HOMEWORK-10

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(1.) Prove that 3SAT (15/16) is NP-Complete.

To prove it is NP-complete, we will prove it is NP first and also NP-hard next.

1. NP: We can count the number of clauses that are satisfied given a truth value assignment and compare it to the value of $\left(\frac{15}{16}\right)k$. This could tell us if it is NP.

2. NP-hard to be proved: $3\text{-SAT} \leq_p 3\text{SAT}(15/16)$.

In order to prove this, we take 8 original clauses along with 8 new clauses formed using 3 new variables and then construct clauses on all possible collection of clauses.

We have two cases now.

Case 1: Number of clauses is a multiple of 8:
We must satisfy all of the original clauses as 7/8 of the new clauses would be satisfied. This would give us that 15/16 clauses are satisfied.

Case 2: Number of clauses is not multiple of 8:
In this case we would satisfy atleast 15/16 clauses. Even if one of the original clauses is not satisfied, then (15/16) clauses would not be satisfied.

Hence, considering the above 2 cases, 3-SAT (15/16) is NP-complete.

NP $\cap$ NP-hard

NP-complete

2: We have the independent set problem and we prove that it can be reduced polynomially to dense subgraph problem.

Independent set problem $\leq_p$ Dense subgraph problem.

Given a graph $G(V, E)$ and an integer K, we use the independent set decision problem and it gives true if there is an independent set of size K and false otherwise.

We now use the concept of clique. A clique is a subset of vertices of an undirected graph G such that every two distinct vertices in the clique are adjacent.

According to the property of clique, if there are k vertices in G, then clique will always contain $\frac{k*(k-1)}{2}$ edges.

An independent set in G is a clique in G_c and vice-versa.

$$m = \frac{k*(k-1)}{2}$$

Claim: There exists an independent set of size k in G iff there exists a subgraph of G_c with at most k vertices and at least $m = \frac{k(k-1)}{2}$

If there exists a clique in G_c of size atleast k, then there exists a subgraph of G_c with atmost k vertices and atleast $\frac{k*(k-1)}{2}$.

If there is a clique of size atleast ~~size~~ k then there is a clique of size ~~atleast~~ exactly k and $\frac{k(k-1)}{2}$ edges.

According to this claim, we can say independent set problem can be polynomially reducible to Dense subgraph problem.

Hence, Dense subgraph problem is NP-complete.

3. Q: SAT' is NP-Complete.

(1) Show SAT' is NP:

(2) SAT $\leq_p$ SAT'

1. When we are given the assignment of values, we could verify the truthness by evaluating SAT'
we can use a linear time algorithm to count the number of satisfied clauses and check if it equal to $m-2$.

2. To show SAT $\leq_p$ SAT' we add 4 clauses say $x_1, x_2, \overline{x_1}, \overline{x_2}$. Now the SAT has $m' = m+2$ clauses.

We need to check whether $m'-2$ are satisfied exactly for SAT'
This will for sure happens are there wont be any possible truth assignment that satisfies x_1 & $\overline{x_1}$ to be true and x_2 and $\overline{x_2}$ to be true.

So there will be for sure $m'-2$ clauses set to true.
Thus, SAT is reduced in polynomial time to SAT'.

$\Rightarrow$ SAT' is NP & SAT' is NP hard.

$\rightarrow$ SAT' is NP-complete.

④ To show that vertex cover is NP-complete with even degree for all vertices.

1) To have a polynomial length certification which is same as vertex cover certification.

To have a polynomial time ~~certification~~ certifier, we can check whether all the vertices in graph have even degree in polynomial time.

2) We choose vertex cover

3) We need to prove that vertex cover $\leq_p$ vertex cover with even degree.

We add an edge to each vertex which have odd degree & connect all these new edges to a new vertex v'. As sum of degree vertices in a graph is always even, so no. of odd degree vertices should be even but to include v' we need to have two or more vertices that form a triangle with v' & let them be v_1' & v_2'. So the new graph has $V+2$ vertices & new graph's vertex cover will include (v', v_1') or (v_1', v_2') or (v_2', v') otherwise one of the edges in triangle will be excluded. So vertex cover of new graph will always contain either of these choices & if doesn't contain v' then we can replace v_1' or v_2' with v' & form a vertex cover of size $V+2$. Thus vertex cover of size V is reduced in polynomial time to vertex cover of size $V+2$. As new vertex cover is NP & NP-hard.

→ vertex cover with even-degree is NP-complete.