Discrete Mathematics and Its Applications

Fifth Edition

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CHAPTER

1 The Foundations: Logic and Proof, Sets, and Functions

This chapter reviews the foundations of discrete mathematics. Three important topics are covered: logic, sets, and functions. The rules of logic specify the meaning of mathematical statements. For instance, these rules help us understand and reason with statements such as "There exists an integer that is not the sum of two squares," and "For every positive integer n the sum of the positive integers not exceeding n is $n(n+1)/2$." Logic is the basis of all mathematical reasoning, and it has practical applications to the design of computing machines, to system specifications, to artificial intelligence, to computer programming, to programming languages, and to other areas of computer science, as well as to many other fields of study.

To understand mathematics, we must understand what makes up a correct mathematical argument, that is, a proof. Moreover, to learn mathematics, a person needs to actively construct mathematical arguments and not just read exposition. In this chapter, we explain what makes up a correct mathematical argument and introduce tools to construct these arguments. Proofs are important not only in mathematics, but also in many parts of computer science, including program verification, algorithm correctness, and system security. Furthermore, automated reasoning systems have been constructed that allow computers to construct their own proofs.

Much of discrete mathematics is devoted to the study of discrete structures, which are used to represent discrete objects. Many important discrete structures are built using sets, which are collections of objects. Among the discrete structures built from sets are combinations, which are unordered collections of objects used extensively in counting; relations, which are sets of ordered pairs that represent relationships between objects; graphs, which consist of sets of vertices and of edges that connect vertices; and finite state machines, which are used to model computing machines.

The concept of a function is extremely important in discrete mathematics. A function assigns to each element of a set precisely one element of a set. Useful structures such as sequences and strings are special types of functions. Functions play important roles throughout discrete mathematics. They are used to represent the computational complexity of algorithms, to study the size of sets, to count objects of different kinds, and in a myriad of other ways.

Logic

INTRODUCTION

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Since a major goal of this book is to teach the reader how to understand and how to construct correct mathematical arguments, we begin our study of discrete mathematics with an introduction to logic.

In addition to its importance in understanding mathematical reasoning, logic has numerous applications in computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. We will discuss each of these applications in the upcoming chapters.

PROPOSITIONS

Our discussion begins with an introduction to the basic building blocks of logic—propositions. A **proposition** is a declarative sentence that is either true or false, but not both.

EXAMPLE 1 All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false. ◀

Some sentences that are not propositions are given in the next example.

EXAMPLE 2 Consider the following sentences.

1. What time is it?
2. Read this carefully.

Links

ARISTOTLE (384 B.C.E.–322 B.C.E.) Aristotle was born in Stargirus in northern Greece. His father was the personal physician of the King of Macedonia. Because his father died when Aristotle was young, Aristotle could not follow the custom of following his father's profession. Aristotle became an orphan at a young age when his mother also died. His guardian who raised him taught him poetry, rhetoric, and Greek. At the age of 17, his guardian sent him to Athens to further his education. Aristotle joined Plato's Academy where for 20 years he attended Plato's lectures, later presenting his own lectures on rhetoric. When Plato died in 347 B.C.E., Aristotle was not chosen to succeed him because his views differed too much from those of Plato. Instead, Aristotle joined the court of King Hermeas where he remained for three years, and married the niece of the King. When the Persians defeated Hermeas, Aristotle moved to Mytilene and, at the invitation of King Philip of Macedonia, he tutored Alexander, Philip's son, who later became Alexander the Great. Aristotle tutored Alexander for five years and after the death of King Philip, he returned to Athens and set up his own school, called the Lyceum.

Aristotle's followers were called the peripatetics, which means "to walk about," because Aristotle often walked around as he discussed philosophical questions. Aristotle taught at the Lyceum for 13 years where he lectured to his advanced students in the morning and gave popular lectures to a broad audience in the evening. When Alexander the Great died in 323 B.C.E., a backlash against anything related to Alexander led to trumped-up charges of impiety against Aristotle. Aristotle fled to Chalcis to avoid prosecution. He only lived one year in Chalcis, dying of a stomach ailment in 322 B.C.E.

Aristotle wrote three types of works: those written for a popular audience, compilations of scientific facts, and systematic treatises. The systematic treatises included works on logic, philosophy, psychology, physics, and natural history. Aristotle's writings were preserved by a student and were hidden in a vault where a wealthy book collector discovered them about 200 years later. They were taken to Rome, where they were studied by scholars and issued in new editions, preserving them for posterity.

3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false, since the variables in these sentences have not been assigned values. Various ways to form propositions from sentences of this type will be discussed in Section 1.3. ◀

Letters are used to denote propositions, just as letters are used to denote variables. The conventional letters used for this purpose are $p, q, r, s, \ldots$. The **truth value** of a proposition is true, denoted by T, if it is a true proposition and false, denoted by F, if it is a false proposition.

The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic.** It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.

Links

We now turn our attention to methods for producing new propositions from those that we already have. These methods were discussed by the English mathematician George Boole in 1854 in his book *The Laws of Thought.* Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions,** are formed from existing propositions using logical operators.

DEFINITION 1 Let p be a proposition. The statement

"It is not the case that p"

is another proposition, called the ***negation*** of p. The negation of p is denoted by $\neg p$. The proposition $\neg p$ is read "not p."

EXAMPLE 3 Find the negation of the proposition

"Today is Friday."

and express this in simple English.

Extra Examples

Solution: The negation is

"It is not the case that today is Friday."

This negation can be more simply expressed by

"Today is not Friday,"

or

"It is not Friday today." ◀

Remark: Strictly speaking, sentences involving variable times such as those in Example 3 are not propositions unless a fixed time is assumed. The same holds for variable places unless a fixed place is assumed and for pronouns unless a particular person is assumed.

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

A **truth table** displays the relationships between the truth values of propositions. Truth tables are especially valuable in the determination of the truth values of propositions constructed from simpler propositions. Table 1 displays the two possible truth values of a proposition p and the corresponding truth values of its negation $\neg p$.

The negation of a proposition can also be considered the result of the operation of the **negation operator** on a proposition. The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called **connectives.**

DEFINITION 2 Let p and q be propositions. The proposition "p and q," denoted $p \wedge q$, is the proposition that is true when both p and q are true and is false otherwise. The proposition $p \wedge q$ is called the *conjunction* of p and q.

The truth table for $p \wedge q$ is shown in Table 2. Note that there are four rows in this truth table, one row for each possible combination of truth values for the propositions p and q.

EXAMPLE 4 Find the conjunction of the propositions p and q where p is the proposition "Today is Friday" and q is the proposition "It is raining today."

Solution: The conjunction of these propositions, $p \wedge q$, is the proposition "Today is Friday and it is raining today." This proposition is true on rainy Fridays and is false on any day that is not a Friday and on Fridays when it does not rain. ◀

DEFINITION 3 Let p and q be propositions. The proposition "p or q," denoted $p \vee q$, is the proposition that is false when p and q are both false and true otherwise. The proposition $p \vee q$ is called the *disjunction* of p and q.

The truth table for $p \vee q$ is shown in Table 3.

The use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, namely, in an inclusive way. A disjunction is true when at least one of the two propositions is true. For instance, the inclusive or is being used in the statement

"Students who have taken calculus or computer science can take this class."

Links

GEORGE BOOLE (1815–1864) George Boole, the son of a cobbler, was born in Lincoln, England, in November 1815. Because of his family's difficult financial situation, Boole had to struggle to educate himself while supporting his family. Nevertheless, he became one of the most important mathematicians of the 1800s. Although he considered a career as a clergyman, he decided instead to go into teaching and soon afterward opened a school of his own. In his preparation for teaching mathematics, Boole—unsatisfied with textbooks of his day—decided to read the works of the great mathematicians. While reading papers of the great French mathematician Lagrange, Boole made discoveries in the calculus of variations, the branch of analysis dealing with finding curves and surfaces optimizing certain parameters.

In 1848 Boole published *The Mathematical Analysis of Logic,* the first of his contributions to symbolic logic. In 1849 he was appointed professor of mathematics at Queen's College in Cork, Ireland. In 1854 he published *The Laws of Thought,* his most famous work. In this book Boole introduced what is now called *Boolean algebra* in his honor. Boole wrote textbooks on differential equations and on difference equations that were used in Great Britain until the end of the nineteenth century. Boole married in 1855; his wife was the niece of the professor of Greek at Queen's College. In 1864 Boole died from pneumonia, which he contracted as a result of keeping a lecture engagement even though he was soaking wet from a rainstorm.

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Here, we mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects. On the other hand, we are using the exclusive or when we say

> "Students who have taken calculus or computer science, but not both, can enroll in this class."

Here, we mean that students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.

Similarly, when a menu at a restaurant states, "Soup or salad comes with an entrée," the restaurant almost always means that customers can have either soup or salad, but not both. Hence, this is an exclusive, rather than an inclusive, or.

EXAMPLE 5 What is the disjunction of the propositions p and q where p and q are the same propositions as in Example 4?

Solution: The disjunction of p and q, $p \vee q$, is the proposition

> "Today is Friday or it is raining today."

This proposition is true on any day that is either a Friday or a rainy day (including rainy Fridays). It is only false on days that are not Fridays when it also does not rain. ◀

Extra Examples

As was previously remarked, the use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, namely, in an inclusive way. Thus, a disjunction is true when at least one of the two propositions in it is true. Sometimes, we use *or* in an exclusive sense. When the exclusive or is used to connect the propositions p and q, the proposition "p or q (but not both)" is obtained. This proposition is true when p is true and q is false, and when p is false and q is true. It is false when both p and q are false and when both are true.

DEFINITION 4 Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

The truth table for the exclusive or of two propositions is displayed in Table 4.

TABLE 4 The Truth Table for the Exclnsive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5 The Truth Table for the Implication $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

IMPLICATIONS

We will discuss several other important ways in which propositions can be combined.

DEFINITION 5 Let p and q be propositions. The *implication* $p \rightarrow q$ is the proposition that is false when p is true and q is false, and true otherwise. In this implication p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

Assessment

The truth table for the implication $p \rightarrow q$ is shown in Table 5. An implication is sometimes called a **conditional statement.**

Because implications play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \rightarrow q$. You will encounter most if not all of the following ways to express this implication:

Extra Examples

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"

The implication $p \rightarrow q$ is false only in the case that p is true, but q is false. It is true when both p and q are true, and when p is false (no matter what truth value q has).

A useful way to understand the truth value of an implication is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is:

"If I am elected, then I will lower taxes."

If the politician is elected, voters would expect this politician to lower taxes. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes. It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true, but q is false in $p \rightarrow q$.

Similarly, consider a statement that a professor might make:

"If you get 100% on the final, then you will get an A."

If you manage to get a 100% on the final, then you would expect to receive an A. If you do not get 100% you may or may not receive an A depending on other factors. However, if you do get 100%, but the professor does not give you an A, you will feel cheated.

Many people find it confusing that "p only if q" expresses the same thing as "if p then q." To remember this, note that "p only if q" says that p cannot be true when q is not true. That is, the statement is false if p is true, but q is false. When p is false, q may be either true or false, because the statement says nothing about the truth value of q. A common error is for people to think that "q only if p" is a way of expressing $p \rightarrow q$. However, these statements have different truth values when p and q have different truth values.

The way we have defined implications is more general than the meaning attached to implications in the English language. For instance, the implication

"If it is sunny today, then we will go to the beach."

is an implication used in normal language, since there is a relationship between the hypothesis and the conclusion. Further, this implication is considered valid unless it is indeed sunny today, but we do not go to the beach. On the other hand, the implication

"If today is Friday, then $2 + 3 = 5$."

is true from the definition of implication, since its conclusion is true. (The truth value of the hypothesis does not matter then.) The implication

"If today is Friday, then $2 + 3 = 6$."

is true every day except Friday, even though $2 + 3 = 6$ is false.

We would not use these last two implications in natural language (except perhaps in sarcasm), since there is no relationship between the hypothesis and the conclusion in either implication. In mathematical reasoning we consider implications of a more general sort than we use in English. The mathematical concept of an implication is independent of a cause-and-effect relationship between hypothesis and conclusion. Our definition of an implication specifies its truth values; it is not based on English usage.

The if-then construction used in many programming languages is different from that used in logic. Most programming languages contain statements such as **if** p **then** S, where p is a proposition and S is a program segment (one or more statements to be executed). When execution of a program encounters such a statement, S is executed if p is true, but S is not executed if p is false, as illustrated in Example 6.

EXAMPLE 6 What is the value of the variable x after the statement

if $2 + 2 = 4$ **then** $x := x + 1$

if $x = 0$ before this statement is encountered? (The symbol $:=$ stands for assignment. The statement $x := x + 1$ means the assignment of the value of $x + 1$ to x.)

Solution: Since $2 + 2 = 4$ is true, the assignment statement $x := x + 1$ is executed. Hence, x has the value $0 + 1 = 1$ after this statement is encountered. ◀

CONVERSE, CONTRAPOSITIVE, AND INVERSE There are some related implications that can be formed from $p \rightarrow q$. The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$. The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$. The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$.

The contrapositive, $\neg q \rightarrow \neg p$, of an implication $p \rightarrow q$ has the same truth value as $p \rightarrow q$. To see this, note that the contrapositive is false only when $\neg p$ is false and $\neg q$ is true, that is, only when p is true and q is false. On the other hand, neither the converse, $q \rightarrow p$, nor the inverse, $\neg p \rightarrow \neg q$, has the same truth value as $p \rightarrow q$ for all possible truth values of p and q. To see this, note that when p is true and q is false, the original implication is false, but the converse and the inverse are both true. When two compound propositions always have the same truth value we call them **equivalent**, so that an implication and its contrapositive are equivalent. The converse and the inverse of an implication are also equivalent, as the reader can verify. (We will study equivalent propositions in Section 1.2.) One of the most common logical errors is to assume that the converse or the inverse of an implication is equivalent to this implication.

We illustrate the use of implications in Example 7.

EXAMPLE 7 What are the contrapositive, the converse, and the inverse of the implication

Extra Examples

"The home team wins whenever it is raining."?

Solution: Because "q whenever p" is one of the ways to express the implication $p \rightarrow q$, the original statement can be rewritten as

"If it is raining, then the home team wins."

Consequently, the contrapositive of this implication is

"If the home team does not win, then it is not raining."

The converse is

"If the home team wins, then it is raining."

The inverse is

"If it is not raining, then the home team does not win."

Only the contrapositive is equivalent to the original statement. ◀

We now introduce another way to combine propositions.

DEFINITION 6 Let p and q be propositions. The *biconditional* $p \leftrightarrow q$ is the proposition that is true when p and q have the same truth values, and is false otherwise.

The truth table for $p \leftrightarrow q$ is shown in Table 6. Note that the biconditional $p \leftrightarrow q$ is true precisely when both the implications $p \rightarrow q$ and $q \rightarrow p$ are true. Because of this, the terminology

"p if and only if q"

is used for this biconditional and it is symbolically written by combining the symbols $\rightarrow$ and $\leftarrow$. There are some other common ways to express $p \leftrightarrow q$:

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

"p is necessary and sufficient for q"
"if p then q, and conversely"
"p iff q".

The last way of expressing the biconditional uses the abbreviation "iff" for "if and only if." Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

EXAMPLE 8 Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

Extra Examples

"You can take the flight if and only if you buy a ticket."

This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket and cannot take the flight (such as when the airline bumps you). ◀

The "if and only if" construction used in biconditionals is rarely used in common language. Instead, biconditionals are often expressed using an "if, then" or an "only if" construction. The other part of the "if and only if" is implicit. For example, consider the statement in English "If you finish your meal, then you can have dessert." What is really meant is "You can have dessert if and only if you finish your meal." This last statement is logically equivalent to the two statements "If you finish your meal, then you can have dessert" and "You can have dessert, only if you finish your meal." Because of this imprecision in natural language, we need to make an assumption whether a conditional statement in natural language implicitly includes its converse. Because precision is essential in mathematics and in logic, we will always distinguish between the conditional statement $p \rightarrow q$ and the biconditional statement $p \leftrightarrow q$.

PRECEDENCE OF LOGICAL OPERATORS

We can construct compound propositions using the negation operator and the logical operators defined so far. We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. For instance, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$. However, to reduce the number of parentheses, we specify that the negation operator is applied before all other logical operators. This means that $\neg p \wedge q$ is the conjunction of $\neg p$ and q, namely, $(\neg p) \wedge q$, not the negation of the conjunction of p and q, namely $\neg(p \wedge q)$.

TABLE 7 Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Another general rule of precedence is that the conjunction operator takes precedence over the disjunction operator, so that $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$. Because this rule may be difficult to remember, we will continue to use parentheses so that the order of the disjunction and conjunction operators is clear.

Finally, it is an accepted rule that the conditional and biconditional operators $\rightarrow$ and $\leftrightarrow$ have lower precedence than the conjunction and disjunction operators, $\wedge$ and $\vee$. Consequently, $p \vee q \rightarrow r$ is the same as $(p \vee q) \rightarrow r$. We will use parentheses when the order of the conditional operator and biconditional operator is at issue, although the conditional operator has precedence over the biconditional operator. Table 7 displays the precedence levels of the logical operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.

TRANSLATING ENGLISH SENTENCES

There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives. In particular, English (and every other human language) is often ambiguous. Translating sentences into logical expressions removes the ambiguity. Note that this may involve making a set of reasonable assumptions based on the intended meaning of the sentence. Moreover, once we have translated sentences from English into logical expressions we can analyze these logical expressions to determine their truth values, we can manipulate them, and we can use rules of inference (which are discussed in Section 1.5) to reason about them.

Extra Examples

To illustrate the process of translating an English sentence into a logical expression, consider Examples 9 and 10.

EXAMPLE 9 How can this English sentence be translated into a logical expression?

> "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: There are many ways to translate this sentence into a logical expression. Although it is possible to represent the sentence by a single propositional variable, such as p, this would not be useful when analyzing its meaning or reasoning with it. Instead, we will use propositional variables to represent each sentence part and determine the appropriate logical connectives between them. In particular, we let a, c, and f represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman," respectively. Noting that "only if" is one way an implication can be expressed, this sentence can be represented as

$$a \rightarrow (c \vee \neg f).$$ ◀

EXAMPLE 10 How can this English sentence be translated into a logical expression?

> "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution: There are many ways to translate this sentence into a logical expression. The simplest but least useful way is simply to represent the sentence by a single propositional variable, say, p. Although this is not wrong, doing this would not assist us when we try to analyze the sentence or reason using it. More appropriately, what we can do is to use propositional variables to represent each of the sentence parts and to decide on the appropriate logical connectives between them. In particular, we let q, r, and s represent

"You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old," respectively. Then the sentence can be translated to

$$(r \wedge \neg s) \rightarrow \neg q.$$

Of course, there are other ways to represent the original sentence as a logical expression, but the one we have used should meet our needs. ◀

SYSTEM SPECIFICATIONS

Translating sentences in natural language (such as English) into logical expressions is an essential part of specifying both hardware and software systems. System and software engineers take requirements in natural language and produce precise and unambiguous specifications that can be used as the basis for system development. Example 11 shows how propositional expressions can be used in this process.

EXAMPLE 11 Express the specification "The automated reply cannot be sent when the file system is full" using logical connectives.

Extra Examples

Solution: One way to translate this is to let p denote "The automated reply can be sent" and q denote "The file system is full." Then $\neg p$ represents "It is not the case that the automated reply can be sent," which can also be expressed as "The automated reply cannot be sent." Consequently, our specification can be represented by the implication $q \rightarrow \neg p$. ◀

System specifications should not contain conflicting requirements. If they did there would be no way to develop a system that satisfies all specifications. Consequently, propositional expressions representing these specifications need to be **consistent.** That is, there must be an assignment of truth values to the variables in the expressions that makes all the expressions true.

EXAMPLE 12 Determine whether these system specifications are consistent:

"The diagnostic message is stored in the buffer or it is retransmitted."
"The diagnostic message is not stored in the buffer."
"If the diagnostic message is stored in the buffer, then it is retransmitted."

Extra Examples

Solution: To determine whether these specifications are consistent, we first express them using logical expressions. Let p denote "The diagnostic message is stored in the buffer" and let q denote "The diagnostic message is retransmitted." The specifications can then be written as $p \vee q$, $\neg p$, and $p \rightarrow q$. An assignment of truth values that makes all three specifications true must have p false to make $\neg p$ true. Since we want $p \vee q$ to be true but p must be false, q must be true. Because $p \rightarrow q$ is true when p is false and q is true, we conclude that these specifications are consistent since they are all true when p is false and q is true. We could come to the same conclusion by use of a truth table to examine the four possible assignments of truth values to p and q. ◀

EXAMPLE 13 Do the system specifications in Example 12 remain consistent if the specification "The diagnostic message is not retransmitted" is added?

Solution: By the reasoning in Example 12, the three specifications from that example are true only in the case when p is false and q is true. However, this new specification is $\neg q$, which is false when q is true. Consequently, these four specifications are inconsistent. ◀

BOOLEAN SEARCHES

Links

Logical connectives are used extensively in searches of large collections of information, such as indexes of Web pages. Because these searches employ techniques from propositional logic, they are called **Boolean searches.**

Extra Examples

In Boolean searches, the connective *AND* is used to match records that contain both of two search terms, the connective *OR* is used to match one or both of two search terms, and the connective *NOT* (sometimes written as *AND NOT*) is used to exclude a particular search term. Careful planning of how logical connectives are used is often required when Boolean searches are used to locate information of potential interest. Example 14 illustrates how Boolean searches are carried out.

EXAMPLE 14 **Web Page Searching.** Most Web search engines support Boolean searching techniques, which usually can help find Web pages about particular subjects. For instance, using Boolean searching to find Web pages about universities in New Mexico, we can look for pages matching NEW *AND* MEXICO *AND* UNIVERSITIES. The results of this search will include those pages that contain the three words NEW, MEXICO, and UNIVERSITIES. This will include all of the pages of interest, together with others such as a page about new universities in Mexico. Next, to find pages that deal with universities in New Mexico or Arizona, we can search for pages matching (NEW *AND* MEXICO *OR* ARIZONA) *AND* UNIVERSITIES. (*Note:* Here the *AND* operator takes precedence over the *OR* operator.) The results of this search will include all pages that contain the word UNIVERSITIES and either both the words NEW and MEXICO or the word ARIZONA. Again, pages besides those of interest will be listed. Finally, to find Web pages that deal with universities in Mexico (and not New Mexico), we might first look for pages matching MEXICO *AND* UNIVERSITIES, but since the results of this search will include pages about universities in New Mexico, as well as universities in Mexico, it might be better to search for pages matching (MEXICO *AND* UNIVERSITIES) *NOT* NEW. The results of this search include pages that contain both the words MEXICO and UNIVERSITIES but do not contain the word NEW. ◀

LOGIC PUZZLES

Links

Puzzles that can be solved using logical reasoning are known as **logic puzzles.** Solving logic puzzles is an excellent way to practice working with the rules of logic. Also, computer programs designed to carry out logical reasoning often use well-known logic puzzles to illustrate their capabilities. Many people enjoy solving logic puzzles, which are published in books and periodicals as a recreational activity.

We will discuss two logic puzzles here. We begin with a puzzle that was originally posed by Raymond Smullyan, a master of logic puzzles, who has published more than a dozen books containing challenging puzzles that involve logical reasoning.

EXAMPLE 15 In [Sm78] Smullyan posed many puzzles about an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if A says "B is a knight" and B says "The two of us are opposite types"?

Extra Examples

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively, so that $\neg p$ and $\neg q$ are the statements that A is a knave and that B is a knave, respectively.

We first consider the possibility that A is a knight; this is the statement that p is true. If A is a knight, then he is telling the truth when he says that B is a knight, so that q is true, and A and B are the same type. However, if B is a knight, then B's statement that A and B are of opposite types, the statement $(p \wedge \neg q) \vee (\neg p \wedge q)$, would have to be true, which it is not, because A and B are both knights. Consequently, we can conclude that A is not a knight, that is, that p is false.

If A is a knave, then because everything a knave says is false, A's statement that B is a knight, that is, that q is true, is a lie, which means that q is false and B is also a knave. Furthermore, if B is a knave, then B's statement that A and B are opposite types is a lie, which is consistent with both A and B being knaves. We can conclude that both A and B are knaves. ◀

We pose more of Smullyan's puzzles about knights and knaves in Exercises 51–55 at the end of this section. Next, we pose a puzzle known as the **muddy children puzzle** for the case of two children.

EXAMPLE 16 A father tells his two children, a boy and a girl, to play in their backyard without getting dirty. However, while playing, both children get mud on their foreheads. When the children stop playing, the father says "At least one of you has a muddy forehead," and then asks the children to answer "Yes" or "No" to the question: "Do you know whether you have a muddy forehead?" The father asks this question twice. What will the children answer each time this question is asked, assuming that a child can see whether his or her sibling has a muddy forehead, but cannot see his or her own forehead? Assume that both children are honest and that the children answer each question simultaneously.

Solution: Let s be the statement that the son has a muddy forehead and let d be the statement that the daughter has a muddy forehead. When the father says that at least

Links

RAYMOND SMULLYAN (BORN 1919) Raymond Smullyan dropped out of high school. He wanted to study what he was really interested in and not standard high school material. After jumping from one university to the next, he earned an undergraduate degree in mathematics at the University of Chicago in 1955. He paid his college expenses by performing magic tricks at parties and clubs. He obtained a Ph.D. in logic in 1959 at Princeton, studying under Alonzo Church. After graduating from Princeton, he taught mathematics and logic at Dartmouth College, Princeton University, Yeshiva University, and the City University of New York. He joined the philosophy department at Indiana University in 1981 where he is now an emeritus professor.

Smullyan has written many books on recreational logic and mathematics, including *Satan, Cantor, and Infinity; What Is the Name of This Book?; The Lady or the Tiger?; Alice in Puzzleland; To Mock a Mockingbird; Forever Undecided;* and *The Riddle of Scheherazade: Amazing Logic Puzzles, Ancient and Modern.* Because his logic puzzles are challenging, entertaining, and thought-provoking, he is considered to be a modern-day Lewis Carroll. Smullyan has also written several books about the application of deductive logic to chess, three collections of philosophical essays and aphorisms, and several advanced books on mathematical logic and set theory. He is particularly interested in self-referenee and has worked on extending some of Godel's results that show that it is impossible to write a computer program that can solve all mathematical problems. He is also particularly interested in explaining ideas from mathematical logic to the public.

Smullyan is a talented musician and often plays piano with his wife, who is a concert-level pianist. Making telescopes is one of his hobbies. He is also interested in optics and stereo photography. He states "I've never had a conflict between teaching and research as some people do because when I'm teaching, I'm doing research."

TABLE 8 Table for the Bit Operators *OR*, *AND*, and *XOR*.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

one of the two children has a muddy forehead he is stating that the disjunction $s \vee d$ is true. Both children will answer "No" the first time the question is asked because each sees mud on the other child's forehead. That is, the son knows that d is true, but does not know whether s is true, and the daughter knows that s is true, but does not know whether d is true.

After the son has answered "No" to the first question, the daughter can determine that d must be true. This follows because when the first question is asked, the son knows that $s \vee d$ is true, but cannot determine whether s is true. Using this information, the daughter can conclude that d must be true, for if d were false, the son could have reasoned that because $s \vee d$ is true, then s must be true, and he would have answered "Yes" to the first question. The son can reason in a similar way to determine that s must be true. It follows that both children answer "Yes" the second time the question is asked. ◀

LOGIC AND BIT OPERATIONS

Truth Value	*Bit*
T	1
F	0

Links

Computers represent information using bits. A **bit** has two possible values, namely, 0 (zero) and 1 (one). This meaning of the word bit comes from *b*inary dig*it*, since zeros and ones are the digits used in binary representations of numbers. The well-known statistician John Tukey introduced this terminology in 1946. A bit can be used to represent a truth value, since there are two truth values, namely, *true* and *false*. As is customarily done, we will use a 1 bit to represent true and a 0 bit to represent false. That is, 1 represents T (true), 0 represents F (false). A variable is called a **Boolean variable** if its value is either true or false. Consequently, a Boolean variable can be represented using a bit.

Computer **bit operations** correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators $\wedge$, $\vee$, and $\oplus$, the tables shown in Table 8 for the corresponding bit operations are obtained. We will also use the notation *OR*, *AND*, and *XOR* for the operators $\vee$, $\wedge$, and $\oplus$, as is done in various programming languages.

Information is often represented using bit strings, which are sequences of zeros and ones. When this is done, operations on the bit strings can be used to manipulate this information.

DEFINITION 7 A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

EXAMPLE 17 101010011 is a bit string of length nine. ◀

We can extend bit operations to bit strings. We define the **bitwise *OR*, bitwise *AND*,** and **bitwise *XOR*** of two strings of the same length to be the strings that have as their bits the *OR, AND,* and *XOR* of the corresponding bits in the two strings, respectively. We use the symbols $\vee$, $\wedge$, and $\oplus$ to represent the bitwise *OR*, bitwise *AND*, and bitwise *XOR* operations, respectively. We illustrate bitwise operations on bit strings with Example 18.

EXAMPLE 18 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

Solution: The bitwise *OR*, bitwise *AND*, and bitwise *XOR* of these strings are obtained by taking the *OR, AND,* and *XOR* of the corresponding bits, respectively. This gives us

$$\begin{array}{ll} 01\ 1011\ 0110 & \\ 11\ 0001\ 1101 & \\ \hline 11\ 1011\ 1111 & \text{bitwise } OR \\ 01\ 0001\ 0100 & \text{bitwise } AND \\ 10\ 1010\ 1011 & \text{bitwise } XOR \end{array}$$

◀

Exercises

1. Which of these sentences are propositions? What are the truth values of those that are propositions?
   a) Boston is the capital of Massachusetts.
   b) Miami is the capital of Florida.
   c) $2 + 3 = 5$.
   d) $5 + 7 = 10$.
   e) $x + 2 = 11$.
   f) Answer this question.
   g) $x + y = y + x$ for every pair of real numbers x and y.

2. Which of these are propositions? What are the truth values of those that are propositions?
   a) Do not pass go.
   b) What time is it?
   c) There are no black flies in Maine.
   d) $4 + x = 5$.
   e) $x + 1 = 5$ if $x = 1$.
   f) $x + y = y + z$ if $x = z$.

Links

JOHN WILDER TUKEY (1915–2000) Tukey, born in New Bedford, Massachusetts, was an only child. His parents, both teachers, decided home schooling would best develop his potential. His formal education began at Brown University, where he studied mathematics and chemistry. He received a master's degree in chemistry from Brown and continued his studies at Princeton University, changing his field of study from chemistry to mathematics. He received his Ph.D. from Princeton in 1939 for work in topology, when he was appointed an instructor in mathematics at Princeton. With the start of World War II, he joined the Fire Control Research Office, where he began working in statistics. Tukey found statistical research to his liking and impressed several leading statisticians with his skills. In 1945, at the conclusion of the war, Tukey returned to the mathematics department at Princeton as a professor of statistics, and he also took a position at AT&T Bell Laboratories. Tukey founded the Statistics Department at Princeton in 1966 and was its first chairman. Tukey made significant contributions to many areas of statistics, including the analysis of variance, the estimation of spectra of time series, inferences about the values of a set of parameters from a single experiment, and the philosophy of statistics. However, he is best known for his invention, with J. W. Cooley, of the fast Fourier transform.

Tukey contributed his insight and expertise by serving on the President's Science Advisory Committee. He chaired several important committees dealing with the environment, education, and chemicals and health. He also served on committees working on nuclear disarmament. Tukey received many awards, including the National Medal of Science.

HISTORICAL NOTE There were several other suggested words for a binary digit, including *binit* and *bigit*, that never were widely accepted. The adoption of the word *bit* may be due to its meaning as a common English word. For an account of Tukey's coining of the word *bit*, see the April 1984 issue of *Annals of the History of Computing*.

3. What is the negation of each of these propositions?

a) Today is Thursday.
b) There is no pollution in New Jersey.
c) $2 + 1 = 3$.
d) The summer in Maine is hot and sunny.

4. Let p and q be the propositions

p : I bought a lottery ticket this week.
q : I won the million dollar jackpot on Friday.

Express each of these propositions as an English sentence.

a) $\neg p$ **b)** $p \vee q$ **c)** $p \to q$
d) $p \wedge q$ **e)** $p \leftrightarrow q$ **f)** $\neg p \to \neg q$
g) $\neg p \wedge \neg q$ **h)** $\neg p \vee (p \wedge q)$

5. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.

a) $\neg q$ **b)** $p \wedge q$ **c)** $\neg p \vee q$
d) $p \to \neg q$ **e)** $\neg q \to p$ **f)** $\neg p \to \neg q$
g) $p \leftrightarrow \neg q$ **h)** $\neg p \wedge (p \vee \neg q)$

6. Let p and q be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of these compound propositions as an English sentence.

a) $\neg p$ **b)** $p \vee q$ **c)** $\neg p \wedge q$
d) $q \to p$ **e)** $\neg q \to \neg p$ **f)** $\neg p \to \neg q$
g) $p \leftrightarrow q$ **h)** $\neg q \vee (\neg p \wedge q)$

7. Let p and q be the propositions

p : It is below freezing.
q : It is snowing.

Write these propositions using p and q and logical connectives.

a) It is below freezing and snowing.
b) It is below freezing but not snowing.
c) It is not below freezing and it is not snowing.
d) It is either snowing or below freezing (or both).
e) If it is below freezing, it is also snowing.
f) It is either below freezing or it is snowing, but it is not snowing if it is below freezing.
g) That it is below freezing is necessary and sufficient for it to be snowing.

8. Let p, q, and r be the propositions

p : You have the flu.
q : You miss the final examination.
r : You pass the course.

Express each of these propositions as an English sentence.

a) $p \to q$ **b)** $\neg q \leftrightarrow r$
c) $q \to \neg r$ **d)** $p \vee q \vee r$
e) $(p \to \neg r) \vee (q \to \neg r)$
f) $(p \wedge q) \vee (\neg q \wedge r)$

9. Let p and q be the propositions

p : You drive over 65 miles per hour.
q : You get a speeding ticket.

Write these propositions using p and q and logical connectives.

a) You do not drive over 65 miles per hour.
b) You drive over 65 miles per hour, but you do not get a speeding ticket.
c) You will get a speeding ticket if you drive over 65 miles per hour.
d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.
f) You get a speeding ticket, but you do not drive over 65 miles per hour.
g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

10. Let p, q, and r be the propositions

p : You get an A on the final exam.
q : You do every exercise in this book.
r : You get an A in this class.

Write these propositions using p, q, and r and logical connectives.

a) You get an A in this class, but you do not do every exercise in this book.
b) You get an A on the final, you do every exercise in this book, and you get an A in this class.
c) To get an A in this class, it is necessary for you to get an A on the final.
d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.
e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

11. Let p, q, and r be the propositions

p : Grizzly bears have been seen in the area.
q : Hiking is safe on the trail.
r : Berries are ripe along the trail.

Write these propositions using p, q, and r and logical connectives.

a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.
b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.
c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.
e) For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.
f) Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

12. Determine whether these biconditionals are true or false.
a) $2+2=4$ if and only if $1+1=2$.
b) $1+1=2$ if and only if $2+3=4$.
c) It is winter if and only if it is not spring, summer, or fall.
d) $1+1=3$ if and only if pigs can fly.
e) $0>1$ if and only if $2>1$.

13. Determine whether each of these implications is true or false.
a) If $1+1=2$, then $2+2=5$.
b) If $1+1=3$, then $2+2=4$.
c) If $1+1=3$, then $2+2=5$.
d) If pigs can fly, then $1+1=3$.
e) If $1+1=3$, then God exists.
f) If $1+1=3$, then pigs can fly.
g) If $1+1=2$, then pigs can fly.
h) If $2+2=4$, then $1+2=3$.

14. For each of these sentences, determine whether an inclusive or an exclusive or is intended. Explain your answer.
a) Experience with C++ or Java is required.
b) Lunch includes soup or salad.
c) To enter the country you need a passport or a voter registration card.
d) Publish or perish.

15. For each of these sentences, state what the sentence means if the or is an inclusive or (that is, a disjunction) versus an exclusive or. Which of these meanings of or do you think is intended?
a) To take discrete mathematics, you must have taken calculus or a course in computer science.
b) When you buy a new car from Acme Motor Company, you get \$2000 back in cash or a 2% car loan.
c) Dinner for two includes two items from column A or three items from column B.
d) School is closed if more than 2 feet of snow falls or if the wind chill is below -100.

16. Write each of these statements in the form "if p, then q" in English. (*Hint:* Refer to the list of common ways to express implications provided in this section.)
a) It is necessary to wash the boss's car to get promoted.
b) Winds from the south imply a spring thaw.
c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.
d) Willy gets caught whenever he cheats.
e) You can access the website only if you pay a subscription fee.
f) Getting elected follows from knowing the right people.
g) Carol gets seasick whenever she is on a boat.

17. Write each of these statements in the form "if p, then q" in English. (*Hint:* Refer to the list of common ways to express implications provided in this section.)
a) It snows whenever the wind blows from the northeast.
b) The apple trees will bloom if it stays warm for a week.
c) That the Pistons win the championship implies that they beat the Lakers.
d) It is necessary to walk 8 miles to get to the top of Long's Peak.
e) To get tenure as a professor, it is sufficient to be world-famous.
f) If you drive more than 400 miles, you will need to buy gasoline.
g) Your guarantee is good only if you bought your CD player less than 90 days ago.

18. Write each of these statements in the form "if p, then q" in English. (*Hint:* Refer to the list of common ways to express implications provided in this section.)
a) I will remember to send you the address only if you send me an e-mail message.
b) To be a citizen of this country, it is sufficient that you were born in the United States.
c) If you keep your textbook, it will be a useful reference in your future courses.
d) The Red Wings will win the Stanley Cup if their goalie plays well.
e) That you get the job implies that you had the best credentials.
f) The beach erodes whenever there is a storm.
g) It is necessary to have a valid password to log on to the server.

19. Write each of these propositions in the form "p if and only if q" in English.
a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.
b) For you to win the contest it is necessary and sufficient that you have the only winning ticket.
c) You get promoted only if you have connections, and you have connections only if you get promoted.
d) If you watch television your mind will decay, and conversely.
e) The trains run late on exactly those days when I take it.

20. Write each of these propositions in the form "p if and only if q" in English.

a) For you to get an A in this course, it is necessary and sufficient that you learn how to solve discrete mathematics problems.

b) If you read the newspaper every day, you will be informed, and conversely.

c) It rains if it is a weekend day, and it is a weekend day if it rains.

d) You can see the wizard only if the wizard is not in, and the wizard is not in only if you can see him.

21. State the converse, contrapositive, and inverse of each of these implications.

a) If it snows today, I will ski tomorrow.

b) I come to class whenever there is going to be a quiz.

c) A positive integer is a prime only if it has no divisors other than 1 and itself.

22. State the converse, contrapositive, and inverse of each of these implications.

a) If it snows tonight, then I will stay at home.

b) I go to the beach whenever it is a sunny summer day.

c) When I stay up late, it is necessary that I sleep until noon.

23. Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$ **b)** $p \vee \neg p$
c) $(p \vee \neg q) \rightarrow q$ **d)** $(p \vee q) \rightarrow (p \wedge q)$
e) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
f) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

24. Construct a truth table for each of these compound propositions.

a) $p \rightarrow \neg p$ **b)** $p \leftrightarrow \neg p$
c) $p \oplus (p \vee q)$ **d)** $(p \wedge q) \rightarrow (p \vee q)$
e) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$
f) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

25. Construct a truth table for each of these compound propositions.

a) $(p \vee q) \rightarrow (p \oplus q)$
b) $(p \oplus q) \rightarrow (p \wedge q)$
c) $(p \vee q) \oplus (p \wedge q)$
d) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
e) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
f) $(p \oplus q) \rightarrow (p \oplus \neg q)$

26. Constrnct a truth table for each of these compound propositions.

a) $p \oplus p$ **b)** $p \oplus \neg p$
c) $p \oplus \neg q$ **d)** $\neg p \oplus \neg q$
e) $(p \oplus q) \vee (p \oplus \neg q)$
f) $(p \oplus q) \wedge (p \oplus \neg q)$

27. Construct a truth table for each of these compound propositions.

a) $p \rightarrow \neg q$ **b)** $\neg p \leftrightarrow q$
c) $(p \rightarrow q) \vee (\neg p \rightarrow q)$
d) $(p \rightarrow q) \wedge (\neg p \rightarrow q)$
e) $(p \leftrightarrow q) \vee (\neg p \leftrightarrow q)$
f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$

28. Construct a truth table for each of these compound propositions.

a) $(p \vee q) \vee r$ **b)** $(p \vee q) \wedge r$
c) $(p \wedge q) \vee r$ **d)** $(p \wedge q) \wedge r$
e) $(p \vee q) \wedge \neg r$ **f)** $(p \wedge q) \vee \neg r$

29. Construct a truth table for each of these compound propositions.

a) $p \rightarrow (\neg q \vee r)$
b) $\neg p \rightarrow (q \rightarrow r)$
c) $(p \rightarrow q) \vee (\neg p \rightarrow r)$
d) $(p \rightarrow q) \wedge (\neg p \rightarrow r)$
e) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$
f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

30. Construct a truth table for $((p \rightarrow q) \rightarrow r) \rightarrow s$.

31. Construct a truth table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$.

32. What is the value of x after each of these statements is encountered in a computer program, if $x = 1$ before the statement is reached?

a) **if** $1 + 2 = 3$ **then** $x := x + 1$
b) **if** $(1 + 1 = 3)$ OR $(2 + 2 = 3)$ **then** $x := x + 1$
c) **if** $(2 + 3 = 5)$ AND $(3 + 4 = 7)$ **then** $x := x + 1$
d) **if** $(1 + 1 = 2)$ XOR $(1 + 2 = 3)$ **then** $x := x + 1$
e) **if** $x < 2$ **then** $x := x + 1$

33. Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of each of these pairs of bit strings.

a) 101 1110, 010 0001
b) 1111 0000, 1010 1010
c) 00 0111 0001, 10 0100 1000
d) 11 1111 1111, 00 0000 0000

34. Evaluate each of these expressions.

a) $1\,1000 \wedge (0\,1011 \vee 1\,1011)$
b) $(0\,1111 \wedge 1\,0101) \vee 0\,1000$
c) $(0\,1010 \oplus 1\,1011) \oplus 0\,1000$
d) $(1\,1011 \vee 0\,1010) \wedge (1\,0001 \vee 1\,1011)$

Fuzzy logic is used in artificial intelligence. In fuzzy logic, a proposition has a truth value that is a number between 0 and 1, inclusive. A proposition with a truth value of 0 is false and one with a truth value of 1 is true. Truth values that are between 0 and 1 indicate varying degrees of truth. For instance, the truth value 0.8 can be assigned to the statement "Fred is happy," since Fred is happy most of the time, and the truth value 0.4 can be assigned to the statement "John is happy," since John is happy slightly less than half the time.

35. The truth value of the negation of a proposition in fuzzy logic is 1 minus the truth value of the proposition. What are the truth values of the statements "Fred is not happy" and "John is not happy"?

36. The truth value of the conjunction of two propositions in fuzzy logic is the minimum of the truth values of the two propositions. What are the truth values of the statements "Fred and John are happy" and "Neither Fred nor John is happy"?

37. The truth value of the disjunction of two propositions in fuzzy logic is the maximum of the truth values of the two propositions. What are the truth values of the statements "Fred is happy, or John is happy" and "Fred is not happy, or John is not happy"?

***38.** Is the assertion "This statement is false" a proposition?

***39.** The nth statement in a list of 100 statements is "Exactly n of the statements in this list are false."

a) What conclusions can you draw from these statements?

b) Answer part (a) if the nth statement is "At least n of the statements in this list are false."

c) Answer part (b) assuming that the list contains 99 statements.

40. An ancient Sicilian legend says that the barber in a remote town who can be reached only by traveling a dangerous mountain road shaves those people, and only those people, who do not shave themselves. Can there be such a barber?

41. Each inhabitant of a remote village always tells the truth or always lies. A villager will only give a "Yes" or a "No" response to a question a tourist asks. Suppose you are a tourist visiting this area and come to a fork in the road. One branch leads to the ruins you want to visit; the other branch leads deep into the jungle. A villager is standing at the fork in the road. What one question can you ask the villager to determine which branch to take?

42. An explorer is captured by a group of cannibals. There are two types of cannibals—those who always tell the truth and those who always lie. The cannibals will barbecue the explorer unless he can determine whether a particular cannibal always lies or always tells the truth. He is allowed to ask the cannibal exactly one question.

a) Explain why the question "Are you a liar?" does not work.

b) Find a question that the explorer can use to determine whether the cannibal always lies or always tells the truth.

43. Express these system specifications using the propositions p "The message is scanned for viruses" and q "The message was sent from an unknown system" together with logical connectives.

a) "The message is scanned for viruses whenever the message was sent from an unknown system."

b) "The message was sent from an unknown system but it was not scanned for viruses."

c) "It is necessary to scan the message for viruses whenever it was sent from an unknown system."

d) "When a message is not sent from an unknown system it is not scanned for viruses."

44. Express these system specifications using the propositions p "The user enters a valid password," q "Access is granted," and r "The user has paid the subscription fee" and logical connectives.

a) "The user has paid the subscription fee, but does not enter a valid password."

b) "Access is granted whenever the user has paid the subscription fee and enters a valid password."

c) "Access is denied if the user has not paid the subscription fee."

d) "If the user has not entered a valid password but has paid the subscription fee, then access is granted."

45. Are these system specifications consistent? "The system is in multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode."

46. Are these system specifications consistent? "Whenever the system software is being upgraded, users cannot access the file system. If users can access the file system, then they can save new files. If users cannot save new files, then the system software is not being upgraded."

47. Are these system specifications consistent? "The router can send packets to the edge system only if it supports the new address space. For the router to support the new address space it is necessary that the latest software release be installed. The router can send packets to the edge system if the latest software release is installed. The router does not support the new address space."

48. Are these system specifications consistent? "If the file system is not locked, then new messages will be queued. If the file system is not locked, then the system is functioning normally, and conversely. If new messages are not queued, then they will be sent to the message buffer. If the file system is not locked, then new messages will be sent to the message buffer. New messages will not be sent to the message buffer."

49. What Boolean search would you use to look for Web pages about beaches in New Jersey? What if you wanted to find Web pages about beaches on the isle of Jersey (in the English Channel)?

50. What Boolean search would you use to look for Web pages about hiking in West Virginia? What if you wanted to find Web pages about hiking in Virginia, but not in West Virginia?

Exercises 51–55 relate to inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people, A and B. Determine, if possible, what A and B are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusions?

51. A says "At least one of us is a knave" and B says nothing.

52. A says "The two of us are both knights" and B says "A is a knave."

53. A says "I am a knave or B is a knight" and B says nothing.

54. Both A and B say "I am a knight."

55. A says "We are both knaves" and B says nothing.

Exercises 56–61 are puzzles that can be solved by translating statements into logical expressions and reasoning from these expressions using truth tables.

56. The police have three suspects for the murder of Mr. Cooper: Mr. Smith, Mr. Jones, and Mr. Williams. Smith, Jones, and Williams each declare that they did not kill Cooper. Smith also states that Cooper was a friend of Jones and that Williams disliked him. Jones also states that he did not know Cooper and that he was out of town the day Cooper was killed. Williams also states that he saw both Smith and Jones with Cooper the day of the killing and that either Smith or Jones must have killed him. Can you determine who the murderer was if

a) one of the three men is guilty, the two innocent men are telling the truth, but the statements of the guilty man may or may not be true?

b) innocent men do not lie?

57. Steve would like to determine the relative salaries of three coworkers using two facts. First, he knows that if Fred is not the highest paid of the three, then Janice is. Second, he knows that if Janice is not the lowest paid, then Maggie is paid the most. Is it possible to determine the relative salaries of Fred, Maggie, and Janice from what Steve knows? If so, who is paid the most and who the least? Explain your reasoning.

58. Five friends have access to a chat room. Is it possible to determine who is chatting if the following information is known? Either Kevin or Heather, or both, are chatting. Either Randy or Vijay, but not both, are chatting. If Abby is chatting, so is Randy. Vijay and Kevin are either both chatting or neither is. If Heather is chatting, then so are Abby and Kevin. Explain your reasoning.

59. A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so is the cook; the cook and the gardener cannot both be telling the truth; the gardener and the handyman are not both lying; and if the handyman is telling the truth then the cook is lying. For each of the four witnesses, can the detective determine whether that person is telling the truth or lying? Explain your reasoning.

60. Four friends have been identified as suspects for an unauthorized access into a computer system. They have made statements to the investigating authorities. Alice said "Carlos did it." John said "I did not do it." Carlos said "Diana did it." Diana said "Carlos lied when he said that I did it."

a) If the authorities also know that exactly one of the four suspects is telling the truth, who did it? Explain your reasoning.

b) If the authorities also know that exactly one is lying, who did it? Explain your reasoning.

***61.** Solve this famous logic puzzle, attributed to Albert Einstein, and known as the **zebra puzzle.** Five men with different nationalities and with different jobs live in consecutive houses on a street. These houses are painted different colors. The men have different pets and have different favorite drinks. Determine who owns a zebra and whose favorite drink is mineral water (which is one of the favorite drinks) given these clues: The Englishman lives in the red house. The Spaniard owns a dog. The Japanese man is a painter. The Italian drinks tea. The Norwegian lives in the first house on the left. The green house is on the right of the white one. The photographer breeds snails. The diplomat lives in the yellow house. Milk is drunk in the middle house. The owner of the green house drinks coffee. The Norwegian's house is next to the blue one. The violinist drinks orange juice. The fox is in a house next to that of the physician. The horse is in a house next to that of the diplomat. (*Hint:* Make a table where the rows represent the men and columns represent the color of their houses, their jobs, their pets, and their favorite drinks and use logical reasoning to determine the correct entries in the table.)

1.2 Propositional Equivalences

INTRODUCTION

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments.

We begin our discussion with a classification of compound propositions according to their possible truth values.

DEFINITION 1 A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. Finally, a proposition that is neither a tautology nor a contradiction is called a *contingency*.

Tautologies and contradictions are often important in mathematical reasoning. The following example illustrates these types of propositions.

EXAMPLE 1 We can construct examples of tautologies and contradictions using just one proposition. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Since $p \vee \neg p$ is always true, it is a tautology. Since $p \wedge \neg p$ is always false, it is a contradiction. ◀

LOGICAL EQUIVALENCES

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Compound propositions that have the same truth values in all possible cases are called **logically equivalent.** We can also define this notion as follows.

DEFINITION 2 The propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark: The symbol $\equiv$ is not a logical connective since $p \equiv q$ is not a compound proposition, but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

Extra Examples

One way to determine whether two propositions are equivalent is to use a truth table. In particular, the propositions p and q are equivalent if and only if the columns giving their truth values agree. The following example illustrates this method.

EXAMPLE 2 Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent. This equivalence is one of *De Morgan's laws* for propositions, named after the English mathematician Augustus De Morgan, of the mid-nineteenth century.

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

TABLE 2 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

TABLE 3 Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Solution: The truth tables for these propositions are displayed in Table 2. Since the truth values of the propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology and that these propositions are logically equivalent. ◀

EXAMPLE 3 Show that the propositions $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Solution: We construct the truth table for these propositions in Table 3. Since the truth values of $\neg p \vee q$ and $p \rightarrow q$ agree, these propositions are logically equivalent. ◀

EXAMPLE 4 Show that the propositions $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

Solution: We construct the truth table for these propositions in Table 4. Since the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ agree, these propositions are logically equivalent. ◀

Remark: A truth table of a compound proposition involving three different propositions requires eight rows, one for each possible combination of truth values of the three propositions. In general, 2^n rows are required if a compound proposition involves n propositions.

Table 5 contains some important equivalences.* In these equivalences, **T** denotes any proposition that is always true and **F** denotes any proposition that is always false. We also display some useful equivalences for compound propositions involving implications and biconditionals in Tables 6 and 7, respectively. The reader is asked to verify the equivalences in Tables 5–7 in the exercises at the end of the section.

*These identities are a special case of identities that hold for any Boolean algebra. Compare them with set identities in Table 1 in Section 1.7 and with Boolean identities in Table 5 in Section 10.1.

TABLE 4 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

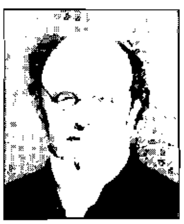
The associative law for disjunction shows that the expression $p \vee q \vee r$ is well defined, in the sense that it does not matter whether we first take the disjunction of p and q and then the disjunction of $p \vee q$ with r, or if we first take the disjunction of q and r and then take the disjunction of p and $q \vee r$. Similarly, the expression $p \wedge q \wedge r$ is well defined. By extending this reasoning, it follows that $p_1 \vee p_2 \vee \cdots \vee p_n$ and $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ are well defined whenever $p_1, p_2, \ldots, p_n$ are propositions. Furthermore, note that De Morgan's laws extend to

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$

and

$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n).$$

(Methods for proving these identities will be given in Section 3.3.)



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AUGUSTUS DE MORGAN (1806–1871) Augustus De Morgan was born in India, where his father was a colonel in the Indian army. De Morgan's family moved to England when he was 7 months old. He attended private schools, where he developed a strong interest in mathematics in his early teens. De Morgan studied at Trinity College, Cambridge, graduating in 1827. Although he considered entering medicine or law, he decided on a career in mathematics. He won a position at University College, London, in 1828, but resigned when the college dismissed a fellow professor without giving reasons. However, he resumed this position in 1836 wheu his successor died, staying there until 1866.

De Morgan was a noted teacher who stressed principles over techniques. His students included many famous mathematicians, ineluding Ada Augusta, Countess of Lovelace, who was Charles Babbage's collaborator in his work on computing machines (see page 25 for biographical notes on Ada Augusta). (De Morgan cautioned the countess against studying too much mathematies, sinee it might interfere with her childbearing abilities!)

De Morgan was an extremely prolific writer. He wrote more than 1000 artieles for more than 15 periodicals. De Morgan also wrote texthooks on many subjects, including logic, probability, ealeulus, and algebra. In 1838 he presented what was perhaps the first clear explanation of an important proof technique known as *mathematical induction* (discussed in Section 3.3 of this text), a term he coined. In the 1840s De Morgan made fundamental contributions to the development of symbolic logic. He invented notations that helped him prove propositional equivalences, such as the laws that are named after him. In 1842 De Morgan presented what was perhaps the first precise definition of a limit and developed some tests for convergence of infinite series. De Morgan was also interested in the history of mathematics and wrote biographies of Newton and Halley.

In 1837 De Morgan married Sophia Frend, who wrote his biography in 1882. De Morgan's research, writing, and teaching left little time for his family or social life. Nevertheless, he was noted for his kindness, humor, and wide range of knowledge.

TABLE 5 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 6 Logical Equivalences Involving Implications.

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 7 Logical Equivalences Involving Biconditionals.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Extra Examples

The logical equivalences in Table 5, as well as any others that have been established (such as those shown in Tables 6 and 7), can be used to construct additional logical equivalences. The reason for this is that a proposition in a compound proposition can be replaced by one that is logically equivalent to it without changing the truth value of the compound proposition. This technique is illustrated in Examples 5 and 6, where we also use the fact that if p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent (see Exercise 50).

EXAMPLE 5 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution: We could use a truth table to show that these compound propositions are equivalent. Instead, we will establish this equivalence by developing a series of logical equivalences, using one of the equivalences in Table 5 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$. We have the following equivalences.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{from the second De Morgan's law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{from the first De Morgan's law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{from the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{from the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{since } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{from the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{from the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent. ◀

EXAMPLE 6 Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to $\mathbf{T}$. (*Note:* This could also be done using a truth table.)

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan's law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative laws for disjunction} \\
&\equiv \mathbf{T} \vee \mathbf{T} && \text{by Example 1 and the commutative law for disjunction} \\
&\equiv \mathbf{T} && \text{by the domination law}
\end{aligned}
$$
◀

A truth table can be used to determine whether a compound proposition is a tautology. This can be done by hand for a proposition with a small number of variables, but when the number of variables grows, this becomes impractical. For instance, there are $2^{20} = 1{,}048{,}576$ rows in the truth value table for a proposition with 20 variables. Clearly, you need a computer to help you determine, in this way, whether a compound proposition in

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ADA AUGUSTA, COUNTESS OF LOVELACE (1815–1852) Ada Augusta was the only child from the marriage of the famous poet Lord Byron and Annabella Millbanke, who separated when Ada was 1 month old. She was raised by her mother, who encouraged her intellectual talents. She was taught by the mathematicians William Frend and Augustus De Morgan. In 1838 she married Lord King, later elevated to Earl of Lovelace. Together they had three children.

Ada Augusta continued her mathematical studies after her marriage, assisting Charles Babbage in his work on an early computing machine, called the Analytic Engine. The most complete accounts of this machine are found in her writings. After 1845 she and Babbage worked toward the development of a system to predict horse races. Unfortunately, their system did not work well, leaving Ada heavily in debt at the time of her death. The programming language Ada is uamed in honor of the Countess of Lovelace.

20 variables is a tautology. But when there are 1000 variables, can even a computer determine in a reasonable amount of time whether a compound proposition is a tautology? Checking every one of the 2^{1000} (a number with more than 300 decimal digits) possible combinations of truth values simply cannot be done by a computer in even trillions of years. Furthermore, no other procedures are known that a computer can follow to determine in a reasonable amount of time whether a compound proposition in such a large number of variables is a tautology. We will study questions such as this in Chapter 2, when we study the complexity of algorithms.

Exercises

1. Use truth tables to verify these equivalences.
   a) $p \wedge \mathbf{T} \equiv p$ **b)** $p \vee \mathbf{F} \equiv p$
   c) $p \wedge \mathbf{F} \equiv \mathbf{F}$ **d)** $p \vee \mathbf{T} \equiv \mathbf{T}$
   e) $p \vee p \equiv p$ **f)** $p \wedge p \equiv p$
2. Show that $\neg(\neg p)$ and p are logically equivalent.
3. Use truth tables to verify the commutative laws
   a) $p \vee q \equiv q \vee p$
   b) $p \wedge q \equiv q \wedge p$
4. Use truth tables to verify the associative laws
   a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
5. Use a trnth table to verify the distributive law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.
6. Use a truth table to verify the equivalence $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
7. Show that each of these implications is a tautology by using truth tables.
   a) $(p \wedge q) \rightarrow p$ **b)** $p \rightarrow (p \vee q)$
   c) $\neg p \rightarrow (p \rightarrow q)$ **d)** $(p \wedge q) \rightarrow (p \rightarrow q)$
   e) $\neg(p \rightarrow q) \rightarrow p$ **f)** $\neg(p \rightarrow q) \rightarrow \neg q$
8. Show that each of these implications is a tautology by using truth tables.
   a) $[\neg p \wedge (p \vee q)] \rightarrow q$
   b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
   c) $[p \wedge (p \rightarrow q)] \rightarrow q$
   d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$
9. Show that each implication in Exercise 7 is a tautology without using truth tables.
10. Show that each implication in Exercise 8 is a tautology without using truth tables.
11. Use truth tables to verify the absorption laws.
   a) $p \vee (p \wedge q) \equiv p$ **b)** $p \wedge (p \vee q) \equiv p$
12. Determine whether $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is a tautology.
13. Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.

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HENRY MAURICE SHEFFER (1883–1964) Henry Maurice Sheffer, born to Jewish parents in the western Ukraine, emigrated to the United States in 1892 with his parents and six siblings. He studied at the Boston Latin School before enteriug Harvard, where he completed his undergraduate degree in 1905, his master's in 1907, and his Ph.D. in philosophy in 1908. After holding a postdoctoral position at Harvard, Henry traveled to Europe on a fellowship. Upon returning to the United States, he became an academic nomad, spending one year each at the University of Washington, Cornell, the University of Minnesota, the University of Missouri, and City College in New York. In 1916 he returned to Harvard as a faculty member in the philosophy department. He remained at Harvard until his retirement in 1952.

Sheffer introduced what is now known as the Sheffer stroke in 1913; it became well known only after its use in the 1925 edition of Whitehead and Russell's *Principia Mathematica*. In this same edition Russell wrote that Sheffer had invented a powerful method that could be used to simplify the *Principia*. Because of this comment, Sheffer was something of a mystery man to logicians, especially because Sheffer, who published little in his career, never published the details of this method, only describing it in mimeographed notes and in a brief published abstract.

Sheffer was a dedicated teacher of mathematical logic. He liked his classes to be small and did not like auditors. When strangers appeared in his classroom, Sheffer would order them to leave, even his colleagues or distinguished guests visiting Harvard. Sheffer was barely five feet tall; he was noted for his wit and vigor, as well as for his nervousness and irritability. Although widely liked, he was quite lonely. He is noted for a quip he spoke at his retirement: "Old professors never die, they just become emeriti." Sheffer is also credited with coining the term "Boolean algebra" (the subject of Chapter 10 of this text). Sheffer was briefly married and lived most of his later life in small rooms at a hotel packed with his logic books and vast files of slips of paper he used to jot down his ideas. Unfortunately, Sheffer suffered from severe depression during the last two decades of his life.

14. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are equivalent.
15. Show that $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not equivalent.
16. Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically eqnivalent.
17. Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.
18. Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.
19. Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically eqnivalent.
20. Show that $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$ are logically equivalent.
21. Show that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent.
22. Show that $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.
23. Show that $(p \rightarrow r) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are logically equivalent.
24. Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent.
25. Show that $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ arc logically equivalent.
26. Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent.
27. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.
28. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tantology.
29. Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.

The **dual** of a compound proposition that contains only the logical operators $\vee$, $\wedge$, and $\neg$ is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each **T** by **F**, and each **F** by **T**. The dual of proposition s is denoted by s^*.

30. Find the dual of each of these propositions.
    a) $p \wedge \neg q \wedge \neg r$
    b) $(p \wedge q \wedge r) \vee s$
    c) $(p \vee \mathbf{F}) \wedge (q \vee \mathbf{T})$
31. Show that $(s^*)^* = s$.
32. Show that the logical equivalences in Table 5, except for the double negation law, come in pairs, where each pair contains propositions that are duals of each other.

**33. Why are the duals of two equivalent compound propositions also equivalent, where these compound propositions contain only the operators $\wedge$, $\vee$, and $\neg$?

34. Find a compound proposition involving the propositions p, q, and r that is true when p and q are true and r is false, but is false otherwise. (*Hint:* Use a conjunction of each proposition or its negation.)
35. Find a compound proposition involving the propositions p, q, and r that is true when exactly two of p, q, and r are true and is false otherwise. (*Hint:* Form a disjunction of conjunctions. Include a conjunction for each combination of values for which the proposition is true. Each conjunction should include each of the three propositions or their negations.)
36. Suppose that a truth table in n propositional variables is specified. Show that a compound proposition with this truth table can be formed by taking the disjunction of conjunctions of the variables or their negations, with one conjunction included for each combination of values for which the compound proposition is true. The resulting compound proposition is said to be in **disjunctive normal form.**

A collection of logical operators is called **functionally complete** if every compound proposition is logically eqnivalent to a componnd proposition involving only these logical operators.

37. Show that $\neg$, $\wedge$, and $\vee$ form a functionally complete collection of logical operators. (*Hint:* Use the fact that every proposition is logically equivalent to one in disjunctive normal form, as shown in Exercise 36.)

*38. Show that $\neg$ and $\wedge$ form a functionally complete collection of logical operators. (*Hint:* First use De Morgan's law to show that $p \vee q$ is equivalent to $\neg(\neg p \wedge \neg q)$.)

*39. Show that $\neg$ and $\vee$ form a functionally complete collection of logical operators.

The following exercises involve the logical operators *NAND* and *NOR*. The proposition p *NAND* q is true when either p or q, or both, are false; and it is false when both p and q are true. The proposition p *NOR* q is true when both p and q are false, and it is false otherwise. The propositions p *NAND* q and p *NOR* q are denoted by $p \mid q$ and $p \downarrow q$, respectively. (The operators $\mid$ and $\downarrow$ are called the **Sheffer stroke** and the **Peirce arrow** after H. M. Sheffer and C. S. Peirce, respectively.)

40. Construct a truth table for the logical operator *NAND*.
41. Show that $p \mid q$ is logically equivalent to $\neg(p \wedge q)$.
42. Construct a truth table for the logical operator *NOR*.
43. Show that $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$.
44. In this exercise we will show that $\{\downarrow\}$ is a functionally complete collection of logical operators.
    a) Show that $p \downarrow p$ is logically equivalent to $\neg p$.
    b) Show that $(p \downarrow q) \downarrow (p \downarrow q)$ is logically equivalent to $p \vee q$.
    c) Conclude from parts (a) and (b), and Exercise 39, that $\{\downarrow\}$ is a functionally complete collection of logical operators.

*45. Find a proposition equivalent to $p \rightarrow q$ using only the logical operator $\downarrow$.

46. Show that $\{\mid\}$ is a functionally complete collection of logical operators.
47. Show that $p \mid q$ and $q \mid p$ are equivalent.

48. Show that $p \mid (q \mid r)$ and $(p \mid q) \mid r$ are not equivalent, so that the logical operator $\mid$ is not associative.

***49.** How many different truth tables of compound propositions are there that involve the propositions p and q?

50. Show that if p, q, and r are compound propositions such that p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent.

51. The following sentence is taken from the specification of a telephone system: "If the directory database is opened, then the monitor is put in a closed state, if the system is not in its initial state." This specification is hard to understand since it involves two implications. Find an equivalent, easier-to-understand specification that involves disjunctions and negations but not implications.

52. How many of the disjunctions $p \vee \neg q, \neg p \vee q, q \vee r, q \vee \neg r, \neg q \vee \neg r$ can be made simultaneously true by an assignment of truth values to p, q, and r?

53. How many of the disjunctions $p \vee \neg q \vee s$, $\neg p \vee \neg r \vee s$, $\neg p \vee \neg r \vee \neg s$, $\neg p \vee q \vee \neg s$, $q \vee r \vee \neg s$, $q \vee \neg r \vee \neg s$, $\neg p \vee \neg q \vee \neg s$, $p \vee r \vee s$, $p \vee r \vee \neg s$ can be made simultaneously true by an assignment of truth values to p, q, r, and s?

A compound proposition is **satisfiable** if there is an assignment of truth values to the variables in the proposition that makes the compound proposition true.

54. Which of these compound propositions are satisfiable?

a) $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg s) \wedge (p \vee \neg r \vee \neg s) \wedge (\neg p \vee \neg q \vee \neg s) \wedge (p \vee q \vee \neg s)$

b) $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg s) \wedge (\neg p \vee \neg r \vee \neg s) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg r \vee \neg s)$

c) $(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg s) \wedge (q \vee \neg r \vee s) \wedge (\neg p \vee r \vee s) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee s) \wedge (\neg p \vee \neg r \vee \neg s)$

55. Explain how an algorithm for determining whether a compound proposition is satisfiable can be used to determine whether a compound proposition is a tautology. (*Hint:* Look at $\neg p$, where p is the proposition that is being examined.)

Predicates and Quantifiers

INTRODUCTION

Statements involving variables, such as

"$x > 3$," "$x = y + 3$," and "$x + y = z$,"

are often found in mathematical assertions and in computer programs. These statements are neither true nor false when the values of the variables are not specified. In this section we will discuss the ways that propositions can be produced from such statements.

The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part—the **predicate**, "is greater than 3"—refers to a property that the subject of the statement can have. We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable. The statement $P(x)$ is also said to be the value of the **propositional function** P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value. Consider Example 1.

EXAMPLE 1 Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Solution: We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$." Hence, $P(4)$, which is the statement "$4 > 3$," is true. However, $P(2)$, which is the statement "$2 > 3$," is false. ◀

We can also have statements that involve more than one variable. For instance, consider the statement "$x = y + 3$." We can denote this statement by $Q(x, y)$, where x and y are variables and Q is the predicate. When values are assigned to the variables x and y, the statement $Q(x, y)$ has a truth value.

EXAMPLE 2 Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Extra Examples

Solution: To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$," which is false. The statement $Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is true. ◀

Similarly, we can let $R(x, y, z)$ denote the statement "$x + y = z$." When values are assigned to the variables x, y, and z, this statement has a truth value.

EXAMPLE 3 What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?

Solution: The proposition $R(1, 2, 3)$ is obtained by setting $x = 1$, $y = 2$, and $z = 3$ in the statement $R(x, y, z)$. We see that $R(1, 2, 3)$ is the statement "$1 + 2 = 3$," which is true. Also note that $R(0, 0, 1)$, which is the statement "$0 + 0 = 1$," is false. ◀

In general, a statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by

$$P(x_1, x_2, \ldots, x_n).$$

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the **propositional function** P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called a **predicate.**

CHARLES SANDERS PEIRCE (1839–1914) Many consider Charles Peirce the most original and versatile intellect from the United States; he was born in Cambridge, Massachusetts. His father, Benjamin Peirce, was a professor of mathematics and natural philosophy at Harvard. Peirce attended Harvard (1855–1859) and received a Harvard master of arts degree (1862) and an advanced degree in chemistry from the Lawrence Scientific School (1863). His father encouraged him to pursue a career in science, but instead he chose to study logic and scientific methodology.

In 1861, Peirce became an aide in the United States Coast Survey, with the goal of better understanding scientific methodology. His service for the Survey exempted him from military service during the Civil War. While working for the Survey, Peirce carried out astronomical and geodesic work. He made fundamental contributions to the design of pendulums and to map projections, applying new mathematical developments in the theory of elliptic functions. He was the first person to use the wavelength of light as a unit of measurement. Peirce rose to the position of Assistant for the Survey, a position he held until he was forced to resign in 1891 when he disagreed with the direction taken by the Survey's new administration.

Links

Although making his living from work in the physical sciences, Peirce developed a hierarchy of sciences, with mathematics at the top rung, in which the methods of one science could be adapted for use by those sciences under it in the hierarchy. He was also the founder of the American philosophical theory of pragmatism.

The only academic position Peirce ever held was as a lecturer in logic at Johns Hopkins University in Baltimore from 1879 to 1884. His mathematical work during this time included contributions to logic, set theory, abstract algebra, and the philosophy of mathematics. His work is still relevant today; some of his work on logic has been recently applied to artificial intelligence. Peirce believed that the study of mathematics could develop the mind's powers of imagination, abstraction, and generalization. His diverse activities after retiring from the Survey included writing for newspapers and journals, contributing to scholarly dictionaries, translating scientific papers, guest lecturing, and textbook writing. Unfortunately, the income from these pursuits was insufficient to protect him and his second wife from abject poverty. He was supported in his later years by a fund created by his many admirers and administered by the philosopher William James, his lifelong friend. Although Peirce wrote and published voluminously in a vast range of subjects, he left more than 100,000 pages of unpublished manuscripts. Because of the difficulty of studying his unpublished writings, scholars have only recently started to understand some of his varied contributions. A group of people is devoted to making his work available over the Internet to bring a better appreciation of Peirce's accomplishments to the world.

Propositional functions occur in computer programs, as Example 4 demonstrates.

EXAMPLE 4 Consider the statement

if $x > 0$ **then** $x := x + 1$.

When this statement is encountered in a program, the value of the variable x at that point in the execution of the program is inserted into $P(x)$, which is "$x > 0$." If $P(x)$ is true for this value of x, the assignment statement $x := x + 1$ is executed, so the value of x is increased by 1. If $P(x)$ is false for this value of x, the assignment statement is not executed, so the value of x is not changed. ◀

QUANTIFIERS

Assessment

When all the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. However, there is another important way, called **quantification,** to create a proposition from a propositional function. Two types of quantification will be discussed here, namely, universal quantification and existential quantification. The area of logic that deals with predicates and quantifiers is called the **predicate calculus.**

THE UNIVERSAL QUANTIFIER Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **universe of discourse** or the **domain.** Such a statement is expressed using a universal quantification. The universal quantification of a propositional function is the proposition that asserts that $P(x)$ is true for all values of x in the universe of discourse. The universe of discourse specifies the possible values of the variable x.

DEFINITION 1 The *universal quantification* of $P(x)$ is the proposition

"$P(x)$ is true for all values of x in the universe of discourse."

The notation

$\forall x P(x)$

denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** The proposition $\forall x P(x)$ is read as

"for all $x P(x)$" or "for every $x P(x)$."

Remark: It is best to avoid using "for any x" since it is often ambiguous as to whether "any" means "every" or "some." In some cases, "any" is unambiguous, such as when it is used in negatives, for example, "there is not any reason to avoid studying."

We illustrate the use of the universal quantifier in Examples 5–10.

EXAMPLE 5 Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the universe of discourse consists of all real numbers?

Extra Examples

Solution: Since $P(x)$ is true for all real numbers x, the quantification

$$\forall x P(x)$$

is true. ◀

EXAMPLE 6 Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the universe of discourse consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, since, for instance, $Q(3)$ is false. Thus

$$\forall x Q(x)$$

is false. ◀

When all the elements in the universe of discourse can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n),$$

since this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

EXAMPLE 7 What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

since the universe of discourse consists of the integers 1, 2, 3, and 4. Since $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false. ◀

EXAMPLE 8 What does the statement $\forall x T(x)$ mean if $T(x)$ is "x has two parents" and the universe of discourse consists of all people?

Solution: The statement $\forall x T(x)$ means that for every person x, that person has two parents. This statement can be expressed in English as "Every person has two parents." This statement is true (except for clones). ◀

Specifying the universe of discourse is important when quantifiers are used. The truth value of a quantified statement often depends on which elements are in this universe of discourse, as Example 9 shows.

EXAMPLE 9 What is the truth value of $\forall x(x^2 \geq x)$ if the universe of discourse consists of all real numbers and what is its truth value if the universe of discourse consists of all integers?

Solution: Note that $x^2 \geq x$ if and only if $x^2 - x = x(x-1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x(x^2 \geq x)$ is false if the universe of discourse consists of all real numbers (since the inequality is false for all real numbers x with $0 < x < 1$). However, if the universe of discourse consists of the integers, $\forall x(x^2 \geq x)$ is true, since there are no integers x with $0 < x < 1$. ◀

To show that a statement of the form $\forall x P(x)$ is false, where $P(x)$ is a propositional function, we need only find one value of x in the universe of discourse for which $P(x)$ is false. Such a value of x is called a **counterexample** to the statement $\forall x P(x)$.

EXAMPLE 10 Suppose that $P(x)$ is "$x^2 > 0$." To show the statement $\forall x\, P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample since $x^2 = 0$ when $x = 0$ so that x^2 is not greater than 0 when $x = 0$. ◀

Looking for counterexamples to universally quantified statements is an important activity in the study of mathematics, as we will see in subsequent sections of this book.

THE EXISTENTIAL QUANTIFIER Many mathematical statements assert that there is an element with a certain property. Such statements are expressed using existential quantification. With existential quantification, we form a proposition that is true if and only if $P(x)$ is true for at least one value of x in the universe of discourse.

DEFINITION 2 The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the universe of discourse such that $P(x)$ is true."

We use the notation

$$\exists x\, P(x)$$

for the existential quantification of $P(x)$. Here $\exists$ is called the **existential quantifier.** The existential quantification $\exists x\, P(x)$ is read as

"There is an x such that $P(x)$,"
"There is at least one x such that $P(x)$,"

or

"For some $x\, P(x)$."

We illustrate the use of the existential quantifier in Examples 11–13.

EXAMPLE 11 Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x\, P(x)$, where the universe of discourse consists of all real numbers?

Extra Examples

Solution: Since "$x > 3$" is true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x\, P(x)$, is true. ◀

EXAMPLE 12 Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x\, Q(x)$, where the universe of discourse consists of all real numbers?

Solution: Since $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x\, Q(x)$, is false. ◀

When all elements in the universe of discourse can be listed—say, $x_1, x_2, \ldots, x_n$—the existential quantification $\exists x\, P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

since this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

EXAMPLE 13 What is the truth value of $\exists x\, P(x)$ where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Since the universe of discourse is $\{1, 2, 3, 4\}$, the proposition $\exists x\, P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Since $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x\, P(x)$ is true. ◀

Table 1 summarizes the meaning of the universal and the existential quantifiers.

It is sometimes helpful to think in terms of looping and searching when determining the truth value of a quantification. Suppose that there are n objects in the universe of discourse for the variable x. To determine whether $\forall x\, P(x)$ is true, we can loop through all n values of x to see if $P(x)$ is always true. If we encounter a value x for which $P(x)$ is false, then we have shown that $\forall x\, P(x)$ is false. Otherwise, $\forall x\, P(x)$ is true. To see whether $\exists x\, P(x)$ is true, we loop through the n values of x searching for a value for which $P(x)$ is true. If we find one, then $\exists x\, P(x)$ is true. If we never find such an x, we have determined that $\exists x\, P(x)$ is false. (Note that this searching procedure does not apply if there are infinitely many values in the universe of discourse. However, it is still a useful way of thinking about the truth values of quantifications.)

BINDING VARIABLES

When a quantifier is used on the variable x or when we assign a value to this variable, we say that this occurrence of the variable is **bound.** An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free.** All the variables that occur in a propositional function must be bound to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier. Consequently, a variable is free if it is outside the scope of all quantifiers in the formula that specifies this variable.

EXAMPLE 14 In the statement $\exists x\, Q(x, y)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable.

In the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x\, R(x)$, all variables are bound. The scope of the first quantifier, $\exists x$, is the expression $P(x) \wedge Q(x)$ because $\exists x$ is applied only to $P(x) \wedge Q(x)$, and not to the rest of the statement. Similarly, the scope of the second quantifier, $\forall x$, is the expression $R(x)$. That is, the existential quantifier binds the variable x in $P(x) \wedge Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$. Observe that we could have written our statement using two different variables x and y, as $\exists x(P(x) \wedge Q(x)) \vee \forall y R(y)$, because the scopes of the two quantifiers do not overlap. The reader should be aware that in common usage, the same letter is often used to represent variables bound by different quantifiers with scopes that do not overlap. ◀

NEGATIONS

We will often want to consider the negation of a quantified expression. For instance, consider the negation of the statement

"Every student in the class has taken a course in calculus."

This statement is a universal quantification, namely,

$$\forall x P(x),$$

where $P(x)$ is the statement "x has taken a course in calculus." The negation of this statement is "It is not the case that every student in the class has taken a course in calculus." This is equivalent to "There is a student in the class who has not taken a course in calculus." And this is simply the existential quantification of the negation of the original propositional function, namely,

$$\exists x \neg P(x).$$

This example illustrates the following equivalence:

$$\neg \forall x P(x) \equiv \exists x \neg P(x).$$

Extra Examples

Suppose we wish to negate an existential quantification. For instance, consider the proposition "There is a student in this class who has taken a course in calculus." This is the existential quantification

$$\exists x Q(x),$$

where $Q(x)$ is the statement "x has taken a course in calculus." The negation of this statement is the proposition "It is not the case that there is a student in this class who has taken a course in calculus." This is equivalent to "Every student in this class has not taken calculus," which is just the universal quantification of the negation of the original propositional function, or, phrased in the language of quantifiers,

$$\forall x \neg Q(x).$$

This example illustrates the equivalence

$$\neg \exists x Q(x) \equiv \forall x \neg Q(x).$$

Negations of quantifiers are summarized in Table 2.

Remark: When the universe of discourse of a predicate $P(x)$ consists of n elements, where n is a positive integer, the rules for negating quantified statements are exactly the same as De Morgan's laws discussed in Section 1.2. This follows because $\neg \forall x P(x)$ is the same as $\neg(P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n))$, which is equivalent to $\neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_n)$ by De Morgan's laws, and this is the same as $\exists x \neg P(x)$. Similarly, $\neg \exists x P(x)$ is the same as $\neg(P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n))$, which by De Morgan's laws is equivalent to $\neg P(x_1) \wedge \neg P(x_2) \wedge \cdots \wedge \neg P(x_n)$, and this is the same as $\forall x \neg P(x)$.

We illustrate the negation of quantified statements in Examples 15 and 16.

TABLE 2 Negating Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

EXAMPLE 15 What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

Solution: Let $H(x)$ denote "x is honest." Then the statement "There is an honest politician" is represented by $\exists x H(x)$, where the universe of diseourse consists of all politicians. The negation of this statement is $\neg\exists x H(x)$, which is equivalent to $\forall x \neg H(x)$. This negation can be expressed as "Every politician is dishonest." (*Note:* In English the statement "All politicians are not honest" is ambiguous. In common usage this statement often means "Not all politicians are honest." Consequently, we do not use this statement to express this negation.)

Extra Examples

Let $C(x)$ denote "x eats cheeseburgers." Then the statement "All Americans eat cheeseburgers" is represented by $\forall x C(x)$, where the universe of discourse consists of all Americans. The negation of this statement is $\neg\forall x C(x)$, which is equivalent to $\exists x \neg C(x)$. This negation can be expressed in several different ways, including "Some American does not eat cheeseburgers" and "There is an American who does not eat cheeseburgers." ◀

EXAMPLE 16 What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x \neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x \neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the universe of discourse. ◀

TRANSLATING FROM ENGLISH INTO LOGICAL EXPRESSIONS

Translating sentences in English (or other natural languages) into logical expressions is a crucial task in mathematics, logic programming, artificial intelligence, software engineering, and many other disciplines. We began studying this topic in Section 1.1, where we used propositions to express sentences in logical expressions. In that discussion, we purposely avoided sentences whose translations required predicates and quantifiers. Translating from English to logical expressions becomes even more complex when quantifiers are needed. Furthermore, there can be many ways to translate a particular sentence. (As a consequence, there is no "cookbook" approaeh that can be followed step by step.) We will use some examples to illustrate how to translate sentences from English into logical expressions. The goal in this translation is to produce simple and useful logical expressions. In this section, we restrict ourselves to sentences that can be translated into logical expressions using a single quantifier; in the next section, we will look at more complicated sentences that require multiple quantifiers.

EXAMPLE 17 Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Extra Examples

Solution: First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use. Doing so, we obtain:

"For every student in this class, that student has studied calculus."

Next, we introduce a variable x so that our statement becomes

"For every student x in this class, x has studied calculus."

Continuing, we introduce the predicate $C(x)$, which is the statement "x has studied calculus." Consequently, if the universe of discourse for x consists of the students in the class, we can translate our statement as $\forall x C(x)$.

However, there are other correct approaches; different universes of discourse and other predicates can be used. The approach we select depends on the subsequent reasoning we want to carry out. For example, we may be interested in a wider group of people than only those in this class. If we change the universe of discourse to consist of all people, we will need to express our statement as

"For every person x, if person x is a student in this class then x has studied calculus."

If $S(x)$ represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x(S(x) \rightarrow C(x))$. [*Caution!* Our statement *cannot* be expressed as $\forall x(S(x) \wedge C(x))$ since this statement says that all people are students in this class and have studied calculus!]

Finally, when we are interested in the background of people in subjects besides calculus, we may prefer to use the two-variable quantifier $Q(x, y)$ for the statement "student x has studied subject y." Then we would replace $C(x)$ by $Q(x, \text{calculus})$ in both approaches we have followed to obtain $\forall x Q(x, \text{calculus})$ or $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$. ◀

In Example 17 we displayed different approaches for expressing the same statement using predicates and quantifiers. However, we should always adopt the simplest approach that is adequate for use in subsequent reasoning.

EXAMPLE 18 Express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.

Solution: The statement "Some student in this class has visited Mexico" means that

"There is a student in this class with the property that the student has visited Mexico."

We can introduce a variable x, so that our statement becomes

"There is a student x in this class having the property x has visited Mexico."

We introduce the predicate $M(x)$, which is the statement "x has visited Mexico." If the universe of discourse for x consists of the students in this class, we can translate this first statement as $\exists x M(x)$.

However, if we are interested in people other than those in this class, we look at the statement a little differently. Our statement can be expressed as

"There is a person x having the properties that x is a student in this class and x has visited Mexico."

In this case, the universe of discourse for the variable x consists of all people. We introduce the predicate $S(x)$, "x is a student in this class." Our solution becomes $\exists x(S(x) \wedge M(x))$ since the statement is that there is a person x who is a student in this class and who has visited Mexico. [*Caution!* Our statement cannot be expressed as $\exists x(S(x) \rightarrow M(x))$, which is true when there is someone not in the class.]

Similarly, the second statement can be expressed as

"For every x in this class, x has the property that x has visited Mexico or x has visited Canada."

(Note that we are assuming the inclusive, rather than the exclusive, *or* here.) We let $C(x)$ be the statement "x has visited Canada." Following our earlier reasoning, we see that if the universe of discourse for x consists of the students in this class, this second statement can be expressed as $\forall x(C(x) \vee M(x))$. However, if the universe of discourse for x consists of all people, our statement can be expressed as

"For every person x, if x is a student in this class, then x has visited Mexico or x has visited Canada."

In this case, the statement can be expressed as $\forall x(S(x) \rightarrow (C(x) \vee M(x)))$.

Instead of using the predicates $M(x)$ and $C(x)$ to represent that x has visited Mexico and x has visited Canada, respectively, we could use a two-place predicate $V(x, y)$ to represent "x has visited country y." In this case, $V(x, \text{Mexico})$ and $V(x, \text{Canada})$ would have the same meaning as $M(x)$ and $C(x)$ and could replace them in our answers. If we are working with many statements that involve people visiting different countries, we might prefer to use this two-variable approach. Otherwise, for simplicity, we would stick with the one-variable predicates $M(x)$ and $C(x)$. ◀

EXAMPLES FROM LEWIS CARROLL

Lewis Carroll (really C. L. Dodgson writing under a pseudonym), the author of *Alice in Wonderland,* is also the author of several works on symbolic logic. His books contain many examples of reasoning using quantifiers. Examples 19 and 20 come from his book

Links

CHARLES LUTWIDGE DODGSON (1832–1898) We know Charles Dodgson as *Lewis Carroll*—the pseudonym he used in his writings on logic. Dodgson, the son of a clergyman, was the third of 11 children, all of whom stuttered. He was uncomfortable in the company of adults and is said to have spoken without stuttering only to young girls, many of whom he entertained, corresponded with, and photographed (often in the nude). Although attracted to young girls, he was extremely puritanical and religious. His friendship with the three young daughters of Dean Liddell led to his writing *Alice in Wonderland,* which brought him money and fame.

Dodgson graduated from Oxford in 1854 and obtained his master of arts degree in 1857. He was appointed lecturer in mathematics at Christ Church College, Oxford, in 1855. He was ordained in the Church of England in 1861 but never practiced his ministry. His writings include articles and books on geometry, determinants, and the mathematics of tournaments and elections. (He also used the pseudonym Lewis Carroll for his many works on recreational logic.)

Symbolic Logic; other examples from that book are given in the exercise set at the end of this section. These examples illustrate how quantifiers are used to express various types of statements.

EXAMPLE 19 Consider these statements. The first two are called *premises* and the third is called the *conclusion.* The entire set is called an *argument.*

"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."

(In Section 1.5 we will discuss the issue of determining whether the conclusion is a valid consequence of the premises. In this example, it is.) Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a lion," "x is fierce," and "x drinks coffee," respectively. Assuming that the universe of discourse is the set of all creatures, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, and $R(x)$.

Solution: We can express these statements as:

$$\forall x(P(x) \rightarrow Q(x)).$$
$$\exists x(P(x) \wedge \neg R(x)).$$
$$\exists x(Q(x) \wedge \neg R(x)).$$

Notice that the second statement cannot be written as $\exists x(P(x) \rightarrow \neg R(x))$. The reason is that $P(x) \rightarrow \neg R(x)$ is true whenever x is not a lion, so that $\exists x(P(x) \rightarrow \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee. Similarly, the third statement cannot be written as

$$\exists x(Q(x) \rightarrow \neg R(x)).$$ ◀

EXAMPLE 20 Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

"All hummingbirds are richly colored."
"No large birds live on honey."
"Birds that do not live on honey are dull in color."
"Hummingbirds are small."

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively. Assuming that the universe of discourse is the set of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

Solution: We can express the statements in the argument as

$$\forall x(P(x) \rightarrow S(x)).$$
$$\neg\exists x(Q(x) \wedge R(x)).$$
$$\forall x(\neg R(x) \rightarrow \neg S(x)).$$
$$\forall x(P(x) \rightarrow \neg Q(x)).$$

(Note we have assumed that "small" is the same as "not large" and that "dull in color" is the same as "not richly colored." To show that the fourth statement is a valid conclusion of the first three, we need to use rules of inference that will be discussed in Section 1.5.) ◀

LOGIC PROGRAMMING

Links

An important type of programming language is designed to reason using the rules of predicate logic. Prolog (from *Pro*gramming in *Log*ic), developed in the 1970s by computer scientists working in the area of artificial intelligence, is an example of such a language. Prolog programs include a set of declarations consisting of two types of statements, **Prolog facts** and **Prolog rules.** Prolog facts define predicates by specifying the elements that satisfy these predicates. Prolog rules are used to define new predicates using those already defined by Prolog facts. Example 21 illustrates these notions.

EXAMPLE 21 Consider a Prolog program given facts telling it the instructor of each class and in which classes students are enrolled. The program uses these facts to answer queries concerning the professors who teach particular students. Such a program could use the predicates $instructor(p, c)$ and $enrolled(s, c)$ to represent that professor p is the instructor of course c and that student s is enrolled in course c, respectively. For example, the Prolog facts in such a program might include:

```
instructor(chan,math273)
instructor(patel,ee222)
instructor(grossman,cs301)
enrolled(kevin,math273)
enrolled(juana,ee222)
enrolled(juana,cs301)
enrolled(kiko,math273)
enrolled(kiko,cs301)
```

(Lowercase letters have been used for entries because Prolog considers names beginning with an uppercase letter to be variables.)

A new predicate $teaches(p, s)$, representing that professor p teaches student s, can be defined using the Prolog rule

```
teaches(P,S) :- instructor(P,C), enrolled(S,C)
```

which means that $teaches(p, s)$ is true if there exists a class c such that professor p is the instructor of class c and student s is enrolled in class c. (Note that a comma is used to represent a conjunction of predicates in Prolog. Similarly, a semicolon is used to represent a disjunction of predicates.)

Prolog answers queries using the facts and rules it is given. For example, using the facts and rules listed, the query

```
?enrolled(kevin,math273)
```

produces the response

```
yes
```

since the fact *enrolled*(kevin, math273) was provided as input. The query

```
?enrolled(X,math273)
```

produces the response

```
kevin
kiko
```

To produce this response, Prolog determines all possible values of X for which *enrolled*(X, math273) has been included as a Prolog fact. Similarly, to find all the professors who are instructors in classes being taken by Juana, we use the query

```
?teaches(X,juana)
```

This query returns

```
patel
grossman
```

◀

Exercises

1. Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values?

a) $P(0)$ **b)** $P(4)$ **c)** $P(6)$

2. Let $P(x)$ be the statement "the word x contains the letter a." What are the truth values?

a) $P(\text{orange})$ **b)** $P(\text{lemon})$
c) $P(\text{true})$ **d)** $P(\text{false})$

3. Let $Q(x, y)$ denote the statement "x is the capital of y." What are these truth values?

a) $Q(\text{Denver, Colorado})$
b) $Q(\text{Detroit, Michigan})$
c) $Q(\text{Massachusetts, Boston})$
d) $Q(\text{New York, New York})$

4. State the value of x after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$," if the value of x when this statement is reached is

a) $x = 0$. **b)** $x = 1$. **c)** $x = 2$.

5. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the universe of discourse for x consists of all students. Express each of these quantifications in English.

a) $\exists x\, P(x)$ **b)** $\forall x\, P(x)$
c) $\exists x\, \neg P(x)$ **d)** $\forall x\, \neg P(x)$

6. Let $N(x)$ be the statement "x has visited North Dakota," where the universe of discourse consists of the students in your school. Express each of these quantifications in English.

a) $\exists x N(x)$ **b)** $\forall x N(x)$ **c)** $\neg \exists x N(x)$
d) $\exists x \neg N(x)$ **e)** $\neg \forall x N(x)$ **f)** $\forall x \neg N(x)$

7. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the universe of discourse consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$ **b)** $\forall x(C(x) \wedge F(x))$
c) $\exists x(C(x) \rightarrow F(x))$ **d)** $\exists x(C(x) \wedge F(x))$

8. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the universe of discourse consists of all animals.

a) $\forall x(R(x) \rightarrow H(x))$ **b)** $\forall x(R(x) \wedge H(x))$
c) $\exists x(R(x) \rightarrow H(x))$ **d)** $\exists x(R(x) \wedge H(x))$

9. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The universe of discourse for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.
b) There is a student at your school who can speak Russiau but who doesn't know C++.
c) Every student at your school either can speak Russian or knows C++.
d) No student at your sehool can speak Russian or knows C++.

10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the universe of discourse consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.
b) All students in your class have a cat, a dog, or a ferret.
c) Some student in your class has a cat and a ferret, but not a dog.
d) No student in your class has a cat, a dog, and a ferret.
e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has one of these animals as a pet.

11. Let $P(x)$ be the statement "$x = x^2$." If the universe of discourse consists of the integers, what are the truth values?

a) $P(0)$ **b)** $P(1)$ **c)** $P(2)$
d) $P(-1)$ **e)** $\exists x P(x)$ **f)** $\forall x P(x)$

12. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the universe of discourse consists of all integers, what are these truth values?

a) $Q(0)$ **b)** $Q(-1)$ **c)** $Q(1)$
d) $\exists x Q(x)$ **e)** $\forall x Q(x)$ **f)** $\exists x \neg Q(x)$
g) $\forall x \neg Q(x)$

13. Determine the truth value of each of these statements if the universe of discourse consists of all integers.

a) $\forall n(n + 1 > n)$ **b)** $\exists n(2n = 3n)$
c) $\exists n(n = -n)$ **d)** $\forall n(n^2 \geq n)$

14. Determine the truth value of each of these statements if the universe of discourse consists of all real numbers.

a) $\exists x(x^3 = -1)$ **b)** $\exists x(x^4 < x^2)$
c) $\forall x((-x)^2 = x^2)$ **d)** $\forall x(2x > x)$

15. Determine the truth value of each of these statements if the universe of discourse for all variables consists of all integers.

a) $\forall n(n^2 \geq 0)$ **b)** $\exists n(n^2 = 2)$
c) $\forall n(n^2 \geq n)$ **d)** $\exists n(n^2 < 0)$

16. Determine the truth value of each of these statements if the universe of discourse of each variable consists of all real numbers.

a) $\exists x(x^2 = 2)$ **b)** $\exists x(x^2 = -1)$
c) $\forall x(x^2 + 2 \geq 1)$ **d)** $\forall x(x^2 \neq x)$

17. Suppose that the universe of discourse of the propositional function $P(x)$ consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ **b)** $\forall x P(x)$ **c)** $\exists x \neg P(x)$
d) $\forall x \neg P(x)$ **e)** $\neg \exists x P(x)$ **f)** $\neg \forall x P(x)$

18. Suppose that the universe of discourse of the propositional function $Q(x)$ consists of the integers $-2, -1, 0, 1$, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ **b)** $\forall x P(x)$ **c)** $\exists x \neg P(x)$
d) $\forall x \neg P(x)$ **e)** $\neg \exists x P(x)$ **f)** $\neg \forall x P(x)$

19. Suppose that the universe of discourse of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ **b)** $\forall x P(x)$
c) $\neg \exists x P(x)$ **d)** $\neg \forall x P(x)$
e) $\forall x((x \neq 3) \rightarrow P(x)) \vee \exists x \neg P(x)$

20. Suppose that the universe of discourse of the propositional function $P(x)$ consists of $-5, -3, -1, 1, 3$, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ **b)** $\forall x P(x)$
c) $\forall x((x \neq 1) \rightarrow P(x))$
d) $\exists x((x \geq 0) \wedge P(x))$
e) $\exists x(\neg P(x)) \wedge \forall x((x < 0) \rightarrow P(x))$

21. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the universe of discourse consist of the students in your class and second, let it consist of all people.

a) Someone in your class can speak Hindi.
b) Everyone in your class is friendly.
c) There is a person in your class who was not born in California.
d) A student in your class has been in a movie.
e) No student in your class has taken a course in logic programming.

22. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the universe of discourse consist of the students in your class and second, let it consist of all people.

a) Everyone in your class has a cellular phone.
b) Somebody in your class has seen a foreign movie.
c) There is a person in your class who cannot swim.
d) All students in your class can solve quadratic equations.
e) Some student in your class does not want to be rich.

23. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) One of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

24. Translate each of these statements into logical expressions in three different ways by varying the universe of discourse and by using predicates with one and with two variables.

a) Someone in your school has visited Uzbekistan.
b) Everyone in your class has studied calculus and C++.
c) No one in your school owns both a bicycle and a motorcycle.
d) There is a person in your school who is not happy.
e) Everyone in your school was born in the twentieth century.

25. Translate each of these statements into logical expressions in three different ways by varying the universe of discourse and by using predicates with one and with two variables.

a) A student in your school has lived in Vietnam.
b) There is a student in your school who cannot speak Hindi.

c) A student in your school knows Java, Prolog, and C++.
d) Everyone in your class enjoys Thai food.
e) Someone in your class does not play hockey.

26. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.
b) All tools are in the correct place and are in excellent condition.
c) Everything is in the correct place and in excellent condition.
d) Nothing is in the correct place and is in excellent condition.
e) One of your tools is not in the correct place, but it is in excellent condition.

27. Express each of these statements using logical operators, predicates, and quantifiers.

a) Some propositions are tautologies.
b) The negation of a contradiction is a tautology.
c) The disjunction of two contingencies can be a tautology.
d) The conjunction of two tautologies is a tautology.

28. Suppose the universe of discourse of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

a) $\exists x\, P(x, 3)$ b) $\forall y\, P(1, y)$
c) $\exists y \neg P(2, y)$ d) $\forall x \neg P(x, 2)$

29. Suppose that the universe of discourse of $Q(x, y, z)$ consists of triples x, y, z, where $x = 0, 1$, or 2, $y = 0$ or 1, and $z = 0$ or 1. Write out these propositions using disjunctions and conjunctions.

a) $\forall y Q(0, y, 0)$
b) $\exists x Q(x, 1, 1)$
c) $\exists z \neg Q(0, 0, z)$
d) $\exists x \neg Q(x, 0, 1)$

30. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")

a) All dogs have fleas.
b) There is a horse that can add.
c) Every koala can climb.
d) No monkey can speak French.
e) There exists a pig that can swim and catch fish.

31. Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")

a) Some old dogs can learn new tricks.
b) No rabbit knows calculus.
c) Every bird can fly.
d) There is no dog that can talk.
e) There is no one in this class who knows French and Russian.

32. Express the negation of these propositions using quantifiers, and then express the negation in English.

a) Some drivers do not obey the speed limit.
b) All Swedish movies are serious.
c) No one can keep a secret.
d) There is someone in this class who does not have a good attitude.

33. Find a counterexample, if possible, to these universally quantified statements, where the universe of discourse for all variables consists of all integers.

a) $\forall x(x^2 \geq x)$ b) $\forall x(x > 0 \vee x < 0)$
c) $\forall x(x = 1)$

34. Find a counterexample, if possible, to these universally quantified statements, where the universe of discourse for all variables consists of all real numbers.

a) $\forall x(x^2 \neq x)$ b) $\forall x(x^2 \neq 2)$
c) $\forall x(|x| > 0)$

35. Express each of these statements using predicates and quantifiers.

a) A passenger on an airline qualifies as an elite flyer if the passenger flies more than 25,000 miles in a year or takes more than 25 flights during that year.
b) A man qualifies for the marathon if his best previous time is less than 3 hours and a woman qualifies for the marathon if her best previous time is less than 3.5 hours.
c) A student must take at least 60 course hours, or at least 45 course hours and write a master's thesis, and receive a grade no lower than a B in all required courses, to receive a master's degree.
d) There is a student who has taken more than 21 credit hours in a semester and received all A's.

Exercises 36–40 deal with the translation between system specification and logical expressions involving quantifiers.

36. Translate these system specifications into English where the predicate $S(x, y)$ is "x is in state y" and where the universe of discourse for x and y consists of all systems and all possible states, respectively.

a) $\exists x S(x, \text{open})$
b) $\forall x(S(x, \text{malfunctioning}) \vee S(x, \text{diagnostic}))$
c) $\exists x S(x, \text{open}) \vee \exists x S(x, \text{diagnostic})$
d) $\exists x \neg S(x, \text{available})$
e) $\forall x \neg S(x, \text{working})$

37. Translate these specifications into English where $F(p)$ is "Printer p is out of service," $B(p)$ is "Printer p is busy," $L(j)$ is "Print job j is lost," and $Q(j)$ is "Print job j is queued."

a) $\exists p(F(p) \wedge B(p)) \rightarrow \exists j L(j)$

b) $\forall p B(p) \rightarrow \exists j Q(j)$
c) $\exists j(Q(j) \wedge L(j)) \rightarrow \exists p F(p)$
d) $(\forall p B(p) \wedge \forall j Q(j)) \rightarrow \exists j L(j)$

38. Express each of these system specifications nsing predicates, quantifiers, and logical connectives.

a) When there is less than 30 megabytes free on the hard disk, a warning message is sent to all users.
b) No directories in the file system can be opened and no files can be closed when system errors have been detected.
c) The file system cannot be backed up if there is a user currently logged on.
d) Video on demand can be delivered when there are at least 8 megabytes of memory available and the connection speed is at least 56 kilobits per second.

39. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) At least one mail message can be saved if there is a disk with more than 10 kilobytes of free space.
b) Whenever there is an active alert, all queued messages are transmitted.
c) The diagnostic monitor tracks the status of all systems except the main console.
d) Each participant on the conference call whom the host of the call did not put on a special list was billed.

40. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) Every user has access to an electronic mailbox.
b) The system mailbox can he accessed by everyone in the group if the file system is locked.
c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.
d) At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy server is not in diagnostic mode.

41. Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ have the same truth value.

42. Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ have the same truth value.

43. Show that $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ have the same truth value.

44. Establish these logical equivalences, where A is a proposition not involving any quantifiers.

a) $(\forall x P(x)) \vee A \equiv \forall x(P(x) \vee A)$
b) $(\exists x P(x)) \vee A \equiv \exists x(P(x) \vee A)$

45. Establish these logical equivalences, where A is a proposition not involving any quantifiers.

a) $(\forall x\ P(x)) \wedge A \equiv \forall x(P(x) \wedge A)$
b) $(\exists x\ P(x)) \wedge A \equiv \exists x(P(x) \wedge A)$

46. Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.

47. Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x(P(x) \wedge Q(x))$ are not logieally equivalent.

48. The notation $\exists! x P(x)$ denotes the proposition

"There exists a nnique x such that $P(x)$ is trne."

If the universe of discourse consists of all integers, what are the truth values?

a) $\exists! x(x > 1)$ **b)** $\exists! x(x^2 = 1)$
c) $\exists! x(x + 3 = 2x)$ **d)** $\exists! x(x = x + 1)$

49. What are the truth values of these statements?

a) $\exists! x P(x) \rightarrow \exists x P(x)$
b) $\forall x P(x) \rightarrow \exists! x P(x)$
c) $\exists! x \neg P(x) \rightarrow \neg \forall x P(x)$

50. Write out $\exists! x P(x)$, where the universe of disconrse consists of the integers 1, 2, and 3, in terms of negations, conjunctions, and disjunctions.

51. Given the Prolog facts in Example 21, what would Prolog return given these queries?

a) `?instructor(chan,math273)`
b) `?instructor(patel,cs301)`
c) `?enrolled(X,cs301)`
d) `?enrolled(kiko,Y)`
e) `?teaches(grossman,Y)`

52. Given the Prolog facts in Example 21, what would Prolog return when given these queries?

a) `?enrolled(kevin,ee222)`
b) `?enrolled(kiko,math273)`
c) `?instructor(grossman,X)`
d) `?instructor(X,cs301)`
e) `?teaches(X,kevin)`

53. Suppose that Prolog facts are used to define the predicates *mother*(M, Y) and *father*(F, X), which represent that M is the mother of Y and F is the father of X, respectively. Give a Prolog rule to define the predicate *sibling*(X, Y), which represents that X and Y are siblings (that is, have the same mother and the same father).

54. Suppose that Prolog facts are used to define the predicates *mother*(M, Y) and *father*(F, X), which represent that M is the mother of Y and F is the father of X, respectively. Give a Prolog rule to define the predicate *grandfather*(X, Y), which represents that X is the grandfather of Y. (*Hint:* You can write a disjunction in Prolog either by using a semicolon to separate predicates or by putting these predicates on separate lines.)

Exercises 55–58 are based on questions found in the book *Symbolic Logic* by Lewis Carroll.

55. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a professor," "x is ignorant," and "x is vain," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the universe of discourse consists of all people.

a) No professors are ignorant.
b) All ignorant people are vain.
c) No professors are vain.

d) Does (c) follow from (a) and (b)? If not, is there a correct conclusion?

56. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the universe of discourse for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

a) All clear explanations are satisfactory.

b) Some excuses are unsatisfactory.

c) Some excuses are not clear explanations.

* **d)** Does (c) follow from (a) and (b)? If not, is there a correct conclusion?

57. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a baby," "x is logical," "x is able to manage a crocodile," and "x is despised," respectively. Suppose that the universe of discourse consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) Babies are illogical.

b) Nobody is despised who can manage a crocodile.

c) Illogical persons are despised.

d) Babies cannot manage crocodiles.

* **e)** Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

58. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a duck," "x is one of my poultry," "x is an officer," and "x is willing to waltz," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) No ducks are willing to waltz.

b) No officers ever decline to waltz.

c) All my poultry are ducks.

d) My poultry are not officers.

* **e)** Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

Nested Quantifiers

INTRODUCTION

In Section 1.3 we defined the existential and uuiversal quantifiers and showed how they can be used to represent mathematical statements. We also explained how they can be used to translate English sentences into logical expressions. In this section we will study **nested quantifiers,** which are quantifiers that occur within the scope of other quantifiers, such as in the statement $\forall x \exists y(x + y = 0)$. Nested quantifiers commonly occur in mathematics and computer science. Although nested quautifiers can sometimes be difficult to understand, the rules we have already studied in Section 1.3 can help us use them.

TRANSLATING STATEMENTS INVOLVING NESTED QUANTIFIERS

Complicated expressions involving quantifiers arise in many contexts. To understand these statements involving many quantifiers, we need to unravel what the quantifiers and predicates that appear mean. This is illustrated in Example 1.

EXAMPLE 1 Assume that the universe of discourse for the variables x and y consists of all real numbers. The statement

Additional Steps

$$\forall x \forall y(x + y = y + x)$$

says that $x + y = y + x$ for all real numbers x and y. This is the commutative law for addition of real numbers. Likewise, the statement

$$\forall x \exists y(x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse. Similarly, the statement

Extra Examples

$$\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers. ◀

EXAMPLE 2 Translate into English the statement

$$\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0)),$$

where the universe of discourse for both variables consists of all real numbers.

Solution: This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$. That is, this statement says that for real numbers x and y, if x is positive and y is negative, then xy is negative. This can be stated more succinctly as "The product of a positive real number and a negative real number is a negative real number." ◀

Expressions with nested quantifiers expressing statements in English can be quite complicated. The first step in translating such an expression is to write out what the quantifiers and predicates in the expression mean. The next step is to express this meaning in a simpler sentence. This process is illustrated in Examples 3 and 4.

EXAMPLE 3 Translate the statement

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$$

into English, where $C(x)$ is "x has a computer," $F(x, y)$ is "x and y are friends," and the universe of discourse for both x and y consists of all students in your school.

Solution: The statement says that for every student x in your school x has a computer or there is a student y such that y has a computer and x and y are friends. In other words, every student in your school has a computer or has a friend who has a computer. ◀

EXAMPLE 4 Translate the statement

$$\exists x \forall y \forall z((F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$$

into English, where $F(a,b)$ means a and b are friends and the universe of discourse for x, y, and z consists of all students in your school.

Solution: We first examine the expression $(F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y, z)$. This expression says that if students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends. It follows that the original statement, which is triply quantified, says that there is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends. In other words, there is a student none of whose friends are also friends with each other. ◀

TRANSLATING SENTENCES INTO LOGICAL EXPRESSIONS

In Section 1.3 we showed how quantifiers can be used to translate sentences into logical expressions. However, we avoided sentences whose translation into logical expressions required the use of nested quantifiers. We now address the translation of such sentences.

EXAMPLE 5 Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a universe of discourse consisting of all people, and logical connectives.

Solution: The statement "If a person is female and is a parent, then this person is someone's mother" can be expressed as "For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y." We introduce the predicates $F(x)$ to represent "x is female," $P(x)$ to represent "x is a parent," and $M(x, y)$ to represent "x is the mother of y." The original statement can be represented as

$$\forall x((F(x) \land P(x)) \rightarrow \exists y M(x, y)).$$

We can move $\exists y$ all the way to the left, because y does not appear in $F(x) \land P(x)$, to obtain an equivalent expression

$$\forall x \exists y((F(x) \land P(x)) \rightarrow M(x, y)).$$ ◀

EXAMPLE 6 Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a universe of discourse consisting of all people, and logical connectives.

Solution: The statement "Everyone has exactly one best friend" can be expressed as "For every person x, person x has exactly one best friend." Introducing the universal quantifier, we see that this statement is the same as "$\forall x$ (person x has exactly one best friend)" where the universe of discourse consists of all people.

To say that x has exactly one best friend means that there is a person y who is the best friend of x, and, furthermore, that for every person z, if person z is not person y, then z is not the best friend of x. When we introduce the predicate $B(x, y)$ to be the statement "y is the best friend of x," the statement that x has exactly one best friend can be represented as

$$\exists y(B(x, y) \land \forall z((z \neq y) \rightarrow \neg B(x, z))).$$

Consequently, our original statement can be expressed as

$$\forall x \exists y(B(x, y) \land \forall z((z \neq y) \rightarrow \neg B(x, z))).$$

(Note that we can write this statement as $\forall x \exists! y B(x, y)$, where $\exists!$ is the "uniqueness quantifier" defined in Exercise 48 of Section 1.3. However, the "uniqueness quantifier" is not really a quantifier; rather, it is a shorthand for expressing certain statements that can be expressed using the quantifiers $\forall$ and $\exists$. The "uniqueness quantifier" $\exists!$ can be thought of as a macro.) ◀

EXAMPLE 7 Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution: Let $P(w, f)$ be "w has taken f" and $Q(f, a)$ be "f is a flight on a." We can express the statement as

$$\exists w \forall a \exists f(P(w, f) \land Q(f, a)),$$

where the universes of discourse for w, f, and a consist of all the women in the world, all airplane flights, and all airlines, respectively.

The statement could also be expressed as

$$\exists w \forall a \exists f R(w, f, a),$$

where $R(w, f, a)$ is "w has taken f on a." Although this is more compact, it somewhat obscures the relationships among the variables. Consequently, the first solution is usually preferable. ◀

Mathematical statements expressed in English can be translated into logical expressions as Examples 8–10 show.

EXAMPLE 8 Translate the statement "The sum of two positive integers is positive" into a logical expression.

Additional Steps

Solution: To translate this statement into a logical expression, we first rewrite it so that the implied quantifiers are shown: "For every two positive integers, the sum of these integers is positive." Next, we introduce the variables x and y to obtain "For all positive integers x and y, $x + y$ is positive." Consequently, we can express this statement as

$$\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0)),$$

where the universe of discourse for both variables consists of all integers. ◀

EXAMPLE 9 Translate the statement "Every real number except zero has a multiplicative inverse."

Extra Examples

Solution: We first rewrite this as "For every real number x except zero, x has a multiplicative inverse." We can rewrite this as "For every real number x, if $x \neq 0$, then there exists a real number y such that $xy = 1$." This can be rewritten as

$$\forall x((x \neq 0) \rightarrow \exists y(xy = 1)).$$ ◀

One example that you may be familiar with is the concept of limit, which is important in calculus.

EXAMPLE 10 **(Calculus required)** Express the definition of a limit using quantifiers.

Solution: Recall that the definition of the statement

$$\lim_{x \to a} f(x) = L$$

is: For every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. This definition of a limit can be phrased in terms of quantifiers by

$$\forall \epsilon \exists \delta \forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon),$$

where the universe of discourse for the variables δ and ϵ consists of all positive real numbers and for x consists of all real numbers.

This definition can also be expressed as

$$\forall \epsilon{>}0\, \exists \delta{>}0\, \forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

when the universe of discourse for the variables ϵ and δ consists of all real numbers, rather than just the positive real numbers. ◀

NEGATING NESTED QUANTIFIERS

Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier. This is illustrated in Examples 11–13.

EXAMPLE 11 Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.

Extra Examples

Solution: By successively applying the rules for negating quantified statements given in Table 2 of Section 1.3, we can move the negation in $\neg\forall x \exists y(xy = 1)$ inside all the quantifiers. We find that $\neg\forall x \exists y(xy = 1)$ is equivalent to $\exists x \neg\exists y(xy = 1)$, which is equivalent to $\exists x \forall y \neg(xy = 1)$. Since $\neg(xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as $\exists x \forall y(xy \neq 1)$. ◀

EXAMPLE 12 Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: This statement is the negation of the statement "There is a woman who has taken a flight on every airline in the world" from Example 7. By Example 7, our statement can be expressed as $\neg\exists w \forall a \exists f(P(w, f) \wedge Q(f, a))$, where $P(w, f)$ is "w has taken f" and $Q(f, a)$ is "f is a flight on a." By successively applying the rules for negating quantified statements from Table 2 of Section 1.3 to move the negation inside successive quantifiers and by applying De Morgan's law in the last step, we find that our statement is equivalent to each of this sequence of statements:

$$\begin{aligned}\forall w \neg\forall a \exists f(P(w, f) \wedge Q(f, a)) &\equiv \forall w \exists a \neg\exists f(P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f \neg(P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f(\neg P(w, f) \vee \neg Q(f, a)).\end{aligned}$$

This last statement states "For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline." ◀

EXAMPLE 13 Use quantifiers and predicates to express the fact that $\lim_{x \to a} f(x)$ does not exist.

Solution: To say that $\lim_{x \to a} f(x)$ does not exist means that for all real numbers L, $\lim_{x \to a} f(x) \neq L$. By using Example 10, the statement $\lim_{x \to a} f(x) \neq L$ can be expressed as

$$\neg\forall \epsilon > 0\, \exists \delta > 0\, \forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon).$$

Successively applying the rules for negating quantified expressions, we construct this sequence of equivalent statements

$$\begin{aligned}&\neg\forall \epsilon > 0\, \exists \delta > 0\, \forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon) \\ &\quad\equiv \exists \epsilon > 0\, \neg\exists \delta > 0\, \forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon) \\ &\quad\equiv \exists \epsilon > 0\, \forall \delta > 0\, \neg\forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon) \\ &\quad\equiv \exists \epsilon > 0\, \forall \delta > 0\, \exists x\, \neg(0 < |x - a| < \delta \to |f(x) - L| < \epsilon) \\ &\quad\equiv \exists \epsilon > 0\, \forall \delta > 0\, \exists x(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).\end{aligned}$$

We use the equivalence $\neg(p \to q) \equiv p \wedge \neg q$, in the last step.

Because the statement $\lim_{x \to a} f(x)$ does not exist means for all real numbers L, $\lim_{x \to a} f(x) \neq L$. This statement can be expressed as

$$\forall L \exists \epsilon{>}0\, \forall \delta{>}0\, \exists x(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).$$

This last statement says that for every real number L there is a real number $\epsilon > 0$ such that for every real number $\delta > 0$, there exists a real number x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$. ◀

THE ORDER OF QUANTIFIERS

Extra Examples

Many mathematical statements involve multiple quantifications of propositional functions involving more than one variable. It is important to note that the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers. These remarks are illustrated by Examples 14–16. In each of these examples the universe of discourse for each variable consists of all real numbers.

EXAMPLE 14 Let $P(x, y)$ be the statement "$x + y = y + x$." What is the truth value of the quantification $\forall x \forall y P(x, y)$?

Solution: The quantification

$$\forall x \forall y P(x, y)$$

denotes the proposition

"For all real numbers x and for all real numbers y, $x + y = y + x$."

Since $P(x, y)$ is true for all real numbers x and y, the proposition $\forall x \forall y P(x, y)$ is true. ◀

EXAMPLE 15 Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$?

Solution: The quantification

$$\exists y \forall x Q(x, y)$$

denotes the proposition

"There is a real number y such that for every real number x, $Q(x, y)$."

No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Since there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y \forall x Q(x, y)$ is false.

The quantification

$$\forall x \exists y Q(x, y)$$

denotes the proposition

"For every real number x there is a real number y such that $Q(x, y)$."

Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall x \exists y Q(x, y)$ is true. ◀

Example 15 illustrates that the order in which quantifiers appear makes a difference. The statements $\exists y \forall x P(x, y)$ and $\forall x \exists y P(x, y)$ are not logically equivalent. The statement $\exists y \forall x P(x, y)$ is true if and only if there is a y that makes $P(x, y)$ true for every x. So, for this statement to be true, there must be a particular value of y for which $P(x, y)$ is true regardless of the choice of x. On the other hand, $\forall x \exists y P(x, y)$ is true if and only if for every value of x there is a value of y for which $P(x, y)$ is true. So, for this statement to be true, no matter which x you choose, there must be a value of y (possibly depending on the x you choose) for which $P(x, y)$ is true. In other words, in the second case y can depend on x, whereas in the first case y is a constant independent of x.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

From these observations, it follows that if $\exists y \forall x P(x, y)$ is true, then $\forall x \exists y P(x, y)$ must also be true. However, if $\forall x \exists y P(x, y)$ is true, it is not necessary for $\exists y \forall x P(x, y)$ to be true. (See Supplementary Exercises 14 and 16 at the end of this chapter.)

THINKING OF QUANTIFICATION AS LOOPS In working with quantifications of more than one variable, it is sometimes helpful to think in terms of nested loops. (Of course, if there are infinitely many elements in the universe of discourse of some variable, we cannot actually loop through all values. Nevertheless, this way of thinking is helpful in understanding nested quantifiers.) For example, to see whether $\forall x \forall y P(x, y)$ is true, we loop through the values for x, and for each x we loop through the values for y. If we find that $P(x, y)$ is true for all values for x and y, we have determined that $\forall x \forall y P(x, y)$ is true. If we ever hit a value x for which we hit a value y for which $P(x, y)$ is false, we have shown that $\forall x \forall y P(x, y)$ is false.

Similarly, to determine whether $\forall x \exists y P(x, y)$ is true, we loop through the values for x. For each x we loop through the values for y until we find a y for which $P(x, y)$ is true. If for all x we hit such a y, then $\forall x \exists y P(x, y)$ is true; if for some x we never hit such a y, then $\forall x \exists y P(x, y)$ is false.

To see whether $\exists x \forall y P(x, y)$ is true, we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y. Once we find such an x, we know that $\exists x \forall y P(x, y)$ is true. If we never hit such an x, then we know that $\exists x \forall y P(x, y)$ is false.

Finally, to see whether $\exists x \exists y P(x, y)$ is true, we loop through the values for x, where for each x we loop through the values for y until we hit an x for which we hit a y for which $P(x, y)$ is true. The statement $\exists x \exists y P(x, y)$ is false only if we never hit an x for which we hit a y such that $P(x, y)$ is true.

Table 1 summarizes the meanings of the different possible quantifications involving two variables.

Quantifications of more than two variables are also common, as Example 16 illustrates.

EXAMPLE 16 Let $Q(x, y, z)$ be the statement "$x + y = z$." What are the truth values of the statements $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$?

Solution: Suppose that x and y are assigned values. Then, there exists a real number z such that $x + y = z$. Consequently, the quantification

$$\forall x \forall y \exists z Q(x, y, z),$$

which is the statement

"For all real numbers x and for all real numbers y there is a real number z such that $x + y = z$,"

is true. The order of the quantification here is important, since the quantification

$$\exists z \forall x \forall y Q(x, y, z),$$

which is the statement

"There is a real number z such that for all real numbers x and for all real numbers y it is true that $x + y = z$,"

is false, since there is no value of z that satisfies the equation $x + y = z$ for all values of x and y. ◀

Exercises

1. Translate these statements into English, where the universe of discourse for each variable consists of all real numbers.

a) $\forall x \exists y (x < y)$
b) $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$
c) $\forall x \forall y \exists z (xy = z)$

2. Translate these statements into English, where the universe of discourse for each variable consists of all real numbers.

a) $\exists x \forall y (xy = y)$
b) $\forall x \forall y (((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$
c) $\forall x \forall y \exists z (x = y + z)$

3. Let $Q(x, y)$ be the statement "x has sent an e-mail message to y," where the universe of discourse for both x and y consists of all students in your class. Express each of these quantifications in English.

a) $\exists x \exists y Q(x, y)$ **b)** $\exists x \forall y Q(x, y)$
c) $\forall x \exists y Q(x, y)$ **d)** $\exists y \forall x Q(x, y)$
e) $\forall y \exists x Q(x, y)$ **f)** $\forall x \forall y Q(x, y)$

4. Let $P(x, y)$ be the statement "student x has taken class y," where the universe of discourse for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantificatious in English.

a) $\exists x \exists y P(x, y)$ **b)** $\exists x \forall y P(x, y)$
c) $\forall x \exists y P(x, y)$ **d)** $\exists y \forall x P(x, y)$
e) $\forall y \exists x P(x, y)$ **f)** $\forall x \forall y P(x, y)$

5. Let $W(x, y)$ mean that student x has visited website y, where the universe of discourse for x consists of all students in your school and the universe of discourse for y consists of all websites. Express each of these statements by a simple English sentence.

a) W(Sarah Smith, www.att.com)
b) $\exists x W(x,$ www.imdb.org)
c) $\exists y W$(Jose Orez, y)
d) $\exists y (W$(Ashok Puri, $y) \wedge W$(Cindy Yoon, y))
e) $\exists y \forall z (y \neq$ (David Belcher) $\wedge$ (W(David Belcher, $z) \rightarrow W(y, z)$))
f) $\exists x \exists y \forall z ((x \neq y) \wedge (W(x, z) \leftrightarrow W(y, z)))$

6. Let $C(x, y)$ mean that student x is enrolled in class y, where the universe of discourse for x consists of all students in your school and the universe of discourse for y is the set of all classes being given at your school. Express each of these statements by a simple English sentence.

a) C(Randy Goldberg, CS 252)
b) $\exists x C(x,$ Math 695)
c) $\exists y C$(Carol Sitea, y)
d) $\exists x (C(x,$ Math 222) $\wedge C(x,$ CS 252))
e) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \rightarrow C(y, z)))$
f) $\exists x \exists y \forall z ((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$

7. Let $T(x, y)$ mean that student x likes cuisine y, where the universe of discourse for x consists of all students at your school and the universe of discourse for y consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T$(Abdallah Hussein, Japanese)
b) $\exists x T(x,$ Korean) $\wedge \forall x T(x,$ Mexican)
c) $\exists y (T$(Monique Arsenault, $y) \vee T$(Jay Johnson, y))
d) $\forall x \forall z \exists y ((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$

e) $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$
f) $\forall x \forall z \exists y(T(x, y) \leftrightarrow T(z, y))$

8. Let $Q(x, y)$ be the statement "student x has been a contestant on quiz show y." Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the universe of discourse for x consists of all students at your school and for y consists of all quiz shows on television.

a) There is a student at your school who has been a contestant on a television quiz show.
b) No student at your school has ever been a contestant on a television quiz show.
c) There is a student at your school who has been a contestant on *Jeopardy* and on *Wheel of Fortune*.
d) Every television quiz show has had a student from your school as a contestant.
e) At least two students from your school have been contestants on *Jeopardy*.

9. Let $L(x, y)$ be the statement "x loves y," where the universe of discourse for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry.
b) Everybody loves somebody.
c) There is somebody whom everybody loves.
d) Nobody loves everybody.
e) There is somebody whom Lydia does not love.
f) There is somebody whom no one loves.
g) There is exactly one person whom everybody loves.
h) There are exactly two people whom Lynn loves.
i) Everyone loves himself or herself.
j) There is someone who loves no one besides himself or herself.

10. Let $F(x, y)$ be the statement "x can fool y," where the universe of discourse consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody can fool Fred.
b) Evelyn can fool everybody.
c) Everybody can fool somebody.
d) There is no one who can fool everybody.
e) Everyone can be fooled by somebody.
f) No one can fool both Fred and Jerry.
g) Nancy can fool exactly two people.
h) There is exactly one person whom everybody can fool.
i) No one can fool himself or herself.
j) There is someone who can fool exactly one person besides himself or herself.

11. Let $S(x)$ be the predicate "x is a student," $F(x)$ the predicate "x is a faculty member," and $A(x,y)$ the predicate "x has asked y a question," where the universe of discourse consists of all people associated with your school. Use quantifiers to express each of these statements.

a) Lois has asked Professor Michaels a question.
b) Every student has asked Professor Gross a question.
c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
d) Some student has not asked any faculty member a question.
e) There is a faculty member who has never been asked a question by a student.
f) Some student has asked every faculty member a question.
g) There is a faculty member who has asked every other faculty member a question.
h) Some student has never been asked a question by a faculty member.

12. Let $I(x)$ be the statement "x has an Internet connection" and $C(x,y)$ be the statement "x and y have chatted over the Internet," where the universe of discourse for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

a) Jerry does not have an Internet connection.
b) Rachel has not chatted over the Internet with Chelsea.
c) Jan and Sharon have never chatted over the Internet.
d) No one in the class has chatted with Bob.
e) Sanjay has chatted with everyone except Joseph.
f) Someone in your class does not have an Internet connection.
g) Not everyone in your class has an Internet connection.
h) Exactly one student in your class has an Internet connection.
i) Everyone except one student in your class has an Internet connection.
j) Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
k) Someone in your class has an Internet connection but has not chatted with anyone else in your class.
l) There are two students in your class who have not chatted with each other over the Internet.
m) There is a student in your class who has chatted with everyone in your class over the Internet.
n) There are at least two students in your class who have not chatted with the same person in your class.
o) There are two students in the class who between them have chatted with everyone else in the class.

13. Let $M(x, y)$ be "x has sent y an e-mail message" and $T(x, y)$ be "x has telephoned y," where the universe of discourse consists of all students in your class. Use quantifiers to express each of these statements. (Assume that all e-mail messages that were sent are received, which is not the way things often work.)

a) Chou has never sent an e-mail message to Koko.
b) Arlene has never sent an e-mail message to or telephoned Sarah.
c) Jose has never received an e-mail message from Deborah.
d) Every student in your class has sent an e-mail message to Ken.
e) No one in your class has telephoned Nina.
f) Everyone in your class has either telephoned Avi or sent him an e-mail message.
g) There is a student in your class who has sent everyone else in your class an e-mail message.
h) There is someone in your class who has either sent an e-mail message or telephoned everyone else in your class.
i) There are two students in your class who have sent each other e-mail messages.
j) There is a student who has sent himself or herself an e-mail message.
k) There is a student in your class who has not received an e-mail message from anyone else in the class and who has not been called by any other student in the class.
l) Every student in the class has either received an e-mail message or received a telephone call from another student in the class.
m) There are at least two students in your class such that one student has sent the other e-mail and the second student has telephoned the first student.
n) There are two students in your class who between them have sent an e-mail message to or telephoned everyone else in the class.

14. Use quantifiers and predicates with more than one variable to express these statements.

a) There is a student in this class who can speak Hindi.
b) Every student in this class plays some sport.
c) Some student in this class has visited Alaska but has not visited Hawaii.
d) All students in this class have learned at least one programming language.
e) There is a student in this class who has taken every course offered by one of the departments in this school.
f) Some student in this class grew up in the same town as exactly one other student in this class.
g) Every student in this class has chatted with at least one other student in at least one chat group.

15. Use quantifiers and predicates with more than one variable to express these statements.

a) Every computer science student needs a course in discrete mathematics.
b) There is a student in this class who owns a personal computer.
c) Every student in this class has taken at least one computer science course.
d) There is a student in this class who has taken at least one course in computer science.
e) Every student in this class has been in every building on campus.
f) There is a student in this class who has been in every room of at least one building on campus.
g) Every student in this class has been in at least one room of every building on campus.

16. A discrete mathematics class contains 1 mathematics major who is a freshman, 12 mathematics majors who are sophomores, 15 computer science majors who are sophomores, 2 mathematics majors who are juniors, 2 computer science majors who are juniors, and 1 computer science major who is a senior. Express each of these statements in terms of quantifiers and then determine its truth value.

a) There is a student in the class who is a junior.
b) Every student in the class is a computer science major.
c) There is a student in the class who is neither a mathematics major nor a junior.
d) Every student in the class is either a sophomore or a computer science major.
e) There is a major such that there is a student in the class in every year of study with that major.

17. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) Every user has access to exactly one mailbox.
b) There is a process that continues to run during all error conditions only if the kernel is working correctly.
c) All users on the campus network can access all websites whose url has a .edu extension.
*__d)__ There are exactly two systems that monitor every remote server.

18. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) At least one console must be accessible during every fault condition.
b) The e-mail address of every user can be retrieved whenever the archive contains at least one message sent by every user on the system.
c) For every security breach there is at least one mechanism that can detect that breach if and only if there is a process that has not been compromised.
d) There are at least two paths connecting every two endpoints on the network.
e) No one knows the password of every user on the system except for the system administrator.

19. Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the universe of discourse consists of all integers.

a) The sum of two negative integers is negative.
b) The difference of two positive integers is not necessarily positive.
c) The sum of the squares of two integers is greater than or equal to the square of their sum.
d) The absolute value of the product of two integers is the product of their absolute values.

20. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the universe of discourse consists of all integers.

a) The product of two negative integers is positive.
b) The average of two positive integers is positive.
c) The difference of two negative integers is not necessarily negative.
d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

21. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that every positive integer is the sum of the squares of four integers.

22. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that there is a positive integer that is not the sum of three squares.

23. Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.

a) The product of two negative real numbers is positive.
b) The difference of a real number and itself is zero.
c) Every positive real number has exactly two square roots.
d) A negative real number does not have a square root that is a real number.

24. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The universe of discourse in each case consists of all real numbers.

a) $\exists x \forall y(x + y = y)$
b) $\forall x \forall y(((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$
c) $\exists x \exists y(((x \leq 0) \wedge (y \leq 0)) \wedge (x - y > 0))$
d) $\forall x \forall y((x \neq 0) \wedge (y \neq 0) \leftrightarrow (xy \neq 0))$

25. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The universe of discourse in each case consists of all real numbers.

a) $\exists x \forall y(xy = y)$
b) $\forall x \forall y(((x < 0) \wedge (y < 0)) \rightarrow (xy > 0))$
c) $\exists x \exists y((x^2 > y) \wedge (x < y))$
d) $\forall x \forall y \exists z(x - y = z)$

26. Let $Q(x, y)$ be the statement "$x + y = x - y$." If the universe of discourse for both variables consists of all integers, what are the truth values?

a) $Q(1, 1)$ **b)** $Q(2, 0)$ **c)** $\forall y Q(1, y)$
d) $\exists x Q(x, 2)$ **e)** $\exists x \exists y Q(x, y)$ **f)** $\forall x \exists y Q(x, y)$
g) $\exists y \forall x Q(x, y)$
h) $\forall y \exists x Q(x, y)$
i) $\forall x \forall y Q(x, y)$

27. Determine the truth value of each of these statements if the universe of discourse for all variables consists of all integers.

a) $\forall n \exists m(n^2 < m)$ **b)** $\exists n \forall m(n < m^2)$
c) $\forall n \exists m(n + m = 0)$ **d)** $\exists n \forall m(nm = m)$
e) $\exists n \exists m(n^2 + m^2 = 5)$ **f)** $\exists n \exists m(n^2 + m^2 = 6)$
g) $\exists n \exists m(n + m = 4 \wedge n - m = 1)$
h) $\exists n \exists m(n + m = 4 \wedge n - m = 2)$
i) $\forall n \forall m \exists p(p = (m + n)/2)$

28. Determine the truth value of each of these statements if the universe of discourse of each variable consists of all real numbers.

a) $\forall x \exists y(x^2 = y)$ **b)** $\forall x \exists y(x = y^2)$
c) $\exists x \forall y(xy = 0)$ **d)** $\exists x \exists y(x + y \neq y + x)$
e) $\forall x(x \neq 0 \rightarrow \exists y(xy = 1))$
f) $\exists x \forall y(y \neq 0 \rightarrow xy = 1)$
g) $\forall x \exists y(x + y = 1)$
h) $\exists x \exists y(x + 2y = 2 \wedge 2x + 4y = 5)$
i) $\forall x \exists y(x + y = 2 \wedge 2x - y = 1)$
j) $\forall x \forall y \exists z(z = (x + y)/2)$

29. Suppose the universe of discourse of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

a) $\forall x \forall y P(x, y)$ **b)** $\exists x \exists y P(x, y)$
c) $\exists x \forall y P(x, y)$ **d)** $\forall y \exists x P(x, y)$

30. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

a) $\neg \exists y \exists x P(x, y)$ **b)** $\neg \forall x \exists y P(x, y)$
c) $\neg \exists y(Q(y) \wedge \forall x \neg R(x, y))$
d) $\neg \exists y(\exists x R(x, y) \vee \forall x S(x, y))$
e) $\neg \exists y(\forall x \exists z T(x, y, z) \vee \exists x \forall z U(x, y, z))$

31. Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z T(x, y, z)$
b) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$
c) $\forall x \exists y(P(x, y) \wedge \exists z R(x, y, z))$
d) $\forall x \exists y(P(x, y) \rightarrow Q(x, y))$

32. Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\exists z\forall y\forall x T(x, y, z)$
b) $\exists x\exists y P(x, y) \wedge \forall x\forall y Q(x, y)$
c) $\exists x\exists y(Q(x, y) \leftrightarrow Q(y, x))$
d) $\forall y\exists x\exists z(T(x, y, z) \vee Q(x, y))$

33. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

a) $\neg\forall x\forall y P(x, y)$ b) $\neg\forall y\exists x P(x, y)$
c) $\neg\forall y\forall x(P(x, y) \vee Q(x, y))$
d) $\neg(\exists x\exists y\neg P(x, y) \wedge \forall x\forall y Q(x, y))$
e) $\neg\forall x(\exists y\forall z P(x, y, z) \wedge \exists z\forall y P(x, y, z))$

34. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")

a) No one has lost more than one thousand dollars playing the lottery.
b) There is a student in this class who has chatted with exactly one other student.
c) No student in this class has sent e-mail to exactly two other students in this class.
d) Some student has solved every exercise in this book.
e) No student has solved at least one exercise in every section of this book.

35. Express each of these statements using quantifiers. Then form the negatiou of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")

a) Every student in this class has taken exactly two mathematics classes at this school.
b) Someone has visited every country in the world except Libya.
c) No one has climbed every mountain in the Himalayas.
d) Every movic actor has either been in a movie with Kevin Bacon or has been in a movie with someone who has been in a movie with Kevin Bacon.

36. Express the negations of these propositions using quantifiers, and in English.

a) Every student in this class likes mathematics.
b) There is a student in this class who has never seen a computer.
c) There is a student in this class who has taken every mathematics course offered at this school.
d) There is a student in this class who has been in at least one room of every building on campus.

37. Find a counterexample, if possible, to these universally quantified statemeuts, where the universe of discourse for all variables consists of all integers.

a) $\forall x\forall y(x^2 = y^2 \rightarrow x = y)$
b) $\forall x\exists y(y^2 = x)$ c) $\forall x\forall y(xy \geq x)$

38. Find a counterexample, if possible, to these universally quantified statements, where the universe of discourse for all variables consists of all integers.

a) $\forall x\exists y(x = 1/y)$ b) $\forall x\exists y(y^2 - x < 100)$
c) $\forall x\forall y(x^2 \neq y^3)$

39. Use quantifiers to express the associative law for multiplication of real numbers.

40. Use quantifiers to express the distributive laws of multiplication over addition for real numbers.

41. Determine the truth value of the statement $\forall x\exists y(xy = 1)$ if the universe of discourse for the variables consists of

a) the nonzero real numbers.
b) the nonzero integers.
c) the positive real numbers.

42. Determine the truth value of the statement $\exists x\forall y(x \leq y^2)$ if the universe of discourse for the variables consists of

a) the positive real numbers.
b) the integers.
c) the nonzero real numbers.

43. Show that the two statements $\neg\exists x\forall y P(x, y)$ and $\forall x\exists y\neg P(x, y)$ have the same truth value.

***44.** Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x\forall y(P(x) \vee Q(y))$ are logically equivalent. (The new variable y is used to combine the quantifications correctly.)

***45.** a) Show that $\forall x P(x) \wedge \exists x Q(x)$ is equivalent to $\forall x\exists y(P(x) \wedge Q(y))$.
b) Show that $\forall x P(x) \vee \exists x Q(x)$ is equivalent to $\forall x\exists y(P(x) \vee Q(y))$.

A statement is in **prenex normal form (PNF)** if and only if it is of the form

$$Q_1x_1Q_2x_2\cdots Q_kx_kP(x_1, x_2, \ldots, x_k),$$

where each $Q_i, i = 1, 2, \ldots, k$, is either the existential quantifier or the universal quantifier, and $P(x_1, \ldots, x_k)$ is a predicate involving no quantifiers. For example, $\exists x\forall y(P(x, y) \wedge Q(y))$ is in prenex normal form, whereas $\exists x P(x) \vee \forall x Q(x)$ is not (since the quantifiers do not all occur first).

Every statement formed from propositional variables, predicates, **T**, and **F** using logical connectives and quantifiers is equivalent to a statement in prenex normal form. Exercise 47 asks for a proof of this fact.

***46.** Put these statements in prenex normal form. (*Hint:* Use logical equivalcncc from Tables 5 and 6 in Section 1.2, Table 2 in Section 1.3, Exercises 42–45 in Section 1.3, and Excrcises 44 and 45 in this section.)

a) $\exists x P(x) \vee \exists x Q(x) \vee A$, where A is a proposition not involving any quantifiers.
b) $\neg(\forall x P(x) \vee \forall x Q(x))$
c) $\exists x P(x) \rightarrow \exists x Q(x)$

****47.** Show how to transform an arbitrary statement to a statement in prenex normal form that is equivalent to the given statement.

48. A real number x is called an **upper bound** of a set S of real numbers if x is greater than or equal to every member of S. The real number x is called the **least upper bound** of a set S of real numbers if x is an upper bound of S and x is less than or equal to every upper bound of S; if the least upper bound of a set S exists, it is unique.

a) Using quantifiers, express the fact that x is an upper bound of S.

b) Using quantifiers, express the fact that x is the least upper bound of S.

***49.** Express the quantification $\exists! x P(x)$ using universal quantifications, existential quantifications, and logical operators.

The statement $\lim_{n\to\infty} a_n = L$ means that for every positive real number ϵ there is a positive integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

50. (Calculus required) Use quantifiers to express the statement that $\lim_{n\to\infty} a_n = L$.

51. (Calculus required) Use quantifiers to express the statement that $\lim_{n\to\infty} a_n$ does not exist.

52. (Calculus required) Use quantifiers to express this definition: A sequence $\{a_n\}$ is a Cauchy sequence if for every real number $\epsilon > 0$ there exists a positive integer N such that $|a_m - a_n| < \epsilon$ for every pair of positive integers m and n with $m > N$ and $n > N$.

53. (Calculus required) Use quantifiers and logical connectives to express this definition: A number L is the **limit superior** of a sequence $\{a_n\}$ if for every real number $\epsilon > 0$, $a_n > L - \epsilon$ for infinitely many n and $a_n > L + \epsilon$ for only finitely many n.

Methods of Proof

INTRODUCTION

Two important questions that arise in the study of mathematics are: (1) When is a mathematical argument correct? (2) What methods can be used to construct mathematical arguments? This section helps answer these questions by describing various forms of correct and incorrect mathematical arguments.

A **theorem** is a statement that can be shown to be true. (Theorems are sometimes called *propositions, facts,* or *results.*) We demonstrate that a theorem is true with a sequence of statements that form an argument, called a **proof.** To construct proofs, methods are needed to derive new statements from old ones. The statements used in a proof can include **axioms** or **postulates,** which are the underlying assumptions about mathematical structures, the hypotheses of the theorem to be proved, and previously proved theorems. The **rules of inference,** which are the means used to draw conclusions from other assertions, tie together the steps of a proof.

Links

In this section rules of inference will be discussed. This will help clarify what makes up a correct proof. Some common forms of incorrect reasoning, called **fallacies,** will also be described. Then various methods commonly used to prove theorems will be introduced.

The terms *lemma* and *corollary* are used for certain types of theorems. A **lemma** (plural **lemmas** or **lemmata**) is a simple theorem used in the proof of other theorems. Complicated proofs are usually easier to understand when they are proved using a series of lemmas, where each lemma is proved individually. A **corollary** is a proposition that can be established directly from a theorem that has been proved. A **conjecture** is a statement whose truth value is unknown. When a proof of a conjecture is found, the conjecture becomes a theorem. Many times conjectures are shown to be false, so they are not theorems.

The methods of proof discussed in this chapter are important not only because they are used to prove mathematical theorems, but also for their many applications to computer science. These applications include verifying that computer programs are correct, establishing that operating systems are secure, making inferences in the area of artificial intelligence, showing that system specifications are consistent, and so on. Consequently, understanding the techniques used in proofs is essential both in mathematics and in computer science.

RULES OF INFERENCE

We will now introduce rules of inference for propositional logic. These rules provide the justification of the steps used to show that a conclusion follows logically from a set of hypotheses. The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called **modus ponens,** or the **law of detachment.** This tautology is written in the following way:

$$\begin{array}{l} \quad p \\ \quad \underline{p \rightarrow q} \\ \therefore\ q \end{array}$$

Using this notation, the hypotheses are written in a column and the conclusion below a bar. (The symbol $\therefore$ denotes "therefore.") Modus ponens states that if both an implication and its hypothesis are known to be true, then the conclusion of this implication is true.

Extra Examples

EXAMPLE 1 Suppose that the implication "if it snows today, then we will go skiing" and its hypothesis, "it is snowing today," are true. Then, by modus ponens, it follows that the conclusion of the implication, "we will go skiing," is true. ◀

EXAMPLE 2 Assume that the implication "if n is greater than 3, then n^2 is greater than 9" is true. Consequently, if n is greater than 3, then, by modus ponens, it follows that n^2 is greater than 9. ◀

Table 1 lists some important rules of inference. The verifications of these rules of inference can be found as exercises in Section 1.2. Here are some examples of arguments using these rules of inference.

EXAMPLE 3 State which rule of inference is the basis of the following argument: "It is below freezing now. Therefore, it is either below freezing or raining uow."

Solution: Let p be the proposition "It is below freezing now" and q the proposition "It is raining now." Then this argument is of the form

$$\begin{array}{l} \quad \underline{p} \\ \therefore\ p \vee q \end{array}$$

This is an argument that uses the addition rule. ◀

EXAMPLE 4 State which rule of inference is the basis of the following argument: "It is below freezing and raining now. Therefore, it is below freezing now."

Solution: Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

$$\begin{array}{l} \quad \underline{p \wedge q} \\ \therefore\ p \end{array}$$

This argument uses the simplification rule. ◀

EXAMPLE 5 State which rule of inference is used in the argument:

> If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$	$p \rightarrow (p \vee q)$	Addition
$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$	$(p \wedge q) \rightarrow p$	Simplification
$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Disjunctive syllogism
$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Solution: Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Hence, this argument is a hypothetical syllogism. ◀

VALID ARGUMENTS

An argument form is called **valid** if whenever all the hypotheses are true, the conclusion is also true. Consequently, showing that q logically follows from the hypotheses $p_1, p_2, \ldots, p_n$ is the same as showing that the implication

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

is true. When all propositions used in a valid argument are true, it leads to a correct conclusion. However, a valid argument can lead to an incorrect conclusion if one or more false propositions are used within the argument. For example, "If $\sqrt{2} > \frac{1}{2}$, then $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently, $\left(\sqrt{2}\right)^2 = 2 > \left(\frac{3}{2}\right)^2 = \frac{9}{4}$."

is a valid argument form based on modus ponens. However, the conclusion of this argument is false, because $2 < \frac{9}{4}$. The false proposition "$\sqrt{2} > \frac{3}{2}$" has been used in the argument, which means that the conclusion of the argument may be false.

Extra Examples

When there are many premises, several rules of iuference are often needed to show that an argument is valid. This is illustrated by the following examples, where the steps of arguments are displayed step by step, with the reason for each step explicitly stated. These examples also show how arguments in English can be analyzed using rules of inference.

EXAMPLE 6 Show that the hypotheses "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the hypotheses become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t.

We construct an argument to show that our hypotheses lead to the desired conclusion as follows.

Step	**Reason**
1. $\neg p \wedge q$	Hypothesis
2. $\neg p$	Simplification using Step 1
3. $r \rightarrow p$	Hypothesis
4. $\neg r$	Modus tollens using Steps 2 and 3
5. $\neg r \rightarrow s$	Hypothesis
6. s	Modus ponens using Steps 4 and 5
7. $s \rightarrow t$	Hypothesis
8. t	Modus ponens using Steps 6 and 7 ◀

EXAMPLE 7 Show that the hypotheses "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the hypotheses are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$. The desired conclusion is $\neg q \rightarrow s$.

This argument form shows that our hypotheses lead to the desired conclusion.

Step	**Reason**
1. $p \rightarrow q$	Hypothesis
2. $\neg q \rightarrow \neg p$	Contrapositive of Step 1
3. $\neg p \rightarrow r$	Hypothesis
4. $\neg q \rightarrow r$	Hypothetical syllogism using Steps 2 and 3
5. $r \rightarrow s$	Hypothesis
6. $\neg q \rightarrow s$	Hypothetical syllogism using Steps 4 and 5 ◀

RESOLUTION

Links

Computer programs have been developed to automate the task of reasoning and proving theorems. Many of these programs make use of a rule of inference known as **resolution.** This rule of inference is based on the tautology

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r).$$

(The verification that this is a tautology was addressed in Exercise 28 in Section 1.2.) The final disjunction in the resolution rule, $q \vee r$, is called the **resolvent.** When we let $q = r$ in this tautology, we obtain $(p \vee q) \wedge (\neg p \vee q) \rightarrow q$. Furthermore, when we let $r = \mathbf{F}$, we obtain $(p \vee q) \wedge (\neg p) \rightarrow q$ (because $q \vee \mathbf{F} \equiv q$), which is the tautology on which the rule of disjunctive syllogism is based.

EXAMPLE 8 Use resolution to show that the hypotheses "Jasmine is skiing or it is not snowing" and "It is snowing or Bart is playing hockey" imply that "Jasmine is skiing or Bart is playing hockey."

Extra Examples

Solution: Let p be the proposition "It is snowing," q the proposition "Jasmine is skiing," and r the proposition "Bart is playing hockey." We can represent the hypotheses as $\neg p \vee q$ and $p \vee r$, respectively. Using resolution, the proposition $q \vee r$, "Jasmine is skiing or Bart is playing hockey," follows. ◀

Resolution plays an important role in programming languages based on the rules of logic, such as Prolog (where resolution rules for quantified statements are applied). Furthermore, it can be used to build automatic theorem proving systems. To construct proofs in propositional logic using resolution as the only rule of inference, the hypotheses and the conclusion must be expressed as **clauses,** where a clause is a disjunction of variables or negations of these variables. We can replace a statement in propositional logic that is not a clause by one or more equivalent statements that are clauses. For example, suppose we have a statement of the form $p \vee (q \wedge r)$. Because $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, we can replace the single statement $p \vee (q \wedge r)$ by two statements $p \vee q$ and $p \vee r$, each of which is a clause. We can replace a statement of the form $\neg(p \vee q)$ by the two statements $\neg p$ and $\neg q$ because De Morgan's law tells us that $\neg(p \vee q) \equiv \neg p \wedge \neg q$. We can also replace an implication $p \rightarrow q$ with the equivalent disjunction $\neg p \vee q$.

EXAMPLE 9 Show that the hypotheses $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

Solution: We can rewrite the hypothesis $(p \wedge q) \vee r$ as two clauses, $p \vee r$ and $q \vee r$. We can also replace $r \rightarrow s$ by the equivalent clause $\neg r \vee s$. Using the two clauses $p \vee r$ and $\neg r \vee s$, we can use resolution to conelude $p \vee s$. ◀

FALLACIES

Links

Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference but are based on contingencies rather than tautologies. These are discussed here to show the distinction between correct and incorrect reasoning.

The proposition $[(p \rightarrow q) \wedge q] \rightarrow p$ is not a tautology, since it is false when p is false and q is true. However, there are many incorrect arguments that treat this as a tautology. This type of incorrect reasoning is called the **fallacy of affirming the conclusion.**

EXAMPLE 10 Is the following argument valid?

> If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.
>
> Therefore, you did every problem in this book.

Solution: Let p be the proposition "You did every problem in this book." Let q be the proposition "You learned discrete mathematics." Then this argument is of the form: if $p \to q$ and q, then p. This is an example of an incorrect argument using the fallacy of affirming the conclusion. Indeed, it is possible for you to learn discrete mathematics in some way other than by doing every problem in this book. (You may learn discrete mathematics by reading, listening to lectures, doing some but not all the problems in this book, and so on.) ◀

The proposition $[(p \to q) \wedge \neg p] \to \neg q$ is not a tautology, since it is false when p is false and q is true. Many incorrect arguments use this incorrectly as a rule of inference. This type of incorrect reasoning is called the **fallacy of denying the hypothesis.**

EXAMPLE 11 Let p and q be as in Example 10. If the implication $p \to q$ is true, and $\neg p$ is true, is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

Solution: It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \to q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis. ◀

RULES OF INFERENCE FOR QUANTIFIED STATEMENTS

We discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the universe of discourse, given the premise $\forall x\, P(x)$. Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the universe of discourse of all women.

Universal generalization is the rule of inference that states that $\forall x\, P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the universe of discourse. Universal generalization is used when we show that $\forall x\, P(x)$ is true by taking an arbitrary element c from the universe of discourse and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the universe of discourse. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly.

Existential instantiation is the rule that allows us to conclude that there is an element c in the universe of discourse for which $P(c)$ is true if we know that $\exists x\, P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Since it exists, we may give it a name (c) and continue our argument.

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the universe of discourse for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

We summarize these rules of inference in Table 2. We will illustrate how one of these rules of inference for quantified statements is used in Example 12.

EXAMPLE 12 Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Extra Examples

Solution: Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \rightarrow C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

◀

EXAMPLE 13 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise

Step	**Reason**
5. $C(a) \to P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8) ◀

Remark: Mathematical arguments often include steps where both a rule of inference for propositions and a rule of inference for quantifiers are used. For example, universal instantiation and modus ponens are often used together. When these rules of inference are combined, the hypothesis $\forall x(P(x) \to Q(x))$ and $P(c)$, where c is a member of the universe of discourse, show that the conclusion $Q(c)$ is true.

Extra Examples

Remark: Many theorems in mathematics state that a property holds for all elements in a particular set, such as the set of integers or the set of real numbers. Although the precise statement of such theorems needs to include a universal quantifier, the standard convention in mathematics is to omit it. For example, the statement "If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$" really means "For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$." Furthermore, when theorems of this type are proved, the law of universal generalization is often used without explicit mention. The first step of the proof usually involves selecting a general element of the universe of discourse. Subsequent steps show that this element has the property in question. Universal generalization implies that the theorem holds for all members of the universe of discourse.

In our subsequent discussions, we will follow the usual conventions and not explicitly mention the use of universal quantification and universal generalization. However, you should always understand when this rule of inference is being implicitly applied.

METHODS OF PROVING THEOREMS

Assessment

Proving theorems can be difficult. We need all the ammunition that is available to help us prove different results. We now introduce a battery of different proof methods. These methods should become part of your repertoire for proving theorems. Because many theorems are implications, the techniques for proving implications are important. Recall that $p \to q$ is true unless p is true but q is false. Note that when the statement $p \to q$ is proved, it need only be shown that q is true if p is true; it is *not* usually the case that q is proved to be true. The following discussion will give the most common techniques for proving implications.

Extra Examples

DIRECT PROOFS The implication $p \to q$ can be proved by showing that if p is true, then q must also be true. This shows that the combination p true and q false never occurs. A proof of this kind is called a **direct proof.** To carry out such a proof, assume that p is true and use rules of inference and theorems already proved to show that q must also be true.

Before we give an example of a direct proof, we need a definition.

DEFINITION 1 The integer n is *even* if there exists an integer k such that $n = 2k$ and it is *odd* if there exists an integer k such that $n = 2k + 1$. (Note that an integer is either even or odd.)

EXAMPLE 14 Give a direct proof of the theorem "If n is an odd integer, then n^2 is an odd integer."

Solution: Assume that the hypothesis of this implication is true, namely, suppose that n is odd. Then $n = 2k + 1$, where k is an integer. It follows that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Therefore, n^2 is an odd integer (it is one more than twice an integer). ◀

INDIRECT PROOFS Since the implication $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$, the implication $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true. This related implication is usually proved directly, but any proof technique can be used. An argument of this type is called an **indirect proof.**

Extra Examples

EXAMPLE 15 Give an indirect proof of the theorem "If $3n + 2$ is odd, then n is odd."

Solution: Assume that the conclusion of this implication is false; namely, assume that n is even. Then $n = 2k$ for some integer k. It follows that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$, so $3n + 2$ is even (since it is a multiple of 2) and therefore not odd. Because the negation of the conclusion of the implication implies that the hypothesis is false, the original implication is true. ◀

VACUOUS AND TRIVIAL PROOFS Suppose that the hypothesis p of an implication $p \rightarrow q$ is false. Then the implication $p \rightarrow q$ is true, because the statement has the form $\mathbf{F} \rightarrow \mathbf{T}$ or $\mathbf{F} \rightarrow \mathbf{F}$, and hence is true. Consequently, if it can be shown that p is false, then a proof, called a **vacuous proof,** of the implication $p \rightarrow q$ can be given. Vacuous proofs are often used to establish special cases of theorems that state that an implication is true for all positive integers [i.e., a theorem of the kind $\forall n\ P(n)$ where $P(n)$ is a propositional function]. Proof techniques for theorems of this kind will be discussed in Section 3.3.

EXAMPLE 16 Show that the proposition $P(0)$ is true where $P(n)$ is the propositional function "If $n > 1$, then $n^2 > n$."

Solution: Note that the proposition $P(0)$ is the implication "If $0 > 1$, then $0^2 > 0$." Since the hypothesis $0 > 1$ is false, the implication $P(0)$ is automatically true. ◀

Remark: The fact that the conclusion of this implication, $0^2 > 0$, is false is irrelevant to the truth value of the implication, because an implication with a false hypothesis is guaranteed to be true.

Suppose that the conclusion q of an implication $p \rightarrow q$ is true. Then $p \rightarrow q$ is true, since the statement has the form $\mathbf{T} \rightarrow \mathbf{T}$ or $\mathbf{F} \rightarrow \mathbf{T}$, which are true. Hence, if it can be shown that q is true, then a proof, called a **trivial proof,** of $p \rightarrow q$ can be given. Trivial proofs are often important when special cases of theorems are proved (see the discussion of proof by cases) and in mathematical induction, which is a proof technique discussed in Section 3.3.

EXAMPLE 17 Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$." Show that the proposition $P(0)$ is true.

Solution: The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$." Since $a^0 = b^0 = 1$, the conclusion of $P(0)$ is true. Hence, $P(0)$ is true. This is an example of a trivial proof. Note that the hypothesis, which is the statement "$a \geq b$," was not needed in this proof. ◀

A LITTLE PROOF STRATEGY We have described both direct and indirect proofs and we have provided an example of how they are used. However, when confronted with an implication to prove, which method should you use? First, quickly evaluate whether a direct proof looks promising. Begin by expanding the definitions in the hypotheses. Then begin to reason using them, together with axioms and available theorems. If a direct proof does not seem to go anywhere, try the same thing with an indirect proof. Recall that in an indirect proof you assume that the conclusion of the implication is false and use a direct proof to show this implies that the hypothesis must be false. Sometimes when there is no obvious way to approach a direct proof, an indirect proof works nicely. We illustrate this strategy in Examples 18 and 19.

Extra Examples

Before we present our next example, we need a definition.

DEFINITION 2 The real number r is *rational* if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational*.

EXAMPLE 18 Prove that the sum of two rational numbers is rational.

Solution: We first attempt a direct proof. To begin, suppose that r and s are rational numbers. From the definition of a rational number, it follows that there are integers p and q, with $q \neq 0$, such that $r = p/q$, and integers t and u, with $u \neq 0$, such that $s = t/u$. Can we use this information to show that $r + s$ is rational? The obvious next step is to add $r = p/q$ and $s = t/u$, to obtain

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + qt}{qu}.$$

Because $q \neq 0$ and $u \neq 0$, it follows that $qu \neq 0$. Consequently, we have expressed $r + s$ as the ratio of two integers, $pu + qt$ and qu, where $qu \neq 0$. This means that $r + s$ is rational. Our attempt to find a direct proof succeeded. ◀

EXAMPLE 19 Prove that if n is an integer and n^2 is odd, then n is odd.

Solution: We first attempt a direct proof. Suppose that n is an integer and n^2 is odd. Then, there exists an integer k such that $n^2 = 2k + 1$. Can we use this information to show that n is odd? There seems to be no obvious approach to show that n is odd because solving for n produces the equation $n = \pm\sqrt{2k + 1}$, which is not terribly useful.

Because this attempt to use a direct proof did not bear fruit, we next attempt an indirect proof. We take as our hypothesis the statement that n is not odd. Because every integer is odd or even, this means that n is even. This implies that there exists an integer k such that $n = 2k$. To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that is, that n^2 is even. Can we use the equation $n = 2k$ to achieve this? By squaring both sides of this equation, we obtain $n^2 = 4k^2 = 2(2k^2)$, which implies that n^2 is also even since $n^2 = 2t$, where $t = 2k^2$. Our attempt to find an indirect proof succeeded. ◀

PROOFS BY CONTRADICTION There are other approaches we can use when neither a direct proof nor an indirect proof succeeds. We now introduce several additional proof techniques.

Suppose that a contradiction q can be found so that $\neg p \rightarrow q$ is true, that is, $\neg p \rightarrow \mathbf{F}$ is true. Then the proposition $\neg p$ must be false. Consequently, p must be true. This technique can be used when a contradiction, such as $r \wedge \neg r$, can be found so that it is possible to show that the implication $\neg p \rightarrow (r \wedge \neg r)$ is true. An argument of this type is called a **proof by contradiction.**

We provide three examples of proof by contradiction. The first is an example of an application of the pigeonhole principle, a combinatorial technique which we will cover in depth in Section 4.2.

EXAMPLE 20 Show that at least four of any 22 days must fall on the same day of the week.

Extra Examples

Solution: Let p be the proposition "At least four of the 22 chosen days are the same day of the week." Suppose that $\neg p$ is true. Then at most three of the 22 days are the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen since three is the most days chosen that could be a particular day of the week. This is a contradiction. ◀

EXAMPLE 21 Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution: Let p be the proposition "$\sqrt{2}$ is irrational." Suppose that $\neg p$ is true. Then $\sqrt{2}$ is rational. We will show that this leads to a contradiction. Under the assumption that $\sqrt{2}$ is rational, there exist integers a and b with $\sqrt{2} = a/b$, where a and b have no common factors (so that the fraction a/b is in lowest terms). Since $\sqrt{2} = a/b$, when both sides of this equation are squared, it follows that

$$2 = a^2/b^2.$$

Hence,

$$2b^2 = a^2.$$

This means that a^2 is even, implying that a is even. Furthermore, since a is even, $a = 2c$ for some integer c. Thus

$$2b^2 = 4c^2,$$

so

$$b^2 = 2c^2.$$

This means that b^2 is even. Hence, b must be even as well.

It has been shown that $\neg p$ implies that $\sqrt{2} = a/b$, where a and b have no common factors, and 2 divides a and b. This is a contradiction since we have shown that $\neg p$ implies both r and $\neg r$ where r is the statement that a and b are integers with no common factors. Hence, $\neg p$ is false, so that p: "$\sqrt{2}$ is irrational" is true. ◀

An indirect proof of an implication can be rewritten as a proof by contradiction. In an indirect proof we show that $p \rightarrow q$ is true by using a direct proof to show that $\neg q \rightarrow \neg p$ is true. That is, in an indirect proof of $p \rightarrow q$ we assume that $\neg q$ is true and show that $\neg p$ must also be true. To rewrite an indirect proof of $p \rightarrow q$ as a proof by contradiction, we suppose that both p and $\neg q$ are true. Then we use the steps from the direct proof of $\neg q \rightarrow \neg p$ to show that $\neg p$ must also be true. This leads to the contradiction $p \wedge \neg p$, completing the proof by contradiction. Example 22 illustrates how an indirect proof of an implication can be rewritten as a proof by contradiction.

EXAMPLE 22 Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd."

Solution: We assume that $3n + 2$ is odd and that n is not odd, so that n is even. Following the same steps as in the solution of Example 15 (an indirect proof of this theorem), we can show that if n is even, then $3n + 2$ is even. This contradicts the assumption that $3n + 2$ is odd, completing the proof. ◀

PROOF BY CASES To prove an implication of the form

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q$$

the tautology

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$$

can be used as a rule of inference. This shows that the original implication with a hypothesis made up of a disjunction of the propositions $p_1, p_2, \ldots, p_n$ can be proved by proving each of the n implications $p_i \rightarrow q$, $i = 1, 2, \ldots, n$, individually. Such an argument is called a **proof by cases.** Sometimes to prove that an implication $p \rightarrow q$ is true, it is convenient to use a disjunction $p_1 \vee p_2 \vee \cdots \vee p_n$ instead of p as the hypothesis of the implication, where p and $p_1 \vee p_2 \vee \cdots \vee p_n$ are equivalent. Consider Example 23.

Extra Examples

EXAMPLE 23 Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers. (Recall that $|x|$, the absolute value of x, equals x when $x \geq 0$ and equals $-x$ when $x \leq 0$.)

Solution: Let p be "x and y are real numbers" and let q be "$|xy| = |x||y|$." Note that p is equivalent to $p_1 \vee p_2 \vee p_3 \vee p_4$, where p_1 is "$x \geq 0 \wedge y \geq 0$," p_2 is "$x \geq 0 \wedge y < 0$," p_3 is "$x < 0 \wedge y \geq 0$," and p_4 is "$x < 0 \wedge y < 0$." Hence, to show that $p \rightarrow q$, we can show that $p_1 \rightarrow q$, $p_2 \rightarrow q$, $p_3 \rightarrow q$, and $p_4 \rightarrow q$. (We have used these four cases because we can remove the absolute value signs by making the appropriate choice of signs within each case.)

We see that $p_1 \rightarrow q$ because $xy \geq 0$ when $x \geq 0$ and $y \geq 0$, so that $|xy| = xy = |x||y|$.

To see that $p_2 \rightarrow q$, note that if $x \geq 0$ and $y < 0$, then $xy \leq 0$, so that $|xy| = -xy = x(-y) = |x||y|$. (Here, because $y < 0$, we have $|y| = -y$.)

To see that $p_3 \rightarrow q$, we follow the same reasoning as the previous case with the roles of x and y reversed.

To see that $p_4 \rightarrow q$, note that when $x < 0$ and $y < 0$, it follows that $xy > 0$. Hence $|xy| = xy = (-x)(-y) = |x||y|$. This completes the proof. ◀

PROOFS OF EQUIVALENCE To prove a theorem that is a biconditional, that is, one that is a statement of the form $p \leftrightarrow q$ where p and q are propositions, the tautology

$$(p \leftrightarrow q) \leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$$

can be used. That is, the proposition "p if and only if q" can be proved if both the implications "if p, then q" and "if q, then p" are proved.

EXAMPLE 24 Prove the theorem "The integer n is odd if and only if n^2 is odd."

Solution: This theorem has the form "p if and only if q," where p is "n is odd" and q is "n^2 is odd." To prove this theorem, we need to show that $p \to q$ and $q \to p$ are true.

Extra Examples

We have already shown (in Example 14) that $p \to q$ is true and (in Example 19) that $q \to p$ is true.

Since we have shown that both $p \to q$ and $q \to p$ are true, we have shown that the theorem is true. ◀

Sometimes a theorem states that several propositions are equivalent. Such a theorem states that propositions $p_1, p_2, p_3, \ldots, p_n$ are equivalent. This can be written as

$$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n,$$

which states that all n propositions have the same truth values, and consequently, that for all i and j with $1 \le i \le n$ and $1 \le j \le n$, p_i and p_j are equivalent. One way to prove these mutually equivalent is to use the tautology

$$[p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n] \leftrightarrow [(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_n \to p_1)].$$

This shows that if the implications $p_1 \to p_2, p_2 \to p_3, \ldots, p_n \to p_1$ can be shown to be true, then the propositions $p_1, p_2, \ldots, p_n$ are all equivalent.

This is much more efficient than proving that $p_i \to p_j$ for all $i \ne j$ with $1 \le i \le n$ and $1 \le j \le n$.

When we prove that a group of statements are equivalent, we can establish any chain of implications we choose as long as it is possible to work through the chain to go from any one of these statements to any other statement. For example, we can show that p_1, p_2, and p_3 are equivalent by showing that $p_1 \to p_3$, $p_3 \to p_2$, and $p_2 \to p_1$.

EXAMPLE 25 Show that these statements are equivalent:

p_1: n is an even integer.
p_2: $n - 1$ is an odd integer.
p_3: n^2 is an even integer.

Solution: We will show that these three statements are equivalent by showing that the implications $p_1 \to p_2$, $p_2 \to p_3$, and $p_3 \to p_1$ are true.

We use a direct proof to show that $p_1 \to p_2$. Suppose that n is even. Then $n = 2k$ for some integer k. Consequently, $n - 1 = 2k - 1 = 2(k - 1) + 1$. This means that $n - 1$ is odd since it is of the form $2m + 1$, where m is the integer $k - 1$.

We also use a direct proof to show that $p_2 \to p_3$. Now suppose $n - 1$ is odd. Then $n - 1 = 2k + 1$ for some integer k. Hence, $n = 2k + 2$ so that $n^2 = (2k + 2)^2 = 4k^2 + 8k + 4 = 2(2k^2 + 4k + 2)$. This means that n^2 is twice the integer $2k^2 + 4k + 2$, and hence is even.

To prove $p_3 \to p_1$, we use an indirect proof. That is, we prove that if n is not even, then n^2 is not even. This is the same as proving that if n is odd, then n^2 is odd, which we have already done in Example 14. This completes the proof. ◀

THEOREMS AND QUANTIFIERS

Many theorems are stated as propositions that involve quantifiers. A variety of methods are used to prove theorems that are quantifications. We will describe some of the most important of these here.

EXISTENCE PROOFS Many theorems are assertions that objects of a particular type exist. A theorem of this type is a proposition of the form $\exists x P(x)$, where P is a predicate. A proof of a proposition of the form $\exists x P(x)$ is called an **existence proof.** There are several ways to prove a theorem of this type. Sometimes an existence proof of $\exists x P(x)$ can be given by finding an element a such that $P(a)$ is true. Such an existence proof is called **constructive.** It is also possible to give an existence proof that is **nonconstructive;** that is, we do not find an element a such that $P(a)$ is true, but rather prove that $\exists x P(x)$ is true in some other way. One common method of giving a nonconstructive existence proof is to use proof by contradiction and show that the negation of the existential quantification implies a contradiction. The concept of a constructive existence proof is illustrated by Example 26.

Extra Examples

EXAMPLE 26 **A Constructive Existence Proof.** Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Solution: After considerable computation (such as a computer search) we find that

$$1729 = 10^3 + 9^3 = 12^3 + 1^3.$$

Because we have displayed a positive integer that can be written as the sum of cubes in two different ways, we are done. ◀

EXAMPLE 27 **A Nonconstructive Existence Proof.** Show that there exist irrational numbers x and y such that x^y is rational.

Solution: By Example 21 we know that $\sqrt{2}$ is irrational. Consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational, we have two irrational numbers x and y with x^y rational, namely, $x = \sqrt{2}$ and $y = \sqrt{2}$. On the other hand if $\sqrt{2}^{\sqrt{2}}$ is irrational, then we can let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ so that $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^2 = 2$.

This proof is an example of a nonconstructive existence proof because we have not found irrational numbers x and y such that x^y is rational. Rather, we have shown that either the pair $x = \sqrt{2}$, $y = \sqrt{2}$ or the pair $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$ have the desired property, but we do not know which of these two pairs works! ◀

UNIQUENESS PROOFS Some theorems assert the existence of a unique element with a particular property. In other words, these theorems assert that there is exactly one element with this property. To prove a statement of this type we need to show that an element with this property exists and that no other element has this property. The two parts of a uniqueness proof are:

Existence: We show that an element x with the desired property exists.
Uniqueness: We show that if $y \neq x$, then y does not have the desired property.

HISTORICAL NOTE The English mathematician G. H. Hardy, when visiting the ailing Indian prodigy Ramanujan in the hospital, remarked that 1729, the number of the cab he took, was rather dull. Ramanujan replied "No, it is a very interesting number; it is the smallest number expressible as the sum of cubes in two different ways." (See the Supplementary Exercises in Chapter 3 for biographies of Hardy and Ramanujan.)

Remark: Showing that there is a unique element x such that $P(x)$ is the same as proving the statement $\exists x(P(x) \wedge \forall y(y \neq x \rightarrow \neg P(y)))$.

EXAMPLE 28 Show that every integer has a unique additive inverse. That is, show that if p is an integer, then there exists a unique integer q such that $p + q = 0$.

Extra Examples

Solution: If p is an integer, we find that $p + q = 0$ when $q = -p$ and q is also an integer. Consequently, there exists an integer q such that $p + q = 0$.

To show that given the integer p, the integer q with $p + q = 0$ is unique, suppose that r is an integer with $r \neq q$ such that $p + r = 0$. Then $p + q = p + r$. By subtracting p from both sides of the equation, it follows that $q = r$, which contradicts our assumption that $q \neq r$. Consequently, there is a unique integer q such that $p + q = 0$. ◀

Extra Examples

COUNTEREXAMPLES In Section 1.3 we mentioned that we can show that a statement of the form $\forall x P(x)$ is false if we can find a counterexample, that is, an example x for which $P(x)$ is false. When we are presented with a statement of the form $\forall x P(x)$, either which we believe to be false or which has resisted all attempts to find a proof, we look for a counterexample. We illustrate the hunt for a counterexample in Example 29.

EXAMPLE 29 Show that the statement "Every positive integer is the sum of the squares of three integers" is false.

Solution: We can show that this statement is false if we can find a counterexample. That is, the statement is false if we can show that there is a particular integer that is not the sum of the squares of three integers. To look for a counterexample, we try to write successive positive integers as a sum of three squares. We find that $1 = 0^2 + 0^2 + 1^2$, $2 = 0^2 + 1^2 + 1^2$, $3 = 1^2 + 1^2 - 1^2$, $4 = 0^2 + 0^2 + 2^2$, $5 = 0^2 + 1^2 + 2^2$, $6 = 1^2 + 1^2 + 2^2$, but we cannot find a way to write 7 as the sum of three squares. To show that there are not three squares that add up to 7, we note that the only possible squares we can use are those not exceeding 7, namely, 0, 1, and 4. Since no three terms where each term is 0, 1, or 4 add up to 7, it follows that 7 is a counterexample. We conclude that the statement "Every positive integer is the sum of the squares of three integers" is false. ◀

Links

A common error is to assume that one or more examples establish the truth of a statement. No matter how many examples there are where $P(x)$ is true, the universal quantification $\forall x P(x)$ may still be false. Consider Example 30.

EXAMPLE 30 Is it true that every positive integer is the sum of 18 fourth powers of integers? That is, is the statement $\forall n P(n)$ a theorem where $P(n)$ is the statement "n can be written as the sum of 18 fourth powers of integers" and the universe of discourse consists of all positive integers?

Solution: To determine whether n can be written as the sum of 18 fourth powers of integers, we might begin by examining whether n is the sum of 18 fourth powers of integers for the smallest positive integers. Because the fourth powers of integers are 0, 1, 16, 81, ..., if we can select 18 terms from these numbers that add up to n, then n is the sum of 18 fourth powers. We can show that all positive integers up to 78 can be written as the sum of 18 fourth powers. (The details are left to the reader.) However, if we decided this was enough checking, we would come to the wrong conclusion. It is not true that every positive integer is the sum of 18 fourth powers because 79 is not the sum of 18 fourth powers (as the reader can verify). ◀

MISTAKES IN PROOFS

There are many common errors made in constructing mathematical proofs. We will briefly describe some of these here. Among the most common errors are mistakes in arithmetic and basic algebra. Even professional mathematicians make such errors, especially when working with complicated formulas. Whenever you use such computations you should check them as carefully as possible. (You should also review any troublesome aspects of basic algebra, especially before you study Section 3.3.)

Each step of a mathematical proof needs to be correct and the conclusion needs to logically follow from the steps that precede it. Many mistakes result from the introduction of steps that do not logically follow from those that precede it. This is illustrated in Examples 31–33.

EXAMPLE 31 What is wrong with this famous supposed "proof" that $1 = 2$?

"Proof:" We use these steps, where a and b are two equal positive integers.

Step	**Reason**
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b

Solution: Every step is valid except for one, step 5 where we divided both sides by $a - b$. The error is that $a - b$ equals zero; division of both sides of an equation by the same quantity is valid as long as this quantity is not zero. ◀

EXAMPLE 32 What is wrong with this "proof"?

"Theorem:" If n^2 is positive, then n is positive.

"Proof:" Suppose that n^2 is positive. Because the implication "If n is positive, then n^2 is positive" is true, we can conclude that n is positive.

Solution: Let $P(n)$ be "n is positive" and $Q(n)$ be "n^2 is positive." Then our hypothesis is $Q(n)$. The statement "If n is positive, then n^2 is positive" is the statement $\forall n(P(n) \rightarrow Q(n))$. From the hypothesis $Q(n)$ and the statement $\forall n(P(n) \rightarrow Q(n))$ we cannot conclude $P(n)$, because we are not using a valid rule of inference. Instead, this is an example of the fallacy of affirming the conclusion. A counterexample is supplied by $n = -1$ for which $n^2 = 1$ is positive, but n is negative. ◀

EXAMPLE 33 What is wrong with this "proof"?

> "Theorem:" If n is not positive, then n^2 is not positive. (This is the contrapositive of the "theorem" in Example 32.)

"Proof:" Suppose that n is not positive. Because the implication "If n is positive, then n^2 is positive" is true, we can conclude that n^2 is not positive.

Solution: Let $P(n)$ and $Q(n)$ be as in the solution of Example 32. Then our hypothesis is $\neg P(n)$ and the statement "If n is positive, then n^2 is positive" is the statement $\forall n(P(n) \rightarrow Q(n))$. From the hypothesis $\neg P(n)$ and the statement $\forall n(P(n) \rightarrow Q(n))$ we cannot conclude $\neg Q(n)$, because we are not using a valid rule of inference. Instead, this is an example of the fallacy of denying the hypothesis. A counterexample is supplied by $n = -1$, as in Example 32. ◀

A common error in making unwarranted assumptions occurs in proofs by cases, where not all cases are considered. This is illustrated in Example 34.

EXAMPLE 34 What is wrong with this "proof"?

> "Theorem:" If x is a real number, then x^2 is a positive real number.

"Proof:" Let p_1 be "x is positive," let p_2 be "x is negative," and let q be "x^2 is positive." To show that $p_1 \rightarrow q$, note that when x is positive, x^2 is positive since it is the product of two positive numbers, x and x. To show that $p_2 \rightarrow q$, note that when x is negative, x^2 is positive since it is the product of two negative numbers, x and x. This completes the proof.

Solution: The problem with the proof we have given is that we missed the case $x = 0$. When $x = 0, x^2 = 0$ is not positive, so the supposed theorem is false. If p is "x is a real number," then we can prove results where p is the hypothesis with three cases, p_1, p_2, and p_3, where p_1 is "x is positive," p_2 is "x is negative," and p_3 is "$x = 0$" because of the equivalence $p \leftrightarrow p_1 \vee p_2 \vee p_3$. ◀

Finally, we briefly discuss a particularly nasty type of error. Many incorrect arguments are based on a fallacy called **begging the question.** This fallacy occurs when one or more steps of a proof are based on the truth of the statement being proved. In other words, this fallacy arises when a statement is proved using itself, or a statement equivalent to it. That is why this fallacy is also called **circular reasoning.**

EXAMPLE 35 Is the following argument correct? It supposedly shows that n is an even integer whenever n^2 is an even integer.

> Suppose that n^2 is even. Then $n^2 = 2k$ for some integer k. Let $n = 2l$ for some integer l. This shows that n is even.

Solution: This argument is incorrect. The statement "let $n = 2l$ for some integer l" occurs in the proof. No argument has been given to show that n can be written as $2l$ for some integer l. This is circular reasoning because this statement is equivalent to the statement being proved, namely, "n is even." Of course, the result itself is correct; only the method of proof is wrong. ◀

Making mistakes in proofs is part of the learning process. When you make a mistake that someone else finds, you should carefully analyze where you went wrong and make sure that you do not make the same mistake again. Even professional mathematicians make mistakes in proofs. More than a few incorrect proofs of important results have fooled people for many years before subtle errors in them were found.

JUST A BEGINNING

We have introduced a variety of methods for proving theorems. Observe that no algorithm for proving theorems has been given here or even mentioned. It is a deep result that no such procedure exists.

There are many theorems whose proofs are easy to find by directly working through the hypotheses and definitions of the terms of the theorem. However, it is often difficult to prove a theorem without resorting to a clever use of an indirect proof or a proof by contradiction, or some other proof technique. Constructing proofs is an art that can be learned only through experience, including writing proofs, having your proofs critiqued, reading and analyzing other proofs, and so on.

We will present a variety of proofs in the rest of this chapter and in Chapter 2 before we return to the subject of proofs. In Chapter 3 we will address some of the art and the strategy in proving theorems and in working with conjectures. We will also introduce several important proof techniques in Chapter 3, including mathematical induction, which can be used to prove results that hold for all positive integers. In Chapter 4 we will introduce the notion of combinatorial proofs.

Exercises

1. What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

2. What rule of inference is used in each of these arguments?

a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.

b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.

c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

d) Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.

e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

3. Construct an argument using rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

4. Construct an argument using rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

5. What rules of inference are used in this famous argument? "All men are mortal. Socrates is a man. Therefore, Socrates is mortal."

6. What rules of inference are used in this argument? "No man is an island. Manhattan is an island. Therefore, Manhattan is not a man."

7. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) "If I take the day off, it either rains or snows." "I took Tuesday off or I took Thursday off." "It was sunny on Tuesday." "It did not snow on Thursday."
b) "If I eat spicy foods, then I have strange dreams." "I have strange dreams if there is thunder while I sleep." "I did not have strange dreams."
c) "I am either clever or lucky." "I am not lucky." "If I am lucky, then I will win the lottery."
d) "Every computer science major has a personal computer." "Ralph does not have a personal computer." "Ann has a personal computer."
e) "What is good for corporations is good for the United States." "What is good for the United States is good for you." "What is good for corporations is for you to buy lots of stuff."
f) "All rodents gnaw their food." "Mice are rodents." "Rabbits do not gnaw their food." "Bats are not rodents."

8. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."
b) "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly snnny on Friday."
c) "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."
d) "Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."
e) "All foods that are healthy to eat do not taste good." "Tofu is healthy to eat." "You only eat what tastes good." "You do not eat tofu." "Cheeseburgers are not healthy to eat."
f) "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."

9. For each of these arguments, explain which rules of inference are used for each step.

a) "Doug, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get a high-paying job. Therefore, someone in this class can get a high-paying job."
b) "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."
c) "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."
d) "Everyone in New Jersey lives within 50 miles of the ocean. Someone in New Jersey has never seen the ocean. Therefore, someone who lives within 50 miles of the ocean has never seen the ocean."

10. For each of these arguments, explain which rules of inference are used for each step.

a) "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."
b) "Each of five roommates, Melissa, Aaron, Ralph, Veneesha, and Keeshawn, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."
c) "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."
d) "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."

11. For each of these arguments determine whether the argument is correct or incorrect and explain why.

a) All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic.
b) Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.
c) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.
d) Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.

12. For each of these arguments determine whether the argument is correct or incorrect and explain why.

a) Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.
b) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.
c) Quincy likes all action movies. Quincy likes the movie *Eight Men Out*. Therefore, *Eight Men Out* is an action movie.

d) All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.

13. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

a) If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.
b) The number $\log_2 3$ is irrational if it is not the ratio of two integers. Therefore, since $\log_2 3$ cannot be written in the form a/b where a and b are integers, it is irrational.
c) If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leq 9$. Then $n \leq 3$.
d) If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

14. Determine whether these are valid arguments.

a) "If x^2 is irrational, then x is irrational. Therefore, if x is irrational, it follows that x^2 is irrational."
b) "If x^2 is irrational, then x is irrational. The number $x = \pi^2$ is irrational. Therefore, the number $x = \pi$ is irrational."

15. What is wrong with this argument? Let $H(x)$ be "x is happy." Given the premise $\exists x H(x)$, we conclude that $H(\text{Lola})$. Therefore, Lola is happy.

16. What is wrong with this argument? Let $S(x, y)$ be "x is shorter than y." Given the premise $\exists s S(s, \text{Max})$, it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.

17. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

18. Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did yon nse?

19. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a + b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

20. Prove that the square of an even number is an even number using

a) a direct proof. **b)** an indirect proof.
c) a proof by contradiction.

21. Prove that if n is an integer and $n^3 + 5$ is odd, then n is even using

a) an indirect proof.
b) a proof by contradiction.

22. Prove that if n is an integer and $3n + 2$ is even, then n is even using

a) an indirect proof.
b) a proof by contradiction.

23. Prove that the sum of two odd integers is even.

24. Prove that the product of two odd numbers is odd.

25. Prove that the sum of an irrational number and a rational number is irrational using a proof by contradiction.

26. Prove that the product of two rational numbers is rational.

27. Prove or disprove that the product of two irrational numbers is irrational.

28. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

29. Prove that if x is irrational, then $1/x$ is irrational.

30. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

31. Show that at least 10 of any 64 days chosen must fall on the same day of the week.

32. Show that at least 3 of any 25 days chosen must fall in the same month of the year.

33. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. (*Hint:* Use a proof by cases, with the two cases corresponding to $x \geq y$ and $x < y$, respectively.)

34. Use a proof by cases to show that $\min(a, \min(b, c)) = \min(\min(a, b), c)$ whenever a, b, and c are real numbers.

35. Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

36. Prove that a square of an integer ends with a 0, 1, 4, 5, 6, or 9. (*Hint:* Let $n = 10k + l$ where $l = 0, 1, \ldots, 9$.)

37. Prove that a fourth power of an integer ends with a 0, 1, 5, or 6.

38. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

39. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

40. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

41. Prove or disprove that if m and n are integers such that $mn = 1$, then either $m = 1$ and $n = 1$, or else $m = -1$ and $n = -1$.

42. Show that these three statements are equivalent where a and b are real numbers: (*i*) a is less than b, (*ii*) the average of a and b is greater than a, and (*iii*) the average of a and b is less than b.

43. Show that these statements are equivalent: (*i*) $3x + 2$ is an even integer, (*ii*) $x + 5$ is an odd integer, (*iii*) x^2 is an even integer.

44. Show that these statements are equivalent: (*i*) x is rational, (*ii*) $x/2$ is rational, and (*iii*) $3x - 1$ is rational.

45. Show that these statements are equivalent: (*i*) x is irrational, (*ii*) $3x + 2$ is irrational, (*iii*) $x/2$ is irrational.

46. Is this reasoning for finding the solutions of the equation $\sqrt{2x^2 - 1} = x$ correct? (*1*) $\sqrt{2x^2 - 1} = x$ is given; (*2*) $2x^2 - 1 = x^2$, obtained by squaring both sides of (1); (*3*) $x^2 - 1 = 0$, obtained by subtracting x^2 from both sides of (2); (*4*) $(x - 1)(x + 1) = 0$, obtained by factoring the left-hand side of $x^2 - 1$; (*5*) $x = 1$

or $x = -1$, which follows since $ab = 0$ implies that $a = 0$ or $b = 0$.

47. Are these steps for finding the solutions of $\sqrt{x+3} = 3 - x$ correct? (*1*) $\sqrt{x+3} = 3 - x$ is given; (*2*) $x + 3 = x^2 - 6x + 9$, obtained by squaring both sides of (1); (*3*) $0 = x^2 - 7x + 6$, obtained by subtracting $x + 3$ from both sides of (2); (*4*) $0 = (x-1)(x-6)$, obtained by factoring the right-hand side of (3); (*5*) $x = 1$ or $x = 6$, which follows from (4) since $ab = 0$ implies that $a = 0$ or $b = 0$.

48. Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?

49. Prove that there are 100 consecutive positive integers that are not perfect squares. Is your proof constructive or nonconstructive?

50. Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square. Is your proof constructive or nonconstructive?

51. Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

52. Show that the product of two of the numbers $65^{1000} - 8^{2001} + 3^{177}$, $79^{1212} - 9^{2399} + 2^{2001}$, and $24^{4493} - 5^{8192} + 7^{1777}$ is nonnegative. Is your proof constructive or nonconstructive? (*Hint:* Do not try to evaluate these numbers!)

53. Show that each of these statements can be used to express the fact that there is a unique element x such that $P(x)$ is true. [Note that by Exercise 48 in Section 1.3, this is the statement $\exists! P(x)$.]

a) $\exists x \forall y (P(y) \leftrightarrow x = y)$
b) $\exists x P(x) \wedge \forall x \forall y (P(x) \wedge P(y) \rightarrow x = y)$
c) $\exists x (P(x) \wedge \forall y (P(y) \rightarrow x = y))$

54. Show that if a, b, and c are real numbers and $a \neq 0$, then there is a unique solution of the equation $ax + b = c$.

55. Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a - c| = |b - c|$.

56. Show that if r is an irrational number, there is a unique integer n such that the distance between r and n is less than $1/2$.

57. Show that if n is an odd integer, then there is a unique integer k such that n is the sum of $k - 2$ and $k + 3$.

58. Prove that given a real number x there exist uniqne numbers n and ϵ snch that $x = n + \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.

59. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n - \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.

60. Use resolution to show the hypotheses "Allen is a bad boy or Hillary is a good girl" and "Allen is a good boy or David is happy" imply the conclusion "Hillary is a good girl or David is happy."

61. Use resolution to show that the hypotheses "It is not raining or Yvette has her umbrella," "Yvette does not have her umbrella or she does not get wet," and "It is raining or Yvette does not get wet" imply that "Yvette does not get wet."

62. Show that the equivalence $p \wedge \neg p \equiv \mathbf{F}$ can be derived using resolution together with the fact that an implication with a false hypothesis is true. (*Hint:* Let $q = r = \mathbf{F}$ in resolution.)

63. Use resolution to show that the compound proposition $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$ is not satisfiable.

64. Prove or disprove that if a and b are rational numbers, then a^b is also rational.

65. Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.

66. Show that the propositions p_1, p_2, p_3, and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4$, $p_2 \leftrightarrow p_3$, and $p_1 \leftrightarrow p_3$.

67. Show that the propositions p_1, p_2, p_3, p_4, and p_5 can be shown to be equivalent by proving that the implications $p_1 \rightarrow p_4$, $p_3 \rightarrow p_1$, $p_4 \rightarrow p_2$, $p_2 \rightarrow p_5$, and $p_5 \rightarrow p_3$ are true.

68. Prove that an 8×8 chessboard can be completely covered using dominos (1×2 pieces).

***69.** Prove that it is impossible to cover completely with dominos the 8×8 chessboard with two squares at opposite corners of the board removed.

***70.** The Logic Problem, taken from *WFF'N PROOF, The Game of Logic*, has these two assumptions:

1. "Logic is difficult or not many students like logic."
2. "If mathematics is easy, then logic is not difficult."

By translating these assumptions into statements involving propositional variables and logical connectives, determine whether each of the following are valid conclusions of these assumptions:

a) That mathematics is not easy, if many students like logic.
b) That not many students like logic, if mathematics is not easy.
c) That mathematics is not easy or logic is difficult.
d) That logic is not difficult or mathematics is not easy.
e) That if not many students like logic, then either mathematics is not easy or logic is not difficult.

71. Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

72. Use Exercise 71 to show that if the first 10 positive integers are placed around a circle, in any order, there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

73. Prove that if n is an integer, these four statements are equivalent: (*i*) n is even, (*ii*) $n+1$ is odd, (*iii*) $3n+1$ is odd, (*iv*) $3n$ is even.

74. Prove that these four statements are equivalent: (*i*) n^2 is odd, (*ii*) $1-n$ is even, (*iii*) n^3 is odd, (*iv*) n^2+1 is even.

75. Which rules of inference are used to establish the conclusion of Lewis Carroll's argument described in Example 19 of Section 1.3?

76. Which rules of inference are used to establish the conclusion of Lewis Carroll's argument described in Example 20 of Section 1.3?

***77.** Determine whether this argument, taken from Backhouse [Ba86], is valid.

> If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

1.6 Sets

INTRODUCTION

We will study a wide variety of discrete structures in this book. These include relations, which consist of ordered pairs of elements; combinations, which are unordered collections of elements; and graphs, which are sets of vertices and edges connecting vertices. Moreover, we will illustrate how these and other discrete structures are used in modeling and problem solving. In particular, many examples of the use of discrete structures in the storage, communication, and manipulation of data will be described. In this section we study the fundamental discrete structure upon which all other discrete structures are built, namely, the set.

Sets are used to group objects together. Often, the objects in a set have similar properties. For instance, all the students who are currently enrolled in your school make up a set. Likewise, all the students currently taking a course in discrete mathematics at any school make up a set. In addition, those students enrolled in your school who are taking a course in discrete mathematics form a set that can be obtained by taking the elements common to the first two collections. The language of sets is a means to study such collections in an organized fashion. We now provide a definition of a set.

DEFINITION 1 A *set* is an unordered collection of objects.

Links

Note that the term *object* has been used without specifying what an object is. This description of a set as a collection of objects, based on the intuitive notion of an object, was first stated by the German mathematician Georg Cantor in 1895. The theory that results from this intuitive definition of a set leads to **paradoxes,** or logical inconsistencies, as the English philosopher Bertrand Russell showed in 1902 (see Exercise 30 for a description of one of these paradoxes). These logical inconsistencies can be avoided by building set theory starting with basic assumptions, called **axioms.** We will use Cantor's original version of set theory, known as **naive set theory,** without developing an axiomatic version of set theory, since all sets considered in this book can be treated consistently using Cantor's original theory.

We now proceed with our discussion of sets.