$$\Rightarrow f(x) \le g(x) \le \sum_{k=1}^{\infty} |f_{n_k}(x)|$$

$$\int f(x) \le \int |g(x)| \le \int |f_{n_1}| + \int \sum \int | f_{n_{k+1}}(x) - f_{n_k}(x)| \le \|f_{n_1}\| + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$$

$\Rightarrow$ By MCT f is integrable

$f < \infty$

$$f_{n_k} = f_{n_1}(x) + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j}) \quad \text{for each } k$$

$$|f - f_{n_k}| \le g$$

$$\|f_{n_k} - f\| \longrightarrow 0$$

Apply DCT

As $n_k \longrightarrow \infty$.

28th March

1. $x_n = (-1)^n + \frac{1}{n} + 2\sin\left(\frac{n\pi}{2}\right)$

$x_{4n} = 1 + 1/4n$

$x_{4n+1} = -1 + 1/4n+1 + 2 = 1 + \frac{1}{4n}$

$x_{4n+2} = 1 + 1/4n+2$

$x_{4n+3} = -3 + 1/4n+3.$

sup $= 2$

inf $= -3$

lim sup $= 1$

lim inf $= -3$

2) $f(0) = 0$

$f(x) = x \sin\frac{1}{x}$, $x > 0$

$m\{x \in [0,1] \mid f(x) \ge 0\}$

$$m\left[\left\{\tfrac{1}{\pi}\right\} \cup \{0\} \cup \left(\tfrac{1}{\pi}, 1\right) \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{(2n+1)\pi}, \frac{1}{2n\pi}\right)\right.$$

$$= 1 - \frac{1}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2n+1}\right)$$

$$= 1 - 1/\pi + 1/\pi \sum_{n=2} \left(\frac{(-1)^n}{n}\right)$$

$$= 1 - \frac{\ln 2}{\pi}$$

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③ $C[a,b]$ is dense in $L^1[a,b]$

$C[a,b]$ is not complete wrt L^1 Norm.

Proof: Take $f \in L^1[a,b]$

Given $\varepsilon > 0$ need to find $g \in C[a,b]$ s.t. $\|f - g\|_1 < \varepsilon$.

$C[a,b] \subseteq R[a,b] \subseteq L_1[a,b]$

$\overline{C[a,b]} = L[a,b]$

$\Rightarrow \overline{R[a,b]} = L_1[a,b]$

Enough to take $f \geq 0$.

$(f = f^+ - f^-,\ f^+ \geq 0,\ f^- \geq 0)$

$g, h \in L(a,b)$

$\|g - f^+\| < \varepsilon/2$

$\|h - f^-\| < \varepsilon/2$

$\|f - (g+h)\|_1 < \varepsilon$.

$s_n \nearrow f$, s_n are simple (non-negative)

$\int s_n \xrightarrow{(MCT)} \int f$.

(simple functions are dense in $L^1[a,b]$)

We can take f to be non negative simple measurable function.

$f = \sum_{i=1}^{n} a_i \chi_{A_i}$, $a_i \geq 0$

$A_i \cap A_j = \phi$

A_i's are measurable

$f = \chi_A$ where A is measurable set

By Littlewood's first principle,

$\exists$ a closed set

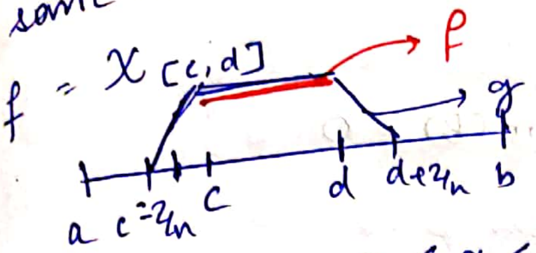
$F = \bigcup_{i=1}^{M} I_i$ s.t. $m(A \Delta F) < \varepsilon$.

$\|\chi_A \setminus \chi_F\|_1 < \varepsilon$.

$\chi_F = \sum_{i=1}^{m} \chi_{I_i}$

Enough to Prove the Theorem for χ_I, I is some interval in $[a,b]$.

$f = \chi_{[c,d]}$



$$g(x) = \begin{cases} 0, & a \le x \le c - 2/n \\ \frac{n}{2}(x - c + 2/n) & c - 2/n \le x \le c \\ 1 & c \le x \le d \\ \frac{n}{2}(d + \frac{2}{n} - x) & d \le x \le d + 2/n \\ 0 & d + \frac{2}{n} \le x \le b \end{cases}$$

$$\int_{c-2/n}^{c} \frac{n}{2}\left(x - c + \frac{2}{n}\right)dx = \frac{2}{n}$$

Egorov's Theorem

$\{f_n\}$ defined on E, where $m(E) = \infty$.

$m(E) < \infty$

$E = \mathbb{R}^+ = [0, \infty)$

$f_n(x) = \chi_{[n, n+1)}(x)$

$f_n \longrightarrow f \quad f = 0 \quad \forall x \in \mathbb{R}^+$

Claim: Let B is any set of finite measure in $\mathbb{R}^+$. Then $f_n \longrightarrow f$ is not uniform on $\mathbb{R}^+ \setminus B$.

$\Rightarrow m(E) < \infty$ is measurable

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$\varepsilon > 0$; $\exists A_\epsilon$ s.t. $f_n \to f$ uniformly on A_ϵ ; $m(E \setminus A_\epsilon) \le \epsilon$. $E \setminus A_\epsilon = B$ $m(B)$

$$\sup_{x \in \mathbb{R}^+ \setminus B} |f_n(x) - f(x)| = \|f_n - f\|_\infty \to 0 \text{ as } n \to$$

1st case: Let $m(B) = 0$

$\sup |f_n(x)|$

(Incomplete)

Lusin's Theorem

If f is measurable and finite valued defined on E s.t. $m(E) < \infty$ $\forall\ \epsilon > 0$, $\exists$ closed set F_ϵ s.t. $f|F_\epsilon$ is cont. & $m(E \setminus F_\epsilon) \leq \epsilon$.

f, $\{f_n\}_{n=1}^{\infty}$ each f_n is cont. s.t. $f_n \to f$ pt. wise $\epsilon = \frac{1}{2^n}$, f_k & a closed set F_k

s.t.

$f_k|_{E_k} = f$ & $m(E - F_k) \leq \frac{1}{2^k}$.

$\{F_k\}_{k=1}^{\infty}$

$E_0 = \limsup_{n \to \infty} F_k^c$

$E_0 = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} F_k^c$.

$m(E_0) \leq m\left(\bigcup_{k=j}^{\infty} F_k^c\right)$ $\forall j \in \mathbb{N}$

$\leq \sum_{k=j}^{\infty} m(F_k^c)$

$\leq \sum_{k=j}^{\infty} \frac{1}{2^k} = \frac{1}{2^{j-1}}$

$m(E_0) \leq \frac{1}{2^{j-1}}$ $\forall j \in \mathbb{N}$

$\Rightarrow m(E_0) = 0$

$x \notin E_0$ $f_k(x) \longrightarrow f$ $\forall\ x \in E_0^c$

$\exists\ j \in \mathbb{N}$ s.t.

$x \notin F_k^c$ $\forall\ k \geq j$

$\Rightarrow x \in F_k$ $\forall\ k \geq j$

$\Rightarrow f_k(x) = f(x)$ $\forall\ k \geq j$ $x \in E_0^c$

$\Rightarrow f_k \longrightarrow f$ pt wise on E_0^c

$\Rightarrow f_k \longrightarrow f$ a.e. on E

$L \int_E f(x)\,dx$

1) simple function.
2) For non-negative.
3) Arbitrary functions

f be a non-negative ~~f~~ measurable simple fn.

$$\int f\,dx = \sum_{i=1}^{n} \alpha_i\, m(E_i) \quad \cancel{\text{range}}$$

range $f = \{\alpha_1, \ldots \alpha_n\}$

$E_i = f^{-1}\{\alpha_i\}$

$0 \cdot \infty = 0$

$$\int f\,dx = \{\sup \int \theta,\ 0 \le \theta \le f \text{ \& } \theta \text{ is simple}\}$$

$$\int f = \int f^+ - \int f^-$$

$f^+ = \max\{f(x), 0\}$

$f^- = \max\{-f(x), 0\}$

$\Rightarrow$ if $\int f^+ < \infty$, $\int f < \infty$

$\Rightarrow \int f < \infty \Longleftrightarrow \int |f| < \infty$

R

① Let $f \geq 0$ & $f = 0$ a.e. in f

$\int f = 0$ & the converse

$f = 0$ a.e $\Leftrightarrow \int f = 0$

$f = 0$ $\quad \phi$ is simple

$0 \leq \phi \leq f \Rightarrow \phi = 0$ a.e

$\int \phi = 0$

$\Rightarrow \sup \int \phi \quad 0 \leq \phi \leq f = 0$

$\Rightarrow \int f = 0.$

$\int f = 0 \Rightarrow f = 0$ a.e.

$A = \{x : f(x) > 0\}$

$A = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \{x : f(x) \geq \frac{1}{n}\}$

$0 = \int_A f = \int_{\bigcup_{n=1}^{\infty} A_n} f \geq \int_{A_n} f = \int f \chi_{A_n}$

$\geq \underbrace{\int \frac{\chi_{A_n}}{n}}_{\text{simple}}$

$= \frac{1}{n} m(A_n)$

Now as $x \in A_n$

$f \chi_{A_n} \geq \frac{1}{n} \chi_{A_n}$

$\therefore m(A) \leq \sum_{n=1}^{\infty} m(A_n)$

$= 0$

$\Rightarrow m(A) = 0$

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$$\Rightarrow \frac{m(A_n)}{n} \leq 0 \quad \forall n$$

$$\Rightarrow m(A_n) = 0 \quad \forall n$$

(6) $f \geq 0$

$f_n(x) = \min\{f(x), n\}$

Then $\int f_n \, dx \uparrow \int f \, dx$

Proof

$f_n \leq f_{n+1}, \forall n \quad f_n \geq 0$

$\lim_{n \to \infty} f_n(x) = f(x)$

By MCT $\int f_n \, dx = \int f \, dx$

$\& \; f_n \leq f \quad \forall n$

$\int f_n \leq \int f$

$\Rightarrow \int f_n \uparrow \int f$

3rd April

Q1 (e) Is $\frac{\sin\frac{1}{x}}{x}$ Lebesgue Integrable?

$$\int_1^\infty \left|\frac{\sin u}{u}\right| du \geq \int_{n(\pi+1)}^{\infty} \left|\frac{\sin u}{u}\right| du.$$

$$= \sum_{n=1}^{\infty} \int_{n\pi}^{} \left|\frac{\sin u}{u}\right| du$$

$$n\pi \leq u \leq n(\pi+1)\ u$$

$$\frac{1}{\pi} \geq \frac{1}{(n+1)u}$$

$$\geq \sum_{n=}^{\infty} \frac{1}{(n+1)\pi} \int_{n\pi}^{\pi(n+1)} |\sin u|\, du$$

$$= \sum_{n=1}^{\infty} \frac{2}{(n+1)\pi} = \infty$$

Not Lebesgue Integrable
(since sum diverges.

Prove that $\lim_{\beta\to\infty} \int_a^b f(x) \sin\beta x\, dx = 0$
where f is a bounded measurable function

$$\lim_{\beta\to\infty} \int_a^b f(x) \sin\beta x\, dx = 0$$

f is bounded and measurable in $[a,b]$

$\Rightarrow f \in L^1[a,b]$.

$\exists\ h$ (simple) ($h = \sum_{i=1}^{n} c_i \chi(a_i, b_i)$)

S.T. $\int_a^b |f-h| < \epsilon$ for any given $\epsilon > 0$.

$\left|\int_a^{b_i} \chi_{(a_i, b_i)} \sin\beta x \, dx\right|$

$= \left|\int_{a_i}^{b_i} \sin\beta x \, dx\right|$

$= \left|\frac{\cos a_i\beta - \cos b_i\beta}{\beta}\right| \leq \frac{2}{\beta} \leq \frac{2\epsilon}{nM}$

for $\beta > \beta_0$

$M = \max\{c_1, \ldots c_n\}$

$\lim_{\beta\to\infty} \int_a^b f(x) \sin\beta x$

$\leq \left|\int_a^b \boxed{(f-h)\sin\beta x}\right| + \left|\int_a^b \sin\beta x\right|$

$\leq \varepsilon + \left|\int_a^b \left(\sum_{i=1}^{n} c_i \chi_{(a_i, b_i)} \sin\beta x\right) dx\right|$

$\leq \varepsilon + nM \int (\chi_{a_i b_i} \sin\beta x) \, dx$

$\leq \varepsilon + 2\varepsilon$

$\leq 3\varepsilon.$

$f \in L^1$

$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x s} \, dx$

Fourier transform

for d dimension

$\left(= \int_{\mathbb{R}^d} f(x) e^{-2\pi i s x_j} \, dx \right)$

If $s \longrightarrow s_0$

then $\hat{f}(s) \longrightarrow \hat{f}(s_0)$

$|f(x) e^{2\pi i s x}| = |f(x)|$

$$\sup_{s} |\hat{f}(s)| \leq \int_{-\infty}^{\infty} |f(x)| = \|f\|$$

As $s \to s_0$

$f(x) e^{-2\pi i x s} \longrightarrow f(x) e^{-2\pi i x s}$

(By continuity)

By DCT

$\Rightarrow \hat{f}(s) \to \hat{f}(s_0)$

as $s \longrightarrow s_0$

Fubini's Theorem

$\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$

$f^y(x) = \{ f(x, y) \mid y \text{ is fixed} \}$

$E \subseteq \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$

$E^x = \{ y \in \mathbb{R}^{d_2} \mid (x,y) \in \mathbb{R}^{d_1} \}$

① $\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{R}^d} f(x-d)\,dx$

② $\int_{\mathbb{R}^d} f(\delta x) = \delta^d \int_{\mathbb{R}^d} f(x)\,dx$

③ $\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$

④ $(f * g)(x)$

$= \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$

$= \int_{\mathbb{R}^d} g(x-y)\, f(y)\, dy$

$= (g * f)(x)$

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Statement

Suppose $f \in L^1(\mathbb{R}^d)$, $(\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$

Then for almost all $y \in \mathbb{R}^{d_2}$, we have

① The slice $f^y(x)$ is integrable on $\mathbb{R}^{d_2}$

② $\int_{\mathbb{R}^{d_2}} f^y(x)\, dy$ is integrable on $\mathbb{R}^{d_1}$

③ $\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^d} f$

Observation

Since the above theorem is symmetric about x & y.

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\, dy \right) dx$$

$$= \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\, dx \right) dy$$

Sketch of the Proof

Let $\mathcal{F}$ be the family of functions that satisfy (i), (ii) and (iii).
We have to show that $L^1[\mathbb{R}^d] \subseteq \mathcal{F}$.

Step 1: $\mathcal{F}$ is closed under finite linear combination & limit.

Step 2: If E is a measurable set of finite measure then $\chi_E \in \mathcal{F}$.

(Q) Show that $f \in L^1(\mathbb{R}^d)$ is need else the conclusion may fail.

Question f is non negative measurable function. Can we apply Fubini's Theorem to compute $\int_{\mathbb{R}^d} f$?

4th April.

$L_2[a,b]$ is complete.

Functional Analysis

$$\ell_p = \left\{ \{x_n\} \;\middle|\; \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} \quad 1 \le p \le \infty.$$

$\|x\|^p$

ℓ_∞ = set of all bounded sequences

i.e. $\left\{ \{x_k\} \;\middle|\; \sup_{1 \le i \le \infty} |x_i| < \infty \right\}$

$\ell_1 \subsetneq \ell_2 \subsetneq \ell_3 \subsetneq \cdots \subsetneq \ell_\infty$

ℓ_2 ↳ Induces an inner product.

$\|x\| = (\langle x, x \rangle)^{1/2}$

Inner Product

$\langle \cdot , \cdot \rangle : X \times X \longrightarrow \mathbb{R}$ or $\mathbb{C}$

(1) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$

(2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(3) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(4) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \alpha \in \mathbb{F}$

Hilbert Spaces

H is said to be an Hilbert space if :

H is complete wrt the norm. $\|\cdot\| = \langle , \rangle^{1/2}$

(i) H is complete wrt the norm

(ii) H is separable (H has a countable dense subsets)

l^p spaces $1 < p < \infty$ are separable

l^∞ is not separable.

$S \subset l^\infty$,

$S = \{ (x_n) \mid x_i = 0 \text{ or } 1 \}$

S is uncountable

$x, y \in S$, $\|x - y\|_\infty = 1$

$\{ x \in S, B(x, 1/2) \}$ collection of disjoint balls. (uncountable)

Cauchy schwartz Inequality

$|\langle x, y \rangle| \leq \|x\| \, \|y\|$

Proof: $y = 0$

$y \neq 0$, $\tilde{y} = \frac{y}{\|y\|}$ $\|y\| = 1$

$0 \leq \langle \langle x, y \rangle x - y, \langle x, y \rangle x - y \rangle$

$\langle \langle x, y \rangle x, \langle x, y \rangle x \rangle - \langle \langle x, y \rangle x, y \rangle$

$- \langle y, \langle x, y \rangle x \rangle + \langle y, y \rangle \geq 0$

$$|\langle x,y\rangle|^2 \langle x,x\rangle - 2\mathrm{Re}\langle\langle x,y\rangle x, y\rangle$$

$$\cancel{-2\langle x,y\rangle}\cancel{\langle x,y\rangle} + \langle y,y\rangle^2$$

$$|\langle x,y\rangle|^2\langle x,x\rangle - 2\mathrm{Re}\langle x,y\rangle$$

$$\langle x - \langle x,y\rangle y,\ x - \langle x,y\rangle y\rangle$$

$$= \langle x,x\rangle + |\langle x,y\rangle|^2 + |x,y|^2\|y\|^2$$

$$\|x\|^2 - 2|\langle x,y\rangle|^2 + |x,y|^2$$

$$4|x,y|^2$$

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + 2\mathrm{Re}\langle y,x\rangle + \|y\|^2$$
$$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{(Cauchy Schwartz)}$$
$$\le (\|x\|+\|y\|)^2$$

$C\ L^1$

$L^\infty[a,b] \leftarrow$ essentially bounded functions

$$\mu\{x : f(x) > M\} = 0$$

$$|f(x)| \le M$$
$\forall$ a.e. x

A-4

5: $f : (0,1) \to \mathbb{R}$

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap (0,1) \\ [1/x]^{-1}, & x \in \mathbb{Q}^c \cap (0,1) \end{cases}$$

$\int_0^1 f\,dx = \infty$

$g(x) = \left[\frac{1}{x}\right]^{-1}, x \in (0,1)$

Then $f = g$ a.e. g is measurable

$\Rightarrow f$ is measurable

$= \int_0^1 g\,dx$

$\geq \int_{1/n+1}^{1} g(x)\,dx \ \forall n \in \mathbb{N}$

$\frac{1}{n+1} \leq x \leq 1$

$1 \leq \frac{1}{x} \leq n+1$

$= \sum_{k=1}^{n} \frac{1}{k}$

$\int_0^1 g\,dx \geq \sum_{n=1}^{k} \frac{1}{n} \ \forall k$

$\longrightarrow \sum_1^\infty \frac{1}{n} = \infty$

$\Rightarrow \int_0^1 f\,dx = \int_0^1 g\,dx = \infty$

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(11) $E_1, E_2 \ldots$ be a sequence of measurable sets s.t. $m(E_n) < 2^{-n}$ $\forall n$. Show that $\chi_{E_n} \to 0$ a.e.

suppose $A = \{x : \chi_{E_n}(x) \not\to 0\}$

$x \in A$ iff $x \in E_n$ for infinitely many n.

$\Rightarrow A = \limsup_{n\to\infty} E_n = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_i$

By Borel Cantelli Theorem

$m(\limsup E_n) = 0$ $\quad \sum_{1}^{\infty} m(E_n) \leq \frac{\varepsilon}{2^n}$

$\Rightarrow m(A) = 0$

$m(E_n) \longrightarrow 0$ as $n \to \infty$, then the above claim does not hold.

$m(E_n) = \frac{1}{n}$

$\sum m(E_n) = \infty$

$f \geq 0$ integrable

f is integrable on S, $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $\int_E f < \varepsilon$ when $m(E) < \delta$ $E \subseteq S$.

It is not true $\Rightarrow \forall n \in \mathbb{N}$ $\exists$ E_n s.t. $\int_{E_n} f \geq \varepsilon$, $m(E_n) < \frac{1}{2^n}$.

$f_n = f \cdot \chi_{E_n}$ $\quad f_n \to 0$ a.e.

$f_n \leq f$ in S; $\Rightarrow f = 0$

$\int_{E_n} f \Rightarrow \int_S f\chi_{E_n} = \int_S f_n \longrightarrow \int f = 0$

(1) $f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}$, $x \in [0,1]$

(i) $f_n(x) \xrightarrow[n \to \infty]{} 0 \quad \forall x \in [0,1]$

(ii) Each f_n is bounded but not uniformly bounded.

$f_n \leq \frac{\sqrt{n}}{2}$. at $x = \frac{1}{n}$ $f_n(1/n) = \frac{\sqrt{n}}{2}$

so each f_n is bounded

$\frac{\sqrt{n}}{2} \leq M$.

$\forall n$

It is not true

(iii) $h(x) = 1 + n^2x^2 - n^{3/2}x^{3/2} \; \forall n$

$\forall x \in [0,1]$.

$h(x) > 0 \; \forall x$.

$\Rightarrow f_n(x) \leq \frac{1}{\sqrt{x}}$

$\int_0^1 \frac{1}{\sqrt{x}}\, dx < \infty$

$|f_n| \leq g$, g is integrable

$f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = \begin{cases} (-1)^n n, & 0 \leq x \leq 1/n \\ 0, & x > \frac{1}{n} \end{cases}$$

$\exists$ no integrable f^n of S.T. $|f_n| \le g \ \forall n$ <u>Check</u>

But each f_n is odd $|f_n(x)| \le n \ \forall n$

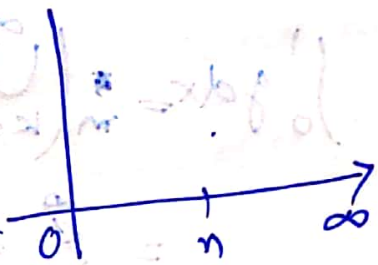
$\int_0^1 f_n(x)\,dx = (-1)^n \cdot n \cdot \frac{1}{n} = (-1)^n < \infty$

$\lim_{n\to\infty} \int_0^1 f_n(x) = \lim_{n\to\infty} (-1)^n$ <u>dne</u>

<u>Q(iv)</u> $f_n(x) = \begin{cases} 1+1/n, & 0 \le x \le n \\ 0, & x > n \end{cases}$

$f_n : [0,\infty) \longrightarrow \mathbb{R}$

$f_n \longrightarrow 1$ as $n \to \infty$



$\int_0^\infty f_n\,dx = \int_0^n (1+1/n)\,dx$

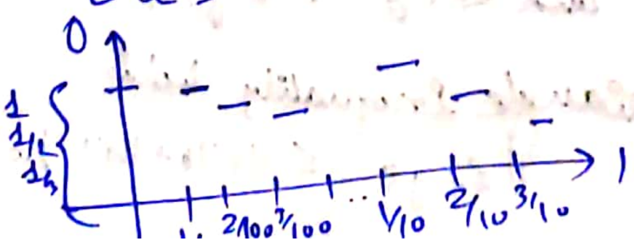
$= n(1+1/n) = n+1 < \infty.$

$\int f\,dx = \int_0^\infty 1\,dx = \infty$

(Q) $f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1/a, & x \in \mathbb{Q}^c \text{ and the first non-zero integer in the decimal representation of } x \text{ is } a \end{cases}$

$f : [0,1] \longrightarrow \mathbb{R}$ show that f is measurable $\int_0^1 f\,dx$?

$g = \{\frac{1}{a}\} \ \forall x \in [0,1]$



$$= \left\{ \frac{1}{i} \quad 1 \le i \le 9 \right.$$

g is simple fn

$$g(x) = \begin{cases} 1, & x \in [10^{-k}, 2\times10^{-k}] \\ \frac{1}{2} & x \in [2\cdot10^{-k}, 3\cdot10^{-k}] \\ \vdots & \vdots \\ \frac{1}{9} & x \in [9\times10^{-k}, 10^{-k+1}] \end{cases}$$

$k \in \mathbb{N}$

$$\int_0^1 g\,dx = m\left(\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{9} [n\cdot10^{-k}, (n+1)10^{-k}]\right)$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{9} (10^{-k})$$

$$=$$

19th April

$$L^2(\mathbb{R}) = \left\{ f \,\middle|\, \int_{\mathbb{R}} |f|^2 < \infty \right\} \qquad \langle f, g\rangle = \int_{R} f(x)\overline{g(x)}$$

(i) $L^2(\mathbb{R})$ is a vector space

(ii) $f(x)\overline{g(x)}$ is always integrable whenever $f, g \in L^2(\mathbb{R})$ & cauchy-schwartz inequality holds $\langle f, g\rangle \le \|f\|\,\|g\|$ holds.

(iii) The map $f \longrightarrow \langle f, g\rangle$ is linear for fixed g and $\langle f, g\rangle = \overline{\langle g, f\rangle}$

(v) The triangle inequality holds

$$\|f+g\| \le \|f\| + \|g\| \text{ holds}$$

1) If either $\|f\|$ or $\|g\| = 0$, then $fg = 0 \Rightarrow \langle f, g\rangle = 0$

2) suppose $g \neq 0$, $f \neq 0$ Let $\|f\| = \|g\| = 1$

$|\langle f, g\rangle| = \left|\int f\bar{g}\right| \leq \frac{1}{2}\left[\|f\|^2 + \|g\|^2\right] \leq 1 = \|f\|\,\|g\|$

Hilbert space H is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) with the following prop.

1) H is equipped with an inner product.

- $x \in H$, $\langle x, x\rangle \geq 0$ & $\langle x, x\rangle = 0 \Leftrightarrow x = 0$
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$
- For fixed y $x \longrightarrow \langle x, y\rangle$ is linear
- $\|x\| = \langle x, x\rangle^{1/2}$

2) $\|x\| = 0 \Leftrightarrow x = 0$

3) H is separable (It has countable dense subset)

An orthonormal set is called a basis of H if every linear combination is dense in H.

Orthogonality

$f \perp g \Leftrightarrow \langle f, g\rangle = 0$

$\|f+g\|^2 = \|f\|^2 + \|g\|^2$ (Pythagoras thm Holds)

orthogonal set $= \{f_1, f_2, \ldots\ldots f_n\}$ $f_i \perp f_j = 0$ $i \neq j$

if $\|f_i\| = 0$, Then orthonormal set $\langle e_i, e_j\rangle = \delta_{ij}$

$\{e_k\}_{k=1}^{\infty}$ is an orthonormal set.

$f = \sum_{k=1}^{m} a_k e_k$

$\|f\|^2 = \sum_{k=1}^{m} |a_k|^2$

For infinite dimensional case:

$f = \sum_{k=1}^{\infty} a_k e_k$

Proposition: suppose $\{e_n\}$ is an orthonormal set in H; then the following are equivalent

1. Finite linear combination of $\{e_k\}$'s are dense in H
2. $f \in H$, $\langle f, e_k \rangle = 0$ $\forall$ $k = 1, 2, \ldots$ then $f = 0$
3. Let $f \in H$ & $S_N(f) = \sum_{k=1}^{N} a_k e_k$, $a_k = \langle f, e_k \rangle$

   Then $S_N(f) \longrightarrow f$ in norm as $N \longrightarrow \infty$.
4. If $a_k = \langle f, e_k \rangle$; then $\|f\|^2 = \sum_{k=1}^{\infty} |a_k|^2$

   (Parseval's Identity)

Proof Assume (1)

Take $f \in H$, $\exists$ g_n (which are finite linear combination of e_k)

S.T. $\|f - g\| \longrightarrow 0$ as $n \to \infty$

$\langle f, g_n \rangle = 0$ for each n ($\because \langle f, e_k \rangle = 0$ for each k)

$\|f\|^2 = \langle f, f \rangle = \langle f - g_n, f \rangle$ or $\langle f, f - g_n \rangle$

$\|f\|^2 = \langle f, f - g_n \rangle \leq \|f\| \, \|f - g_n\| \to 0$ as $n \to \infty$

Cauchy Schwartz

$\Rightarrow \|f\|^2 = 0$

$\Rightarrow f = 0.$

Assume 2 is true

$S_N(f) \perp f - S_N(f)$

$1 \le j \le N$

$\langle f - \sum_{k=1}^{N} a_k e_k , e_j \rangle$

$= \langle f, e_j \rangle - \sum_{k=1}^{N} a_k \langle e_k, e_j \rangle$

$= a_j - a_j = 0$

Pythagoras theorem gives,

$\|f\|^2 = \|S_N(f)\|^2 + \|f - S_N(f)\|^2 \geq \|S_N(f)\|^2$

$= \sum_{k=1}^{N} |a_k|^2$

$\forall N$

Letting $N \to \infty$

$\sum_{k=1}^{\infty} |a_k|^2 \le \|f\|^2 \longrightarrow$ (Bessel's Inequality)

$\Rightarrow \sum_{k=1}^{\infty} |a_k|^2$ converges

$\|S_N(f) - S_M(f)\|^2 \quad N > M$

$= \|\sum_{k=M+1}^{N} a_k e_k\|^2 = \sum_{k=M+1}^{N} |a_k|^2 \longrightarrow 0$

as $M, N \to \infty$

$\{S_N\}$ is a cauchy sequence & since H is complete

$S_N \longrightarrow g \in H$

Fix j, for large N we have

$\langle f - S_N(f), e_j \rangle = a_j - a_j = 0$

As $N \to \infty$

$\langle f - g, e_j \rangle = 0 \quad \forall j$

By property (2) $f - g = 0 \Rightarrow \underline{f = g}$

$S_N(f) \longrightarrow f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k$