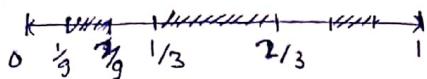


7.1.18

Textbook: Measure Theory and Integration.
G. de Barra

Cantor Set:



$$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Step 1 $[0, \frac{1}{3}] \quad [\frac{2}{3}, 1]$

Step 2 $[0, \frac{1}{9}] \quad [\frac{2}{9}, \frac{1}{3}], \quad [\frac{2}{3}, \frac{8}{9}], \quad [\frac{8}{9}, 1]$

Step n: 2^n intervals

$$J_{n,1}, J_{n,2}, \dots, J_{n,2^n}$$

$$\text{Let } J_n = \bigcup_{i=1}^{2^n} J_{n,i}$$

Cantor set defined as (or Cantor Ternary Set)

$$C = \bigcap_{n=1}^{\infty} J_n$$

Remark:-

Cantor set consists of the points x which can be given an expansion to the base 3 with the form $x = 0.x_1x_2x_3\dots$ with $x_n = 0$ or $2 \quad \forall n \geq 1$

Proof to show uncountable:

Suppose C is countable, say $x^{(1)}, x^{(2)}, \dots, x^{(n)}$

For each $n \geq 1$ if $x_n^{(n)} = 0$ Let $x_n = 2$ and if $x_n^{(n)} = 2$

then let $x_n = 0$. $x = 0.x_1x_2\dots \in C$

but $x \neq x^{(n)} \quad \forall n \geq 1$

$\therefore C$ cannot be countable. $\therefore C$ is an uncountable set.

Def : The Lebesgue ^{outer} measure or an outer measure of a subset A of \mathbb{R} is defined as

$$m^*(A) = \inf \left(\sum_{n \geq 1} l(I_n) \right)$$

Where infimum is taken over all finite or countable collection of intervals $\{I_n\}_{n \geq 1}$ such that

$$A \subseteq \bigcup_{n \geq 1} I_n, \text{ where } I_n = [a_n, b_n]$$

Theorem Let $A \subseteq \mathbb{R}$

- (i) $m^*(A) \geq 0$
- (ii) $m^*(\emptyset) = 0$
- (iii) If $A \subseteq B$, then $m^*(A) \leq m^*(B)$
- (iv) $m^(\{x\}) = 0 \quad \forall x \in \mathbb{R}$

Proof: (i), (ii), (iii) follows from defⁿ. Ex.

(iv) Let $J_n = [x, x + \frac{1}{n}]$ $\forall n \geq 1$

The $\{x\} \subseteq J_n$

$$l(J_n) = \frac{1}{n}$$

$$\begin{aligned} \text{By (iii)} \quad m^*(\{x\}) &\leq m^*(J_n) \\ &= l(J_n) \Rightarrow \forall n \geq 1 \\ &= \frac{1}{n}. \end{aligned}$$

$$\therefore m^*(\{x\}) \leq 0$$

$$\text{By (i)} \quad m^*(\{x\}) = 0.$$

Remark: $m^*(I) = l(I)$ for any finite interval $I \subset \mathbb{R}$

Proposition: - Let $A \subset \mathbb{R}$ be $x \in \mathbb{R}$. Then $m^*(A+x) = m^*(A)$
where $A+x = \{a+x \mid \forall a \in A\}$.

Note: Let a set be S , $l = \inf(S)$ $\epsilon > 0$

Then $l + \epsilon$ is not a lower bound.

$\therefore \exists x \in S$ such that $x \leq l + \epsilon$

Let $\epsilon > 0$

Then \exists a collection $\{I_n\}_{n \geq 1}$ such that $A \subseteq \bigcup_{n \geq 1} I_n$ &

$$m^*(A) + \epsilon \geq \sum_{n \geq 1} l(I_n) \quad (\text{by using inf. property})$$

$$\text{We have } A+x \subseteq \bigcup_{n \geq 1} (I_n + x)$$

$$\begin{aligned} m^*(A+x) &\leq \sum_{n \geq 1} l(I_n + x) = \sum_{n \geq 1} l(I_n) \\ &\leq m^*(A) + \epsilon \end{aligned}$$

$$\therefore m^*(A+x) \leq m^*(A)$$

$$\text{We have } A = (A+x) - x$$

$$m^*(A) = m^*((A+x) - x) \leq m^*(A+x) \quad \therefore m^*(A) = m^*(A+x)$$

EoC

9.1.19]

Thm: The outer measure of an interval is equal to its length.

Proof: Case 1.

Suppose $I = [a, b]$. For each $\epsilon > 0$, there exists $\{I_n\}$ intervals such that $I \subset \bigcup_{n \geq 1} I_n$ and

$$m^*(I) + \epsilon \geq \sum_{n \geq 1} l(I_n)$$

$$\text{Let } I_n = [a_n, b_n] \quad \forall n \geq 1$$

$$\text{Let } I'_n = (a_n - \frac{\epsilon}{2^n}, b_n) \quad \forall n \geq 1 \quad \text{open interval}$$

$$\text{We have } I \subseteq \bigcup_{n \geq 1} I'_n$$

Recall: Heine - Borel Theorem.

If A is closed and bounded set in \mathbb{R} and $\& A \subseteq \bigcup_{\alpha \in J} G_\alpha$, where G_α are open sets. then there

exists a finite subcollection $G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}$ such that

$$A \subseteq \bigcup_{i=1}^n G_{\alpha_i}$$

Continuing,

\therefore By using Heine Borel theorem, there exists a finite subcollection J_1, J_2, \dots, J_N , of $\{I'_n\}_{n \geq 1}$

such that $I \subseteq \bigcup_{i=1}^N J_i$. Say $J_i = (c_i, d_i)$

We can assume that no J_i is contained in another i.e.

$$c_1 < c_2 < c_3 < \dots < c_N$$

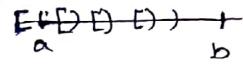
$$\text{Now, } d_N - c_1 = \sum_{k=1}^N (d_k - c_k) - \underbrace{\sum_{k=1}^N (d_k - c_{k+1})}_{\geq 0}$$

Note $d_k \geq c_{k+1}$ $\left\{ \begin{array}{l} \text{Otherwise they cannot cover } I. \\ \text{i.e. } J_i \text{ are intersecting} \end{array} \right\}$

$$\leq \sum_{k=1}^N (d_k - c_k)$$

$$= \sum_{k=1}^N l(J_k)$$

$$d_N - c_1 \leq \sum_{k=1}^N l(J_k)$$



We have

$$\begin{aligned} m^*(I) &\geq \sum_{n \geq 1} l(I_n) - \varepsilon \\ &\geq \sum_{n \geq 1} \left(l(I'_n) - \frac{\varepsilon}{2^n} \right) - \varepsilon \\ &\geq \sum_{n \geq 1} l(I'_n) - \varepsilon \sum_{n \geq 1} \frac{1}{2^n} - \varepsilon \\ &\geq \sum_{n \geq 1} l(I'_n) - 2\varepsilon \\ &\geq \sum_{n=1}^N l(J_n) - 2\varepsilon \\ &\geq d_N - c_1 - 2\varepsilon \\ &\geq b - a - 2\varepsilon \\ &= l(I) - 2\varepsilon \end{aligned}$$

$$\underline{m^*(I)} \geq \underline{l(I) - 2\varepsilon} \quad (1)$$

$$\begin{aligned} m^*(I) &= m^*([a, b]) \leq m^*([a, b + \varepsilon]) \quad \left\{ : [a, b] \subseteq [a, b + \varepsilon] \right\} \\ &\leq b - a + \varepsilon \end{aligned}$$

$$\underline{m^*(I)} \leq \underline{l(I) + \varepsilon} \quad (2)$$

$$(1) \& (2) \Rightarrow m^*(I) = l(I)$$

Case 2 Suppose $I = [a, b]$

Let $\epsilon > 0$ & $\epsilon < b - a$

Let $I' = [a + \epsilon, b]$

$I \subseteq I'$

$$m^*(I') \leq m^*(I)$$

||

$$l(I') = b - a - \epsilon = l(I) - \epsilon$$

∴

$$\therefore \underline{m^*(I)} \geq l(I) - \epsilon \quad (II) \quad \forall \epsilon > 0$$

Let $I'' = [a, b + \epsilon]$

$I \subseteq I''$

$$\therefore m^*(I) \leq m^*(I'') = b + \epsilon - a = l(I) + \epsilon$$

$$m^*(I) \leq l(I) + \epsilon \quad \forall \epsilon > 0 \quad (IV)$$

From ^(II) & ^(IV)

$$m^*(I) = l(I)$$

Similarly for (a, b)

Case 3 :- Suppose I is an infinite interval

Say $I = (-\infty, a]$

For any $M > 0 \exists k$ s.t. $I_k = [k, k+M] \subseteq I$

$$\therefore m^*(I_k) \leq m^*(I)$$

$$k+M - k$$

$$\frac{M}{M}$$

$$m^*(I) \geq M \quad \forall M > 0$$

$$m^*(I) = +\infty$$

Theorem :- For any sequence $\{E_i\}_{i \geq 1}$ of subsets of \mathbb{R}

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

Proof: Let $\epsilon > 0$

For each i , there exists a sequence of intervals

$\{I_{i,n}\}_{n \geq 1}$ such that $E_i \subseteq \bigcup_{n=1}^{\infty} I_{i,n}$

$$\& m^*(E_i) + \frac{\epsilon}{2^i} \geq \sum_{n=1}^{\infty} l(I_{i,n}) \quad (*)$$

$$\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \left(\bigcup_{n=1}^{\infty} I_{i,n} \right)$$

Thus $\{I_{i,n}\}_{i,n \geq 1}$ is a countable collection of intervals

that covers $\bigcup_{i=1}^{\infty} E_i$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} l(I_{i,n}) \quad [\text{from def}^n]$$

$$\leq \sum_{i=1}^{\infty} \left(m^*(E_i) + \frac{\epsilon}{2^i} \right) \quad [\text{by } (*)]$$

$$= \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon$$

$$\therefore m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

Proposition :

For any $E \subseteq \mathbb{R}$ any $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}$ such that $E \subseteq U$ & $m^*(U) \leq m^*(E) + \varepsilon$

Proof :- $\varepsilon > 0$ There exists seq. of intervals $\{I_n\}_{n=1}^{\infty}$

$$\text{s.t. } E \subseteq \bigcup_{n=1}^{\infty} I_n \text{ & } m^*(E) + \frac{\varepsilon}{2} \geq \sum_{n=1}^{\infty} l(I_n)$$

$$\text{Let } I_n = [a_n, b_n] \quad \forall n \geq 1.$$

$$\text{Let } I'_n = \left(a_n - \frac{\varepsilon}{2^{n+1}}, b_n\right) \quad \forall n \geq 1 \text{ open}$$

$$I_n \subseteq I'_n \quad \forall n \geq 1$$

$$\text{hence } E \subseteq \bigcup_{n=1}^{\infty} I'_n$$

$$\text{Let } U = \bigcup_{n=1}^{\infty} I'_n$$

$$U \text{ is an open set & } E \subseteq U$$

$$\begin{aligned}
 m^*(E) &\leq m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) \\
 &\leq \sum_{n=1}^{\infty} m^*(I_n') \quad (\text{by countable subadditivity property}) \\
 &= \sum_{n=1}^{\infty} l(I_n') \\
 &= \sum_{n=1}^{\infty} \left(b_n - a_n + \frac{\epsilon}{2^{n+1}} \right) \\
 &= \sum_{n=1}^{\infty} l(I_n) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} \\
 &= \sum_{n=1}^{\infty} l(I_n) + \frac{\epsilon}{2} \leq m^*(E) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= m^*(E) + \epsilon
 \end{aligned}$$

Thus $m^*(E) \leq m^*(U) \leq m^*(E) + \epsilon$

Proposition

In the definition of outer measure, $m^*(E)$

$$= \inf \left(\sum l(I_n) \right), \quad E \subseteq \bigcup_{n \geq 1} I_n$$

We stipulate : (i) $I_n = (a_n, b_n)$

(ii) $I_n = [a_n, b_n)$

(iii) $I_n = (a_n, b_n]$

(iv) $I_n = [a_n, b_n]$

(v) mixture of above

For different n , of various types of intervals their inf. is the same.

Proof: Denote m_0^*

Proof:- Denote m_0^* the outer measure obtained from (i)

We denote m_m^* the outer measure obtained from (v)

Let $E \subseteq \mathbb{R}$

$$m_0^*(E) = \inf_{\substack{E \subseteq \bigcup_{n=1}^{\infty} O_n \\ O_n \text{ open}}} \left(\sum_{n=1}^{\infty} l(O_n) \right)$$

$$\& m_m^*(E) = \inf_{\substack{E \subseteq \bigcup_{n=1}^{\infty} I_n \\ I_n \text{ any interval}}} \left(\sum_{n=1}^{\infty} l(I_n) \right)$$

By defⁿ of infimum $m_m^*(E) \leq m_0^*(E)$ --- (i)

Let $\epsilon > 0$ there exists $\{I_n\}_{n=1}^{\infty}$ intervals of type (v)

such that $E \subseteq \bigcup_{n=1}^{\infty} I_n$ and

$$\begin{aligned} m_m^*(E) + \epsilon &\geq \sum_{n=1}^{\infty} l(I_n) \\ &\geq \sum_{n=1}^{\infty} (1+\epsilon)^{-1} l(I_n') \end{aligned}$$

But $E \subseteq \bigcup_{n=1}^{\infty} I_n'$

$$m_0^*(E) \leq \sum_{n=1}^{\infty} l(I_n')$$

$$\text{Thus } m_m^*(E) + \epsilon \geq (1+\epsilon)^{-1} m_0^*(E)$$

$$\Rightarrow m_0^*(E) \leq (1+\epsilon) (m_m^*(E) + \epsilon)$$

$$\Rightarrow m_0^*(E) \leq m_m^*(E) + \epsilon (m_m^*(E) + 1 + \epsilon) \quad \forall \epsilon > 0$$

$$\text{Thus } m_0^*(E) \leq m_m^*(E) \quad \text{--- (ii)}$$

$$\text{Thus } m_0^*(E) = m_m^*(E) \quad \text{① and ②}$$

Definition:-

Let X be any non empty set. Let \mathcal{F} be a family of subsets of X . ~~\mathcal{F} be a family of sub~~ \mathcal{F} is said to be a σ algebra if it satisfies :-

- (i) $X \in \mathcal{F}$
- (ii) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
- (iii) If $E_1, E_2, \dots \in \mathcal{F}$ $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$

Eg ① $\{\emptyset, X\} \rightarrow \sigma$ algebra

② $\wp(X) \rightarrow \sigma$ algebra

Definition :-

Let $E \subseteq \mathbb{R}$. We say E is said to be Lebesgue measurable or simply measurable if for each subset $A \subseteq \mathbb{R}$,

the following holds:-

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

Remark $A = (A \cap E) \cup (A \cap E^c)$

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c))$$

$$\leq m^*(A \cap E) + m^*(A \cap E^c)$$

By subadditive property of m^* .

In def'n of measurable set we can take only one

inequality $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \quad \forall A \subseteq \mathbb{R}$

EoC

Recall.

A subset $E \subseteq \mathbb{R}$ is said to be measurable if $\forall A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad (*)$$

Ex ① Suppose $E \subseteq \mathbb{R}$ and $m^*(E) = 0$. Then E is measurable.

Proof: For $A \subseteq \mathbb{R}$, $A \cap E \subseteq E$ $m^*(A \cap E) \leq m^*(E) = 0$

$$\therefore m^*(A \cap E) = 0$$

$$\& A \cap E^c \subseteq A$$

$$\therefore m^*(A \cap E^c) \leq m^*(A)$$

$$\therefore m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A)$$

The other inequality is always true.

$$\therefore m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

$\therefore E$ is measurable.

② Let M be the set of all measurable subsets of \mathbb{R} .

Thm: M is a σ -algebra.

(i) $\mathbb{R} \in M$. For $A \subseteq \mathbb{R}$ $A \cap \mathbb{R} = A$, $A \cap \mathbb{R}^c = \emptyset$

$$\begin{aligned} \therefore m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^c) &= m^*(A) + m^*(\emptyset) \\ &= m^*(A) \end{aligned}$$

$\therefore (*)$ holds for $E = \mathbb{R}$.

$\therefore \mathbb{R}$ is measurable, $\mathbb{R} \in M$

(ii) Let $E \in M$. To show $E^c \in M$

$$\begin{aligned} \text{For } A \subseteq \mathbb{R}, m^*(A) &= m^*(A \cap E) + m^*(A \cap E^c) \\ &= m^*(A \cap (E^c)^c) + m^*(A \cap E^c) \end{aligned}$$

$$\Rightarrow E^c \in M$$

iii) Let $E_i \in \mathcal{M} \quad \forall i \geq 1$

To show $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$

Enough to show, for any $A \subseteq \mathbb{R}$

$$m^*(A) \geq m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) + m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c)$$

Given $E_1 \in \mathcal{M}$.

$$\therefore m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$$

$$\text{Let } B = A \cap E_1^c$$

Since E_2 is measurable.

$$m^*(B) = m^*(B \cap E_2) + m^*(B \cap E_2^c)$$

$$\begin{aligned} \therefore m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c) \\ &= m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap (E_1 \cup E_2)^c) \end{aligned}$$

Continuing, for any $n \geq 2$

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1 \cap \dots \cap \bigcap_{j < i} E_j^c) \\ &\quad + m^*(A \cap (\bigcup_{j=1}^n E_j)^c) \end{aligned}$$

We have $\bigcup_{i=1}^n E_i \subseteq \bigcup_{i=1}^{\infty} E_i$

$$\text{or, } \left(\bigcup_{i=1}^n E_i\right)^c \supseteq \left(\bigcup_{i=1}^{\infty} E_i\right)^c, \text{ or, } \left(\bigcup_{i=1}^n E_i\right) \cap A \supseteq \left(\bigcup_{i=1}^{\infty} E_i\right)^c \cap A$$

$$\text{or, } m\left(\left(\bigcup_{i=1}^n E_i\right)^c \cap A\right) \geq m\left(\left(\bigcup_{i=1}^{\infty} E_i\right)^c \cap A\right)$$

$$\begin{aligned} m^*(A) &\geq m^*(A \cap E_1) + m^*(A \cap E_1 \cap \dots \cap \bigcap_{j < i} E_j^c) + m^*\left(\left(\bigcup_{i=1}^{\infty} E_i\right)^c \cap A\right) \\ &\geq m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) + m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c) \end{aligned}$$

$$m^*\left(\bigcup_{i=1}^{\infty} (E_i \cap \bigcap_{j < i} E_j^c) \cap A\right) \leq \sum_{i=1}^{\infty} m^*(E_i \cap \bigcap_{j < i} E_j^c \cap A) \quad (+)$$

[count. subadd.]

$$\begin{aligned}
 m^*(A) &\geq m^*(A \cap E_1) + m^*\left(\bigcup_{i=1}^{\infty}(A \cap E_i \cap (\bigcup_{j < i} E_j)^c)\right) + m^*\left((\bigcup_{i=1}^{\infty} E_i)^c \cap A\right) \\
 &= m^*\left(\bigcup_{i=1}^{\infty}(A \cap E_i \cap (\bigcup_{j < i} E_j)^c)\right) + m^*\left((\bigcup_{i=1}^{\infty} E_i)^c \cap A\right)
 \end{aligned}$$

$$\begin{aligned}
 m^*(A) &\geq m^*\left(\bigcup_{i=1}^{\infty} A \cap E_i\right) = + m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c) \\
 &= m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) + m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c)
 \end{aligned}$$

Other inequality holds obviously.

$$\begin{aligned}
 &\therefore \bigcup_{i=1}^{\infty} E_i \text{ is measurable} \\
 &\Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}
 \end{aligned}$$

(i)(ii)(iii) $\Rightarrow \mathcal{M}$ is a σ -algebra

Theorem: Suppose $\{E_i\}_{i \geq 1}$ is a family of measurable subsets of \mathbb{R} and they are disjoint. Then $m^*(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E_i)$

Take $A = \bigcup_{i=1}^{\infty} E_i$ in the inequality (+) of prev. proof.

$$\begin{aligned}
 \text{Then, } m^*(\bigcup_{i=1}^{\infty} E_i) &\geq m^*((\bigcup_{i=1}^{\infty} E_i) \cap E_1) + \sum_{j=2}^{\infty} m^*\left(\bigcup_{i=1}^{\infty} E_i \cap E_j \cap (\bigcup_{i < j} E_i)^c\right) \\
 &\quad + \underbrace{m^*((\bigcup_{i=1}^{\infty} E_i) \cap (\bigcup_{i < j} E_i))}_{\emptyset}
 \end{aligned}$$

$$\begin{aligned}
 m^*(\bigcup_{i=1}^{\infty} E_i) &\geq m^*(E_1) + \sum_{i=2}^{\infty} m^*(E_i) \\
 &\geq \sum_{i=1}^{\infty} m^*(E_i)
 \end{aligned}$$

Subadditivity is obvious

$$\therefore \sum m^*(E_i) = m^*(\bigcup_{i=1}^{\infty} E_i)$$

EOC

14.1.19

Suppose S is a σ -algebra of all subsets of a set X .

Then if $E_1, E_2, \dots \in S \Rightarrow \bigcap_{i=1}^{\infty} E_i \in S$

Proof: Given $E_1, E_2, \dots \in S$

$\Rightarrow E_1^c, E_2^c, \dots \in S$

$\Rightarrow \bigcup_{i=1}^{\infty} E_i^c \in S$

$\Rightarrow ((\bigcup_{i=1}^{\infty} E_i)^c)^c \in S$

$\Rightarrow \bigcap_{i=1}^{\infty} E_i \in S$.

Corollary: Let \mathcal{M} denote the σ -algebra of measurable subsets of \mathbb{R} . Then for $E_i \in \mathcal{M} \quad \forall i \geq 1$, we have

$$\bigcap_{i=1}^{\infty} E_i \in \mathcal{M}.$$

Proposition: Suppose $F \in \mathcal{M}$ & $m^*(C_F \setminus G_1) \cup (G_1 \setminus F) = 0$ for a subset $G_1 \in \mathbb{R}$. Then $G_1 \in \mathcal{M}$, i.e. G_1 is also measurable.

Proof: ~~Suppose~~ E is measurable & $F \subseteq E$. Then F is measurable. (Wrong!) 回

Proof:- To show

$$m^*(A) \geq m^*(A \cap F) + m^*(A \cap F^c)$$

Since E is measurable,

$$\begin{aligned} m^*(E) &\geq m^*(A \cap E) + m^*(A \cap E^c) \\ &\geq m^*(A \cap F) + m^*(A \cap F^c) \end{aligned}$$

We know \mathbb{R} is measurable, but every subset of \mathbb{R} need not be measurable because \exists non measurable sets. [Remark is false]

Proof:

$$F \Delta G = (F \setminus G) \cup (G \setminus F)$$

$$\text{&} \quad F \setminus G \subseteq F \Delta G, \quad G \setminus F \subseteq F \Delta G$$

$$\Rightarrow m^*(F \setminus G) \leq m^*(F \Delta G) \quad \& \quad m^*(G \setminus F) \leq m^*(F \Delta G) = 0$$

$$\therefore m^*(F \setminus G) = m^*(G \setminus F) = 0.$$

$F \setminus G_1, G \setminus F$ are measurable

i.e. $F \setminus G_1, G \setminus F \in \mathcal{M}$. Since \mathcal{M} is a σ -algebra

Thm. Every interval is measurable.

Let us probe for $[a, \infty)$

Note

$$\begin{aligned}[a, b] &= \bigcap_{n=1}^{\infty} [a, b + \frac{1}{n}) \\ &= [a, \infty) \cap (-\infty, b]\end{aligned}$$
$$(a, b) = \dots$$
$$[a, b) = \dots$$
$$(a, b] = \dots$$

To show:

For $A \subset \mathbb{R}$

$$\begin{aligned}m^*(A) &\geq m^*(A \cap [a, \infty)) \\ &\quad + m^*(A \cap (-\infty, a])\end{aligned}$$

$$\text{Let } A_1 = A \cap (-\infty, a)$$

$$\& A_2 = A \cap [a, \infty)$$

Given $\epsilon > 0$, there exists $\{I_n\}_{n \geq 1}$ intervals such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ & $m^*(A) + \epsilon \geq \sum_{n=1}^{\infty} l(I_n)$

Let $I_n' = I_n \cap (-\infty, a)$

$$I_n'' = I_n \cap [a, \infty)$$

Since $A \subseteq \bigcup_{n=1}^{\infty} I_n$ therefore

$$A \cap (-\infty, a) \subseteq \left(\bigcup_{n=1}^{\infty} I_n' \right) \cap (-\infty, a)$$

$$\& A \cap [a, \infty) \subseteq \left(\bigcup_{n=1}^{\infty} I_n'' \right) \cap [a, \infty)$$

$$\Rightarrow A_1 \subseteq \bigcup_{n=1}^{\infty} (I_n \cap (-\infty, a))$$

$$\& A_2 \subseteq \bigcup_{n=1}^{\infty} (I_n \cap [a, \infty))$$

$$A_1 \subseteq \bigcup_{n=1}^{\infty} I_n'$$

$$\& A_2 \subseteq \bigcup_{n=1}^{\infty} I_n''$$

$$\Rightarrow m^*(A_1) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) \leq \sum_{n=1}^{\infty} m^*(I_n')$$

$$\& m^*(A_2) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n''\right) \leq \sum_{n=1}^{\infty} m^*(I_n'')$$

$$m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} (m^*(I_n') + m^*(I_n''))$$

$$= \sum_{n=1}^{\infty} \left(l(I_n') + l(I_n'') \right)$$

disjoint, union to I_n .

$$= \sum_{n=1}^{\infty} l(I_n)$$

$$m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} l(I_n)$$

$$\leq m^*(A) + \epsilon$$

True for any $\epsilon > 0$

$$m^*(A_1) + m^*(A_2) \leq m^*(A)$$

$[a, \infty)$ is measurable.

Definition ..

Let \mathcal{M} = set of all measurable subsets of \mathbb{R} .

Define a map $m : \mathcal{M} \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$

$$\text{as } m(E) = m^*(E) \quad \forall E \in \mathcal{M}$$

Known as the Lebesgue measure of E

m is countably additive.

Eoc

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Let \mathcal{A} be a family of subsets of a set X .

Theorem Then there exists a smallest σ algebra containing \mathcal{A} . This σ algebra is known as the σ algebra generated by \mathcal{A} .

Proof This σ algebra is precisely, the intersection of all σ -algebras (on subsets of X) containing \mathcal{A} .

Say $\{S_\alpha\}_{\alpha \in I}$ be the family of all σ algebras containing \mathcal{A} . Then $\bigcap_{\alpha \in I} S_\alpha$ is the smallest σ algebra containing \mathcal{A} .

Defn. Let B be the σ algebra generated by the intervals of the form $[a, b)$ $a, b \in \mathbb{R}$.

i.e. $A = \{[a, b) \mid a, b \in \mathbb{R}\}$

$B = \sigma$ algebra generated by A .

Then the members of B are called Borel Sets.

Theorem:

(i) Every Borel set is measurable i.e. $B \subseteq M$

(ii) B is generated by each of the following classes:

open intervals, open sets, G_δ sets, F_σ sets.

Proof (i) Note $[a, b) \in M \quad \forall a, b \in \mathbb{R}$

$\Rightarrow B \subseteq M$ ($\because B$ is the smallest σ algebra containing $[a, b) \forall a, b \in \mathbb{R}$)

(ii) Let B_1 be the σ algebra generated by open intervals.
Since any open interval $(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b) \in B_1$

$\therefore B_1 \subseteq B$
Also $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b) \in B_1 \Rightarrow B \subseteq B_1 \quad B = B_1$

Any open set is the countable union of open intervals.

Thus implies $B_1 = B_2$, where $B_2 = \sigma\text{alg. generated by open}$

Defn

i) G_δ defined as countable intersection of open sets.

(ii) F_σ defined as countable union of closed sets.

Let $B_3 = \sigma$ algebra generated by G_δ sets in \mathbb{R} .

Any G_δ set $A = \bigcap_{n=1}^{\infty} O_n$ O is open $\in B_2$

$$B_3 \subseteq B_2$$

Let U be any open set in \mathbb{R}

$$\text{Let } O_n = U \quad \forall n \geq 1$$

$\bigcap_{n=1}^{\infty} O_n = U$ is a G_δ set.

$$B_2 \subseteq B_3$$

$$\therefore B_2 = B_3$$

F_σ sets - Exercise

Remark: $B \not\subseteq M$

$\therefore \exists$ a measurable set which is not Borel set

Proposition: For any subset $A \subseteq \mathbb{R}$, \exists a measurable set E containing A & $m^*(A) = m(E)$

Prof: For $\epsilon = \frac{1}{n}$. There exists an open set O_n s.t.

$$m^*(O_n) \leq m^*(A) + \epsilon = m^*(A) + \frac{1}{n}$$

Let $E = \bigcap_{n=1}^{\infty} O_n$ E is G_δ set

E is measurable.

$$\& m^*(E) = m^*(\bigcap_{n=1}^{\infty} E_n) \leq m^*(E_n) \leq m^*(A) + \frac{1}{n} \quad \forall n \geq 1$$

$$\therefore \underline{m^*(E) \leq m^*(A)}.$$

On the other hand

$$A \subseteq E_n \quad \forall n$$

$$\Rightarrow A \subseteq \bigcap_{n=1}^{\infty} E_n = E$$

$$\Rightarrow m^*(A) \leq m^*(E) = m(E)$$

$$\therefore \underline{m^*(A) = m^*(E) = m(E)}$$

Let $\{E_i\}_{i \geq 1}$ be a sequence of subsets of X .

$$\text{Def: } \limsup(E_i) = \bigcap_{n=1}^{\infty} \left(\bigcup_{i \geq n} E_i \right)$$

$$\liminf(E_i) = \bigcup_{n=1}^{\infty} \left(\bigcap_{i \geq n} E_i \right)$$

$E_N \cap E_{N+1} \cap \dots$
etc

$$\text{Remark. } \liminf(E_i) \subseteq \limsup(E_i)$$

Let $x \in \liminf(E_i)$

$$\Rightarrow \exists N \text{ st. } x \in \bigcap_{i \geq N} E_i$$

$$\Rightarrow x \in E_i \quad \forall i \geq N$$

$$\Rightarrow x \in \bigcup_{i \geq N} E_i \quad \forall$$

\Rightarrow

$$\text{for } n \geq 1 \quad (E_n \cap E_{n+1} \cap \dots) \subseteq \limsup(E_i)$$

$$\Rightarrow \bigcup_{n \geq 1} (E_n \cap E_{n+1} \cap \dots) \subseteq \limsup(E_i)$$

$$\Rightarrow \liminf(E_i) \subseteq \limsup(E_i)$$



If $\liminf(E_i) = \limsup(E_i)$
 Then this set is defined as $\lim(E_i)$

Examples :-

① Suppose $E_1 \subset E_2 \subset \dots$

$$\begin{aligned}\liminf(E_i) &= \bigcup_{n=1}^{\infty} \left(\bigcap_{i \geq n} E_i \right) \\ &= \bigcup_{n=1}^{\infty} (E_n \cap E_{n+1} \cap \dots) = \bigcup_{n=1}^{\infty} E_n.\end{aligned}$$

$$\begin{aligned}\limsup(E_i) &= \bigcap_{n=1}^{\infty} \left(\bigcup_{i \geq n} E_i \right) \subseteq \bigcup_{i \geq n} E_i \\ &\quad // \subseteq \bigcup_{i \geq 1} E_i \\ &= \liminf.\end{aligned}$$

$$\therefore \limsup(E_i) = \liminf(E_i)$$

② Suppose.

$$\lim(E_i) = \bigcup_{n=1}^{\infty} E_n.$$

$$E_1 \supset E_2 \supset \dots$$

$$\begin{aligned}\limsup &= \bigcap_{n=1}^{\infty} (E_n \cup E_{n+1} \cup \dots) \\ &= \bigcap_{n=1}^{\infty} E_n \subseteq \bigcap_{i \geq n} E_i \forall n.\end{aligned}$$

$$\liminf = \bigcup_{n=1}^{\infty} \left(\bigcap_{i \geq n} E_i \right)$$

$$\Rightarrow \bigcap_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \left(\bigcap_{i \geq n} E_i \right)$$

$$\limsup(E_i) \quad \liminf(E_i)$$

$$\therefore \lim(E_i) = \bigcap_{n=1}^{\infty} E_i$$

Theorem

Every nonempty open subset U of \mathbb{R} is the union of disjoint open intervals, at most countable in number.

Proof: Define an ~~injection~~ \sim on \mathbb{R} as follows.

$$a, b \in U, a \sim b \Leftrightarrow$$

the closed interval $[a, b]$ or $[b, a]$ lies in U .
 $a \leq b$ $b \leq a$

To show \sim is an equivalence relation.

$$(a) a \sim a (\because \{a\} = [a, a] \subseteq U)$$

(b) Suppose $a \sim b$. To show $b \sim a$.

Suppose $a < b$ then $[a, b] \subseteq U$.

(c) Suppose $a \sim b$ and $b \sim c$ and hence

$$[b, a] = \emptyset \subseteq U.$$

To show $a \sim c$.

Say $a \leq b \leq c$.

$$[a, b], [b, c] \subseteq U \Rightarrow [a, b] \cup [b, c] \subseteq U.$$

||

$$[a, c]$$

U = Disjoint union of equivalence classes of \sim .

For $a \in U$,

Let $C(a)$ denote the equivalence class containing a

Then $C(a)$ is an interval. For $b \in C(a)$ we have

$$a \sim b \Rightarrow [a, b] \text{ or } [b, a] \subseteq U$$

\Rightarrow all others $b/w a$ or b also in $C(a)$

$C(a)$ is an interval.

To show that $C(\alpha)$ is open.

For $k \in C(\alpha) \subseteq U$ there is $\varepsilon > 0$

such that $(k - \varepsilon, k + \varepsilon) \subseteq U \rightarrow U$ open.

but $(k - \varepsilon, k + \varepsilon) \subseteq C(\alpha)$ for ε suff. small.

$\therefore C(a)$ is open.

$$\therefore U = \bigcup_{\alpha} C(\alpha)$$

Recall: Lindelöf Theorem

Suppose $\mathcal{F} = \{I_\alpha | \alpha \in A\}$ is a collection of open intervals. \exists a subcollection, say $\{I_1 | i=1, 2, \dots\}$ at most countable number such that

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{i=1}^{\infty} I_i$$

\therefore By Lindelöf Thm.

\exists at most countable a_1, a_2, \dots

such that $U = \bigcup_{i=1}^{\infty} C(a_i)$

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NOTE

Theorem:

Let $\{E_j\}_{j \geq 1}$ be a sequence of measurable subsets of \mathbb{R} . Then (i) If $E_1 \subseteq E_2 \subseteq \dots$ Then $m(\lim E_i) = \lim m(E_i)$
(ii) If $E_1 \supseteq E_2 \supseteq \dots$ then $m(\lim (E_i)) = \lim (m(E_i))$

Proof

(i) We have $\lim E_i = \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.
is measurable.

Let $F_1 = E_1$.

$$F_2 = E_2 \setminus F_1$$

$$F_3 = E_3 \setminus F_2$$

\vdots

$$F_i = E_i \setminus F_{i-1}$$

$$\text{Then } \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

$$m(\lim (E_i)) = m\left(\bigcup_{i=1}^{\infty} E_i\right).$$

$$= m\left(\bigcup_{i=1}^{\infty} F_i\right) = m^*\left(\bigcup_{i=1}^{\infty} F_i\right).$$

$$= \sum_{i=1}^{\infty} m^*(F_i) \quad [\text{additive property}]$$

$$= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n m^*(F_i) \right)$$

$$= \lim_{n \rightarrow \infty} \left(m^*\left(\bigcup_{i=1}^n F_i\right) \right)$$

$$= \lim_{n \rightarrow \infty} \left(m^*\left(\bigcup_{i=1}^n E_i\right) \right)$$

$$= \lim_{n \rightarrow \infty} m^*(E_n)$$

$$= \lim_{n \rightarrow \infty} m(E_n)$$

$$\therefore m(\lim (E_i)) = \lim (m(E_i)).$$

(11) Assume $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$

Look at $E_1 \setminus E_2 \subseteq E_1 \setminus E_3 \subseteq E_1 \setminus E_4$
where $E_1 \setminus E_i$ are measurable.

by (ii)

$$m(\lim(E_1 \setminus E_i)) = \lim(m(E_1 \setminus E_i))$$

Since $E_1, E_1 \setminus E_i, E_i$ are measurable.

$$\text{therefore } m(E_1 \setminus E_i) = m(E_1) - m(E_i)$$

$$E_1 = E_i \cup (E_1 \setminus E_i)$$

$$\begin{aligned} m(\lim(E_1 \setminus E_i)) &= \lim(m(E_1) - m(E_i)) \\ &= m(E_1) - \lim m(E_i). \end{aligned}$$

$$\begin{aligned} \lim(E_1 \setminus E_i) &= \bigcup_{i=1}^{\infty} (E_1 \setminus E_i) \\ &= E_1 \setminus \bigcap_{i=1}^{\infty} E_i = E_1 \setminus \lim E_i. \end{aligned}$$

$$\begin{aligned} m(\lim(E_1 \setminus E_i)) &= m(E_1 \setminus \lim E_i) \\ &= m(E_1) - m(\lim E_i). \end{aligned}$$

From *

$$\begin{aligned} m(E_1) - m(\lim E_i) \\ = m(E_1) - \lim(m(E_i)) \end{aligned}$$

$$\text{So, } m(\lim E_i) = \lim(m(E_i)) \quad m(E_1) < \infty.$$

~~Proposition~~

1) Every non-empty set in \mathbb{R} has the measure.

2) Suppose $Q = \{q_1, q_2, \dots\}$.

$$\text{& } Q = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2} \right).$$

Then for any closed set $F \subseteq \mathbb{R}$ we have
 $m(Q \Delta F) > 0$.

where $G \Delta F = (G \setminus F) \cup (F \setminus G)$.

Proof :- 1) Since every non empty open set of \mathbb{R} is a countable disjoint union of open intervals,

① follows.

2) If $m(G \setminus F) > 0$, then nothing to prove.

Assume $m(G \setminus F) = 0$

$G \setminus F = G \cap F^c$ which is open.

Then, by ② if $G \setminus F \neq \emptyset$

$$m(G \setminus F) > 0.$$

$$\Rightarrow G \setminus F = \emptyset.$$

$$\Rightarrow G \subseteq F.$$

But $Q \subseteq G$ (by defⁿ of Q).

$$\Rightarrow Q \subseteq G \subseteq F.$$

$$\Rightarrow \overline{Q} \subseteq \overline{G} \subseteq \overline{F}$$

∴
 \mathbb{R} :

$$\Rightarrow \overline{F} = \mathbb{R}$$

If
 F :
 $\Rightarrow F = \mathbb{R}$.

$$m(F) = m(\mathbb{R}) = \infty$$

$$m(G) \leq \sum_{n=1}^{\infty} \left((q_n + \frac{1}{n^2}) - (q_n - \frac{1}{n^2}) \right)$$

$$= \sum \frac{2}{n^2} < \infty.$$

$$\therefore m(F \setminus G) = m(F) - m(G) \\ = \infty > 0.$$

$$\therefore m(G \Delta F) > 0.$$

① Let $k > 0$ & $A \subseteq \mathbb{R}$

Denote $kA = \{x \in \mathbb{R} \mid k^{-1}x \in A\}$.

To show that

$$\text{i)} m^*(kA) = k m^*(A).$$

ii) A is measurable $\iff kA$ is measurable.

Sol:-

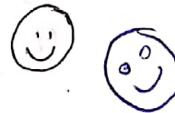
$$m^*(kA) = \inf \left\{ \sum l(I_n) \mid kA \subseteq \bigcup I_n \right\}.$$

Suppose $I_n = [a_n, b_n]$

$$kA \subseteq \bigcup_{n=1}^{\infty} [a_n, b_n)$$

$$\Rightarrow A \subseteq \bigcup_{n=1}^{\infty} [k^{-1}a_n, k^{-1}b_n)$$

$$\begin{aligned}\therefore m^*(kA) &= \inf \left\{ \sum (b_n - a_n) \mid A \subseteq \bigcup_{n=1}^{\infty} [k^{-1}a_n, k^{-1}b_n) \right\} \\ &= k \inf \left\{ \sum (k^{-1}b_n - k^{-1}a_n) \mid A \subseteq \bigcup_{n=1}^{\infty} [k^{-1}a_n, k^{-1}b_n) \right\} \\ &= k \inf \left\{ \sum l([k^{-1}a_n, k^{-1}b_n]) \mid A \subseteq \bigcup_{n=1}^{\infty} [k^{-1}a_n, k^{-1}b_n) \right\}.\end{aligned}$$



Thanks L^3

Proposition

There exists an uncountable set whose measure is zero.

Proof

Look at the cantor set $C = \bigcap_{n=1}^{\infty} P_n$

where $P_n = \bigcup_{k=1}^n J_{n,k}$, where each $J_{n,k}$ is an interval.

Note C is uncountable.

Each $J_{n,k}$ is an interval. Thus each P_n is measurable.

& hence C is measurable and hence C^c is measurable

$$C^c \in \mathcal{M} \quad (C \in \mathcal{M})$$

$$\text{Look at } m(C^c) = m([0, 1] \setminus C)$$

$$= m([0, 1]) - m(C)$$

$$= 1 - m(C)$$

$$C^c = (\bigcap_{n=1}^{\infty} P_n)^c = \bigcup_{n=1}^{\infty} P_n^c = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{2^{n-1}} I_{n,k} \right) \quad \begin{array}{l} I_{n,k} \text{ is interval} \\ l(I_{n,k}) = \frac{1}{3^n} \end{array}$$

$$\text{But } \bigcup_{n=1}^{\infty} P_n^c = \bigcup_{k=1}^{2^{n-1}} I_{n,k} \Rightarrow l(I_{n,k}) = \frac{1}{3^n}$$

$$\begin{aligned} m(C^c) &= m\left(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}\right) = \sum_{n=1}^{\infty} m\left(\bigcup_{k=1}^{2^{n-1}} I_{n,k}\right) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} m(I_{n,k}) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} l(I_{n,k}) \\ &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1. \end{aligned}$$

$$m(C^c) = 1.$$

$$\therefore m(C) = 0$$

Theorem

Let $E \subseteq \mathbb{R}$. Then the following are equivalent.

(i) E is measurable.

(ii) Given $\epsilon > 0$, \exists an open set $U \subseteq \mathbb{R}$

such that $U \supseteq E$ & $m^*(U \setminus E) < \epsilon$

(iii) There exists a G_δ set $G \subseteq \mathbb{R}$ such that

$G \supseteq E$ & $m^*(G \setminus E) = 0$ ($\because m^*(\emptyset) = m^*(\mathbb{R})$)

(\star) Given $\epsilon > 0$, \exists a closed set $F \subseteq \mathbb{R}$ such that

$F \subseteq E$ & $m^*(E \setminus F) < \epsilon$

($\star\star$) There exists an F_σ $F \subseteq E$ such that $m^*(E \setminus F)$

If a measure satisfies

(A non negative, countably additive set function satisfying above equivalent conditions is said to be a regular measure)

Proof

(i) \Rightarrow (ii).

Assume E is measurable.

Case 1. Suppose $m(E) < \infty$

Then given $\epsilon > 0$, we have shown that there exists U open in \mathbb{R} such that $E \subseteq U$ & $m^*(U \setminus E) < \epsilon$

Case 2. Suppose $m(E) = \infty$

Case 2 Suppose $m(E) = \infty$

Write $\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$ where I_n 's are open intervals

$$\Rightarrow E = \bigcup_{n=1}^{\infty} I_n \cap E$$

Let $E_n = I_n \cap E$ & $n \geq 1$

$$m(E_n) < \infty \quad (\because E_n \subseteq I_n \text{ & } l(I_n) < \infty)$$

Note E_n is measurable.

Let $\epsilon > 0$

There exists open set $U_n \subseteq \mathbb{R}$.

such that $U_n \supseteq E_n$ & $m^*(U_n \setminus E_n) < \frac{\epsilon}{2^n}$

$$\begin{aligned} \text{Let } U = \bigcup_{n=1}^{\infty} U_n. \text{ Then } U \text{ is open &} U = \bigcup_{n=1}^{\infty} U_n \supseteq \bigcup_{n=1}^{\infty} E_n \\ = \bigcup_{n=1}^{\infty} I_n \cap E = E \end{aligned}$$

$$\Rightarrow U \supseteq E$$

$$\text{Look at } U \setminus E = \bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} (U_n \setminus E_n)$$

$$\begin{aligned} m^*(U \setminus E) &\leq m^*\left(\bigcup_{n=1}^{\infty} (U_n \setminus E_n)\right) \\ &\leq \sum_{n=1}^{\infty} m^*(U_n \setminus E_n) \\ &\leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon \left(\frac{1/2}{1-1/2} \right) = \epsilon \end{aligned}$$

Proved that (i) \Rightarrow (ii)

(ii) \Rightarrow (iii)

To show - There exist a G_S -set

G such that $G \supseteq E$ & $m^*(G \setminus E) = 0$

By (i) For each $n > 0$, \exists open $U_n \subseteq \mathbb{R}$ such that

$$U_n \supseteq E \text{ & } m^*(U_n \setminus E) < \frac{1}{n}$$

$$\text{Let } G_1 = \bigcap_{n=1}^{\infty} U_n$$

G_1 is a G_S set & $G \supseteq E$

$$\text{now } m^*(G \setminus E) \leq m^*(U_n \setminus E)$$

$$\leftarrow G_1 \setminus E \subseteq U_n \setminus E \forall n$$

$$m^*(G_1 \setminus E) < \frac{1}{n} \quad \forall n \geq 1$$

$$\Rightarrow m^*(G_1 \setminus E) = 0$$

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(iii) \Rightarrow (i) Assume $\exists G_1$ set $G_1 \subset \mathbb{R}$ such that $G_1 \supseteq E$ & $m^*(G_1 \setminus E) = 0$

To show E is measurable.

We know $G_1 \setminus E$ is measurable because

$$m^*(G_1 \setminus E) = 0$$

Also every G_{1j} set is measurable hence G_1 is measurable.

Now

$$E = G_1 \setminus (G_1 \setminus E) \in \mathcal{M} \quad \therefore E \text{ is measurable.}$$

(i) \Rightarrow (ii)' Assume E is measurable.

To show: Given $\varepsilon > 0 \exists F$ closed in \mathbb{R} s.t.

$$F \subseteq E \text{ & } m^*(E \setminus F) < \varepsilon$$

Let $\varepsilon > 0$.

We have E^c also measurable.

$\therefore \exists U$ open in \mathbb{R} . such that $m^*(U \setminus E^c) < \varepsilon$

$$\& U \supseteq E^c \quad (\text{by (ii)})$$

$$\text{Now } U \setminus E^c = E \setminus U^c$$

$$(\text{Since } U \cap E = E \cap U)$$

$$\text{Then } m^*(E \setminus U^c) < \varepsilon \quad \& \quad U^c \subseteq E$$

$$\text{Let } F = U^c$$

$$\text{Then } F \text{ is a closed set} \\ \& F \subseteq E \quad \& \quad m^*(E \setminus F) < \varepsilon$$

(ii)'-(iii)'

Assume (ii)'

To show, \exists a F_σ set $F \subseteq E$ & $m^*(E \setminus F) = 0$

For each n , \exists a closed set F_n such that

$F_n \subseteq E$ & $m^*(E \setminus F_n) < \frac{1}{n}$ (By (ii)').

Let $F = \bigcup_{i=1}^{\infty} F_n \Rightarrow F$ is F_σ

$F \subseteq E$

$$\& m^*(E \setminus F) = m^*(E \setminus \bigcup_{i=1}^{\infty} F_n)$$

$$= m^*(E \cap (\bigcup_{i=1}^{\infty} F_n)^c)$$

$$= m^*(E \cap (\bigcap_{i=1}^{\infty} F_n^c))$$

$$\leq m^*(E \cap F_n^c) \quad \forall n$$

$$= m^*(E \setminus F_n)$$

$$\leq \frac{1}{n} \quad \forall n \geq 1$$

$$\Rightarrow m^*(E \setminus F) = 0$$

(iii) \Rightarrow (i)

There exists an F_σ -set $F \subseteq \mathbb{R}$ such that

$$F \subseteq E \& m^*(E \setminus F) = 0$$

$\Rightarrow E \setminus F$ is measurable

& F is measurable.

$$E = (E \setminus F) \cup F$$

$\Rightarrow E$ is measurable.

Corollary:

Lebesgue measure is a regular measure.

Definition:

Let $f: E \rightarrow \mathbb{R} \cup \{\pm\infty\}$

where $E \subseteq \mathbb{R}$ measurable be a function. Then f is said to be Lebesgue measurable function or simply a measurable function.
If for each $\alpha \in \mathbb{R}$, the set $\{x \in E | f(x) > \alpha\}$ is measurable.

Examples:

① $f: \mathbb{R} \rightarrow \mathbb{R}$.

$$f(x) = c \quad \forall x \quad (c \in \mathbb{R})$$

$$c \in \mathbb{R}$$

$$\{x \in \mathbb{R} | f(x) > c\} = \mathbb{R} \text{ or } \emptyset$$

$\Rightarrow f$ is measurable.

② Let $E \subseteq \mathbb{R}$ be measurable

Define $\chi_E: \mathbb{R} \rightarrow [0,1]$.

$$\chi_E(x) = \begin{cases} 0, & x \notin E \\ 1, & x \in E \end{cases}$$

For $\alpha \in \mathbb{R}$

$$\{x \in \mathbb{R} | \chi_E(x) > \alpha\} = A_\alpha \quad (\text{say})$$

$$A_\alpha = \begin{cases} \mathbb{R} & \text{if } \alpha < 0 \\ \emptyset & \text{if } \alpha \geq 1 \\ E & \text{if } \alpha = 0 \\ F & 0 < \alpha < 1 \end{cases}$$

In each value of α

A_α is measurable.

$\therefore \chi_E$ is a measurable function.

Thm: Every continuous function is measurable.

Proof: $\alpha \in \mathbb{R} \Rightarrow f: E \rightarrow \mathbb{R}$ is continuous.

$$\{x \in E \mid f(x) > \alpha\} = f^{-1}((\alpha, \infty))$$

\Rightarrow preimage of (α, ∞)
open set

\Rightarrow open set in \mathbb{R} .

($\because f$ is continuous)

$\Rightarrow U$ is measurable & $\therefore F$ is measurable

Thm

Let $f: E \rightarrow \mathbb{R}$ be a function & $E \subseteq \mathbb{R}$ measurable

Then the following statements are equivalent.

(i) f is measurable

(ii) For any $\alpha \in \mathbb{R}$, $\{x \mid f(x) \geq \alpha\}$ is measurable

(iii) For any $\{x \mid f(x) < \alpha\}$ is measurable

(iv) For any $\alpha \in \mathbb{R}$, $\{x \mid f(x) \leq \alpha\}$ is measurable.

(v) For any $\alpha \in \mathbb{R}$, $\{x \mid f(x) = \alpha\}$

Proof:

(i) \Rightarrow (ii) $\alpha \in \mathbb{R}$

$$\{x \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) > \alpha - \frac{1}{n}\}$$

measurable

\therefore measurable.

(ii) \Rightarrow (iii), $\alpha \in \mathbb{R}$

$$\{x \mid f(x) < \alpha\} = \{x \mid f(x) \geq \alpha^c\}$$

by (ii) $\{x \mid f(x) \geq \alpha^c\} \subseteq \mathcal{M}$

\therefore (iii) follows.

(iii) \Rightarrow (iv)

Let $\alpha \in \mathbb{R}$

$$\{x \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) < \alpha + \frac{1}{n}\}$$

(M (iii))

$\therefore \{x \mid f(x) \leq \alpha\}$ is measurable.

$$\{x \mid f(x) > \alpha\} = \{x \mid f(x) \leq \alpha\}^c$$

Proposition

Let $f, g: E \rightarrow \mathbb{R}$ be measurable functions & $c \in \mathbb{R}$ then $f+c, f+g, cf, fg$ are measurable

Proof. $\alpha \in \mathbb{R}$.

(i) To show $f+c$ is measurable.

$$\begin{aligned} & \{x \in E \mid (f+c)(x) > \alpha\} \\ &= \{x \in E \mid f(x) + c > \alpha\} \\ &= \{x \in E \mid f(x) > \alpha - c\} \end{aligned}$$

is measurable, As f is measurable.

(ii) To show cf is measurable.

$$\begin{aligned} \{x \mid (cf)(x) > \alpha\} &= \{x \mid cf(x) > \alpha\} \\ &= \{x \mid f(x) < \frac{\alpha}{c}\} \end{aligned}$$

$$= \begin{cases} \emptyset \text{ or } E, c=0 \\ \{x \mid f(x) > \frac{\alpha}{c}\} & c > 0 \\ \{x \mid f(x) < \frac{\alpha}{c}\} & c < 0 \end{cases}$$

In all cases, set is measurable.

To show: $f+g$ is measurable

Let $\alpha \in \mathbb{R}$

$$\{x \in E \mid (f+g)(x) > \alpha\}$$

$$= \{x \in E \mid f(x) + g(x) > \alpha\}$$

$$= \{x \in E \mid f(x) > \alpha - g(x)\}$$

Let $\{\gamma_1, \gamma_2, \dots\}$ be the set of all rational numbers. i.e., b/w $f(x)$ & $\alpha - g(x)$
i.e. $f(x) > \gamma_i > \alpha - g(x) \quad \forall i \geq 1$

$$\{x \in E \mid f(x) > \alpha - g(x)\}$$

$$= \bigcup_{i=1}^{\infty} \{x \in E \mid f(x) > \gamma_i\} \cap \{x \in E \mid \gamma_i > \alpha - g(x)\}$$

Thus set is measurable because

~~f, g~~ are measurable

$f+g$ is measurable.

$$f-g = f+(-g)$$

$$f(g) = \frac{1}{4} ((f+g)^2 - (f-g)^2)$$

To show f^2 is also measurable.

$$\{x | f^2(x) > \alpha\}$$

$$= \{x | [f^2(x)]^2 > \alpha\}$$

$$= \begin{cases} \emptyset & \text{if } \alpha < 0 \\ \mathbb{R} & \text{if } \alpha \geq 0 \end{cases}$$

Say $\alpha > 0$

$$= \{x | f(x) > \sqrt{\alpha}\} \cup \{x | f(x) < -\sqrt{\alpha}\}$$

EOC

24.1.19

Theorem.

Let $\{f_n\}_{n \geq 1}$ be a sequence of measurable functions defined on a measurable set $E \subseteq \mathbb{R}$.

Then

- (i) $\sup_{1 \leq i \leq n} (f_i)$ is a measurable f^n . $\forall n \geq 1$
- (ii) if (f_i) is a measurable f_h . $\forall n \geq 1$
- (iii) $\sup_{n \geq 1} (f_n)$ is measurable
- (iv) $\inf_n (f_n)$ is measurable
- (v) $\limsup (f_n)$ "
- (vi) $\liminf (f_n)$ "

Proof:

(i) Let $\alpha \in \mathbb{R}$

$$\begin{aligned}
 & \text{Consider } \left\{ x \in E \mid \sup_{1 \leq i \leq n} (f_i) \stackrel{<}{\rightarrow} \alpha \right\} \\
 & = \left\{ x \in E \mid \sup (f_i(x)) \stackrel{<}{\rightarrow} \alpha \right\} \\
 & = \left\{ x \in E \mid f_i(x) \stackrel{<}{\rightarrow} \alpha \quad \forall i, 1 \leq i \leq n \right\} \\
 & = \bigcap_{i=1}^n \left\{ x \in E \mid f_i(x) < \alpha \right\} \quad | \text{ Each } f_i \in \mathcal{M} \\
 & \qquad \qquad \qquad \mathcal{S}^n \qquad \qquad \mathcal{M}
 \end{aligned}$$

$\sup_{1 \leq i \leq n} \{f_i\}$ is measurable.

ii) Prove using greater than α as above.

OR

$$\inf_n (f_n) = - \sup_n (-f_n)$$

" R is measurable by (i)

$$(iii) \left\{ x \in E \mid \sup_n (f_n)(x) < \alpha \right\}$$

$$= \left\{ x \in E \mid f_n(x) < \alpha \quad \forall \alpha \right\}$$

$$= \bigcap_{n=1}^{\infty} \left\{ x \in E \mid f_n(x) < \alpha \right\} \subset M$$

$\therefore \sup(f_n)$ is a measurable fn.

$$iv) \text{ Use } \inf_n (f_n) = - \sup_n (-f_n)$$

v) Recall.

$$\limsup (f_n) = \inf_{n \geq 1} (\sup_{i \geq n} (f_i))$$

By (iii) $\sup(f_i)$ is measurable $\forall n \geq 1$

& by (iv) $\inf_{n \geq 1} (\sup_{i \geq n} (f_i))$ is measurable.

$$vi) \liminf (f_n) = - \limsup (-f_n)$$

$f_n \rightarrow f$ p.w except on a set of

$$\text{is } m(\{x / f_n(x) \neq f(x)\}) = 0$$

$$\frac{f}{g} \quad g(x) \neq 0 \quad \forall x \in E$$

$$\left\{ x \mid \frac{1}{g(x)} > x \right\}$$

$$= \left\{ x \mid \frac{1}{g(x)} > x \right\}$$

$$= \left\{ x \mid \frac{1}{x} > g(x) \right\} \text{ if } x > 0$$

$$= \left\{ x \mid \frac{1}{x} < g(x) \right\} \text{ if } x < 0$$

$$\left\{ x \mid \frac{1}{g(x)} > 0 \right\} = \emptyset \text{ or } E$$

or subset of E

Let $E \subseteq [0, 1]$ be a non measurable set

$$g: [0, 1] \rightarrow \mathbb{R}$$

$$g(x) = \begin{cases} 1 & x \in E \\ -1 & x \in [0, 1] \setminus E \end{cases}$$

g is not measurable.

$$\{x \mid g^{-1}(x) > \alpha\}$$

$$\subseteq \{x \mid g(g^{-1}(x)) \geq g(\alpha)$$

$$\{x \in g^{-1} \cup \{x \mid g(g^{-1}(x)) \leq g(\alpha)\}$$

monotonie der Funktionen und der Umkehrfunktion

Zeigt die Monotonie

$$2^{\alpha} < x < 2^{\beta} \text{ dann } \alpha < \beta$$

Für $x \in (2^{\alpha}, 2^{\beta})$ gilt $\alpha < \beta$

aus Monotonie

$\alpha < \beta \Rightarrow \alpha < g(\alpha) < g(\beta) < \beta$

Monotonie einer Funktion ist eine wichtige Voraussetzung um
die Existenz eines Extremwerts zu gewährleisten.

Bei einer monotonen Funktion kann es sich nur in der

Extremwerte und gleichzeitig auf

der Monotonie der Funktionen und der Umkehrfunktion

Zeigt die Monotonie

aus Monotonie der Funktionen und der Umkehrfunktion

28.1.19

f, g measurable & $g(x) \neq 0 \quad \forall x \in E$ on E where
 E is measurable then f/g is measurable.

Proof: Let $\alpha \in \mathbb{R}$

$$\left\{ x \in E \mid \frac{f(x)}{g(x)} > \alpha \right\} = \left\{ x \in E \mid f(x) > \alpha g(x) \right\}$$

Let $\{r_1, r_2, \dots\}$ be the set of all rational numbers in E

$$= \bigcup_{i=1}^{\infty} \left\{ x \in E \mid f(x) > r_i > \alpha g(x) \right\}$$

$$= \bigcup_{i=1}^{\infty} \left\{ x \in E \mid f(x) > r_i \right\} \cap \left\{ x \in E \mid \alpha g(x) < r_i \right\}$$

\nwarrow measurable

$\therefore f/g$ is measurable.

Defn: We say a function is defined on a measurable set is said to be Borel measurable or Borel function if for all $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) > \alpha\}$ is a Borel set.

Borel set.

Theorem:- Let $f: E \rightarrow \mathbb{R}$ is a function

The following are equivalent:

- ① f is Borel measurable
- ② for all $\alpha \in \mathbb{R}$, $\{x \in E \mid f(x) > \alpha\}$ is Borel set

is Borel measurable (Borel set)

Brief: Ex

Theorem

rem
 Let $f, g : E \rightarrow \mathbb{R}$ be Borel measurable functions where
 E is a measurable set. Let $c \in \mathbb{R}$. Then c.f.
 $f \pm g, fg, f+c$ are all Borel measurable functions.

Proof Ex

Theorem: Let $f_n: E \rightarrow \mathbb{R}$ be a sequence of Borel measurable functions. Then $\sup_{1 \leq i \leq n} (f_i)$, $\inf_{1 \leq i \leq n} (f_i)$, $\sup_n (f_n)$, $\inf_n (f_n)$, $\limsup_n (f_n)$, $\liminf_n (f_n)$ are Borel measurable functions.

Proof. Ex.

Defn.: If a property holds except on a set of measure zero, we say that it holds "almost everywhere" simply written as a.e.

$$\begin{aligned}
 \underline{\text{Ex.:-}} \quad A &= [0, 1] \\
 B &= [0, 1] \cup \{2\} \\
 B \setminus A &= \{2\} \\
 m(B \setminus A) &= 0 \\
 A \setminus B &= \emptyset
 \end{aligned}$$

$$f(B) \rightarrow B, \quad g: B \rightarrow B$$

$$f(x) = \begin{cases} \sin x & \text{if } x \neq 2 \\ 3 & \text{if } x = 2. \end{cases}$$

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 2 \\ 2 & \text{if } x = 2. \end{cases}$$

$$\Rightarrow f = g \quad \text{a.e.}$$

$$\{x \in E \mid f(x) \neq g(x)\} = \{2\}$$

which has measure zero $\Rightarrow f = g$ a.e.

Proposition: Let $f: E \rightarrow \mathbb{R}$ be measurable & let $g: E \rightarrow \mathbb{R}$ be any map such that $f = g$ a.e. Then g is also measurable.

Proof: Recall: Suppose F is measurable & $m^*(F \Delta G_1) = 0$ for some set G_1 . Then G_1 is measurable. Let $\alpha \in \mathbb{R}$.

$$F = \{x \in E \mid f(x) > \alpha\}$$

$$G_1 = \{x \in E \mid g(x) > \alpha\}$$

To show. G_1 is measurable set.

$$\begin{aligned} F \Delta G_1 &= (F \setminus G_1) \cup (G_1 \setminus F) \\ &= \{x \in E \mid f(x) > \alpha \text{ & } g(x) \leq \alpha\} \cup \\ &\quad \{x \in E \mid g(x) > \alpha \text{ & } f(x) \leq \alpha\} \\ &\subseteq \{x \in E \mid f(x) \neq g(x)\} \end{aligned}$$

By assumption $m(\{x \in E \mid f(x) \neq g(x)\}) = 0$

$$m^*(F \Delta G_1) = 0 \Rightarrow G_1 \text{ is measurable.}$$

Corollary.

Let $f_n: E \rightarrow \mathbb{R}$ be a measurable functions & $\{f_n\}$ converges to f a.e. Then f is measurable.

Proof: $f_n \rightarrow f$ a.e.
i.e. $\{x \in E \mid f_n \rightarrow f(x)\}$ has measure zero

$$f = \limsup(f_n) \text{ a.e.}$$

We know $\limsup f_n$ is measurable.
 $\Rightarrow f$ is measurable (my above prop).

FOC

30.1.19

Let $f: E \rightarrow \mathbb{R}$ be a measurable function.

$$\textcircled{1} f^+ = \max\{f, 0\}$$

\textcircled{2} $f^- = -\min\{f, 0\}$ are measurable.

Proposition

The set of points to which a sequence of measurable functions $\{f_n\}$ converges, is measurable
i.e. $\{x \in E \mid f_n(x) \text{ converges}\}$ is a measurable

set.

Proof

$$\{x \in E \mid f_n(x) \text{ converges}\}$$

$$= \{x \in E \mid \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}$$

$$= \{x \in E \mid \limsup_n f_n(x) - \liminf_n f_n(x) = 0\}$$

Since $\limsup_n f_n$ and $\liminf_n f_n$ are measurable \Rightarrow the above set is also measurable.

Defⁿ

Let f be a measurable function defined on a measurable set E , then the essential supremum of f is defined as

$$\text{ess sup}(f) = \inf \{\alpha \in \mathbb{R} \mid f \leq \alpha \text{ a.e.}\}$$

where $f \leq \alpha$ a.e.

② $\chi_{[0,1]}$

$$\text{ess sup}(\chi_{[0,1]}) = \inf \{\alpha \mid \chi_{[0,1]} \leq \alpha \text{ a.e.}\} = 1$$

Suppose $\alpha_0 < 1$

$$\inf \{\alpha \mid \chi_{[0,1]} \leq \alpha \text{ a.e.}\}$$

$$\begin{aligned} m(\{x \in \mathbb{R} \mid \chi_{[0,1]}(x) > \alpha_0\}) \\ = m([0,1]) = 1 \neq 0. \end{aligned}$$

Propⁿ
Let $f: E \rightarrow \mathbb{R}$ ~~not~~ be measurable.
 $f \leq \text{ess sup}(f)$ a.e.

Proof If $\text{ess sup}(f) = +\infty$ Then we are done.

Suppose $\text{ess sup}(f) = -\infty$
For each $n \in \mathbb{Z}$, $f \leq n$ a.e. (by def)

& we have nothing to prove.

Assume $\text{ess sup}(f)$ is a finite number.

Let $E_n = \{x \in E \mid f(x) > \frac{1}{n} + \text{ess sup}(f)\}$

& $A = \{x \in E \mid f(x) > \text{ess sup}(f)\}$

Clearly $E_n \subseteq A \quad \forall n \geq 1$.

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq A.$$

$$f(x) > \text{ess sup}_{\mathbb{R}}$$

$$\Rightarrow \exists n \text{ s.t. } f(x) > \frac{1}{n} + \text{ess sup}(f)$$

$$\Rightarrow x \in E_n \exists n.$$

$$\therefore A \subseteq \bigcup_{n=1}^{\infty} E_n$$

$$\Rightarrow A = \bigcup_{n=1}^{\infty} E_n$$

By defn $\text{ess sup}(f_n) = \inf \{ \alpha \mid f \leq \alpha \text{ a.e.} \}$

$$m^*(E_n) = m^*\left(\{x \in E \mid f(x) \notin \frac{1}{n} + \text{ess sup}(f)\}\right)$$

To show $m^*(E_n) = 0$

~~Def~~ $\text{ess sup}(f) \leq \alpha = \text{ess sup}(f) + \frac{1}{n}$ (let)

The $f \leq \alpha$ a.e. by defn

$$\Rightarrow m(\{x \in E \mid f(x) > \alpha\})$$

$$\rightarrow m(E_n) = 0.$$

$$m^*(A) = m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n) = 0.$$

$$\therefore m^*(A) = 0$$

$$f \leq \text{ess sup } f \text{ a.e.}$$

Propn Let $f, g : E \rightarrow \mathbb{R}$ be measurable fns.

Then $\text{ess sup}(f+g) \leq \text{ess sup}(f) + \text{ess sup}(g)$ a.e.

Proof

From above

$$f+g \leq \text{ess sup}(f) + \text{ess sup}(g) \text{ a.e.}$$

$$\Rightarrow \text{ess sup}(f+g) \leq \text{ess sup}(f) + \text{ess sup}(g) \text{ a.e.}$$

Example

$$\text{Let } f = \chi_{[-1, 0]} - \chi_{[0, 1]}$$

$$\& g = -f$$

Then

$$\text{ess sup}(f+g) = 0$$

$$\text{ess sup}(f) = 1$$

$$\text{ess sup}(g) = 1$$

$$\text{ess sup}(f) + \text{ess sup}(g) = 2 \neq 0$$

Defn

Let $f : E \rightarrow \mathbb{R}$ be a measurable fn. Then the essential infimum of f is defined as:

$$\text{ess inf}(f) = \sup \{ \alpha \in \mathbb{R} \mid f \geq \alpha \text{ a.e.} \}$$

Remark $\text{ess sup}(f) = -\text{ess inf}(-f)$

$$\begin{aligned}
 \text{Pf: } \text{ess sup}(f) &= \inf \{ \alpha \in \mathbb{R} \mid f \leq \alpha \text{ a.e.} \} \\
 &= \inf \{ \alpha \in \mathbb{R} \mid -f \geq -\alpha \text{ a.e.} \} \\
 &= -\sup \{ -\alpha \mid -f \geq -\alpha \text{ a.e.} \} \\
 &= -\sup \{ f \in \mathbb{R} \mid -f \geq \beta \text{ a.e.} \} \\
 &= -\text{ess sup}(-f)
 \end{aligned}$$

Defn

Let
We say a measurable function $f: E \rightarrow \mathbb{R}$ is

essentially bounded, if
 $\text{ess sup}(|f|) < \infty$

Proposition

Let $f: E \rightarrow \mathbb{R}$ be a measurable $f^{-1}(B)$ be a

Borel set. Then $f^{-1}(B)$ is measurable.

Proof: We have for any $A_i \subseteq \mathbb{R} \forall i \geq 1$

$$(i) f^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$$

$$\text{& (ii)} f^{-1}(A_i^c) = (f^{-1}(A_i))^c$$

To prove

Let $x \in \text{LHS}$.

$$\Rightarrow f(x) \in \bigcup_{i=1}^{\infty} A_i$$

$$\Rightarrow f(x) \in A_i \exists i$$

$$\Rightarrow x \in f^{-1}(A_i)$$

$$\Rightarrow x \in \text{RHS.}$$

Let $x \in \text{RHS}$

$$\Rightarrow x \in f^{-1}(A_i) \exists i$$

$$\Rightarrow f(x) \in A_i \subseteq \bigcup_{j=1}^{\infty} A_j$$

$$\Rightarrow x \in f^{-1}(\bigcup_{j=1}^{\infty} A_j)$$

(i) $x \in \text{LHS}$

$$\rightarrow f(x) \in A_i^c$$

$\Rightarrow f(x) \notin A_i$ (since $A_i \subset A_i^c$) $\Rightarrow x \notin f^{-1}(A_i)$

$$\left\{ \text{Therefore} \right. \rightarrow x \notin f^{-1}(A_i) \left. \Rightarrow x \in (f^{-1}(A_i))^c = \text{RHS} \right.$$

Let $x \in \text{RHS}$

($x \notin f^{-1}(A_i)$)

$$\Rightarrow f(x) \notin A_i$$

$$\Rightarrow f(x) \in A_i^c$$

$$\therefore x \in f^{-1}(A_i^c) = \text{LHS}$$

Then look at the class \mathcal{C} = sets whose preimage under f is measurable.

$\forall i \geq 1, b_i \in \mathbb{R}, A_i \in \mathcal{C}$

$$\text{Then } \bigcup A_i, A_i^c \in \mathcal{C}$$

$\therefore \mathcal{C}$ is a σ -algebra

($f^{-1}(\mathbb{R})$ is measurable)

\mathcal{C} is a σ -algebra.

Also all intervals belong to \mathcal{C} .

$$\Rightarrow \mathcal{B} \subseteq \mathcal{C}$$

Borel σ -algebra

$\Rightarrow f^{-1}(\mathcal{B})$ is measurable \forall Borel in \mathcal{B} .

31.1.19

Theorem:-

Let $E \subseteq \mathbb{R}$ is a measurable set. Then for any $y \in \mathbb{R}$, The set $E+y = \{x+y \mid x \in E\}$ is measurable & $m(E) = m(E+y)$.

Proof

Given $\epsilon > 0$, there exists an open set $U \subseteq \mathbb{R}$ such that $U \supseteq E$ & $m(U \setminus E) < \epsilon$ (E is measurable).

Let $y \in \mathbb{R}$

Look at $U+y \subseteq \mathbb{R}$ which is open set

$$\text{&} U+y \supseteq E+y$$

$$\text{Now, } (U+y) \setminus (E+y) = (U \setminus E)+y$$

$$\Rightarrow m^*((U \setminus E)+y) = m^*(U \setminus E) = m(U \setminus E) < \epsilon$$

By one of equivalent condition we get $E+y$ is measurable.

$$\begin{aligned} m(E+y) &= m^*(E+y) \\ &= m^*(E) \\ &= m(E). \end{aligned}$$

Theorem: \exists a non measurable set.

Proof

Let $x, y \in [0, 1]$

Define $x \sim y$ if $y - x \in \mathbb{Q} \cap [-1, 1]$

\sim is an equivalence relation.

$x \sim x$ [$x - x = 0 \in \mathbb{Q} \cap [-1, 1]$]

$x \sim y \Rightarrow y - x \in \mathbb{Q} \cap [-1, 1]$

$\Rightarrow x - y \in \mathbb{Q} \cap [-1, 1]$

$\Rightarrow y \sim x$

$x \sim y, y \sim z$

$\Rightarrow y - x, z - y \in \mathbb{Q} \cap [-1, 1]$

$\Rightarrow y - x + z - y \in \mathbb{Q} \cap [-1, 1]$

$\Rightarrow z - x \in \mathbb{Q} \cap [-1, 1]$

$\Rightarrow x \sim z$

$\therefore \sim$ is an equivalence relation

$$[0, 1] = \bigcup_{\alpha} E_{\alpha}$$

where E_{α} is an equivalence relation \sim

Suppose $x, y \in E_{\alpha} \Rightarrow x \sim y \Rightarrow y \sim x \in \mathbb{Q} \cap [-1, 1]$

$\Rightarrow E_{\alpha}$ is countable for some $x \in E_{\alpha}$.

$$y \in [x]$$

$$y \sim x$$

Thus each E_{α} is a countable set.

Since $[0, 1]$ is uncountable & every E_α is countable therefore there are uncountably many ~~equat~~^{equivalence} classes.

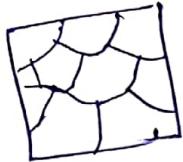
Recall - Axiom of choice:

Let X be a nonempty set. Suppose $\{E_\alpha \mid \alpha \in I\}$ is a nonempty collection of nonempty disjoint subsets of X .

Then there exists a set $V \subseteq X$ containing just one element from each E_α .

By axiom of choice.

There exists a set $V \subseteq [0, 1]$ containing exactly one element x_α from each E_α such that $V = \{x_\alpha \mid x_\alpha \in E_\alpha\} \subseteq [0, 1]$



Claim :

V is not measurable.

Pf: Let $\varnothing \cap [-1, 1] = \{r_1, r_2, \dots\}$

Denote $V_n = V + r_n \quad \forall n \geq 1$.

To show $V_n \cap V_m = \varnothing \quad \forall n \neq m$, suppose

Suppose $x \in V_n \cap V_m$

$$\Rightarrow x = v_1 + r_n \quad \text{for some } v_1 \in V$$

$$= v_2 + r_m \quad \text{for some } v_2 \in V$$

$$v_2 - v_1 = r_m - r_n \in \varnothing \cap [-1, 1]$$

$$\Rightarrow v_1 \sim v_2$$

$$\Rightarrow v_1 = v_2 \quad (\text{By Def of } V)$$

$$\cancel{\Rightarrow v_1 = v_2}$$

$$\Rightarrow r_m = r_n$$

$$\Rightarrow m = n$$

Thus if $m \neq n \Rightarrow V_n \cap V_m = \varnothing$

~~Prove~~

Now $[0, 1] \subseteq \bigcup_{n=1}^{\infty} V_n$. $\therefore [0, 1] = \bigcup_{n=1}^{\infty} V_n$

$$\begin{aligned} \varnothing &= \bigcup_{\alpha} E_{\alpha} \\ &= \bigcup_{\alpha} [x_{\alpha}, z_{\alpha}] \end{aligned}$$

$$x \notin [0, 1]$$

$$\Rightarrow x \in E_{\alpha} \text{ from d}$$

$$\begin{aligned} x &\sim x_{\alpha} \\ x &= x_{\alpha} + r_n \text{ for some } n \in V_n. \end{aligned}$$

Then $\forall z \in [0, 1]$

Then $[0, 1] \subseteq \bigcup_{n=1}^{\infty} V_n$

Also $\bigcup_{n=1}^{\infty} V_n \subseteq [-1, 2]$ [By Axiom of Choice?]

Suppose V is measurable. Then by above, then each $V_n = V + Y_n$ is measurable & $m(V_n) = m(V)$ $\forall n \geq 1$.

Look at $[0, 1] \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq [-1, 2]$

$$\Rightarrow m([0, 1]) \leq m\left(\bigcup_{n=1}^{\infty} V_n\right) \leq m([-1, 2])$$

$$\sum_{n=1}^{\infty} m(V_n)$$

$$\Rightarrow 1 \leq \underbrace{\sum_{n=1}^{\infty} m(V)}_{0 \text{ or } +\infty} \leq 3.$$

The possible value for $\sum m(V)$

either 0 or ∞

which gives a contradiction

V is not measurable.

Q.E.D.

4.2.19

Theorem -

There exists a measurable set which is not a

Borel Set i.e., $B \not\subseteq M$

Proof

For $x \in [0, 1]$

$$x = \sum_{n=1}^{\infty} \frac{e_n}{2^n} \quad (\text{Binary exp})$$

$e_n = 0 \text{ or } 1$.

Define a function

$$f : [0, 1] \rightarrow \mathbb{R}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2e_n}{3^n} \quad \text{when } x = \sum_{n=1}^{\infty} \frac{e_n}{2^n} \in [0, 1]$$

Note $\sum_{n=1}^{\infty} \frac{2e_n}{3^n} \in P$, the Cantor set

$f(x)$ is well defined because,
 x is uniquely defined determines
 e_n 's and conversely

& f is an injective map.
(or 1-1 map)

$$\& \text{Im}(f) \subseteq P$$

$$f([0, 1]) = \text{Im}(f) \subseteq P$$

f is measurable

(Since each e_n is a measurable function
of x , i.e.)

Suppose $f_n : [0, 1] \rightarrow \mathbb{R}$

$x \in [0, 1]$ with $x = 0.x_1 x_2 \dots$ decimal exp.

Defined $f_n(x) = x_n (\forall n \geq 1)$ then
each f_n is measurable (Ex))

Let $V \subseteq [0,1]$ be a non-measurable set

Look at $f(V) \subseteq P$

Let $U = f(V)$

Claim: U is measurable & ~~$f^{-1}(U)$ is V~~ is not a Borel set

Proof: Since $m^*(P) = 0$ therefore

$m^*(U) = 0$ & hence U is measurable

We know

We know preimage of a Borel set is measurable under a measurable function

Look at $f^{-1}(U) = V$ which is not measurable
 $\Rightarrow U$ is not a Borel set

$U \notin \mathcal{B}$

& $U \in \mathcal{M}$

Proposition

Suppose $f: E \rightarrow \mathbb{R}$ is a monotonically increasing function then f is measurable.

Proof:

Monotonic function: ($\text{If } x < y \text{ in } E, \text{ then } f(x) \leq f(y)$)

To show: Let $\alpha \in \mathbb{R}$

$\{x \in E \mid f(x) > \alpha\}$ is measurable

Consider: $\{x \in E \mid f(x) > \alpha\} = f^{-1}((\alpha, \infty)) \cap E$

If choose $x, y \in f^{-1}((\alpha, \infty))$

$$x < y \Rightarrow f(x) \leq f(y)$$

$$\Rightarrow \alpha < f(x) \leq f(y)$$

Consider $z \in [f(x), f(y)] \cap \text{Im}(E)$ for some $w \in E$

($\overline{\text{?}}$) Then $w \in \{x \in E \mid f(x) > \alpha\}$

$$f(w) \in [f(x), f(y)]$$

To show $w \in [x, y] \cap E \subseteq \{x \in E \mid f(x) > \alpha\}$

$$\Rightarrow x \leq w \leq y$$

$$\Rightarrow f(x) \leq f(w) \leq f(y)$$

$$\Rightarrow f(w) \in [f(x), f(y)]$$

$$\Rightarrow w \in \{x \in E \mid f(x) > \alpha\}$$

Then $\{x \in E \mid f(x) > \alpha\}$
= $I \cap E$ for some Int. $I \subset \mathbb{R}$
which is measurable

$\therefore f$ is measurable.

6.2.19.

Let $x \in [0, 1)$

Suppose $x = a_0 \cdot x_1 x_2 x_3 \dots$ (base d)

$$x = \frac{x_1}{d} + \frac{x_2}{d^2} + \frac{x_3}{d^3} + \dots$$

$$d^{n-1}x = (x_1 d^{n-2} + x_2 d^{n-3} + \dots + x_{n-1}) + \frac{x_n}{d} + \dots$$

$$\lfloor d^{n-1}x \rfloor = x_1 d^{n-2} + x_2 d^{n-3} + \dots + x_{n-1} \in \mathbb{Z}_+$$

$$d^{n-1}x - \lfloor d^{n-1}x \rfloor = \frac{x_n}{d} + \frac{x_{n+1}}{d^2} + \dots$$

$$d(d^{n-1}x - \lfloor d^{n-1}x \rfloor) = x_n + \frac{x_{n+1}}{d} + \dots$$

$$\lfloor d(d^{n-1}x - \lfloor d^{n-1}x \rfloor) \rfloor = x_n \quad \forall n \geq 1$$

To show f_n is a measurable function.

+ Consider $g: \mathbb{R} \rightarrow \mathbb{R}$
 $g(x) = \lfloor x \rfloor$ is measurable (monotonic)
~~monotonically increasing~~ $\Rightarrow f_n$ is measurable.

Definition:
A non-negative finite valued function taking only a finite number of different values is called

a simple function.

i.e. If $a_1, a_2, \dots, a_n \in \mathbb{R}$, distinct values

taken by a fn. $\phi: \mathbb{R} \rightarrow \mathbb{R}$

$A_i = \{x \in \mathbb{R} \mid \phi(x) = a_i\}$

~~$\mathbb{R} \rightarrow \mathbb{R}$~~ $\mathbb{R} \rightarrow \mathbb{R}$.

$$\varphi = \sum_{i=1}^{\infty} a_i \chi_{A_i} \quad \text{is } a_i \geq 0 \quad \forall i=1, 2, \dots n.$$

$$\varphi(x) = \begin{cases} a_i & \text{if } x \in A_i \\ 0 & \dots \end{cases} \quad \forall i=1, 2, \dots$$

Example:

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{else} \end{cases}$$

Step f^n

$$f = \chi_{[0, 1]} + \chi_{[1, 2]} + \chi_{[2, 3]}$$

f is simple

Note $\varphi = \sum a_i \chi_{A_i}$ is measurable if each A_i is measurable.

Defn: Let $\varphi = \sum a_i \chi_{A_i}$ be a simple function & φ is a measurable function. Then the Lebesgue integral of φ is defined as

$$\int \varphi d\mu = \sum_{i=1}^{\infty} a_i m(A_i)$$

Ques: Suppose $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ is a simple fn.

Then $A_i \cap A_j = \emptyset \quad \forall i \neq j$

$$A_i = \{x \in \mathbb{R} \mid \varphi(x) = a_i\}$$

$$A_j = \{x \in \mathbb{R} \mid \varphi(x) = a_j\}$$

$$\& A_1 \cup A_2 \cup \dots \cup A_n = \mathbb{R}$$

Defⁿ For any non negative measurable function $f: E \rightarrow \mathbb{R}$
the Lebesgue integral of f is defined as

$$\int f dx = \sup_{\varphi \in F} \left(\int \varphi dx \right)$$

φ is simple
measurable fn

7.2.19

1. Show that every countable set has outer measure

$$\text{Let } A = \{a_i | i \in \mathbb{N}\}$$

$$m^*(\{a_i\}) = 0 \quad \forall i \in \mathbb{N}$$

$$A = \bigcup_{i=1}^{\infty} \{a_i\}$$

$$m^*(A) \leq \sum_{i=1}^{\infty} m^*(\{a_i\})$$

$$= 0$$

2. Let I_1, \dots, I_n be intervals such that

$$Q \cap [0, 1] \subset I_1 \cup I_2 \cup \dots \cup I_n$$

Show that $\ell(I_1) + \ell(I_2) + \dots + \ell(I_n) \geq 1$.

Let $J = [a_i, b_i]$

WLOG assume

$$a_1 < a_2 < \dots < a_n.$$

Consider $[a_1, b_1] \times [a_2, b_2]$

Claim $a_2 \leq b_1$

Suppose $b_1 < a_2$

$$(b_1, a_2) \cap Q \neq \emptyset$$

Let $y \in (b_1, a_2)$

$y \notin [a_1, b_1]$

$$a_1 < a_2$$

$\nexists [a_i, b_i] \text{ for any } i \in \{1, 2, \dots, n\}$

~~$a_2 < b_1$~~

$$g \notin \bigcup_{j=1}^n I_j$$

which is a contradiction.

$$\text{So } a_2 \leq b_1$$

Similarly, for any $j \in \{1, \dots, n-1\}$

$$a_{j+1} \leq b_j$$

$$\text{Let } x \in [0, 1]$$

Let i be the largest of $\{1, \dots, n\}$ such that

$$a_i \leq x$$

Since i is a rational number

$i \in [a_i, b_i]$ for some

$$a_i \leq b_i \leq \dots \leq b_n$$

If $i = n$

$$\text{Then } a_n \leq x \leq b_n$$

$$x \in [a_n, b_n]$$

If $i < n$

$$\text{Then } a_i \leq x < a_{i+1} \leq b_i$$

$$\text{So } x \in [a_i, b_i]$$

$$\text{Then } x \in \bigcup_{j=1}^n I_j$$

$$[0, 1] \subset \bigcup_{j=1}^n I_j$$

$$m(\mathbb{Q} \cap [0, 1]) = m(\mathbb{Q}) - m(\mathbb{Q} \setminus [0, 1])$$

$$1 = L(0, 1) \leq \sum_{j=1}^n L(I_j)$$

$$\begin{aligned} m(\mathbb{Q}) \\ = m\mathbb{Q} - (0, 1) \\ + m\mathbb{Q} \cap [0, 1] \end{aligned}$$

Theorem

If $\{E_i\}$ collection of measurable sets

$$\text{set } E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

$$m(E_i) < \infty \text{ for all } i$$

$$\text{Then } m(\bigcap E_i) = \lim_{i \rightarrow \infty} m(E_i)$$

Show that the above holds if $m(E_i) < \infty$ then

$$m^*(E_n) < \infty \forall n \geq 1$$

Consider

$$E_i \supseteq E_{i+1} \supseteq E_{i+2} \supseteq \dots$$

$$m(\bigcap_{n=1}^{\infty} E_n) = \lim_{m \rightarrow \infty} m(E_{i+m})$$

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=i}^{\infty} E_n$$

$$m(\bigcap_{n=i}^{\infty} E_n) = m(\bigcap_{n=1}^{\infty} E_n)$$

$$= \lim_{n \rightarrow \infty} m(E_{i+n})$$

$$= \lim_{n \rightarrow \infty} m(E_n)$$

The theorem does not hold if $m(E_i) = \infty \forall i$

Example

$$E_i = [i, \infty) \quad \forall i$$

$$m(E_i) = \infty \quad \forall i$$

$$\bigcap_{i=1}^{\infty} E_i = \emptyset$$

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = 0$$



In
ZFC

11.2.19

Outer measure

$$A \subseteq \mathbb{R}$$

$$m^*(A) = \inf \left\{ \sum_n l(I_n) \mid \begin{array}{l} \{I_n\} \text{ countable collection of} \\ \text{intervals and } A \subseteq \bigcup_n I_n \end{array} \right\}$$

Suppose D is a dense subset of \mathbb{R}
Then $m_D^*(A) = \inf \left\{ \sum_n l(I_n) \mid \begin{array}{l} \{I_n\} \text{ is a finite / countable} \\ \text{collection of intervals with} \\ \text{end pts. in } D \text{ and } A \subseteq \bigcup_n I_n \end{array} \right\}$

$$\text{Show } m^*(A) = m_D^*(A)$$

Note second sets $\bigcup I_n$ contained in first
Then their infimums also have same relation

It is obvious

$$m_D^*(A) \geq m^*(A)$$

$$\text{To show: } m^*(A) \leq m_D^*(A)$$

Let $\{I_n\}$ be a finite or countable collection
of intervals in \mathbb{R} . Let $I_n = [a_n, b_n]$

$$\overline{D} = \mathbb{R}$$

$$(a - \frac{\epsilon}{2^{n+1}}, b_n) \cap D \neq \emptyset$$

$$\text{and } (b_n, b_n + \frac{\epsilon}{2^{n+1}}) \cap D \neq \emptyset$$

$$\text{Let } p_n \in (a_n - \frac{\epsilon}{2^{n+1}}, a_n) \cap D \quad \forall n.$$

$$q_n \in (b_n, b_n + \frac{\epsilon}{2^{n+1}}) \cap D \quad \forall n$$

$$a_n - p_n < \frac{\epsilon}{2^{n+1}}$$

$$q_n - b_n < \frac{\epsilon}{2^{n+1}}$$

Let $J_n = [p_n, q_n]$

$$\begin{aligned}l(J_n) &= q_n - p_n \\&= q_n - b_n + b_n - a_n + a_n - p_n \\&< \frac{\varepsilon}{2^{n+1}} + l(I_n) + \frac{\varepsilon}{2^{n+1}},\end{aligned}$$

$$l(J_n) < l(I_n) + \frac{\varepsilon}{2^n}.$$

$$\sum l(J_n) < \sum l(I_n) + \varepsilon.$$

$$m_D^*(A) \leq \sum l(I_n) + \varepsilon.$$

$$\begin{aligned}m_D^*(A) - \varepsilon &\leq \sum l(I_n) \\&\leq m^*(A)\end{aligned}$$

$$m_D^*(A) - m^*(A) \leq 0$$

$$m_D^*(A) \leq m^*(A)$$

Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be measurable

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous

Then $f \circ g$ is measurable

Proof: Let $\alpha \in \mathbb{R}$

$$\{x \in \mathbb{R} | (f \circ g)(x) > \alpha\} = (f \circ g)^{-1}(\alpha, \infty) = g^{-1}(f^{-1}(\alpha, \infty))$$

f is continuous $\Rightarrow f^{-1}(\alpha, \infty)$ is open which is
Borel set (measurable). Also g is measurable
 $g^{-1}(f^{-1}(\alpha, \infty))$ is measurable

Thm

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous

Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be measurable

and $f = g$ a.e. Then

$$\text{ess sup}(g) = \sup(f)$$

$$m(\{x \in \mathbb{R} \mid \alpha < f(x)\}) = 0$$

If $m(\{x \in \mathbb{R} \mid \alpha < f(x)\}) = 0$
Then we say α is an essential upper bound

for f . Let M_f be the set of all essential

upper bounds of f .

$$\text{ess sup } f = \inf M_f$$

$f = g$ a.e. f is continuous;

$$m(\{x \in \mathbb{R} \mid f(x) \neq g(x)\}) = 0$$

$$f(x) \leq \sup(f) \quad \forall x$$

$$\text{sup}(f) < g(x)$$

Let $x \in \mathbb{R}$ suppose

$$\text{sup}(f) < g(x)$$

Then $f(x) < g(x)$

$$\begin{aligned} \{x \in \mathbb{R} \mid \sup(f) < g(x)\} &\subseteq \{x \in \mathbb{R} \mid f(x) < g(x)\} \\ &\subseteq \{x \in \mathbb{R} \mid f(x) \neq g(x)\} \end{aligned}$$

$$m(\{x \in \mathbb{R} \mid f(x) \neq g(x)\}) = 0$$

$$\Rightarrow \sup(f) \in M_g$$

Next step: prove

$$\sup(f) \leq \alpha$$

get a contradiction.

$\forall \alpha \in M_g, \alpha \in \text{sup}(f)$