Logic of Quantified Statements

Venkata Khandrika

Outline

Topic

Subtopic

1. Predicates and Quantified Statements I
   - Predicates
   - Quantifiers
   - Universal Conditional Statements
   - Implicit Quantification

Proposition vs. Predicate

Consider the following:

(1) $5 + 5 = 11$
(2) $x^2 = 16$

In the list above, (1) is a proposition, but (2) is not. Why?

It's because (2) may be either true or false depending on the value of x.

Predicate

In grammar, 'predicate' refers to the part of a sentence that gives information about the subject.

The predicate is the part of the sentence from which the subject has been removed. In logic, predicates generally have two parts.

Consider the statement "x is greater than 3." The first part, the variable x, is the subject of the statement. The second part—the predicate, "is greater than 3"—refers to a property that the subject of the statement can have.

We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable. Once a value has been assigned to the variable x, the statement $P(x)$ (also called the propositional function) becomes a proposition and has a truth value.

Predicate

Definition

A **predicate** is a sentence that contains a finite number of variables and becomes a proposition when specific values are substituted for the variables.

The *domain* of a predicate variable is the set of all values that may be substituted in place of the variable.

The *truth set* of a predicate, $P(x)$, is the set of all elements in its domain that make $P(x)$ true when they are substituted for x.

Example

Let $A(c, n)$ denote the statement "Computer c is connected to network n," where c is a variable representing a computer and n is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of $A(\text{MATH1}, \text{CAMPUS1})$ and $A(\text{MATH1}, \text{CAMPUS2})$?

Solution:

$A(\text{MATH1}, \text{CAMPUS1}) = \text{False}$
$A(\text{MATH1}, \text{CAMPUS2}) = \text{True}.$

Subtopic

1. Predicates and Quantified Statements I
   - Predicates
   - Quantifiers
   - Universal Conditional Statements
   - Implicit Quantification

Quantifiers

A propositional function (predicate) $P(x)$ is not a proposition until it has a truth value. Until now, we could convert a propositional function into a proposition by assigning a value to the variable.

Now, we will convert a propositional function into a proposition using **quantifiers**.

The two most widely used quantifiers are:

(1) Universal Quantifier

(2) Existential Quantifier

Universal Quantifier

The symbol $\forall$ is called the universal quantifier. Depending on the context, it is read as either "for every" or "for each" or "for all."

For example, another way to express the sentence "All human beings are mortal" is to write:

$$\forall \text{ humans } x, x \text{ is mortal.}$$

If you let H be the set of all human beings, then you can symbolize the statement more formally by writing:

$$\forall x \in H, x \text{ is mortal.}$$

Universal Quantifier

Definition

Let $P(x)$ be a predicate and let D be the domain of the variable x. The **universal quantification** of $P(x)$ is the statement

$$\forall x \in D,\ P(x).$$

This tells us that the proposition $P(x)$ must be true *for all* values of x in the domain D.

Example 1

Let $P(x)$ be the statement $x + 1 > x$. What is the truth value of the quantification $\forall x \in \mathbb{R}, P(x)$?

Solution:

The predicate $P(x)$ is $x + 1 > x$. If x is subtracted from both sides, this is equivalent to $1 > 0$. Needless to say, $1 > 0$ is true $\forall x \in \mathbb{R}$.

Therefore, truth value of the quantification $\forall x \in \mathbb{R}, P(x)$ is true.

Example 2

Let $Q(x)$ be the statement $x < 2$. What is the truth value of the quantification $\forall x \in \mathbb{R}, Q(x)$?

Solution:

The predicate $Q(x)$ is $x < 2$. The domain of x is the set of all real numbers. If the domain is $\mathbb{R}$, then there exists at least one real number x whose value is less than 2.

A *counterexample* to the quantification $\forall x \in \mathbb{R}, Q(x)$ is $x = 0$.

Therefore, truth value of the quantification $\forall x \in \mathbb{R}, Q(x)$ is false.

Existential Quantifier

The symbol $\exists$ denotes "there exists" and is called the existential quantifier. For example, the sentence "There is a student in CS50" can be written as:

$$\exists \text{ a person } p \text{ such that } p \text{ is a student in CS50.}$$

If we let P be the set of all people, then we can symbolize the statement more formally by writing:

$$\exists p \in P \text{ such that } p \text{ is a student in CS50.}$$

Existential Quantifier

Definition

Let $P(x)$ be a predicate and let D be the domain of the variable x. The **existential quantification** of $P(x)$ is the statement

$$\exists x \in D \text{ such that } P(x).$$

This tells us that the proposition $P(x)$ is true for at least one value of x in the domain D.

Example 1

Let $P(x)$ denote the statement $x > 3$. What is the truth value of the quantification $\exists x \in \mathbb{R}, P(x)$?

Solution:

The predicate $P(x)$ is $x > 3$. We need to check if there exists at least one number x in the set of all real numbers that satisfies the predicate.

Needless to say, an $x \in \mathbb{R}$ does exist that satisfies $x > 3$, like $x = 4$.

Therefore, the truth value of the quantification $\exists x \in \mathbb{R}, P(x)$ is true.

Example 2

Let $Q(x)$ denote the statement $x = x + 1$. What is the truth value of the quantification $\exists x \in \mathbb{R}, Q(x)$?

<u>Solution:</u>

The predicate $Q(x)$ is $x = x + 1$. If x is subtracted from both sides, we have $0 = 1$. This is not true $\forall x \in \mathbb{R}$.

Therefore, the truth value of the quantification $\exists x \in \mathbb{R}, Q(x)$ is false.

Quantifiers over Finite Domains

When the domain of a quantifier is finite, that is, when all its elements can be listed, quantified statements can be expressed using propositional logic.

In particular, when the elements of the domain D are $x_1, x_2, \ldots, x_n$, the universal quantification $\forall x \in D,\ P(x)$ is the same as the conjunction: $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$.

Similarly, when the elements of the domain D are $x_1, x_2, \ldots, x_n$, the existential quantification $\exists x \in D$ such that $Q(x)$ is the same as the disjunction: $P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$.

Summary of Quantifiers

Universal Quantifier	**Existential Quantifier**
$\forall$	$\exists$
"For all"	"There exists"
When true? When $P(x)$ is true for every x in the domain	*When true?* There is an x in the domain for which $P(x)$ is true
When false? There is an x in the domain for which $P(x)$ is false	*When false?* When $P(x)$ is false for every x in the domain
$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$	$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$

Subtopic

1. Predicates and Quantified Statements I
   - Predicates
   - Quantifiers
   - Universal Conditional Statements
   - Implicit Quantification

Universal Conditional Statements

One of the most important form of mathematical statements is the **universal conditional statement**. It has the form:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

More formally, it is written:

$$\forall x, P(x) \to Q(x).$$

It is common to omit explicit identification of the domain of predicate variables in universal conditional statements.

Examples

Rewrite the following statements more formally:

(1) If a real number is an integer, then it is a rational number.

(2) All bytes have eight bits.

Solution:

(1) $\forall x$, if $x \in \mathbb{Z}$, then $x \in \mathbb{Q}$.

(2) $\forall b$, if b is a byte, then b has 8 bits.

Subtopic

1. Predicates and Quantified Statements I
   - Predicates
   - Quantifiers
   - Universal Conditional Statements
   - Implicit Quantification

Equivalent Universal Statements

Consider the statement:

All squares are rectangles.

Its universal quantification is:

$$\forall \text{ squares } x, \; x \text{ is a rectangle.}$$

This can also be written as a universal conditional statement:

$$\forall x, \; \text{if } x \text{ is a square, then } x \text{ is a rectangle.}$$

Implicit Quantification

Consider the statement:

If a number is an integer, then it is a rational number.

This statement is an example of *implicit* universal quantification. Now, we can convert this conditional statement into a universal statement.

$$\forall x,\ \text{if } x \in \mathbb{Z} \text{ then } x \in \mathbb{Q}.$$

Mathematicians use a double arrow to indicate implicit quantification:

$$x \in \mathbb{Z} \ \Rightarrow\ x \in \mathbb{Q}.$$

Implicit Quantification

Definition

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x,\ P(x) \rightarrow Q(x)$.
- The notation $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x,\ P(x) \leftrightarrow Q(x)$.

Example

Let

$$Q(n) \text{ be "} n \text{ is a factor of } 8,\text{"}$$
$$R(n) \text{ be "} n \text{ is a factor of } 4,\text{"}$$
$$S(n) \text{ be "} n < 5 \text{ and } n \neq 3.\text{"}$$

Suppose the domain of n is $\mathbb{Z}^+$. Use the $\Rightarrow$ and $\Leftrightarrow$ symbols to indicate true relationships among $Q(n)$, $R(n)$, and $S(n)$.

Solution:

- $R(n) \Rightarrow Q(n)$.
- $S(n) \Rightarrow Q(n)$.
- $R(n) \Leftrightarrow S(n)$.

Topic

Subtopic

2. Predicates and Quantified Statements II
   - Negations of Quantified Statements
   - Negations of Universal Conditional Statements
   - Relation among $\forall$, $\exists$, $\wedge$, and $\vee$
   - Vacuous Truth of Universal Statements
   - Variants of Universal Conditional Statements
   - Necessary and Sufficient Conditions, Only If

Negation of Universal Quantifier

Consider the proposition "All roads lead to Rome." Many people would say that the negation is "No road leads to Rome." However, this is not true.

If even one road does not lead to Rome, then the sweeping statement that *all* roads lead to Rome is false.

Therefore, a correct negation is "It is not the case that all roads lead to Rome." Equivalently, we can also say that "There is a road that does not lead to Rome." Formally, the negation is "There exists at least one road that does not lead to Rome."

Negation of Universal Quantifier

Theorem

The negation of a statement of the form

$$\forall x \in D, P(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D \text{ such that } \neg P(x).$$

Symbolically,

$$\neg [\forall x \in D, P(x)] \equiv \exists x \in D \text{ such that } \neg P(x).$$

Negation of Existential Quantifier

Consider the proposition "A student in this class has taken Calculus." One might be tempted to say that the negation is "All students in the class have taken Calculus." However, this is not true.

If no student in the class has taken Calculus, then the sweeping statement that *at least one* student has taken Calculus is false.

Therefore, a correct negation is "It is not the case that a student in this class has taken Calculus." Equivalently, we say that "No student in this class has taken Calculus." Formally, the negation is "All students in the class have not taken Calculus."

Negation of Existential Quantifier

Theorem

The negation of a statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \neg Q(x).$$

Symbolically,

$$\neg\left[\exists x \in D \text{ such that } Q(x)\right] \equiv \forall x \in D, \neg Q(x).$$

Examples

Rewrite the following statements formally. Then write their negations.

1. No politicians are honest.
2. The number 1,357 is not divisible by any integer between 1 and 37.

Solution:

1. No politicians are honest.
   $\Rightarrow$ For all politicians p, there is no honest p.
   $\Rightarrow \forall p \in P,$ there is no honest p.
   Negation is: $\exists p \in P$ such that p is honest.
2. The number 1,357 is not divisible by any integer between 1 and 37.
   $\Rightarrow \forall x \in \{1, 2, 3, \ldots, 37\},$ 1,357 is not divisible by x.
   Negation is: $\exists x \in \{1, 2, 3, \ldots, 37\}$ such that 1,357 is divisible by x.

Negations Made Easy!

An easier way to remember the negation of a quantified statement is to think of the negation like an operator (which it is) such as $-$.

For instance, $-(5-2) = -5+2 = -3$. Similarly, we apply the distributive property when we see the negation operator.

$$\neg\left[\forall x\ P(x)\right] = \exists x\ \neg P(x)$$

$$\neg\left[\forall x\ \neg P(x)\right] = \exists x\ P(x)$$

$$\neg\left[\exists x\ Q(x)\right] = \forall x\ \neg Q(x)$$

$$\neg\left[\exists x\ \neg R(x)\right] = \forall x\ R(x)$$

Subtopic

2. Predicates and Quantified Statements II
   - Negations of Quantified Statements
   - Negations of Universal Conditional Statements
   - Relation among $\forall$, $\exists$, $\wedge$, and $\vee$
   - Vacuous Truth of Universal Statements
   - Variants of Universal Conditional Statements
   - Necessary and Sufficient Conditions, Only If

Negation of Universal Conditional Statements

Recall that a conditional proposition $p \to q$ can be rewritten as $\neg p \vee q$. Therefore, we can find the negation of a universally quantified conditional statement relatively easily.

$$\neg\,[\forall x, P(x) \to Q(x)] \equiv \exists x \text{ such that } \neg\,(P(x) \to Q(x))$$

$$\neg\,(P(x) \to Q(x)) \equiv \neg\,(\neg P(x) \vee Q(x)) \equiv P(x) \wedge \neg Q(x)$$

$$\neg\,[\forall x, P(x) \to Q(x)] \equiv \exists x \text{ such that } \;(P(x) \wedge \neg Q(x))$$

Subtopic

2. Predicates and Quantified Statements II
   - Negations of Quantified Statements
   - Negations of Universal Conditional Statements
   - Relation among $\forall$, $\exists$, $\wedge$, and $\vee$
   - Vacuous Truth of Universal Statements
   - Variants of Universal Conditional Statements
   - Necessary and Sufficient Conditions, Only If

Relation among $\forall$, $\exists$, $\wedge$, and $\vee$

Suppose that the domain of x is $D = \{x_1, x_2, \ldots, x_n\}$

$$\forall x \in D, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n).$$

$$\exists x \text{ such that } Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n).$$

Subtopic

2 Predicates and Quantified Statements II

- Negations of Quantified Statements
- Negations of Universal Conditional Statements
- Relation among $\forall$, $\exists$, $\wedge$, and $\vee$
- Vacuous Truth of Universal Statements
- Variants of Universal Conditional Statements
- Necessary and Sufficient Conditions, Only If

Vacuous Truth of Universal Statements

Consider a room with no balloons. Now, consider the statement:

"All the balloons in the room are blue."

Is this statement true or false? The statement is false if, and only if, its negation is true. Its negation is:

"There exists a balloon in the room that is not blue."

The only way this negation can be true is for there to actually be a non-blue balloon in the room, and there is not! Hence the negation is false, and so the statement is true "by default."

Vacuous Truth of Universal Statements

In general, a statement of the form

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if, $P(x)$ is false for every x in D.

In mathematics, the phrase "in general" signals that what is to follow is a generalization of an example that always holds true.

Subtopic

2. Predicates and Quantified Statements II
   - Negations of Quantified Statements
   - Negations of Universal Conditional Statements
   - Relation among $\forall$, $\exists$, $\wedge$, and $\vee$
   - Vacuous Truth of Universal Statements
   - **Variants of Universal Conditional Statements**
   - Necessary and Sufficient Conditions, Only If

Variants of Universal Conditional Statements

Consider a statement of the form $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

A universal conditional statement is logically equivalent to its contrapositive.

Subtopic

2. Predicates and Quantified Statements II
   - Negations of Quantified Statements
   - Negations of Universal Conditional Statements
   - Relation among $\forall$, $\exists$, $\wedge$, and $\vee$
   - Vacuous Truth of Universal Statements
   - Variants of Universal Conditional Statements
   - Necessary and Sufficient Conditions, Only If

Necessary and Sufficient Conditions, Only If

"$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means
"$\forall x$, if $r(x)$ then $s(x)$."

"$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means
"$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."

"$\forall x$, $r(x)$ **only if** $s(x)$" means
"$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

Topic

Subtopic

3. Multiple or Nested Quantifiers
   - Interpreting Statements with Multiple Quantifiers
   - Negation of Statements with Multiple Quantifiers
   - Formal Logical Notation

Meaning of Multiple Quantifiers

Suppose $P(x, y)$ denotes "$xy = yx$" (the commutative property). Assume that the domain is the set of real numbers. Then, the statement:

$$\forall x \forall y \; P(x, y)$$

means "for all real numbers x and for all real numbers y, $xy = yx$."
Similarly,

$$\forall y \forall x \; P(x, y)$$

means "for all real numbers y and for all real numbers x, $xy = yx$."

Fortunately, the order does *not* matter in this case. However, the order will matter when two *different* quantifiers are used in the same statement.

Meaning of Multiple Quantifiers

Consider the true proposition "All real numbers have an additive inverse."

(1) What does $\forall x \in \mathbb{R} \quad \exists y \in \mathbb{R} \quad (x + y = 0)$ mean?

(2) What does $\exists y \in \mathbb{R} \quad \forall x \in \mathbb{R} \quad (x + y = 0)$ mean?

Here, (1) means "for all real numbers x, there exists a real number y such that, y is the additive inverse of x."

Here, (2) means "there exists a real number y such that, for all real numbers x, y is the additive inverse of x."

It is imperative to note that (1) and (2) do not mean the same thing.

Truth with Multiple Similar Quantifiers

If you want to establish the truth of a statement of the form

$$\forall x \in D \quad \forall y \in E \quad P(x, y)$$

your challenge is to allow someone else to pick any element $x \in D$ and any element $y \in E$, and then you must show that $P(x, y)$ holds for that arbitrary pair.

If you want to establish the truth of a statement of the form

$$\exists x \in D \quad \exists y \in E \quad P(x, y)$$

your job is to find one particular $x \in D$ and one particular $y \in E$ for which $P(x, y)$ is true.

Truth with Multiple Different Quantifiers

If you want to establish the truth of a statement of the form

$$\forall x \in D \quad \exists y \in E \quad P(x, y)$$

your challenge is to allow someone else to pick whatever element $x \in D$ they wish and then you must find an element $y \in E$ that "works" for that particular x. You are allowed to find a different value of y for each different x you are given.

If you want to establish the truth of a statement of the form

$$\exists x \in D \text{ such that } \forall y \in E,\ P(x, y)$$

your job is to find one particular $x \in D$ that will "work" no matter what $y \in E$ anyone might choose to challenge you with. You are not allowed to change your x once you have specified it initially.

Universal and Existential Quantifiers

Consider the true proposition "All real numbers have an additive inverse."

(1) Is $\forall x \in \mathbb{R} \quad \exists y \in \mathbb{R} \quad (x + y = 0)$ true?

(2) Is $\exists y \in \mathbb{R} \quad \forall x \in \mathbb{R} \quad (x + y = 0)$ true?

We can say that (1) is true as it means that we are allowed to find a different value of y for each different x we are given.

However, (2) is not true as it means that we have to find one value of y that satisfies $x + y = 0$, as x changes.

Subtopic

3. Multiple or Nested Quantifiers
   - Interpreting Statements with Multiple Quantifiers
   - Negation of Statements with Multiple Quantifiers
   - Formal Logical Notation

Negation of Statements with Multiple Quantifiers

To find the negation of a statement with multiple quantifiers, we can use the same rules that we used to negate simpler quantified statements. Suppose the domain of x is D, then recall that:

$$\neg(\forall x \ P(x)) \equiv \exists x \ \neg P(x)$$

and

$$\neg(\exists x \ P(x)) \equiv \forall x \ \neg P(x).$$

To find the negation of a statement with multiple quantifiers we apply the negation in steps like we do with an operator like $(-)$. For instance

$$-(5+2-3) = -5-(2-3) = -5-2+3.$$

Negation of Statements with Multiple Quantifiers

$$\begin{aligned}
\neg\left[\forall x \in D \quad \forall y \in E \quad P(x,y)\right] &\equiv \exists x \in D \quad \neg\left[\forall y \in E \quad P(x,y)\right] \\
&\equiv \exists x \in D \quad \exists y \in E \quad \neg P(x,y)
\end{aligned}$$

$$\begin{aligned}
\neg\left[\forall x \in D \quad \exists y \in E \quad P(x,y)\right] &\equiv \exists x \in D \quad \neg\left[\exists y \in E \quad P(x,y)\right] \\
&\equiv \exists x \in D \quad \forall y \in E \quad \neg P(x,y)
\end{aligned}$$

$$\begin{aligned}
\neg\left[\exists x \in D \quad \forall y \in E \quad P(x,y)\right] &\equiv \forall x \in D \quad \neg\left[\forall y \in E \quad P(x,y)\right] \\
&\equiv \forall x \in D \quad \exists y \in E \quad \neg P(x,y)
\end{aligned}$$

$$\begin{aligned}
\neg\left[\exists x \in D \quad \exists y \in E \quad P(x,y)\right] &\equiv \forall x \in D \quad \neg\left[\exists y \in E \quad P(x,y)\right] \\
&\equiv \forall x \in D \quad \forall y \in E \quad \neg P(x,y)
\end{aligned}$$

Subtopic

3. Multiple or Nested Quantifiers
   - Interpreting Statements with Multiple Quantifiers
   - Negation of Statements with Multiple Quantifiers
   - Formal Logical Notation

Formal Logical Notation

The formal notation for logic involves using predicates to describe all properties of variables and omitting the words "such that" in existential statements.

"$\forall x$ in D, $P(x)$" can be written as "$\forall x \, (x \in D \rightarrow P(x))$."

"$\exists x$ in D such that $P(x)$" can be written as "$\exists x \, (x \in D \land P(x))$."

Example

Translate the statement "Every real number except zero has a multiplicative inverse." (A multiplicative inverse of a real number x is a real number y such that $xy = 1$.)

Solution:

"For every real x, there exists y, such that $xy = 1$."

$$\forall x \in \mathbb{R} - \{0\}, \exists y \in \mathbb{R} - \{0\} \text{ such that } xy = 1.$$

Here, $\exists y$ such that $P(x, y) \equiv \exists y\, [y \in D \wedge P(x, y)]$.
Furthermore, $\forall x\ P(x, y) \equiv \forall x\, [x \in D \to P(x, y)]$.

$$\forall x\, [(x \in \mathbb{R} \wedge x \neq 0) \to \exists y (xy = 1)]\,.$$

Topic

Subtopic

4. Arguments with Quantified Statements
   - Rules of Inference for Quantified Statements
   - Combining Rules of Inference for Propositions and Quantified Statements

Rules of Inference for Quantified Statements

We have discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments.

Here, we will discuss four rules:

(1) Universal Instantiation
(2) Universal Generalization
(3) Existential Instantiation
(4) Existential Generalization

Universal Instantiation

Definition

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$.

Universal instantiation means that the truth of a property in a particular case follows as a special instance of its more general or universal truth. Consider one of the most famous examples of universal instantiation:

$$\begin{aligned} &\text{All men are mortal.}\\ &\text{Socrates is a man.}\\ \therefore\ &\text{Socrates is mortal.}\end{aligned}$$

Universal Generalization

Definition

Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain.

If you prove something without assuming anything special about the element, you may conclude it holds universally.

The element c that we select must be an arbitrary, and not a specific, element of the domain. We have no control over c and cannot make any other assumptions about c other than it comes from the domain.

For example, we can prove that $x + 0 = x$ for any real number. To do so, we make no specific assumptions about x.

Existential Instantiation

Definition

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true.

Usually we have no knowledge of what c is, only that it exists. However, because it exists, we may give it a name (c) and continue our argument.

Existential Generalization

Definition

Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known.

If we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

Summary of Rules of Inference for Quantified Statements

Universal Instantiation	**Universal Generalization**
$\forall x P(x)$ $\therefore P(c)$	$P(c)$ for all c in the domain $\therefore \forall x P(x)$
Existential Instantiation	**Existential Generalization**
$\exists x P(x)$ $\therefore P(c)$	$P(c)$ for some c in the domain $\therefore \exists x P(x)$

Example 1

Consider the following exercise:

Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Suppose $D(x)$ is "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \rightarrow C(x))$ and $D(Marla)$. The conclusion is $C(Marla)$.

How do we get from the premises to the conclusion? We use rules of inference for quantified statements and modus ponens.

Example 1

Step	**Reason**
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal Instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus Ponens from (2) and (3)

Subtopic

4. Arguments with Quantified Statements
   - Rules of Inference for Quantified Statements
   - Combining Rules of Inference for Propositions and Quantified Statements

Universal Modus Ponens

Note that in our argument in Example 1 we used both universal instantiation, a rule of inference for quantified statements, and modus ponens, a rule of inference for propositional logic.

We will often need to use this combination of rules of inference. Because universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called **universal modus ponens**.

$$\begin{array}{rl} & \forall x(P(x) \rightarrow Q(x)) \\ & P(c) \\ \therefore & Q(c) \end{array}$$

Example

Assume that "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true. Use universal modus ponens to show that $100^2 < 2^{100}$.

<u>Solution</u>:

Suppose that $P(x)$ is "$x > 4$," and $Q(x)$ is "$x^2 < 2^x$." Furthermore, let $c = 100$. Then by universal modus ponens:

$$\begin{aligned} & \forall x(P(x) \to Q(x)) \\ & P(100) \\ \therefore\ & Q(100). \end{aligned}$$

Universal Modus Tollens

Another useful combination of a rule of inference from propositional logic and a rule of inference for quantified statements is **universal modus tollens**.

Universal modus tollens combines universal instantiation and modus tollens and can be expressed in the following way:

$$\begin{aligned} &\forall x(P(x) \to Q(x)) \\ &\neg Q(c) \\ \therefore\ &\neg P(c) \end{aligned}$$

Example

$$\begin{aligned} & \text{All human beings are mortal.} \\ & \text{Zeus is not mortal.} \\ \therefore\ & \text{Zeus is not a human being.} \end{aligned}$$

Topic

References

- Epp, Susanna S. (2019). Discrete Mathematics with Applications (5th ed.). Cengage Learning. ISBN: 9781337694193.
- Rosen, Kenneth H. (2012). Discrete Mathematics and Its Applications (7th ed.). McGraw-Hill Education. ISBN: 9781260912784.