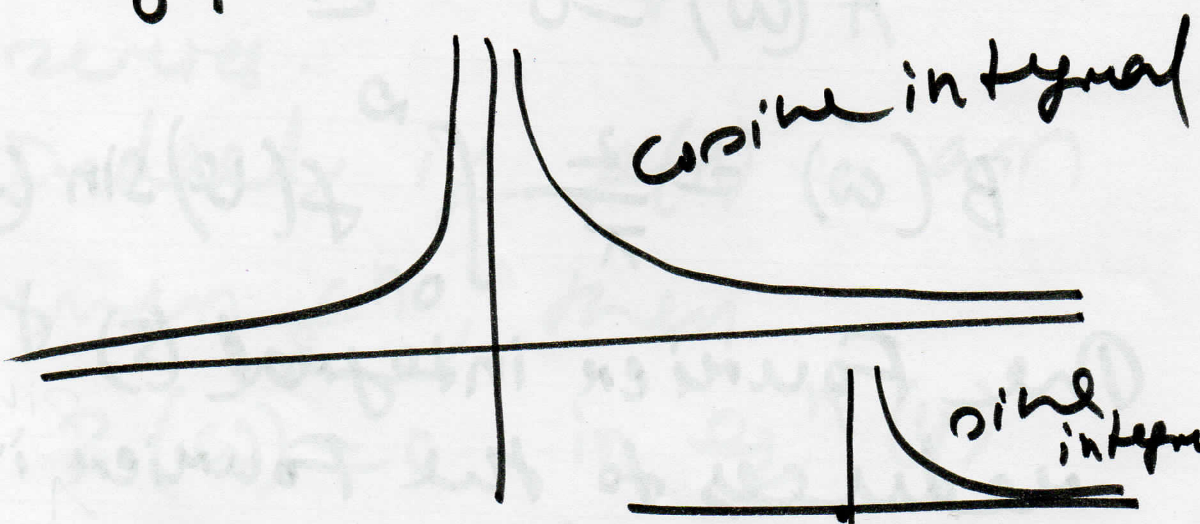
(f odd)

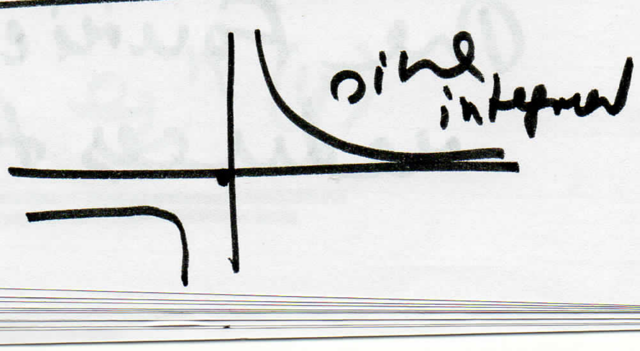
Ex1/ Find the Fourier cosine & sine integrals of

$$f(x) = e^{-kx}, \quad (x > 0,\ k > 0)$$

Sol^n:-







Sol^n:- We have

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos(\omega v)\,dv$$

$$= \frac{2}{\pi}\int_0^\infty \underbrace{e^{-kv}\cos(\omega v)\,dv}$$

Now, integration by parts,

$$\int e^{-kv}\cos(\omega v)\,dv$$

$$= \frac{-k}{k^2+\omega^2}e^{-kv}\left(-\frac{\omega}{k}\sin\omega v + \cos\omega v\right)$$

(EX)

$$= \left[\frac{-k}{(k^2+\omega^2)}e^{-kv}\left(-\frac{\omega}{k}\sin(\omega v) + \cos(\omega v)\right)\right]_0^\infty$$

$$= \frac{-k}{k^2+\omega^2}[0-1] = \left(\frac{k}{k^2+\omega^2}\right)$$

Thus $$\boxed{A(\omega) = \frac{2k/\pi}{k^2+\omega^2}}$$

$\therefore$ the Fourier cosine integral representation is

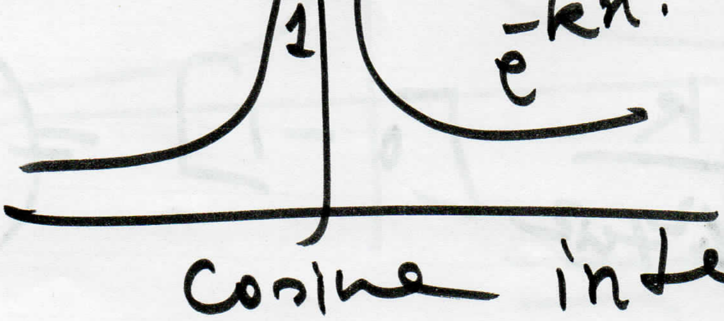
$$f(x) = \int_0^\infty A(\omega) \cos(\omega x)\, d\omega$$

$$\Rightarrow e^{-kx} = \frac{2k}{\pi} \int_0^\infty \frac{\cos \omega x}{k^2+\omega^2}\, dx$$

$$\Rightarrow \boxed{\int_0^\infty \frac{\cos \omega x}{k^2+\omega^2}\, d\omega = \frac{\pi}{2k} e^{-kx}}$$

$(x>0, k>0)$

e^{-kx}.



cosine integral.

b) S'lly, we have

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin(\omega v)\,dv$$

$$= \frac{2}{\pi}\int_0^\infty e^{-kv}\sin(\omega v)\,dv$$

✓ Sec 4.

~~Roughwork~~

Integrating by parts.

$$\int e^{-kv}\sin(\omega v)\,dv = \frac{-\omega}{k^2+\omega^2}e^{-kv}\left(\frac{k}{\omega}\sin\omega v + \cos\omega v\right)$$

$$= \left[\frac{-\omega}{(k^2+\omega^2)}\cdot e^{-kv}\left(\frac{k}{\omega}\sin(\omega v)+\cos(\omega v)\right)\right]_0^\infty$$

$$= -\frac{\omega}{k^2+\omega^2}\left[0-1\right] = \frac{\omega}{k^2+\omega^2}.$$

Thus, $\boxed{B(\omega) = \frac{2\omega/\pi}{k^2+\omega^2}}$

$\therefore$ the Fourier sine integral representation is

$$f(x) = \int_0^\infty B(x)\sin(\omega x)\,d\omega$$

$$\Rightarrow e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{\omega \sin(\omega x)}{k^2+\omega^2}\,d\omega$$

$$\Rightarrow \boxed{\int_0^\infty \frac{\omega \sin(\omega x)}{k^2+\omega^2}\,d\omega = \frac{\pi}{2}e^{-kx}}$$

$x>0, k>0$

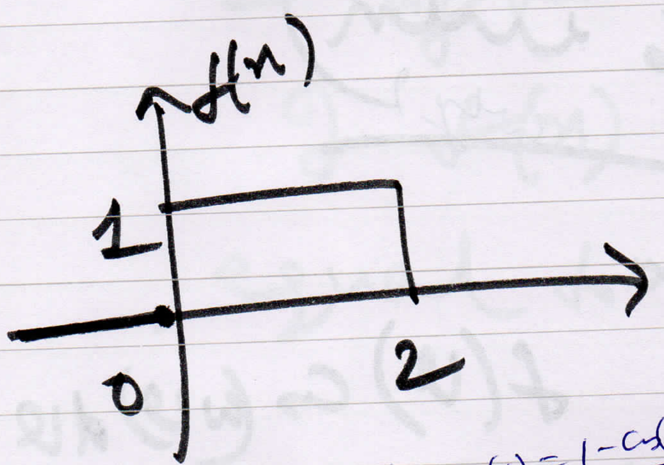
$\rightarrow$ (2)

The above integrals are called the (1) & (2) Laplace integrals.

Ex

) <u>Find</u> the Fourier integral representation of the piece-wise continuous f^n

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$



Hence evaluate $\int_0^\infty \frac{\sin\alpha}{\alpha}\, d\alpha$ $(= \pi/2)$

<u>Hint</u> :- $A(\alpha) = \frac{\sin 2\alpha}{\alpha}$, $B(\alpha) = \frac{1-\cos(2\alpha)}{\alpha}$

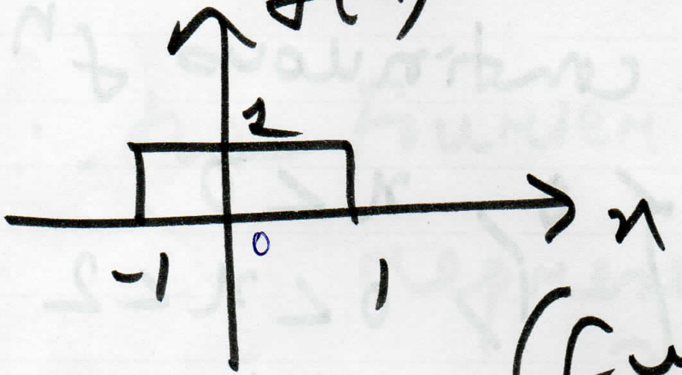
$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha \cos\{\alpha(x-1)\}}{\alpha}\, d\alpha$

CW

Ex2 <u>Single pulse, sine integral</u>

Q) Find the Fourier integral representation of the f^n

$$f(x) = \begin{cases} 1, & \text{if } |x| < 1 \quad \text{i.e., } -1 < x < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$



EX: Find Fourier integral representation for

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

(f is even) why?

we have

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv$$

$$= \frac{1}{\pi}\int_{-1}^{1} 1 \cdot \cos(\omega v)\,dv$$

$$= \left[\frac{\sin(\omega v)}{\pi v}\right]_{-1}^{1} = \boxed{\frac{2\sin\omega}{\pi\omega}}$$

$$B(\omega) = 0$$

$$\therefore f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos\omega x \sin\omega}{\omega}\,d\omega$$

$$\Rightarrow \int_0^\infty \frac{\cos \omega x \ \sin \omega}{\omega}\, d\omega = \tfrac{\pi}{2} f(x) \quad \rightarrow (1)$$

The average of the left & right hand limits of $f(x)$ at $x=1$ is equal to $\left(\frac{1+0}{2} = \tfrac{1}{2}\right)$

$\therefore$ from (1) & (h-1) ?? we get

$$\int_0^\infty \frac{\cos \omega x \ \sin \omega}{\omega}\, d\omega = \begin{cases} \tfrac{\pi}{2} \cdot 1 = \tfrac{\pi}{2}, & 0 \le x < 1 \\ \tfrac{\pi}{2} \cdot \tfrac{1}{2} = \tfrac{\pi}{4} & \text{if } x = 1. \\ \tfrac{\pi}{2} \cdot 0 = 0 & \text{if } x > 1 \end{cases}$$

This integral is called

Dirichlet's discontinuous
factor.

Let $x = 0$,

then $$\boxed{\int_0^\infty \frac{\sin \omega}{\omega}\, d\omega = \pi/2}$$

This integral is the limit of the so-called sine integral.

sine integral

$$Si(u) = \int_0^u \frac{\sin \omega}{\omega}\, d\omega$$

as $u \to \infty$.

— x —