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**CSCI 570**

# Homework 1

**1. Solve Kleinberg and Tardos, Chapter 1, Exercise 1. (5pts)**

**Question:** In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

**Solution/Answer:** <u>FALSE</u>

I would like to prove that this statement is false by using a counter example. Let us consider a simple example of 'n' men and 'n' women with n=2, (men: m1 and m2 & women: w1 and w2).

<u>Preference lists:</u>

**Men preferences:**

| | | |
|---|---|---|
| **m1** | w1 | w2 |
| **m2** | w2 | w1 |

**Women preferences:**

| | | |
|---|---|---|
| **w1** | m2 | m1 |
| **w2** | m1 | m2 |

By using the G-S algorithm to find the stable matching, we could get the following:

i) m1 proposes w1 and gets accepted. (m1 → w1)

ii) m2 proposes w2 and gets accepted (m2 → w2)

From the above two, we get the stable matching as (m1, w1) and (m2, w2).

As per the above example, we could find that m is not ranked first on the preference list of w and w is not ranked first on the preference list of m. It is not always true that there would be stable matching only if the preference list is as per the given statement. From this it can be proved that the above given statement is False.

**2. Solve Kleinberg and Tardos, Chapter 1, Exercise 2. (5pts)**

**Question:** Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.

**Solution/Answer:** <u>TRUE</u>

<u>Preference lists:</u>

**Men preferences:**

| | | |
|---|---|---|
| **m1** | w1 | w2 |
| **m2** | w1 | w2 |

**Women preferences:**

| | | |
|---|---|---|
| **w1** | m2 | m1 |
| **w2** | m1 | m2 |

By using the G-S algorithm to find the stable matching, we could get the following:

i) m1 proposes w1 and gets rejected.

ii) m1 proposes w2 and gets accepted. (m1 → w2)

iii) m2 proposes w1 and gets accepted (m2 → w1)

From the above three, we get the stable matching as (m1, w2) and (m2, w1).

This is the only possible stable matching possible for this scenario. If we consider the following matching instead of the matching given by G-S algorithm,

m1 → w1 **INSTABILITY**

m2 → w2

There is an instability. Hence, every stable matching pair should contain the pair (m, w) which belongs to S.

3. **Determine whether the following statement is true or false. If it is true, give an example. If it is false, give a short explanation. (5pts)**

   **For some $n \geq 2$, there exists a set of preferences for $n$ men and $n$ women such that in the stable matching returned by the G-S algorithm, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.**

**Solution/Answer:** <u>TRUE</u>
Let us consider n=3 (3 men- m1, m2 and m3 and 3 women- w1, w2 and w3). Their preference lists are given below.

<u>Preference lists:</u>

**Men preferences:**

| | | | |
|---|---|---|---|
| **m1** | w1 | w3 | w2 |
| **m2** | w2 | w1 | w3 |
| **m3** | w1 | w2 | w3 |

**Women preferences:**

| | | | |
|---|---|---|---|
| **w1** | m2 | m1 | m3 |
| **w2** | m3 | m2 | m1 |
| **w3** | m1 | m3 | m2 |

Applying the G-S algorithm,

1. m1 proposes w1 and gets accepted.
2. m2 proposes w2 and gets accepted.
3. m3 proposes w1 and gets rejected.
4. m3 proposes w2 and gets accepted. m2 becomes free.
5. m2 proposes w1 and gets accepted. m1 becomes free.
6. m1 proposes w3 and gets accepted.

After the G-S algorithm, the stable matching pairs formed are (m1, w3); (m2,w1); (m3,w2). From these pairs it is evident that the women got their first preference. i.e., their best valid partners.

Hence, the statement every woman is matched with their most preferred man, even though that man does not prefer that woman the most is True.

4. **Four students, $a$, $b$, c, and $d$ are rooming in a dormitory. Each student ranks the others in strict order of preference. A *roommate matching* is defined as a partition of the students into two groups of two roommates each. A roommate matching is *stable* if no two students who are not roommates prefer each other over their roommate.**

   **Does a stable roommate matching always exist? If yes, give a proof. Otherwise, give an example of roommate preferences where no stable roommate matching exists. (8pts)**

**Solution/Answer:** From this statement we can say that the **<u>stable matching always does not exist.</u>**
As per the question, a, b, c, and d have their own preferences. Let us say 'a' is of least priority in everyone's preferences. Then nobody would be interested in pairing up with 'a'. Even if there is a pair formed, there would always be an instability. This can be proved with an example below.

<u>Preference list:</u>

| | | | |
|---|---|---|---|
| **a** | b | c | d |
| **b** | c | d | a |
| **c** | d | b | a |
| **d** | b | c | a |

'a' is Least preferred

Now if the 4 roommates are divided into 2 pairs, say (a, b), (c, d); (a, c), (b, d); (a, d), (b, c) in every case there would be an instability as one prefers some other one over their assigned roommate. From this we can conclude that the stable matching does not always exist.

**5. Solve Kleinberg and Tardos, Chapter 1, Exercise 4. (15pts)**

**Question:** Gale and Shapley published their paper on the Stable Matching Problem in 1962; but a version of their algorithm had already been in use for ten years by the National Resident Matching Program, for the problem of assigning medical residents to hospitals.

Show that there is always a stable assignment of students to hospitals and give an algorithm to find one.

**Answer/Solution:** There is always a stable assignment of students to hospitals and stable matching always exists.

**Algorithm:**

**Let s represent a student and h represent a hospital.**

*Initially all $s \in S$ and $h \in H$ are free*

*While there is a hospital h with a vacant position to be admitted*

    *Choose such a hospital h*

    *Let s be the highest-ranked student in h's preference list*

    *If s is free then*

        *(h, s) become a match*

    *Else s is having an offer from h'*

        *If s prefers h' to h then*

            *h's position remains free*

        *Else s prefers h to h' then*

            *(h, s) become a match*

            *one position of h' becomes free*

        *Endif*

    *Endif*

*Endwhile*

*Return the set S of finally formed pairs*

In a hospital as there are greater than 0 positions available, after this algorithm runs, all the positions will be filled. Every hospital admits students and if there are any vacancies at any hospital, then student must have chosen the other one over this and the total positions are higher than the students count. But as per the given assumption all positions in hospital are occupied.

**First type of instability:** There are students s and s', and a hospital h, so that

– s is assigned to h, and

– s' is assigned to no hospital, and

– h prefers s' to s.

From these set of statements, s is assigned to h and s' is free. But h prefers s' to s and should have chosen s' before s. But this is a contradiction. This is not first type of instability.

**Second type of instability:** There are students s and s', and hospitals h and h', so that

– s is assigned to h, and

– s' is assigned to h', and

– h prefers s' to s, and

– s prefers h to h'

From these set of statements, s is assigned to h and s' to h'. As h prefers s' to s but it is assigned to s, h did not take s' when s' looked for h and later it took s. This tells us that s' was looking for h'. But this is clearly a contradiction. This is not second type of instability.

Hence, as there is no instability, a stable matching exists.

6. **Solve Kleinberg and Tardos, Chapter 1, Exercise 8. (10pts)**

**Question:** For this problem, we will explore the issue of truthfulness in the Stable Matching Problem and specifically in the Gale-Shapley algorithm. The basic question is: Can a man or a woman end up better off by lying about his or her preferences? More concretely, we suppose each participant has a true preference order. Now consider a woman w. Suppose w prefers man m to m' but both m and m' are low on her list of preferences. Can it be the case that by switching the order of m and m' on her list of preferences (i.e., by falsely claiming that she prefers m to m) and running the algorithm with this false preference list, w will end up with a man m'' that she truly prefers to both m and m'?

**Solution/Answer:** TRUE

Let us consider n=3 (3 men- m1, m2 and m3 and 3 women- w1, w2 and w3). Their preference lists are given below.

Preference lists:

**Men preferences:**

| | | | |
|---|---|---|---|
| **m1** | w3 (w3') | w1 | w2 |
| **m2** | w1 | w3 (w3') | w2 |
| **m3** | w1 | w3 (w3') | w2 |

**Women preferences:**

| | | | |
|---|---|---|---|
| **w1** | m1 | m2 | m3 |
| **w2** | m1 | m2 | m3 |
| **w3** | m2 | m1 | m3 |
| **w3'** | m2 | m3 | m1 |

Applying the G-S algorithm,

1. m1 proposes w3 and gets accepted. (m1→w3). w3 got her second preference.
2. m2 proposes w1 and gets accepted. (m2→w1)
3. m3 proposes w1 and gets rejected. He then proposes w3 and gets rejected again. Finally, he proposes w2 and gets accepted. (m3→w2)

Now, as per the statement if w3 changes the preference list, m3 gets second preference and m1 gets first preference.

Applying the G-S algorithm,

1. m1 proposes w3' and gets accepted.
2. m2 proposes w1 and gets accepted.
3. m3 proposes w1 and gets rejected. He then proposes w3' and gets accepted.
4. m1 proposes w1 and gets accepted. m2 is free now. (m1→w1)
5. m2 proposes w3' and gets accepted. m3 is free now. (m2→w3'). w3' got her first preference now.
6. m3 proposes w2 and gets accepted. (m3→w2)
7. w3 got her second preference.

From this we can conclude that w3' got her first preference when she changed her preference list. So when a women changed her preference list, she got her most valid/preferred partner.

7. Determine whether the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. (5pts)

For all $n \geq 2$, there exists a set of preferences for $n$ men and $n$ women such that in the stable matching returned by the G-S algorithm, every man is matched with their most preferred woman.

**Solution/Answer:** TRUE

This can be proved using an example mentioned below.

Preference lists:

**Men preferences:**

| | | | |
|---|---|---|---|
| **m1** | w1 | w3 | w2 |
| **m2** | w2 | w1 | w3 |
| **m3** | w3 | w2 | w1 |

**Women preferences:**

| | | | |
|---|---|---|---|
| **w1** | m2 | m1 | m3 |
| **w2** | m3 | m2 | m1 |
| **w3** | m1 | m3 | m2 |

Applying the G-S algorithm,

1. m1 proposes w1 and gets accepted. (m1→w1)
2. m2 proposes w2 and gets accepted. (m2→w2)
3. m3 proposes w3 and gets accepted. (m3→w3)

Considering the above example, we could say that every man is matched with their most preferred woman when each man has a different most preferred woman on their preference list. In this case every man is matched with their most preferred woman.

8. Consider a stable marriage problem where the set of men is given by $M = m_1, m_2, ..., m_N$ and the set of women is $W = w_1, w_2, ..., w_N$. Consider their preference lists to have the following properties:

$$\forall w_i \in W : w_i \text{ prefers } m_i \text{ over } m_j \quad \forall j > i$$

$$\forall m_i \in M : m_i \text{ prefers } w_i \text{ over } w_j \quad \forall j > i$$

Prove that a unique stable matching exists for this problem. Note: the $\forall$ symbol means "for all". (12pts)

**Solution/Answer:** A stable matching exists.

We could prove that a unique stable matching exists for this problem using the following proof.

i. As per the condition mentioned in the question each man, $m_i$ for i=1, $m_1$ will pair up with $w_i$ where i=1, $w_1$ only.
ii. Next for i=2, man $m_2$ has possibility of pairing up with $w_2$, $w_3$, … as $w_2$ is available, m2 prefers w2 as his partner.
iii. A man cannot be pair up with the previous woman as the position would be paired up with the person of same index. Also, a man cannot choose a woman from greater index.
iv. If $m_i$ is paired up with $w_j$ or if $m_j$ is paired up with $w_i$ then there would be an instability as each of them would choose one from same index.
v. Hence, a man $m_i$ would choose a woman $w_i$ only. A stable matching exists for this problem.