Sketching for low-rank matrix approximation

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▶ Setting

Skecthing is a popular method for low-rank approximation (that we did not cover in the lectures); see e.g. [1]. The basic idea is that instead of working with a large matrix $A \in \mathbb{R}^{n \times n}$, we use a *sketch* $A\Omega$ where Ω is a suitable random matrix with $m \ll n$ columns; an advantage of this representation is that if A is updated by a low-rank matrix B, the sketch can be updated by multiplying B with Ω without accessing A again. The goal of this project is to implement and analyze a sketch-based algorithm for low-rank approximation, taken from [2].

▶ Tasks

1. Given a matrix $A \in \mathbb{R}^{n \times m}$ and integers r and $\ell \geq 0$, the following algorithm computes a rank-r approximation $A \approx BC^T$ for $B \in \mathbb{R}^{n \times r}$ and $C \in \mathbb{R}^{m \times r}$.

Algorithm 1

1: **procedure** SKECTHING(A, r, ℓ)
2: Draw Gaussian random matrices $X \in \mathbb{R}^{m \times r}$ and $Y \in \mathbb{R}^{n \times (r+\ell)}$
3: Compute the products AX and $Y^T A$
4: Compute an economy QR factorization $QR = Y^T AX$
5: Set $B \leftarrow AXR^\dagger$ and $C \leftarrow A^T YQ$
6: **end procedure**

$R^\dagger$ denotes the Moore-Penrose pseudo-inverse of the matrix R.

Implement this algorithm. What is the asymptotic time complexity of the algorithm in function of n, m, r, ℓ and $\mathrm{nnz}(A)$?

2. Prove that if A has rank exactly r, then `SKETCHING`$(A, r, 0)$ returns an exact low-rank factorization of A with probability 1.

   Hint: A square Gaussian random matrix is invertible with probability 1.

3. Find and correct the mistake in [2, Lemma 3.1].

4. Consider the three following 200×200 matrices:

   - The Hilbert matrix $A_1(i, j) = \frac{1}{i+j-1}$;
   - A matrix A_2 constructed in the following way:

   ```
   d = 0.8.^(1:n);
   [Q, ~] = qr(randn(n));
   [V, ~] = qr(randn(n));
   A = Q*diag(d)*V;
   ```

   - A matrix A_3 constructed similarly as before, with `d = (1:n).^(-1.5)`.

   Plot the error of the low-rank approximations that you obtain with ranks $1, \ldots, 100$, for $\ell = 0, \ell = 3, \ell =$ `floor`$(r/2)$. What do you notice?

5. Explore the effect of randomness in the algorithm: For a fixed instance of matrix A_3, for $r = 1, \ldots, 100$, apply the algorithm 1000 times and plot the average error and the 95% confidence interval, in the case $\ell = 3$ and then in the case $\ell =$ `floor`$(r/2)$.

▶ References

[1] Woodruff, David P. Sketching as a tool for numerical linear algebra. arXiv preprint arXiv:1411.4357 (2014). https://arxiv.org/pdf/1411.4357

[2] Nakatsukasa, Yuji. Fast and stable randomized low-rank matrix approximation. arXiv preprint arXiv:2009.11392 (2020). https://arxiv.org/pdf/2009.11392