# Lemma 3.1 - derivations

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Let us look at the Lemma as described in the paper.

**Lemma 3.1** *Let $G$ be a $m \times n$ Gaussian with $m - 1 \geq n \geq 2$. Then*

$$\mathbb{E}\|G^\dagger\|_2^2 \leq \frac{\mathrm{e}^2 m}{(m-n)^2 - 1} \tag{1}$$

Obviously, we would have a problem if $m - 1 = n$, because in this case, we would be dividing by zero. Now, for the derivation. Since $\|G^\dagger\|$ is nonnegative, it holds that for each $t > 0$,

$$\mathbb{P}\left[\|G^\dagger\|_2^2 > t\right] = \mathbb{P}\left[\|G^\dagger\|_2 > \sqrt{t}\right] \tag{2}$$

From Proposition A.3, we have that:

$$\mathbb{P}\left[\|G^\dagger\|_2 > \sqrt{t}\right] \leq \frac{1}{\sqrt{2\pi(m-n+1)}} \left(\frac{\mathrm{e}\sqrt{m}}{m-n+1}\right)^{m-n+1} t^{-(m-n+1)/2} \tag{3}$$

and therefore:

$$\mathbb{P}\left[\|G^\dagger\|_2^2 > t\right] \leq \frac{1}{\sqrt{2\pi(m-n+1)}} \left(\frac{\mathrm{e}\sqrt{m}}{m-n+1}\right)^{m-n+1} t^{-(m-n+1)/2} \tag{4}$$

Using arguments similar to those in Proposition A.4, we get:

$$\mathbb{E}\|G^\dagger\|_2^2 = \int_0^\infty \mathbb{P}\left[\|G^\dagger\|_2^2 > t\right] \mathrm{d}t \leq E + \int_E^\infty \mathbb{P}\left[\|G^\dagger\|_2^2 > t\right] \mathrm{d}t \tag{5}$$

Write $C := \frac{1}{\sqrt{2\pi(m-n+1)}}\left(\frac{\mathrm{e}\sqrt{m}}{m-n+1}\right)^{m-n+1}$, and let us calculate the integral in eq. (5).

$$
\begin{aligned}
\int_E^\infty \mathbb{P}\left[\|G^\dagger\|_2^2 > t\right] \mathrm{d}t &= \lim_{y\to\infty} \int_E^y \mathbb{P}\left[\|G^\dagger\|_2^2 > t\right] \mathrm{d}t && (6)\\
&= \lim_{y\to\infty} \int_E^y \frac{1}{\sqrt{2\pi(m-n+1)}}\left(\frac{\mathrm{e}\sqrt{m}}{m-n+1}\right)^{m-n+1} t^{-(m-n+1)/2}\,\mathrm{d}t && (7)\\
&= \lim_{y\to\infty} \int_E^y C t^{-(m-n+1)/2}\,\mathrm{d}t && (8)\\
&= \lim_{y\to\infty} \left[C\left(\frac{1}{\frac{-(m-n+1)}{2}+1}\right) t^{\frac{-(m-n+1)}{2}+1}\right]_E^y && (9)\\
&= C\left(\frac{1}{\frac{-(m-n+1)}{2}+1}\right)\lim_{y\to\infty}\left(y^{\frac{-(m-n+1)}{2}+1} - E^{\frac{-(m-n-1)}{2}}\right) && (10)\\
&= C\left(\frac{1}{\frac{-(m-n+1)}{2}+1}\right)\left(0 - E^{\frac{-(m-n+1)}{2}+1}\right) && (11)\\
&= C\left(\frac{1}{(-1)\cdot\left(\frac{(m-n+1)}{2}-1\right)}\right)(-1)\cdot E^{\frac{-(m-n+1)}{2}+1} && (12)\\
&= C\left(\frac{1}{\frac{(m-n+1)}{2}-1}\right)\cdot E^{\frac{-(m-n+1)}{2}+1} && (13)
\end{aligned}
$$

which, substituted into eq. (5), yields:

$$
\mathbb{E}\|G^\dagger\|_2^2 \leq E + C\left(\frac{1}{\frac{(m-n+1)}{2}-1}\right)\cdot E^{\frac{-(m-n+1)}{2}+1} \tag{14}
$$

Taking the derivative wrt. $E$ gives us:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}E}\left(E + C\left(\frac{1}{\frac{(m-n+1)}{2}-1}\right)\cdot E^{\frac{-(m-n+1)}{2}+1}\right) &= 1 + C\left(\frac{1}{\frac{(m-n+1)}{2}-1}\right)\left(\frac{-(m-n+1)}{2}+1\right)E^{\frac{-(m-n+1)}{2}} && (15)\\
&= 1 + C\left(\frac{1}{\frac{(m-n+1)}{2}-1}\right)\left(\frac{(m-n+1)}{2}-1\right)(-1)E^{\frac{-(m-n+1)}{2}} && (16)\\
&= 1 - CE^{\frac{-(m-n+1)}{2}} && (17)
\end{aligned}
$$

Setting equal to zero, we get:

$$
1 - CE^{\frac{-(m-n+1)}{2}} = 0 \quad\to\quad 1 = CE^{\frac{-(m-n+1)}{2}} \quad\to\quad \frac{1}{C} = \frac{1}{E^{\frac{(m-n+1)}{2}}} \quad\to\quad C = E^{\frac{(m-n+1)}{2}} \quad\to\quad C^{\frac{2}{(m-n+1)}} = E \tag{18}
$$

Substituting this expression for $C$ into Eq. (14) gives:

$$\mathbb{E}\|G^{\dagger}\|_2^2 \leq E + E^{\frac{(m-n+1)}{2}} \left( \frac{1}{\frac{(m-n+1)}{2} - 1} \right) E^{\frac{-(m-n+1)}{2}+1} \tag{19}$$

$$= E + \left( \frac{1}{\frac{(m-n+1)}{2} - 1} \right) E \tag{20}$$

$$= E \left( 1 + \left( \frac{1}{\frac{(m-n-1)}{2}} \right) \right) \tag{21}$$

By substituting $E = C^{\frac{2}{(m-n+1)}}$ back in, we get:

$$\mathbb{E}\|G^{\dagger}\|_2^2 \leq C^{\frac{2}{(m-n+1)}} \left( 1 + \left( \frac{1}{\frac{(m-n-1)}{2}} \right) \right) \tag{22}$$

$$= \left( \frac{1}{\sqrt{2\pi(m-n+1)}} \left( \frac{\mathrm{e}\sqrt{m}}{m-n+1} \right)^{m-n+1} \right)^{\frac{2}{(m-n+1)}} \left( 1 + \left( \frac{1}{\frac{(m-n-1)}{2}} \right) \right) \tag{23}$$

$$= \left( \frac{1}{\sqrt{2\pi(m-n+1)}} \right)^{\frac{2}{(m-n+1)}} \left( \left( \frac{\mathrm{e}\sqrt{m}}{m-n+1} \right)^{m-n+1} \right)^{\frac{2}{(m-n+1)}} \left( 1 + \left( \frac{1}{\frac{(m-n-1)}{2}} \right) \right) \tag{24}$$

$$= (2\pi(m-n+1))^{-\frac{1}{2} \cdot \frac{2}{(m-n+1)}} \left( \frac{\mathrm{e}\sqrt{m}}{m-n+1} \right)^{(m-n+1) \cdot \frac{2}{(m-n+1)}} \left( 1 + \left( \frac{1}{\frac{(m-n-1)}{2}} \right) \right) \tag{25}$$

$$= (2\pi(m-n+1))^{-\frac{1}{(m-n+1)}} \left( \frac{\mathrm{e}\sqrt{m}}{m-n+1} \right)^{2} \left( 1 + \left( \frac{1}{\frac{(m-n-1)}{2}} \right) \right) \tag{26}$$

$$= \frac{1}{(2\pi(m-n+1))^{\frac{1}{(m-n+1)}}} \left( \frac{\mathrm{e}\sqrt{m}}{m-n+1} \right)^{2} \left( 1 + \frac{2}{m-n-1} \right) \tag{27}$$

$$= \underbrace{\left( \frac{1}{2\pi(m-n+1)} \right)^{\frac{1}{(m-n+1)}}}_{<1} \left( \frac{\mathrm{e}\sqrt{m}}{m-n+1} \right)^{2} \left( 1 + \frac{2}{m-n-1} \right) \tag{28}$$

Introducing $p := m - n$, we have:

$$\mathbb{E}\|G^{\dagger}\|_2^2 \leq \mathrm{e}^2 m \cdot \left( \frac{1}{(p+1)^2} \right) \left( 1 + \frac{2}{p-1} \right) \tag{29}$$

$$= \mathrm{e}^2 m \cdot \frac{1}{p+1} \cdot \frac{1}{p+1} \cdot \left( \frac{p-1}{p-1} + \frac{2}{p-1} \right) \tag{30}$$

$$= \mathrm{e}^2 m \cdot \frac{1}{p+1} \cdot \frac{1}{p+1} \cdot \left( \frac{p+1}{p-1} \right) \tag{31}$$

$$= \mathrm{e}^2 m \cdot \frac{1}{(p+1)(p-1)} \tag{32}$$

$$= \frac{\mathrm{e}^2 m}{p^2 - 1} = \frac{\mathrm{e}^2 m}{(m-n)^2 - 1} \tag{33}$$

The error in Lemma 3.1 is in Equation (28), but since we discard this term it does not make a difference for the end result. Furthermore, instead of having $m-1 \geq n \geq 2$, we should have $m-2 \geq n \geq 2$.