# Logic of Compound Statements

Venkata Khandrika

# Outline

# Topic

# Subtopic

1. Logical Form and Logical Equivalence
   - Overview
   - Propositions
   - Negation, Disjunction, and Conjunction
   - Translating English into Logic
   - Truth Values
   - Propositional Form
   - Logical Equivalence
   - De Morgan's Laws
   - Tautologies and Contradictions
   - Logical Equivalence Theorems

# Arguments, Form, and Conclusions

An **argument** is a sequence of statements aimed at proving the truth of an assertion. The assertion at the end of the sequence is called the **conclusion**, and the preceding statements are called **premises**.

In logic, the form of an argument is distinguished from its content. This means that logical analysis won't help you determine the truth of an argument's content, but it will help you analyze an argument's form to determine whether the truth of the conclusion follows necessarily from the truth of the premises.

# Arguments, Form, and Conclusions

Consider the following two arguments.

## Argument 1

If $\underbrace{\text{the bell rings}}_{p}$ or $\underbrace{\text{the flag drops}}_{q}$, then $\underbrace{\text{the race is over}}_{r}$.

$\therefore$ If $\underbrace{\text{the race is not over}}_{\text{not } r}$, then $\underbrace{\text{the bell hasn't rung}}_{\text{not } p}$ and $\underbrace{\text{the flag hasn't dropped}}_{\text{not } q}$.

## Argument 2

If $\underbrace{x = 2}_{p}$ or $\underbrace{x = -2}_{q}$, then $\underbrace{x^2 = 4}_{r}$.

$\therefore$ If $\underbrace{x^2 \neq 4}_{\text{not } r}$, then $\underbrace{x \neq 2}_{\text{not } p}$ and $\underbrace{x \neq -2}_{\text{not } q}$.

# Arguments, Form, and Conclusions

They have very different content but their logical form is the same. To help make this clear, we use letters like $p$, $q$, and $r$ to represent component sentences. We let the expression "not $p$" refer to the sentence "It is not the case that $p$ is true" or "$p$ is false".

The two arguments have a common *logical form* of:

$$\begin{array}{l} \text{If } p \text{ or } q\text{, then } r. \\ \therefore \text{ If not } r\text{, then not } p \text{ and not } q. \end{array}$$

# Subtopic

1. Logical Form and Logical Equivalence

# Proposition

## Definition

A **proposition** is a sentence that is either true or false but not both.

## Example

For example, "$2 + 2 = 4$" and "$2 + 2 = 5$" are both propositions, the first because it is true and the second because it is false.

## Definition

A **theorem** is a proposition that can be shown to be true.

## Example

For example, Pythagoras' theorem "$a^2 + b^2 = c^2$" can be shown to be true in a multitude of ways.

# Propositions

Here are some examples of sentences that are *not* propositions:

(1) $x^2 + 2 = 11$

(2) $x + y > 0$

(1) is not a proposition because its truth or falsity depends on the value of $x$. For some values of $x$, it is true, whereas for other values it is false.

(2) is not a proposition because its truth or falsity depends on the values of $x$ and $y$. For instance, when $x = -1$ and $y = 2$ it is true, whereas when $x = -1$ and $y = 1$ it is false.

# Subtopic

1. Logical Form and Logical Equivalence
   - Overview
   - Propositions
   - Negation, Disjunction, and Conjunction
   - Translating English into Logic
   - Truth Values
   - Propositional Form
   - Logical Equivalence
   - De Morgan's Laws
   - Tautologies and Contradictions
   - Logical Equivalence Theorems

# Compound Propositions

We have seen that propositions are sentences that are either true or false but not both.

Naturally, we would next like to build more complex sentences. We now introduce three symbols that are used to build more complicated logical expressions out of simpler ones.

# Negation

The symbol $\neg$ denotes "not". Given a proposition $p$, the sentence $\neg p$ is read "not $p$" or "it is not the case that $p$ is true" or "$p$ is false."

More formally, it called the **negation** of $p$. It can be written in many different ways including: $\sim p$, $\bar{p}$, $!p$, or even $p'$.

# Disjunction

The symbol $\vee$ denotes "or". Given propositions $p$ and $q$, the sentence $p \vee q$ is read "$p$ or $q$."

More formally, $p \vee q$ is called the **disjunction** of $p$ and $q$. This can also be written as: $p + q$, $p \mid q$, and $p \mid\mid q$.

Programming languages often use $p \mid\mid q$.

# Conjunction

The symbol $\wedge$ denotes "and". Given propositions $p$ and $q$, the sentence $p \wedge q$ is read "$p$ and $q$."

More formally, $p \wedge q$ is called the **conjunction** of $p$ and $q$. This can also be written as: $p \times q$, $p \ \& \ q$, $p \ \&\& \ q$.

Programming languages often use $p \ \&\& \ q$.

# Inequalities

The notation for inequalities involves *and* and *or* statements. For instance, if $x$, $a$, and $b$ are particular real numbers, then:

$$\begin{aligned} x \leq a \quad &\text{means} \quad x < a \quad \text{or} \quad x = a \\ a \leq x \leq b \quad &\text{means} \quad a \leq x \quad \text{and} \quad x \leq b. \end{aligned}$$

# Order of Operations

In expressions that include the symbol $\neg$, as well as $\wedge$ or $\vee$, the order of operations becomes extremely important.

The order is as follows:

(1) $\neg$
(2) $\wedge$ and $\vee$

The symbols $\wedge$ and $\vee$ are considered coequal in order of operation, and an expression such as $p \wedge q \vee r$ is considered ambiguous. This expression must be written as either $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$ to have meaning.

# Subtopic

1. Logical Form and Logical Equivalence
   - Overview
   - Propositions
   - Negation, Disjunction, and Conjunction
   - **Translating English into Logic**
   - Truth Values
   - Propositional Form
   - Logical Equivalence
   - De Morgan's Laws
   - Tautologies and Contradictions
   - Logical Equivalence Theorems

# But

A variety of words translate into logic as $\wedge$, $\vee$, and $\neg$.

The word ‘but’ translates the same as ‘and.’ Generally, the word ‘but’ is used in place of ‘and’ when the part of the sentence that follows is, in some way, unexpected.

For example, consider the sentence “Jim is tall but he is not heavy.” If we let “Jim is tall” be the proposition $t$ and “he is heavy” be $h$, then we can rewrite the sentence in logic as $t \wedge \neg h$.

# Neither-Nor

Another example involves the translating the words ‘neither-nor’ into logic.

For example, consider the sentence “Neither a borrower nor a lender be.” If we let “Be a borrower” be the proposition $b$ and “be a lender” be $d$, then we can rewrite the sentence in logic as $\neg b \wedge \neg d$.

# Subtopic

1. Logical Form and Logical Equivalence
   - Overview
   - Propositions
   - Negation, Disjunction, and Conjunction
   - Translating English into Logic
   - **Truth Values**
   - Propositional Form
   - Logical Equivalence
   - De Morgan's Laws
   - Tautologies and Contradictions
   - Logical Equivalence Theorems

# Truth Value

Recall that a proposition is a sentence that is either true or false but not both. If the sentence is true, then its **truth value** is *true*, otherwise it is *false*.

We now define compound sentences as propositions by specifying their truth values in terms of the propositions that compose them. We use letters to denote *propositional variables* (or sentential variables), that is, variables that represent propositions, just as letters are used to denote numerical variables.

To do this, we make some definitions about our logical operators.

# Negation

## Definition

If $p$ is a propostional variable, then the **negation** of $p$ is "not $p$" or "it is not the case that $p$" and is denoted $\neg p$. It has the opposite truth value from $p$: if $p$ is true, $\neg p$ is false; if $p$ is false, $\neg p$ is true.

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

Table 1: Truth Table for the Negation of a Proposition

# Conjunction

## Definition

If $p$ and $q$ are propostional variables, then the **conjunction** of $p$ and $q$ is "$p$ and $q$" denoted "$p \wedge q$". It is true when, and only when, both $p$ and $q$ are true. If either $p$ or $q$ or both are false, then $p \wedge q$ is false.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Table 2: Truth Table for Conjunction

# Disjunction

## Definition

If $p$ and $q$ are propostional variables, then the **disjunction** of $p$ and $q$ is "$p$ or $q$" denoted "$p \vee q$". It is false when, and only when, both $p$ and $q$ are false. If either $p$ or $q$ or both are true, then $p \vee q$ is true.

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Table 3: Truth Table for Disjunction

# Inclusive or Exclusive Or

In ordinary language, ‘or’ is sometimes used in an exclusive sense ($p$ or $q$ but not both) and sometimes in an inclusive sense ($p$ or $q$ or both).

For instance, a waiter who says you may have “coffee, tea, or milk” uses the word or in an exclusive sense; extra payment is generally required if you want more than one beverage.

On the other hand, a waiter who offers “cream or sugar” uses the word or in an inclusive sense; you are entitled to both cream and sugar if you wish to have them.

Inclusive ‘or’ is denoted by $\vee$, while $\oplus$ is reserved for exclusive ‘or’.

# Subtopic

1. Logical Form and Logical Equivalence
   - Overview
   - Propositions
   - Negation, Disjunction, and Conjunction
   - Translating English into Logic
   - Truth Values
   - Propositional Form
   - Logical Equivalence
   - De Morgan's Laws
   - Tautologies and Contradictions
   - Logical Equivalence Theorems

# Propositional Form

Now that truth values have been assigned to $\neg p$, $p \wedge q$, and $p \vee q$, consider the question of assigning truth values to more complicated expressions such as $\neg p \vee q$ and $(p \vee q) \wedge \neg(p \wedge q)$.

Such expressions are called propositional forms.

**Definition**

A **propositional form** is a proposition made up of propositional variables variables (like $p$, $q$, and $r$) and logical connectives (like $\wedge$, $\vee$, and $\neg$).

It is common to refer to a propositional form to as a compound proposition.

# Example

Consider the truth table for the compound proposition $(p \wedge q) \vee \neg r$.

| $p$ | $q$ | $r$ | $p \wedge q$ | $\neg r$ | $(p \wedge q) \vee \neg r$ |
|---|---|---|---|---|---|
| T | T | T | T | F | T |
| T | T | F | T | T | T |
| T | F | T | F | F | F |
| T | F | F | F | T | T |
| F | T | T | F | F | F |
| F | T | F | F | T | T |
| F | F | T | F | F | F |
| F | F | F | F | T | T |

# Subtopic

# Logical Equivalence

Consider the sentences:

(1) Dogs bark and cats meow
(2) Cats meow and dogs bark

These are two different ways of saying the same thing, the reason has nothing to do with the definition of the words. It has to do with the logical form of the statements.

Any two propositions whose logical forms are related in the same way as (1) and (2) would either both be true or both be false.

# Logical Equivalence

## Definition

Two compound propositions $P$ and $Q$ are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of propositions for their propositional variables. It is denoted by writing:

$$P \equiv Q$$

How to test whether two compound propositions are logically equivalent?

(1) Construct a truth table with one column for the truth values of $P$ and another column for the truth values of $Q$.

(2) If the truth value of $P$ in each row is the same as the truth value of $Q$, then $P$ and $Q$ are logically equivalent.

# Example 1

Consider the double negation of a proposition $\neg(\neg p)$. This is logically equivalent to the proposition $p$.

This can be proved using the truth table for $p$, $\neg p$, and $\neg(\neg p)$:

| $p$ | $\neg p$ | $\neg(\neg p)$ |
|---|---|---|
| T | F | T |
| F | T | F |

The truth value of $p$ in each row is the same as the truth value of $\neg(\neg p)$. Therefore, $p$ and $\neg(\neg p)$ are logically equivalent.

# Example 2

Consider the compound propositions $\neg(p \wedge q)$ and $\neg p \wedge \neg q$. We can show that these two are *not* logically equivalent using their truth tables.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \wedge q$ | $\neg(p \wedge q)$ | | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | | F |
| T | F | F | T | F | T | $\neq$ | F |
| F | T | T | F | F | T | $\neq$ | F |
| F | F | T | T | F | T | | T |

Let $P$ and $Q$ be the compound propositions $\neg(p \wedge q)$ and $\neg p \wedge \neg q$, respectively. The truth value of $P$ in each row is *not* the same as the truth value of $Q$. Therefore, $P$ and $Q$ are *not* logically equivalent.

# Subtopic

# De Morgan's Laws 1

In Example 2, we showed that $\neg(p \wedge q)$ and $\neg p \wedge \neg q$ are not logically equivalent. Naturally, one might ask what is the compound proposition $\neg(p \wedge q)$ logically equivalent to.

To answer the question posed above, consider the proposition "John is tall and Jim is redheaded." For this proposition to be true, both components must be true. So for it to be false, one or both components must be false. Thus the negation can be written as "John is not tall or Jim is not redheaded."

In general, the negation of the conjunction of two statements is logically equivalent to the disjunction of their negations.

# De Morgan's Laws 1

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p \vee \neg q$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | F | T | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | T | T |

$$\therefore \quad \neg(p \wedge q) \equiv \neg p \vee \neg q$$

# De Morgan's Laws 2

Similarly, the negation of the disjunction of two propositions is logically equivalent to the conjunction of their negations.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | T | F | F |
| F | T | T | F | T | F | F |
| F | F | T | T | F | T | T |

$$\therefore \quad \neg(p \vee q) \equiv \neg p \wedge \neg q$$

# Example

Use De Morgan's laws to write the negation of $-1 < x \leq 4$.

Firstly, the inequality means $x > -1$ *and* $x \leq 4$. Therefore, the negation of this would be $\neg(x > -1 \wedge x \leq 4)$. This is logically equivalent to the disjunction of the negations.

$$\begin{aligned} \neg(x > -1 \wedge x \leq 4) &\equiv \neg(x > -1) \vee \neg(x \leq 4) \\ &\equiv (x \leq -1) \vee (x > 4) \end{aligned}$$

$$\therefore \quad x \in (-\infty, -1] \cup (4, \infty)$$

# Subtopic

1. Logical Form and Logical Equivalence
   - Overview
   - Propositions
   - Negation, Disjunction, and Conjunction
   - Translating English into Logic
   - Truth Values
   - Propositional Form
   - Logical Equivalence
   - De Morgan's Laws
   - **Tautologies and Contradictions**
   - Logical Equivalence Theorems

# Tautologies

## Definition

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**.

The compound proposition $p \vee \neg p$ is a tautology. This can be shown using the truth table for $p$, $\neg p$, and $p \vee \neg p$:

| $p$ | $\neg p$ | $p \vee \neg p$ |
|---|---|---|
| T | F | T |
| F | T | T |

# Contradictions

## Definition

A compound proposition that is always false, no matter what the truth values of the propositional variables that occur in it, is called a **contradiction**.

The compound proposition $p \wedge \neg p$ is a contradiction. This can be shown using the truth table for $p$, $\neg p$, and $p \wedge \neg p$:

| $p$ | $\neg p$ | $p \wedge \neg p$ |
|---|---|---|
| T | F | F |
| F | T | F |

# Example

If **t** is a tautology and **c** is a contradiction, it can be shown that $p \wedge t \equiv p$ and $p \wedge c \equiv c$.

| $p$ | $t$ | $p \wedge t$ | $p$ | $c$ | $p \wedge c$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| F | T | F | F | F | F |

As evident, $p \wedge t \equiv p$ and $p \wedge c \equiv c$.

# Subtopic

1. Logical Form and Logical Equivalence
   - Overview
   - Propositions
   - Negation, Disjunction, and Conjunction
   - Translating English into Logic
   - Truth Values
   - Propositional Form
   - Logical Equivalence
   - De Morgan's Laws
   - Tautologies and Contradictions
   - Logical Equivalence Theorems

# Theorems

Given any statement variables $p$, $q$, and $r$, a tautology $t$ and a contradiction $c$, the following logical equivalences hold.

## Theorem

**Commutative Laws**

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

## Theorem

**Associative Laws**

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

# Theorems

## Theorem

**Distributive Laws**

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

## Theorem

**Identity Laws**

$$p \wedge t \equiv p$$

$$p \vee c \equiv p$$

# Theorems

## Theorem

**Negation Laws**

$$p \vee \neg p \equiv t$$

$$p \wedge \neg p \equiv c$$

## Theorem

**Double Negative Law**

$$\neg(\neg p) \equiv p$$

## Theorem

**Idempotent Laws**

$$p \wedge p \equiv p$$

$$p \vee p \equiv p$$

# Theorems

### Theorem

**Universal Bound Laws**

$$p \vee t \equiv t$$

$$p \wedge c \equiv c$$

### Theorem

**De Morgan's Laws**

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

# Theorems

## Theorem

**Absorption Laws**

$$p \wedge (p \vee q) \equiv p$$

$$p \vee (p \wedge q) \equiv p$$

## Theorem

**Negations of Tautologies and Contradictions**

$$\neg t \equiv c$$

$$\neg c \equiv t$$

# Topic

# Subtopic

2 Conditional Propositions

- Conditional
- Negation of Conditional
- Contrapositive
- Converse and Inverse
- Biconditional
- Necessary and Sufficient Conditions

# Conditional

## Definition

If $p$ and $q$ are propositional variables, the the conditional of $q$ by $p$ is "If $p$, then $q$." It is denoted by $p \to q$. It is false when $p$ is true and $q$ is false; otherwise it is true. We call $p$ the hypothesis (or antecedent) of the conditional and $q$ the conclusion (or consequent).

The notation $p \to q$ indicates that $\to$ is a connective, like $\wedge$ or $\vee$, which can be used to join propositions to create new propositions.

To define $p \to q$ as a statement, therefore, we must specify the truth values for $p \to q$ as we specified truth values for $p \wedge q$ and for $p \vee q$.

# Truth Table for Conditional

To understand the truth table for $p \to q$, we can think of the conditional as a promise. For instance, suppose you go to interview for a job at a store and the owner of the store makes you the promise: "If you show up for work Monday morning, then you will get the job."

Under what circumstances are you justified in saying the owner spoke falsely? That is, under what circumstances is the above sentence false? The answer is: You *do* show up for work Monday morning and you *do not* get the job. Nothing is specified about what happens if you do not show up. Therefore, it is not false regardless of whether you get the job or not.

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Vacuously True

A conditional statement that is true by virtue of the fact that its hypothesis is false is often called **vacuously true** or **true by default**.

In the previous example, the store owner's promise only says you will get the job if a certain condition (showing up for work Monday morning) is met; it says nothing about what will happen if the condition is not met.

For example, consider the proposition "If $0 = 1$, then $1 = 2$." This proposition is true. Why? The hypothesis $0 = 1$ is false, therefore the conditional is vacuously true.

# Updated Order of Operations

The order is as follows:

(1) $\neg$

(2) $\wedge$ and $\vee$

(3) $\rightarrow$

# Representation of If-Then as Or

Consider the conditional $p \to q$. We can rewrite this as $\neg p \vee q$.

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $p \to q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

As evident, $\neg p \vee q$ and $p \to q$ have the same truth values for all possible combinations of truth values of $p$ and $q$. Therefore, they are logically equivalent.

# Subtopic

2. Conditional Propositions
   - Conditional
   - Negation of Conditional
   - Contrapositive
   - Converse and Inverse
   - Biconditional
   - Necessary and Sufficient Conditions

# Negation of a Conditional

We now know that $p \to q \equiv \neg p \vee q$. Thus, we can now find the negation of $p \to q$ by taking the negation of $\neg p \vee q$.

$$\neg(\neg p \vee q) \equiv \neg(\neg p) \wedge \neg q \equiv p \wedge \neg q$$
$$\therefore \quad \neg(p \to q) \equiv p \wedge \neg q$$

The negation of "if $p$ then $q$" is logically equivalent to "$p$ and not $q$."

# Example

Write down the negations of the following conditionals:

(1) If my car is in the repair shop, then I cannot get to class.
(2) If it is sunny, then I will go to the beach.

Answers:

(1) My car is in the repair shop and I can get to class.
(2) It is sunny and I will not go to the beach.

# Subtopic

2. Conditional Propositions
   - Conditional
   - Negation of Conditional
   - Contrapositive
   - Converse and Inverse
   - Biconditional
   - Necessary and Sufficient Conditions

# Contrapositive

**Definition**

The **contrapositive** of a conditional proposition of form "If $p$ then $q$" is:

$$\text{If } \neg q \text{ then } \neg p.$$

Symbolically, the contrapositive of $p \to q$ is $\neg q \to \neg p$.

A conditional proposition is logically equivalent to its contrapositive.

The above stated result is a fundamental result in logic. It is used very often in mathematical proofs as a proof technique called **proof by contrapositive**, which will be discussed later.

# Subtopic

2. Conditional Propositions
   - Conditional
   - Negation of Conditional
   - Contrapositive
   - Converse and Inverse
   - Biconditional
   - Necessary and Sufficient Conditions

# Converse

**Definition**

Suppose a conditional proposition of the form "If p then q" is given. Its **converse** is:

$$\text{If } q \text{ then } p.$$

Symbolically, the converse of $p \to q$ is $q \to p$.

**Definition**

Suppose a conditional proposition of the form "If p then q" is given. Its **inverse** is:

$$\text{If } \neg p \text{ then } \neg q.$$

Symbolically, the inverse of $p \to q$ is $\neg p \to \neg q$.

# Example

Write the converse and inverse of each of the following propositions:

(1) If Howard can swim across the lake, then Howard can swim to the island.

(2) If today is Easter, then tomorrow is Monday.

Answers:

- Converse (1): If Howard can swim to the island, then Howard can swim across the lake.
- Inverse (1): If Howard cannot swim across the lake, then Howard cannot swim to the island.
- Converse (2): If tomorrow is Monday, then today is Easter.
- Inverse (2): If today is not Easter, then tomorrow is not Monday.

# Converse and Inverse

Note that the converse and inverse of a conditional proposition are logically equivalent. For this, suppose that we have a conditional proposition $p \to q$.

Its converse is: $q \to p$. The contrapositive of the converse is: $\neg p \to \neg q$. Note the contrapositive of the converse is the inverse.

Since a conditional proposition and its contrapositive are logically equivalent, the converse and the inverse of a conditional are logcially equivalent.

# Subtopic

2 Conditional Propositions

- Conditional
- Negation of Conditional
- Contrapositive
- Converse and Inverse
- **Biconditional**
- Necessary and Sufficient Conditions

# Only If

It is important to understand "only if" before studying the biconditional.

To say "$p$ only if $q$" means that $p$ can take place only if $q$ takes place also. That is, if $q$ does not take place, then $p$ cannot take place.

Symbolically, "$p$ only if $q$" means $\neg q \to \neg p$. Furthermore, we know that a conditional proposition is logically equivalent to its contrapositive.

Therefore, "$p$ only if $q$" means $p \to q$.

# Caution!

Remember, "$p$ only if $q$" does *not* mean "$p$ if $q$."

$$p \text{ only if } q \quad \text{means} \quad p \to q$$

$$p \text{ if } q \quad \text{means} \quad q \to p$$

# Biconditional

## Definition

Given propositional variables $p$ and $q$, the **biconditional** of $p$ and $q$ is "$p$ if, and only if, $q$." It is denoted by $p \leftrightarrow q$. It is true if both $p$ and $q$ have the same truth values and is false if $p$ and $q$ have opposite truth values.

Note that the first part of the biconditional is "$p$ if $q$" this means $q \rightarrow p$. The second part is "$p$ only if $q$" this means $p \rightarrow q$.

The phrase "if and only if" is abbreviated **iff**.

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# Order of Operations

The order is as follows:

(1) $\neg$

(2) $\wedge$ and $\vee$

(3) $\rightarrow$ and $\leftrightarrow$

Note that the first part of the biconditional is "$p$ if $q$" this means $q \to p$. The second part is "$p$ only if $q$" this means $p \to q$. Therefore, the biconditional is simply the conjunction of the two conditionals.

| $p$ | $q$ | $p \to q$ | $q \to p$ | $p \leftrightarrow q$ | $(p \to q) \wedge (q \to p)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | T | F | F | F |
| F | F | T | T | T | T |

# Subtopic

2 Conditional Propositions

- Conditional
- Negation of Conditional
- Contrapositive
- Converse and Inverse
- Biconditional
- Necessary and Sufficient Conditions

# Necessary and Sufficient Conditions

## Definition

If $p$ and $q$ are propositional variables:

- "$p$ is a **sufficient condition** for $q$" means that $p \to q$
- "$p$ is a **necessary condition** for $q$" means that $q \to p$

The sufficient condition is equivalent to "$p$ only if $q$" and the necessary condition is equivalent to "$p$ if $q$."

Therefore, saying that "$p$ is a necessary and sufficient condition for $q$" means "$p$ if and only if $q$." Symbolically, it means $p \leftrightarrow q$.

# Topic

# Subtopic

3 Valid and Invalid Arguments

- Arguments
- Validity
- Modus Ponens and Modus Tollens
- Rules of Inference
- Fallacy
- Contradictions and Valid Arguments

# Arguments and Argument Form

**Definition**

An **argument** in logic is a sequence of propositions. All but the final proposition in the argument are called **premises** and the final proposition is called the **conclusion**. An argument is **valid** if the truth of all its premises implies that the conclusion is true.

**Definition**

An **argument form** in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is **valid** if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

# Subtopic

3 Valid and Invalid Arguments

- Arguments
- **Validity**
- Modus Ponens and Modus Tollens
- Rules of Inference
- Fallacy
- Contradictions and Valid Arguments

# Valid Argument

The crucial fact about a valid argument is that the truth of its conclusion follows *necessarily* or *inescapably* or *by logical form alone* from the truth of its premises.

It is impossible to have a valid argument with all true premises and a false conclusion.

When an argument is valid and its premises are true, the truth of the conclusion is said to be *inferred* or *deduced* from the truth of its premises.

# Testing Validity

We now understand that when an argument is valid, the truth of the conclusion is inferred from the truth of its premises. However, we might sometimes want to confirm whether an argument is valid. To achieve this, we use truth tables.

To test an argument form for validity:

(1) Identify the premises and conclusion of the argument form.
(2) Construct a truth table showing the truth values of all the premises and the conclusion.
(3) A row of the truth table in which all the premises are true is called a *critical row*. If the conclusion in every critical row is true, then the argument form is valid. Otherwise, it is invalid.

# Example

Prove that the following argument form is valid:

$$\begin{aligned} &\text{If } p \text{ then } q. \\ &p. \\ \therefore\ &q. \end{aligned}$$

| | | premises | | conclusion |
|---|---|---|---|---|
| $p$ | $q$ | $p \to q$ | $p$ | $q$ |
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | F | F |

As evident from the highlighted critical row, the conclusion in every critical row is true, therefore the argument form is valid.

# Validity and Truth Tables

We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be a tedious approach.

For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires $2^{10} = 1024$ different rows.

Fortunately, we do not always have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**.

We will now introduce *modus ponens* and *modus tollens*, the most important rules of inference in propositional logic.

# Subtopic

3. Valid and Invalid Arguments
   - Arguments
   - Validity
   - Modus Ponens and Modus Tollens
   - Rules of Inference
   - Fallacy
   - Contradictions and Valid Arguments

# Syllogism

**Definition**

An argument form consisting of two premises and a conclusion is called a **syllogism**. The first and second premises are called the major premise and minor premise, respectively.

The two most famous forms of syllogism in logic are modus ponens and modus tollens.

# Modus Ponens

Modus Ponens has the form:

$$\boxed{\begin{array}{rl} & p \to q. \\ & p. \\ \therefore & q. \end{array}}$$

Here is an argument of this form:

$$\begin{array}{rl} & \text{If the sum of the digits of 371,487 is divisible by 3,} \\ & \text{then 371,487 is divisible by 3.} \\ & \text{The sum of the digits of 371,487 is divisible by 3.} \\ \therefore & \text{371,487 is divisible by 3.} \end{array}$$

The term modus ponens is Latin meaning "method of affirming" (the conclusion is an affirmation).

# Proving Validity of Modus Ponens

| | | premises | | conclusion |
|---|---|---|---|---|
| $p$ | $q$ | $p \to q$ | $p$ | $q$ |
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | F | F |

The first row is the only one in which both premises are true. Furthermore, the conclusion in that row is also true. Hence, the argument form is valid.

# Modus Tollens

Modus Tollens has the form:

$$\boxed{\begin{array}{rl} & p \to q. \\ & \neg q. \\ \therefore & \neg p. \end{array}}$$

Here is an argument of this form:

$$\begin{array}{rl} & \text{If Zeus is human, then Zeus is mortal.} \\ & \text{Zeus is not mortal.} \\ \therefore & \text{Zeus is not human.} \end{array}$$

The term modus tollens is Latin meaning "method of denying" (the conclusion is a denial).

# Proving Validity of Modus Tollens

| | | premises | | conclusion |
|---|---|---|---|---|
| $p$ | $q$ | $p \to q$ | $\neg q$ | $\neg p$ |
| T | T | T | F | F |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | T | T |

The last row is the only one in which both premises are true. Furthermore, the conclusion in that row is also true. Hence, the argument form is valid.

# Subtopic

3 Valid and Invalid Arguments

- Arguments
- Validity
- Modus Ponens and Modus Tollens
- Rules of Inference
- Fallacy
- Contradictions and Valid Arguments

# Validity of an Argument

As we have seen, the rules of inferences called modus ponens and modus tollens are valid forms of argument. Now, we look at more rules of inference.

Before that, it is imperative to understand that from the definition of a valid argument form, we see that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion $q$ is valid exactly when $p_1 \wedge p_2 \wedge \cdots \wedge p_n \to q$ is a tautology.

# Rules of Inference

| Rule of Inference | Tautology | Name |
|---|---|---|
| $p \to q$<br>$p$<br>$\therefore\ q$ | $(p \land (p \to q)) \to q$ | Modus ponens |
| $p \to q$<br>$\neg q$<br>$\therefore\ \neg p$ | $(\neg q \land (p \to q)) \to \neg p$ | Modus tollens |
| $p \to q$<br>$q \to r$<br>$\therefore\ p \to r$ | $((p \to q) \land (q \to r)) \to (p \to r)$ | Hypothetical syllogism (Transitivity) |
| $p \lor q$<br>$\neg p$<br>$\therefore\ q$ | $((p \lor q) \land \neg p) \to q$ | Disjunctive syllogism (Elimination) |
| $p$<br>$\therefore\ p \lor q$ | $p \to (p \lor q)$ | Addition (Generalization) |
| $p \land q$<br>$\therefore\ p$ | $(p \land q) \to p$ | Simplification (Specialization) |
| $p$<br>$q$<br>$\therefore\ p \land q$ | $(p \land q) \to (p \land q)$ | Conjunction |
| $p \lor q$<br>$\neg p \lor r$<br>$\therefore\ q \lor r$ | $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$ | Resolution |

Table 4: Rules of Inference

# Subtopic

3. Valid and Invalid Arguments
   - Arguments
   - Validity
   - Modus Ponens and Modus Tollens
   - Rules of Inference
   - Fallacy
   - Contradictions and Valid Arguments

# Fallacy

A **fallacy** is an error in reasoning that results in an invalid argument.

Some common fallacies include:

- using ambiguous premises (and treating them as if they were unambiguous)
- circular reasoning (assuming what is to be proved without having derived it from the premises)
- jumping to a conclusion (without adequate grounds)
- converse error (i.e. fallacy of affirming the conclusion)
- inverse error (i.e. fallacy of denying the hypothesis)

# Converse Error

Consider the following argument:

> If you do every problem in the book, then you will learn discrete mathematics.
>
> You learned discrete mathematics.
>
> Therefore, you did every problem in the book.

Is this argument valid? The answer is **no**. Why?

The argument's premises are $p \to q$ and $q$. Its conclusion is $p$. This argument form is valid when $((p \to q) \wedge q) \to p$ is a tautology. However, we can prove that this is *not* a tautology using truth tables.

This type of incorrect reasoning is called the fallacy of affirming the conclusion or converse error.

# Inverse Error

Consider the following argument:

> If you do every problem in the book, then you will learn discrete mathematics.
>
> You did not do every problem in the book.
>
> Therefore, you did not learn discrete mathematics.

Is this argument valid? The answer is **no**. Why?

The argument's premises are $p \to q$ and $\neg p$. Its conclusion is $\neg q$. This argument form is valid when $((p \to q) \wedge \neg p) \to \neg q$ is a tautology. However, we can prove that this is *not* a tautology using truth tables.

This type of incorrect reasoning is called the fallacy of denying the hypothesis or inverse error.

# Subtopic

3 Valid and Invalid Arguments

- Arguments
- Validity
- Modus Ponens and Modus Tollens
- Rules of Inference
- Fallacy
- Contradictions and Valid Arguments

# Contradiction

Contradiction can be used to make inferences through a technique of reasoning called the *contradiction rule*.

## Definition

If it can be shown that the proposition $\neg p$ leads to a contradiction, then we can conclude that $p$ is true. Symbolically, we must show that $\neg p \to c$.

Now, it is natural to ask whether or not this is a valid argument form. It is. This can be shown using the truth table for the following argument form which represents the contradiction rule:

$$\neg p \to c$$
$$\therefore p$$

# Contradiction Rule

Recall that an argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion $q$ is valid when $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ is a tautology.

Here, the premise is $\neg p \to c$ and the conclusion is $p$. Hence, we must show that $(\neg p \to c) \to p$ is a tautology to prove that the contradiction rule is valid.

| $p$ | $\neg p$ | $c$ | $\neg p \to c$ | $(\neg p \to c) \to p$ |
|---|---|---|---|---|
| T | F | F | T | T |
| F | T | F | F | T |

As evident, $(\neg p \to c) \to p$ is a tautology. Therefore, the argument form of the contradiction rule is valid.

The contradiction rule is the logical heart of the method of proof by contradiction.

# Topic

# Boolean Variables and Boolean Expressions

In logic, variables such as $p$, $q$, and $r$ represent propositions. Furthermore, a proposition can have one of only two truth values: T (true) or F (false).

A propositional form (or compound proposition) is an expression, such as $p \wedge (\neg q \vee r)$, composed of propositional variables and logical connectives.

One of the founders of symbolic logic was the English mathematician George Boole. In his honor, any variable, such as a propositional variable or an input signal, that can take one of only two values is called a **Boolean variable**.

An expression composed of Boolean variables and the connectives $\neg$ , $\wedge$, and *lor* is called a **Boolean expression**.

# Topic

# References


- Epp, Susanna S. (2019). Discrete Mathematics with Applications (5th ed.). Cengage Learning. ISBN: 9781337694193.