FIRST SEMESTER EXAMINATION - 2008
MATHEMATICS - I

1. Answer the following questions:

(a) What do you mean by integrating factor of a differential equation?

(b) What do you mean by general solution and particular solution of a differential equation? What is the practical significance of these two concepts?

(c) What is the wronskian? What role does it play in getting solution of a differential equation?

(d) What does the convergence of a power series means? Why is it important?

(e) Using Rodrigue's formula find $P_2(x)$.

(f) What is the relationship between $J_n(x)$ and $J_{-n}(x)$?

(g) Is there exist any asymptote for the curve $x^2+y^2=25$? If so, find one.

(h) Fing the radius of curvature at the origin for the curve $x^3+y^3-2x^2+6y=0$.

(i) How can you say a real square matrix is orthogonal?
What is the unit step function? Why is it important in Laplace transform of a function?

(j) Explain the conditions for which a system of linear equations will possess more than one solution.

OR

What is the Laplace transform of the function $f(t)=t^2$ if $0<t<1$
$=0$ otherwise

2.(a) Find an integrating factor and solve the following differential equation :

$$2xy\,dx+3x^2dy=0$$

(b) Reduce the following differential equation into linear form and solve :

$$\frac{dy}{dx}+\frac{y}{3}=\frac{(1-2x)y^4}{3}$$

3.(a) Solve the following differential equation :

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0 \quad \text{where } y(0)=1.5 \text{ and } y'(0)=1.$$

(b) Using method of variation of parameter, solve the following differential equation :

$$\frac{d^2y}{dx^2}+9y=\sec 3x$$

4.(a) Show that the radius of curvature at any point of the astroid $x=a\cos^3\theta$, $y=a\sin^3\theta$ is equal to three times the length of the perpendicular from the origin to the tangent.

(b) Find all asymptote of the following curve : $x^2(x-y)^2+a^2(x^2-y^2)=a^2xy$

5.(a) Find a power series solution in powers of x of the following differential equation :

$$(1-x^2)y''=2xy$$

(b) Show that $G(u,x)=(1-2ux+u^2)^{\frac{-1}{2}}$ is the generating polynomial for the Legendre polynomial $P_n(x)$.

6.(a) Show that $\int J_{v+1}(x)dx=\int J_{v-1}(x)dx-2J_v(x)$ where $J_v(x)$ is the Bessel's function of the first kind of order v.

(b) Reduce the following differential equation in to Bessel's differential equation and find a general solution in terms of Bessel's function : $y''+x^2y=0$

7.(a) Solve the following system of linear equations by Gauss elimination :

$$4x-3y+3z=21$$
$$-x+2y-5z=-21$$
$$3x-6y+z=7$$

(b) Show that the rank of a matrix a is equal to the maximum number of linearly independent columns of A.

OR

(a) Find the Laplace transform of the following function :

$$f(t)=\begin{cases}2 & \text{if } 0<t<\pi\\ 0 & \text{if } \pi<t<2\pi\\ \sin t & \text{if } t>2\pi\end{cases}$$

(b) Using convolution theorem, find the inverse Laplace transform of the following function:

$$F(s)=\frac{1}{s(s^2+4)}$$

8.(a) Find the inverse of the following matrix by using Guass-Jordan elimination method :

$$\begin{bmatrix}1 & 8 & -7\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}$$

(b) Find a basis of eigen vector and diagonalize the matrix. $\begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}$

OR

(a) Solve the following differential equation by using Laplace transformation :

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+8y=e^{-3t}-e^{-5t} \quad \text{where } y(0)=0 \text{ and } y'(0)=0$$

(b) Using Laplace transform solve the following integral equation :

$$y(t)=1-\int_0^t (t-\tau)y(\tau)d\tau$$

ANSWERS - 2008

1.(a) $M(x,y)dx + N(x,y)\,dy = 0$, if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then. To make it exact choose a function is called an integrating factor.

(b) General solution will have arbitrary constants particular solution obtained from general solution by choosing particular values of the arbitrary constants.

(c) $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y_2y'_1$

If it is linear independent ($W \neq 0$), we get general solution.
If it is linear dependent (W=0), we cannot get solution.

(d) $S_n = \sum_{n=0}^{\infty} a_n x_n$, converges if $\lim_{n\to\infty} a_n =$ finite , diverges if $\lim_{n\to\infty} a_n = +\infty$ or $-\infty$

(e) $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

(f) $I_{-n}(x) = (-1)^n J_n(x)$

(g) No asymptotes to the curve $x^2 + y^2 = 25$

(h) The radius of curvature at origin $f = \frac{1}{3}$

(i) $A^T = A^{-1}$ or $A^TA = I$ or eigenvalue real on complex conjugate its absolute value =1

Or,

$U_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$

(j) If two lines are coincide or, rank (A) = rank $(\tilde{A}) = r < n$, then infinitely many solution.

Or,

$L(f(t)) = t^2\{U(t-0) - U(t-1)\} + 0 = t^2U(t) - (t-1)^2U(t-1) - 2(t-1)U(t-1) + U(t-1)$

$= \frac{2}{s^3} - \frac{2}{s^3}e^{-s} - \frac{2}{s^2}e^{-s} + \frac{e^{-s}}{s}$

2.(a) $2xy\,dx + 3x^2dy = 0$

Here $M = 2xy$, $N = 3x^2$, $\frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = 6x$, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-4x}{2xy} = -\frac{2}{y}$

$\mu = e^{\int \frac{2}{y}dy} = e^{2\ell n y} = y^2$, $\quad 2xy^3dx + 3x^2y^2dy = 0$

$\int_{y=c} M\,dx + \int_{\text{term's not contain 'x'}} N\,dy = c$, $\quad \int_{y=c} 2xy^3\,dx + 0 = c$, $\quad x^2y^3 = c$

(b) $\frac{dy}{dx} + \frac{y}{3} = \frac{(1-2x)}{3}y^4$, $\quad 3y^{-4}\frac{dy}{dx} + y^{-3} = (1-2x)$, Put $y^{-3} = t$

$-\frac{dt}{dx} + t = (1-2x)$, $-3y^{-4}\frac{dy}{dx} = \frac{dt}{dx}$, $\frac{dt}{dx} - t = 2x - 1$, $P = -1$, $Q = 2x - 1$, $\mu = e^{-x}$

The solution is $\mu \cdot t = \int \mu Q(x)dx + c$

$e^x \cdot t = \int (2x-1)e^{-x}dx + c$, $\quad e^{-x}y^{-3} = (2x-1)\frac{e^{-x}}{-1} - 2e^{-x} + c$, $\quad \frac{1}{y^3} = -2x - 1 + ce^x$

3.(a) It is an Euler cauchy equation

Put $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$m(m-1) - 2m + 2 = 0$, $\quad (m-1)(m-2) = 0$, $m = 1, 2$

The solution is $y = c_1x + c_2x^2$

$y(0) = 1.5$, $1.5 = c_10 + c_20$ (Not satisfied)

$y' = c_1 + 2c_2x$, $y'(0) = 1$, $c_1 = 1$

Hence $y = x + c_2x^2$ $\qquad$ ($\because$ c_2 is arbitrary)

(b) $\lambda^2 + 9 = 0$, $\lambda = \pm 3i$, $r(x) = \sec 3x$

$y_h = c_1\cos 3x + c_2\sin 3x$, $\quad W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$

$y_p = -y_1\int \frac{y_2 r(x)}{w}dx + y_2\int \frac{y_1 r(x)}{w}dx$

$= -\cos 3x \int \frac{\sin 3x \sec 3x}{3}dx + \sin 3x \int \frac{\cos 3x \sec 3x}{3}dx = -\frac{1}{4}\cos 3x\,\ell n \sec 3x + \frac{1}{3}x\sin 3x$

$y = y_h + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{4}\cos 3x\,\ell n\sec 3x + \frac{1}{3}x\sin 3x$

4.(a) $x = a\cos^3\theta$, $\quad y = a\sin^3\theta$, $\quad \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$\frac{d^2x}{d\theta^2} = -3a\cos^3\theta + 6a\cos\theta\sin^2\theta$, $\quad \frac{d^2y}{d\theta^2} = -3a\sin^3\theta + 6a\sin\theta\cos^2\theta$

$$\rho=\frac{\left[x'^2+y'^2\right]^{\frac{3}{2}}}{x'y''-y'x''}=3a\sin\theta\cos\theta=3(axy)^{1/3}$$

So here (0, 0) to (x, y), $(x,y)=\left(a\cos^3\theta, a\sin^3\theta\right)$

$y-a\cos^3\theta=\frac{dy}{dx}\left(x-a\cos^3\theta\right)$, $\quad y-a\cos^2\theta=-\tan\theta\left(x-a\cos^3\theta\right)$ $\left(\because \frac{dy}{dx}=-\tan\theta\right)$

$x\sin\theta+y\cos\theta-a\cos\theta\sin\theta=0$

Distance $(P)=\frac{x\sin\theta+y\cos\theta-a\sin\theta\cos\theta}{\sqrt{\cos^2\theta+\sin^2\theta}}$

At (0, 0), $P=a\cos\theta\sin\theta$

So distance $=(axy)^{1/3}$

(b) $\phi_4(m)=(1-m)^2, \phi'_4(m)=-2(1-m), \phi''_4(m)=2$

$\phi_3(m)=0$, $\phi'_3(m)=0$, $\phi_2(m)=a^2\left(1-m^2\right)-a^2m$, $\phi_1(m)=0$, $\phi_0(m)=0$

Put $\phi_4(m)=0$, $\Rightarrow (1-m)^2=0$, $m=1, 1$

$\frac{c^2}{2!}\phi''_4(m)+c\phi'_3(m)+\phi_2(m)=0$, $\frac{c^2}{2}(2)+c(0)+a^2\left(1-m^2-m\right)=0$

$c^2+a^2\left(1-m^2-m\right)=0$ $\quad$ Put $m=1$, $c^2-a^2=0$, $c=\pm a$

The required asymptotes is $y=mx+c$ i.e. $y=x\pm a$

5.(a) Put $\quad y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots\ldots$

$y'=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots\ldots$, $\quad y''=2a_2+6a_3x+12a_4x^2+\ldots\ldots$

Putting it, $\left(1-x^2\right)\left(2a_2+6a_3x+12a_4x^2+\ldots\right)-2x\left(a_0+a_1x+a_2x^2+a_3x^3+\ldots\right)=0$

$2a_2+\left(6a_3-2a_0\right)x+\left(12a_4-2a_2-2a_1\right)x^2+\ldots\ldots=0$

Equating the coefficient of 'x' to zero

$2a_2=0$, $a_2=0$, $6a_3-2a_0=0$, $a_3=\frac{a_0}{3}$, $\quad 12a_4-2a_2-2a_1=0$, $a_4=\frac{a_1}{6}$

Hence the required solution is

$$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots\ldots$$

$$=a_0+a_1x+\frac{a_0}{3}x^3+\frac{a_1}{6}x^4+\ldots\ldots \quad =a_0\left(1+\frac{x^3}{3}+\ldots\ldots\right)+a_1\left(x+\frac{x^4}{6}+\ldots\ldots\right)$$

(b) $\left(1-2xu+u^2\right)^{-\frac{1}{2}}=1-\left(-\frac{1}{2}\right)\left(2xu-u^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{\underline{|2}}\left(2xu-u^2\right)^2+\ldots\ldots$

$$=1+\frac{2xu-u^2}{2}+\frac{1}{2}\cdot\frac{3}{2}\frac{\left(2xu-u^2\right)^2}{\underline{|2}}+\ldots\ldots \quad =1+ux+u^2\left(\frac{3x^2-1}{2}\right)+u^3\left(\frac{5x^3-3x}{2}\right)+\ldots\ldots$$

$$=P_0(x)+P_1(x)u+P_2(x)u^2+P_3(x)u^3+\ldots\ldots\ldots =\sum_{n=0}^{\infty}P_n(x)u^3$$

6.(a) We know that $\frac{d}{dx}\left[x^VJ_V(x)\right]=x^VJ_{V-1}(x)$

$x^VJ'_V(x)+Vx^{V-1}J_V(x)=x^VJ_{V-1}(x)$

$J'_V(x)+\frac{V}{x}J_V(x)=J_{V-1}(x)$ $\quad$ ...(1)

We know that $\frac{d}{dx}\left[x^{-V}J_V(x)\right]=-x^VJ_{V+1}(x)$

$x^{-V}J'_V(x)-Vx^{-V-1}J_V(x)=-x^VJ_{V+1}(x)$

$J'_V(x)-\frac{V}{x}J_V(x)=-J_{V+1}(x)$ $\quad$ ...(2)

Adding (1) and (2) we get, $2J'_V(x)=J_{V-1}(x)-J_{V+1}(x)$

Integrate $2J_V(x)=\int J_{V-1}(x)dx-\int J_{V+1}(x)dx$

Hence $\int J_{V+1}(x)dx=\int J_{V-1}(x)dx-2J_V(x)$

(b) Put $y=U\sqrt{x}$, $\frac{1}{2}x^2=z$, $\frac{1}{2}x^2=z$, $x=(2z)^{1/2}$, $\frac{dz}{dx}=x$

$$\frac{dy}{dx}=\frac{u}{2}x^{-\frac{1}{2}}+x^{\frac{1}{2}}\frac{du}{dx}=\frac{u}{2}(2z)^{-1/4}+(2z)^{1/4}\frac{du}{dz}\cdot\frac{dz}{dx} \quad =\frac{u}{2}(2z)^{1/4}+(2z)^{3/4}\frac{du}{dz}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dz}\left(\frac{dy}{dx}\right)\cdot\frac{dz}{dx} \quad =\frac{d}{dz}\left(\frac{u}{2}(2z)^{-1/4}+(2z)^{3/4}\frac{du}{dz}\right)\cdot\frac{dz}{dx}$$

$$=\left(\frac{u}{2}\left(-\frac{1}{4}\right)(2z)^{-\frac{5}{4}}\cdot 2+(2z)^{3/4}\frac{d^2u}{dz^2}\div\frac{3}{4}(2z)^{-1/4}=\frac{du}{dz}\right)$$

$$=\frac{u}{4}(2z)^{-\frac{3}{4}}+2(2z)^{\frac{1}{4}}\frac{du}{dz}+(2z)^{\frac{5}{4}}\frac{d^2u}{dz^2}(2z)^{\frac{1}{2}}$$

Putting it, $-\frac{u}{4}(2z)^{-\frac{3}{4}}+2(2z)^{\frac{1}{4}}\frac{du}{dz}+(2z)^{\frac{5}{4}}\frac{d^2u}{dz^2}+(2z)^{\frac{5}{4}}u=0$

$$(2z)^{\frac{5}{4}}\frac{d^2u}{dz^2}+2(2z)^{1/4}\frac{du}{dz}+u(2z)^{\frac{5}{4}}-\frac{(2z)^{-\frac{3}{4}}}{u}=0$$

Multiplying $2^{-5/4}z^{3/4}$, we get, $z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+u\left(z^2-\left(\frac{1}{4}\right)^2\right)=0$

Hence $u=AJ_1/4(z)+BY_{1/4}(z)$, $y=\left(AJ_{1/4}\left(\frac{x^2}{2}\right)+BY_{1/4}\left(\frac{x^2}{2}\right)\right)\sqrt{x}$

7.(a) $4x-3y+3z=21$
$-x+2y-5z=-21$
$3x-6y+z=7$

$$\begin{bmatrix}4&-3&3&21\\-1&2&-5&-21\\3&-6&1&7\end{bmatrix}$$

$4x-3y+3z=21$
$\frac{5}{4}y-\frac{17}{4}z=-\frac{63}{4}$
$-\frac{15}{4}y-\frac{5}{4}z=\frac{-35}{4}$

$$\begin{bmatrix}4&-3&3&21\\0&\frac{5}{4}&-\frac{17}{4}&-\frac{63}{4}\\0&-\frac{15}{4}&-\frac{5}{4}&-\frac{35}{4}\end{bmatrix}$$

$4x-3y+3z=21$
$\frac{5}{4}y-\frac{17}{4}z=-\frac{63}{4}$
$-\frac{56}{4}z=-\frac{224}{4}$

$$\begin{bmatrix}4&-3&3&21\\0&\frac{5}{4}&-\frac{17}{4}&-\frac{63}{4}\\0&0&-\frac{56}{4}&-\frac{224}{4}\end{bmatrix}$$

Hence $z=4$, $y=1$ and $x=3$

(b) Let A be an m x m matrix and let rank A = r.
Then by definition, A has a linearly independent set of r row vector, called them v_1, v_2,...v_r and all row vectors $a_1, a_2, \ldots a_m$ of A are linear combinations of those independent ones

$a_1=c_{11}v_1+c_{12}v_2+\ldots\ldots+c_{1r}v_r$

$a_2=c_{21}v_1+c_{22}v_2+\ldots\ldots+c_{2r}v_r$

....

$a_m=c_{m_1}v_1+c_{m_2}v_2+\ldots\ldots+c_{m_r}v_r$

These are vector equations. Each of them is equivalent to n equations for corresponding components. Denoting the components of $V_{(1)}$ by $V_{11}, V_{12}, \ldots\ldots, V_{1n}$ the component of

$V_{(2)}$ by $V_{21}, V_{22}, \ldots\ldots, V_{2n}$ etc. and similarly for the vectors on the left side, we thus have for $k=1, 2, \ldots\ldots, n$.

This can be written as

$$\begin{bmatrix}a_{1k}\\a_{2k}\\\vdots\\a_{mk}\end{bmatrix}=V_{1k}\begin{bmatrix}c_{11}\\c_{12}\\\vdots\\c_{m1}\end{bmatrix}+V_{2k}\begin{bmatrix}c_{12}\\c_{22}\\\vdots\\c_{m2}\end{bmatrix}+\ldots\ldots+V_{rk}\begin{bmatrix}c_{1r}\\c_{2r}\\\vdots\\c_{mr}\end{bmatrix}$$

where $k=1, 2, \ldots\ldots n$.

The vector on the left is the kth column of A.

Hence the equation shows that each column vector of A is linear combination of the r vectors on the right. Hence the maximum number of L.I column vector of A cannot exceed r. which is the maximum number of L.I row vector of A by the definition of rank

OR

(a) $(f(t))=2(u(t-0)-u(t-\pi))+0(u(t-\pi)-u(t-2\pi))+\sin t\, u(t-2\pi)=\frac{2}{s}-\frac{2}{s}e^{-\pi s}+\frac{e^{-2\pi s}}{s^2+1}$

(b) $f(t)=1\times\frac{\sin 2t}{2}=\int_0^t 1.\frac{\sin 2(t-\tau)}{2}d\tau=\left[\frac{1}{4}\cos 2(t-\tau)\right]_0^t=\frac{1}{4}(1-\cos 2t)$

8.(a) $$\begin{bmatrix}1&8&-7\\0&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&8&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&7\\0&1&-3\\0&0&1\end{bmatrix}\begin{matrix}R_1=R_1+7R_3\\R_2=R_2-3R_3\end{matrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&-8&31\\0&1&-3\\0&0&1\end{bmatrix}R_1=R_1-8R_2$$

$$A^{-1}=\begin{bmatrix}1&-8&31\\0&1&-3\\0&0&1\end{bmatrix},\qquad AA^{-1}=\begin{bmatrix}1&8&-7\\0&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}1&-8&31\\0&1&-3\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

(b) The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix}2-\lambda&1\\2&1-\lambda\end{vmatrix}=0,\ (2-\lambda)(1-\lambda)-2=0,\ \lambda^2-3\lambda=0,\ \lambda=0,3$$

For $\lambda=0$, apply for eigenvector $[A-\lambda I]x=\bar{0}$

$$\begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\bar{0},\quad 2x_1+x_2=0,\ 2x_1+x_2=0$$

After solving we get $x_2=-2x_1$

The eigenvector is $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-2x_1\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$

For $\lambda=3$ apply for eigen vector $\left[\bar{A}-\lambda\bar{I}\right]x=\bar{0}$, $\begin{bmatrix}-1 & 1\\2 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\bar{0}$,

$-x_1+x_2=0,\ 2x_1-2x_2=0$

After solving, we get $x_1=x_2$

The eigen vector is $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_2\\x_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}\forall x_2\neq 0$

$$x=\begin{bmatrix}1 & 1\\-2 & 1\end{bmatrix},|x|=3,\ x^{-1}=\frac{\operatorname{adj}x}{|x|}=\frac{1}{3}\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}$$

$$D=x^{-1}Ax=\frac{1}{3}\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\-2 & 1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}0 & 3\\0 & 3\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 3\end{bmatrix}$$

OR

(a) $S^2L(y)-Sy(0)+6(SL(y)-y(0))+8L(y)=\dfrac{1}{S+3}-\dfrac{1}{S+5}$

$$(s^2+6s+8)L(y)=\frac{2}{(s+3)(s+5)}$$

$$L(y)=\frac{2}{(s+2)(s+3)(s+4)(s+5)}=\frac{A}{s+2}+\frac{B}{s+3}+\frac{C}{s+4}+\frac{D}{s+5}$$

After solving we get $A=\frac{1}{3},B=-1,C=1,D=-\frac{1}{3}$

$$y=L^{-1}\left(\frac{\frac{1}{3}}{s+2}+\frac{-1}{s+3}+\frac{1}{s+4}+\frac{\frac{-1}{3}}{s+5}\right)=\frac{1}{3}e^{-2t}-e^{-3t}+e^{-4t}-\frac{1}{3}e^{-5t}$$

(b) $y(t)=1-y(t)*t,\qquad L(y)=\frac{1}{s}-L(y)\frac{1}{s^2},\qquad L(y)\left(1+\frac{1}{s^2}\right)=\frac{1}{s}$

$$L(y)=\frac{s}{s^2+1}\Rightarrow y=L^{-1}\left(\frac{s}{s^2+1}\right)=\cos t$$

FIRST SEMESTER EXAMINATION - 2009
MATHEMATICS - I

Answer the following questions:

1.
(a) Find the general solution of $y'+y\sin x=xe^{\cos x}$
(b) Solve : $y''+4y=0,\ y(0)=3,y(\pi/2)=0$
(c) Find the general solution of $x^2y''+xy'+y=0$
(d) Find the solution of $(y+1)dx-(x+1)dy=0$
(e) Show that : $2P_2(x)=3xP_1(x)-P_0(x)$ where $P_n(x)$ is the Legendre Polynomial.
(f) Find : $\int J_3(x)dx$ where $J_n(x)$ is the Bessel function.
(g) Find the asymptotes the curve, $y(x-y)^2=x+y$
(h) Find the rank of the following matrix :

$$\begin{bmatrix}3 & -1 & 5\\2 & -4 & 6\\10 & 0 & 14\end{bmatrix}$$

(i) Let , $A=\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$, $P=\begin{bmatrix}1 & 3\\3 & 6\end{bmatrix}$ find an eigenvector of $P^{-1}AP$.

(j) Show that the determinant of an orthogonal matrix can take only two values.

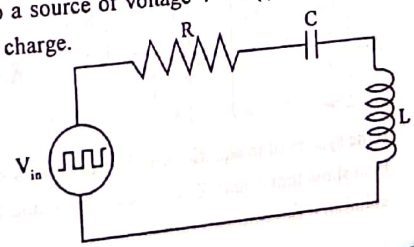
2.(a) Find the general solution of : $(D^2+2D+2)y=4e^{-x}\sec^3 x$

(b) Solve the equation : $x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0$ by reducing the order and using $y=x^{-\frac{1}{2}}\cos x$ as a solution.

3.(a) Solve the differential equation : $y''+9y=6\cos 3x,y(0)=1,y'(0)=0$ by the method of undetermined coefficients.

(b) Solve : $2xy'+(x-1)y=y^{-1}x^2e^x,y(0)=1$.

4.(a) Find the current (t) in RLC-circuit below with R = 80 ohms, L = 10 henrys, C = 0.004 farad and which is connected to a source of voltage V = E(t) = 240.5 sin 10t volts assuming zero initial current and charge.



(b) Find a power series solution in powers of x of the following differential equation :

$(1-x^2)y''-2xy'+2y=0$.

5.(a) Prove that for the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the radius of curvature $\rho=\frac{a^2b^2}{p^3}$

where p is the perpendicular from the centre of the ellipse on the tangent to the ellipse at the point (x, y).

(b) Show that for Legendre Polynomial $P_n(x)$, $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left[x^2-1\right]^n\right]$

6.(a) Show that : $J_{v-1}(x)+J_{v+1}(x)=\frac{2V}{x}J_v(x)$, $J_{v-1}(x)-J_{v+1}(x)=2J'_v(x)$,

where $J_v(x)$ is the Bessel's function.

(b) (i) Find the conic section represented by the quadratic form :

$$-11x_1^2+84x_1x_2+24x_2^2=156$$

(ii) Find the general solution of the following equation in terms of Bessel's function after reducing it to Bessel's differential equation :

$$4x^2y''+4xy'+(100x^2-9)y=0$$

7.(a) Diagonalize the following matrix after finding the basis vectos :

$$\begin{bmatrix}-43 & 77\\ 13 & 93\end{bmatrix}$$

(b) (i) Show that the eigenvalues of a Hermitian matrix are real.

(ii) Show that eigenvalues of a unitary matrix have absolute value 1.

8.(a) Solve the following system of equations by Gauss elimination method :

$10x+4y-2z=-4$

$17x+y+2z=14$

$34x+5y-3z=-8$

Also find the echelon form of the corresponding coefficient matrix.

(b) Let

$$A=\begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \cdot & \cdots & \cdot\\ \cdot & \cdots & \cdot\\ a_{m1} & \cdots & a_{mn}\end{bmatrix},\bar{x}=\begin{bmatrix}x_1\\ \cdot\\ \cdot\\ x_n\end{bmatrix},\bar{b}=\begin{bmatrix}b_1\\ \cdot\\ \cdot\\ b_m\end{bmatrix},\quad \bar{A}=\begin{bmatrix}a_{11} & \cdots & a_{1n} & b_1\\ \cdot & \cdots & \cdot & \cdot\\ \cdot & \cdots & \cdot & \cdot\\ a_{m1} & \cdots & a_{mn} & b_m\end{bmatrix}$$

If the system of m equations and n unknowns $x_1, \ldots\ldots\ldots, x_n$ given by $A\bar{x}=\bar{b}$ has solutions, then show that A and $\bar{A}$ have same rank and the system of equations has precisely one solution if the common rank is n.

ANSWERS - 2009

1.(a) $P=\sin x,\ a=xe^{\cos x},\ \mu=e^{\int \sin x\,dx}=e^{-\cos x},\quad e^{-\cos x}\cdot y=\int x\,dx+c,\ y=e^{\cos x}\left(\frac{x^2}{2}+c\right)$

(b) The characteristic equation is $\lambda^2+4=0,\ \lambda=\pm 2i$

$y=c_1\cos 2x+c_2\sin 2x,\ y(0)=3,\ c_1=3,\ y\left(\frac{\pi}{2}\right)=0,\ c_1=0$

(Not possible two values of c_1)

(c) The characteristic equation for Euler cauchy is

$m(m-1)+m+1=0,\qquad m^2+1=0,\ m=\pm i,\ y=c_1\cos(\ell nx)+c_2\sin(\ell nx)$

(d) $(y+1)dx-(x+1)dy=0,\ \frac{dy}{y+1}=\frac{dx}{x+1}$

After taking integration $(y+1)=c(x+1)$

(e) LHS $=3xP_1(x)-P_1(x)\ =3x(x)-1\ =3x^2-1\ =2\left(\frac{3x^2}{2}-\frac{1}{2}\right)\ =2P_2(x)$

(f) $\int J_3(x)dx=\int J_1(x)dx-2J_2(x)\qquad (\because J_3(x)=J_1(x)-2J'_2(x))$

$=-J_0(x)-2J_2(x)\qquad (\because J_1(x)=-J'_0(x))$

(g) $\phi_3(m)=m(1-m)^2,\ \phi'_{3m}(m)=-2m(1-m)+(1-m)^2,\phi''_3(m)=2m-4(1-m)=cm-4$

$\phi_2(m)=0,\ \phi'_2(m)=0,\ \phi_1(m)=-(1+m),\ \phi_0(m)=0$

Pu $\phi_3(m)=0,\ m(1-m)^2=0, m=0, 1, 1$

For $m=0,\ c\phi'_3(m)+\phi_2(m)=0, c=0$

The asymptotes is $y=0$

For $m=1,\ \frac{c^2}{\lfloor 2}\phi''_3(m)+c\phi'_3(m)+\phi_2(m)=0$

$\frac{c^2}{2}(6m-4)+c(0)+(-|c|+m)=0,\ \frac{c^2}{2}(2)-2=0,\ c^2=2, c=\pm\sqrt{2}$

The asymptotes are $y=x+\sqrt{2}, y=x-\sqrt{2}$

(h) $$\begin{bmatrix}3 & -1 & 5\\ 2 & -4 & 6\\ 10 & 0 & 14\end{bmatrix} = \begin{bmatrix}3 & -1 & 5\\ 0 & -\frac{10}{3} & \frac{8}{3}\\ 0 & \frac{10}{3} & -\frac{8}{3}\end{bmatrix} \begin{matrix}R_2 = R_2 - \frac{2}{3}R_1\\ R_3 = R_3 - \frac{10}{3}R_1\end{matrix} = \begin{bmatrix}3 & -1 & 5\\ 0 & -\frac{10}{3} & \frac{8}{3}\\ 0 & 0 & 0\end{bmatrix} R_3 = R_3 + R_2$$

Rank = 2

(i) $A = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ $\quad P = \begin{bmatrix}1 & 3\\ 3 & 6\end{bmatrix}, |P| = -3, P^{-1} = \frac{\text{adj}P}{|P|} = \frac{1}{-3}\begin{bmatrix}6 & -3\\ -3 & 1\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 1 & -\frac{1}{3}\end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix}-2 & 1\\ 1 & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}1 & 3\\ 3 & 6\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 1 & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}7 & 15\\ 14 & 30\end{bmatrix} = \begin{bmatrix}0 & 0\\ \frac{7}{3} & 5\end{bmatrix}$$

The eigenvalues are $\begin{vmatrix}0-\lambda & 0\\ \frac{7}{3} & 5-\lambda\end{vmatrix} = 0$

$\lambda = 0, 5$

For $\lambda = 0$, apply for eigenvector $(A - \lambda J)x = \bar{0}$

$$\begin{bmatrix}0 & 0\\ \frac{7}{3} & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \bar{0}$$

The eigenvector is $x = \begin{bmatrix}7\\ -15\end{bmatrix}$

For $\lambda = 5$, apply for eigenvector $[A - \lambda I]x = \bar{0}$

$$\begin{bmatrix}-5 & 0\\ \frac{7}{3} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \bar{0}$$

The eigenvector is $x = \begin{bmatrix}0\\ 1\end{bmatrix}$

(j) The determinant of an orthogonal matrix is + 1 or −1

Example : $A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $A^T = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $A^TA = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

It is orthogonal. $|A| = 1$

Example : $A = \begin{bmatrix}0 & 0 & -1\\ 0 & +1 & 0\\ -1 & 0 & 0\end{bmatrix}, A^T = \begin{bmatrix}0 & 0 & -1\\ 0 & +1 & 0\\ 1 & 0 & 0\end{bmatrix}$

$$A^TA = \begin{bmatrix}0 & 0 & -1\\ 0 & +1 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & -1\\ 0 & +1 & 0\\ -1 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

It is orthogonal. $|A| = -1$

Hence it has two values.

2.(a) The characteristic equation is

$$\lambda^2 + 2\lambda + 2 = 0, \quad \lambda = -1 \pm i, \quad y_h = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$$W = \begin{vmatrix}e^{-x}\cos x & e^{-x}\sin x\\ -e^{-x}\sin x - e^{-x}\cos x & e^{-x}\cos x - e^{-x}\sin x\end{vmatrix} = e^{-2x},\ r(x) = 4e^{-x}\sec^3 x$$

$$y_p = -y_1\int\frac{y_2 r(x)}{W}dx + y_2\int\frac{y_1 r(x)}{W}dx$$

$$= -e^{-x}\cos x\int\frac{(e^{-x}\sin x)(4e^{-x}\sec^3 x)}{e^{-2x}}dx + e^{-x}\sin x\int\frac{(e^{-x}\cos x)(4e^{-x}\sec^3 x)}{e^{-2x}}dx$$

$$= -4e^{-x}\cos x\int\frac{\sin x}{\cos^3 x}dx + 4e^{-x}\sin x\int\sec^2 x\,dx$$

$$= -4e^{-x}\cos x\left(-\frac{1}{2\cos^2 x}\right) + 4e^{-x}\sin x\tan x \quad = 2e^{-x}\sec x + 4e^{-x}\sin^2 x\cos x$$

Hence $y = y_h + y_p = e^{-x}(c_1\cos x + c_2\sin x) + 2e^{-x}\sec x + 4e^{-x}\sin^2 x\cos x$

(b) First we write it in linear form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0, \text{ Hence } P = \frac{1}{x}, Q = 1 - \frac{1}{4x^2}$$

Apply $y_2 = y_1\int M\,dx$, $y_1 = x^{-\frac{1}{2}}\cos x$

$$M = \frac{1}{y_1^2}e^{-\int P\,dx} = \frac{x}{\cos x}e^{-\int\frac{1}{x}dx} = \frac{1}{\cos x}$$

$$y_2 = x^{-\frac{1}{2}}\cos x\int\sec x\,dx = x^{-\frac{1}{2}}\cos x\,\ell n(\sec x + \tan x)$$

3.(a) The characteristic equation is

$$\lambda^2+9=0,\quad \lambda=\pm 3i,\quad y_h=c_1\cos 3x+c_2\sin 3x$$

$y_p=M\cos 3x+M\sin 3x$, $y'_p=-3K\sin 3x+3M\cos 3x$

$y''_p=-9K\cos 3x-9M\sin 3x$

Putting it we get LHS is zero apply case - 1

$y_p=x(K\cos 3x+M\sin 3x)$, $y'_p=x(-3K\sin 3x+3M\cos 3x)+(K\cos 3x+M\sin 3x)$

$y''=x(-9K\cos 3x-9M\sin 3x)+(-6K\sin 3x+6M\cos 3x)$

Putting it

$x(-9K\cos 3x-9M\sin 3x)+(-6K\sin 3x+6M\cos 3x)+9x(K\cos 3x+M\sin 3x)=6\cos 3x$

After equating we get $K=0, M=1$

$y_p=x\sin 3x$, $y=y_h+y_p=c_1\cos 3x+c_2\sin 3x+x\sin 3x$

$y(0)=1, c_1=1$, $y'=-3c_1\sin 3x+3c_2\cos 3x+\sin 3x-3x\cos 3x$, $y'(0)=0$, $c_2=0$

Hence $y=\cos 3x+x\sin 3x$

(b) $2xy'+(x-1)y=y^{-1}x^2e^x$, $2xyy'+(x-1)y^2=x^2e^x$

$$2yy'+\left(1-\frac{1}{x}\right)y^2=xe^x,\ \text{put } y^2=t, 2yy'=\frac{dt}{dx}$$

$$\frac{dt}{dx}+\left(1-\frac{1}{x}\right)t=xe^x, P=\left(1-\frac{1}{x}\right), Q=xe^x,\ \mu=e^{\int\left(1-\frac{1}{x}\right)dx}=e^{x-\ell nx}=\frac{e^x}{x}$$

The solution is $\mu\cdot t=\int\mu\cdot Q(x)dx+c$

$$\frac{e^x}{x}\cdot y^2=\int\frac{e^x}{x}\cdot xe^x dx+c,\quad \frac{e^x}{x}y^2=\frac{1}{2}e^{2x}+c,\quad y^2=\frac{1}{2}e^x+cx\,e^{-x}$$

4.(a) The R-L-C circuit equation is

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{c}=\frac{dE}{dt},\qquad 10\frac{d^2I}{dt^2}+80\frac{dI}{dt}+\frac{I}{0.004}=2405\cos 10t$$

$$\frac{d^2I}{dt^2}+8\frac{dI}{dt}+25I=240.5\cos 10t$$

The characteristic equation is $\lambda^2+8\lambda+25=0$, $\lambda=-4\pm 3i$, $I_h=e^{-4t}(c_1\cos 3t+c_2\sin 3t)$

choose $I_p=K\cos 10t+M\sin 10t$, $y'_p=-10K\sin 10t+10M\cos t$

$I''_p=-100K\cos 10t-100M\sin t$

putting it, $(-100K\cos 10t-100M\sin t)+8(-10K\sin t+10M\cos t)$

$+25(K\cos 10t+M\sin 10t)=240.5\cos 10t$

$(-75K+80M)\cos t+(-75M-80K)\sin t=240.5\cos 10t$

Equating we get, $-75K+80M=240.5$, $-75M-80K$

After solving $K=-1.5, M=1.6$, $I_p=-1.5\cos 10t+1.6\sin 10t$

$I=I_h+I_p=e^{-4t}(c_1\cos 3t+c_2\sin 3t)-1.5\cos 10t+1.6\sin 10t$, $I(0)=0, c_1=1.5$

$I'=e^{-4t}(-3c_1\sin 3t+3c_2\cos 3t)-4e^{-4t}(c_1\cos 3t+c_2\sin 3t)-15\sin 10t+16\cos 10t$

$I'(0)=0$, $c_2=-10/3$

$$I=e^{-4t}\left(1.5\cos 3t-\frac{10}{3}\sin 3t\right)-1.5\cos 10t+1.6\sin 10t$$

(b) $(1-x^2)y''-2xy'+2y=0$

Put $y=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots$

$y'=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots$, $y''=2a_2+6a_3x+12a_4x^2+\ldots$

Putting it $(1-x^2)(2a_2+6a_3x+12a_4x^2+\ldots\ldots)$

$-2x(a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots\ldots)+2(a_0+a_1x+a_2x^2+a_3x^3+\ldots)=0$

Equating the coefficient of 'x' to zero

$2a_2+2a_0=0$, $a_2=-a_0$

$a_3=0$, $a_3=0$

$12a_4-4a_2=0$, $a_4=\frac{a_2}{3}=-\frac{a_0}{3}$

$20a_5-10a_3=0$, $a_5=0$

Hence the required solution is

$$y=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots=a_0+a_1x-a_0x^2-\frac{a_0}{3}x^4\ldots\ldots$$

$$=a_1x+a_0\left(1-x^2-\frac{x^4}{3}-\ldots\ldots\right)$$

5.(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, Put $x=a\cos\theta$, $y=b\sin\theta$

$$\frac{dx}{d\theta}=-a\sin\theta,\ \frac{dy}{d\theta}=b\cos\theta,\qquad \frac{d^2x}{d\theta^2}=-a\cos\theta,\ \frac{d^2y}{d\theta^2}=-b\sin\theta$$

$$\rho=\frac{\left[x'^2+y'^2\right]^{3/2}}{|x'y''-y'x''|}=\frac{\left[a^2\sin^2\theta+b^2\cos^2\right]^{3/2}}{|ab|}$$

So here (0, 0) to (x, y), $(x,y)=(a\cos\theta, b\sin\theta)$

$$y-b\sin\theta=\frac{dy}{dx}(x-a\cos\theta),\ y-b\sin\theta=-\frac{b}{a}\cot\theta(x-a\cos\theta),\ ay\sin\theta+bx\cos\theta-ab=0$$

$$\text{Distance }(P)=\frac{|xb\cos\theta+ya\sin\theta-ab|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

At (0, 0), $P=\frac{ab}{(b^2\cos^2\theta+a^2\sin^2\theta)^{1/2}}$, Hence $f=\frac{\frac{1}{ab}}{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}=\frac{a^2b^2}{p^3}$

(b) $(x^2-1)^n=nc_0x^{2n}-nc_1x^{2n-2}+\ldots\ldots+nc_n(-1)^n$

$$=\sum_{m=0}^{n}(-1)^m nc_m(x^2)^{n-m}=\sum_{m=0}^{n}(-1)^m\frac{n!}{m!(n-m)!}x^{2n-2m}$$

$$=\sum_{m=0}^{M}\frac{(-1)^m n!}{m!(n-m)!}x^{2n-2m}+\sum_{n=M+1}^{n}\frac{(-1)^m n!}{m!(n-m)!}x^{2n-2m}$$

$0\le m\le M,\quad 2n-2m\ge n$

$M+1\le m\le n,\quad 2n-2m<n$

$$\text{R.H.S.}=\frac{d^n\left[(x^2-1)^n\right]}{dx^n}=\frac{d^n}{dx^n}\left\{\sum_{m=0}^{M}\frac{(-1)^m n!}{m!(n-m)!}x^{2n-2m}+\sum_{n=M+1}^{n}\frac{(-1)^m n!x^{2n-2m}}{m!(n-m)!}\right\}$$

$$=\sum_{m=0}^{M}\frac{(-1)^m n!(2n-2m)!}{m!(n-m)!(n-2m)!}x^{n-2m}+0$$

Multiplying $\frac{1}{2^n n!}$ on both the sides

$$\frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}=\sum_{m=0}^{M}\frac{(-1)^m(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}=P_n(x)$$

Hence $P_n(x)=\frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}$

6.(a) We know that $\frac{d}{dx}\left[x^vJ_v(x)\right]=x^vJ_{v-1}(x)$

Differentiating w.r.t. 'x' we get, $x^vJ'_v(x)+Vx^{v-1}J_v(x)=x^vJ_{v-1}(x)$

$J'_v(x)+\frac{v}{x}J_v(x)=J_{v-1}(x)$ ....(1)

We know that $\frac{d}{dx}\left[x^{-v}J_v(x)\right]=-x^vJ_{v+1}(x)$, $x^{-v}J'_v(x)+(-v)x^{-v-1}J_v(x)=-x^vJ_{v+1}(x)$

$J'_v(x)-\frac{V}{x}J_v(x)=-J_{v+1}(x)$ ....(2)

Substrating (1) and (2) we get, $J_{v-1}(x)+J_{v+1}(x)=\frac{2}{x}vJ_v(x)$

Adding (1) and (2) we get, $J_{v-1}(x)-J_{v+1}(x)=2J'_v(x)$

(b) (i) $Q=x^TAx$, $-11x^2+84x_1x_2+24x_2^2=156$

$Q=156,\ A=\begin{bmatrix}-11 & 42\\ 42 & 24\end{bmatrix}, X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix}-11-\lambda & 42\\ 42 & 24-\lambda\end{vmatrix}=0$$

$(-11-\lambda)(24-\lambda)-1764=0$, $\lambda=52,-39$

The equation for conic section is

$Q=\lambda_1y_1^2+\lambda_2y_2^2+\ldots\ldots+\lambda_ny_n^2$

$52y_1^2-39y_2^2=156$ It is an Hyperbola.

(ii) Put $5x=z$, $\frac{dz}{dx}=5$, $\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=5\frac{dy}{dz}$, $\frac{d^2y}{dx^2}=25\frac{d^2y}{dz^2}$

Putting it, $100x^2\frac{d^2y}{dz^2}+20x\frac{dy}{dz}+(100x^2-9)y=0$

$$25x^2\frac{d^2y}{dz^2}+5x\frac{dy}{dz}+\left(25x^2-\frac{9}{4}\right)y=0,\quad z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+\left(z^2-\left(\frac{3}{z}\right)^2\right)y=0$$

Bessel equation of first kind and second kind solution

$y=AJ_{\frac{3}{2}}(x)+BJ_{-\frac{3}{2}}(x)$, $y=A\,J_{\frac{3}{2}}(x)+B\,y_{\frac{3}{2}}(x)$

7.(a) The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix} -43-\lambda & 77 \\ 13 & 93-\lambda \end{vmatrix}=0$$

$(-43-\lambda)(93-\lambda)-77\times 13=0$, $\lambda=100,-50$

For $\lambda=100$, we find the eigenvector $[A-\lambda I]x=\bar{0}$

$$\begin{bmatrix} -143 & 77 \\ 13 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\bar{0}$$

$-143x_1+77x_2=0, 13x_1-7x_2=0$

After solving, we get $x_2=\frac{13}{7}x_1$

The eigenvector is $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 7 \\ 13 \end{bmatrix}\forall x_1\neq 0$

For $\lambda=-50$, find eigenvector of $[A-\lambda I]x=\bar{0}$

$$\begin{bmatrix} 7 & 77 \\ 13 & 143 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\bar{0}$$

$7x_1+77x_2=0,\quad 13x_1+143x_2=0$

After solving, we get $x_1=-11x_2$

The eigenvector is $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} -11 \\ 1 \end{bmatrix}$

Basis of vectors $=\begin{bmatrix} 7 & -11 \\ 13 & 1 \end{bmatrix}|x|=150,\quad x^{-1}=\frac{\operatorname{adj}x}{|x|}=\frac{1}{150}\begin{bmatrix} 1 & 11 \\ -13 & 7 \end{bmatrix}$

$$D=x^{-1}Ax=\frac{1}{150}\begin{bmatrix} 1 & 11 \\ -13 & 7 \end{bmatrix}\begin{bmatrix} -43 & 77 \\ 13 & 93 \end{bmatrix}\begin{bmatrix} 7 & -11 \\ 13 & 1 \end{bmatrix}=\begin{bmatrix} 100 & 0 \\ 0 & -50 \end{bmatrix}$$

(b) (i) Eigenvalues of Hermitian matrix are real.

Example : $A=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}, \bar{A}=\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

$$\bar{A}^T=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}, \bar{A}^T=A$$

It is Hermitian.

The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix} 4-\lambda & 1-3i \\ 1+3i & 7-\lambda \end{vmatrix}=0$$

$(4-\lambda)(7-\lambda)-(1-3i)(1+3i)$, $\lambda=4,2$(It is real)

(ii) Eigenvalues of unitary matrix has absolute value 1.

Example : $A=\begin{bmatrix} \frac{1}{2} & i\sqrt{\frac{3}{4}} \\ i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}$ $\quad \bar{A}=\begin{bmatrix} \frac{1}{2} & -i\sqrt{\frac{3}{4}} \\ -i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}$

$$\bar{A}^T=\begin{bmatrix} \frac{1}{2} & -i\sqrt{\frac{3}{4}} \\ -i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}, \quad \bar{A}^T A=\begin{bmatrix} \frac{1}{2} & -i\sqrt{\frac{3}{4}} \\ -i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & i\sqrt{\frac{3}{4}} \\ i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix} \frac{1}{2}-\lambda & i\sqrt{\frac{3}{4}} \\ i\sqrt{\frac{3}{4}} & \frac{1}{2}-\lambda \end{vmatrix}=0$$

$\left(\frac{1}{2}-\lambda\right)^2+\frac{3}{4}=0$, $\lambda=-\frac{1}{2}\pm i\sqrt{\frac{3}{4}}$, $\lambda_1=-\frac{1}{2}+i\sqrt{\frac{3}{4}}$, $|\lambda_1|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\sqrt{\frac{3}{4}}\right)^2}=1$

$\lambda_2=-\frac{1}{2}-i\sqrt{\frac{3}{4}}$, $|\lambda_2|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(-\sqrt{\frac{3}{4}}\right)^2}=1$ Its absolute value = 1

8.(a) $10x+4y-2z=-4$

$17x+y+2z=14$

$34x+5y-3z=-8$

$$\begin{bmatrix} 10 & 4 & -2 & -4 \\ 17 & 1 & 2 & 14 \\ 34 & 5 & -3 & -8 \end{bmatrix}$$

Scanned by CamScanner

Soln. **Step - 1 :** Elimination of 'x' from second to onwards,

$$10x+4y-2z=-4 \qquad \left[\begin{array}{cccc|l} 10 & 4 & -2 & -4 & R_2=R_2-\frac{17}{10}R_1 \\ 0 & -\frac{58}{10} & \frac{54}{10} & \frac{208}{10} & \\ 0 & -\frac{86}{10} & \frac{38}{10} & \frac{56}{10} & R_3=R_3\frac{-34}{10}R_1 \end{array}\right]$$

$$-\frac{58}{10}y+\frac{54}{10}z=\frac{208}{10}$$

$$-\frac{86}{10}y+\frac{38}{10}z=\frac{56}{10}$$

Step -2 : Elimination of y from third ,

$$10x+4y-2z=-4 \qquad \begin{bmatrix} 10 & 4 & -2 & -4 \\ 0 & -\frac{58}{10} & \frac{54}{10} & \frac{208}{10} \\ 0 & 0 & \frac{2440}{580} & \frac{14640}{580} \end{bmatrix}_{R_3=R_3-\frac{86}{58}R_2}$$

$$-\frac{58}{10}y+\frac{54}{10}z=\frac{208}{10}$$

$$\frac{2440}{580}z=\frac{14640}{580}$$

Hence $z=6, y=2, x=0$

(b) Let ,
$$A=\begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}, \quad \bar{X}=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \bar{b}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\bar{A}=\begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix}$$

If the system of m equations and n unknowns $x_1, x_2 \ldots x_n$ given by $A\bar{x}=\bar{b}$ has solutions, then show that A and $\bar{A}$ have same rank and the system of equations has precisely one solution if the common rank in n.

Taking suitable we can prove it.

First Semester Examination - 2010
Mathematics - I

1. **Answer the following questions:**

(a) Find the solution of the differential equation $y'-2xy=2x$

(b) Find the solution of the initial value problem $y'-e^x y=0$ with $y(0)=e$.

(c) Find the integrating factor of the differential equation $(x+xy)dx+(y+xy)d(y)=0$

(d) Write the particular solution of the differential equation $y''+y'=2x$ in general form using method of undetermined coefficient.

(e) If $y_1(x)$ and $y_2(x)$ are solutions of the differential equation $y''+p(x)y'+q(x)y=0$, then what is the relation among $y_1(x), y_2(x)$ and $p(x)$.

(f) What is the radius of convergence of the power series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$?

(g) What necessary condition should p(x) satisfy in order to take $y(x)=\sum_{n=0}^{\infty}a_n x^{n+r}$ as the series solution of the differential equation $y''+p(x)y'+y=0$

(h) If $J_n(x)$ is the Bessel function of order n, then what is the relation between $J_n(x)$ and $J_{-n}(x)$?

(i) If $P_n(x)$ is the Legendre polynomial of degree n, then what is the relation between $P_n(-x)$ and $P_n(x)$?

(j) What is the relation between diagonal matrix of order n and identify matrix of order n?

2. Solve the following problems :

(a) Find the radius of curvature of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ at the point (0,a).

(b) Find the asymptote to the curve $x^3+y^3+3axy=0$.

3. Answer the following questions according to the instruction :

(a) Solve the Bernoulli's equation $x^3y'+4x^2\tan(y)=e^x\sec(y)$

(b) Solve the non-linear differential equation $(3x-2y+2)dx=(2x-4y+5)dy$.

4. Solve the following initial value problems :

(a) $y''+y=\sin(x)$ with $y(0)=0$ and $y'(0)=2$ using method of undetermined coefficient.

(b) $y''+2y'+y=xe^x$ with $y(0)=0$ and $y'(0)=0$ using method of variation of parameter.

5. Answer the following questions according to the instruction :

(a) Find the general solution of the differential equation $y''+y=\sin(x)$ using $y(x)=\sin(x)$ as a solution of the homogeneous differential equation by the method of reduction order.

(b) Solve Cauchy-Euler equation $x^2y''-xy'+y=\ln(x)$

6. Answer according to the instruction :

(a) Show that $J_{\frac{1}{2}}(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin(x)$

(b) Show that $=\int_{-1}^{1}P_n(x)P_m(x)dx=0$ for $m\neq n$.

7. Answer according to the instruction :

(a) State the conditions under which a system of equations $Ax=b$ has unique solution, no solution and infinitely many solution. Find a solution of the system of equations in integer form.

$$x+y+z=3,\quad 2x+3y+z=4,\quad 2x+2y+2z=k$$

(b) What is the relation between the algebraic multiplicity and the geometric multiplicity of an eigenvalue of any square matrix? Find the eigenvalue and the corresponding eigenvectors of the matrix $I_{3\times3}$.

OR

(a) Find the Laplace transform of the function

$$f(t)=\begin{cases}0, & t<2\\ 2t, & 2<t<4\\ 0, & \text{otherwise}\end{cases}\quad \text{using unit step function.}$$

(b) If $f(t)*g(t)=\int_0^t f(t-x)g(x)dx$, then show that $f(t)*g(t)=g(t)*f(t)$.

8. Answer the following questions according to the instruction :

(a) Find the name of the conic section which is represented by the quadratic form $9x^2+y^2-6xy=40$, and the corresponding transform which transforms the quadratic form to that conic section.

(b) Find the orthogonal matrix P such that PAP^T is a diagonal matrix where the matrix A is

$$A=\begin{pmatrix}-1 & 2 & 0\\ 2 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$$

OR

(a) Solve the integral equation $y(t)=e^t+\int_0^t y(x)e^{t-x}dx$ using Laplace transform.

(b) Solve the initial value problem $y''+y=2$ with $y(0)=0$ and $y'(0)=2$ using Laplace transform.

ANSWERS - 2010

1.(a) $P=-2x,\ Q=2x,\ \mu=e^{-x^2},\ e^{-x^2}y=\int 2xe^{-x^2}dx+c,\ y=-1+ce^{x^2}$

(b) $\frac{dy}{y}=e^x dx,\ \ell xy=e^x+c,\ y(0)=e$, when $x=0, y=e$, $c=\ell ne-1$

(c) $M=x+xy, N=y+xy,\ \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x},\ \mu=\frac{1}{(1+x)(1+y)}$

(d) $y_p=k_1x+k_0\qquad y'_p=k_1\qquad y''_p=0$

putting it $k_1=2x$ apply case = 1

$y_p=k_1x^2+k_0x\quad y'_p=2k_1x+k_0, y''_p=2k_1$

putting it $2k_1+2k_1x+k_0=2x$, $(2k_1+k_0)+2k_1x=2x$

Equating $k_1=1$, $k_0=-2$, $y_p=x^2-2x$

(e) $y_2=y_1\int M dx$, $M=\frac{1}{y_1^2}e^{-\int P(x)dx}$

(f) $a_n=\frac{1}{n!}, a_{n+1}=\frac{1}{(n+1)!}, \frac{1}{R}=\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right|\ \Rightarrow R=\infty$

(g) $P(x)=$ is analytic

(h) $J_{-n}(x)=(-1)^n J_n(x)$

(i) $P_n(-x)=(-1)^n P_n(x)$

(j) Diagonal elements are exist it may or may not be 1. but in case if identify matrix the diagonal elements are 1.

2.(a) $x^{2/3}+y^{2/3}=a^{2/3}$ at the point $(0, a)$

put $x=a\cos^3\theta$, $y=a\sin^3\theta$, $\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$

$\frac{d^2x}{d\theta^2}=-3a\cos^3\theta+6a\cos\theta\sin^2\theta$, $\frac{d^2y}{d\theta^2}=-3a\sin^3\theta+6a\sin\theta\cos^2\theta$

$\rho=\frac{\left[x'^2+y'^2\right]^{3/2}}{x'y''-y'x''}=3a\sin\cos\theta=3(axy)^{1/3}$ at the point $(0, a)$, $\rho=0$

(b) $x^3+y^3+3axy=0$, $\phi_3(m)=1+m^3, \phi'_3(m)=3m^2$, $\phi_2(m)=3am$

$\phi_3(m)=0$, $1+m^3=0, m=-1$

For $m=-1$, $c\phi'_3(m)+\phi_2(m)=0$, $c(3m^2)+3am=0$, $c(3)+(-3a)=0$, $c=a$

Hence the required asymptotes is $y+x=a$

3.(a) $y'+\frac{4}{x}\tan y=\frac{e^x}{x^3}\sec y$, $\cos y\, y'+\frac{4}{x}\sin y=\frac{e^x}{x^3}$

Put $\sin y=t, \cos y y'=\frac{dt}{dx}$, $P=\frac{4}{x}, Q=\frac{e^x}{x^3}, \mu=x^4$

$x^4\cdot t=\int\frac{e^x}{x^3}x^4dx+c$, $x^4\cdot\sin y=xe^x-e^x+c$

(b) $(3x-2y+2)dx=(2x-4y+5)dy$

$\frac{dy}{dx}=\frac{3x-2y+2}{2x-4y+5}$, $\frac{a_1}{a_2}=\frac{3}{2}\neq\frac{b_1}{b_2}=\frac{1}{2}$

Put $x=X+h, y=Y+k$, $\frac{dy}{dx}=\frac{dY}{dX}$, $\frac{dY}{dX}=\frac{3X-2Y+(3h-2k+2)}{2X-4Y+(2h-4k+5)}$

$3h-2x+2=0$, $2h-4k+5=0$, $h=\frac{1}{4}$, $k=\frac{11}{8}$

Put $Y=VX, \frac{dy}{dx}=V+X\frac{dV}{dx}$

Putting hand k, $\frac{dY}{dX}=\frac{3X-2Y}{2x-4y}$, $V+X\frac{dV}{dx}=\frac{3X-2VX}{2X-4VX}$, $X\frac{dV}{dX}=\frac{3-2V}{2-4V}-V$

$\frac{2-4V}{3-4V+4V^2}dV=\frac{dX}{X}$, $\frac{(8V-4)dV}{4V^2-4V+3}=-2\frac{dx}{X}$

$4V^2-4V+3=C/X^2$, $4Y^2-4YX+3X^2=C$

$4\left(y-\frac{11}{8}\right)^2-4\left(y-\frac{11}{8}\right)\left(x-\frac{1}{4}\right)+3\left(x-\frac{1}{4}\right)^2=C$

4.(a) The characteristic equation is $\lambda^2+1=0$, $\lambda=\pm i$, $y_h=c_1\cos x+c_2\sin x$

$y_p=K\cos x+M\sin x$, $y'_p=-K\sin x+M\cos x$, $y''_p=-K\cos x-M\sin x$

Putting it in the equation LHS = 0, apply case - 1

$y_p=x(K\cos x+M\sin x)$, $y'_p=(K\cos x+M\sin x)+x(-K\sin x+M\cos x)$

$y''_p=x(-K\cos x-M\sin x)+(-2K\sin x+2M\cos x)$

Putting it $x(-K\cos x-M\sin x)+(-2K\sin x+2M\cos x)+x(K\cos x+M\sin x)=\sin x$

$K=-1/2, M=0$, $y_P=-\frac{1}{2}x\cos x$, $y=y_h+y_p=c_1\cos x+c_2\sin x-\frac{1}{2}x\cos x$

$y(0)=0$, $c_1=0$, $y'=-c_1\sin x+c_2\cos x+\frac{1}{2}x\sin x-\frac{1}{2}\cos x$

$y'(0)=0$, $c_2=\frac{1}{2}$, $y=\frac{1}{2}\sin x-\frac{1}{2}x\cos x$

(b) $y''+2y'+y=xe^{-x}$, $r(x)=xe^{-x}$

The characteristic equation is $\lambda^2+2\lambda+1=0$, $\lambda=-1,-1$

$y_h=(c_1+c_2x)e^{-x}$, $W=\begin{vmatrix}e^{-x} & xe^{-x}\\ -e^{-x} & e^{-x}-xe^{-x}\end{vmatrix}=e^{-2x}$

$y_p=-y_1\int\frac{y_2r(x)}{W}dx+y_2\int\frac{y_1r(x)}{W}dx$ $=-e^{-x}\int\frac{xe^{-x}xe^{-x}}{e^{-2x}}dx+xe^{-x}\int\frac{e^{-x}xe^{-x}}{e^{-2x}}dx$

$=-e^{-x}\left(\frac{x^3}{3}\right)+xe^{-x}\left(\frac{x^2}{2}\right)=\frac{x^3}{6}e^{-x}$

$y=(c_1+c_2x)e^{-x}+\frac{x^3}{6}e^{-x}$, $y(0)=0$, $c_1=0$

$y'=(c_1+c_2x)(-e^{-x})+c_2e^{-x}-\frac{x^3}{6}e^{-x}+\frac{x^2}{2}e^{-x}$, $y'(0)=0$, $c_2=0$

Hence $y=\frac{x^3}{6}e^{-x}$

5.(a) $y_2=uy_1=u\sin(x)$, $y'_2=u\cos x+\sin x\frac{du}{dx}$, $y''_2=-u\sin x+2\cos x\frac{du}{dx}+\sin x\frac{d^2u}{dx^2}$

Putting it

$\left(-u\sin x+2\cos x\frac{du}{dx}+\sin x\frac{d^2u}{dx^2}\right)+u\sin x=\sin x$

$\sin x\frac{d^2u}{dx^2}+2\cos x\frac{du}{dx}=\sin x$, $\sin x\left(\frac{d^2u}{dx^2}-1\right)=-2\cos x\frac{du}{dx}$, $\frac{du}{dx}=P, \frac{d^2u}{dx^2}=\frac{dp}{dx}$

$\sin x\left(\frac{dp}{dx}-1\right)=-2\cos xp$, $\frac{dp}{dx}-1=-2\cot xp$, $\frac{dp}{dx}+2\cot x\,p=1$, $\mu=\operatorname{cosec}^2x$

$$\operatorname{cosec}^2 x\cdot p=\int \operatorname{cosec}^2 x\,dx+c,\quad \operatorname{cosec}^2 x\cdot\frac{du}{dx}=-\cot x$$

$$du=(-\cos x\sin x)dx,\quad du=\left\{-\frac{1}{2}\sin 2x\right\}dx,\quad u=\frac{1}{4}\cos 2x,\qquad y_2=uy_1=\frac{1}{4}\sin x\cos 2x$$

(b) $x^2y''-xy'+y=\ell nx,\ y''-\frac{1}{x}y'+\frac{1}{x^2}y=\frac{\ell nx}{x^2},\ r(x)=\frac{\ell nx}{x^2}$

The characteristic equation is

$m(m-1)-m+1=0,\ (m-1)(m-1)=0,\quad m=1,1$

$y_h=(c_1+c_2\ell nx)x$

$$W=\begin{vmatrix} x & x\ell nx\\ 1 & \ell nx+1\end{vmatrix}=x$$

$$y_2=-y_1\int\frac{y_2r(x)}{W}dx+y_2\int\frac{y_1r(x)}{W}dx\quad=-x\int\frac{x\ell nx\cdot\left(\frac{\ell nx}{x^2}\right)}{x}+x\,\ell nx\int\frac{x\left(\frac{\ell nx}{x^2}\right)}{x}dx$$

$$=-x\int\frac{(\ell nx)^2}{x^2}dx+x\ell nx\int\frac{\ell nx}{x^2}dx\quad=-x\int t^2e^{-t}dt+x\ell nx\int te^{-t}dt$$

$$=-x\left((\ell nx)^2\left(-\frac{1}{x}\right)-2\ell nx\left(\frac{1}{x}\right)+2\left(-\frac{1}{x}\right)\right)+x\ell nx\left(\ell nx\left(-\frac{1}{x}\right)-\frac{1}{x}\right)$$

$$=(\ell nx)^2+2\ell nx+2-(\ell nx)^2-\ell nx$$

$y_p=\ell nx+2,\quad y=y_h+y_p=(c_1+c_2\ell nx)x+\ell nx+2$

6.(a) LHS : Putting it in $J_v(x)$

$$J_{1/2}(x)=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}m!\gamma\left(m+\frac{1}{2}+1\right)}=\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\gamma\left(\frac{3}{2}\right)}-\frac{x^{\frac{5}{2}}}{2^{\frac{5}{2}}\gamma\left(\frac{5}{2}\right)}+\frac{x^{9/2}}{2^{9/2}\gamma\left(\frac{7}{2}\right)2!}\dots\dots$$

$$=\sqrt{\frac{2}{x}}\left(\sqrt{\frac{x}{2}}\left(\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\gamma\left(\frac{3}{2}\right)}-\frac{x^{5/2}}{2^{5/2}\gamma\left(\frac{5}{2}\right)}+\frac{x^{9/2}}{2^{9/2}\gamma\left(\frac{5}{2}\right)\frac{1}{2}2!}\dots\dots\right)\right)$$

$$=\sqrt{\frac{2}{x}}\left(\frac{x}{2\cdot\frac{1}{2}\sqrt{x}}-\frac{x^3}{2^3\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}+\frac{x^5}{2^5\frac{5.3.1}{2^3}\sqrt{\pi}}2!\dots\dots\right)=\sqrt{\frac{2}{\pi x}}\left(x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\dots\dots\right)=\sqrt{\frac{2}{\pi x}}\sin x$$

(b) $\int_{-1}^{1}P_n(x)P_m(x)dx=0$ if $m\neq n$

$P_n(x)$ satisfies the equation

$$(1-x^2)\frac{d^2P_m}{dx^2}-2x\frac{dP_m}{dx}+m(m+1)P_m=0$$

which can be written as

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right]=-m(m+1)P_m$$

On multiplication of both sides of P_n and integration the above equation becomes

$$\int_{-1}^{1}P_n\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right]dx=-m(m+1)\int_{-1}^{1}P_nP_m dx$$

Integrating by parts the left hand integral in the above equation

$$\int_{-1}^{1}(1-x^2)\frac{dP_n}{dx}\cdot\frac{dP_m}{dx}dx=m(m+1)\int_{-1}^{1}P_nP_m dx$$

Interchanging m and n, we get

$$\int_{-1}^{1}(1-x^2)\frac{dP_m}{dx}\frac{dP_n}{dx}dx=n(n+1)\int_{-1}^{1}P_mP_n dx$$

Substraction of equation

$$(n-m)(n+m+1)\int_{-1}^{1}P_nP_m dx=0,\quad \int_{-1}^{1}P_nP_m dx=0,\ \text{for } m\neq n$$

7.(a) $x+y+z=3$

$2x+3y+z=4$

$2x+2y+2z=k$

$$\begin{bmatrix}1 & 1 & 1 & 3\\ 2 & 3 & 1 & 4\\ 2 & 2 & 2 & k\end{bmatrix}$$

$x+y+z=3$

$y-z=-2$

$0=k-3$

$$\begin{bmatrix}1 & 1 & 1 & 3\\ 0 & 1 & -1 & -2\\ 0 & 0 & 0 & k-3\end{bmatrix}\begin{matrix}R_2=R_2-2R_1\\ R_3=R_3-2R_1\end{matrix}$$

If $k-3=0$, $k=3$, infinitely many solution $k\neq 3$, no solution.

In system of equation

I. If rank of A and rank of the augumented matrix $\tilde{A}$ are equal, then the system is consistent.

(a) $r(A)=r(\tilde{A})=n$, then unique solution exists

(b) If $r(A)=r(\tilde{A})=r<n$, then infinitely many solution exist.

II. If rank of A is not equal to rank of $\bar{A}$, then the system is in consistent and has no solution at all.

(b) The order M_λ of an eigenvalue λ as a root of the characteristic polynomial is called the algebraic multiplicity of λ. The number m_λ of linearly independent eigenvectors corresponding to λ is called the geometric multiplicity of λ.

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda&0&0\\0&1-\lambda&0\\0&0&1-\lambda\end{vmatrix}=0$$

$(1-\lambda)^3=0,\ \lambda=1,1,1$

The eigenvectors are $=\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$

OR

(a) $f(t)=0(u(t-0)-u(t-2))+2t(u(t-2)-u(t-4))$

$=2(t-2)u(t-2)+4u(t-2)+2(t-4)$

$U(t-4)+8u(t-4)\quad =\frac{2}{s^2}e^{-2s}+\frac{4}{s}e^{-2s}+\frac{2}{s^2}e^{-4s}+\frac{8}{5}e^{-4s}$

(b) LHS $f(t)\times g(t)=\int_0^t f(\lambda)g(t-\lambda)d\lambda\ =\int_0^t g(k)f(t-k)dk\ =g(t)\times f(t)$

Hence $f\times g=g\times f$

8.(a) $Q=X^TAX,\ Q=40,\ A=\begin{bmatrix}9&-3\\-3&1\end{bmatrix}$

The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix}9-\lambda&-3\\-3&1-\lambda\end{vmatrix}=0$$

$(9-\lambda)(1-\lambda)-9=0,\ \lambda=0,10$

Hence the equation for conic section

$\lambda_1y_1^2+\lambda_2y_2^2+\dots\dots+\lambda_ny_n^2=Q$

$y_2=40$

(b) Find the orthogonal matrix P such that PAP^T is a diagonal where the matrix

$$A=\begin{bmatrix}-1&2&0\\2&2&0\\0&0&2\end{bmatrix}$$

Soln. Try by yourself.

OR

(a) Applying Laplace Transformation

$\alpha(y)=\frac{1}{s-1}+\alpha(y)\cdot\frac{1}{s-1},\qquad \alpha(y)\left(1-\frac{1}{s-1}\right)=\frac{1}{s-1}$

$\alpha(y)\left(\frac{s-2}{s-1}\right)=\frac{1}{s-1},\qquad y=\alpha^{-1}\left[\frac{1}{s-2}\right]=e^{2t}$

(b) $\alpha(y''+y)=\alpha(2),\ y(0)=0$ and $y'(0)=2$

$s^2\alpha(y)-sy(0)-y'(0)+\alpha(y)=\frac{2}{5},\ s^2\alpha(y)-2+\alpha(y)=\frac{2}{5},\ (s^2+1)\alpha(y)=2+\frac{2}{5}$

$\alpha(y)=\frac{2}{s^2+1}+\frac{2}{s(s^2+1)},\ y=\alpha^{-1}\left[\frac{2}{s^2+1}+\frac{2}{s}-\frac{2s}{s^2+1}\right]$

$y=2\sin t+2t-2\cos t$

First Semester Examination - 2011
Mathematics - I

1. Answer the following questions:

(a) Find the differential equation of the family of curves $y = ce^{2x}$.

(b) What do you mean by general solution and particular solution of a differential equation? What is the practical significance of these two concepts ?

(c) State all rules for the method of undetermined coefficients.

(d) Find all solutions of $\frac{dy}{dx} = x|y|$

(e) What are the asymptotes parallel to the co-ordinate axes for the curve $x^2y^2 - a^2(x^2 + y^2) = 0$?

(f) What is the radius of curvature at any point of the catenary $s = c \tan \psi$?

(g) What is the radius of convergence of the power series $\sum_{m=0}^{\infty} \frac{x^{2m}}{m!}$

(h) 'Show that $J_0(x) = -J_1(x)$ where $J_n(x)$ represents Bessel function.

(i) What is a basis of eigenvectors ? When does it exist ?

(j) What do you mean by unitary matrix ?. What is the value of the determinant of an unitary matrix?

2. Solve the following differential equations:

(a) $y(x^3e^{xy} - y)dx + x(xy + x^3e^{xy})dy = 0$

(b) $\frac{dy}{dx} = \frac{1}{1+x^2}\left(e^{\tan^{-1}x} - y\right)$ such that $y(0) = 1$

3.(a) Find the general solution of the second order differential equation $(D^2 + 2)y = x^2e^{3x} + e^x \sin 3x$

(b) Using variation of parameters, solve the differential equation $\frac{d^2y}{dx^2} + y = \sec x$

4.(a) Find the power series solution of the differential equation $(1 - x^2)y' = 2xy$

(b) Show that $\int J_{v+1}(x)dx = \int J_{v-1}(x)dx - 2J_v(x)$

5.(a) Solve the following system of equation by Gauss elimination method:
$x + y + z = 6,\ 3x + 3y + 4z = 20,\ 2x + y + 3z = 13$

(b) Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

6.(a) Diagonalize the matrix

$$\begin{bmatrix} 18 & 0 & 0 \\ 24 & -4 & 0 \\ 42 & -12 & 2 \end{bmatrix}$$

(b) Find out what section of conic section the following quadratic form represents and transform it to principal axes
$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$

7.(a) Find the centre of curvature and the equation of the circle of curvature at (0, –3) for the curve $y = 3x^3 + 2x^2 - 3$

(b) Find all the asymptotes of the curve $x^3 + x^2y - xy^2 - y^3 - 3x - y - 1 = 0$

8.(a) Prove that the eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal.

(b) Find the radius of curvature for the cycloid $x = a(t + \sin t), y = a(1 - \cos t)$

ANSWERS - 2011

1.(a) $y = ce^{2x}, y' = 2ce^{2x}$

$y' = 2y$ This is required differential equation.

(b) The general solution of an ODE is a family of infinitely many solution curves, one of or each value of the constant c. If we choose a specific c, we obtain what is called a particular solution of the ODE. A particular solution does not contain any arbitrary constants.

(c) 1.If $r(x) = Ke^{ax}$, $y_p = ce^{ax}$

Case-1: If LHS = 0, after putting $y_p = c'e^{ax}$, $y = cxe^{ax}$

Case-2: If again L.H.S = 0

after putting $y_p = cxe^{ax}$, $y_p = cx^2e^{ax}$

2. $r(x) = ke^{ax} + Me^{bx}$, $y_p = Ae^{ax} + Be^{bx}$

3. $r(x) = kx^n\ (n = 0,1,2......)$, $y_p = k_n x^n + k_{n-1}x^{n-1} + ... + x_1 x + x_0$

Case 1: If LHS = 0 after putting y_p, $y_p = x\left(k_n x^n + k_{n-1}x^{n-1} + ... + k_1 x + k_0\right)$

Case2: If LHS = 0 after putting y_p; $y_p = x^2\left(k_n x^n + k_{n-1}n^{n-1} + ... + k_1 x + k_0\right)$

4. $\gamma(x)\begin{cases} k\cos\omega x \\ k\sin\omega x \end{cases}$ $y_p = k\cos\omega x + M\sin\omega x$

5. $\gamma(x) = \begin{cases} ke^{ax}\cos\omega x \\ ke^{ax}\sin\omega x \end{cases}$ $y_p = e^{ax}\left(k\cos\omega x + M\sin\omega x\right)$

(d) $\dfrac{dy}{y} = xdx, y = ce^{\frac{x^2}{2}}$, $\dfrac{dy}{-y} = xdx, \dfrac{1}{y} = ce^{\frac{x^2}{2}}$

There are two solutions

(e) $y = \pm a$ and $x = \pm a$ are the parallel asymptotes

(f) $s = c\tan\psi\ \dfrac{ds}{d\psi} = c\sec^2\psi$, $\dfrac{d^2s}{d\psi^2} = 2c\sec^2\psi\tan x$

$$f = \frac{\left[1 + \left(\frac{ds}{d\psi}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2s}{d\psi^2}\right|} = \frac{\left(1 + c^2\sec^4\psi\right)^{\frac{3}{2}}}{2c\sec^2\psi\tan\psi}$$

(g) $a_m = \left(\dfrac{1}{m!}\right)^2\ a_{m+1} = \left(\dfrac{1}{(m+1)!}\right)^2$ $\dfrac{1}{R} = \lim\limits_{m\to\infty}\left|\dfrac{a_m + 1}{a_m}\right| = \lim\limits_{m\to\infty}\left|\left(\dfrac{1}{m+1}\right)^2\right| = 0$

$R = \infty$

(h) $J_0(x) = \sum\limits_{m=0}^{\infty}\dfrac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$

$$J'_0(x) = \sum_{m=1}^{\infty}\frac{(-1)^m 2mx^{2m-1}}{2^{2m}(m!)^2} = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}2(m+1)x^{2m+1}}{2^{2m+2}((m+1)!)^2} = (-1)\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}(m!)^2(m+1)} = -J_1(x)$$

(i) A Hermitian, skew Hermitian or unitary matrix has a basis of eigenvectors for c^n that is a unitary system.

(j) A square matrix $A = [a_{kj}]$ is called unitary if $(\bar{A})^T = A^{-1}$ or, $\bar{A}^T A = I$. The determinant of unitary matrix has absolute value is 1.

2.(a) $y\left(x^3e^{xy} - y\right)dx + x\left(xy + x^3e^{xy}\right)dy = 0$ $yx^3e^{xy}dx - y^2dx + x^2ydy + x^4e^{xy}d_y = 0$

$x^3e^{xy}\left(ydx + xdy\right) + y\left(xdy - ydx\right) = 0$ $x^3 \cdot d\left(e^{xy}\right) + yx^2\dfrac{(xdy - ydx)}{x^2} = 0$

$d\left(e^{xy}\right) + \dfrac{y}{x}d\left(\dfrac{y}{x}\right) = 0$, $e^{xy} + \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = c$

(b) $\dfrac{dy}{dx} - \dfrac{y}{1+x^2} = \dfrac{e^{\tan^{-1}x}}{1+x^2}$ $p = -\dfrac{1}{1+x^2}, Q = \dfrac{e^{\tan^{-1}x}}{1+x^2}$ $\mu - e^{\int -\frac{1}{1+x^2}d_x} = e^{-\tan^{-1}x}$

The solution is $\mu.y = \int \mu Q(x)dx + c$

$e^{-\tan^{-1}x}, y = \int \dfrac{1}{1+x^2}dx + c$, $y = e^{\tan^{-1}x}\left(\tan^{-1}x + c\right)$

Q.3.(a) The characteristic equation is $\lambda^2 + 2 = 0$, $\lambda = \pm\sqrt{2}i$

$y_h = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}i$

$$y_p = \frac{e^{3x}x^2}{D^2+2} + \frac{e^x\sin 3x}{D^2+2} = e^{3x}\frac{x^2}{(D+3)^2+2} + e^x\frac{\sin 3x}{(D+1)^2+2}$$

$$= e^{3x}\cdot\frac{x^2}{11+6D+D^2} + e^x\cdot\frac{\sin 3x}{D^2+2D+3} \qquad = \frac{e^{3x}}{11}\left(1 + \frac{6D+D^2}{11}\right)(x^2) + e^x\frac{\sin 3x}{2D-6}$$

$$= \frac{e^{3x}}{11}\left(1 \mp \left(\frac{6D+D^2}{11}\right) + \left(\frac{6D+D^2}{11}\right)^2 .....\right)(x^2) + e^x\frac{(2D+6)\sin 3x}{4D^2-6}$$

$$=\frac{e^{3x}}{11}\left(x^2-\frac{(12x+2)}{11}+\frac{12}{121}\right)+\frac{e^x(6\cos 3x+6\sin 3x)}{-42}$$

$$y_p=\frac{1}{11}x^2e^{3x}-\frac{12x}{121}e^{3x}-\frac{10e^{3x}}{1331}-\frac{e^x}{7}\cos 3x-\frac{e^x}{7}\sin 3x$$

$$y=y_h+y_p=c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x+\frac{1}{11}x^2e^{3x}$$

$$-\frac{12x}{121}e^{3x}-\frac{10e^{3x}}{1331}-\frac{e^x}{7}\cos 3x-\frac{e^x}{7}\sin 3x$$

(b) The characteristic equation is $\lambda^2+1=0,\ \lambda=\pm i$

$$y_h=c_1\cos x+c_2\sin x,\qquad W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$$

$$y_p=-y_1\frac{\int y_2 r(x)}{W}dx+y_2\frac{\int y_1 r(x)}{W}dx$$

$$=-\cos x\int\tan x\ dx+\sin x\int dx \qquad =-\cos x\ell n\sec x+x\sin x$$

$$y=y_x+y_p=c_1\cos x+c_2\sin x+x\sin x-\cos x\ell n\sec x$$

4.(a) Let $y=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots$

$$y'=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots\ldots$$

Putting it $(1-x^2)(a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots\ldots)$

$$-2x(a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots)=0$$

$$a_1+(2a_2-2a_0)x+(3a_3-a_1-2a_1)x^2+(4a_4-2a_2-2a_2)x^3+\ldots\ldots=0$$

Equating coefficient of 'x' to zero

$a_1=0,\qquad a_1=0$

$2a_2-2a_0=0,\qquad a_2=a_0$

$3a_3-3a_1=0,\qquad a_3=a_1=0$

$4a_4-4a_2=0,\qquad a_4=a_2=a_0$

Hence the required solution is

$$y=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots \qquad =a_0+a_0x^2+a_0x^4+\ldots\ldots$$

$$=a_0(1+x^2+x^4+\ldots\ldots)=a_0(1-x^2)^{-1}$$

(b) We know that, $J_{v-1}(x)-J_{v+1}(x)=2J_v'(x)$

Integrate $\int J_{v-1}(x)dx-\int J_{v+1}(x)dx=2J_v(x)$

Q.5.(a)

$$\begin{aligned}x+y+z&=6\\3x+3y+4z&=20\\2x+y+3z&=13\end{aligned}\qquad\approx\begin{bmatrix}1&1&1&6\\3&3&4&20\\2&1&3&13\end{bmatrix}$$

$$\begin{aligned}x+y+z&=6\\z&=2\\-y+z&=1\end{aligned}\qquad\approx\begin{bmatrix}1&1&1&6\\0&0&1&2\\0&-1&1&1\end{bmatrix}$$

$$\begin{aligned}x+y+z&=6\\-y+z&=1\\z&=2\end{aligned}\qquad\approx\begin{bmatrix}1&1&1&6\\&-1&1&1\\&&1&2\end{bmatrix}$$

Hence $z=3,\ y=1,\ x=3$

(b) The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=0$$

$\lambda=5,-3,-3$

For $\lambda=5$, apply for eigen vector $[A-\lambda I]x=\bar{0}$

$$\begin{bmatrix}-7&2&-3\\2&-4&-6\\-1&-2&-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\bar{0}$$

$-7x_1+2x_2-3x_3=0,\ 2x_1-4x_2-6x_3=0,\ -x_1-2x_2-5x_3=0$

After solving we get $x_1=-x_3,\ x_2=-2x_3$

The eigen vector is $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_3\\-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}-1\\-2\\1\end{bmatrix}\forall x_3\neq 0$

For $\lambda=-3$, apply for eigen vector $[A-\lambda I]x=\bar{0}$

$$\begin{bmatrix}1&2&-3\\2&4&-6\\-1&-2&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\bar{0}$$

$x_1+2x_2-3x_3=0,\ 2x_1+4x_2-6x_3=0,\ -x_1-2x_2+3x_3=0$

After solving we get $x_1 = -2x_2 + 3x_3$

The eigen vector is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

The eigen values are = 5, – 3, –3

The eigen vectors are $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Q.6.(a) The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 18-\lambda & 0 & 0 \\ 24 & -4-\lambda & 0 \\ 42 & -12 & 2-\lambda \end{vmatrix} = 0$$

$(18-\lambda)(-4-\lambda)(2-\lambda) = 0$, $\lambda = 2, -4, 18$

For $\lambda = 2$, apply for eigenvector $(A - \lambda I)x = 0$

$$\begin{vmatrix} 16 & 0 & 0 \\ 24 & -6 & 0 \\ 42 & -12 & 0 \end{vmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \bar{0}$$

$16x_1 = 0$, $24x_1 - 6x_2 = 0$, $42x_1 - 12x_2 = 0$

After solving we get $x_1 = 0$, $x_2 = 0$

The eigen vector is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \forall\ x_3 \neq 0$

For $\lambda = -4$, apply for eigen vector $[A - \lambda I]x = \bar{0}$

$$\begin{bmatrix} 22 & 0 & 0 \\ 24 & 0 & 0 \\ 42 & -12 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \bar{0}$$

$22x_1 = 0$, $24x_2 = 0$, $42x_1 - 12x_2 + 6x_3 = 0$

After solving we get $x_1 = 0, x_3 = 2x_2$

The eigen vector is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \forall x_2 \neq 0$

For $\lambda = 18$, apply for eigen vector $[A - \lambda I]x = \bar{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 24 & -22 & 0 \\ 42 & -12 & -16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \bar{0}$$

$24x_1 = 22x_2 = 0$, $42x_1 - 12x_2 - 16x_3 = 0$

After solving we get $x_2 = \frac{12}{11}x_1, x_3 = \frac{318}{176}x_1$

The eigen vector is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 176 \\ 132 \\ 318 \end{bmatrix}$

Basis of eigen vector $x = \begin{bmatrix} 0 & 0 & 176 \\ 0 & 1 & 132 \\ 1 & 2 & 318 \end{bmatrix} 1 \times 1 = -176$

$\text{cof } x = \begin{bmatrix} 54 & 132 & 1 \\ 352 & -176 & 0 \\ -176 & 0 & 0 \end{bmatrix}$ $\text{adj}x = \begin{bmatrix} 54 & 352 & -176 \\ 132 & -176 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$x^{-1} = \frac{\text{adj } x}{|x|} = -\frac{1}{176}\begin{bmatrix} 54 & 352 & -176 \\ 132 & -176 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$D = x^{-1}Ax = -\frac{1}{176}\begin{bmatrix} 54 & 352 & -176 \\ 132 & -176 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 18 & 0 & 0 \\ 24 & -4 & 0 \\ 42 & -12 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 176 \\ 0 & 1 & 132 \\ 1 & 2 & 318 \end{bmatrix}$$

$$= -\frac{1}{176}\begin{bmatrix} 54 & 352 & -176 \\ 132 & -176 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 3168 \\ 0 & -4 & 3696 \\ 2 & -8 & 6444 \end{bmatrix} = -\frac{1}{176}\begin{bmatrix} -352 & 0 & 337920 \\ 0 & 704 & -232320 \\ 0 & 0 & 3168 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1920 \\ 0 & 4 & 1320 \\ 0 & 0 & -18 \end{bmatrix}$$

(b) $Q = x^T Ax, Q = 128$

$$A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{vmatrix} = 0$$

$\lambda = 2, 32$

The equation for conic section is

$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2$

$2y_1^2 + 32y_2^2 = 128$

For $\lambda = 2$, apply for eigen vector $[A - \lambda I]x = \bar{0}$

$$\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{0}$$

$15x_1 - 15x_2 = 0, -15x_1 + 15x_2 = 0$

After solving $x_1 = x_2$

The eigen vector is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \forall x_2 \neq 0$

The eigen basis are $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

For $\lambda = 32$, apply for eigenvector

$[A - \lambda I]x = \bar{0}$

$$\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{0}$$

$-15x_1 - 15x_2 = 0, -15x_1 + 15x_2 = 0$

$$x = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \ x = xy$$

After solving, we get $x_1 = -x_2$

The eigenvector is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \forall x_2 \neq 0$

The eigen basis are $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$x = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, x = Xy$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad x_1 = \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{2}}y_2, x_2 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2$$

Q.7.(a)Center of curvature at is (x, y) to the curvature $y = 3x^3 + 2x^2 - 3$ is

$$x = x - \frac{y_1(1+y_1^2)}{y_2}, \ y = y + \frac{1+y_1^2}{y_2}$$

$$y_1 = \frac{d_y}{d_x} = 9x^2 + 4x, \ y_2 = \frac{d^2y}{dx^2} = 18x + 4 \qquad x = x - \frac{(9x^2+4x)(1+(9x^2+4x))}{18x+4}$$

$$= \frac{(18x^2+4x) - (9x^2+4x) + (9x^2+4x)^3}{18x+4} = \frac{9x^2 + (9x^2+4x)^3}{18x+4}$$

$$y = y - \frac{1+(9x^2+4x)^2}{18x+4} = \frac{y(18x+4) - 1 - (9x^2+4x)^2}{18x+4}$$

The equation for circle of curvature is $(x-X)^2 + (y-Y)^2 = f^2$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y^2} = \frac{\left[1+(9x^2+4x)^2\right]^{\frac{3}{2}}}{18x+4}$$

$$\left(x - \frac{9x^2+(9x^2+4x)^3}{18x+4}\right)^2 + \left(y - \frac{y(18x+4)-1-(9x^2+4x)^2}{18x+4}\right)^2 = \frac{\left[1+(9x^2+4x)^2\right]^3}{(18x+4)^2}$$

$$\left[x - 9x^2 + (9x^2+4x)^3\right]^2 + \left[y - y(18x+4) - 1 - (9x^2+4x)^2\right]^2 = \left[1+(9x^2+4x)^2\right]^3$$

(b) $\phi_3(m) = 1 + m - m^2 - m^3, \ \phi_3'(m) = 1 - 2m - 3m^2, \phi_3'(m) = 1 - 2m - 3m^2, \ \phi_3'(m) = -2 - 6m$

$\phi_2(m) = 0, \ \phi_1(m) = -3 - m, \ \phi_0(m) = -1$

Put $\phi_3(m) = 0, \ 1 + m - m^2 - m^3 = 0, \ m = -1, -1, 1$

For $m = 1, \ x\phi_3'(m) + \phi_2(m)0, \ c(1 - 2m - 3m^2) + 0 = 0, c = 0$

The required asymptotes is $y = x$

For $m = -1$, $\frac{C^2}{L^2}\phi_3''(m) + c\phi_2(m) + \phi_1(m) = 0$

$\frac{C^2}{2}(-2-6m) + c \cdot 0 + (-3-m) = 0$, $c^2(-1-3m) = 3+m$

$c^2 = \frac{3+m}{-1-3m}$, $c^2 = 1, x = \pm 1$

The required asymptotes is $y = -x \pm 1$

Hence $y = x$, $y + x = 1$, $y + mx = -1$ are asymptotes.

Q.8.(a) $A = \begin{bmatrix} 4 & 0 \\ 0 & -6 \end{bmatrix}$, $A^T = \begin{bmatrix} 4 & 0 \\ 0 & -6 \end{bmatrix}$

$A^T = A$ It is an symmetric matrix

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 0 \\ 0 & -6-\lambda \end{vmatrix} = 0 \quad (4-\lambda)(-6-\lambda) = 0, \lambda = 4, -6$$

For $\lambda - 4$, apply for eigen vector $[A - \lambda I]x = \overline{0}$

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overline{0}, \quad -2x_2 = 0, x_2 = 0$$

The eigen vector is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \forall x_1 \neq 0$

For $\lambda = -6$, apply for eigen vector $[A - \lambda I]x = \overline{0}$

$$\begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0, \quad 10x_1 = 0, x_1 = 0$$

The eigen vector is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \forall x_2 \neq 0$

Basis of eigen vector and $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $|x| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ hence it is orthogonal.

(b) $x' = a(1+\cos t)$ $y' = a\sin t$

$x'' = -a\sin t$, $y'' = a\cos t$

$$\rho = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{|x'y'' - y'x''|} = \frac{\left[a^2(1+\cos t)^2 + a^2\sin^2 t\right]^{\frac{3}{2}}}{a^2(\cos t + 1)} = a\, 2^{\frac{3}{2}}[1+\cos t]^{\frac{1}{2}} = 4a\cos\frac{t}{2}$$