Curves

Bezier, Curvature

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Curvature

In mathematics, **curvature** refers to any of a number of loosely related concepts in different areas of geometry. Intuitively, curvature is the amount by which a geometric object deviates from being *flat,* or *straight* in the case of a line, but this is defined in different ways depending on the context. There is a key distinction between **extrinsic curvature**, which is defined for objects embedded in another space (usually a Euclidean space) in a way that relates to the radius of curvature of circles that touch the object, and *intrinsic curvature*, which is defined at each point in a Riemannian manifold. This article deals primarily with the first concept.

The canonical example of extrinsic curvature is that of a circle, which everywhere has curvature equal to the reciprocal of its radius. Smaller circles bend more sharply, and hence have higher curvature. The curvature of a smooth curve is defined as the curvature of its osculating circle at each point.

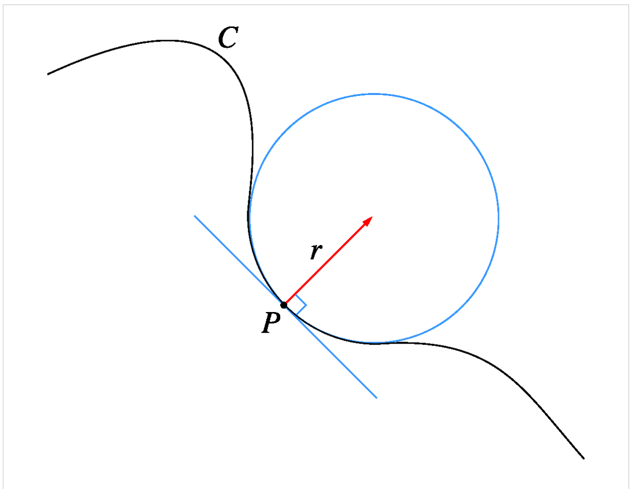
In a plane, this is a scalar quantity, but in three or more dimensions it is described by a curvature vector that takes into account the direction of the bend as well as its sharpness. The curvature of more complex objects (such as surfaces or even curved *n*-dimensional spaces) is described by more complex objects from linear algebra, such as the general Riemann curvature tensor.

The remainder of this article discusses, from a mathematical perspective, some geometric examples of curvature: the curvature of a curve embedded in a plane and the curvature of a surface in Euclidean space. See the links below for further reading.

Curvature of plane curves

Cauchy defined the center of curvature C as the intersection point of two infinitely close normals to the curve, the radius of curvature as the distance from the point to C, and the curvature itself as the inverse of the radius of curvature.[1]

Let C be a plane curve (the precise technical assumptions are given below). The curvature of C at a point is a measure of how sensitive its tangent line is to moving the point to other nearby points. There are a number of equivalent ways that this idea can be made precise.



One way is geometrical. It is natural to define the curvature of a straight line to be identically zero. The curvature of a circle of radius R should be large if R is small and small if R is large. Thus the curvature of a circle is defined to be the reciprocal of the radius:

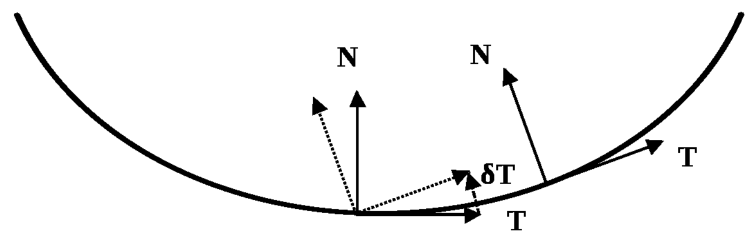
$$\kappa = \frac{1}{R}.$$

Given any curve C and a point P on it, there is a unique circle or line which most closely approximates the curve near P, the osculating circle at P. The curvature of C at P is then defined to be the curvature of that circle or line. The radius of curvature is defined as the reciprocal of the curvature.

Another way to understand the curvature is physical. Suppose that a particle moves along the curve with unit speed. Taking the time s as the parameter for C, this provides a natural parametrization for the curve. The unit tangent vector $\mathbf{T}$ (which is also the velocity vector, since the particle is moving with unit speed) also depends on time. The curvature is then the magnitude of the rate of change of $\mathbf{T}$. Symbolically,

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

This is the magnitude of the acceleration of the particle and the vector $d\mathbf{T}/ds$ is the acceleration vector. Geometrically, the curvature κ measures how fast the unit tangent vector to the curve rotates. If a curve keeps close to the same direction, the unit tangent vector changes very little and the curvature is small; where the curve undergoes a tight turn, the curvature is large.



The **T** and **N** vectors at two points on a plane curve, a translated version of the second frame (dotted), and the change in **T**: δ**T**'. δs is the distance between the points. In the limit $\frac{d\mathbf{T}}{ds}$ will be in the direction **N** and the curvature describes the speed of rotation of the frame.

These two approaches to the curvature are related geometrically by the following observation. In the first definition, the curvature of a circle is equal to the ratio of the angle of an arc to its length. Likewise, the curvature of a plane curve at any point is the limiting ratio of $d\theta$, an infinitesimal angle (in radians) between tangents to that curve at the ends of an infinitesimal segment of the curve, to the length of that segment ds, i.e., $d\theta/ds$. If the tangents at the ends of the segment are represented by unit vectors, it is easy to show that in this limit, the magnitude of the difference vector is equal to $d\theta$, which leads to the given expression in the second definition of curvature.

Precise definition

Suppose that C is a twice continuously differentiable immersed plane curve, which here means that there exists parametric representation of C by a pair of functions $\gamma(t) = (x(t), y(t))$ such that the first and second derivatives of x and y both exist and are continuous, and

$$\|\gamma'\|^2 = x'(t)^2 + y'(t)^2 \neq 0$$

throughout the domain. For such a plane curve, there exists a reparametrization with respect to arc length s. This is a parametrization of C such that

$$\|\gamma'\|^2 = x'(s)^2 + y'(s)^2 = 1.$$ [2]

The velocity vector $\mathbf{T}(s)$ is the unit tangent vector. The unit normal vector $\mathbf{N}(s)$, the **curvature** $\kappa(s)$, the **oriented** or **signed curvature** $k(s)$, and the **radius of curvature** $R(s)$ are given by

$$\mathbf{T}(s) = \gamma'(s), \quad \mathbf{T}'(s) = k(s)\mathbf{N}(s), \quad \kappa(s) = \|\mathbf{T}'(s)\| = \|\gamma''(s)\| = |k(s)|, \quad R(s) = \frac{1}{\kappa(s)}.$$

Expressions for calculating the curvature in arbitrary coordinate systems are given below.

Signed curvature

The sign of the signed curvature k indicates the direction in which the unit tangent vector rotates as a function of the parameter along the curve. If the unit tangent rotates counterclockwise, then $k > 0$. If it rotates clockwise, then $k < 0$.

The signed curvature depends on the particular parametrization chosen for a curve. For example the unit circle can be parametrised by $(\cos(\theta),\sin(\theta))$ (counterclockwise, with $k > 0$), or by $(\cos(-\theta),\sin(-\theta))$ (clockwise, with $k < 0$). More precisely, the signed curvature depends only on the choice of orientation of an immersed curve. Every immersed curve in the plane admits two possible orientations.

Local expressions

For a plane curve given parametrically in Cartesian coordinates as γ(t) = ($x(t)$,$y(t)$), the curvature is

$$\kappa = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}},$$

where primes refer to derivatives with respect to parameter t . The signed curvature k is

$$k = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}.$$

These can be expressed in a coordinate-independent manner via

$$k = \frac{\det(\gamma', \gamma'')}{\|\gamma'\|^3}, \quad \kappa = \frac{|\det(\gamma', \gamma'')|}{\|\gamma'\|^3}.$$

Curvature of a graph

For the less general case of a plane curve given explicitly as $y = f(x)$, and now using primes for derivatives with respect to coordinate x , the curvature is

$$\kappa = \frac{|y''|}{(1 + y'^2)^{3/2}},$$

and the signed curvature is

$$k = \frac{y''}{(1 + y'^2)^{3/2}}.$$

This quantity is common in physics and engineering; for example, in the equations of bending in beams, the 1D vibration of a tense string, approximations to the fluid flow around surfaces (in aeronautics), and the free surface boundary conditions in ocean waves. In such applications, the assumption is almost always made that the slope is small compared with unity, so that the approximation:

$$\kappa \approx \left| \frac{d^2y}{dx^2} \right|$$

may be used. This approximation yields a straightforward linear equation describing the phenomenon.

If a curve is defined in polar coordinates as $r(\theta)$, then its curvature is

$$\kappa(\theta) = \frac{|r^2 + 2r'^2 - rr''|}{\left(r^2 + r'^2\right)^{3/2}}$$

where here the prime now refers to differentiation with respect to θ .

Example

Consider the parabola $y = x^2$. We can parametrize the curve simply as $\gamma(t) = (t,t^2) = (x,y)$. If we use primes for derivatives with respect to parameter t , then

$$x' = 1, \quad x'' = 0, \quad y' = 2t, \quad y'' = 2.$$

Substituting and dropping unnecessary absolute values, get

$$\kappa(t) = \left| \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \right| = \frac{1 \cdot 2 - (2t)(0)}{(1 + (2t)^2)^{3/2}} = \frac{2}{(1 + 4t^2)^{3/2}}.$$

Curvature of space curves

As in the case of curves in two dimensions, the curvature of a regular space curve *C* in three dimensions (and higher) is the magnitude of the acceleration of a particle moving with unit speed along a curve. Thus if $\gamma(s)$ is the arclength parametrization of *C* then the unit tangent vector $\mathbf{T}(s)$ is given by

$$\mathbf{T}(s) = \gamma'(s)$$

and the curvature is the magnitude of the acceleration:

$$\kappa(s) = \|\mathbf{T}'(s)\| = \|\gamma''(s)\|.$$

The direction of the acceleration is the unit normal vector $\mathbf{N}(s)$, which is defined by

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}.$$

The plane containing the two vectors $\mathbf{T}(s)$ and $\mathbf{N}(s)$ is called the osculating plane to the curve at $\gamma(s)$. The curvature has the following geometrical interpretation. There exists a circle in the osculating plane tangent to $\gamma(s)$ whose Taylor series to second order at the point of contact agrees with that of $\gamma(s)$. This is the osculating circle to the curve. The radius of the circle $R(s)$ is called the radius of curvature, and the curvature is the reciprocal of the radius of curvature:

$$\kappa(s) = \frac{1}{R(s)}.$$

The tangent, curvature, and normal vector together describe the second-order behavior of a curve near a point. In three-dimensions, the third order behavior of a curve is described by a related notion of torsion, which measures the extent to which a curve tends to perform a corkscrew in space. The torsion and curvature are related by the Frenet–Serret formulas (in three dimensions) and their generalization (in higher dimensions).

Local expressions

For a parametrically defined space curve in three-dimensions given in Cartesian coordinates by $\gamma(t) = (x(t),y(t),z(t))$, the curvature is

$$\kappa = \frac{\sqrt{(z''y' - y''z')^2 + (x''z' - z''x')^2 + (y''x' - x''y')^2}}{(x'^2 + y'^2 + z'^2)^{3/2}}.$$

where the prime denotes differentiation with respect to time *t*. This can be expressed independently of the coordinate system by means of the formula

$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3}$$

where $\times$ is the vector cross product. Equivalently,

$$\kappa = \frac{\sqrt{\det\left((\gamma', \gamma'')^t (\gamma', \gamma'')\right)}}{\|\gamma'\|^3}.$$

Here the *t* denotes the matrix transpose. This last formula is also valid for the curvature of curves in a Euclidean space of any dimension.

Curvature from arc and chord length

Given two points P and Q on C, let $s(P,Q)$ be the arc length of the portion of the curve between P and Q and let $d(P,Q)$ denote the length of the line segment from P to Q. The curvature of C at P is given by the limit

$$\kappa(P) = \lim_{Q\to P} \sqrt{\frac{24\left(s(P,Q) - d(P,Q)\right)}{s(P,Q)^3}}$$

where the limit is taken as the point Q approaches P on C. The denominator can equally well be taken to be $d(P,Q)^3$. The formula is valid in any dimension. Furthermore, by considering the limit independently on either side of P, this definition of the curvature can sometimes accommodate a singularity at P. The formula follows by verifying it for the osculating circle.

Curves on surfaces

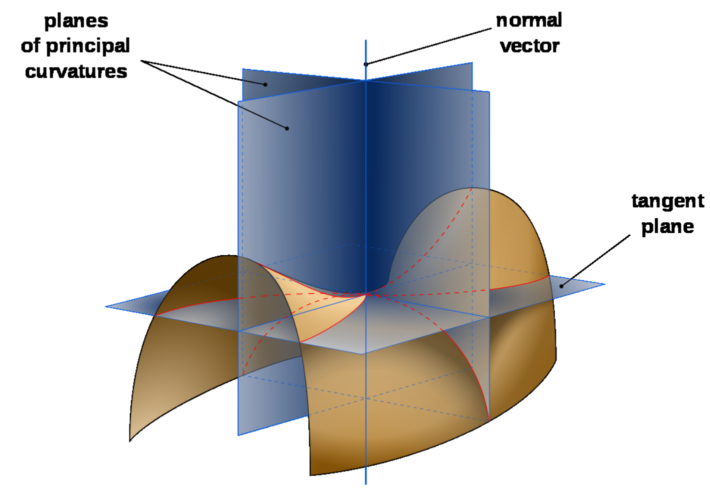
When a one dimensional curve lies on a two dimensional surface embedded in three dimensions $\mathbf{R}^3$, further measures of curvature are available, which take the surface's unit-normal vector, **u** into account. These are the normal curvature, geodesic curvature and geodesic torsion. Any non-singular curve on a smooth surface will have its tangent vector **T** lying in the tangent plane of the surface orthogonal to the normal vector. The **normal curvature**, k_n, is the curvature of the curve projected onto the plane containing the curve's tangent **T** and the surface normal **u**; the **geodesic curvature**, k_g, is the curvature of the curve projected onto the surface's tangent plane; and the **geodesic torsion** (or **relative torsion**), τ_r, measures the rate of change of the surface normal around the curve's tangent.

Let the curve be a unit speed curve and let $\mathbf{t} = \mathbf{u} \times \mathbf{T}$ so that **T**, **u**, **t** form an orthonormal basis: the **Darboux frame**. The above quantities are related by:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{t}' \\ \mathbf{u}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_r \\ -\kappa_n & -\tau_r & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{t} \\ \mathbf{u} \end{pmatrix}$$

Principal curvature

All curves with the same tangent vector will have the same normal curvature, which is the same as the curvature of the curve obtained by intersecting the surface with the plane containing **T** and **u**. Taking all possible tangent vectors then the maximum and minimum values of the normal curvature at a point are called the **principal curvatures**, k_1 and k_2, and the directions of the corresponding tangent vectors are called **principal directions**.



Saddle surface with normal planes in directions of principal curvatures

Two dimensions: Curvature of surfaces

Gaussian curvature

In contrast to curves, which do not have intrinsic curvature, but do have extrinsic curvature (they only have a curvature given an embedding), surfaces can have intrinsic curvature, independent of an embedding. The Gaussian curvature, named after Carl Friedrich Gauss, is equal to the product of the principal curvatures, k_1k_2. It has the dimension of $1/\text{length}^2$ and is positive for spheres, negative for one-sheet hyperboloids and zero for planes. It determines whether a surface is locally convex (when it is positive) or locally saddle (when it is negative).

This definition of Gaussian curvature is *extrinsic* in that it uses the surface's embedding in $\mathbf{R}^3$, normal vectors, external planes etc. Gaussian curvature is however in fact an *intrinsic* property of the surface, meaning it does not depend on the particular embedding of the surface; intuitively, this means that ants living on the surface could determine the Gaussian curvature. For example, an ant living on a sphere could measure the sum of the interior angles of a triangle and determine that it was greater than 180 degrees, implying that the space it inhabited had positive curvature. On the other hand, an ant living on a cylinder would not detect any such departure from Euclidean geometry, in particular the ant could not detect that the two surfaces have different mean curvatures (see below) which is a purely extrinsic type of curvature.

Formally, Gaussian curvature only depends on the Riemannian metric of the surface. This is Gauss's celebrated Theorema Egregium, which he found while concerned with geographic surveys and mapmaking.

An intrinsic definition of the Gaussian curvature at a point *P* is the following: imagine an ant which is tied to *P* with a short thread of length *r*. She runs around *P* while the thread is completely stretched and measures the length C(*r*) of one complete trip around *P*. If the surface were flat, she would find C(*r*) = 2π*r*. On curved surfaces, the formula for C(*r*) will be different, and the Gaussian curvature *K* at the point *P* can be computed by the Bertrand–Diquet–Puiseux theorem as

$$K = \lim_{r \to 0^+} 3\frac{2\pi r - C(r)}{\pi r^3}.$$

The integral of the Gaussian curvature over the whole surface is closely related to the surface's Euler characteristic; see the Gauss-Bonnet theorem.

The discrete analog of curvature, corresponding to curvature being concentrated at a point and particularly useful for polyhedra, is the (angular) defect; the analog for the Gauss-Bonnet theorem is Descartes' theorem on total angular defect.

Because curvature can be defined without reference to an embedding space, it is not necessary that a surface be embedded in a higher dimensional space in order to be curved. Such an intrinsically curved two-dimensional surface is a simple example of a Riemannian manifold.

Mean curvature

The mean curvature is equal to half the sum of the principal curvatures, $(k_1+k_2)/2$. It has the dimension of 1/length. Mean curvature is closely related to the first variation of surface area, in particular a minimal surface such as a soap film, has mean curvature zero and a soap bubble has constant mean curvature. Unlike Gauss curvature, the mean curvature is extrinsic and depends on the embedding, for instance, a cylinder and a plane are locally isometric but the mean curvature of a plane is zero while that of a cylinder is nonzero.

Second fundamental form

The intrinsic and extrinsic curvature of a surface can be combined in the second fundamental form. This is a quadratic form in the tangent plane to the surface at a point whose value at a particular tangent vector *X* to the surface is the normal component of the acceleration of a curve along the surface tangent to *X*; that is, it is the normal curvature to a curve tangent to *X* (see above). Symbolically,

$$I\!I(X,X) = N \cdot (\nabla_X X)$$

where *N* is the unit normal to the surface. For unit tangent vectors *X*, the second fundamental form assumes the maximum value k_1 and minimum value k_2, which occur in the principal directions u_1 and u_2, respectively. Thus, by the principal axis theorem, the second fundamental form is

$$I\!I(X,X) = k_1(X \cdot u_1)^2 + k_2(X \cdot u_2)^2.$$

Thus the second fundamental form encodes both the intrinsic and extrinsic curvatures.

A related notion of curvature is the shape operator, which is a linear operator from the tangent plane to itself. When applied to a tangent vector *X* to the surface, the shape operator is the tangential component of the rate of change of the normal vector when moved along a curve on the surface tangent to *X*. The principal curvatures are the eigenvalues of the shape operator, and in fact the shape operator and second fundamental form have the same matrix representation with respect to a pair of orthonormal vectors of the tangent plane. The Gauss curvature is thus the determinant of the shape tensor and the mean curvature is half its trace.

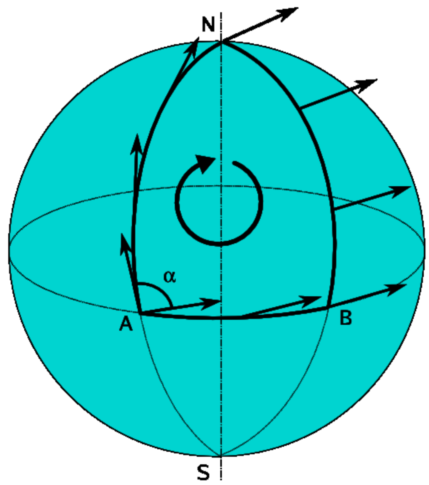
Higher dimensions: Curvature of space

By extension of the former argument, a space of three or more dimensions can be intrinsically curved. The curvature is *intrinsic* in the sense that it is a property defined at every point in the space, rather than a property defined with respect to a larger space that contains it. In general, a curved space may or may not be conceived as being embedded in a higher-dimensional ambient space; if not then its curvature can only be defined intrinsically.

After the discovery of the intrinsic definition of curvature, which is closely connected with non-Euclidean geometry, many mathematicians and scientists questioned whether ordinary physical space might be curved, although the success of Euclidean geometry up to that time meant that the radius of curvature must be astronomically large. In the theory of general relativity, which describes gravity and cosmology, the idea is slightly generalised to the "curvature of space-time"; in relativity theory space-time is a pseudo-Riemannian manifold. Once a time coordinate is defined, the three-dimensional space corresponding to a particular time is generally a curved Riemannian manifold; but since the time coordinate choice is largely arbitrary, it is the underlying space-time curvature that is physically significant.

Although an arbitrarily curved space is very complex to describe, the curvature of a space which is locally isotropic and homogeneous is described by a single Gaussian curvature, as for a surface; mathematically these are strong conditions, but they correspond to reasonable physical assumptions (all points and all directions are indistinguishable). A positive curvature corresponds to the inverse square radius of curvature; an example is a sphere or hypersphere. An example of negatively curved space is hyperbolic geometry. A space or space-time with zero curvature is called **flat**. For example, Euclidean space is an example of a flat space, and Minkowski space is an example of a flat space-time. There are other examples of flat geometries in both settings, though. A torus or a cylinder can both be given flat metrics, but differ in their topology. Other topologies are also possible for curved space. See also shape of the universe.

Generalizations



Parallel transporting a vector from $A \to N \to B \to A$ yields a different vector. This failure to return to the initial vector is measured by the holonomy of the surface.

The mathematical notion of *curvature* is also defined in much more general contexts.[3] Many of these generalizations emphasize different aspects of the curvature as it is understood in lower dimensions.

One such generalization is kinematic. The curvature of a curve can naturally be considered as a kinematic quantity, representing the force felt by a certain observer moving along the curve; analogously, curvature in higher dimensions can be regarded as a kind of tidal force (this is one way of thinking of the sectional curvature). This generalization of curvature depends on how nearby test particles diverge or converge when they are allowed to move freely in the space; see Jacobi field.

Another broad generalization of curvature comes from the study of parallel transport on a surface. For instance, if a vector is moved around a loop on the surface of a sphere keeping parallel throughout the motion, then the final position of the vector may not be the same as the initial position of the vector. This phenomenon is known as holonomy. Various generalizations capture in an abstract form this idea of curvature as a measure of holonomy; see curvature form. A closely related notion of curvature comes from gauge theory in physics, where the curvature represents a field and a vector potential for the field is a quantity that is in general path-dependent: it may change if an observer moves around a loop.

Two more generalizations of curvature are the scalar curvature and Ricci curvature. In a curved surface such as the sphere, the area of a disc on the surface differs from the area of a disc of the same radius in flat space. This difference (in a suitable limit) is measured by the scalar curvature. The difference in area of a sector of the disc is measured by the Ricci curvature. Each of the scalar curvature and Ricci curvature are defined in analogous ways in three and higher dimensions. They are particularly important in relativity theory, where they both appear on the side of Einstein's field equations that represents the geometry of spacetime (the other side of which represents the presence of matter and energy). These generalizations of curvature underlie, for instance, the notion that curvature can be a property of a measure; see curvature of a measure.

Another generalization of curvature relies on the ability to compare a curved space with another space that has *constant* curvature. Often this is done with triangles in the spaces. The notion of a triangle makes senses in metric spaces, and this gives rise to CAT(k) spaces.

Notes

[1] *Borovik, Alexandre; Katz, Mikhail G. (2011), "Who gave you the Cauchy--Weierstrass tale? The dual history of rigorous calculus", *Foundations of Science*, arXiv:1108.2885, doi:10.1007/s10699-011-9235-x

[2] *Kennedy, John (2011), *The ArcLength Parametrization of a Curve* (http://homepage.smc.edu/kennedy_john/ArcLengthParametrization.PDF),

[3] See e.g. S.Kobayashi and K.Nomizu, "Foundations of Differential Geometry", Chapters 2 and 3, Vol.I, Wiley-Interscience.

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External links

- Create your own animated illustrations of moving Frenet-Serret frames and curvature (http://www.math.uni-muenster.de/u/urs.hartl/gifs/CurvatureAndTorsionOfCurves.mw) (Maple-Worksheet)
- The History of Curvature (http://www3.villanova.edu/maple/misc/history_of_curvature/k.htm)
- Curvature, Intrinsic and Extrinsic (http://www.mathpages.com/rr/s5-03/5-03.htm) at MathPages

Mean curvature

In mathematics, the **mean curvature** H of a surface S is an *extrinsic* measure of curvature that comes from differential geometry and that locally describes the curvature of an embedded surface in some ambient space such as Euclidean space.

The concept was introduced by Sophie Germain in her work on elasticity theory.[1][2]

Definition

Let p be a point on the surface S. Consider all curves C_i on S passing through p. Every such C_i has an associated curvature K_i given at p. Of those curvatures K_i, at least one is characterized as maximal κ_1 and one as minimal κ_2, and these two curvatures κ_1, κ_2 are known as the *principal curvatures* of S.

The **mean curvature** at $p \in S$ is then the average of the principal curvatures (Spivak 1999, Volume 3, Chapter 2), hence the name:

$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

More generally (Spivak 1999, Volume 4, Chapter 7), for a hypersurface T the mean curvature is given as

$$H = \frac{1}{n}\sum_{i=1}^{n} \kappa_i.$$

More abstractly, the mean curvature is the trace of the second fundamental form divided by *n* (or equivalently, the shape operator).

Additionally, the mean curvature H may be written in terms of the covariant derivative ∇ as

$$H\vec{n} = g^{ij}\nabla_i\nabla_j X,$$

using the *Gauss-Weingarten relations,* where $X(x,t)$ is a family of smoothly embedded hypersurfaces, $\vec{n}$ a unit normal vector, and g_{ij} the metric tensor.

A surface is a minimal surface if and only if the mean curvature is zero. Furthermore, a surface which evolves under the mean curvature of the surface S, is said to obey a heat-type equation called the mean curvature flow equation.

The sphere is the only embedded surface of constant positive mean curvature without boundary or singularities. However, the result is not true when the condition "embedded surface" is weakened to "immersed surface".[3]

Surfaces in 3D space

For a surface defined in 3D space, the mean curvature is related to a unit normal of the surface:

$$2H = \nabla \cdot \hat{n}$$

where the normal chosen affects the sign of the curvature. The sign of the curvature depends on the choice of normal: the curvature is positive if the surface curves "away" from the normal. The formula above holds for surfaces in 3D space defined in any manner, as long as the divergence of the unit normal may be calculated.

For the special case of a surface defined as a function of two coordinates, e.g. $z = S(x,y)$, and using downward pointing normal the (doubled) mean curvature expression is

$$\begin{aligned} 2H &= \nabla \cdot \left(\frac{\nabla(S-z)}{|\nabla(S-z)|} \right) \\ &= \nabla \cdot \left(\frac{\nabla S}{\sqrt{1+(\nabla S)^2}} \right) \\ &= \frac{\left(1+\left(\frac{\partial S}{\partial x}\right)^2\right)\frac{\partial^2 S}{\partial y^2} - 2\frac{\partial S}{\partial x}\frac{\partial S}{\partial y}\frac{\partial^2 S}{\partial x \partial y} + \left(1+\left(\frac{\partial S}{\partial y}\right)^2\right)\frac{\partial^2 S}{\partial x^2}}{\left(1+\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2\right)^{3/2}}. \end{aligned}$$

If the surface is additionally known to be axisymmetric with $z = S(r)$,

$$2H = \frac{\frac{\partial^2 S}{\partial r^2}}{\left(1+\left(\frac{\partial S}{\partial r}\right)^2\right)^{3/2}} + \frac{\partial S}{\partial r}\frac{1}{r\left(1+\left(\frac{\partial S}{\partial r}\right)^2\right)^{1/2}},$$

where $\frac{\partial S}{\partial r}\frac{1}{r}$ comes from the derivative of $z = S(r) = S\left(\sqrt{x^2+y^2}\right)$.

Mean curvature in fluid mechanics

An alternate definition is occasionally used in fluid mechanics to avoid factors of two:

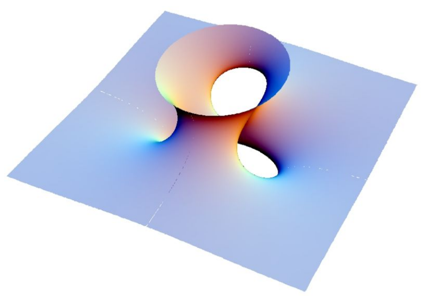
$$H_f = (\kappa_1 + \kappa_2).$$

This results in the pressure according to the Young-Laplace equation inside an equilibrium spherical droplet being surface tension times H_f; the two curvatures are equal to the reciprocal of the droplet's radius

$$\kappa_1 = \kappa_2 = r^{-1}.$$

Minimal surfaces

A **minimal surface** is a surface which has zero mean curvature at all points. Classic examples include the catenoid, helicoid and Enneper surface. Recent discoveries include Costa's minimal surface and the Gyroid.



A rendering of Costa's minimal surface.

An extension of the idea of a minimal surface are surfaces of constant mean curvature.

Notes

[1] Dubreil-Jacotin on Sophie Germain (http://www-groups.dcs.st-and.ac.uk/~history/Extras/Dubreil-Jacotin_Germain.html)

[2] Lodder, J. (2003). "Curvature in the Calculus Curriculum". *The American Mathematical Monthly* **110** (7): 593–605. doi:10.2307/3647744.

[3] http://projecteuclid.org/DPubS/Repository/1.0/Disseminate?view=body&id=pdf_1&handle=euclid.pjm/1102702809

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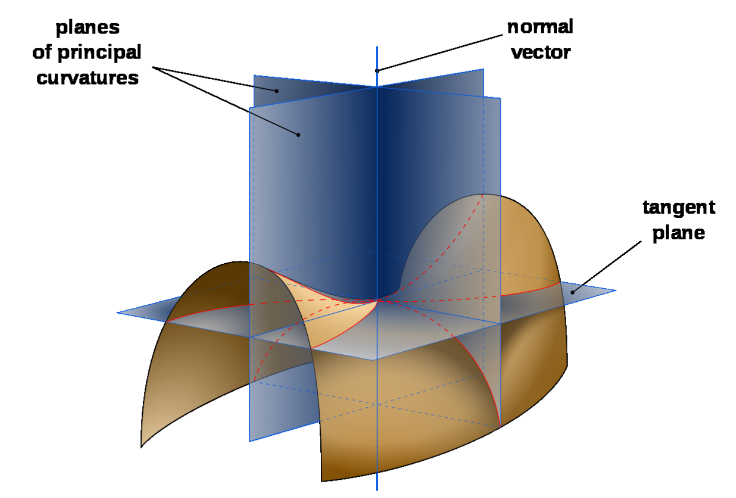
- Spivak, Michael (1999), *A comprehensive introduction to differential geometry (Volumes 3-4)* (3rd ed.), Publish or Perish Press, ISBN 0-914098-72-1, (Volume 3), (Volume 4).

Principal curvature

In differential geometry, the two **principal curvatures** at a given point of a surface are the eigenvalues of the shape operator at the point. They measure how the surface bends by different amounts in different directions at that point.



Saddle surface with normal planes in directions of principal curvatures

Discussion

At each point p of a differentiable surface in 3-dimensional Euclidean space one may choose a unit normal vector. A normal plane at p is one that contains the normal, and will therefore also contain a unique direction tangent to the surface and cut the surface in a plane curve. This curve will in general have different curvatures for different normal planes at p. The **principal curvatures** at p, denoted k_1 and k_2, are the maximum and minimum values of this curvature.

Here the curvature of a curve is by definition the reciprocal of the radius of the osculating circle. The curvature is taken to be positive if the curve turns in the same direction as the surface's chosen normal, and otherwise negative. The directions of the normal plane where the curvature takes its maximum and minimum values are always perpendicular, if k_1 does not equal k_2, a result of Euler (1760), and are called **principal directions**. From a modern perspective, this theorem follows from the spectral theorem because they can be given as the principal axes of a symmetric tensor—the second fundamental form. A systematic analysis of the principal curvatures and principal directions was undertaken by Gaston Darboux, using Darboux frames.

The product k_1k_2 of the two principal curvatures is the Gaussian curvature, K, and the average $(k_1+k_2)/2$ is the mean curvature, H.

If at least one of the principal curvatures is zero at every point, then the Gaussian curvature will be 0 and the surface is a developable surface. For a minimal surface, the mean curvature is zero at every point.

Formal definition

Let M be a surface in Euclidean space with second fundamental form $II(X,Y)$. Fix a point $p \in M$, and an orthonormal basis X_1, X_2 of tangent vectors at p. Then the principal curvatures are the eigenvalues of the symmetric matrix

$$[II_{ij}] = \begin{bmatrix} II(X_1, X_1) & II(X_1, X_2) \\ II(X_2, X_1) & II(X_2, X_2) \end{bmatrix}.$$

If X_1 and X_2 are selected so that the matrix $[II_{ij}]$ is a diagonal matrix, then they are called the **principal directions**. If the surface is oriented, then one often requires that the pair (X_1, X_2) to be positively oriented with respect to the given orientation.

Without reference to a particular orthonormal basis, the principal curvatures are the eigenvalues of the shape operator, and the principal directions are its eigenvectors.

Generalizations

For hypersurfaces in higher dimensional Euclidean spaces, the principal curvatures may be defined in a directly analogous fashion. The principal curvatures are the eigenvalues of the matrix of the second fundamental form $II(X_i,X_j)$ in an orthonormal basis of the tangent space. The principal directions are the corresponding eigenvectors.

Similarly, if M is a hypersurface in a Riemannian manifold N, then the principal curvatures are the eigenvalues of its second-fundamental form. If k_1, ..., k_n are the n principal curvatures at a point $p \in M$ and X_1, ..., X_n are corresponding orthonormal eigenvectors (principal directions), then the sectional curvature of M at p is given by

$$K(X_i, X_j) = k_i k_j.$$

Classification of points on a surface

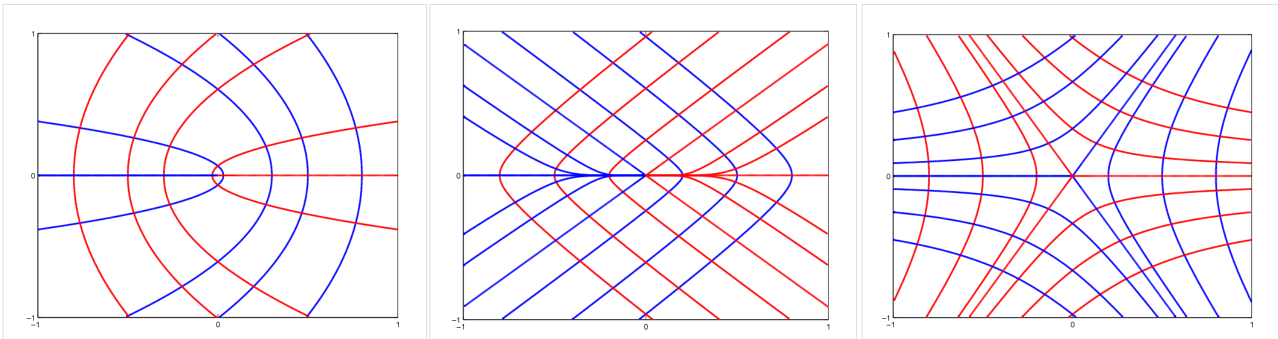
- At **elliptical** points, both principal curvatures have the same sign, and the surface is locally convex.
  - At **umbilic** points, both principal curvatures are equal and every tangent vector can be considered a principal direction. These typically occur in isolated points.
- At **hyperbolic** points, the principal curvatures have opposite signs, and the surface will be locally saddle shaped.
- At **parabolic** points, one of the principal curvatures is zero. Parabolic points generally lie in a curve separating elliptical and hyperbolic regions.
  - At **flat umbilic** points both principal curvatures are zero. A generic surface will not contain flat umbilic points. The Monkey saddle is one surface with an isolated flat umbilic.

Lines of curvature

The **lines of curvature** or **curvature lines** are curves which are always tangent to a principal direction (they are integral curves for the principal direction fields). There will be two lines of curvature through each non-umbilic point and the lines will cross at right angles.

In the vicinity of an umbilic the lines of curvature form one of three configurations **star**, **lemon** and **monstar** (derived from *lemon-star*)[1]. These points are also called Darbouxian Umbilics, in honor to Gaston Darboux, the first to make a systematic study in Vol. 4, p455, of his Leçons (1896).

configurations of lines of curvature near umbilics



Lemon Monstar Star

In these figures, the red curves are the lines of curvature for one family of principal directions, and the blue curves for the other.

When a line of curvature has a local extremum of the same principal curvature then the curve has a **ridge point**. These ridge points form curves on the surface called **ridges**. The ridge curves pass through the umbilics. For the star pattern either 3 or 1 ridge line pass through the umbilic, for the monstar and lemon only one ridge passes through.[2]

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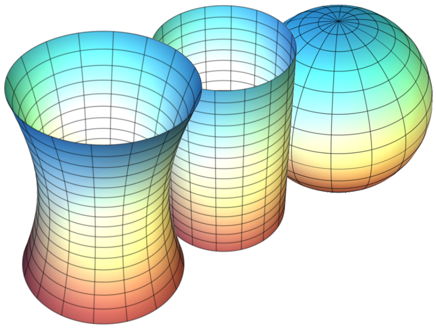
[4] http://www.hti.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=ABV4153.0002.001
[5] http://www.hti.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=ABV4153.0003.001
[6] http://www.hti.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=ABV4153.0004.001

External links

- Historical Comments on Monge's Ellipsoid and the Configuration of Lines of Curvature on Surfaces Immersed in $\mathbf{R}^3$ (http://front.math.ucdavis.edu/0411.5403)

Gaussian curvature

In differential geometry, the **Gaussian curvature** or **Gauss curvature** of a point on a surface is the product of the principal curvatures, κ_1 and κ_2, of the given point. It is an *intrinsic* measure of curvature, i.e., its value depends only on how distances are measured on the surface, not on the way it is isometrically embedded in space. This result is the content of Gauss's Theorema egregium.



From left to right: a surface of negative Gaussian curvature (hyperboloid), a surface of zero Gaussian curvature (cylinder), and a surface of positive Gaussian curvature (sphere).

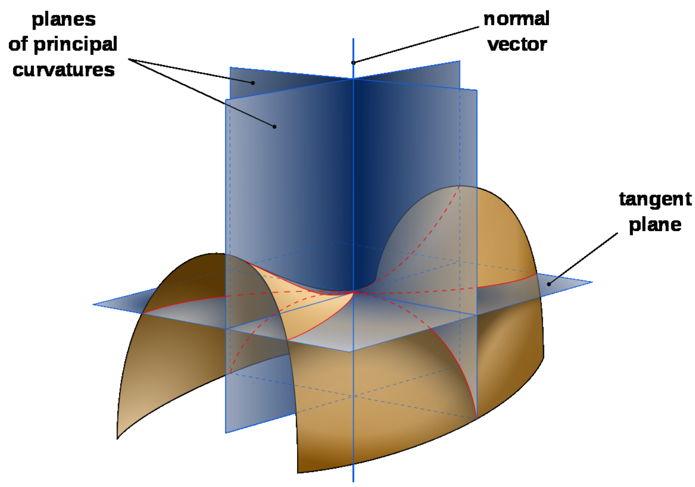
Symbolically, the Gaussian curvature K is defined as

$$\mathrm{K} = \kappa_1 \kappa_2.$$

where $\kappa_1$and $\kappa_2$are the principal curvatures.

Informal definition

At any point on a surface we can find a normal vector which is at right angles to the surface, the intersection of a plane containing the normal with the surface will form a curve called a *normal section* and the curvature of this curve is the *normal curvature*. For most points on most surfaces, different sections will have different curvatures; the maximum and minimum values of these are called the principal curvatures, call these κ_1, κ_2. The **Gaussian curvature** is the product of the two principal curvatures $\mathrm{K} = \kappa_1 \kappa_2$.



Saddle surface with normal planes in directions of principal curvatures

The sign of the Gaussian curvature can be used to characterise the surface.

- If both principal curvatures are the same sign $\kappa_1 > 0, \kappa_2 > 0$or $\kappa_1 < 0, \kappa_2 < 0$then the Gaussian curvature is positive and the surface is said to have an elliptic point. At such points the surface will be dome like, locally lying on one side of its tangent plane. All sectional curvatures will have the same sign.
- If the principal curvatures have different signs $\kappa_1 > 0, \kappa_2 < 0$then the Gaussian curvature is negative and the surface is said to have a hyperbolic point. At such points the surface will be saddle shaped and intersect its tangent plane in two curves. For two directions the sectional curvatures will be zero giving the asymptotic directions.
- If one principal curvature is zero $\kappa_1 \neq 0, \kappa_2 = 0$or $\kappa_1 = 0, \kappa_2 \neq 0$the Gaussian curvature is zero and the surface is said to have a parabolic point.

Most surfaces will contain regions of positive Gaussian curvature (elliptical points) and regions of negative Gaussian curvature separated by a curve of points with zero Gaussian curvature called a parabolic line.

Further informal discussion

In differential geometry, the two **principal curvatures** at a given point of a surface are the eigenvalues of the shape operator at the point. They measure how the surface bends by different amounts in different directions at that point. We represent the surface by the implicit function theorem as the graph of a function, *f*, of two variables, in such a way that the point *p* is a critical point, i.e., the gradient of *f* vanishes (this can always be attained by a suitable rigid motion). Then the Gaussian curvature of the surface at *p* is the determinant of the Hessian matrix of *f* (being the product of the eigenvalues of the Hessian). (Recall that the Hessian is the 2-by-2 matrix of second derivatives.) This definition allows one immediately to grasp the distinction between cup/cap *versus* saddle point.

Alternative definitions

It is also given by

$$\mathrm{K} = \frac{\langle (\nabla_2 \nabla_1 - \nabla_1 \nabla_2)\mathbf{e}_1, \mathbf{e}_2 \rangle}{\det g},$$

where $\nabla_i = \nabla_{\mathbf{e}_i}$ is the covariant derivative and g is the metric tensor.

At a point **p** on a regular surface in $\mathbf{R}^3$, the Gaussian curvature is also given by

$$K(\mathbf{p}) = \det(S(\mathbf{p})),$$

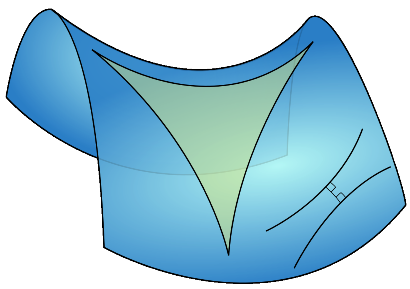
where S is the shape operator.

A useful formula for the Gaussian curvature is Liouville's equation in terms of the Laplacian in isothermal coordinates.

Total curvature

The surface integral of the Gaussian curvature over some region of a surface is called the **total curvature**. The total curvature of a geodesic triangle equals the deviation of the sum of its angles from π. The sum of the angles of a triangle on a surface of positive curvature will exceed π, while the sum of the angles of a triangle on a surface of negative curvature will be less than π. On a surface of zero curvature, such as the Euclidean plane, the angles will sum to precisely π.

$$\sum_{i=1}^{3} \theta_i = \pi + \iint_T K\, dA.$$



The sum of the angles of a triangle on a surface of negative curvature is less than that of a plane triangle.

A more general result is the Gauss–Bonnet theorem.

Important theorems

Theorema egregium

Gauss's **Theorema Egregium** (Latin: "remarkable theorem") states that Gaussian curvature of a surface can be determined from the measurements of length on the surface itself. In fact, it can be found given the full knowledge of the first fundamental form and expressed via the first fundamental form and its partial derivatives of first and second order. Equivalently, the determinant of the second fundamental form of a surface in $\mathbf{R}^3$ can be so expressed. The "remarkable", and surprising, feature of this theorem is that although the *definition* of the Gaussian curvature of a surface *S* in $\mathbf{R}^3$ certainly depends on the way in which the surface is located in space, the end result, the Gaussian curvature itself, is determined by the inner metric of the surface without any further reference to the ambient space: it is an intrinsic invariant. In particular, the Gaussian curvature is invariant under isometric deformations of the surface.

In contemporary differential geometry, a "surface", viewed abstractly, is a two-dimensional differentiable manifold. To connect this point of view with the classical theory of surfaces, such an abstract surface is embedded into $\mathbf{R}^3$ and endowed with the Riemannian metric given by the first fundamental form. Suppose that the image of the embedding is a surface S in $\mathbf{R}^3$. A *local isometry* is a diffeomorphism $f\colon U \to V$ between open regions of $\mathbf{R}^3$ whose restriction to $S \cap U$ is an isometry onto its image. **Theorema Egregium** is then stated as follows:

> The Gaussian curvature of an embedded smooth surface in $\mathbf{R}^3$ is invariant under the local isometries.

For example, the Gaussian curvature of a cylindrical tube is zero, the same as for the "unrolled" tube (which is flat).[1] On the other hand, since a sphere of radius R has constant positive curvature R^{-2} and a flat plane has constant curvature 0, these two surfaces are not isometric, even locally. Thus any planar representation of even a part of a sphere must distort the distances. Therefore, no cartographic projection is perfect.

Gauss–Bonnet theorem

The Gauss-Bonnet theorem links the total curvature of a surface to its Euler characteristic and provides an important link between local geometric properties and global topological properties.

Surfaces of constant curvature

- **Minding's theorem** (1839) states that all surfaces with the same constant curvature K are locally isometric. A consequence of Minding's theorem is that any surface whose curvature is identically zero can be constructed by bending some plane region. Such surfaces are called developable surfaces. Minding also raised the question whether a closed surface with constant positive curvature is necessarily rigid.
- **Liebmann's theorem** (1900) answered Minding's question. The only regular (of class C^2) closed surfaces in $\mathbf{R}^3$ with constant positive Gaussian curvature are spheres.[2]
- **Hilbert's theorem** (1901) states that there exists no complete analytic (class C^ω) regular surface in $\mathbf{R}^3$ of constant negative Gaussian curvature. In fact, the conclusion also holds for surfaces of class C^2 immersed in $\mathbf{R}^3$, but breaks down for C^1-surfaces. The pseudosphere has constant negative Gaussian curvature except at its singular cusp.[3]

Alternative Formulas

- Gaussian curvature of a surface in $\mathbf{R}^3$ can be expressed as the ratio of the determinants of the second and first fundamental forms:

$$K = \frac{\det II}{\det I} = \frac{LN - M^2}{EG - F^2}.$$

- The **Brioschi formula** gives Gaussian curvature solely in terms of the first fundamental form:

$$K = \left(\det \begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \det \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right) \Big/ (EG-F^2)^2$$

- For an **orthogonal parametrization** (i.e., $F = 0$), Gaussian curvature is:

$$K = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial u} \frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v} \frac{E_v}{\sqrt{EG}} \right).$$

- For a surface described as graph of a function $z = F(x, y)$, Gaussian curvature is:

$$K = \frac{F_{xx} \cdot F_{yy} - {F_{xy}}^2}{(1 + {F_x}^2 + {F_y}^2)^2}$$

- For a surface $F(x,y,z)=0$, Gaussian curvature is:[4]

$$K=\frac{[F_z(F_{xx}F_z-2F_xF_{xz})+F_x^2F_{zz}][F_z(F_{yy}F_z-2F_yF_{yz})+F_y^2F_{zz}]-[F_z(-F_xF_{yz}+F_{xy}F_z-F_{xz}F_y)+F_xF_yF_{zz}]^2}{F_z^2(F_x^2+F_y^2+F_z^2)^2}$$

- For a surface with metric conformal to the Euclidean one, so $F=0$ and $E=G=e^\sigma$, the Gauss curvature is given by

$$K=-\frac{1}{2e^\sigma}\Delta\sigma,$$

being Δ the usual Laplace operator.

- Gaussian curvature is the limiting difference between the **circumference of a geodesic circle** and a circle in the plane[5]:

$$K=\lim_{r\to 0^+}3\frac{2\pi r-C(r)}{\pi r^3}$$

- Gaussian curvature is the limiting difference between the **area of a geodesic disk** and a disk in the plane[5]:

$$K=\lim_{r\to 0^+}12\frac{\pi r^2-A(r)}{\pi r^4}$$

- Gaussian curvature may be expressed with the **Christoffel symbols**:[6]

$$K=-\frac{1}{E}\left(\frac{\partial}{\partial u}\Gamma_{12}^2-\frac{\partial}{\partial v}\Gamma_{11}^2+\Gamma_{12}^1\Gamma_{11}^2-\Gamma_{11}^1\Gamma_{12}^2+\Gamma_{12}^2\Gamma_{12}^2-\Gamma_{11}^2\Gamma_{22}^2\right)$$

External links

- Curvature in two spacelike dimensions [7]

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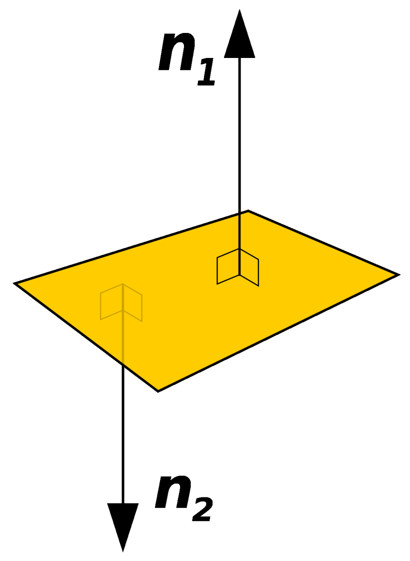
[7] http://www.lightandmatter.com/html_books/genrel/ch05/ch05.html#Section5.3

Normal (geometry)

In geometry, an object such as a line or vector is called a **normal** to another object if they are perpendicular to each other. For example, in the two-dimensional case, the **normal line** to a curve at a given point is the line perpendicular to the tangent line to the curve at the point.

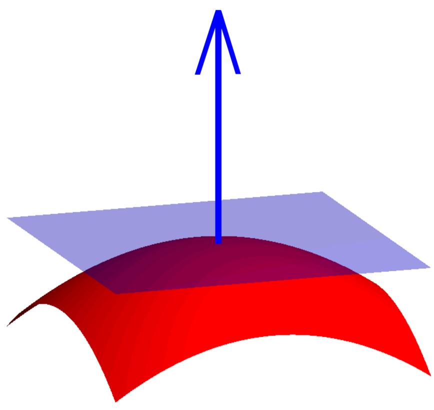
A **surface normal**, or simply **normal**, to a surface at a point P is a vector that is perpendicular to the tangent plane to that surface at P. The word "normal" is also used as an adjective: a line normal to a plane, the normal component of a force, the **normal vector**, etc. The concept of **normality** generalizes to orthogonality.



A polygon and two of its normal vectors

The concept has been generalized to differential manifolds of arbitrary dimension embedded in a Euclidean space. The **normal vector space** or **normal space** of a manifold at a point P is the set of the vectors which are orthogonal to the tangent space at P. In the case of differential curves, the curvature vector is a normal vector of special interest.

The **normal** is often used in computer graphics to determine a surface's orientation toward a light source for flat shading, or the orientation of each of the corners (vertices) to mimic a curved surface with Phong shading.


A normal to a surface at a point is the same as a normal to the tangent plane to that surface at that point.

Normal to surfaces in 3D space

Calculating a surface normal

For a convex polygon (such as a triangle), a surface normal can be calculated as the vector cross product of two (non-parallel) edges of the polygon.

For a plane given by the equation $ax + by + cz + d = 0$, the vector (a, b, c) is a normal.

For a plane given by the equation

$$\mathbf{r}(\alpha, \beta) = \mathbf{a} + \alpha\mathbf{b} + \beta\mathbf{c},$$

i.e., **a** is a point on the plane and **b** and **c** are (non-parallel) vectors lying on the plane, the normal to the plane is a vector normal to both **b** and **c** which can be found as the cross product $\mathbf{b} \times \mathbf{c}$.

For a hyperplane in n+1 dimensions, given by the equation

$$\mathbf{r} = \mathbf{a}_0 + \alpha_1\mathbf{a}_1 + \cdots + \alpha_n\mathbf{a}_n,$$

where $\mathbf{a}_0$ is a point on the hyperplane and $\mathbf{a}_i$ for i = 1, ..., n are non-parallel vectors lying on the hyperplane, a normal to the hyperplane is any vector in the null space of A where A is given by

$$A = [\mathbf{a}_1 \ldots \mathbf{a}_n].$$

That is, any vector orthogonal to all in-plane vectors is by definition a surface normal.

If a (possibly non-flat) surface S is parameterized by a system of curvilinear coordinates $\mathbf{x}(s, t)$, with s and t real variables, then a normal is given by the cross product of the partial derivatives

$$\frac{\partial \mathbf{x}}{\partial s} \times \frac{\partial \mathbf{x}}{\partial t}.$$

If a surface *S* is given implicitly as the set of points (x, y, z) satisfying $F(x, y, z) = 0$, then, a normal at a point (x, y, z) on the surface is given by the gradient

$$\nabla F(x, y, z).$$

since the gradient at any point is perpendicular to the level set, and $F(x, y, z) = 0$ (the surface) is a level set of F.

For a surface *S* given explicitly as a function $f(x, y)$ of the independent variables x, y (e.g., $f(x, y) = a_{00} + a_{01}y + a_{10}x + a_{11}xy$), its normal can be found in at least two equivalent ways. The first one is obtaining its implicit form $F(x, y, z) = z - f(x, y) = 0$, from which the normal follows readily as the gradient

$$\nabla F(x, y, z).$$

(Notice that the implicit form could be defined alternatively as

$$F(x, y, z) = f(x, y) - z;$$

these two forms correspond to the interpretation of the surface being oriented upwards or downwards, respectively, as a consequence of the difference in the sign of the partial derivative $\partial F / \partial z$.) The second way of obtaining the normal follows directly from the gradient of the explicit form,

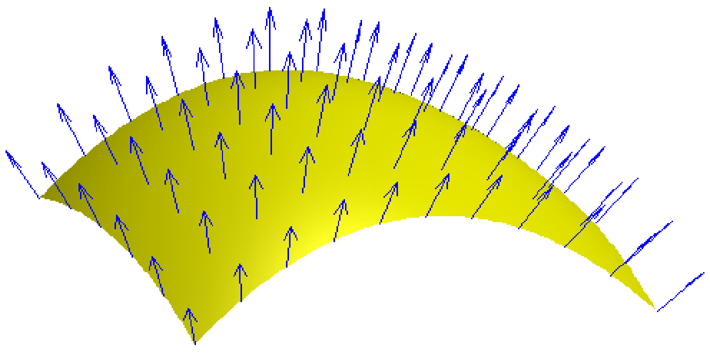
$$\nabla f(x, y);$$

by inspection,

$$\nabla F(x, y, z) = \hat{\mathbf{k}} - \nabla f(x, y),$$ where $\hat{\mathbf{k}}$ is the upward unit vector.

If a surface does not have a tangent plane at a point, it does not have a normal at that point either. For example, a cone does not have a normal at its tip nor does it have a normal along the edge of its base. However, the normal to the cone is defined almost everywhere. In general, it is possible to define a normal almost everywhere for a surface that is Lipschitz continuous.

Uniqueness of the normal



A vector field of normals to a surface

A normal to a surface does not have a unique direction; the vector pointing in the opposite direction of a surface normal is also a surface normal. For a surface which is the topological boundary of a set in three dimensions, one can distinguish between the **inward-pointing normal** and **outer-pointing normal**, which can help define the normal in a unique way. For an oriented surface, the surface normal is usually determined by the right-hand rule. If the normal is constructed as the cross product of tangent vectors (as described in the text above), it is a pseudovector.

Transforming normals

When applying a transform to a surface it is sometimes convenient to derive normals for the resulting surface from the original normals. All points P on tangent plane are transformed to P'. We want to find $\mathbf{n'}$ perpendicular to P. Let $\mathbf{t}$ be a vector on the tangent plane and M_l be the upper 3x3 matrix (translation part of transformation does not apply to normal or tangent vectors).

$$t' = M_l \times t$$
$$n^T \times t = n^T \times M_l^{-1} M_l \times t$$
$$n^T \times t = n^T \times M_l^{-1} M_l \times t = (M_l^{-1T} \times n)^T (M_l \times t)$$
$$n^T \times t = (M_l^{-1T} \times n)^T \times t'$$
$$n' = M_l^{-1T} \times n$$

So use the inverse transpose of the linear transformation (the upper 3x3 matrix) when transforming surface normals.

Hypersurfaces in *n*-dimensional space

The definition of a normal to a surface in three-dimensional space can be extended to $(n-1)$-dimensional hypersurfaces in a n-dimensional space. A *hypersurface* may be locally defined implicitly as the set of points $(x_1, x_2, \ldots, x_n)$ satisfying an equation $F(x_1, x_2, \ldots, x_n) = 0$, where F is a given scalar function. If F is continuously differentiable then the hypersurface is a differentiable manifold in the neighbourhood of the points where the gradient is not null. At these points the **normal vector space** has dimension one and is generated by the gradient

$$\nabla F(x_1, x_2, \ldots, x_n) = \left(\tfrac{\partial F}{\partial x_1}, \tfrac{\partial F}{\partial x_2}, \ldots, \tfrac{\partial F}{\partial x_n} \right) .$$

The **normal line** at a point of the hypersurface is defined only if the gradient is not null. It is the line passing through the point and having the gradient as direction.

Varieties defined by implicit equations in *n*-dimensional space

A **differential variety** defined by implicit equations in the n-dimensional space is the set of the common zeros of a finite set of differential functions in n variables

$$f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n).$$

The Jacobian matrix of the variety is the $k \times n$ matrix whose i-th row is the gradient of f_i. By implicit function theorem, the variety is a manifold in the neighborhood of a point of it where the Jacobian matrix has rank k. At such a point P, the **normal vector space** is the vector space generated by the values at P of the gradient vectors of the f_i.

In other words, a variety is defined as the intersection of k hypersurfaces, and the normal vector space at a point is the vector space generated by the normal vectors of the hypersurfaces at the point.

The **normal (affine) space** at a point P of the variety is the affine subspace passing through P and generated by the normal vector space at P.

These definitions may be extended *verbatim* to the points where the variety is not a manifold.

Example

Let V be the variety defined in the 3-dimensional space by the equations

$$x\,y = 0, \quad z = 0\,.$$

This variety is the union of the x-axis and the y-axis.

At a point $(a, 0, 0)$ where $a \neq 0$, the rows of the Jacobian matrix are (0, 0, 1) and $(0, a, 0)$. Thus the normal affine space is the plane of equation $x=a$. Similarly, if $b \neq 0$, the normal plane at $(0, b, 0)$ is the plane of equation $y=b$.

At the point (0, 0, 0) the rows of the Jacobian matrix are (0, 0, 1) and (0,0,0). Thus the normal vector space and the normal affine space have dimension 1 and the normal affine space is the z-axis.

Uses

- Surface normals are essential in defining surface integrals of vector fields.
- Surface normals are commonly used in 3D computer graphics for lighting calculations; see Lambert's cosine law.
- Surface normals are often adjusted in 3D computer graphics by normal mapping.
- Render layers containing surface normal information may be used in Digital compositing to change the apparent lighting of rendered elements.

Normal in geometric optics

The **normal** is the line perpendicular to the surface[1] of an optical medium. In reflection of light, the angle of incidence and the angle of reflection are respectively the angle between the normal and the incident ray and the angle between the normal and the reflected ray.

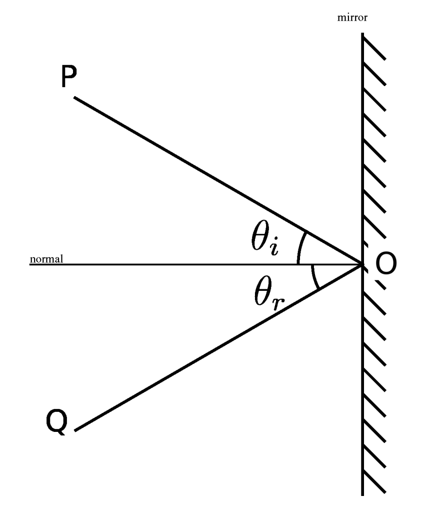


Diagram of specular reflection

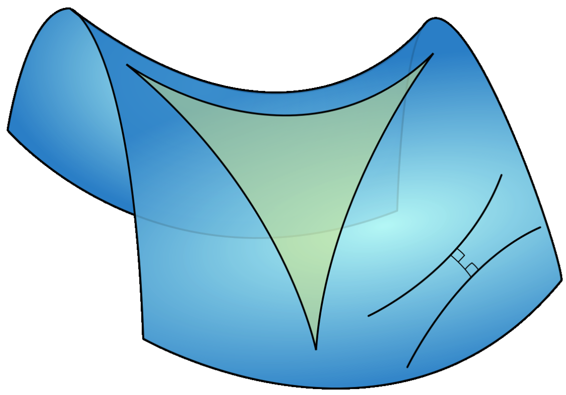
References

[1] "The Law of Reflection" (http://www.glenbrook.k12.il.us/gbssci/phys/Class/refln/u13l1c.html). *The Physics Classroom Tutorial*. . Retrieved 2008-03-31.

External links

- An explanation of normal vectors (http://msdn.microsoft.com/en-us/library/bb324491(VS.85).aspx) from Microsoft's MSDN
- Clear pseudocode for calculating a surface normal (http://www.opengl.org/wiki/Calculating_a_Surface_Normal) from either a triangle or polygon.

Differential geometry



A triangle immersed in a saddle-shape plane (a hyperbolic paraboloid), as well as two diverging ultraparallel lines.

Differential geometry is a mathematical discipline that uses the techniques of differential calculus and integral calculus, as well as linear algebra and multilinear algebra, to study problems in geometry. The theory of plane and space curves and of surfaces in the three-dimensional Euclidean space formed the basis for development of differential geometry during the 18th century and the 19th century. Since the late 19th century, differential geometry has grown into a field concerned more generally with the geometric structures on differentiable manifolds. Differential geometry is closely related to differential topology, and to the geometric aspects of the theory of differential equations. Grigori Perelman's proof of the Poincaré conjecture using the techniques of Ricci flows demonstrated the power of the differential-geometric approach to questions in topology and it highlighted the important role played by its analytic methods. The differential geometry of surfaces captures many of the key ideas and techniques characteristic of this field.

Branches of differential geometry

Riemannian geometry

Riemannian geometry studies Riemannian manifolds, smooth manifolds with a *Riemannian metric*. This is a concept of distance expressed by means of a smooth positive definite symmetric bilinear form defined on the tangent space at each point. Riemannian geometry generalizes Euclidean geometry to spaces that are not necessarily flat, although they still resemble the Euclidean space at each point "infinitesimally", i.e. in the first order of approximation. Various concepts based on length, such as the arc length of curves, area of plane regions, and volume of solids all possess natural analogues in Riemannian geometry. The notion of a directional derivative of a function from multivariable calculus is extended in Riemannian geometry to the notion of a covariant derivative of a tensor. Many concepts and techniques of analysis and differential equations have been generalized to the setting of Riemannian manifolds.

A distance-preserving diffeomorphism between Riemannian manifolds is called an isometry. This notion can also be defined *locally*, i.e. for small neighborhoods of points. Any two regular curves are locally isometric. However, the Theorema Egregium of Carl Friedrich Gauss showed that already for surfaces, the existence of a local isometry imposes strong compatibility conditions on their metrics: the Gaussian curvatures at the corresponding points must be the same. In higher dimensions, the Riemann curvature tensor is an important pointwise invariant associated to a Riemannian manifold that measures how close it is to being flat. An important class of Riemannian manifolds is the Riemannian symmetric spaces, whose curvature is not necessarily constant. These are the closest analogues to the "ordinary" plane and space considered in Euclidean and non-Euclidean geometry.

Pseudo-Riemannian geometry

Pseudo-Riemannian geometry generalizes Riemannian geometry to the case in which the metric tensor need not be positive-definite. A special case of this is a Lorentzian manifold, which is the mathematical basis of Einstein's general relativity theory of gravity.

Finsler geometry

Finsler geometry has the *Finsler manifold* as the main object of study. This is a differential manifold with a Finsler metric, i.e. a Banach norm defined on each tangent space. A Finsler metric is a much more general structure than a Riemannian metric. A Finsler structure on a manifold M is a function $F : \mathrm{T}M \to [0,\infty)$ such that:

1. $F(x, my) = |m|F(x,y)$ for all x, y in TM,
2. F is infinitely differentiable in $\mathrm{T}M - \{0\}$,
3. The vertical Hessian of F^2 is positive definite.

Symplectic geometry

Symplectic geometry is the study of symplectic manifolds. An **almost symplectic manifold** is a differentiable manifold equipped with a smoothly varying non-degenerate skew-symmetric bilinear form on each tangent space, i.e., a nondegenerate 2-form ω, called the *symplectic form*. A symplectic manifold is an almost symplectic manifold for which the symplectic form ω is closed: $d\omega = 0$.

A diffeomorphism between two symplectic manifolds which preserves the symplectic form is called a symplectomorphism. Non-degenerate skew-symmetric bilinear forms can only exist on even dimensional vector spaces, so symplectic manifolds necessarily have even dimension. In dimension 2, a symplectic manifold is just a surface endowed with an area form and a symplectomorphism is an area-preserving diffeomorphism. The phase space of a mechanical system is a symplectic manifold and they made an implicit appearance already in the work of Joseph Louis Lagrange on analytical mechanics and later in Carl Gustav Jacobi's and William Rowan Hamilton's formulations of classical mechanics.

By contrast with Riemannian geometry, where the curvature provides a local invariant of Riemannian manifolds, Darboux's theorem states that all symplectic manifolds are locally isomorphic. The only invariants of a symplectic manifold are global in nature and topological aspects play a prominent role in symplectic geometry. The first result in symplectic topology is probably the Poincaré-Birkhoff theorem, conjectured by Henri Poincaré and then proved by G.D. Birkhoff in 1912. It claims that if an area preserving map of an annulus twists each boundary component in opposite directions, then the map has at least two fixed points.[1]

Contact geometry

Contact geometry deals with certain manifolds of odd dimension. It is close to symplectic geometry and like the latter, it originated in questions of classical mechanics. A *contact structure* on a (2n + 1) - dimensional manifold M is given by a smooth hyperplane field H in the tangent bundle that is as far as possible from being associated with the level sets of a differentiable function on M (the technical term is "completely nonintegrable tangent hyperplane distribution"). Near each point p, a hyperplane distribution is determined by a nowhere vanishing 1-form α, which is unique up to multiplication by a nowhere vanishing function:

$$H_p = \ker \alpha_p \subset T_pM.$$

A local 1-form on M is a *contact form* if the restriction of its exterior derivative to H is a non-degenerate two-form and thus induces a symplectic structure on H_p at each point. If the distribution H can be defined by a global one-form α then this form is contact if and only if the top-dimensional form

$$\alpha \wedge (d\alpha)^n$$

is a volume form on **M**, i.e. does not vanish anywhere. A contact analogue of the Darboux theorem holds: all contact structures on an odd-dimensional manifold are locally isomorphic and can be brought to a certain local normal form by a suitable choice of the coordinate system.

Complex and Kähler geometry

Complex differential geometry is the study of complex manifolds. An almost complex manifold is a *real* manifold M, endowed with a tensor of type (1 , 1), i.e. a vector bundle endomorphism (called an *almost complex structure*)

$J : TM \to TM$, such that $J^2 = -1$.

It follows from this definition that an almost complex manifold is even dimensional.

An almost complex manifold is called *complex* if $N_J = 0$, where N_J is a tensor of type (2 , 1) related to J, called the Nijenhuis tensor (or sometimes the *torsion*). An almost complex manifold is complex if and only if it admits a holomorphic coordinate atlas. An *almost Hermitian structure* is given by an almost complex structure *J*, along with a Riemannian metric *g*, satisfying the compatibility condition

$$g(JX, JY) = g(X, Y).$$

An almost Hermitian structure defines naturally a differential two-form

$$\omega_{J,g}(X, Y) := g(JX, Y).$$

The following two conditions are equivalent:

1. $N_J = 0 \text{ and } d\omega = 0$
2. $\nabla J = 0$

where ∇ is the Levi-Civita connection of g. In this case, (J, g) is called a *Kähler structure*, and a *Kähler manifold* is a manifold endowed with a Kähler structure. In particular, a Kähler manifold is both a complex and a symplectic manifold. A large class of Kähler manifolds (the class of Hodge manifolds) is given by all the smooth complex projective varieties.

CR geometry

CR geometry is the study of the intrinsic geometry of boundaries of domains in complex manifolds.

Differential topology

Differential topology is the study of (global) geometric invariants without a metric or symplectic form. It starts from the natural operations such as Lie derivative of natural vector bundles and de Rham differential of forms. Beside Lie algebroids, also Courant algebroids start playing a more important role.

Lie groups

A Lie group is a group in the category of smooth manifolds. Beside the algebraic properties this enjoys also differential geometric properties. The most obvious construction is that of a Lie algebra which is the tangent space at the unit endowed with the Lie bracket between left-invariant vector fields. Beside the structure theory there is also the wide field of representation theory.

Bundles and connections

The apparatus of vector bundles, principal bundles, and connections on bundles plays an extraordinarily important role in modern differential geometry. A smooth manifold always carries a natural vector bundle, the tangent bundle. Loosely speaking, this structure by itself is sufficient only for developing analysis on the manifold, while doing geometry requires, in addition, some way to relate the tangent spaces at different points, i.e. a notion of parallel transport. An important example is provided by affine connections. For a surface in $\mathbf{R}^3$, tangent planes at different points can be identified using a natural path-wise parallelism induced by the ambient Euclidean space, which has a well-known standard definition of metric and parallelism. In Riemannian geometry, the Levi-Civita connection serves a similar purpose. (The Levi-Civita connection defines path-wise parallelism in terms of a given arbitrary Riemannian metric on a manifold.) More generally, differential geometers consider spaces with a vector bundle and an arbitrary affine connection which is not defined in terms of a metric. In physics, the manifold may be the space-time continuum and the bundles and connections are related to various physical fields.

Intrinsic versus extrinsic

From the beginning and through the middle of the 18th century, differential geometry was studied from the *extrinsic* point of view: curves and surfaces were considered as lying in a Euclidean space of higher dimension (for example a surface in an ambient space of three dimensions). The simplest results are those in the differential geometry of curves and differential geometry of surfaces. Starting with the work of Riemann, the *intrinsic* point of view was developed, in which one cannot speak of moving "outside" the geometric object because it is considered to be given in a free-standing way. The fundamental result here is Gauss's theorema egregium, to the effect that Gaussian curvature is an intrinsic invariant.

The intrinsic point of view is more flexible. For example, it is useful in relativity where space-time cannot naturally be taken as extrinsic (what would be "outside" of it?). With the intrinsic point of view it is harder to define the central concept of curvature and other structures such as connections, so there is a price to pay.

These two points of view can be reconciled, i.e. the extrinsic geometry can be considered as a structure additional to the intrinsic one. (See the Nash embedding theorem.)

Applications of differential geometry

Below are some examples of how differential geometry is applied to other fields of science and mathematics.

- In physics, three uses will be mentioned:
  - Differential geometry is the language in which Einstein's general theory of relativity is expressed. According to the theory, the universe is a smooth manifold equipped with a pseudo-Riemannian metric, which describes the curvature of space-time. Understanding this curvature is essential for the positioning of satellites into orbit around the earth. Differential geometry is also indispensable in the study of gravitational lensing and black holes.
  - Differential forms are used in the study of electromagnetism.
  - Differential geometry has applications to both Lagrangian mechanics and Hamiltonian mechanics. Symplectic manifolds in particular can be used to study Hamiltonian systems.
- In economics, differential geometry has applications to the field of econometrics.[2]
- Geometric modeling (including computer graphics) and computer-aided geometric design draw on ideas from differential geometry.
- In engineering, differential geometry can be applied to solve problems in digital signal processing.[3]
- In probability, statistics, and information theory, one can interpret various structures as Riemannian manifolds, which yields the field of information geometry, particularly via the Fisher information metric.
- In structural geology, differential geometry is used to analyze and describe geologic structures.
- In computer vision, differential geometry is used to analyze shapes.[4]
- In image processing, differential geometry is used to process and analyse data on non-flat surfaces.[5]

References

[1] It is easy to show that the area preserving condition (or the twisting condition) cannot be removed. Note that if one tries to extend such a theorem to higher dimensions, one would probably guess that a volume preserving map of a certain type must have fixed points. This is false in dimensions greater than 3.

[2] Paul Marriott and Mark Salmon (editors), "Applications of Differential Geometry to Econometrics", Cambridge University Press; 1 edition (September 18, 2000).

[3] Jonathan H. Manton, "On the role of differential geometry in signal processing" (http://ieeexplore.ieee.org/iel5/9711/30654/01416480.pdf?arnumber=1416480).

[4] Mario Micheli, "The Differential Geometry of Landmark Shape Manifolds: Metrics, Geodesics, and Curvature", http://www.math.ucla.edu/~micheli/PUBLICATIONS/micheli_phd.pdf

[5] Anand A. Joshi, "Geometric methods for image processing and signal analysis", (http://users.loni.ucla.edu/~ajoshi/final_thesis.pdf)

Further reading

- Wolfgang Kühnel (2002). *Differential Geometry: Curves - Surfaces - Manifolds* (2nd ed. ed.). ISBN 0-8218-3988-8.
- Theodore Frankel (2004). *The geometry of physics: an introduction* (2nd ed. ed.). ISBN 0-521-53927-7.
- Spivak, Michael (1999). *A Comprehensive Introduction to Differential Geometry (5 Volumes)* (3rd Edition ed.).
- do Carmo, Manfredo (1976). *Differential Geometry of Curves and Surfaces*. ISBN 0-13-212589-7. Classical geometric approach to differential geometry without tensor analysis.
- Kreyszig, Erwin (1991). *Differential Geometry*. ISBN 0-486-66721-9. Good classical geometric approach to differential geometry with tensor machinery.
- do Carmo, Manfredo Perdigao (1994). *Riemannian Geometry*.
- McCleary, John (1994). *Geometry from a Differentiable Viewpoint*.
- Bloch, Ethan D. (1996). *A First Course in Geometric Topology and Differential Geometry*.
- Gray, Alfred (1998). *Modern Differential Geometry of Curves and Surfaces with Mathematica* (2nd ed. ed.).
- Burke, William L. (1985). *Applied Differential Geometry*.
- ter Haar Romeny, Bart M. (2003). *Front-End Vision and Multi-Scale Image Analysis*. ISBN 1-4020-1507-0.

External links

- B. Conrad. Differential Geometry handouts, Stanford University (http://math.stanford.edu/~conrad/diffgeomPage/)
- Michael Murray's online differential geometry course, 1996 (http://www.maths.adelaide.edu.au/michael.murray/teaching_old.html)
- A Modern Course on Curves and Surface, Richard S Palais, 2003 (http://VirtualMathMuseum.org/Surface/a/bk/curves_surfaces_palais.pdf)
- Richard Palais's 3DXM Surfaces Gallery (http://VirtualMathMuseum.org/)
- Balázs Csikós's Notes on Differential Geometry (http://www.cs.elte.hu/geometry/csikos/dif/dif.html)
- N. J. Hicks, Notes on Differential Geometry, Van Nostrand. (http://www.wisdom.weizmann.ac.il/~yakov/scanlib/hicks.pdf)
- MIT OpenCourseWare: Differential Geometry, Fall 2008 (http://ocw.mit.edu/courses/mathematics/18-950-differential-geometry-fall-2008/)

Chord (geometry)

A **chord** of a circle is a geometric line segment whose endpoints both lie on the circumference of the circle. A **secant** or a **secant line** is the line extension of a chord. More generally, a chord is a line segment joining two points on any curve, such as but not limited to an ellipse. A chord that passes through the circle's center point is the circle's diameter.

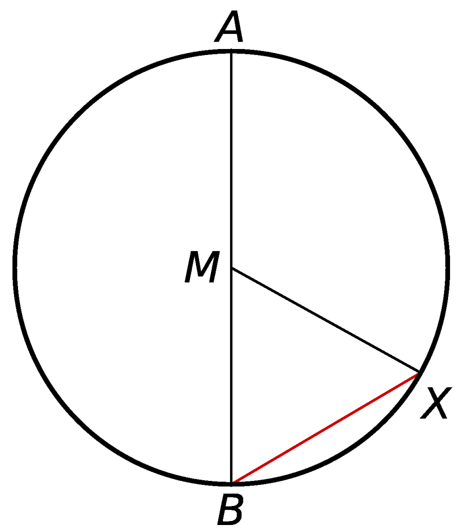
Chords of a circle

Further information: Chord properties

Among properties of chords of a circle are the following:

1. Chords are equidistant from the center only if their lengths are equal.
2. A chord's perpendicular bisector passes through the centre.
3. If the line extensions (secant lines) of chords AB and CD intersect at a point P, then their lengths satisfy AP·PB = CP·PD (power of a point theorem).

The area that a circular chord "cuts off" is called a circular segment.

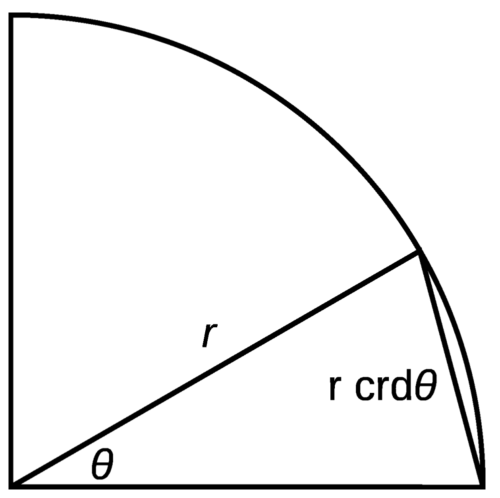


The red line *BX* is a **chord** (as is the diameter line *AB*).

Chords of an ellipse

The midpoints of a set of parallel chords of an ellipse are collinear.[1]:p.147

Chords in trigonometry



Chords were used extensively in the early development of trigonometry. The first known trigonometric table, compiled by Hipparchus, tabulated the value of the chord function for every 7.5 degrees. Ptolemy of Alexandria compiled a more extensive table of chords in his book on astronomy, giving the value of the chord for angles ranging from 1/2 degree to 180 degrees by increments of half a degree.

The chord function is defined geometrically as in the picture to the left. The chord of an angle is the length of the chord between two points on a unit circle separated by that angle. The chord function can be related to the modern sine function, by taking one of the points to be (1,0), and the other point to be (cos θ , sin θ), and then using the Pythagorean theorem to calculate the chord length:

$$\operatorname{crd}\ \theta = \sqrt{(1-\cos\theta)^2+\sin^2\theta} = \sqrt{2-2\cos\theta} = 2\sqrt{\frac{1-\cos\theta}{2}} = 2\sin\frac{\theta}{2}.$$

The last step uses the half-angle formula. Much as modern trigonometry is built on the sine function, ancient trigonometry was built on the chord function. Hipparchus is purported to have written a twelve volume work on chords, all now lost, so presumably a great deal was known about them. The chord function satisfies many identities analogous to well-known modern ones:

Name	Sine-based	Chord-based
Pythagorean	$\sin^2\theta + \cos^2\theta = 1$	$\mathrm{crd}^2\theta + \mathrm{crd}^2(180^\circ - \theta) = 4$
Half-angle	$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$	$\mathrm{crd}\ \frac{\theta}{2} = \pm\sqrt{2 - \mathrm{crd}(180^\circ - \theta)}$

The half-angle identity greatly expedites the creation of chord tables. Ancient chord tables typically used a large value for the radius of the circle, and reported the chords for this circle. It was then a simple matter of scaling to determine the necessary chord for any circle. According to G. J. Toomer, Hipparchus used a circle of radius 3438' (= 3438/60 = 57.3). This value is extremely close to $180/\pi$ (= 57.29577951...). One advantage of this choice of radius was that he could very accurately approximate the chord of a small angle as the angle itself. In modern terms, it allowed a simple linear approximation:

$$\frac{3438}{60}\mathrm{crd}\ \theta = 2\frac{3438}{60}\sin\frac{\theta}{2} \approx 2\frac{3438}{60}\frac{\pi}{180}\frac{\theta}{2} = \left(\frac{3438}{60}\frac{\pi}{180}\right)\theta \approx \theta.$$

Calculating circular chords

The chord of a circle can be calculated using other information:[2]

Initial data	Radius (r)	Diameter (D)
Sagitta (s)	$c = 2\sqrt{s(2r-s)}$	$c = 2\sqrt{s(D-s)}$
Apothem (a)	$c = 2\sqrt{r^2 - a^2}$	$c = \sqrt{D^2 - 4a^2}$
Angle (θ)	$c = 2r\sin\left(\frac{\theta}{2}\right)$	$c = D\sin\left(\frac{\theta}{2}\right)$

References

[1] Chakerian, G. D. "A Distorted View of Geometry." Ch. 7 in *Mathematical Plums* (R. Honsberger, editor). Washington, DC: Mathematical Association of America, 1979.

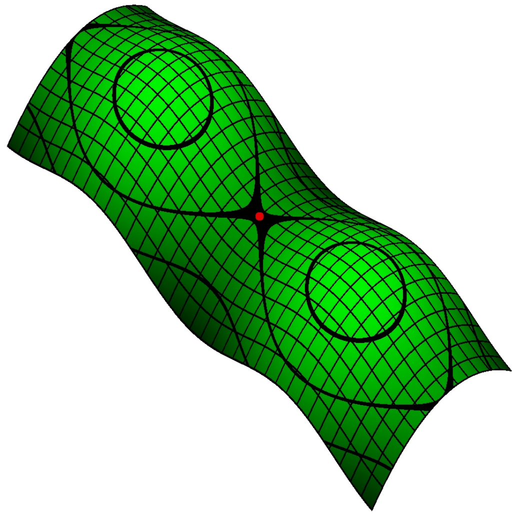
[2] Déplanche, Y.,*Diccio fórmulas*, 1996, Edunsa (publ.), p. 29. (http://books.google.com/books?id=1HVHOwAACAAJ), isbn=978-84-7747-119-6

External links

- History of Trigonometry Outline (http://aleph0.clarku.edu/~djoyce/ma105/trighist.html)
- Trigonometric functions (http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Trigonometric_functions.html), focusing on history
- Chord (of a circle) (http://www.mathopenref.com/chord.html) With interactive animation

Surface

In mathematics, specifically in topology, a **surface** is a two-dimensional topological manifold. The most familiar examples are those that arise as the boundaries of solid objects in ordinary three-dimensional Euclidean space $\mathbf{R}^3$ — for example, the surface of a ball. On the other hand, there are surfaces, such as the Klein bottle, that cannot be embedded in three-dimensional Euclidean space without introducing singularities or self-intersections.



An open surface with *X*-, *Y*-, and *Z*-contours shown.

To say that a surface is "two-dimensional" means that, about each point, there is a *coordinate patch* on which a two-dimensional coordinate system is defined. For example, the surface of the Earth is (ideally) a two-dimensional sphere, and latitude and longitude provide two-dimensional coordinates on it (except at the poles and along the 180th meridian).

The concept of surface finds application in physics, engineering, computer graphics, and many other disciplines, primarily in representing the surfaces of physical objects. For example, in analyzing the aerodynamic properties of an airplane, the central consideration is the flow of air along its surface.

Definitions and first examples

A *(topological) surface* is a nonempty second countable Hausdorff topological space in which every point has an open neighbourhood homeomorphic to some open subset of the Euclidean plane $\mathbf{E}^2$. Such a neighborhood, together with the corresponding homeomorphism, is known as a *(coordinate) chart*. It is through this chart that the neighborhood inherits the standard coordinates on the Euclidean plane. These coordinates are known as *local coordinates* and these homeomorphisms lead us to describe surfaces as being *locally Euclidean*.

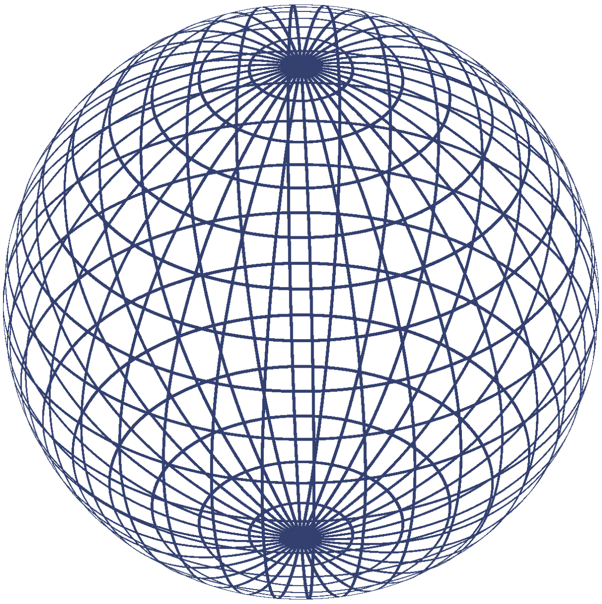
More generally, a *(topological) surface with boundary* is a Hausdorff topological space in which every point has an open neighbourhood homeomorphic to some open subset of the upper half-plane $\mathbf{H}^2$. These homeomorphisms are also known as *(coordinate) charts*. The boundary of the upper half-plane is the x-axis. A point on the surface mapped via a chart to the x-axis is termed a *boundary point*. The collection of such points is known as the *boundary* of the surface which is necessarily a one-manifold, that is, the union of closed curves. On the other hand, a point mapped to above the x-axis is an *interior point*. The collection of interior points is the *interior* of the surface which is always non-empty. The closed disk is a simple example of a surface with boundary. The boundary of the disc is a circle.

The term *surface* used without qualification refers to surfaces without boundary. In particular, a surface with empty boundary is a surface in the usual sense. A surface with empty boundary which is compact is known as a 'closed' surface. The two-dimensional sphere, the two-dimensional torus, and the real projective plane are examples of closed surfaces.

The Möbius strip is a surface with only one "side". In general, a surface is said to be *orientable* if it does not contain a homeomorphic copy of the Möbius strip; intuitively, it has two distinct "sides". For example, the sphere and torus are orientable, while the real projective plane is not (because deleting a point or disk from the real projective plane produces the Möbius strip).

In differential and algebraic geometry, extra structure is added upon the topology of the surface. This added structures detects singularities, such as self-intersections and cusps, that cannot be described solely in terms of the underlying topology.

Extrinsically defined surfaces and embeddings



A sphere can be defined parametrically (by $x = r \sin \theta \cos \varphi$, $y = r \sin \theta \sin \varphi$, $z = r \cos \theta$) or implicitly (by $x^2 + y^2 + z^2 - r^2 = 0$.)

Historically, surfaces were initially defined as subspaces of Euclidean spaces. Often, these surfaces were the locus of zeros of certain functions, usually polynomial functions. Such a definition considered the surface as part of a larger (Euclidean) space, and as such was termed *extrinsic*.

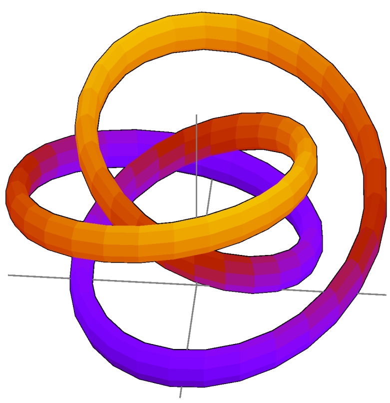
In the previous section, a surface is defined as a topological space with certain property, namely Hausdorff and locally Euclidean. This topological space is not considered as being a subspace of another space. In this sense, the definition given above, which is the definition that mathematicians use at present, is *intrinsic*.

A surface defined as intrinsic is not required to satisfy the added constraint of being a subspace of Euclidean space. It seems possible at first glance that there are surfaces defined intrinsically that are not surfaces in the extrinsic sense. However, the Whitney embedding theorem asserts that every surface can in fact be embedded homeomorphically into Euclidean space, in fact into $\mathbf{E}^4$. Therefore the extrinsic and intrinsic approaches turn out to be equivalent.

In fact, any compact surface that is either orientable or has a boundary can be embedded in $\mathbf{E}^3$; on the other hand, the real projective plane, which is compact, non-orientable and without boundary, cannot be embedded into $\mathbf{E}^3$ (see Gramain). Steiner surfaces, including Boy's surface, the Roman surface and the cross-cap, are immersions of the real projective plane into $\mathbf{E}^3$. These surfaces are singular where the immersions intersect themselves.

The Alexander horned sphere is a well-known pathological embedding of the two-sphere into the three-sphere.

The chosen embedding (if any) of a surface into another space is regarded as extrinsic information; it is not essential to the surface itself. For example, a torus can be embedded into $\mathbf{E}^3$ in the "standard" manner (that looks like a bagel) or in a knotted manner (see figure). The two embedded tori are homeomorphic but not isotopic; they are topologically equivalent, but their embeddings are not.



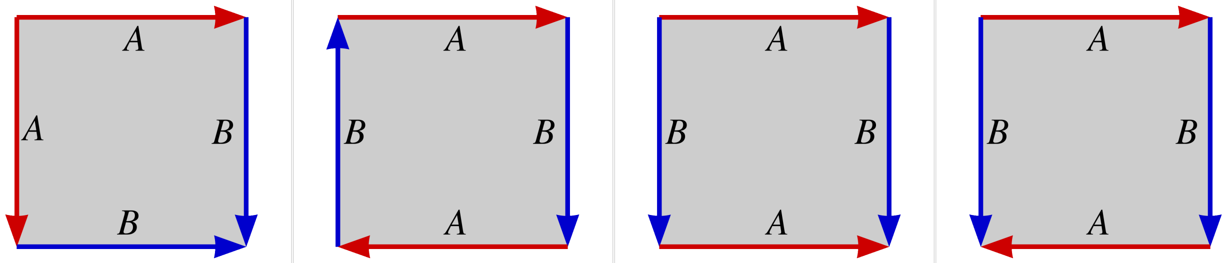
A knotted torus.

The image of a continuous, injective function from $\mathbf{R}^2$ to higher-dimensional $\mathbf{R}^n$ is said to be a parametric surface. Such an image is so-called because the *x*- and *y*- directions of the domain $\mathbf{R}^2$ are 2 variables that parametrize the image. Be careful that a parametric surface need not be a topological surface. A surface of revolution can be viewed as a special kind of parametric surface.

If f is a smooth function from $\mathbf{R}^3$ to $\mathbf{R}$ whose gradient is nowhere zero,
Then the locus of zeros of f does define a surface, known as an *implicit surface*. If the condition of non-vanishing gradient is dropped then the zero locus may develop singularities.

Construction from polygons

Each closed surface can be constructed from an oriented polygon with an even number of sides, called a fundamental polygon of the surface, by pairwise identification of its edges. For example, in each polygon below, attaching the sides with matching labels (A with A, B with B), so that the arrows point in the same direction, yields the indicated surface.



sphere | real projective plane | torus | Klein bottle

Any fundamental polygon can be written symbolically as follows. Begin at any vertex, and proceed around the perimeter of the polygon in either direction until returning to the starting vertex. During this traversal, record the label on each edge in order, with an exponent of -1 if the edge points opposite to the direction of traversal. The four models above, when traversed clockwise starting at the upper left, yield

- sphere: $ABB^{-1}A^{-1}$
- real projective plane: $ABAB$
- torus: $ABA^{-1}B^{-1}$
- Klein bottle: $ABAB^{-1}$.

Note that the sphere and the projective plane can both be realized as quotients of the 2-gon, while the torus and Klein bottle require a 4-gon (square).

The expression thus derived from a fundamental polygon of a surface turns out to be the sole relation in a presentation of the fundamental group of the surface with the polygon edge labels as generators. This is a consequence of the Seifert–van Kampen theorem.

Gluing edges of polygons is a special kind of quotient space process. The quotient concept can be applied in greater generality to produce new or alternative constructions of surfaces. For example, the real projective plane can be obtained as the quotient of the sphere by identifying all pairs of opposite points on the sphere. Another example of a quotient is the connected sum.

Connected sums

The connected sum of two surfaces M and N, denoted $M \# N$, is obtained by removing a disk from each of them and gluing them along the boundary components that result. The boundary of a disk is a circle, so these boundary components are circles. The Euler characteristic χ of $M \# N$ is the sum of the Euler characteristics of the summands, minus two:

$$\chi(M\#N) = \chi(M) + \chi(N) - 2.$$

The sphere $\mathbf{S}$ is an identity element for the connected sum, meaning that $\mathbf{S} \# M = M$. This is because deleting a disk from the sphere leaves a disk, which simply replaces the disk deleted from M upon gluing.

Connected summation with the torus $\mathbf{T}$ is also described as attaching a "handle" to the other summand M. If M is orientable, then so is $\mathbf{T} \# M$. The connected sum is associative, so the connected sum of a finite collection of surfaces is well-defined.

The connected sum of two real projective planes, **P** # **P**, is the Klein bottle **K**. The connected sum of the real projective plane and the Klein bottle is homeomorphic to the connected sum of the real projective plane with the torus; in a formula, **P** # **K** = **P** # **T**. Thus, the connected sum of three real projective planes is homeomorphic to the connected sum of the real projective plane with the torus. Any connected sum involving a real projective plane is nonorientable.

Closed surfaces

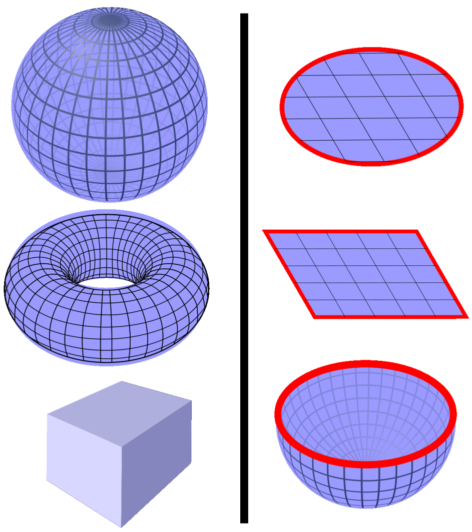
A **closed surface** is a surface that is compact and without boundary. Examples are spaces like the sphere, the torus and the Klein bottle. Examples of non-closed surfaces are: an open disk, which is a sphere with a puncture; a cylinder, which is a sphere with two punctures; and the Möbius strip.

Classification of closed surfaces

The *classification theorem of closed surfaces* states that any connected closed surface is homeomorphic to some member of one of these three families:

1. the sphere;
2. the connected sum of g tori, for $g \geq 1$;
3. the connected sum of k real projective planes, for $k \geq 1$.

The surfaces in the first two families are orientable. It is convenient to combine the two families by regarding the sphere as the connected sum of 0 tori. The number g of tori involved is called the *genus* of the surface. The sphere and the torus have Euler characteristics 2 and 0, respectively, and in general the Euler characteristic of the connected sum of g tori is $2 - 2g$.



Some examples of orientable closed surfaces (left) and surfaces with boundary (right). Left: Some orientable closed surfaces are the surface of a sphere, the surface of a torus, and the surface of a cube. (The cube and the sphere are topologically equivalent to each other.) Right: Some surfaces with boundary are the disk surface, square surface, and hemisphere surface. The boundaries are shown in red. All three of these are topologically equivalent to each other.

The surfaces in the third family are nonorientable. The Euler characteristic of the real projective plane is 1, and in general the Euler characteristic of the connected sum of k of them is $2 - k$.

It follows that a closed surface is determined, up to homeomorphism, by two pieces of information: its Euler characteristic, and whether it is orientable or not. In other words, Euler characteristic and orientability completely classify closed surfaces up to homeomorphism.

For closed surfaces with multiple connected components, they are classified by the class of each of their connected components, and thus one generally assumes that the surface is connected.

Monoid structure

Relating this classification to connected sums, the closed surfaces up to homeomorphism form a monoid with respect to the connected sum, as indeed do manifolds of any fixed dimension. The identity is the sphere, while the real projective plane and the torus generate this monoid, with a single relation **P** # **P** # **P** = **P** # **T**, which may also be written **P** # **K** = **P** # **T**, since **K** = **P** # **P**. This relation is sometimes known as **Dyck's theorem** after Walther von Dyck, who proved it in (Dyck 1888), and the triple cross surface **P** # **P** # **P** is accordingly called **Dyck's surface**.[1]

Geometrically, connect-sum with a torus (# **T**) adds a handle with both ends attached to the same side of the surface, while connect-sum with a Klein bottle (# **K**) adds a handle with the two ends attached to opposite sides of the surface; in the presence of a projective plane (# **P**), the surface is not orientable (there is no notion of side), so there is no difference between attaching a torus and attaching a Klein bottle, which explains the relation.

Surfaces with boundary

Compact surfaces, possibly with boundary, are simply closed surfaces with a number of holes (open discs that have been removed). Thus, a connected compact surface is classified by the number of boundary components and the genus of the corresponding closed surface – equivalently, by the number of boundary components, the orientability, and Euler characteristic. The genus of a compact surface is defined as the genus of the corresponding closed surface.

This classification follows almost immediately from the classification of closed surfaces: removing an open disc from a closed surface yields a compact surface with a circle for boundary component, and removing *k* open discs yields a compact surface with *k* disjoint circles for boundary components. The precise locations of the holes are irrelevant, because the homeomorphism group acts *k*-transitively on any connected manifold of dimension at least 2.

Conversely, the boundary of a compact surface is a closed 1-manifold, and is therefore the disjoint union of a finite number of circles; filling these circles with disks (formally, taking the cone) yields a closed surface.

The unique compact orientable surface of genus *g* and with *k* boundary components is often denoted $\Sigma_{g,k},$ for example in the study of the mapping class group.

Riemann surfaces

A closely related example to the classification of compact 2-manifolds is the classification of compact Riemann surfaces, i.e., compact complex 1-manifolds. (Note that the 2-sphere and the torus are both complex manifolds, in fact algebraic varieties.) Since every complex manifold is orientable, the connected sums of projective planes are not complex manifolds. Thus, compact Riemann surfaces are characterized topologically simply by their genus. The genus counts the number of holes in the manifold: the sphere has genus 0, the one-holed torus genus 1, etc.

Non-compact surfaces

Non-compact surfaces are more difficult to classify. As a simple example, a non-compact surface can be obtained by puncturing (removing a finite set of points from) a closed manifold. On the other hand, any open subset of a compact surface is itself a non-compact surface; consider, for example, the complement of a Cantor set in the sphere, otherwise known as the Cantor tree surface. However, not every non-compact surface is a subset of a compact surface; two canonical counterexamples are the Jacob's ladder and the Loch Ness monster, which are non-compact surfaces with infinite genus.

Proof

The classification of closed surfaces has been known since the 1860s,[1] and today a number of proofs exist.

Topological and combinatorial proofs in general rely on the difficult result that every compact 2-manifold is homeomorphic to a simplicial complex, which is of interest in its own right. The most common proof of the classification is (Seifert & Threlfall 1934),[1] which brings every triangulated surface to a standard form. A simplified proof, which avoids a standard form, was discovered by John H. Conway circa 1992, which he called the "Zero Irrelevancy Proof" or "ZIP proof" and is presented in (Francis & Weeks 1999).

A geometric proof, which yields a stronger geometric result, is the uniformization theorem. This was originally proven only for Riemann surfaces in the 1880s and 1900s by Felix Klein, Paul Koebe, and Henri Poincaré.

Surfaces in geometry

Polyhedra, such as the boundary of a cube, are among the first surfaces encountered in geometry. It is also possible to define *smooth surfaces*, in which each point has a neighborhood diffeomorphic to some open set in $\mathbf{E}^2$. This elaboration allows calculus to be applied to surfaces to prove many results.

Two smooth surfaces are diffeomorphic if and only if they are homeomorphic. (The analogous result does not hold for higher-dimensional manifolds.) Thus closed surfaces are classified up to diffeomorphism by their Euler characteristic and orientability.

Smooth surfaces equipped with Riemannian metrics are of fundational importance in differential geometry. A Riemannian metric endows a surface with notions of geodesic, distance, angle, and area. It also gives rise to Gaussian curvature, which describes how curved or bent the surface is at each point. Curvature is a rigid, geometric property, in that it is not preserved by general diffeomorphisms of the surface. However, the famous Gauss-Bonnet theorem for closed surfaces states that the integral of the Gaussian curvature K over the entire surface S is determined by the Euler characteristic:

$$\int_S K\;dA = 2\pi\chi(S).$$

This result exemplifies the deep relationship between the geometry and topology of surfaces (and, to a lesser extent, higher-dimensional manifolds).

Another way in which surfaces arise in geometry is by passing into the complex domain. A complex one-manifold is a smooth oriented surface, also called a Riemann surface. Any complex nonsingular algebraic curve viewed as a complex manifold is a Riemann surface.

Every closed orientable surface admits a complex structure. Complex structures on a closed oriented surface correspond to conformal equivalence classes of Riemannian metrics on the surface. One version of the uniformization theorem (due to Poincaré) states that any Riemannian metric on an oriented, closed surface is conformally equivalent to an essentially unique metric of constant curvature. This provides a starting point for one of the approaches to Teichmüller theory, which provides a finer classification of Riemann surfaces than the topological one by Euler characteristic alone.

A *complex surface* is a complex two-manifold and thus a real four-manifold; it is not a surface in the sense of this article. Neither are algebraic curves defined over fields other than the complex numbers, nor are algebraic surfaces defined over fields other than the real numbers.

Notes

[1] (Francis & Weeks 1999)

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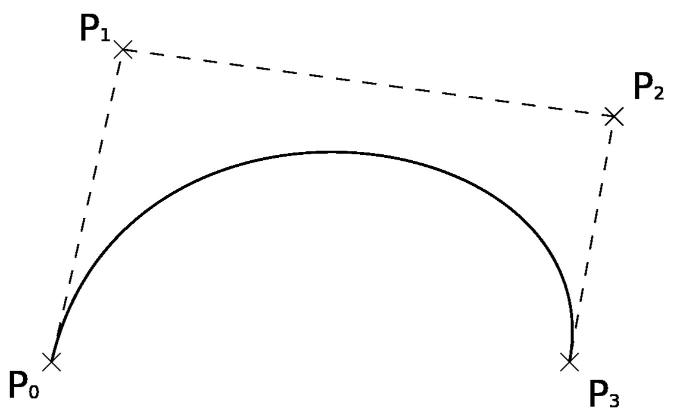
- Francis, George K.; Weeks, Jeffrey R. (May 1999), "Conway's ZIP Proof" (http://new.math.uiuc.edu/zipproof/zipproof.pdf), *American Mathematical Monthly* **106** (5), page discussing the paper: On Conway's ZIP Proof (http://new.math.uiuc.edu/zipproof/)

External links

- The Classification of Surfaces and the Jordan Curve Theorem (http://www.maths.ed.ac.uk/~aar/jordan/) in Home page of Andrew Ranicki
- Math Surfaces Gallery, with 60 ~surfaces and Java Applet for live rotation viewing (http://xahlee.org/surface/gallery.html)
- Math Surfaces Animation, with JavaScript (Canvas HTML) for tens surfaces rotation viewing (http://wokos.nethium.pl/surfaces_en.net)
- The Classification of Surfaces (http://www.math.ohio-state.edu/~fiedorow/math655/classification.html) Lecture Notes by Z.Fiedorowicz
- History and Art of Surfaces and their Mathematical Models (http://maxwelldemon.com/2009/03/21/surfaces-1-the-ooze-of-the-past/)

Bézier curve

A **Bézier curve** is a parametric curve frequently used in computer graphics and related fields. Generalizations of Bézier curves to higher dimensions are called Bézier surfaces, of which the Bézier triangle is a special case.



Cubic Bézier curve

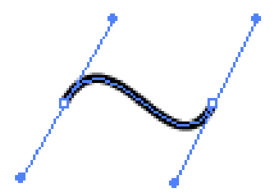
In vector graphics, Bézier curves are used to model smooth curves that can be scaled indefinitely. "Paths," as they are commonly referred to in image manipulation programs,[1] are combinations of linked Bézier curves. Paths are not bound by the limits of rasterized images and are intuitive to modify. Bézier curves are also used in animation as a tool to control motion.[2]

Bézier curves are also used in the time domain, particularly in animation and interface design, e.g., a Bézier curve can be used to specify the velocity over time of an object such as an icon moving from A to B, rather than simply moving at a fixed number of pixels per step. When animators or interface designers talk about the "physics" or "feel" of an operation, they may be referring to the particular Bézier curve used to control the velocity over time of the move in question.

Bézier curves were widely publicized in 1962 by the French engineer Pierre Bézier, who used them to design automobile bodies. But the study of these curves was first developed in 1959 by mathematician Paul de Casteljau using de Casteljau's algorithm, a numerically stable method to evaluate Bézier curves.

Applications

Computer graphics



Bézier path in Adobe Illustrator

Bézier curves are widely used in computer graphics to model smooth curves. As the curve is completely contained in the convex hull of its control points, the points can be graphically displayed and used to manipulate the curve intuitively. Affine transformations such as translation, and rotation can be applied on the curve by applying the respective transform on the control points of the curve.

Quadratic and cubic Bézier curves are most common; higher degree curves are more computationally expensive to evaluate. When more complex shapes are needed, low order Bézier curves are patched together. This is commonly referred to as a "path" in vector graphics standards (like SVG) and vector graphics programs (like Adobe Illustrator and Inkscape). To guarantee smoothness, the control point at which two curves meet must be on the line between the two control points on either side.

The simplest method for scan converting (rasterizing) a Bézier curve is to evaluate it at many closely spaced points and scan convert the approximating sequence of line segments. However, this does not guarantee that the rasterized output looks sufficiently smooth, because the points may be spaced too far apart. Conversely it may generate too many points in areas where the curve is close to linear. A common adaptive method is recursive subdivision, in which a curve's control points are checked to see if the curve approximates a line segment to within a small tolerance. If not, the curve is subdivided parametrically into two segments, $0 \le t \le 0.5$ and $0.5 \le t \le 1$, and the same procedure is applied recursively to each half. There are also forward differencing methods, but great care must be taken to analyse error propagation. Analytical methods where a spline is intersected with each scan line involve finding roots of cubic polynomials (for cubic splines) and dealing with multiple roots, so they are not often used in practice.

Animation

In animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline, for example, movement. Users outline the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For 3D animation Bézier curves are often used to define 3D paths as well as 2D curves for keyframe interpolation.

Fonts

TrueType fonts use Bézier splines composed of **quadratic** Bézier curves. Modern imaging systems like PostScript, Asymptote, Metafont, and SVG use Bézier splines composed of **cubic** Bézier curves for drawing curved shapes. OpenType fonts can use either types, depending on the flavor of the font.

The internal rendering of all Bézier curves in font or vector graphics renderers will split them recursively up to the point where the curve is flat enough to be drawn as a series of linear or circular segments. The exact splitting algorithm is implementation dependent, only the flatness criteria must be respected to reach the necessary precision and to avoid non-monotonic local changes of curvature. The "smooth curve" feature of charts in Microsoft Excel also uses this algorithm.[3]

Because arcs of circles and ellipses cannot be exactly represented by Bézier curves, they are first approximated by Bézier curves, which are in turn approximated by arcs of circles. This is inefficient as there exists also approximations of all Bézier curves using arcs of circles or ellipses, which can be rendered incrementally with arbitrary precision. Another approach, used by modern hardware graphics adapters with accelerated geometry, can convert exactly all Bézier and conic curves (or surfaces) into NURBS, that can be rendered incrementally without first splitting the curve recursively to reach the necessary flatness condition. This approach also allows preserving the curve definition under all linear or perspective 2D and 3D transforms and projections.

Font engines, like FreeType, draw the font's curves (and lines) on a pixellated surface, in a process called Font rasterization.[4]

Examination of cases

A Bézier curve is defined by a set of *control points* $\mathbf{P}_0$ through $\mathbf{P}_n$, where n is called its order (n = 1 for linear, 2 for quadratic, etc.). The first and last control points are always the end points of the curve; however, the intermediate control points (if any) generally do not lie on the curve.

Linear Bézier curves

Given points $\mathbf{P}_0$ and $\mathbf{P}_1$, a linear Bézier curve is simply a straight line between those two points. The curve is given by

$$\mathbf{B}(t) = \mathbf{P}_0 + t(\mathbf{P}_1 - \mathbf{P}_0) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1 \mbox{ , } t \in [0,1]$$

and is equivalent to linear interpolation.

Quadratic Bézier curves

A quadratic Bézier curve is the path traced by the function **B**(t), given points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$,

$$\mathbf{B}(t) = (1-t)[(1-t)\mathbf{P}_0 + t\mathbf{P}_1] + t[(1-t)\mathbf{P}_1 + t\mathbf{P}_2] \mbox{ , } t \in [0,1],$$

which can be interpreted as the linear interpolant of corresponding points on the linear Bézier curves from $\mathbf{P}_0$ to $\mathbf{P}_1$ and from $\mathbf{P}_1$ to $\mathbf{P}_2$ respectively. More explicitly it can be written as:

$$\mathbf{B}(t) = (1-t)^2\mathbf{P}_0 + 2(1-t)t\mathbf{P}_1 + t^2\mathbf{P}_2 \mbox{ , } t \in [0,1].$$

It departs from $\mathbf{P}_0$ in the direction of $\mathbf{P}_1$, then bends to arrive at $\mathbf{P}_2$ in the direction from $\mathbf{P}_1$. In other words, the tangents in $\mathbf{P}_0$ and $\mathbf{P}_2$ both pass through $\mathbf{P}_1$. This is directly seen from the derivative of the Bézier curve:

$$\mathbf{B}'(t) = 2(1-t)(\mathbf{P}_1 - \mathbf{P}_0) + 2t(\mathbf{P}_2 - \mathbf{P}_1) \mbox{ .}$$

A quadratic Bézier curve is also a parabolic segment. As a parabola is a conic section, some sources refer to quadratic Béziers as "conic arcs".[4]

Cubic Bézier curves

Four points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ in the plane or in higher-dimensional space define a cubic Bézier curve. The curve starts at $\mathbf{P}_0$ going toward $\mathbf{P}_1$ and arrives at $\mathbf{P}_3$ coming from the direction of $\mathbf{P}_2$. Usually, it will not pass through $\mathbf{P}_1$ or $\mathbf{P}_2$; these points are only there to provide directional information. The distance between $\mathbf{P}_0$ and $\mathbf{P}_1$ determines "how long" the curve moves into direction $\mathbf{P}_2$ before turning towards $\mathbf{P}_3$.

Writing $\mathbf{B}_{\mathbf{P}}$i,$\mathbf{P}_j$,$\mathbf{P}_k$(t) for the quadratic Bézier curve defined by points $\mathbf{P}_i$, $\mathbf{P}_j$, and $\mathbf{P}_k$, the cubic Bézier curve can be defined as a linear combination of two quadratic Bézier curves:

$$\mathbf{B}(t)=(1-t)\mathbf{B}_{\mathbf P_0,\mathbf P_1,\mathbf P_2}(t)+t\mathbf{B}_{\mathbf P_1,\mathbf P_2,\mathbf P_3}(t) \mbox{ , } t \in [0,1].$$

The explicit form of the curve is:

$$\mathbf{B}(t)=(1-t)^3\mathbf{P}_0+3(1-t)^2t\mathbf{P}_1+3(1-t)t^2\mathbf{P}_2+t^3\mathbf{P}_3 \mbox{ , } t \in [0,1].$$

For some choices of $\mathbf{P}_1$ and $\mathbf{P}_2$ the curve may intersect itself, or contain a cusp.

Generalization

Bézier curves can be defined for any degree *n*.

Recursive definition

A recursive definition for the Bézier curve of degree *n* expresses it as a point-to-point linear combination of a pair of corresponding points in two Bézier curves of degree *n* – 1.

Let $\mathbf{B}_{\mathbf{P}_0\mathbf{P}_1\ldots\mathbf{P}_n}$ denote the Bézier curve determined by the points $\mathbf{P}_0$, $\mathbf{P}_1$, ..., $\mathbf{P}_n$. Then

$$\mathbf{B}_{\mathbf{P}_0}(t) = \mathbf{P}_0 \text{ to start, and}$$

$$\mathbf{B}(t) = \mathbf{B}_{\mathbf{P}_0\mathbf{P}_1\ldots\mathbf{P}_n}(t) = (1-t)\mathbf{B}_{\mathbf{P}_0\mathbf{P}_1\ldots\mathbf{P}_{n-1}}(t) + t\mathbf{B}_{\mathbf{P}_1\mathbf{P}_2\ldots\mathbf{P}_n}(t)$$

This recursion is elucidated in the animations below.

Explicit definition

The formula can be expressed explicitly as follows:

$$\begin{aligned}\mathbf{B}(t) &= \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i \mathbf{P}_i \\ &= (1-t)^n \mathbf{P}_0 + \binom{n}{1}(1-t)^{n-1} t \mathbf{P}_1 + \cdots \\ &\cdots + \binom{n}{n-1}(1-t)t^{n-1}\mathbf{P}_{n-1} + t^n \mathbf{P}_n, \quad t \in [0,1],\end{aligned}$$

where $\binom{n}{i}$ are the binomial coefficients.

For example, for $n = 5$:

$$\begin{aligned}\mathbf{B}_{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3\mathbf{P}_4\mathbf{P}_5}(t) = \mathbf{B}(t) &= (1-t)^5\mathbf{P}_0 + 5t(1-t)^4\mathbf{P}_1 + 10t^2(1-t)^3\mathbf{P}_2 \\ &+ 10t^3(1-t)^2\mathbf{P}_3 + 5t^4(1-t)\mathbf{P}_4 + t^5\mathbf{P}_5, \quad t \in [0,1].\end{aligned}$$

Terminology

Some terminology is associated with these parametric curves. We have

$$\mathbf{B}(t) = \sum_{i=0}^{n} \mathbf{b}_{i,n}(t)\mathbf{P}_i, \quad t \in [0,1]$$

where the polynomials

$$\mathbf{b}_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, \ldots n$$

are known as Bernstein basis polynomials of degree *n*.

Note that $t^0 = 1$, $(1 - t)^0 = 1$, and that the binomial coefficient, $\binom{n}{i}$, also expressed as ${}^n\mathbf{C}_i$ or $\mathbf{C}_i^n$ is:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

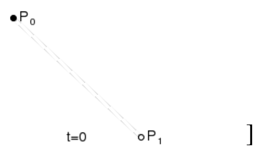
The points $\mathbf{P}_i$ are called *control points* for the Bézier curve. The polygon formed by connecting the Bézier points with lines, starting with $\mathbf{P}_0$ and finishing with $\mathbf{P}_n$, is called the *Bézier polygon* (or *control polygon*). The convex hull of the Bézier polygon contains the Bézier curve.

Properties

- The curve begins at $\mathbf{P}_0$ and ends at $\mathbf{P}_n$; this is the so-called *endpoint interpolation* property.
- The curve is a straight line if and only if all the control points are collinear.
- The start (end) of the curve is tangent to the first (last) section of the Bézier polygon.
- A curve can be split at any point into two subcurves, or into arbitrarily many subcurves, each of which is also a Bézier curve.
- Some curves that seem simple, such as the circle, cannot be described exactly by a Bézier or piecewise Bézier curve; though a four-piece cubic Bézier curve can approximate a circle (see Bézier spline), with a maximum radial error of less than one part in a thousand, when each inner control point (or offline point) is the distance $\frac{4\left(\sqrt{2}-1\right)}{3}$ horizontally or vertically from an outer control point on a unit circle. More generally, an n-piece cubic Bézier curve can approximate a circle, when each inner control point is the distance $\frac{4}{3}\tan(t/4)$ from an outer control point on a unit circle, where t is 360/n degrees, and $n > 2$.
- The curve at a fixed offset from a given Bézier curve, often called an *offset curve* (lying "parallel" to the original curve, like the offset between rails in a railroad track), cannot be exactly formed by a Bézier curve (except in some trivial cases). However, there are heuristic methods that usually give an adequate approximation for practical purposes.
- Every quadratic Bézier curve is also a cubic Bézier curve, and more generally, every degree n Bézier curve is also a degree m curve for any $m > n$. In detail, a degree n curve with control points $\mathbf{P}_0, \ldots, \mathbf{P}_n$ is equivalent (including the parametrization) to the degree $n + 1$ curve with control points $\mathbf{P}'_0, \ldots, \mathbf{P}'_{n+1}$, where $\mathbf{P}'_k = \tfrac{k}{n+1}\mathbf{P}_{k-1} + \left(1 - \tfrac{k}{n+1}\right)\mathbf{P}_k$.

Constructing Bézier curves

Linear curves



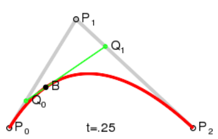
Animation of a linear Bézier curve, t in [0,1]

The t in the function for a linear Bézier curve can be thought of as describing how far $\mathbf{B}(t)$ is from $\mathbf{P}_0$ to $\mathbf{P}_1$. For example when t=*0.25*, $\mathbf{B}(t)$ is one quarter of the way from point $\mathbf{P}_0$ to $\mathbf{P}_1$. As t varies from 0 to 1, $\mathbf{B}(t)$ describes a straight line from $\mathbf{P}_0$ to $\mathbf{P}_1$.

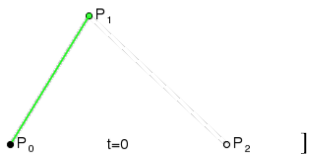
Quadratic curves

For quadratic Bézier curves one can construct intermediate points $\mathbf{Q}_0$ and $\mathbf{Q}_1$ such that as t varies from 0 to 1:

- Point $\mathbf{Q}_0$ varies from $\mathbf{P}_0$ to $\mathbf{P}_1$ and describes a linear Bézier curve.
- Point $\mathbf{Q}_1$ varies from $\mathbf{P}_1$ to $\mathbf{P}_2$ and describes a linear Bézier curve.
- Point $\mathbf{B}(t)$ varies from $\mathbf{Q}_0$ to $\mathbf{Q}_1$ and describes a quadratic Bézier curve.



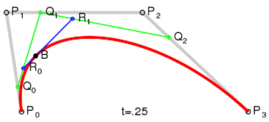
Construction of a quadratic Bézier curve



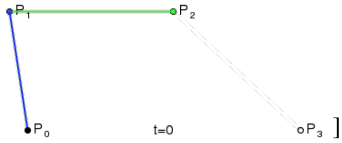
Animation of a quadratic Bézier curve, t in [0,1]

Higher-order curves

For higher-order curves one needs correspondingly more intermediate points. For cubic curves one can construct intermediate points $\mathbf{Q}_0$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ that describe linear Bézier curves, and points $\mathbf{R}_0$ & $\mathbf{R}_1$ that describe quadratic Bézier curves:

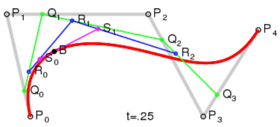


Construction of a cubic Bézier curve

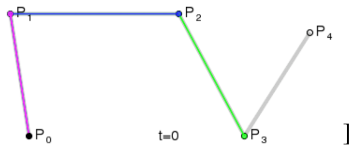


Animation of a cubic Bézier curve, *t* in [0,1]

For fourth-order curves one can construct intermediate points $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$ & $\mathbf{Q}_3$ that describe linear Bézier curves, points $\mathbf{R}_0$, $\mathbf{R}_1$ & $\mathbf{R}_2$ that describe quadratic Bézier curves, and points $\mathbf{S}_0$ & $\mathbf{S}_1$ that describe cubic Bézier curves:

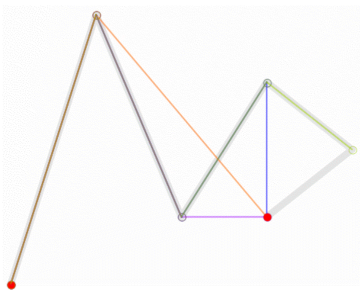


Construction of a quartic Bézier curve



Animation of a quartic Bézier curve, *t* in [0,1]

For fifth-order curves, one can construct similar intermediate points.



Animation of a fifth order Bézier curve, *t* in [0,1]

Degree elevation

A Bézier curve of degree *n* can be converted into a Bézier curve of degree *n* + 1 **with the same shape**. This is useful if software supports Bézier curves only of specific degree. For example, you can draw a quadratic Bézier curve with Cairo, which supports only cubic Bézier curves.

To do degree elevation, we use the equality $\mathbf{B}(t) = (1-t)\mathbf{B}(t) + t\mathbf{B}(t)$. Each component $\mathbf{b}_{i,n}(t)\mathbf{P}_i$ is multiplied by (1 – *t*) or *t*, thus increasing a degree by one. Here is the example of increasing degree from 2 to 3.

$$\begin{aligned}
&(1-t)^2\mathbf{P}_0 + 2(1-t)t\mathbf{P}_1 + t^2\mathbf{P}_2 \\
&= (1-t)^3\mathbf{P}_0 + (1-t)^2t\mathbf{P}_0 + 2(1-t)^2t\mathbf{P}_1 \\
&\quad + 2(1-t)t^2\mathbf{P}_1 + (1-t)t^2\mathbf{P}_2 + t^3\mathbf{P}_2 \\
&= (1-t)^3\mathbf{P}_0 + 3(1-t)^2t\frac{\mathbf{P}_0 + 2\mathbf{P}_1}{3} + 3(1-t)t^2\frac{2\mathbf{P}_1 + \mathbf{P}_2}{3} + t^3\mathbf{P}_2
\end{aligned}$$

For arbitrary *n* we use equalities

$$\binom{n+1}{i}(1-t)\mathbf{b}_{i,n} = \binom{n}{i}\mathbf{b}_{i,n+1}, \quad (1-t)\mathbf{b}_{i,n} = \frac{n+1-i}{n+1}\mathbf{b}_{i,n+1}$$

$$\binom{n+1}{i+1} t\mathbf{b}_{i,n} = \binom{n}{i} \mathbf{b}_{i+1,n+1}, \quad t\mathbf{b}_{i,n} = \frac{i+1}{n+1}\mathbf{b}_{i+1,n+1}$$

$$\begin{aligned}\mathbf{B}(t) &= (1-t)\sum_{i=0}^{n} \mathbf{b}_{i,n}(t)\mathbf{P}_i + t\sum_{i=0}^{n} \mathbf{b}_{i,n}(t)\mathbf{P}_i \\ &= \sum_{i=0}^{n} \frac{n+1-i}{n+1}\mathbf{b}_{i,n+1}(t)\mathbf{P}_i + \sum_{i=0}^{n} \frac{i+1}{n+1}\mathbf{b}_{i+1,n+1}(t)\mathbf{P}_i \\ &= \sum_{i=0}^{n+1} \left(\frac{i}{n+1}\mathbf{P}_{i-1} + \frac{n+1-i}{n+1}\mathbf{P}_i\right) \mathbf{b}_{i,n+1}(t) = \sum_{i=0}^{n+1} \mathbf{b}_{i,n+1}(t)\mathbf{P'}_i\end{aligned}$$

introducing arbitrary $\mathbf{P}_{-1}$ and $\mathbf{P}_{n+1}$.

Therefore new control points are [5]

$$\mathbf{P'}_i = \frac{i}{n+1}\mathbf{P}_{i-1} + \frac{n+1-i}{n+1}\mathbf{P}_i, \quad i = 0, \ldots, n+1.$$

Polynomial form

Sometimes it is desirable to express the Bézier curve as a polynomial instead of a sum of less straightforward Bernstein polynomials. Application of the binomial theorem to the definition of the curve followed by some rearrangement will yield:

$$\mathbf{B}(t) = \sum_{j=0}^{n} t^j \mathbf{C}_j$$

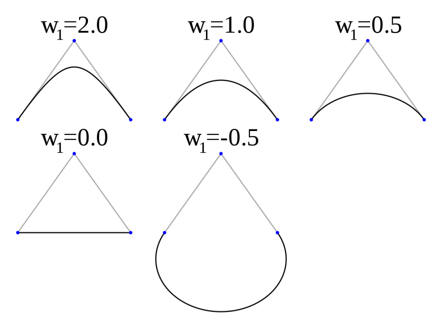
where

$$\mathbf{C}_j = \frac{n!}{(n-j)!} \sum_{i=0}^{j} \frac{(-1)^{i+j}\mathbf{P}_i}{i!(j-i)!} = \prod_{m=0}^{j-1} (n-m) \sum_{i=0}^{j} \frac{(-1)^{i+j}\mathbf{P}_i}{i!(j-i)!}.$$

This could be practical if $\mathbf{C}_j$ can be computed prior to many evaluations of $\mathbf{B}(t)$; however one should use caution as high order curves may lack numeric stability (de Casteljau's algorithm should be used if this occurs). Note that the empty product is 1.

Rational Bézier curves

The rational Bézier curve adds adjustable weights to provide closer approximations to arbitrary shapes. The numerator is a weighted Bernstein-form Bézier curve and the denominator is a weighted sum of Bernstein polynomials. Rational Bézier curves can, among other uses, be used to represent segments of conic sections exactly.[6]



Sections of conic sections represented exactly by rational Bézier curves

Given $n + 1$ control points $\mathbf{P}_i$, the rational Bézier curve can be described by:

$$\mathbf{B}(t) = \frac{\sum_{i=0}^{n} b_{i,n}(t)\mathbf{P}_i w_i}{\sum_{i=0}^{n} b_{i,n}(t) w_i}$$

or simply

$$\mathbf{B}(t) = \frac{\sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i}\mathbf{P}_i w_i}{\sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} w_i}.$$

Notes

[1] Image manipulation programs such as Inkscape, Adobe Photoshop, and GIMP.

[2] In animation applications such as Adobe Flash, Adobe After Effects, Microsoft Expression Blend, Blender, Autodesk Maya and Autodesk 3ds max.

[3] http://www.xlrotor.com/resources/files.shtml

[4] FreeType Glyph Conventions (http://www.freetype.org/freetype2/docs/glyphs/glyphs-6.html), David Turner + Freetype Development Team, Freetype.org, retr May 2011

[5] Farin, Gerald (1997), *Curves and surfaces for computer-aided geometric design* (4 ed.), Elsevier Science & Technology Books, ISBN 978-0-12-249054-5

[6] Neil Dodgson (2000-09-25). "Some Mathematical Elements of Graphics: Rational B-splines" (http://www.cl.cam.ac.uk/teaching/2000/AGraphHCI/SMEG/node5.html). . Retrieved 2009-02-23.

References

- Paul Bourke: *Bézier Surfaces (in 3D)*, http://local.wasp.uwa.edu.au/~pbourke/geometry/bezier/index.html
- Donald Knuth: *Metafont: the Program*, Addison-Wesley 1986, pp. 123–131. Excellent discussion of implementation details; available for free as part of the TeX distribution.
- Dr Thomas Sederberg, BYU *Bézier curves*, http://www.tsplines.com/resources/class_notes/Bezier_curves.pdf
- J.D. Foley *et al.*: *Computer Graphics: Principles and Practice in C* (2nd ed., Addison Wesley, 1992)

External links

- Weisstein, Eric W., " Bézier Curve (http://mathworld.wolfram.com/BezierCurve.html)" from MathWorld.
- Finding All Intersections of Two Bézier Curves. (http://www.truetex.com/bezint.htm) – Locating all the intersections between two Bézier curves is a difficult general problem, because of the variety of degenerate cases. By Richard J. Kinch.
- From Bézier to Bernstein (http://www.ams.org/featurecolumn/archive/bezier.html#2) Feature Column from American Mathematical Society
- A Primer on Bézier Curves (http://processingjs.nihongoresources.com/bezierinfo) A detailed explanation of implementing Bezier curves and associated graphics algorithms, with interactive graphics

Second fundamental form

In differential geometry, the **second fundamental form** (or **shape tensor**) is a quadratic form on the tangent plane of a smooth surface in the three dimensional Euclidean space, usually denoted by $\mathrm{I\!I}$ (read "two"). Together with the first fundamental form, it serves to define extrinsic invariants of the surface, its principal curvatures. More generally, such a quadratic form is defined for a smooth hypersurface in a Riemannian manifold and a smooth choice of the unit normal vector at each point.

Surface in $\mathbf{R}^3$

Motivation

The second fundamental form of a parametric surface *S* in $\mathbf{R}^3$ was introduced and studied by Gauss. First suppose that the surface is the graph of a twice continuously differentiable function, $z = f(x,y)$, and that the plane $z = 0$ is tangent to the surface at the origin. Then *f* and its partial derivatives with respect to *x* and *y* vanish at (0,0). Therefore, the Taylor expansion of *f* at (0,0) starts with quadratic terms:

$$z = L\frac{x^2}{2} + Mxy + N\frac{y^2}{2} + \text{ higher order terms,}$$

and the second fundamental form at the origin in the coordinates *x*, *y* is the quadratic form

$$L\,\mathrm{d}x^2 + 2M\,\mathrm{d}x\,\mathrm{d}y + N\,\mathrm{d}y^2.$$

For a smooth point *P* on *S*, one can choose the coordinate system so that the coordinate *z*-plane is tangent to *S* at *P* and define the second fundamental form in the same way.

Classical notation

The second fundamental form of a general parametric surface is defined as follows. Let $\mathbf{r} = \mathbf{r}(u,v)$ be a regular parametrization of a surface in $\mathbf{R}^3$, where $\mathbf{r}$ is a smooth vector valued function of two variables. It is common to denote the partial derivatives of $\mathbf{r}$ with respect to *u* and *v* by $\mathbf{r}_u$ and $\mathbf{r}_v$. Regularity of the parametrization means that $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent for any (u,v) in the domain of $\mathbf{r}$, and hence span the tangent plane to *S* at each point. Equivalently, the cross product $\mathbf{r}_u \times \mathbf{r}_v$ is a nonzero vector normal to the surface. The parametrization thus defines a field of unit normal vectors $\mathbf{n}$:

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

The second fundamental form is usually written as

$$\mathrm{I\!I} = L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2,$$

its matrix in the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ of the tangent plane is

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix}.$$

The coefficients *L*, *M*, *N* at a given point in the parametric *uv*-plane are given by the projections of the second partial derivatives of $\mathbf{r}$ at that point onto the normal line to *S* and can be computed with the aid of the dot product as follows:

$$L = \mathbf{r}_{uu} \cdot \mathbf{n}, \quad M = \mathbf{r}_{uv} \cdot \mathbf{n}, \quad N = \mathbf{r}_{vv} \cdot \mathbf{n}.$$

Physicist's notation

The second fundamental form of a general parametric surface S is defined as follows: Let $\mathbf{r}=\mathbf{r}(u^1,u^2)$ be a regular parametrization of a surface in $\mathbf{R}^3$, where $\mathbf{r}$ is a smooth vector valued function of two variables. It is common to denote the partial derivatives of $\mathbf{r}$ with respect to u^α by $\mathbf{r}_\alpha$, $\alpha = 1, 2$. Regularity of the parametrization means that $\mathbf{r}_1$ and $\mathbf{r}_2$ are linearly independent for any (u^1,u^2) in the domain of $\mathbf{r}$, and hence span the tangent plane to S at each point. Equivalently, the cross product $\mathbf{r}_1 \times \mathbf{r}_2$ is a nonzero vector normal to the surface. The parametrization thus defines a field of unit normal vectors $\mathbf{n}$:

$$\mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|}.$$

The second fundamental form is usually written as

$$\mathrm{I\!I} = b_{\alpha,\beta}\, \mathrm{d}u^\alpha\, \mathrm{d}u^\beta.$$

The equation above uses the Einstein Summation Convention. The coefficients $b_{\alpha,\beta}$ at a given point in the parametric (u^1, u^2)-plane are given by the projections of the second partial derivatives of $\mathbf{r}$ at that point onto the normal line to S and can be computed with the aid of the dot product as follows:

$$b_{\alpha,\beta} = \mathbf{r}_{\alpha,\beta} \cdot \mathbf{n}.$$

Hypersurface in a Riemannian manifold

In Euclidean space, the second fundamental form is given by

$$\mathrm{I\!I}(v,w) = \langle d\nu(v), w\rangle$$

where ν is the Gauss map, and $d\nu$ the differential of ν regarded as a vector valued differential form, and the brackets denote the metric tensor of Euclidean space.

More generally, on a Riemannian manifold, the second fundamental form is an equivalent way to describe the shape operator (denoted by S) of a hypersurface,

$$\mathrm{I\!I}(v,w) = \langle S(v), w\rangle = -\langle \nabla_v n, w\rangle = \langle n, \nabla_v w\rangle,$$

where $\nabla_v w$ denotes the covariant derivative of the ambient manifold and n a field of normal vectors on the hypersurface. (If the affine connection is torsion-free, then the second fundamental form is symmetric.)

The sign of the second fundamental form depends on the choice of direction of n (which is called a co-orientation of the hypersurface - for surfaces in Euclidean space, this is equivalently given by a choice of orientation of the surface).

Generalization to arbitrary codimension

The second fundamental form can be generalized to arbitrary codimension. In that case it is a quadratic form on the tangent space with values in the normal bundle and it can be defined by

$$\mathrm{I\!I}(v,w) = (\nabla_v w)^\perp,$$

where $(\nabla_v w)^\perp$ denotes the orthogonal projection of covariant derivative $\nabla_v w$ onto the normal bundle.

In Euclidean space, the curvature tensor of a submanifold can be described by the following formula:

$$\langle R(u,v)w, z\rangle = \langle \mathrm{I\!I}(u,z), \mathrm{I\!I}(v,w)\rangle - \langle \mathrm{I\!I}(u,w), \mathrm{I\!I}(v,z)\rangle.$$

This is called the **Gauss equation**, as it may be viewed as a generalization of Gauss's Theorema Egregium. The eigenvalues of the second fundamental form, represented in an orthonormal basis, are the **principal curvatures** of the surface. A collection of orthonormal eigenvectors are called the **principal directions**.

For general Riemannian manifolds one has to add the curvature of ambient space; if N is a manifold embedded in a Riemannian manifold (M,g) then the curvature tensor R_N of N with induced metric can be expressed using the second fundamental form and R_M, the curvature tensor of M:

$$\langle R_N(u,v)w,z\rangle = \langle R_M(u,v)w,z\rangle + \langle \mathrm{II}(u,z),\mathrm{II}(v,w)\rangle - \langle \mathrm{II}(u,w),\mathrm{II}(v,z)\rangle.$$

References

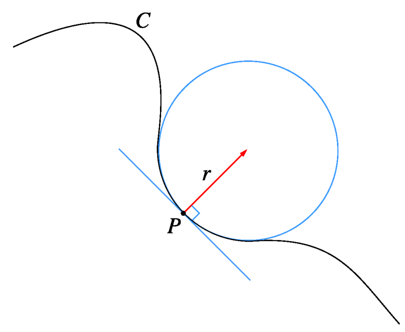
- Guggenheimer, Heinrich (1977). "Chapter 10. Surfaces". *Differential Geometry*. Dover. ISBN 0-486-63433-7.
- Kobayashi, Shoshichi and Nomizu, Katsumi (1996 (New edition)). *Foundations of Differential Geometry, Vol. 2*. Wiley-Interscience. ISBN 0-471-15732-5.
- Spivak, Michael (1999). *A Comprehensive introduction to differential geometry (Volume 3)*. Publish or Perish. ISBN 0-914098-72-1.

External links

- A PhD thesis about the geometry of the second fundamental form by Steven Verpoort: https://repository.libis.kuleuven.be/dspace/bitstream/1979/1779/2/hierrrissiedan!.pdf

Osculating circle

In differential geometry of curves, the **osculating circle** of a sufficiently smooth plane curve at a given point p on the curve has been traditionally defined as the circle passing through p and a pair of additional points on the curve infinitesimally close to p. Its center lies on the inner normal line, and its curvature is the same as that of the given curve at that point. This circle, which is the one among all **tangent circles** at the given point that approaches the curve most tightly, was named *circulus osculans* (Latin for "kissing circle") by Leibniz.



An osculating circle

The center and radius of the osculating circle at a given point are called **center of curvature** and **radius of curvature** of the curve at that point. A geometric construction was described by Isaac Newton in his *Principia*:

> There being given, in any places, the velocity with which a body describes a given figure, by means of forces directed to some common centre: to find that centre.
>
> — Isaac Newton, *Principia*; PROPOSITION V. PROBLEM I.

Description in lay terms

Imagine a car moving along a curved road on a vast flat plane. Suddenly, at one point along the road, the steering wheel locks in its present position. Thereafter, the car moves in a circle that "kisses" the road at the point of locking. The curvature of the circle is equal to that of the road at that point. That circle is the osculating circle of the road curve at that point.

Mathematical description

Let $\gamma(s)$ be a regular parametric plane curve, where s is the arc length, or natural parameter. This determines the unit tangent vector T, the unit normal vector N, the signed curvature $k(s)$ and the radius of curvature at each point:

$$T(s)=\gamma'(s),\quad T'(s)=k(s)N(s),\quad R(s)=\frac{1}{|k(s)|}.$$

Suppose that P is a point on C where $k \neq 0$. The corresponding center of curvature is the point Q at distance R along N, in the same direction if k is positive and in the opposite direction if k is negative. The circle with center at Q and with radius R is called the **osculating circle** to the curve C at the point P.

If C is a regular space curve then the osculating circle is defined in a similar way, using the principal normal vector N. It lies in the *osculating plane*, the plane spanned by the tangent and principal normal vectors T and N at the point P.

The plane curve can also be given in a different regular parametrization $\gamma(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ where regular means that $\gamma'(t) \neq 0$ for all t. Then the formulas for the signed curvature $k(t)$, the normal unit vector $N(t)$, the radius of curvature $R(t)$, and the center $Q(t)$ of the osculating cicle are

$$k(t) = \frac{x_1'(t) \cdot x_2''(t) - x_1''(t) \cdot x_2'(t)}{\left(x_1'(t)^2 + x_2'(t)^2\right)^{\frac{3}{2}}} \qquad N(t) = \frac{1}{||\gamma'(t)||} \cdot \begin{pmatrix} -x_2'(t) \\ x_1'(t) \end{pmatrix},$$

$$R(t) = \left| \frac{\left(x_1'(t)^2 + x_2'(t)^2\right)^{\frac{3}{2}}}{x_1'(t) \cdot x_2''(t) - x_1''(t) \cdot x_2'(t)} \right| \qquad \text{and} \qquad Q(t) = \gamma(t) + \frac{1}{k(t) \cdot ||\gamma'(t)||} \cdot \begin{pmatrix} -x_2'(t) \\ x_1'(t) \end{pmatrix}.$$

Properties

For a curve C given by a sufficiently smooth parametric equations (twice continuously differentiable), the osculating circle may be obtained by a limiting procedure: it is the limit of the circles passing through three distinct points on C as these points approach P.[1] This is entirely analogous to the construction of the tangent to a curve as a limit of the secant lines through pairs of distinct points on C approaching P.

The osculating circle S to a plane curve C at a regular point P can be characterized by the following properties:

- The circle S passes through P.
- The circle S and the curve C have the common tangent line at P, and therefore the common normal line.
- Close to P, the distance between the points of the curve C and the circle S in the normal direction decays as the cube or a higher power of the distance to P in the tangential direction.

This is usually expressed as "the curve and its osculating circle have the third or higher order contact" at P. Loosely speaking, the vector functions representing C and S agree together with their first and second derivatives at P.

If the derivative of the curvature with respect to s is nonzero at P then the osculating circle crosses the curve C at P. Points P at which the derivative of the curvature is zero are called vertices. If P is a vertex then C and its osculating circle have contact of order at least four. If, moreover, the curvature has a non-zero local maximum or minimum at P then the osculating circle touches the curve C at P but does not cross it.

The curve C may be obtained as the envelope of the one-parameter family of its osculating circles. Their centers, i.e. the centers of curvature, form another curve, called the *evolute* of C. Vertices of C correspond to singular points on its evolute.

Examples

Parabola

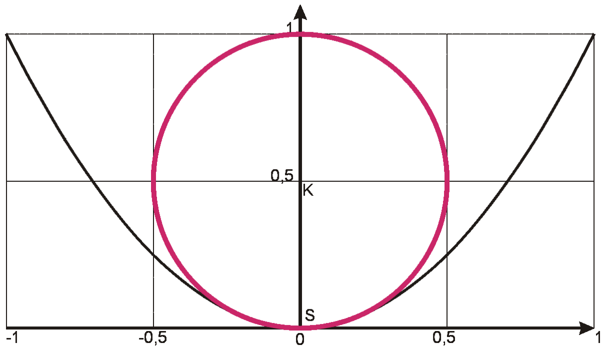
For the parabola

$$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

the radius of curvature is

$$R(t) = \left| \frac{(1 + 4 \cdot t^2)^{\frac{3}{2}}}{2} \right|$$

At the vertex $\gamma(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the radius of curvature equals *R(0)=0.5* (see figure). The parabola has fourth order contact with its osculating circle there. For large *t* the radius of curvature increases ~ t^3, that is, the curve straightens more and more.



The osculating circle of the parabola at its vertex has radius 0.5 and fourth order contact.

Lissajous curve

A Lissajous curve with ratio of frequencies (3:2) can be parametrized as follows

$$\gamma(t) = \begin{pmatrix} \cos(3t) \\ \sin(2t) \end{pmatrix}.$$

It has signed curvature *k*(*t*), normal unit vector *N*(*t*) and radius of curvature *R*(*t*) given by

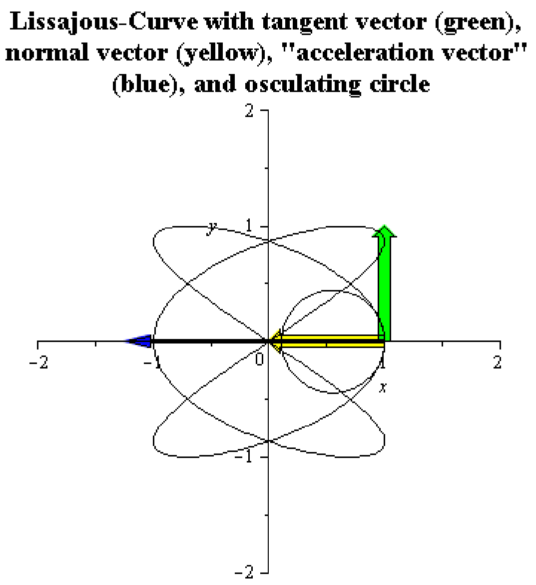
$$k(t) = \frac{6\cos(t)(8\cos(t)^4 - 10\cos(t)^2 + 5)}{(232\cos(t)^4 - 97\cos(t)^2 + 13 - 144\cos(t)^6)^{3/2}},$$

$$N(t) = \frac{1}{||\gamma'(t)||} \cdot \begin{pmatrix} -2\cos(2t) \\ -3\sin(3t) \end{pmatrix}$$

and

$$R(t) = \left| \frac{(232\cos(t)^4 - 97\cos(t)^2 + 13 - 144\cos(t)^6)^{3/2}}{6\cos(t)(8\cos(t)^4 - 10\cos(t)^2 + 5)} \right|.$$

See the figure for an animation. There the "acceleration vector" is the second derivative $\frac{\mathrm{d}^2\gamma(s)}{\mathrm{d}s^2}$ with respect to the arc length s.



Animation of the osculating circle to a Lissajous curve

Notes

[1] Actually, point *P* plus two additional points, one on either side of *P* will do. See Lamb (on line): Horace Lamb (1897). *An Elementary Course of Infinitesimal Calculus* (http://books.google.com/books?id=eDM6AAAAMAAJ&pg=PA406&dq="osculating+circle"&lr=&as_brr=0). University Press. p. 406. .

Further reading

For some historical notes on the study of curvature, see

- Grattan-Guinness & H. J. M. Bos (2000). *From the Calculus to Set Theory 1630-1910: An Introductory History* (http://books.google.com/books?id=OLNeNIbD3jUC&pg=PA72&vq=curvature&dq=Leibniz+Calculus+Bos&lr=&source=gbs_search_s&sig=ACfU3U1ksyW8pE2DkRUdMQ6-Dw5jo4WdJA). Princeton University Press. p. 72. ISBN 0-691-07082-2.
- Roy Porter, editor (2003). *The Cambridge History of Science: v4 - Eighteenth Century Science* (http://books.google.com/books?id=KDSqLsOHc9UC&pg=PA313&dq=motion+"center+of+curvature"&lr=&as_brr=0&sig=ACfU3U1xXmwMgWfk0Is_CToKzTbY591kXQ#PPA313,M1). Cambridge University Press. p. 313. ISBN 0-521-57243-6.

For application to maneuvering vehicles see

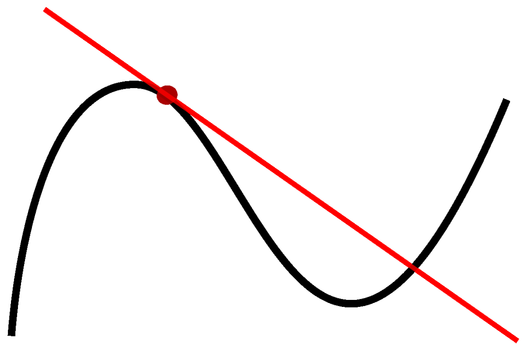
- JC Alexander and JH Maddocks: *On the maneuvering of vehicles* (https://drum.umd.edu/dspace/bitstream/1903/4630/1/TR_87-122.pdf)
- Murray S. Klamkin (1990). *Problems in Applied Mathematics: selections from SIAM review* (http://books.google.com/books?id=-b2hQ7_ARocC&pg=PA1&dq=motion+"center+of+curvature"&lr=&as_brr=0&sig=ACfU3U0e0jN1k65IAWtXyKbYwYxtdeGlRw#PPA1,M1). Society for Industrial and Applied Mathematics. p. 1. ISBN 0-89871-259-9.

External links

- Create your own animated illustrations of osculating circles (http://www.math.uni-muenster.de/u/urs.hartl/gifs/CurvatureAndTorsionOfCurves.mw) (Maple-Worksheet)
- Weisstein, Eric W., " Osculating Circle (http://mathworld.wolfram.com/OsculatingCircle.html)" from MathWorld.
- Module for Curvature (http://math.fullerton.edu/mathews/n2003/CurvatureMod.html)

Tangent

In geometry, the **tangent line** (or simply the **tangent**) to a plane curve at a given point is the straight line that "just touches" the curve at that point—that is, coincides with the curve at that point and, near that point, is closer to the curve that any other line passing through that point. More precisely, a straight line is said to be a tangent of a curve $y = f(x)$ at a point $x = c$ on the curve if the line passes through the point $(c, f(c))$ on the curve and has slope $f'(c)$ where f' is the derivative of f. A similar definition applies to space curves and curves in n-dimensional Euclidean space.

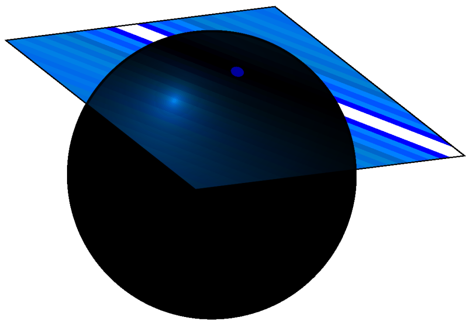


Tangent to a curve. The red line is tangential to the curve at the point marked by a red dot.

As it passes through the point where the tangent line and the curve meet, called the *point of tangency*, the tangent line is "going in the same direction" as the curve, and is thus the best straight-line approximation to the curve at that point.

Similarly, the **tangent plane** to a surface at a given point is the plane that "just touches" the surface at that point. The concept of a tangent is one of the most fundamental notions in differential geometry and has been extensively generalized; see Tangent space.

The word *tangent* comes from the Latin *tangere*, *to touch*.



Tangent plane to a sphere

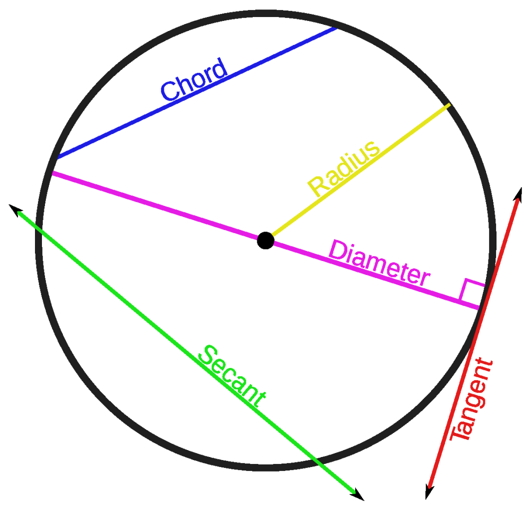
History

The first definition of a tangent was "a line which touch a curve, but which when produced, does not cut it". [1] This old definition prevent inflection points to have any tangent. It has been dismissed and the modern definitions are equivalent to that of Leibniz.

Pierre de Fermat developed a general technique for determining the tangents of a curve using his method of adequality in the 1630s.

Leibniz defined the tangent line as the line through a pair of infinitely close points on the curve.

Tangent line to a curve

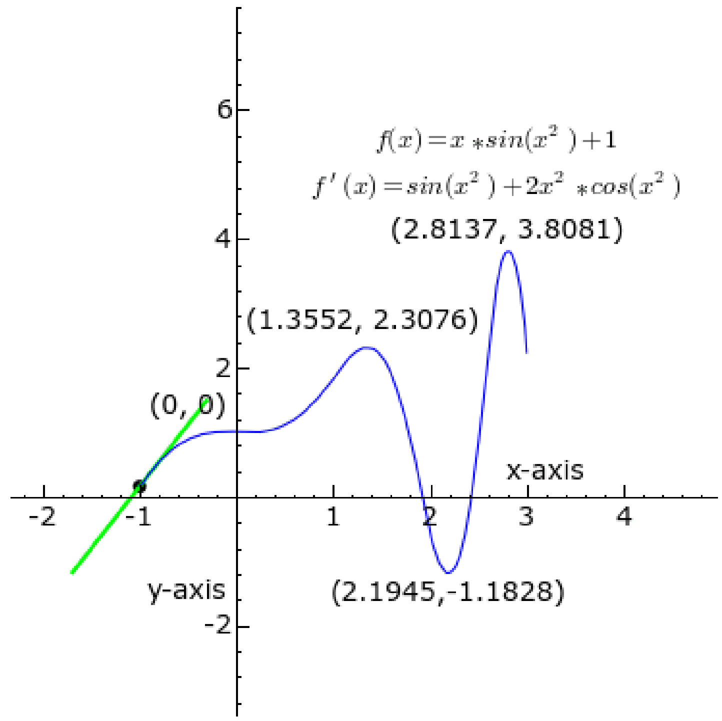


A tangent, a chord, and a secant to a circle

The intuitive notion that a tangent line "touches" a curve can be made more explicit by considering the sequence of straight lines (secant lines) passing through two points, A and B, those that lie on the function curve. The tangent at A is the limit when point B approximates or tends to A. The existence and uniqueness of the tangent line depends on a certain type of mathematical smoothness, known as "differentiability." For example, if two circular arcs meet at a sharp point (a vertex) then there is no uniquely defined tangent at the vertex because the limit of the progression of secant lines depends on the direction in which "point B" approaches the vertex.

At most points, the tangent touches the curve without crossing it (though it may, when continued, cross the curve at other places away from the point of tangent). A point where the tangent (at this point) crosses the curve is called an *inflection point*. Circles, parabolas, hyperbolas and ellipses do not have any inflection point, but more complicated curve do have, like the graph of a cubic function, which has exactly one inflection point.

Conversely, it may happen that the curve lies entirely on one side of a straight line passing through a point on it, and yet this straight line is not a tangent line. This is the case, for example, for a line passing through the vertex of a triangle and not intersecting the triangle—where the tangent line does not exist for the reasons explained above. In convex geometry, such lines are called supporting lines.



At each point, the line is always tangent to the curve. Its slope is the derivative; the positivity, negativity and zeroes of the derivative are marked by green, red and black respectively.

Analytical approach

The geometric idea of the tangent line as the limit of secant lines serves as the motivation for analytical methods that are used to find tangent lines explicitly. The question of finding the tangent line to a graph, or the **tangent line problem,** was one of the central questions leading to the development of calculus in the 17th century. In the second book of his *Geometry*, René Descartes[2] said of the problem of constructing the tangent to a curve, "And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know".[3]

Intuitive description

Suppose that a curve is given as the graph of a function, $y = f(x)$. To find the tangent line at the point $p = (a, f(a))$, consider another nearby point $q = (a + h, f(a + h))$ on the curve. The slope of the secant line passing through p and q is equal to the difference quotient

$$\frac{f(a+h)-f(a)}{h}.$$

As the point q approaches p, which corresponds to making h smaller and smaller, the difference quotient should approach a certain limiting value k, which is the slope of the tangent line at the point p. If k is known, the equation of the tangent line can be found in the point-slope form:

$$y - f(a) = k(x - a).$$

More rigorous description

To make the preceding reasoning rigorous, one has to explain what is meant by the difference quotient approaching a certain limiting value k. The precise mathematical formulation was given by Cauchy in the 19th century and is based on the notion of limit. Suppose that the graph does not have a break or a sharp edge at p and it is neither plumb nor too wiggly near p. Then there is a unique value of k such that, as h approaches 0, the difference quotient gets closer and closer to k, and the distance between them becomes negligible compared with the size of h, if h is small enough. This leads to the definition of the slope of the tangent line to the graph as the limit of the difference quotients for the function f. This limit is the derivative of the function f at $x = a$, denoted $f'(a)$. Using derivatives, the equation of the tangent line can be stated as follows:

$$y = f(a) + f'(a)(x - a).$$

Calculus provides rules for computing the derivatives of functions that are given by formulas, such as the power function, trigonometric functions, exponential function, logarithm, and their various combinations. Thus, equations of the tangents to graphs of all these functions, as well as many others, can be found by the methods of calculus.

How the method can fail

Calculus also demonstrates that there are functions and points on their graphs for which the limit determining the slope of the tangent line does not exist. For these points the function f is *non-differentiable*. There are two possible reasons for the method of finding the tangents based on the limits and derivatives to fail: either the geometric tangent exists, but it is a vertical line, which cannot be given in the point-slope form since it does not have a slope, or the graph exhibits one of three behaviors that precludes a geometric tangent.

The graph $y = x^{1/3}$ illustrates the first possibility: here the difference quotient at $a = 0$ is equal to $h^{1/3}/h = h^{-2/3}$, which becomes very large as h approaches 0. This curve has a tangent line at the origin, that is vertical.

The graph $y = x^{2/3}$ illustrates another possibility: this graph has a *cusp* at the origin. This means that, when h approaches 0, the difference quotient at $a = 0$ approaches plus or minus infinity depending on the sign of x. Thus both branches of the curve are near to the half vertical line for which y=0, but none is near to the negative part of this line. Basically, there is no tangent at the origin in this case, but in some context one may consider this line as a tangent, and even, in algebraic geometry, as a *double tangent*.

The graph $y = |x|$ of the absolute value function consists of two straight lines with different slopes joined at the origin. As a point q approaches the origin from the right, the secant line always has slope 1. As a point q approaches the origin from the left, the secant line always has slope −1. Therefore, there is no unique tangent to the graph at the origin. Having two different (but finite) slopes is called a *corner*.

Finally, since differentiability implies continuity, the contrapositive states *discontinuity* implies non-differentiability. Any such jump or point discontinuity will have no tangent line. This includes cases where one slope approaches positive infinity while the other approaches negative infinity, leading to an infinite jump discontinuity

Equations

When the curve is given by $y = f(x)$ then the slope of the tangent is $\frac{dy}{dx}$, so by the point–slope formula the equation of the tangent line at (X, Y) is

$$y - Y = \frac{dy}{dx}(X) \cdot (x - X)$$

where (x, y) are the coordinates of any point on the tangent line, and where the derivative is evaluated at $x = X$.[4]

When the curve is given by $y = f(x)$, the tangent line's equation can also be found[5] by using polynomial division to divide $f(x)$ by $(x - X)^2$; if the remainder is denoted by $g(x)$, then the equation of the tangent line is given by

$$y = g(x).$$

When the equation of the curve is given in the form $f(x, y) = 0$ then the value of the slope can be found by implicit differentiation, giving

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

The equation of the tangent line at a point (X,Y) such that $f(X,Y) = 0$ is then[4]

$$\frac{\partial f}{\partial x}(X,Y) \cdot (x - X) + \frac{\partial f}{\partial y}(X,Y) \cdot (y - Y) = 0.$$

This equation remains true if $\frac{\partial f}{\partial y}(X,Y) = 0$ but $\frac{\partial f}{\partial x}(X,Y) \neq 0$ (in this case the slope of the tangent is infinite). If $\frac{\partial f}{\partial y}(X,Y) = \frac{\partial f}{\partial x}(X,Y) = 0,$ the tangent line is not defined and the point (X,Y) is said singular.

For algebraic curves, computations may be simplified somewhat by converting to homogeneous coordinates. Specifically, let the homogeneous equation of the curve be $g(x, y, z) = 0$ where g is a homogeneous function of degree n. Then, if (X, Y, Z) lies on the curve, Euler's theorem implies

$$\frac{\partial g}{\partial x} \cdot X + \frac{\partial g}{\partial y} \cdot Y + \frac{\partial g}{\partial z} \cdot Z = ng(X, Y, Z) = 0.$$

It follows that the homogeneous equation of the tangent line is

$$\frac{\partial g}{\partial x}(X, Y, Z) \cdot x + \frac{\partial g}{\partial y}(X, Y, Z) \cdot y + \frac{\partial g}{\partial z}(X, Y, Z) \cdot z = 0.$$

The equation of the tangent line in Cartesian coordinates can be found by setting z=1 in this equation.[6]

To apply this to algebraic curves, write $f(x, y)$ as

$$f = u_n + u_{n-1} + \ldots + u_1 + u_0$$

where each u_r is the sum of all terms of degree r. The homogeneous equation of the curve is then

$$g = u_n + u_{n-1}z + \ldots + u_1 z^{n-1} + u_0 z^n = 0.$$

Applying the equation above and setting z=1 produces

$$\frac{\partial f}{\partial x}(X, Y) \cdot x + \frac{\partial f}{\partial y}(X, Y) \cdot y + \frac{\partial g}{\partial z}(X, Y, 1) = 0$$

as the equation of the tangent line.[7] The equation in this form is often simpler to use in practice since no further simplification is needed after it is applied.[6]

If the curve is given parametrically by

$$x = x(t), \quad y = y(t)$$

then the slope of the tangent is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

giving the equation for the tangent line at $t = T,\ X = x(T),\ Y = y(T)$ as[8]

$$\frac{dx}{dt}(T) \cdot (y - Y) = \frac{dy}{dt}(T) \cdot (x - X).$$

If $\frac{dx}{dt}(T) = \frac{dy}{dt}(T) = 0,$, the tangent line is not defined. However, it may occur that the tangent line exists and may be computed from an implicit equation of the curve.

Normal line to a curve

The line perpendicular to the tangent line to a curve at the point of tangency is called the *normal line* to the curve at that point. The slopes of perpendicular lines have product −1, so if the equation of the curve is $y = f(x)$ then slope of the normal line is

$$-\frac{1}{\frac{dy}{dx}}$$

and it follows that the equation of the normal line is

$$(X - x) + \frac{dy}{dx}(Y - y) = 0.$$

Similarly, if the equation of the curve has the form $f(x, y) = 0$ then the equation of the normal line is given by[9]

$$\frac{\partial f}{\partial y}(X - x) - \frac{\partial f}{\partial x}(Y - y) = 0.$$

If the curve is given parametrically by

$$x = x(t), \quad y = y(t)$$

then the equation of the normal line is[8]

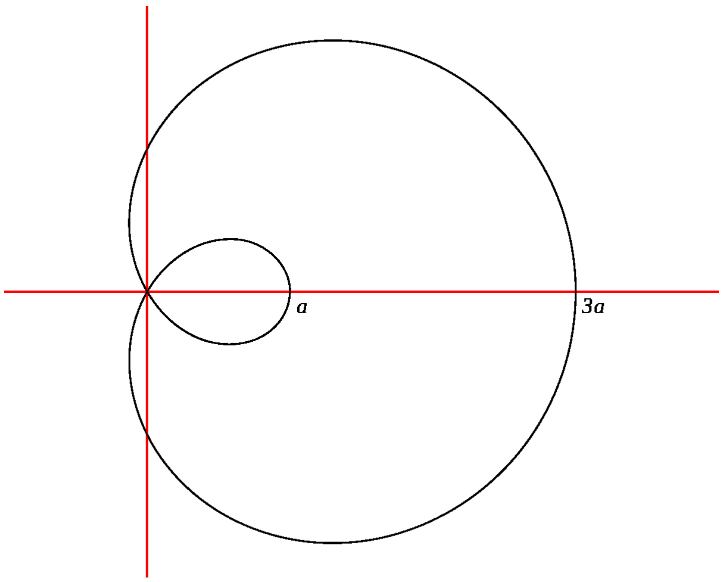
$$\frac{dx}{dt}(X - x) + \frac{dy}{dt}(Y - y) = 0.$$

Angle between curves

The angle between two curves at a point where they intersect is defined as the angle between their tangent lines at that point. More specifically, two curves are said to be tangent at a point if they have the same tangent at a point, and orthogonal if their tangent lines are orthogonal.[10]

Multiple tangents at the origin

The formulas above fail when the point is a singular point. In this case there may be two or more branches of the curve which pass through the point, each branch having its own tangent line. When the point is the origin, the equations of these lines can be found for algebraic curves by factoring the equation formed by eliminating all but the lowest degree terms from the original equation. Since any point can be made the origin by a change of variables, this gives a method for finding the tangent lines at any singular point.



The limaçon trisectrix: a curve with two tangents at the origin.

For example, the equation of the limaçon trisectrix shown to the right is

$$(x^2 + y^2 - 2ax)^2 = a^2(x^2 + y^2).$$

Expanding this and eliminating all but terms of degree 2 gives

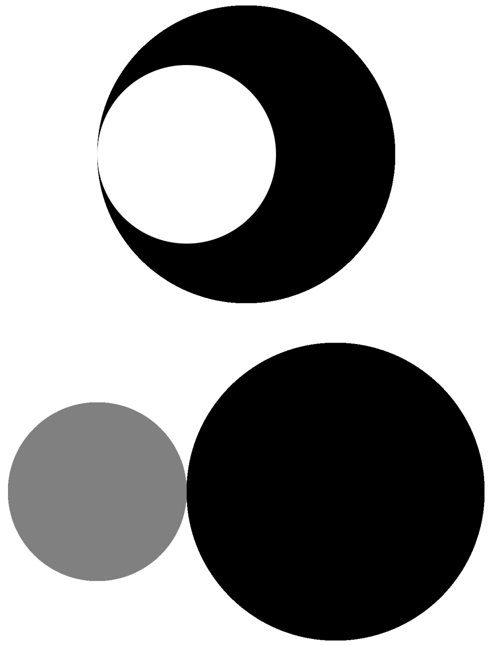
$$a^2(3x^2 - y^2) = 0$$

which, when factored, becomes

$$y = \pm\sqrt{3}x.$$

So these are the equations of the two tangent lines through the origin.[11]

Tangent circles



Two pairs of tangent circles. Above internally and below externally tangent

Two circles of non-equal radius, both in the same plane, are said to be tangent to each other if they meet at only one point. Equivalently, two circles, with radii of r_i and centers at (x_i, y_i), for $i = 1, 2$ are said to be tangent to each other if

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (r_1 \pm r_2)^2 .$$

- Two circles are **externally tangent** if the distance between their centres is equal to the sum of their radii.

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (r_1 + r_2)^2 .$$

- Two circles are **internally tangent** if the distance between their centres is equal to the difference between their radii.[12]

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (r_1 - r_2)^2 .$$

Surfaces and higher-dimensional manifolds

The *tangent plane* to a surface at a given point *p* is defined in an analogous way to the tangent line in the case of curves. It is the best approximation of the surface by a plane at *p*, and can be obtained as the limiting position of the planes passing through 3 distinct points on the surface close to *p* as these points converge to *p*. More generally, there is a *k*-dimensional tangent space at each point of a *k*-dimensional manifold in the *n*-dimensional Euclidean space.

References

[1] Noah Webster, *American Dictionary of the English Language* (New York: S. Converse, 1828), vol.2, p.733, (http://archive.org/stream/americandictiona02websrich#page/n733/mode/2up)
[2] Descartes, René (1954). *The geometry of René Descartes*. Courier Dover (http://store.doverpublications.com/0486600688.html). pp. 95. ISBN 0-486-60068-8.
[3] R. E. Langer (October 1937). "Rene Descartes". *American Mathematical Monthly* (Mathematical Association of America) **44** (8): 495–512. doi:10.2307/2301226. JSTOR 2301226.
[4] Edwards Art. 191
[5] Strickland-Constable, Charles, "A simple method for finding tangents to polynomial graphs", *Mathematical Gazette*, November 2005, 466-467.
[6] Edwards Art. 192
[7] Edwards Art. 193
[8] Edwards Art. 196
[9] Edwards Art. 194
[10] Edwards Art. 195
[11] Edwards Art. 197
[12] Circles For Leaving Certificate Honours Mathematics by Thomas O'Sullivan 1997 (http://homepage.eircom.net/~phabfys/circles.html)

- J. Edwards (1892). *Differential Calculus* (http://books.google.com/books?id=unltAAAAMAAJ&pg=PA143#v=onepage&q&f=false). London: MacMillan and Co.. pp. 143 ff..

External links

- Weisstein, Eric W., " Tangent Line (http://mathworld.wolfram.com/TangentLine.html)" from MathWorld.
- Tangent to a circle (http://www.mathopenref.com/tangent.html) With interactive animation
- Tangent and first derivative (http://www.vias.org/simulations/simusoft_difftangent.html) - An interactive simulation
- The Tangent Parabola by John H. Mathews (http://math.fullerton.edu/mathews/n2003/TangentParabolaMod.html)

Tangent lines to circles

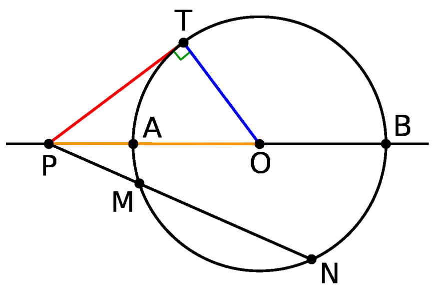
In Euclidean plane geometry, **tangent lines to circles** form the subject of several theorems, and play an important role in many geometrical constructions and proofs. Since the tangent line to a circle at a point **P** is perpendicular to the radius to that point, theorems involving tangent lines often involve radial lines and orthogonal circles.

Tangent lines to one circle

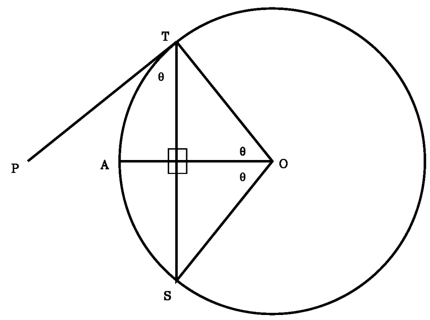
A tangent line *t* to a circle *C* intersects the circle at a single point **T**. For comparison, secant lines intersect a circle at two points, whereas another line may not intersect a circle at all. This property of tangent lines is preserved under many geometrical transformations, such as scalings, rotation, translations, inversions, and map projections. In technical language, these transformations do not change the incidence structure of the tangent line and circle, even though the line and circle may be deformed.

The radius of a circle is perpendicular to the tangent line through its endpoint on the circle's circumference. Conversely, the perpendicular to a radius through the same endpoint is a tangent line. The resulting geometrical figure of circle and tangent line has a reflection symmetry about the axis of the radius.

No tangent line can be drawn through a point in the interior of a circle, since any such line must be a secant line. However, *two* tangent lines can be drawn to a circle from a point **P** outside of the circle. The geometrical figure of a circle and both tangent lines likewise has a reflection symmetry about the radial axis joining **P** to the center point **O** of the circle. Thus the lengths of the segments from **P** to the two tangent points are equal. By the secant-tangent theorem, the square of this tangent length equals the power of the point P in the circle *C*. This power equals the product of distances from **P** to any two intersection points of the circle with a secant line passing through **P**.



By the power-of-a-point theorem, the product of lengths PM·PN for any ray PMN equals to the square of PT, the length of the tangent line segment (red).



The angle PTO between a tangent line and a radial line segment is a right angle. By the inscribed angle theorem, the angle PTS between the tangent line and another point on the circumference equals half of the central angle TOS subtended by the two points **T** and **S**.

The tangent line *t* and the tangent point **T** have a conjugate relationship to one another, which has been generalized into the idea of pole points and polar lines. The same reciprocal relation exists between a point **P** outside the circle and the secant line joining its two points of tangency.

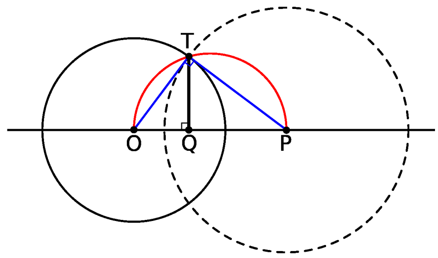
If a point P is exterior to a circle with center O, and if the tangent lines from P touch the circle at points T and Q, then ∠TPQ and ∠TOQ are supplementary (sum to 180°).

If a chord TM is drawn from the a tangency point T of exterior point P, then ∠PTM = (1/2)∠MOT.

Geometrical constructions

It is relatively straightforward to construct a line *t* tangent to a circle at a point **T** on the circumference of the circle. A line *a* is drawn from **O**, the center of the circle, through the radial point **T**; the line *t* is the perpendicular line to *a*. One method for constructing this perpendicular is as follows. Placing the compass point on **T** with the circle's radius *r*, a second point **G** is identified on the radial line *a*; thus, **T** is the midpoint of the line segment OG. Two intersecting circles of the same radius *R*>*r* are drawn, centered on **O** and **G**, respectively. The line drawn through their two points of intersection is the tangent line.

Thales' theorem may be used to construct the tangent lines to a point **P** external to the circle *C*. A circle is drawn centered on **Q**, the midpoint of the line segment OP, where **O** is again the center of the circle *C*. The intersection points $\mathbf{T}_1$ and $\mathbf{T}_2$ are the tangent points for lines passing through **P**, by the following argument. The line segments OT_1 and OT_2 are radii of the circle *C*; since both are inscribed in a semicircle, they are perpendicular to the line segments PT_1 and PT_2, respectively. But only a tangent line is perpendicular to the radial line. Hence, the two lines from **P** and passing through $\mathbf{T}_1$ and $\mathbf{T}_2$ are tangent to the circle *C*.

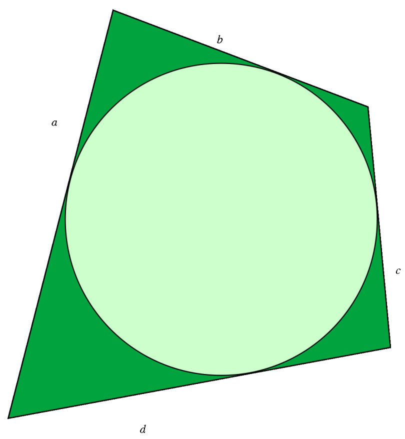


Thales' theorem allows the construction of a tangent line PT to a given circle (solid black). A semicircle with diameter OP intersects the given circle at the desired point **T**. The other tangent point is the second intersection of the dashed and given circles.

Tangent quadrilateral theorem and inscribed circles

A tangential quadrilateral ABCD is a closed figure of four straight sides that are tangent to a given circle *C*. Equivalently, the circle *C* is inscribed in the quadrilateral ABCD. By the Pitot theorem, the sums of opposite sides of any such quadrilateral are equal, i.e.,

$$\overline{AB} + \overline{CD} = \overline{BC} + \overline{DA}.$$



Tangential quadrilateral.

This conclusion follows from the equality of the tangent segments from the four vertices of the quadrilateral. Let the tangent points be denoted as **P** (on segment AB), **Q** (on segment BC), **R** (on segment CD) and **S** (on segment DA). The symmetric tangent segments about each point of ABCD are equal, e.g., BP=BQ=b, CQ=CR=c, DR=DS=d, and AS=AP=a. But each side of the quadrilateral is composed of two such tangent segments

$$\overline{AB}+\overline{CD}=(a+b)+(c+d)=\overline{BC}+\overline{DA}=(b+c)+(d+a)$$

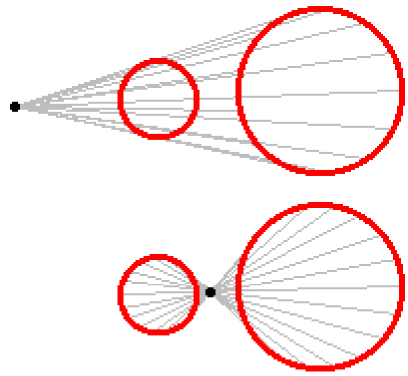
proving the theorem.

The converse is also true: a circle can be inscribed into every quadrilateral in which the lengths of opposite sides sum to the same value.[1]

This theorem and its converse have various uses. For example, they show immediately that no rectangle can have an inscribed circle unless it is a square, and that every rhombus has an inscribed circle, whereas a general parallelogram does not.

Tangent lines to two circles

For two circles, there are generally four bitangent lines that are tangent to both. For two of these, the external tangent lines, the circles fall on the same side of the line; for the two others, the internal tangent lines, the circles fall on opposite sides of the line. The external tangent lines intersect in the external homothetic center, whereas the internal tangent lines intersect at the internal homothetic center. The internal homothetic center always lies on the line joining the centers of the two circles, in the segment between the two circles. The exterior center lies on the same line, and is closer to the center of the smaller than to the center of the larger. If the two circles have equal radius, the external tangent lines are parallel, and — at least in inversive geometry — the external homothetic center, lies at infinity.[2]


The external (above) and internal (below) homothetic centers of the two circles (red) are shown as black points.

Geometrical constructions

Let $\mathbf{O}_1$ and $\mathbf{O}_2$ be the centers of the two circles, C_1 and C_2 and let r_1 and r_2 be their radii, with $r_1 > r_2$; in other words, circle C_1 is defined as the larger of the two circles. Two different methods may be used to construct the external and internal tangent lines.

External tangents

A new circle C_3 of radius $r_1 - r_2$ is drawn centered on $\mathbf{O}_1$. Using the method above, two lines are drawn from $\mathbf{O}_2$ that are tangent to this new circle. These lines are parallel to the desired tangent lines, because the situation corresponds to shrinking both circles C_1 and C_2 by a constant amount, r_2, which shrinks C_2 to a point. Two radial lines may be drawn from the center $\mathbf{O}_1$ through the tangent points on C_3; these intersect C_1 at the desired tangent points. The desired external tangent lines are the lines perpendicular to these radial lines at those tangent points, which may be constructed as described above.

Internal tangents

A new circle C_3 of radius $r_1 + r_2$ is drawn centered on $\mathbf{O}_1$. Using the method above, two lines are drawn from $\mathbf{O}_2$ that are tangent to this new circle. These lines are parallel to the desired tangent lines, because the situation corresponds to shrinking C_2 to a point while expanding C_1 by a constant amount, r_2. Two radial lines may be drawn from the center $\mathbf{O}_1$ through the tangent points on C_3; these intersect C_1 at the desired tangent points. The desired internal tangent lines are the lines perpendicular to these radial lines at those tangent points, which may be constructed as described above.

Algebraic solutions

Let the circles have centres $c_1 = (x_1,y_1)$ and $c_2 = (x_2,y_2)$ with radius r_1 and r_2 respectively. If the tangent line is (a,b,c) where $a^2 + b^2 = 1$ then $a*x_1 + b*y_1 + c = r_1$ and $a*x_2 + b*y_2 + c = r_2$ and we subtract the first from the second to get $a*x + b*y = r$ where $x = x_2 - x_1$, $y = y_2 - y_1$ and $r = r_2 - r_1$ If d is the distance from c_1 to c_2 we can use X = x/d, Y = y/d and R = r/d to make things easier, a*X + b*Y = R and $a^2 + b^2 = 1$, solve these to get two solutions (k = +/-1).

$a = R*X - k*Y*sqrt(1 - R^2)$ *This solution is beautiful. The unit vector (X,Y) points from c_1 to c_2.*

$b = R*Y + k*X*sqrt(1 - R^2)$ *The (unit) R matrices rotate it to point along the tangent lines.*

$c = r_1 - (a*x_1 + b*y_1)$

k = 1 is the tangent line to the right of the circles looking from c_1 to c_2.

k = -1 is the tangent line to the right of the circles looking from c_2 to c_1.

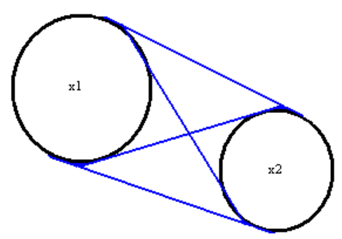
The above assumes each circle has positive radius. If r_1 is positive and r_2 negative then c_1 will lie to the left of each line and c_2 to the right, and the two tangent lines will cross. In this way all 4 solutions are obtained.

Belt problem

The internal and external tangent lines are useful in solving the *belt problem*, which is to calculate the length of a belt or rope needed to fit snugly over two pulleys. If the belt is considered to be a mathematical line of negligible thickness, and if both pulleys are assumed to lie in exactly the same plane, the problem devolves to summing the lengths of the relevant tangent line segments with the lengths of circular arcs subtended by the belt. If the belt is wrapped about the wheels so as to cross, the interior tangent line segments are relevant. Conversely, if the belt is wrapped exteriorly around the pulleys, the exterior tangent line segments are relevant; this case is sometimes called the *pulley problem*.

Illustration

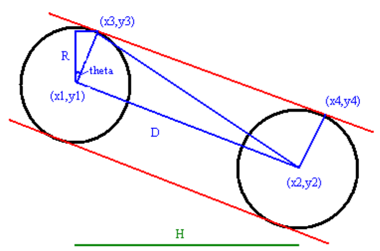
In general the points of tangency t_1 and t_2 for the four lines tangent to two circles with centers x_1 and x_2 and radii r_1 and r_2 are given by solving the simultaneous equations:



Finding Outer Tangent. Circle tangents.

$$(t_2 - x_2).(t_2 - t_1) = 0$$
$$(t_1 - x_1).(t_2 - t_1) = 0$$
$$(t_1 - x_1).(t_1 - x_1) = r_1^2$$
$$(t_2 - x_2).(t_2 - x_2) = r_2^2$$

Outer tangent

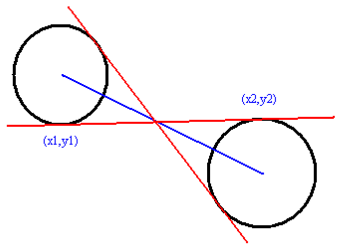


Finding Outer Tangent. Two circles' outer tangents.

The red line joining the points (x3,y3) and (x4,y4) is the outer tangent between the two circles. Given points (x1,y1), (x2,y2) the points (x3,y3), (x4,y4) can easily be calculated by equating the angle theta and adding the x,y coordinates of the triangle(theta) to the original coordinates (x1,y1) as shown in the figure.[3]

Inner tangent

An inner tangent is a tangent that intersects the segment joining two circles' centers. Note that the inner tangent will not be defined for cases when the two circles overlap.



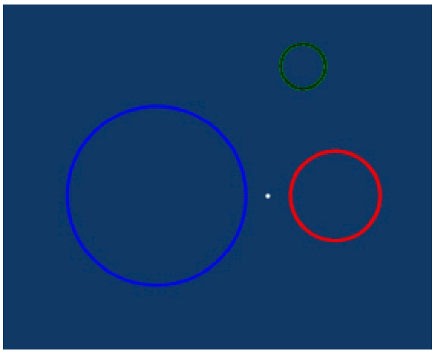
Inner Tangent. The internal tangent lines pass through the internal homothetic center.

Tangent lines to three circles: Monge's theorem

For three circles denoted by C_1, C_2, and C_3, there are three pairs of circles (C_1C_2, C_2C_3, and C_1C_3). Since each pair of circles has two homothetic centers, there six homothetic centers altogether. Gaspard Monge showed in the early 19th century that these six points lie on four lines, each line having three collinear points.

Problem of Apollonius

Many special cases of Apollonius' problem involve finding a circle that is tangent to one or more lines. The simplest of these is to construct circles that are tangent to three given lines (the **LLL** problem). To solve this problem, the center of any such circle must lie on an angle bisector of any pair of the lines; there are two angle-bisecting lines for every intersection of two lines. The intersections of these angle bisectors give the centers of solution circles. There are four such circles in general, the inscribed circle of the triangle formed by the intersection of the three lines, and the three exscribed circles.

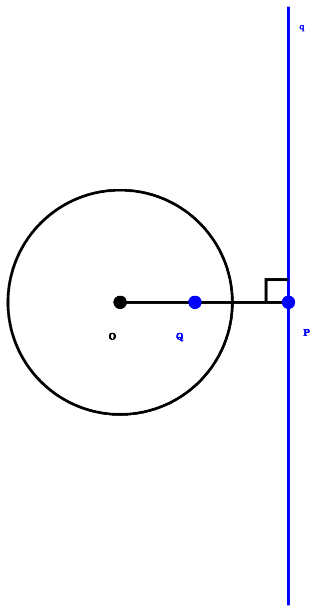


Animation showing the inversive transformation of an Apollonius problem. The blue and red circles swell to tangency, and are inverted in the grey circle, producing two straight lines. The yellow solutions are found by sliding a circle between them until it touches the transformed green circle from within or without.

A general Apollonius problem can be transformed into the simpler problem of circle tangent to one circle and two parallel lines (itself a special case of the **LLC** special case). To accomplish this, it suffices to scale two of the three given circles until they just touch, i.e., are tangent. An inversion in their tangent point with respect to a circle of appropriate radius transforms the two touching given circles into two parallel lines, and the third given circle into another circle. Thus, the solutions may be found by sliding a circle of constant radius between two parallel lines until it contacts the transformed third circle. Re-inversion produces the corresponding solutions to the original problem.

Generalizations

The concept of a tangent line to one or more circles can be generalized in several ways. First, the conjugate relationship between tangent points and tangent lines can be generalized to pole points and polar lines, in which the pole points may be anywhere, not only on the circumference of the circle. Second, the union of two circles is a special (reducible) case of a quartic plane curve, and the external and internal tangent lines are the bitangents to this quartic curve. A generic quartic curve has 28 bitangents.



The concept of a tangent line and tangent point can be generalized to a pole point **Q** and its corresponding polar line *q*. The points **P** and **Q** are inverses of each other with respect to the circle.

A third generalization considers tangent circles, rather than tangent lines; a tangent line can be considered as a tangent circle of infinite radius. In particular, the external tangent lines to two circles are limiting cases of a family of circles which are internally or externally tangent to both circles, while the internal tangent lines are limiting cases of a family of circles which are internally tangent to one and externally tangent to the other of the two circles.[4]

In Möbius or inversive geometry, lines are viewed as circles through a point "at infinity" and for any line and any circle, there is a Möbius transformation which maps one to the other. In Möbius geometry, tangency between a line and a circle becomes a special case of tangency between two circles. This equivalence is extended further in Lie sphere geometry.

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[1] Alexander Bogomolny, "When A Quadrilateral Is Inscriptible?" at Cut-the-knot (http://www.cut-the-knot.org/proofs/InscriptibleQuadrilateral.shtml#anotherProof)

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External links

- Weisstein, Eric W., " Tangent lines to one circle (http://mathworld.wolfram.com/CircleTangentLine.html)" from MathWorld.
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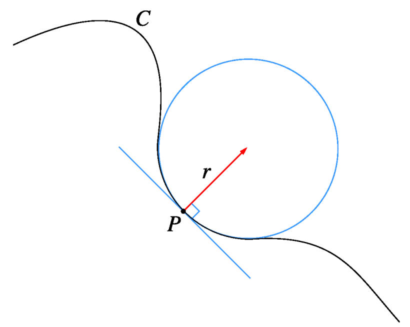
Osculating curve

In mathematics and geometry, an **osculating curve** is an extension of the concept of tangent. A tangent line to a curve is the straight line that shares the location and direction of the curve, while an osculating circle to the same curve shares the location, direction, and curvature.



A curve *C* containing a point *P* where the radius of curvature equals *r*, together with the tangent line and the osculating circle touching *C* at *P*

Two curves are said to be osculating at a particular point if they share the same osculating circle, just as they are said to be tangent if they share the same tangent line. The term derives from the Latinate root "osculate", to kiss, because the two curves contact one another in a more intimate way than simple tangency.

If two smooth curves are tangent at a point and also cross there, they are not only tangent but also osculating. The converse – osculating curves cross at the point of osculation – is not necessarily true, but holds in almost all cases.

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Contact (mathematics)

In mathematics, **contact** of order *k* of functions is an equivalence relation, corresponding to having the same value at a point P and also the same derivatives there, up to order *k*. The equivalence classes are generally called jets. The point of osculation is also called the double cusp.

One speaks also of curves and geometric objects having *k*-th order contact at a point: this is also called *osculation* (i.e. kissing), generalising the property of being tangent. See for example osculating circle and osculating orbit.

Contact forms are particular differential forms of degree 1 on odd-dimensional manifolds; see contact geometry. Contact transformations are related changes of co-ordinates, of importance in classical mechanics. See also Legendre transformation.

Contact between manifolds is often studied in singularity theory, where the type of contact are classified, these include the *A* series (A_0: crossing, A_1: tangent, A_2: osculating, ...) and the umbilic or *D*-series where there is a high degree of contact with the sphere.

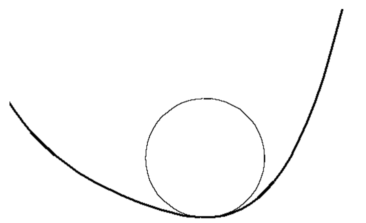
Contact between curves

Two curves in the plane intersecting at a point *p* are said to have:

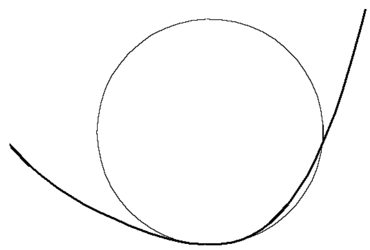
- 1-point contact if the curves have a simple crossing (not tangent).
- 2-point contact if the two curves are tangent.
- 3-point contact if the curvatures of the curves are equal. Such curves are said to be osculating.
- 4-point contact if the derivatives of the curvature are equal.
- 5-point contact if the second derivatives of the curvature are equal.

Contact between a curve and a circle

For a smooth curve *S* in the plane then for each point, *S*(*t*) on the curve then there is always exactly one **osculating circle** which has radius $^1/_{\kappa(t)}$ where $\kappa(t)$ is the curvature of the curve at *t*. If the curve has zero curvature (i.e. an inflection point on the curve) then the osculating circle will be a straight line. The set of the centers of all the osculating circles form the evolute of the curve.



A circle with 2-point contact (tangent)

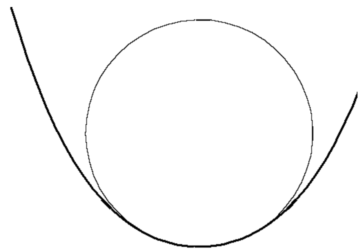


A circle with 3-point contact (osculating)

If the derivative of curvature $\kappa'(t)$ is zero, then the osculating circle will have 4-point contact and the curve is said to have a vertex. The evolute will have a cusp at the center of the circle. The sign of the second derivative of curvature determines whether the curve has a local minimum or maximum of curvature. All closed curves will have at least four vertices, two minima and two maxima (the four-vertex theorem).

In general a curve will not have 5-point with any circle. However, 5-point contact can occur generically in a 1-parameter family of curves, where two vertices (one maximum and one minimum) come together and annihilate. At such points the second derivative of curvature will be zero.

Bi-tangents



A circle with 4-point contact at a vertex of a curve

It is also possible to consider circles which have two point contact with two points $S(t_1)$, $S(t_2)$ on the curve. Such circles are *bi-tangent* circles. The centers of all bi-tangent circles form the symmetry set. The medial axis is a sub set of the symmetry set. These sets have been used as a method of characterising the shapes of biological objects.

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