Books

1) Introduction to Real Analysis by Bertle and Sherbert.

2) Mathematical Analysis by Apostal.

3) Principles of Mathematical Analysis by Walter Rudin.

The Real Numbers ($\mathbb{R}$)

First we give ~~the~~ a formal construction of real numbers on the basis of a more primitive set (such as the natural numbers ($\mathbb{N}$) or rational numbers ($\mathbb{Q}$)).

Later we shall give the axiomatic definition of the real numbers.

Natural numbers:-

The set of natural numbers ($\mathbb{N}$) consists of elts $\{1, 2, 3, 4, 5, \ldots\}$.

It has two algebraic operations, addition (+) and multiplication (.).

If we take $a, b \in \mathbb{N}$, then $a+b$ and $a.b \in \mathbb{N}$,
(these operations are closed in $\mathbb{N}$)

We also have a 'linear order' (a partial order

~~is an order~~ such that any two elements are comparable ie. for two elements a and b, either ~~$a \le b$ or $b \le a$~~ $a < b$ or $b < a$ or $a = b$.

We discuss the following two properties of the set $\mathbb{N}$ of natural numbers.

1) Wellordering property:-

Every non-empty subset of $\mathbb{N}$ has a least element. This means if S is a non-empty set of $\mathbb{N}$, then there is $m \in S$ such that $m \le s \ \forall s \in S$

Proof:- Let $k \in S$. Then define $T := \{s \in S : s \le k\}$.
Then, T is a non-empty finite set. So, it has a least element say m.
If m is not the least element in S, then $\exists \ t \in S$ s.t. $t < m$ Then $t \in T$ and it contradicts the fact that m is the least element of T. So, m is the least element of S.

2) Principle of Induction:-

Let $P(n)$ be a statement about $n \in \mathbb{N}$. Then $P(n)$ may be true for some n but false for other n (like $n^2 = n$ is a statement which is true for $n = 1$ but false for $n > 1$) But it may happen that $P(n)$ is true for all $n \in \mathbb{N}$ (like $n^2 \geq 1$). In this context, we can apply the following theorem

Theorem:-

Let $P(n)$ be a statement involving a natural number n. If

(i) $P(1)$ is true and

(ii) $P(k+1)$ is true whenever $P(k)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Example:-

For each $n \in \mathbb{N}$, $1+2+ \quad +n = \frac{n(n+1)}{2}$.

Proof:- The statement is true for $n=1$ as

$$1 = \frac{1(1+1)}{2}.$$

Suppose the statement is true ~~for~~ k, i.e.

$$1+2+ - - +k = \frac{k(k+1)}{2}.$$

Then, $1+2+ \quad - \quad +k+(k+1)$

$$= \frac{k(k+1)}{2} + (k+1) = (k+1)\left(1+\frac{k}{2}\right) = \frac{(k+1)(k+2)}{2}.$$

That is, the statement is true for the natural number $k+1$.

Then by the previous theorem, the statement is true for all natural numbers n.

To prove the previous theorem we first prove the following well-known property of the natural numbers

Principle of Mathematical Induction:-

Let S be a subset of $\mathbb{N}$ that possesses the two properties:-

(i) $1 \in S$,

(ii) for every $k \in \mathbb{N}$, if $k \in S$, then $k+1 \in S$.

Then $S = \mathbb{N}$.

Proof:- Suppose to the contrary $S \neq \mathbb{N}$. Then the set $S^c := \mathbb{N} - S$ is a non-empty subset of $\mathbb{N}$. Then by well-ordering property of $\mathbb{N}$, S^c has a least element m.

Clearly $m > 1$. Also $m-1$ is a natural number which is in S (as m is the least elt of S^c). So by hypothesis $(m-1)+1 = m \in S$, which is a contradiction.

So, $S = \mathbb{N}$.

Proof of the previous theorem:-

Let S be the set of all natural numbers for which $P(n)$ is true, i.e. $S := \{n \mid P(n) \text{ is true}\}$

Then, (i) $1 \in S$

(ii) for every $k \in S$, $k+1 \in S$.

So, by Principle of Mathematical Induction

$S = \mathbb{N}$.

So $P(n)$ is true for all $n \in \mathbb{N}$.

Remark:- We can prove the following slightly generalisation of the previous theorem, which is also very important—

Let a statement $P(n)$ satisfy the following conditions:—

(i) $\exists$ a natural number m s.t. $P(m)$ is true and for $n<m$, $P(n)$ is not true.

(ii) For any $k \geq m$, if $P(k)$ is true, then $P(k+1)$ is also true.

Then $P(n)$ is true for all natural numbers $\geq m$.

Integers:-

Addition and multiplication are defined on $\mathbb{N}$. But, substraction is not defined, i.e. $a, b \in \mathbb{N}$ ~~$\Rightarrow a-b$~~ doesn't imply that $a-b \in \mathbb{N}$.

Now, we enlarge the set $\mathbb{N}$ by including 0 and negative of natural numbers (denoted by $-n$), then we get the set of integers (denoted by $\mathbb{Z}$), which is the set

$$\mathbb{Z} = \{0, 1, 2, 3, \cdots, -1, -2, -3, \cdots\}.$$

We define ~~substraction~~ addition on $\mathbb{Z}$ by

~~$m+(n):=$~~

$$m+(-n) := \begin{cases} m-n & \text{if } n<m \\ -(n-m) & \text{if } n>m \\ 0 & \text{if } n=m. \end{cases}$$

In $\mathbb{Z}$, we can substract two natural numbers defined by

$$a-b := a+(-b).$$

$\mathbb{Z}$ forms a group with the 'addition' operation.

Rational Numbers:-

Multiplication is defined on $\mathbb{Z}$, namely

$$m.(-n) = (-m).n := -mn$$

and $(-m).(-n) = mn$.

But we can't do the inverse operation of multiplication i.e. division on $\mathbb{Z}$ as if $a \in \mathbb{Z}, b \in \mathbb{Z}$ then $\frac{a}{b}$ is not alway in $\mathbb{Z}$. So, we again need to enlarge $\mathbb{Z}$ by adding elts of the form $\frac{a}{b}$.

(Here suppose $\frac{a}{0}$ is a number, then multiplying both sides by 0 we get $a=0$. Also, $\frac{0}{0}$ should be equal to $0.\frac{1}{0}$, but $\frac{1}{0}$ is not defined as above).

So a rational number is of the form $\frac{p}{q}$ where $q \neq 0$. We also say $\frac{p}{q} = \frac{m}{n}$ where $q \neq 0$, $n \neq 0$

if $pn = qm$.

The set of all rational numbers is denoted by

$\mathbb{Q}$ and has two binary operations 'addition' (+) and 'multiplication' $(\cdot)$ with the following properties:-

(i) $a, b \in \mathbb{Q} \Rightarrow a+b \in \mathbb{Q}$

(ii) $a, b, c \in \mathbb{Q} \Rightarrow (a+b)+c = a+(b+c)$ (Associativity)

(iii) $a+0 = a \;\forall a \in \mathbb{Q}$.

(iv) for each $a \in \mathbb{Q}$, there exists $(-a) \in \mathbb{Q}$ such that

$$a+(-a)=0$$

(v) $a+b = b+a \;\forall a, b \in \mathbb{Q}$. (commutativity)

(vi) $a, b \in \mathbb{Q} \Rightarrow a.b \in \mathbb{Q}$.

(vii) $(a.b).c = a.(b.c)$, (associativity)

(viii) $a.1 = a \;\forall a \in \mathbb{Q}$.

(ix) For each $a \neq 0$ in $\mathbb{Q}$, there exists an elt b s.t. $a.b = b.a = 1$, we denote b by $1/a$.

(x) $a.b = b.a \;\forall a, b \in \mathbb{Q}$, (commutativity)

(xi) $a.(b+c) = a.b + a.c \;\forall a, b, c \in \mathbb{Q}$.

$-a$ (negative of a) is the additive inverse of a. $\frac{1}{a}$ is the multiplicative inverse of a.

(multiplication is distributive over addition).

$\mathbb{Q}$ forms a field under addition and multiplication.

Order properties of $\mathbb{Q}$:-

In $\mathbb{Q}$, a rational number of the p/q is said to a positive number if either both p and q are natural numbers or both $(-p)$ and $(-q)$ are natural numbers. The set of all positive rational numbers have the following properties:-

(i) If a and b are positive numbers, then so is $a+b$.

(ii) If a and b are positive numbers, then $a.b$ is again a positive number.

(iii) If a is a rational number, then exactly one of the following is true-

a is a positive number,

$a = 0$,

$-a$ is a positive number.

We can give a linear (or total) partial) order on $\mathbb{Q}$ by definning '$a < b$' if $b - a$ is a positive rational number. This partial order satisfies the following properties:-

(a) $a < b, b < c \Rightarrow a < c$ (transitivity),

(b) $a < b \Rightarrow a+c < b+c$,

(c) $a < b, 0 < c \Rightarrow ac < bc$,

(d) If $a, b \in \mathbb{Q}$, then exactly one of ~~the~~ the following is true –

$a < b$, $a = b$ or $b < a$ (law of trichotomy).

The field $\mathbb{Q}$ with the order relation is called <u>an ordered ~~field~~ field</u>.

<u>Density Property of $\mathbb{Q}$:–</u>

If x and y are two rational numbers, then $x < y$

$\Rightarrow x + x < x + y$

$\Rightarrow 2x < x + y$

$\Rightarrow \frac{1}{2} \cdot 2x < \frac{1}{2} \cdot (x+y)$ as $\frac{1}{2} > 0$

$\Rightarrow x < \frac{x+y}{2}$.

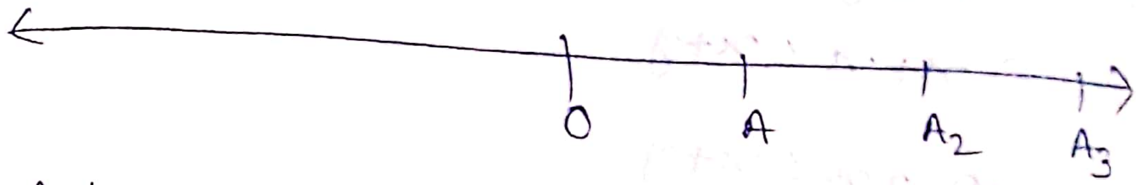
Similarly, $\frac{x+y}{2} < y$. So, $x < \frac{x+y}{2} < y$.

So, between any two rational number x and y with $x < y$, there exists another rational number $\frac{x+y}{2}$ with $x < \frac{x+y}{2} < y$.

Again between x and $\frac{x+y}{2}$, there exists another rational number and this process continues indefinitely. So, between any two rational numbers x and y, there are infinitely many rational numbers. This is the density property of $\mathbb{Q}$ and we say that '$\mathbb{Q}$ is dense'.

Geometric Representation of Rational Numbers

Rational numbers can be represented by points on a straight line. Let $X'X$ be a directed line. Fix a point O. O divides the line into two parts, the part on the right of O is called positive side of O and the part on the left of O is called negative side of O.



Let A be a point on the right of O. We represent 0 by the point O and 1 by the point A. Then, let A_n be a point on the right of O such that $OA_n = n\,OA$, where OA_n is the length of the line segment from O to A_n. Then A_n represents n, natural number. For representing a negative natural number $-n$, choose a point A'_n on the left of O such that $OA'_n = OA_n$.

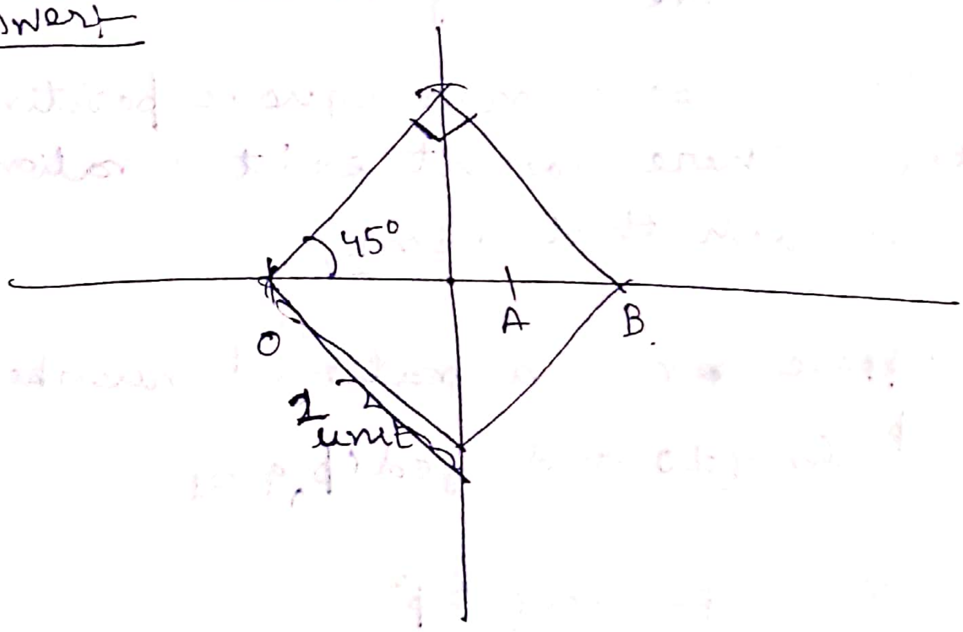
To represent a positive rational number p/q, first choose ~~first~~ a point A_p on the right of O such that $p\,OA = OA_p$, then fix ~~first~~ a point B_q on the right of O such that $q\,OB_q = OA_p$.

Observe that if a is represented by P and b is represented by Q, and $a < b$, then b is on the right side of A.

Thus every rational numbers can be made to correspond to a point on the line. These points are called rational points, So between any two rational points there are infinitely many rational points.

Q: Is the whole line composed of rational points?

Answer:



Here OB has lenghth $\sqrt{2}$.

But $\sqrt{2}$ is not a rational number as

if possible suppose $\sqrt{2} = \frac{p}{q}$, $q \neq 0$

We also suppose that p and q are coprime, ie $\gcd(p, q) = 1$.

$\therefore p = \sqrt{2}q \therefore p^2 = 2q^2$.

$\because 2 \mid p^2 \therefore 2 \mid p. \therefore p = 2m$

$\therefore p^2 = 2q^2 \Rightarrow 4m^2 = 2q^2 \Rightarrow 2m^2 = q^2$

$\Rightarrow 2 \mid q^2$

$\Rightarrow 2 \mid q.$

So, this is a contradiction to the fact that $\gcd(p,q) = 1$.

$\therefore \sqrt{2}$ is not a rational number.

These numbers which correspond to a place on the line, but is not rational are said to be irrational number.

In fact there are many irrational numbers

<u>Theorem</u>:- Let m be a non-square positive ~~num~~ integer. There doesn't exist a rational number r such that $r^2 = m$.

<u>Proof</u>:- Suppose r is a rational number. Then, $r = \frac{p}{q}$ for $q \neq 0$ and $\gcd(p,q) = 1$.

$\therefore r^2 q^2 = p^2 \therefore mq^2 = p^2$

Let p_1 be a prime divisor of m with odd power.

$\therefore p_1 \mid p^2 \therefore p_1 \mid p$ (as for a prime number p_1, $p_1 \mid ab \Rightarrow p_1 \mid a$ or $p_1 \mid b$ or both)

$\therefore p = p_1 m_1$

$\therefore mq^2 = p_1^2 m_1^2.$

If the power of p_1 in m is 1, ~~then $p_1 \nmid q^2$~~

then $p_1 \mid \frac{m}{p_1} \cdot q^2 \Rightarrow p_1 \mid q^2$.

$\Rightarrow p_1 \mid q$.

$\Rightarrow p_1 \mid g.c.d(p, q)$, a contradiction.

If the power of p_1 in $m > 2$, then

$$\frac{m}{p_1^2} q^2 = m_1^2 \text{ and as } p_1 \mid \frac{m}{p_1^2}, \text{ so}$$

$$p_1 \mid m_1^2 \Rightarrow p_1 \mid m_1.$$

Continuing this way ultimately we shall get $m' \cdot q^2 = p_1^2 m_r^2$ where $p_1 \mid m'$ but $p_1^2 \nmid m'$.

$$\therefore \frac{m'}{p_1} \cdot q^2 = p_1 m_r^2$$

$\therefore p_1 \mid q^2 \Rightarrow p_1 \mid q \Rightarrow p_1 \mid gcd(p, q)$, a contradiction.

Real numbers ($\mathbb{R}$):-

The set of all rational and irrational numbers is called the set of all real numbers. We denote the set by $\mathbb{R}$.

Axiomatic Definition of $\mathbb{R}$:-

$\mathbb{R}$, the set of all real numbers is a complete ordered field.

We begin with a brief description of the 'algebraic structure' of the real number system. We give a short list of basic properties (axioms) of addition and multiplication from which all other algebraic properties can be deduced.

In the terminology of abstract algebra, the set of real numbers forms a 'field' with respect to addition and multiplication.

Algebraic Properties of $\mathbb{R}$:-

On the set $\mathbb{R}$ of real numbers, two binary operations, addition and multiplication (denoted by '+' and '.' respectively) are defined which satisfy the following properties -

~~(A1) a+b = b+a (commutativity of addition)~~

(A1) $a, b \in \mathbb{R} \Rightarrow a+b \in \mathbb{R}$

(A2) $a+b = b+a$ (commutativity of addition)

(A3) $(a+b)+c = a+(b+c)$ (associativity of addition).

(A4) there exists an element 0 in $\mathbb{R}$ such that $a+0 = a \quad \forall a \in \mathbb{R}$ (existence of a zero element)

(A5) for each $a \in \mathbb{R}$ there exists an element (denoted by $-a$) in $\mathbb{R}$ such that $a+(-a)=0$ (existence of negative elements)

(M1) $a, b \in \mathbb{R} \Rightarrow a.b \in \mathbb{R}$

(M2) $a.b = b.a$ (commutativity of multiplication)

(M3) $(a.b).c = a.(b.c)$ (associativity of multiplication)

(M4) there exists an element 1 in $\mathbb{R}$, distinct from 0, such that $a.1 = a$ (existence of a unit element).

(M5) for each $a \neq 0$ in $\mathbb{R}$, there exists $\frac{1}{a}$ in $\mathbb{R}$ such that $a.\left(\frac{1}{a}\right) = 1$ (existence of reciprocal)

(D4) $a.(b+c) = a.b + a.c$ ~~and (b+c).a~~
(distributive property of multiplication over addition).

Remark With respect to the algebraic properties, there is no difference between $\mathbb{R}$ (the set of real numbers) and $\mathbb{Q}$ (the set of rational numbers).

We now state some basic algebraic results on $\mathbb{R}$ which follow easily from the previous axioms

Theorem:-

Let $a, b, c \in \mathbb{R}$. Then

(i) $a+b = a+c$ ~~implies~~ implies $b = c$ (cancellation law for addition).

(ii) $a \neq 0$ and $a.b = a.c$ implies $b = c$ (cancellation law for multiplication)

(iii) $a.0 = 0$

(iv) $-(-a) = a$

(v) $\frac{1}{(1/a)} = a \quad \forall a \neq 0$

(vi) $a.b = 0$, then either $a = 0$ or $b = 0$

(vii) If $a \neq 0$ and $a.b = 1$, then $b = \frac{1}{a}$.

(viii) $\cancel{(-1).a = -a}$, $(-a).b = a.(-b) = -ab$

(ix) $(-a).(-b) = ab$

Proof:- We shall prove (vi) and (ix)

(vi) Suppose, $a \neq 0$. Then $a.\frac{1}{a} = \frac{1}{a}.a = 1$.

We have
$$a.b = 0$$
$$\Rightarrow \frac{1}{a}.(a.b) = \frac{1}{a}.0$$
$$\Rightarrow \left(\frac{1}{a}.a\right).b = 0 \text{ (by associativity of multiplication and (iii))}$$
$$\Rightarrow 1.b = 0 \Rightarrow b = 0$$

(ix) Let $c = -a$
$$\therefore c.(-b) = -cb = -((-a).b) \text{ (by (viii))}$$
$$= -(-a.b)$$
$$= a.b \text{ (by (iv))}$$

Order Properties of $\mathbb{R}$:-

On the set $\mathbb{R}$, a linear or total partial order relation '$<$' is defined ~~by "a<b~~ which satisfies the following conditions:-

(O1). $a<b$ and $b<c$ implies $a<c$ (transitivity)

(O2) $a<b$ implies $a+c<b+c$

(O3) $a<b$ and $0<c \Rightarrow ac<bc$.

(O4) If $a,b \in \mathbb{R}$, then exactly one of the following statements hold –

$a<b$, or $a=b$, or $b<a$ (law of trichotomy).

Remark:

(i) $a<b$ is equivalently expressed $b>a$

(ii) The symbol $a \leq b$ ~~iff~~ means either $a<b$ or $a=b$.

(iii) The set $\mathbb{Q}$ of rational numbers also has the same order axioms.

(iv) The "order properties" of $\mathbb{R}$ refer to the notions of positivity and inequalities between real numbers.

The simplest way to define order relation on $\mathbb{R}$ is to identify a special ~~set~~ subset of $\mathbb{R}$ by using the notion of "positivity".

Set of positive real numbers:

There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers, that ~~satisfy~~ satisfy the following properties:

(i) If $a, b \in \mathbb{P}$, then $a+b \in \mathbb{P}$

(ii) If $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$

(iii) If $a \in \mathbb{R}$, then exactly one of the following holds:–

$a \in \mathbb{P}$, $a=0$, $-a \in \mathbb{P}$.

Definition:-

Let $a, b \in \mathbb{R}$.

(a) If $a-b \in \mathbb{P}$, then we write ~~$a<b$~~ $a>b$ or $b<a$

(b) If $a-b \in \mathbb{P} \cup \{0\}$, then we write $a \geq b$ or $b \leq a$.

Then (O1), (O2), (O3), (O4) fallow from the property of $\mathbb{P}$. We shall prove (O3)

$a<b \Rightarrow b-a \in \mathbb{P}$. and $0<c \Rightarrow c-0 \in \mathbb{P}$.
$\Rightarrow c \in \mathbb{P}$.

$\therefore$ By property (iii), $(b-a)c \in \mathbb{P} \Rightarrow bc-ac \in \mathbb{P}$
$\Rightarrow ac<bc$

We can prove the fallowing inequalities:-

~~Fthe~~ (a) $a>0 \Rightarrow -a<0$

(b) $a<0 \Rightarrow -a>0$

(c) $a>0, b>0 \Rightarrow a+b>0$

(d) $a<0, b<0 \Rightarrow a+b<0$

(~~Pa~~ Proof:- $a+b<0+b \Rightarrow a+b<b<0$)

(e) $a>0, b>0 \Rightarrow a.b>0$

(f) $a<0, b<0 \Rightarrow a.b>0$ ~~$\Leftarrow a.b$~~

(Proof:- $b<0 \Rightarrow -b>0$ (by (b))

~~$\Rightarrow a.(-b) > a.0(-b)$~~

and $a<0 \Rightarrow a.(-b) < 0.(-b)$

$\Rightarrow -ab < 0 \Rightarrow ab > 0$)

(g) $a>0, b<0 \Rightarrow a.b<0$

(h) $a>b, c>d \Rightarrow a+b > c+d$.

Now, we wish to find positive elements.

Theorem:-

(a) If $a \in \mathbb{R}$, and $a \neq 0$, then $a^2 > 0$.

(b) $1 > 0$

(c) If $n \in \mathbb{N}$, then $n > 0$.

Proof:- (a) As $a \neq 0$, so by Trichotomy property of $\mathbb{P}$,

$a \in \mathbb{P}$ or $-a \in \mathbb{P}$.

If $a \in \mathbb{P}$, then ~~$a>0 \Rightarrow a.a>0$~~ $a.a \in \mathbb{P} \Rightarrow a^2 \in \mathbb{P}$

$\Rightarrow a^2 > 0$

If $-a \in \mathbb{P}$, then $(-a).(-a) \in \mathbb{P} \Rightarrow a^2 \in \mathbb{P} \Rightarrow a^2 > 0$

(b) $1 = 1^2 > 0$

(c) When $n=1$, this is true by (b).

Suppose for $n=k$, $k>0$

$\therefore k>0$ and $1>0 \Rightarrow k+1>0$

$\therefore$ So, by mathematical induction, $n>0$ for all $n \in \mathbb{N}$.

We can prove the following results:-

Theorem:-

(a) If $ab>0$, then either (i) $a>0, b>0$ or

(ii) $a<0, b<0$.