Utility

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Outline

1. Utility
2. Marginal Rate of Substitution (MRS)
3. References

Topic

1. Utility
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Subtopic

1. Utility
   - Consumer Preferences
   - Monotonic Transformations
   - Indifference Curves and Utility
   - Different Utility Functions

Utility and Preferences

Preferences allow us to rank bundles by choosing between two bundles. Now, however, we want to move towards finding the bundle that gives us the most satisfaction or pleasure.

To do this, we introduce utility. In essence, utility is the satisfaction derived from consuming a good or service.

We can construct a utility function to compare bundles:

- A utility function assigns a number to every possible consumption bundle. More preferred bundles are assigned a higher number than less preferred bundles.
- $(x_1, x_2) \succeq (y_1, y_2)$ if and only if $u(x_1, x_2) \geq u(y_1, y_2)$.

Ordinal and Cardinal Utility

In these slides, the utility is an ordinal concept, not a cardinal concept. Meaning that the order matters, not the magnitude.

Theories of utility that attach a significance to the magnitude of utility are known as **cardinal utility**. The size of the utility difference between two bundles of goods is supposed to have some sort of significance.

On the other hand, when we order two bundles based on which is chosen, we assign an **ordinal utility** to the two bundles of goods: we just assign a higher utility to the chosen bundle than to the rejected bundle. Any assignment that does this will be a utility function.

We will stick with a purely ordinal utility framework.

Ordinal Utility

If bundle X is weakly preferred to Y, then any valid utility function must assign a number at least as high to X as to Y.

For example, $U(1 \text{ unit of coffee}) = 10$, $U(1 \text{ unit of cookies}) = 0$, and $U(1 \text{ unit of oatmeal}) = -5$ represents the same preferences as $X \succeq Y$, $Y \succeq Z$, and $X \succeq Z$.

We only care about the rank of utilities; the actual numbers are less important. Furthermore, a utility function that preserves the rank will preserve the underlying preferences:

$$U(coffee) = 1000, U(cookies) = 999, U(oatmeal) = 998$$

is just as good as

$$U(coffee) = 10, U(cookies) = 0, U(oatmeal) = -5.$$

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Monotonic Transformations

An increasing monotonic transformation is a function $g(x)$ such that $g(a) \geq g(b) \iff a \geq b$.

Any strictly increasing monotonic transformation of a utility function represents the same preferences as the original utility function.

This is because an increasing function will preserve the rank of the utilities, and therefore preserve the same preferences.

If $X \succ Y$ and $g(X) > g(Y)$, where g is a monotone increasing function, then $g(u(x_1, x_2))$ is an equivalent utility function to $u(x_1, x_2)$.

Example

Consider the "doubling function" which is a positive monotonic transformation: $v(\cdot) = 2u(\cdot)$. This preserves the ranking of the bundles.

If $u(x_1, x_2) = 10$ and $u(y_1, y_2) = 2$, then $g(u(x_1, x_2)) = 20$ and $g(u(x_1, x_2)) = 4$. Needless to say, $g(u(x_1, x_2)) > g(u(y_1, y_2))$.

Therefore, $X \succ Y$, and the ordering (or ranking) is preserved.

Other examples include: $\ln(x)$, e^x, cx where $c > 0$, etc. In essence, any function whose first derivative is positive, can be considered a monotonic transformation.

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Indifference Curves and Utility

Consider a utility function $u(x_1, x_2) = x_1 x_2$.

- Fix a level of utility k and find all combinations of x_1 and x_2 that give that level of utility.
- Changing the level k produces different indifference curves.
- To draw the indifference curves, we draw *level sets* of the utility function, that is, the set of all (x_1, x_2) such that $u(x_1, x_2)$ is constant.

$$u(x_1, x_2) = k = x_1 x_2$$

$$x_2 = \frac{k}{x_1}$$

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Utility of Perfect Substitutes

If two goods are perfect substitutes, then it does not matter which good I buy as they will provide me with the same utility.

All that matters to me is the total quantity I have: $u(x_1, x_2) = x_1 + x_2$.

More generally, they will provide the same rate of utility:

$$u(x_1, x_2) = ax_1 + bx_2$$

Here, a and b measure the value of goods x_1 and x_2 to the consumer.

The slope of a typical indifference curve is: $\frac{-a}{b}$.

Utility of Perfect Complements

If two goods are perfect complements, then I consume them in a certain proportion.

The general form of the indifference curves is:

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

Here, a and b are positive and represent the proportion in which the goods are consumed. For example, $a = 1$ and $b = 1$ for right and left shoes.

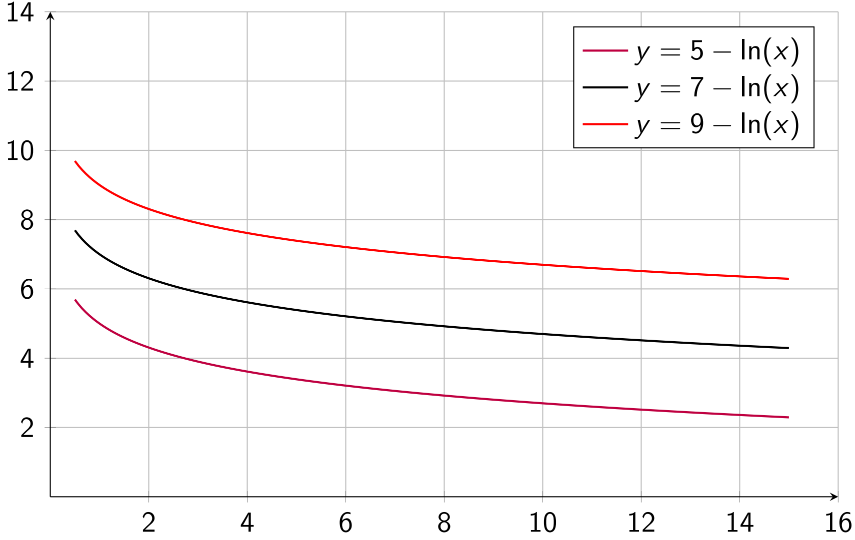
Quasi-Linear Utility Functions

Quasilinear utility functions have two a linear component and a non-linear component. The general form is:

$$u(x_1, x_2) = v(x_1) + x_2$$

Here, $v(x_1)$ is the non-linear component. This function is linear in good 2 and non-linear in good 1.

Quasi-Linear Utility Functions



Cobb-Douglas Utility Functions

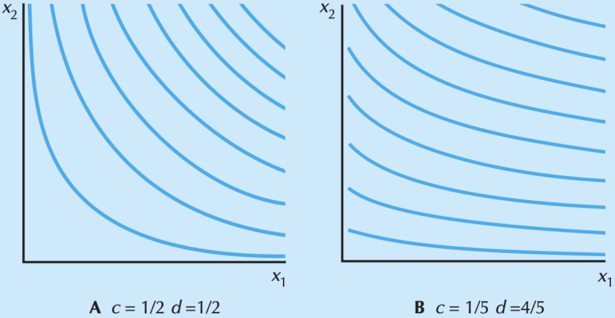
Cobb-Douglas utility functions embody the ‘well-behaved’ preferences assumption. They have the general form:

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$$

This is, arguably, the most important form of the utility function.

Its slope is the marginal rate of substitution (MRS) for a consumer with ‘well-behaved’ preferences.

Cobb-Douglas Utility Functions



Cobb-Douglas indifference curves. Panel A shows the case where $c = 1/2$, $d = 1/2$ and panel B shows the case where $c = 1/5$, $d = 4/5$.

Cobb-Douglas Utility Functions

To get the slope of the function, we must differentiate:

$$u(x_1, x_2) = x_1^\alpha x_2^\beta$$

This is unnecessarily tedious. It is much easier to differentiate it after applying a monotonic transformation like the natural logarithm.

Let's call the new function $v(x_1, x_2) = g(u(x_1, x_2))$, where g is a monotonic transformation, i.e. the natural logarithm.

$$v(x_1, x_2) = \ln(x_1^\alpha x_2^\beta) = \ln(x_1^\alpha) + \ln(x_2^\beta) = \alpha \ln(x_1) + \beta \ln(x_2)$$

Now, we can find the rate of change in utility with respect to x_1 and x_2. The next topic (MRS) will discuss precisely this.

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1 Utility

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2. Marginal Rate of Substitution (MRS)
   - Marginal Utility
   - Calculus
   - MRS of Cobb-Douglas

Marginal Utility and MRS

Consider a consumer who is consuming some bundle of goods, (x, y).

How does this consumer's utility change as we give him/her a little more of good x (while holding consumption of good y constant)?

This rate of change is called the marginal utility with respect to good x.

$$MU_x = \frac{\Delta U}{\Delta x} = \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

It measures the rate of change in utility (Δu) associated with a small change in the amount of good x (Δx).

To calculate the change in utility associated with a small change in consumption of good x:

$$\Delta U = MU_x \Delta x$$

Marginal Utility and MRS

The marginal utility with respect to good y is:

$$MU_y = \frac{\Delta U}{\Delta y} = \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}$$

It measures the rate of change in utility (Δu) associated with a small change in the amount of good y (Δy).

To calculate the change in utility associated with a small change in consumption of good y:

$$\Delta U = MU_y \Delta y$$

Marginal Utility and MRS

We have two different concepts:

1. Marginal Rate of Substitution: slope of an indifference curve.
2. Marginal Utility: rate of change of utility given a small change in the consumption of a good.

Now, we bring them together.

Marginal Utility and MRS

Consider a change in the consumption of each good, Δx and Δy, that keeps utility constant — a change in consumption that moves us along the indifference curve. Then, we must have:

$$\Delta U = 0$$

Here, we need to consider the change in utility due to a change in the amount of good x and a change in the amount of good y:

$$\Delta U = MU_x \Delta x + MU_y \Delta y = 0$$

$$\text{MRS} = \frac{\Delta y}{\Delta x} = -\frac{MU_x}{MU_y}$$

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2. Marginal Rate of Substitution (MRS)
   - Marginal Utility
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   - MRS of Cobb-Douglas

Marginal Utility and MRS

Given a particular bundle, a consumer's MRS is equal to the slope of the tangent line to the level set of their utility function at that bundle.

Suppose I have the utility function $U(x, y)$ where x is cookies and y is coffee. The additional utility I experience from consuming one more unit of a good is my **marginal utility** for that good. Mathematically:

$$MU_x = \frac{\partial U}{\partial x}$$

$$MU_y = \frac{\partial U}{\partial y}$$

Marginal Utility and MRS

So far, we have the utility function $U(x, y)$ and the marginal utilities:

$$MU_x = \frac{\partial U}{\partial x} \quad \text{and} \quad MU_y = \frac{\partial U}{\partial y}$$

We can now take the total derivative of the utility function:

$$dU = \left(\frac{\partial U}{\partial x}\right) dx + \left(\frac{\partial U}{\partial y}\right) dy$$

We can rewrite this using the marginal utilities:

$$dU = (MU_x)dx + (MU_y)dy$$

If we want to stay on the same indifference curve, we need $dU = 0$:

$$(MU_x)dx + (MU_y)dy = 0$$

Marginal Utility and MRS

From $dU = 0$, we have:

$$-\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{MU_x}{MU_y}$$

We also know that:

$$MRS = \frac{\mathrm{d}y}{\mathrm{d}x}$$

Therefore:

$$MRS = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{MU_x}{MU_y}$$

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2. Marginal Rate of Substitution (MRS)
   - Marginal Utility
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MRS and Cobb-Douglas Utility Function

The Cobb-Douglas utility function has the form:

$$u(x, y) = x^a y^b$$

The marginal utilities, then, are:

$$MU_x = \frac{\partial U}{\partial x} = ax^{a-1}y^b \quad \text{and} \quad MU_y = \frac{\partial U}{\partial y} = bx^a y^{b-1}$$

The MRS, then, is:

$$MRS = -\frac{MU_x}{MU_y} \quad \Rightarrow \quad MRS = -\frac{ay}{bx}$$

Monotonic Transformation and MRS

Consider a monotonic transformation of the Cobb-Douglas utility function with the natural logarithm:

$$u(x, y) = a\ln(x) + b\ln(y)$$

$$MU_x = \frac{\partial U}{\partial x} = \frac{a}{x} \quad \text{and} \quad MU_y = \frac{\partial U}{\partial y} = \frac{b}{y}$$

$$MRS = -\frac{MU_x}{MU_y} = -\frac{ay}{bx}$$

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References

Varian, H. R. (2014). Intermediate Microeconomics: A Modern Approach (9th ed.). W. W. Norton & Company.