(Set-A_1)

B.Tech-1st

Mathematics-I

Full Marks : 70

Time : 3 hours

Answer six questions including Q.No.1 which is compulsory.

The figures in the right-hand margin indicate marks.

Symbols carry usual meaning.

1. Answer *all* questions : 2×10

(*a*) Define closed set and show that the set $S = \{x : 0 < x < 1, x \in R\}$ is open but not closed.

(*b*) Show that $\lim_{x \to 1} 2^{1/x-1}$ does not exist.

(*c*) If a function f is continuous at an interior point c of an interval $[a, b]$ and $f(c) \neq 0$ then show that there exist a $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$ for every $x \in]c - \delta, c + \delta[$.

(*d*) For what value of k, the following vectors (1,2,3) (4,5,6) (7,8, k) are linearly independent.

(*e*) Show that rank of A^T = rank of A

(*f*) Evaluate $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$

(*g*) For what value of λ does the following system

$$(a_{11}-\lambda)x_1+a_{12}x_2=0$$
$$a_{21}x_1+(a_{22}-\lambda)x_2=0$$

have a nontrivial solution.

(*h*) Define an eigenvalue. Show that $\lambda=0$ is an eigenvalue of a square matrix if A is singular.

(*i*) Find a solution of $f(x)=x^3+x-1=0$ by iteration method.

(*j*) Define trapezoidal and Simpson rule.

2. (a) Show that the Intersection of an arbitrary family of closed sets is closed. 5

(b) Show that $\lim_{x \to 0} \frac{1}{x} \sin \frac{1}{x}$ does not exist. 5

3. (a) A function f is defined on R by

$$f(x) = \begin{cases} -x^2, & \text{if } x \le 0 \\ 5x-4, & \text{if } 0 < x \le 1 \\ 4x^2-3x, & \text{if } 1 < x < 2 \\ 3x+4, & \text{if } x \ge 2 \end{cases}$$

Examine the continuity of f at $x = 0,1,2$. 5

(b) Suppose that f is continuous on a closed interval $I = [a, b]$ and that f has a derivative in the open interval (a, b) then show that there exist at least one point c in (a, b) such that

$$f(b) - f(a) = f'(c)(b-a)$$ 5

4. (a) Solve the system

$$-x_1 + x_2 + 2x_3 = 2$$
$$3x_1 - x_2 + x_3 = 6$$
$$-x_1 + 3x_2 + 4x_3 = 4$$

by using Gauss elimination method. 5

(*b*) Show that the rank of matrix A be equals the maximum number of linearly independent column vectors of A. 5

5. (*a*) Prove that a system of m linear equation

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

in n unknowns $x_1, x_2, \dots, x_n$ has solution if and only if the coefficient matrix A and the augmented matrix B that is

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$ and

$$B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_2 \end{pmatrix}$$

have the same rank. 5

(*b*) Consider the vector space p_n of polynomials of degree $\leq n$. Let f and g be elements of p_n. Prove that the following function

$$\langle f, g\rangle = \int_0^1 f(x).g(x)dx$$

defines an inner product on p_n. Determine the inner product of the polynomials

$f(x) = x^2 + 2x - 1$ and $g(x) = 4x + 1$. 5

6. (*a*) If A has the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ then show that

$$k_m A^m + k_{m-1} A^{m-1} + \ldots\ldots + k_1 A + k_0 I$$

has the eigenvalues

$$k_m \lambda_j^m + k_{m-1} \lambda_j^{m-1} + \ldots\ldots + k_1 \lambda_j + k_0 . (j = 1, 2, \ldots n)$$ 5

(*b*) Find the eigenvalues and the eigenvectors of

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$ 5

7. (*a*) Set up a Newton iteration for computing the square root x of a given positive number c and apply it to $c = 2$. 5

(*b*) Define Lagrangian interpolation and use it to find Ln 9.2 with $n = 3$. 5

8. (*a*) Evaluate $J = \int_0^1 e^{-x^2} dx$ by Simpson's rule with $2n = 10$ and estimate the error. 5

(*b*) Consider $f(x) = x^4$ for $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$. Calculate f_2' by using Lagrangian 3 point and five point formula. Determine the errors and compare. 5
