Date
12/09/2017

Lecture 17

(Sawtooth wave)

Sol^n:- We may write

$$f = f_1 + f_2 ,$$

where $f_1(x) = x$ & $f_2(x) = \pi$ (even)

$f_1(-x) = -f_1(x)$ (odd)

as $f_2(-x) = f_2(x)$

The Fourier co-efficients of f_2 are zero, except for the first one (i.e., the constant term a_0)

as $a_0 = \frac{1}{\pi}\int_0^\pi f_2(x)\,dx = \frac{1}{\cancel{\pi}}\int_0^\pi \cancel{\pi}\,dx$

$= \int_0^\pi dx = [x]_0^\pi = \boxed{\pi}$.

$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx$

$= \frac{2}{\cancel{\pi}}\int_0^\pi \cancel{\pi}\cos(nx)\,dx$ & $b_n = 0$ (why?)

$\Rightarrow a_n = 2\int_0^{\pi} \cos(nx)\,dx$

$$= 2\left[\frac{\sin(nx)}{n}\right]_0^{\pi}$$

$$= \boxed{0}$$

Also, $b_n = 0$ (why?)

Now, the Fourier co-efficients a_n, b_n are those from $f_1(x)$, except for a_0, which is π: Since f_1 is odd, $a_n = 0$, for $n = 1, 3 \ldots$

& $b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin(nx)\,dx$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\,dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right]$$

$$= \frac{2}{\cancel{\pi}}\left\{\left[\frac{-\cancel{\pi}\cos n\pi}{n}\right] + \left[\frac{1}{n^2}\sin(nx)\right]_0^{\pi}\right\}$$

$$\Rightarrow b_n = \boxed{-\frac{2\cos n\pi}{n}}, \quad n = 1, 2, \ldots$$

Hence

$b_1 = 2$, $b_2 = -2/2 = -1$

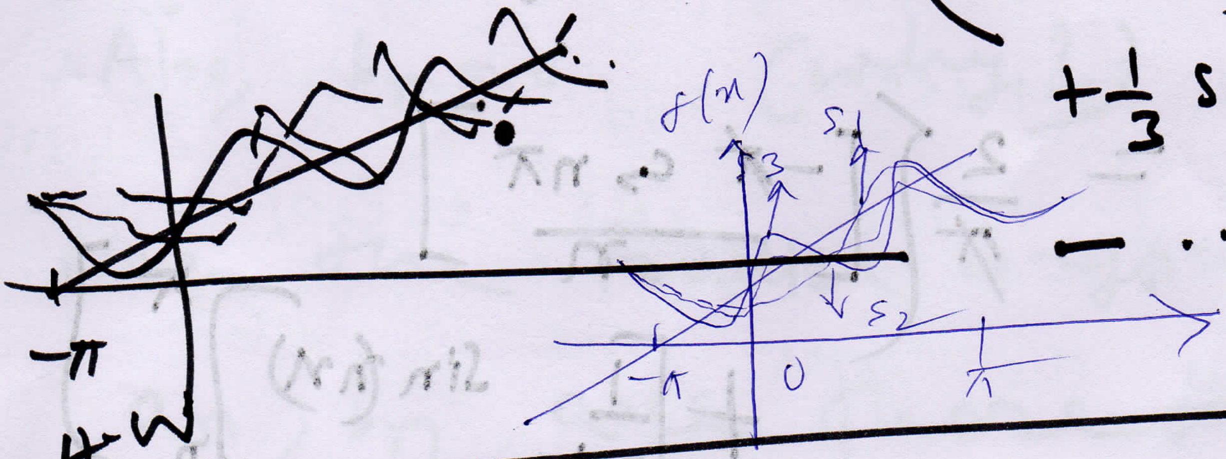
$b_3 = 2/3$, $b_4 = -2/4$, ...

& the Fourier series

If $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$



H.W

EX/ Rectangular pulse

The fn $f^*(x)$ in Fig2 is the sum of the fn $f(x)$ in Fig1 & the constant k.

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$

$$f(x+2\pi) = f(x)$$

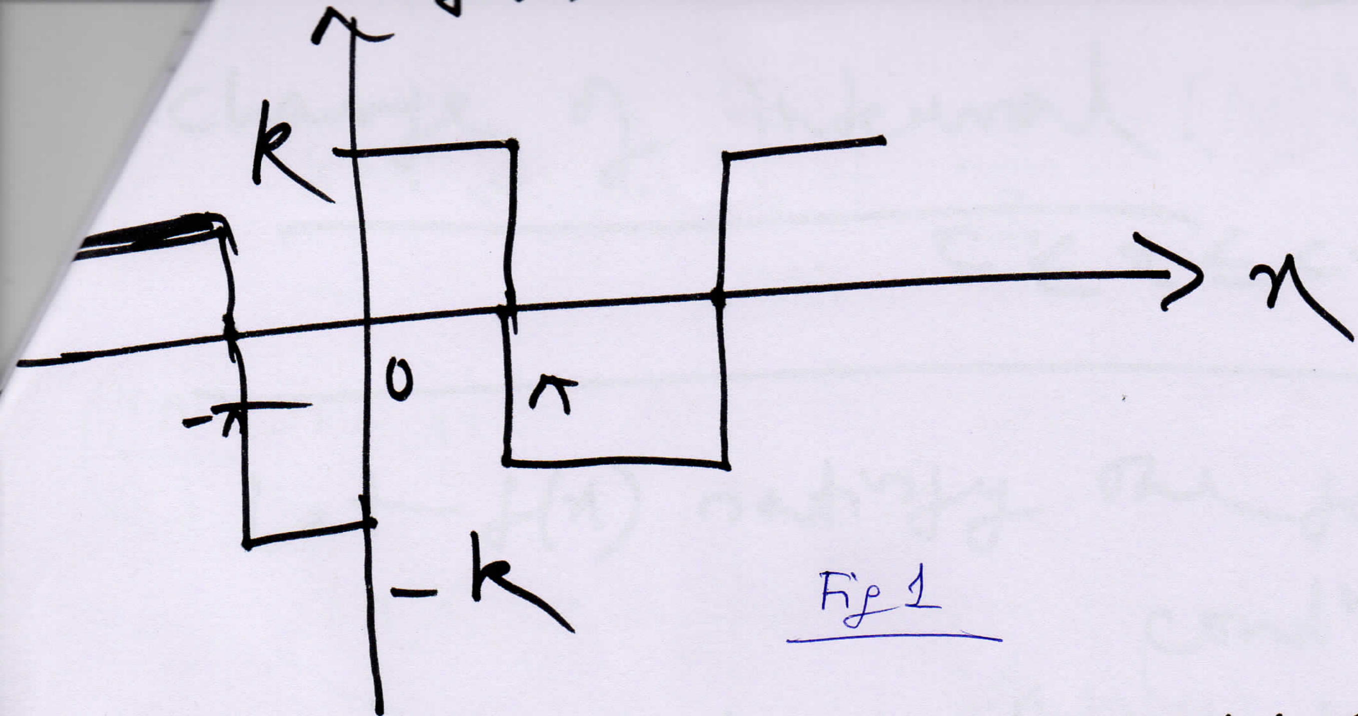


Fig 1

$$f^*(x) = \begin{cases} \underbrace{-k+k}_{=0}, & -\pi < x < 0 \\ \underbrace{k+k}_{=2k}, & 0 < x < \pi \end{cases}$$

∴ the Fourier series for $f^*(x)$ is given by

$$= k + \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

(how?)

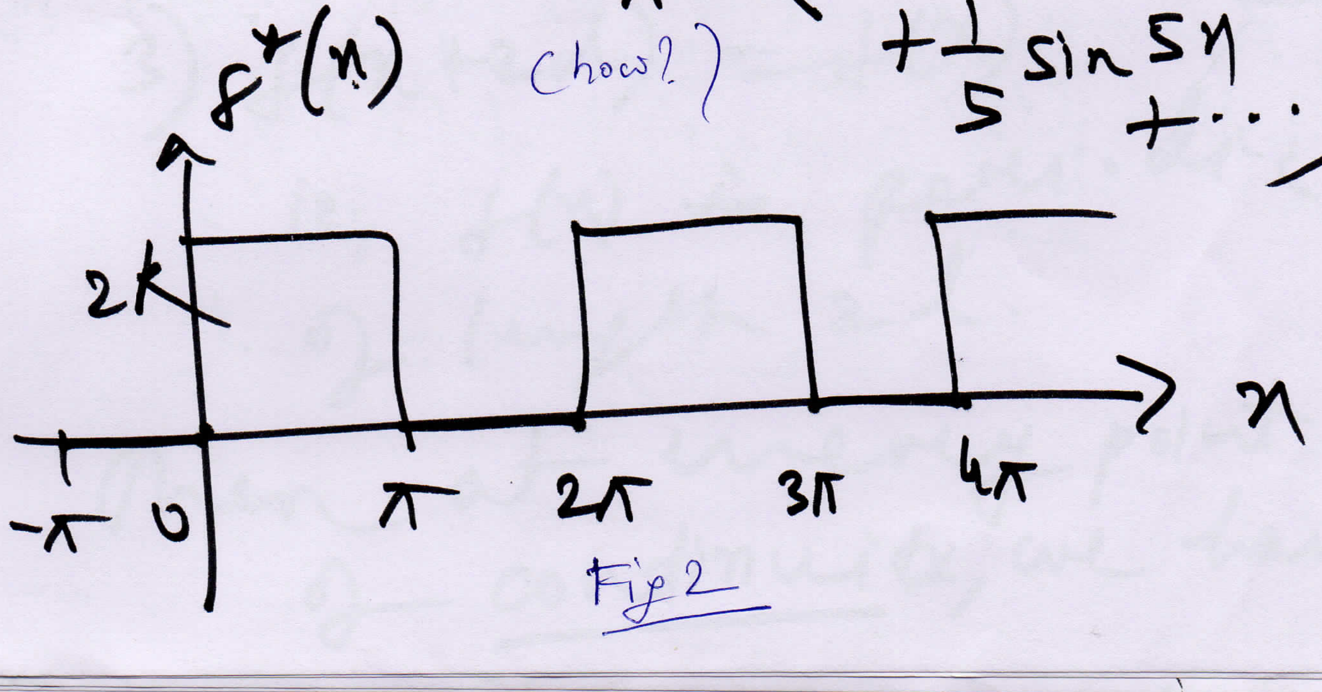


Fig 2

Change of interval :

$c \le x \le c+2l$

Let $f(x)$ satisfy the following condns:

1) $f(x)$ is defined in the interval $c < x < c+2l$

2) $f(x)$ & $f'(x)$ are piece-wise cont. fn in $c < x < c+2l$

3) $f(x+2l) = f(x)$
ie, $f(x)$ is periodic of length $2l$.

Then at every point of <u>continuity</u>, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \rightarrow (1)$$

where

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx.$$

(2)

At a point of discontinuity, the L.H.S ($f(x)$) is replaced by

$$\frac{1}{2} \left\{ f(x+0) + f(x-0) \right\}$$

i.e., the mean value at the discontinuity

The series (1) with coefficients (2) is called the Fourier series of $f(x)$. For many problems,

$c = 0$, or $c = -l$.

In case $l = \pi$, $f(x)$ has period 2π & eq^n (1) & (2) are simplified.

The above cond^{n}s are often called Dirichlet cond^{n}s & are sufficient (but not necessary) cond^{n}s for convergence of Fourier series.

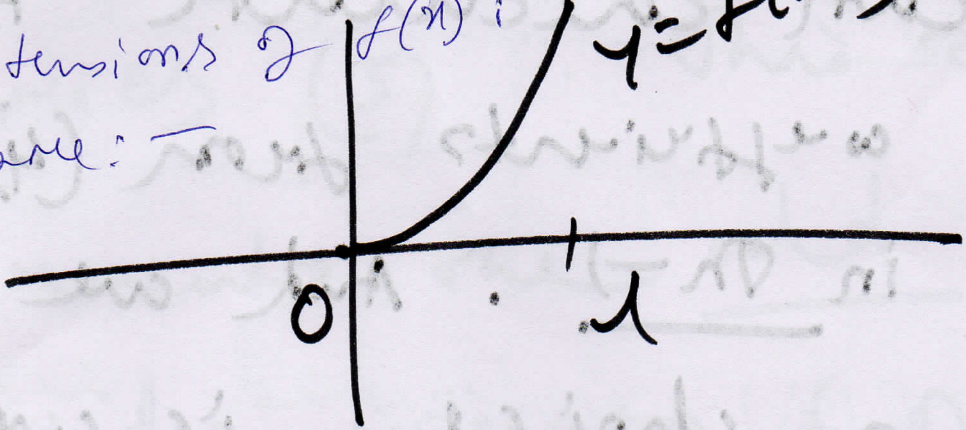
$0 \leq x \leq l$.

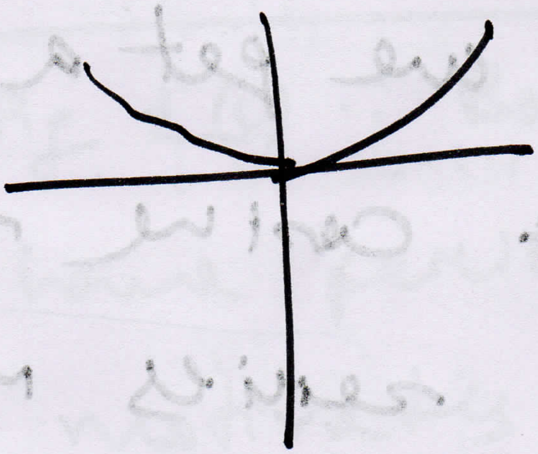
l

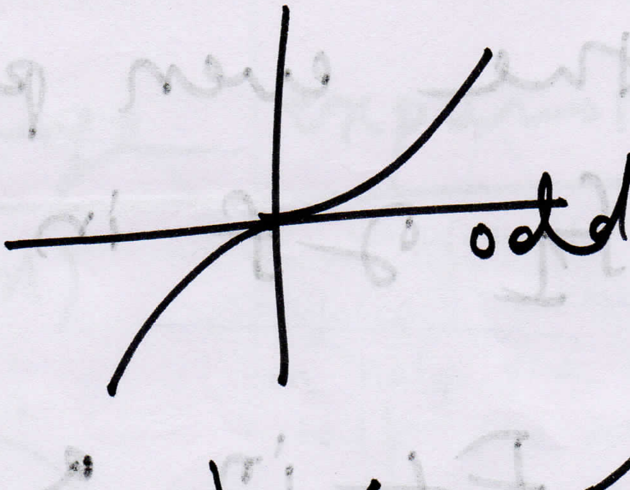
There are 3 kinds of extensions of $f(x)$:

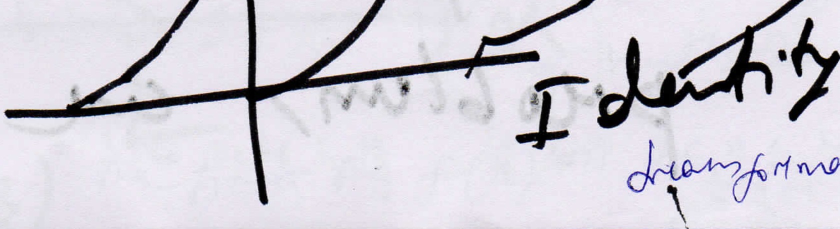
They are:

$y = f(x^2)$



1)  even extension

2)  odd extension

3)  Identity transformation

Half – Range Expansions :-

In applications we often want to employ a F.S. for a fn f that is given only on some interval

~~The~~ fn f is given only on $0 \le x \le L$

For our fn f we

E.g., the fn f can be the displacement of a violin string of (undistorted) length L or, the temp. in a metal bar of length L etc.

can calculate Fourier

coefficients from (4) or (6)

in Th-1 . And we have

a choice , if we use

eqn (4), we get a Fourier

~~coeffici~~ cosine series (3)

This series represents

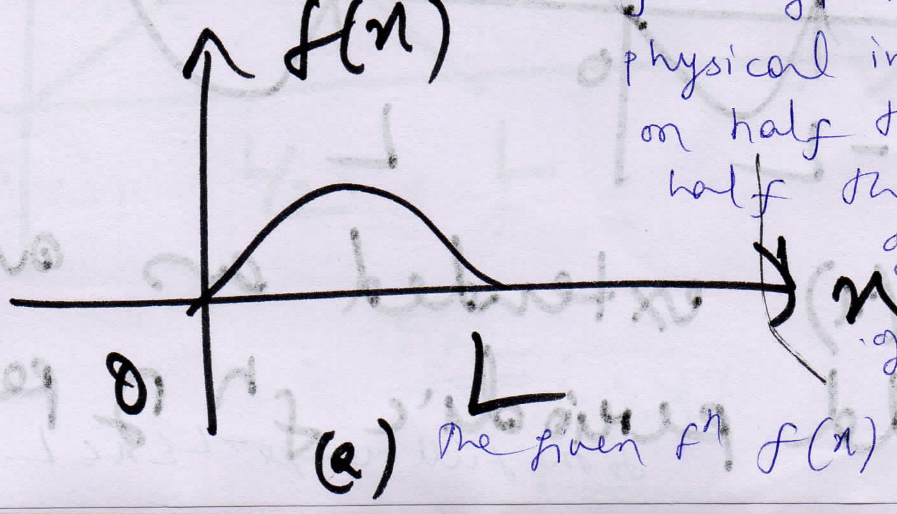
the even periodic extension

f_1 of f in in Fig 1(b)

If in a practical

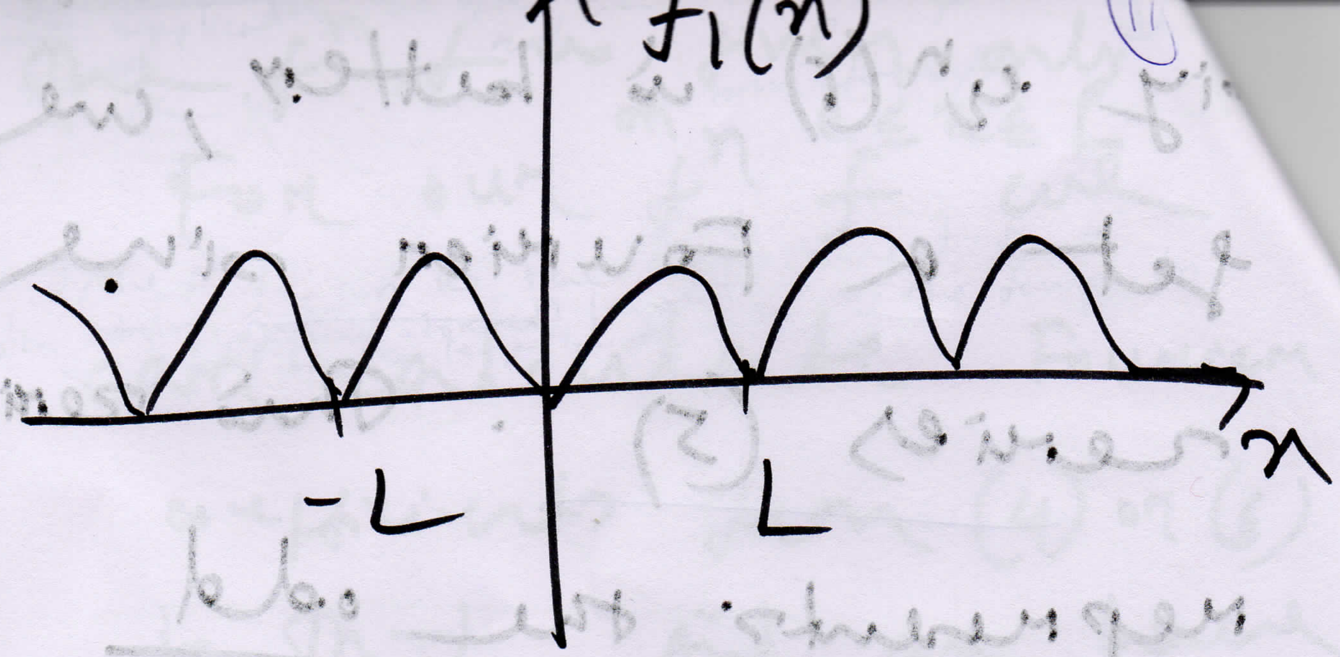
problem, we think that

only eqn (6) is better, we get a Fourier sine series (5). This series represents the odd periodic extension f_2 of f in Fig. 1(c). Both extensions will have period 2L. This motivates the name half-range expansion.

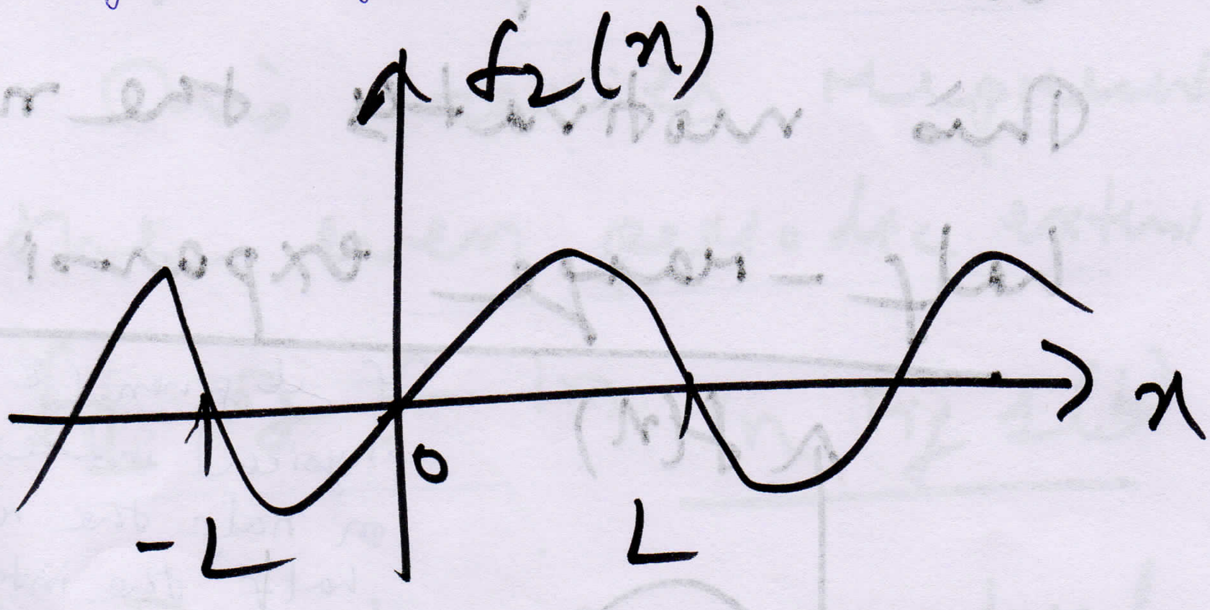


f is given (& of physical interest) only on half the range, half the interval of periodicity of length 2L.

(a) The given fn $f(x)$ on $0 \le x \le L$



(b) $f(x)$ extended as an even periodic f^n of period $2L$. to the full range $-L \le x \le L$.



(c) $f(x)$ extended as an odd periodic f^n of period $2L$. to the full range $-L \le x \le L$.

Fig 1

Triangle & its half-range expansions

Q) Find the two half-range expansions of the f^n

$$f(x) = \begin{cases} \frac{2k}{L}x, & \text{if } 0 < x < L/2 \\ \frac{2k}{L}(L-x), & \text{if } L/2 < x < L \end{cases}$$

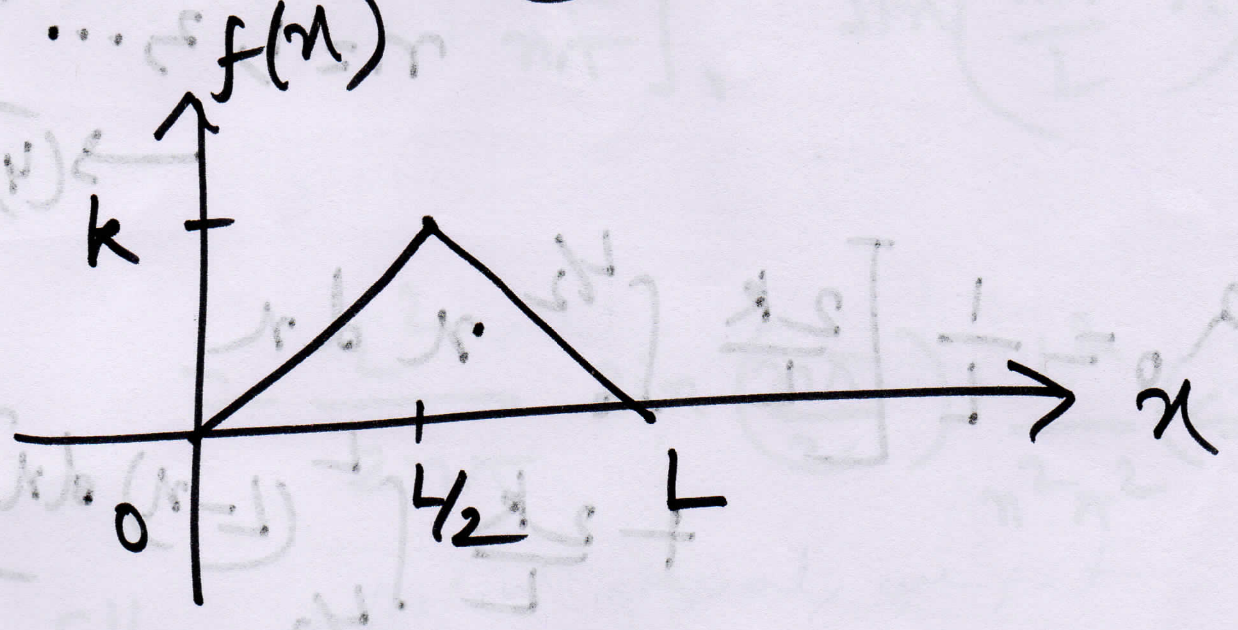


Fig 2

Soln :- (a) <u>Even periodic extension</u>

Using
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x,$$
where
$$a_0 = \frac{1}{L}\int_0^L f(x)\, dx \rightarrow (3)$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx, \quad n = 1, 2 \cdots \rightarrow (4)$$

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\, dx + \frac{2k}{L}\int_{L/2}^{L} (L-x)\, dx\right]$$

$$= \boxed{k/2}$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\frac{n\pi}{L}x\,dx\right]$$

Now $\int_0^{L/2} x \cos\frac{n\pi}{L}x\,dx$

$$= \left[\frac{Lx}{n\pi}\sin\frac{n\pi}{L}x\right]_0^{L/2}$$

$$- \frac{L}{n\pi}\int_0^{L/2}\sin\left(\frac{n\pi}{L}x\right)dx.$$

$$= \frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right).$$

S'lly, for the 2nd integral, we get $\int_{L/2}^{L}(L-x)\cos\frac{n\pi}{L}x\,dx$

$= 0 - L/n\pi\,(L - L/2)\sin n\pi/2 - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos n\pi/2\right)$

Hence, (after insertion of these two results into the formula for a_n), we get

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus, $a_2 = \frac{-16k}{2^2\pi^2}$, $a_6 = \frac{-16k}{6^2\pi^2}$, $a_{10} = \frac{-16k}{10^2\pi^2}$ etc.

& $a_n = 0$ if $n \neq 2, 6, 10, 14, \ldots$

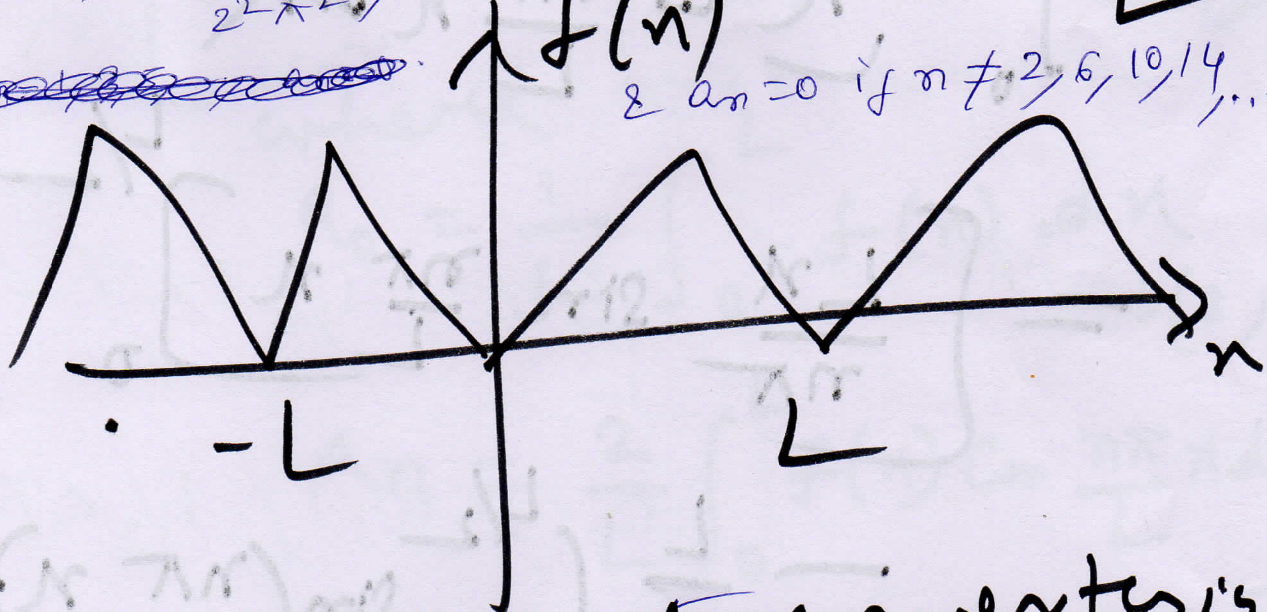


Fig 3 (a). Even extension

(Periodic extension of $f(x)$ in Fig 3)

EX. odd extension.

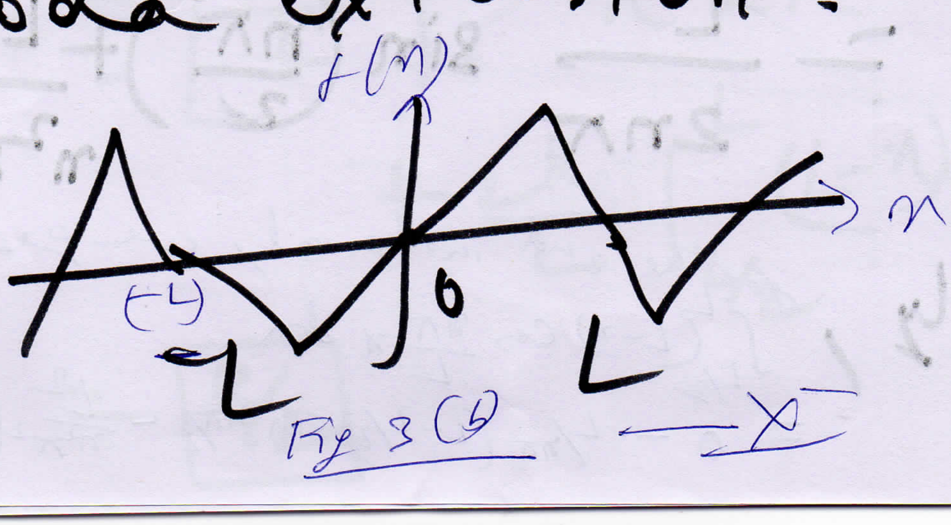


Fig 3 (b)