Group Theory

Lecture 11

2nd Isomorphism Thm :

For a non-empty subset A of G define the normalizer as

$$N_G(A) := \{ g \in G \mid gAg^{-1} = A \}$$

Thm: Let G be a group. Let A, B subgps of G and assume $A \subseteq N_G(B)$. Then $A \cap B$ is a normal subgp of A. and

$$AB/B \cong A/A\cap B.$$

Pf: WTS $A \cap B$ is a normal subgp of A.

Let $x \in A \cap B$ then WTS $gxg^{-1} \in A \cap B$ $\forall g \in A$.

$gxg^{-1} \in A$.

$gxg^{-1} \in B$ $\left(\because A \subseteq N_G(B). \right)$

$(gBg^{-1} = B \quad \forall g \in A.)$

$\therefore \quad gxg^{-1} \in A \cap B \quad \forall g \in A.$

Hence $A \cap B$ is a normal subgp of A.

Check: AB is subgp $(\because A \leq N_G(B))$.

Now define $\varphi : AB \longrightarrow A/A \cap B$

$$\varphi(ab) = a(A \cap B)$$

<u>WTS</u> φ is well-defined

Let $ab = a_1 b_1$ | WTS $\varphi(ab) = \varphi(a_1 b_1)$

$\Rightarrow \underset{A}{\underset{\cap}{a_1^{-1}a}} = \underset{B}{\underset{\cap}{b_1 b^{-1}}}$ | $a(A \cap B) = a_1(A \cap B)$

WTS $a_1^{-1} a \in A \cap B$.

$\therefore \quad a_1^{-1} a \in A \cap B$

<u>WTS</u> φ is a group homo

WTS $\varphi((ab)\cdot(a_1 b_1)) = \varphi(ab)\cdot\varphi(a_1 b_1)$

$$\varphi((ab)\cdot(a_1 b_1)) = \varphi(ab)\cdot\varphi(a_1 b_1)$$

$$\varphi(ab)\cdot\varphi(a_1 b_1) = a(A\cap B)\cdot a_1(A\cap B) = aa_1(A\cap B)$$

$$\varphi((ab)\cdot(a_1 b_1)) = \varphi(a b a_1 b_1) = \varphi(a(a_1 \underbrace{a_1^{-1}) b a_1}) b_1) = \varphi(a a_1 b_2 b_1) = aa_1(A\cap B)$$

where $b_2 = a_1^{-1} b a_1$

$\therefore$ φ is a group homomorphism.

$$\varphi(ab) = a(A\cap B)$$

φ is surjective because $\varphi(a) = a(A\cap B)$

$$\ker\varphi = \{ab \mid \varphi(ab) = (A\cap B)\}.$$

$$= \{ab \mid a(A\cap B) = (A\cap B)\}.$$

$$= \{ab \mid a \in A\cap B\}.$$

<u>WTS</u> $\ker\varphi \overset{\subseteq}{\underset{\supseteq}{=}} B.$

Note that $B \subseteq \{ab \mid a \in A\cap B\}.$

(because $1 \in A$).

$\because$ $A\cap B \subseteq B$ we have reverse inclusion.

$\therefore$ $B = \ker\varphi$

Hence by 1st isomorphism Thm,

$$AB/B \cong A/A\cap B.$$

Third isomorphism Thm :

Let G be a gp and let H & K be normal subgps of G with $H \leq K$.

Then K/H is a normal subgp of G/H

and $\dfrac{G/H}{K/H} \cong G/K$.

$$\begin{array}{c} G \\ | \\ K \\ | \\ H \end{array}$$

Pf: G/H is the quotient gp.

K/H is a subgp of G/H

WTS K/H is a normal subgp of G/H.

Let $gH \in G/H$. WTS $gH\, kH\, (gH)^{-1} \in K/H$.

when $kH \in K/H$.

$$gH\,kH\,(gH)^{-1} = gH \cdot kH \cdot g^{-1}H$$
$$= \underbrace{gkg^{-1}}_{K} H \in K/H.$$

$[\because K \text{ is normal subg}]$.

$\therefore K/H \triangleleft G/H.$

G
$|$
K
$|$
H

__Define__ $\varphi: G/H \longrightarrow G/K$

$$\varphi(gH) = gK$$

check that φ is a gp homo.

__WTS__ φ is well defined.

Let $g_1H = g_2H$ WTS $\varphi(g_1H) = \varphi(g_2H)$

$\Downarrow$ __WTS__ $g_1K = g_2K$

$g_2^{-1}g_1 \in H \subseteq K$, WTS $g_2^{-1}g_1 \in K$.

$\therefore$ φ is well defined.

φ is surjective.

$$\ker \varphi = \{ gH \mid \varphi(gH) = K \}$$

$$= \{ gH \mid gK = K \}$$

$$= \{ gH \mid g \in K \}$$

$$= K/H.$$

$\therefore$ By 1st isomorphism Thm

$$\frac{G/H}{K/H} \cong G/K.$$

Thm: Let $f : G \to \tilde{G}$ be a surj gp homo and K be the kernel of f. Then there is a bijective correspondence

$$\left\{\begin{array}{l}\text{subgps of } G \\ \text{containing } K\end{array}\right\} \longleftrightarrow \left\{\text{subgps of } \tilde{G}\right\}$$

given by $\quad H \longmapsto f(H)$

$$f^{-1}(\tilde{H}) \longleftarrow\!\shortmid \ \tilde{H}$$

and the bijection restrict to

$$\left\{\begin{array}{l}\text{normal subgps of} \\ G \text{ containing } K\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{normal subgps} \\ \text{of } \tilde{G}\end{array}\right\}$$

Pf: If H is a subgp of G containg K then $f(H)$ is a subgp of $\tilde{G}$ (check)

WTS $$f^{-1}\left(f(H)\right) \underset{\subseteq}{\underset{\supseteq}{=}} H. \quad (\text{Ex})$$

WTS $$f\left(f^{-1}(\tilde{H})\right) \underset{\subseteq}{\underset{\supseteq}{=}} \tilde{H}, \quad (\text{Ex}).$$