B.S Grewal Higher Engineering Mathematics

1

Solution of Equations

1. Introduction. 2. General Properties. 3. Relations between roots and co-efficients. 4. Transformations of equations. 5. Reciprocal equations. 6. Solution of cubic equations–Cardan's method. 7. Solution of biquadratic equations–Ferrari's method ; Descarte's method. 8. Graphical solution of equations. 9. Objective Type of Questions.

1.1. INTRODUCTION

The expression $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$

where a's are constants ($a_0 \neq 0$) and n is a positive integer, is called a *polynomial* in x of degree n. The polynomial $f(x) = 0$ is called an *algebraic equation of degree n*. If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc. ; then $f(x) = 0$ is called a *transcendental equation*.

The value of x which satisfies $f(x) = 0$, ...(1)

is called its root. Geometrically, a root of (1) is that value of x where the graph of $y = f(x)$ crosses the x-axis. The process of finding the roots of an equation is known as *solution* of that equation. This is a problem of basic importance in applied mathematics. We often come across problems in deflection of beams, electrical circuits and mechanical vibrations which depend upon the solution of equations. As such, a brief account of solution of equations is given in this chapter.

1.2. GENERAL PROPERTIES

I. *If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $x - \alpha$ and conversely.*

For instance, 3 is a root of the equation $x^4 - 6x^2 - 8x - 3 = 0$, because $x = 3$ satisfies this equation.

$\therefore$ $x - 3$ divides $x^4 - 6x^2 - 8x - 3$ completely, *i.e.* $x - 3$ is its factor.

II. *Every equation of the nth degree has n roots (real or imaginary).*

Conversely if $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ be the roots of the nth degree equation $f(x) = 0$, then

$$f(x) = A(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \text{ where } A \text{ is a constant.}$$

Obs. If a polynomial of degree n vanishes for more than n values of x, it must be identically zero.

Example 1.1. *Solve the equation $2x^3 + x^2 - 13x + 6 = 0$.*

Sol. By inspection, we find $x = 2$ satisfies the given equation.

$\therefore$ 2 is its root, *i.e.* $x - 2$ is a factor of $2x^3 + x^2 - 13x + 6$. Dividing this polynomial by $x - 2$, we get the quotient $2x^2 + 5x - 3$ and remainder 0.

Equating the quotient to zero, we get $2x^2 + 5x - 3 = 0$.

Solving this quadratic, we get $x = \dfrac{-5 \pm \sqrt{[5^2 - 4\ (2)\ (-3)]}}{2 \times 2} = \dfrac{-5 \pm 7}{4} = -3, \dfrac{1}{2}$

∴ the roots of the given equation are $2, -3, 1/2$.

Note. *The labour of dividing the polynomial by $x-2$ can be saved considerably by the following simple device called* **synthetic division :**

2	1	–13	6	2
	4	10	–6	
2	5	–3	0	

[*Explanation* : (*i*) Write down the co-efficients of the powers of x in order (supplying the missing powers of x by zero co-efficients) and write 2 on extreme right.

(*ii*) Put 2 as the first term of 3rd row and multiply it by 2, write 4 under 1 and add, giving 5.

(*iii*) Multiply 5 by 2, write 10 under -13 and add, giving -3.

(*iv*) Multiply -3 by 2, write -6 under 6 and add giving zero].

Thus the quotient is $2x^2 + 5x - 3$ and remainder is zero.

Obs. To divide a polynomial by $x + h$, we write $-h$ on the extreme right.

III. Intermediate value property. *If $f(a)$ and $f(b)$ have different signs, then the equation $f(x) = 0$ has atleast one root between $x = a$ and $x = b$.*

The polynomial $f(x)$ is a continuous function of x.

So while x changes from a to b, $f(x)$ must pass through all the values from $f(a)$ to $f(b)$. But since one of these quantities $f(a)$ or $f(b)$ is positive and the other negative, it follows that atleast for one value of x (say α) lying between a and b, $f(x)$ must be zero. Then α is the required root.

IV. *In an equation with real coefficients, imaginary roots occur in conjugate pairs, i.e.* if $\alpha + i\beta$ is a root of the equation $f(x) = 0$, then $\alpha - i\beta$ must also be its root. (See p. 592)

Similarly if $a + \sqrt{b}$ is an irrational root of an equation, then $a - \sqrt{b}$ must also be its root.

Obs. *Every equation of the odd degree has atleast one real root.*

This follows from the fact that imaginary roots occur in conjugate pairs.

Example 1·2. *Solve the equation $3x^3 - 4x^2 + x + 88 = 0$, one root being $2 + \sqrt{7}i$*

Sol. Since one root is $2 + \sqrt{7}i$, the other root must be $2 - \sqrt{7}i$.

∴ The factors corresponding to these roots are

$$(x - 2 - \sqrt{7}i) \text{ and } (x - 2 + \sqrt{7}i)$$

or $$(x - 2 - \sqrt{7}i)(x - 2 + \sqrt{7}i) = (x-2)^2 + 7 = x^2 - 4x + 11$$

is a divisor of $3x^3 - 4x^2 + x + 88$ ...(*i*)

∴ Division of (*i*) by $x^2 - 4x + 11$ gives $3x + 8$ as the quotient.

Thus the depressed equation is $3x + 8 = 0$. Its root is $-8/3$. Hence the roots of the given equation are $2 \pm \sqrt{7}i, -8/3$.

V. Descarte's rule of signs.* *The equation $f(x) = 0$ cannot have more positive roots than the changes of signs in $f(x)$; and more negative roots than the changes of signs in $f(-x)$.*

For instance, consider the equation $f(x) = 2x^7 - x^5 + 4x^3 - 5 = 0$ ...(1)

Signs of $f(x)$ are + − + −

Clearly $f(x)$ has 3 changes of signs (from + to − or − to +).

Thus (*i*) cannot have more than 3 positive roots.

Also $$f(-x) = 2(-x)^7 - (-x)^5 + 4(-x)^3 - 5$$
$$= -2x^7 + x^5 - 4x^3 - 5$$

This shows that $f(-x)$ has 2 changes of signs. Thus (*i*) cannot have more than 2 negative roots.

* After the French mathematician and philosopher *Rene Descartes* (1596–1650), who invented Analytic geometry in 1637.

Obs. Existence of imaginary roots. *If an equation of the nth degree has at the most p positive roots and at the most q negative roots, then it follows that the equation has at least n − (p + q) imaginary roots.*

Evidently (1) above is an equation of the 7th degree and has at the most 3 positive roots and 2 negative roots. Thus (1) has atleast 2 imaginary roots.

1·3. RELATIONS BETWEEN ROOTS AND COEFFICIENTS

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ *be the roots of the equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

then

$$\Sigma\alpha_1 = -\frac{a_1}{a_0}, \quad \Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}, \quad \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

....

$$\alpha_1\alpha_2\alpha_3 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

Proof. Since $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ are the roots of (1), we have

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$$
$$= a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots(x-\alpha_n)$$
$$= a_0\,[x^n - (\alpha_1+\alpha_2+\ldots\ldots+\alpha_n)\,x^{n-1} + (\alpha_1\alpha_2+\alpha_2\alpha_3+\ldots)\,x^{n-2}$$
$$- (\alpha_1\alpha_2\alpha_3+\alpha_2\alpha_3\alpha_4+\ldots\ldots)\,x^{n-3} + \ldots\ldots + (-1)^n\,\alpha_1\alpha_2\alpha_3\ldots\alpha_n]$$

Dividing both sides by a_0 and writing

$$\alpha_1+\alpha_2+\alpha_3+\ldots\ldots+\alpha_n = \Sigma\alpha_1$$
$$\alpha_1\alpha_2+\alpha_2\alpha_3+\ldots\ldots = \Sigma\alpha_1\alpha_2$$
$$\alpha_1\alpha_2\alpha_3+\alpha_2\alpha_3\alpha_4+\ldots\ldots = \Sigma\alpha_1\alpha_2\alpha_3$$ and so on, we get

$$x^n + \left(\frac{a_1}{a_0}\right)x^{n-1} + \left(\frac{a_2}{a_0}\right)x^{n-2} + \ldots\ldots + \left(\frac{a_{n-1}}{a_0}\right)x + \frac{a_n}{a_0}$$
$$= x^n - (\Sigma\alpha_1)\,x^{n-1} + (\Sigma\alpha_1\alpha_2)\,x^{n-2} - \ldots\ldots + (-1)^n\,\alpha_1\alpha_2\alpha_3\ldots\ldots\alpha_n$$

Equating coefficients of the like powers of x from both sides, we get the desired results.

Example 1·3. *Solve the equation $x^3 - 7x^2 + 36 = 0$, given that one root is double of another.*

Sol. Let the roots be α, β, γ such that $\beta = 2\alpha$.

Also

$$\alpha+\beta+\gamma = 7,\ \alpha\beta+\beta\gamma+\gamma\alpha = 0,\ \alpha\beta\gamma = -36$$

$\therefore$

$$3\alpha+\gamma = 7$$
$$2\alpha^2 + 3\alpha\gamma = 0 \qquad \ldots(i)$$
$$2\alpha^2\gamma = -36 \qquad \ldots(ii)$$

Solving (*i*) and (*ii*), we get $\alpha = 3, \gamma = -2$. ...(*iii*)

[The values $\alpha = 0, \gamma = 7$ are inadmissible, as they do not satisfy (*iii*)].

Hence the roots are 3, 6, and − 2.

Example 1·4. *Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, given that the sum of two of its roots is zero.*

(*Cochin, 2005 ; Madras, 2003*)

Sol. Let the roots be $\alpha, \beta, \gamma, \delta$ such that $\alpha+\beta = 0$.

Also

$$\alpha+\beta+\gamma+\delta = 2 \quad \therefore \quad \gamma+\delta = 2$$

Thus the quadratic factor corresponding to α, β is of the form $x^2 - 0x + p$, and that corresponding to γ, δ is of the form of $x^2 - 2x + q$.

$$\therefore \quad x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 + p)(x^2 - 2x + q) \qquad \ldots(i)$$

Equating coefficients of x^2 and x from both sides of (*i*), we get

$$4 = p + q, \quad 6 = -2p.$$

$$\therefore \quad p = -3, \quad q = 7.$$

Hence the given equation is equivalent to $(x^2 - 3)(x^2 - 2x + 7) = 0$

$\therefore$ The roots are $x = \pm\sqrt{3}, 1 \pm i\sqrt{6}$.

Example 1·5. *Find the condition that the cubic $x^3 - lx^2 + mx - n = 0$ should have its roots in* (Madras, 2000 S)

(a) arithmetical progression.

(b) geometrical progression.

Sol. (*a*) Let the roots be $a - d, a, a + d$ so that the sum of the roots $= 3a = l$ i.e. $a = l/3$

Since a is the root of the given equation

$$\therefore \quad a^3 - la^2 + ma - n = 0$$

Substituting $a = l/3$, we get $(l/3)^3 - l\,(l/3)^2 + m\,(l/3) - n = 0$.

or $2l^3 - 9lm + 27n = 0$ which is the required condition.

(*b*) Let the roots be $a/r, a, ar$, then the product of the roots $= a^3 = n$.

Putting $a = (n)^{1/3}$, in (*i*), we get $n - ln^{2/3} + mn^{1/3} - n = 0$ or $m = ln^{1/3}$

Cubing both sides, we get $m^3 = l^3 n$, which is the required condition.

Example 1·6. *Solve the equation $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$ whose roots are in A.P.*

Sol. Let the roots be $a - 3d, a - d, a + d, a + 3d$, so that the sum of the roots $= 4a = 2$.

$$\therefore \quad a = 1/2$$

Also product of the roots $= (a^2 - 9d^2)(a^2 - d^2) = 40$

or $\left(\frac{1}{4} - 9d^2\right)\left(\frac{1}{4} - d^2\right) = 40$ or $144d^4 - 40d^2 - 639 = 0$

$$\therefore \quad d^2 = 9/4 \quad \text{or} -7/36$$

Thus $d = \pm 3/2$, the other value is not admissible.

Hence the required roots are $-4, -1, 2, 5$.

Example 1·7. *Solve the equation $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0$, whose roots are in G.P.*

Sol. Let the roots be $a/r^3, a/r, ar, ar^3$, so that product of the roots $= a^4 = 4$.

Also the products of $a/r^3, ar^3$ and $a/r, ar$ are each $= a^2 = 2$.

$\therefore$ The factors corresponding to $a/r^3, ar^3$ and $a/r, ar$ are $x^2 + px + 2, x^2 + qx + 2$.

Thus $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 2(x^2 + px + 2)(x^2 + qx + 2)$

Equating coefficients of x^3 and x^2, we get

$$-15 = 2p + 2q \quad \text{and} \quad 35 = 8 + 2pq$$

$$\therefore \quad p = -9/2, \quad q = -3.$$

Thus the given equation is $2\left(x^2 - \frac{9}{2}x + 2\right)(x^2 - 3x + 2) = 0$

Hence the roots are $1/2, 4$ and $1, 2$ *i.e.*, $\frac{1}{2}, 1, 2, 4$.

Example 1·8. *If α, β, γ be the roots of the equation $x^3 + px + q = 0$, find the value of (a) $\Sigma\alpha^2\beta$, (b) $\Sigma\alpha^4$, (c) $\Sigma\alpha^3\beta$.*

Sol. We have

$$\alpha + \beta + \gamma = 0$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = p$$

$$\alpha\beta\gamma = -q$$

(*a*) Multiplying (*i*) and (*ii*), we get

$$\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + 3\alpha\beta\gamma = 0$$

or $$\Sigma\alpha^2\beta = -3\alpha\beta\gamma = 3q \quad \text{[by } (iii)]$$

(*b*) Multiplying the given equation by x, we get $x^4 + px^2 + qx = 0$

Putting $x = \alpha, \beta, \gamma$ successively and adding, we get $\Sigma\alpha^4 + p\Sigma\alpha^2 + q\Sigma\alpha = 0$

or $$\Sigma\alpha^4 = -p\Sigma\alpha^2 - q(0) \quad \ldots(iv)$$

Now squaring (*i*), we get $\alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 0$

or $$\Sigma\alpha^2 = -2p \quad \text{[by } (ii)]$$

Hence, substituting the value of $\Sigma\alpha^2$ in (*iv*), we obtain

$$\Sigma\alpha^4 = -p(-2p) = 2p^2.$$

(*c*) $\Sigma\alpha^3\beta = \Sigma\alpha^2\,\Sigma\alpha\beta - \alpha\beta\gamma\Sigma\alpha = -2p(p) - (-q)(0) = -2p^2.$

Problems 1.1

1. Form the equation of the fourth degree whose roots are $3 + i$ and $\sqrt{7}$. (*Madras, 2000 S*)
2. Solve the equation (*i*) $x^3 + 6x + 20 = 0$, one root being $1 + 3i$.
   (*ii*) $x^4 - 2x^3 - 22x^2 + 62x - 15 = 0$, given that $2 + \sqrt{3}$ is a root. (*Kottayam, 1996*)
3. Show that $x^7 - 3x^4 + 2x^3 - 1 = 0$ has atleast four imaginary roots. (*Cochin, 2005*)
4. Show that the equation $x^4 + 15x^2 + 7x - 11 = 0$ has one positive, one negative and two imaginary roots.
5. Find the number and position of real roots of $x^4 + 4x^3 - 4x - 13 = 0$.
6. Solve the equation $3x^3 - 11x^2 + 8x + 4 = 0$, given that two of its roots are equal.
7. The equation $x^4 - 4x^3 + ax^2 + 4x + b = 0$ has two pairs of equal roots. Find the values of a and b.

Solve the equations 8–14 :

8. $x^3 - 3x^2 - 16x + 48 = 0$, the sum of two of its roots being equal to zero.
9. $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$, given that it has two pairs of equal roots. (*Madras, 2003*)
10. $x^3 - 9x^2 + 14x + 24 = 0$, given that two of its roots are in the ratio 3 : 2.
11. $x^3 - 4x^2 - 20x + 48 = 0$ given that the roots α and β are connected by the relation $\alpha + 2\beta = 0$. (*S.V.T.U., 2007*)
12. $x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$ given that the sum of two of the roots is equal to the sum of the other two.
13. $x^3 - 12x^2 + 39x - 28 = 0$, roots being in arithmetical progression. (*Madras, 2001 S*)
14. $8x^3 - 14x^2 + 7x - 1 = 0$, roots being in geometrical progression. (*Osmania, 1999*)
15. O, A, B, C are the four points on a straight line such that the distances of A, B, C from O are the roots of equation $ax^3 + 3bx^2 + 3cx + d = 0$. If B is the middle point of AC, show that $a^2d - 3abc + 2b^3 = 0$ (*S.V.T.U., 2006*)
16. Solve the equations (*i*) $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ whose roots are in A.P.
    (*ii*) $x^4 + 5x^3 - 30x^2 + 40x + 64 = 0$ whose roots are in G.P.
17. If α, β, γ be the roots of the equation $x^3 - lx^2 + mx - n = 0$, find the value of
    (*i*) $\Sigma\alpha^2\beta^2$, (*ii*) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$
18. Find the sum of the cubes of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$.
19. If α, β, γ are the roots of [illegible] $= 0$, find the value of (*i*) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$
    (*ii*) $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$

20. If α, β, γ be the roots of $x^3 + px + q = 0$, show that
    (*i*) $\alpha^5 + \beta^5 + \gamma^5 = 5\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)$. (*ii*) $3\Sigma\alpha^2\,\Sigma\alpha^5 = 5\Sigma\alpha^3\,\Sigma\alpha^4$

1.4. TRANSFORMATION OF EQUATIONS

(1) To find an equation whose roots are m times the roots of the given equation, *multiply the second term by m, third term by m^2 and so on* (all missing terms supplied with zero coefficients).

For instance, let the given equation be

$$3x^4 + 6x^3 + 4x^2 - 8x + 11 = 0 \quad ...(i)$$

To multiply its roots by m, put $y = mx$ (or $x = y/m$) in (i).

Then $$3(y/m)^4 + 6(y/m)^3 + 4(y/m)^2 + 8(y/m) + 11 = 0$$

or multiplying by m^4, we get $3y^4 + m(6y^3) + m^2(4y^2) - m^3(8y) + m^4(11) = 0$

This is same as multiplying the second term by m, third term by m^2 and so on in (i).

Cor. To find an equation whose roots are with opposite signs to those of the given equation, *change the signs of the every alternative term of the given equation beginning with the second.*

Changing the signs of the roots of (i) is same as multiplying its roots by -1

The required equation will be

$$3x^4 + (-1)6x^3 + (-1)^2 4x^2 - (-1)^3 8x + (-1)^4 11 = 0$$

or $$3x^4 - 6x^3 + 4x^3 + 8x + 11 = 0$$

which is (i) with signs of every alternate term changed beginning with the second.

(2) To find an equation whose roots are reciprocal of the roots of the given equation, *change x to 1/x.*

Example 1.9. *Solve $6x^3 - 11x^2 - 3x + 2 = 0$, given that its roots are in harmonic progression.* (Triputi, 1998)

Sol. Since the roots of the given equation are in H.P., the roots of the equation having reciprocal roots will be in A.P.

The equation with reciprocal roots is $6(1/x)^3 - 11(1/x)^2 - 3(1/x) + 2 = 0$

or $$2x^3 - 3x^2 - 11x + 6 = 0 \quad ...(i)$$

Since the roots of the given equation are in H.P., therefore, the roots of (i) are in A.P. Let the roots be $a - d, a, a + d$. Then

$$3a = 3/2 \text{ and } a(a^2 - d^2) = -3.$$

Solving these equations, we get $a = 1/2, d = 5/2$.

Thus the roots of (1) are $-2, 1/2, 3$.

Hence the roots of the given equation are $-1/2, 2, 1/3$.

Example 1.10. *If α, β, γ be the roots of the cubic $x^3 - px^2 + qx - r = 0$, form the equation whose roots are $\beta\gamma + 1/\alpha, \gamma\alpha + 1/\beta, \alpha\beta + 1/\gamma$.*

Hence evaluate $\Sigma(\alpha\beta + 1/\gamma)(\beta\gamma + 1/\alpha)$.

Sol. If x is a root of the given equation and y a root of the required equation, then

$$y = \beta\gamma + 1/\alpha = \frac{\alpha\beta\gamma + 1}{\alpha} = \frac{r+1}{\alpha}$$

$$= \frac{r+1}{x} \quad [\because \alpha\beta\gamma = r]$$

Thus substituting $x = (r+1)/y$ in the given equation, we get

$$\left(\frac{r+1}{y}\right)^3 - p\left(\frac{r+1}{y}\right)^2 + q\left(\frac{r+1}{y}\right) - r = 0$$

or $$ry^3 - q(r+1)y^2 + p(r+1)^2 y - (r+1)^3 = 0,$$ which is the required equation.

Hence $\Sigma(\alpha\beta + 1/\gamma)(\beta\gamma + 1/\alpha) = p(r+1)^2/r$.

Example 1.11. *Form an equation whose roots are cubes of the roots of $x^3 - 3x^2 + 1 = 0$* ...(i)

Sol. If y be a root of the required equation, then $y = x^3$ ...(ii)

Now we have to eliminate x from (i) and (ii)

$\therefore$ Rewriting (i) as $x^3 + 1 = 3x^2$

Cubing both sides, $x^9 + 3x^6 + 3x^3 + 1 = 27x^6$

Substituting $x^3 = y$, we get $y^3 - 24y^2 + 3y + 1 = 0$

which is the required equation.

(3) To diminish the roots of an equation $f(x) = 0$ by h, *divide $f(x)$ by $x - h$ successively. Then the successive remainders determine the coefficients of the required equation.*

Let the given equation be

$$a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n = 0 \quad \ldots(i)$$

To diminish its roots by h, put $y = x - h$ (or $x = y + h$) in (i) so that

$$a_0(y+h)^n + a_1(y+h)^{n-1} + \ldots + a_n = 0 \quad \ldots(ii)$$

On simplification, it takes the form

$$A_0y^n + A_1y^{n-1} + \ldots + A_n = 0 \quad \ldots(iii)$$

Its coefficients $A_0, A_1, \ldots\ldots A_n$ can easily be found with the help of *synthetic division* (p. 2). For this, we put $y = x - h$ in (iii) so that

$$A_0(x-h)^n + A_1(x-h)^{n-1} + \ldots\ldots + A_n = 0 \quad \ldots(iv)$$

Clearly (i) and (iv) are identical. If we divide L.H.S. of (iv) by $x - h$, the remainder is A_n and the quotient $Q = A_0(x-h)^{n-1} + A_1(x-h)^{n-2} + \ldots\ldots + A_{n-1}$. Similarly if we divide Q by $x - h$, the remainder is A_{n-1} and the quotient is Q_1 (say). Again dividing Q_1 by $x - h$, A_{n-2} will be obtained and so on.

Obs. To increase the roots by h, we take h negative.

Example 1.12. *Transform the equation $x^3 - 6x^2 + 5x + 8 = 0$ into another in which the second term is missing. Hence find the equation of its squared differences.* (Cochin, 2005)

Sol. Sum of the roots of the given equation = 6.

In order that the second term in the transformed equation is missing, the sum of the roots is to be zero.

Since the equation has 3 roots, if we decrease each root by 2, the sum of the roots of the new equation will become zero.

$\therefore$ Dividing $x^3 - 6x^2 + 5x + 8$ by $x - 2$ successively, we have

1	– 6	5	8	(2
	2	– 8	– 6	
	– 4	– 3	2	
	2	– 4		
	– 2	– 7		
	2			
1	0			

Thus the transformed equation is $x^3 - 7x + 2 = 0$. ...(i)

If α, β, γ be the roots of the given equation, then the roots of (i) are $\alpha - 2, \beta - 2, \gamma - 2$

Let these be denoted by a, b, c.

Then $b - c = \beta - \gamma$. Also $a + b + c = 0$, $abc = -2$.

Now $(b - c)^2 = (b + c)^2 - 2bc = (a + b + c - a)^2 - \dfrac{2abc}{a} = a^2 + 4/a$

$\therefore$ The equation of squared differences of (i) is given by the transformation $y = x^2 + 4/x$

or $\qquad x^3 - xy + 4 = 0$ ...(ii)

Subtracting (ii) from (i), we get

$$-7x + xy - 2 = 0 \quad \text{or} \quad x = 2/(y - 7)$$

Substituting for x in (i), the equation becomes

$$[2/(y - 7)]^3 - 7\,[2/(y - 7)] + 2 = 0$$

or $\qquad y^3 - 28y^2 + 245y - 682 = 0$ ...(iii)

Roots of this equation are $(b - c)^2, (c - a)^2, (a - b)^2$ *i.e.* $(\beta - \gamma)^2, (\gamma - \alpha)^2, (\alpha - \beta)^2$.

Hence (iii) is the required equation.

1·5. RECIPROCAL EQUATIONS

If an equation remains unaltered on changing x to 1/x, it is called a **reciprocal equation.** *Such equations are of the following standard types :*

I. *A reciprocal equation of an odd degree having coefficients of terms equidistant from the beginning and end equal.* It has a root $= -1$.

II. *A reciprocal equation of an odd degree having coefficient of terms equidistant from the beginning and end equal but opposite in sign.* It has root $= 1$.

III. *A reciprocal equation of an even degree having coefficients of terms equidistant from the beginning and end equal but opposite in sign.* Such an equation has two roots $= 1$ and -1.

The substitution $x + 1/x = y$ reduces the degree of the equation to half its former degree.

Example 1·13. *Solve* $6x^5 - 41x^4 + 97x^3 - 97x^2 + 41x - 6 = 0$. *(Coimbatore, 2001 S)*

Sol. This is a reciprocal equation of odd degree with opposite signs. $\therefore x = 1$ is a root.

Dividing L.H.S. by $x - 1$, the given equation reduces to

$$6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$$

Dividing by x^2, we have

$$6\,(x^2 + 1/x^2) - 35\,(x + 1/x) + 62 = 0$$

Putting $x + 1/x = y$ and $x^2 + 1/x^2 = y^2 - 2$, we get

$$6\,(y^2 - 2) - 35y + 62 = 0 \quad \text{or} \quad 6y^2 - 35y + 50 = 0$$

or $\qquad (3y - 1)(2y - 5) = 0$

$\therefore \qquad x + 1/x = y = 1/3 \quad \text{or} \quad 5/2$

i.e., $\qquad 3x^2 - 10x + 3 = 0 \quad \text{or} \quad 2x^2 - 5x + 2 = 0$

i.e., $\qquad (3x - 1)(x - 3) = 0 \quad \text{or} \quad (2x - 1)(x - 2) = 0$

$\therefore \qquad x = 1/3, 3 \quad \text{or} \quad 1/2, 2$

Hence the roots are 1, 1/3, 3, 1/2, 2.

Example 1·14. Solve $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$. *(Madras, 2003)*

Sol. This is a reciprocal equation of even degree with opposite signs. $\therefore x = 1, -1$ are its roots

Dividing L.H.S. by $x - 1$ and $x + 1$, the given equation reduces to

$$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$$

Dividing by x^2, we get

$$6\,(x^2 + 1/x^2) - 25\,(x + 1/x) + 37 = 0.$$

Putting $x + 1/x = y$ and $x^2 + 1/x^2 = y^2 - 2$, it becomes

$$6\,(y^2 - 2) - 25y + 37 = 0 \quad \text{or} \quad 6y^2 - 25y + 25 = 0$$

or $\quad (2y - 5)(3y - 5) = 0$

$\therefore \quad x + 1/x = y = 5/2 \quad \text{or} \quad 5/3.$

i.e., $\quad 2x^2 - 5x + 2 = 0 \quad \text{or} \quad 3x^2 - 5x + 3 = 0$

$\therefore \quad x = 2,\ 1/2 \quad \text{or} \quad x = \dfrac{5 \pm i\sqrt{11}}{6}$

Hence the roots of the given equation are $1, -1, 2, 1/2, \dfrac{5 \pm i\sqrt{11}}{6}$.

Problems 1.2

1. Find the equation whose roots are 3 times the roots of $x^3 + 2x^2 - 4x + 1 = 0$.
2. Find the equation whose roots are the reciprocals of the roots of $x^4 - 7x^3 + 8x^2 + 9x - 6 = 0$.
3. Find the equation whose roots are the negative reciprocals of the roots of
$$x^4 + 7x^3 + 8x^2 - 9x + 10 = 0.$$
4. Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$, given that its roots are in H.P.
5. Find the equation whose roots are the roots of
   (i) $x^3 - 6x^2 + 11x - 6 = 0$ each increased by 1.
   (ii) $x^4 + x^3 - 3x^2 - x + 2 = 0$ each diminished by 3.
   (iii) $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x + 6 = 0$ each diminished by 1.
6. Find the equation whose roots are the squares of the roots of $x^3 - x^2 + 8x - 6 = 0$.
7. Find the equation whose roots are the cubes of the roots of $x^3 + px^2 + q = 0$.
8. If α, β, γ are the roots of the equation $2x^3 + 3x^2 - x - 1 = 0$, form the equation whose roots are $(1-\alpha)^{-1}, (1-\beta)^{-1}$ and $(1-\gamma)^{-1}$.
9. If a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are ab, bc and ca. (*Madras, 2003*)
10. If α, β, γ be the roots of $x^3 + mx + n = 0$, form the equation whose roots are
    (a) $\alpha + \beta - \gamma,\ \beta + \gamma - \alpha,\ \gamma + \alpha - \beta$,
    (b) $\beta\gamma/\alpha,\ \gamma\alpha/\beta,\ \alpha\beta/\gamma$ $\qquad$ (c) $\dfrac{1}{\beta} + \dfrac{1}{\gamma}, \dfrac{1}{\gamma} + \dfrac{1}{\alpha}, \dfrac{1}{\alpha} + \dfrac{1}{\beta}$.
11. Find the equation of squared differences of the roots of the cubic $x^3 + 6x^2 + 7x + 2 = 0$.
12. Solve the equations :
    (i) $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$.
    (ii) $6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0$. (*Madras, 2000 S*)
    (iii) $8x^5 - 22x^4 - 55x^3 + 55x^2 + 22x - 8 = 0$. (*S.V.T.U., 2006*)
    (iv) $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$. (*Madras, 2000*)
    (v) $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$. (*Madras, 2003*)
13. Show that the equation $x^4 - 10x^3 + 23x^2 - 6x - 15 = 0$ can be transformed into reciprocal equation by diminishing the roots by 2. Hence solve the equation.

14. By suitable transformation, reduce the equation $x^4 + 16x^3 + 83x^2$ [illegible] $= 0$ to an equation in which [illegible] is absent and hence solve it. (*Madras, 2002*)

1.6. SOLUTION OF CUBIC EQUATIONS – CARDAN'S METHOD*

Consider the equation $\qquad ax^3 + bx^2 + cx + d = 0 \qquad$ ...(1)

Dividing by a, we get an equation of the form $x^3 + lx^2 + mx + n = 0$.

To remove the x^2 term, put $y = x - (-l/3)$ or $x = y - l/3$ so that the resulting equation is of the form

$$y^3 + py + q = 0 \qquad \text{...(2)}$$

To solve (2), put $\qquad y = u + v$

so that $\qquad y^3 = u^3 + v^3 + 3uv(u + v) = u^3 + v^3 + 3uvy$

or $\qquad y^3 - 3uvy - (u^3 + v^3) = 0 \qquad$ ...(3)

Comparing (2) and (3), we get

$$uv = -p/3,\ u^3 + v^3 = -q \text{ or } u^3 + v^3 = -q \text{ and } u^3v^3 = -p^3/27$$

$\therefore u^3, v^3$ are the roots of the equation $t^2 + qt - p^3/27 = 0$

which gives $\qquad u^3 = \frac{1}{2}(-q + \sqrt{q^2 + 4p^3/27}) = \lambda^3$ (say)

and $\qquad v^3 = \frac{1}{2}(-q - \sqrt{q^2 + 4p^3/27})$

$\therefore$ The three values of u are $\lambda, \lambda\omega, \lambda\omega^2$, where ω is one of the imaginary cube roots of unity.

From $\qquad uv = -p/3$, we have $v = -p/3u$

$\therefore$ when $\qquad u = \lambda, \lambda\omega$ and $\lambda\omega^2$,

$$v = -\frac{p}{3\lambda}, -\frac{p\omega^2}{3\lambda} \text{ and } -\frac{p\omega}{3\lambda}. \qquad [\because \omega^3 = 1]$$

Hence the three roots of (2) are $\lambda - \frac{p}{3\lambda},\ \lambda\omega - \frac{p\omega^2}{3\lambda},\ \lambda\omega^2 - \frac{p\omega}{3\lambda}$ $\qquad$ (being $= u + v$)

Having known y, the corresponding values of x can be found from the relation $x = y - l/3$.

Obs. 1. *If one value of u is found to be a rational number, find the corresponding value of v giving one root $y = u + v$. Then find the corresponding root $x = \alpha$ (say). Finally, divide the left hand side of (1) by $x - \alpha$, giving the remaining quadratic equation from which the other two roots can be found readily.*

Obs. 2. *If u^3 and v^3 turn out to be conjugate complex numbers, the roots of the given cubic can be obtained in neat forms by employing De Moivre's theorem.* (§ 19.5)

Example 1.15. *Solve by Cardan's method* $x^3 - 3x^2 + 12x + 16 = 0$. $\qquad$ *(U.P.T.U., 2001)*

Sol. Given equation is $x^3 - 3x^2 + 12x + 16 = 0 \qquad$ ...(i)

To remove the second term from (i), diminish each root of (i) by 3/3 = 1, *i.e* put $y = x - 1$ or $x = y + 1$ [$\because$ Sum of roots = 3]. Then (i) becomes

$$(y + 1)^3 - 3(y + 1) + 12(y + 1) + 16 = 0 \text{ or } y^3 + 9y^2 + 26 = 0 \qquad \text{...(ii)}$$

To solve (ii), put $y = u + v$ so that $y^3 - 3uvy - (u^3 + v^3) = 0 \qquad$ ...(iii)

Comparing (ii) and (iii), we get $uv = -3$ and $u^3 + v^3 = -26$

$\therefore u^3, v^3$ are the roots of the equation $t^2 + 26t - 27 = 0$

or $\qquad (t + 27)(t - 1) = 0$ whence $t = -27, t = 1$.

or $\qquad u^3 = -27$ *i.e.*, $u = -3$ and $v^3 = 1$ *i.e.*, $v = 1$

$\therefore \qquad y = u + v = -3 + 1 = -2$ and ... 1

*Named after an Italian mathematician *Girolamo Cardan* (1501–1576) who was the first to use complex numbers as roots of an equation.

Dividing L.H.S of (*i*) by $x + 1$, we obtain $x^2 - 4x + 16 = 0$

or
$$x = \frac{4 \pm \sqrt{(16 - 64)}}{2} = 2 \pm i\, 2\sqrt{3}$$

Hence the roots of the given equation are $-1, 2 \pm i\, 2\sqrt{3}$

Example 1·16. *Solve the cubic $28x^3 - 9x^2 + 1 = 0$ by Cardan's method.*

Sol. Since the term in x is missing, let us put $x = 1/y$ in the given equation so that the transformed equation is $y^3 - 9y + 28 = 0$ ...(*i*)

To solve (*i*), put $y = u + v$ so that $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(*ii*)

Comparing (*ii*) and (*iii*), we get $uv = 3$ and $u^3 + v^3 = -28$.

$\therefore u^3, v^3$ are the roots of $t^2 + 28t + 27 = 0$

or
$$(t + 1)(t + 27) = 0 \text{ or } t = -1, -27 \text{ or } u = -1, v = -3$$

$\therefore y = u + v = -4$. Dividing L.H.S. of (*i*) by $y + 4$, we obtain $y^2 - 4y + 7 = 0$ whence $y = 2 \pm i\sqrt{3}$.

$\therefore$ roots of (*i*) are $-4, 2 \pm i\sqrt{3}$.

Hence the roots of the given cubic are $-\frac{1}{4}, \frac{1}{2 \pm i\sqrt{3}}$ or $-\frac{1}{4}, (2 - i\sqrt{3})/7, (2 + i\sqrt{3})/7$.

Example 1·17. *Solve the equation $x^3 + x^2 - 16x + 20 = 0$*

Sol. Instead of diminishing the roots of the given equation by $-1/3$, we first multiply its roots by 3, so that the equation becomes

$$x^3 + 3x^2 - 144x + 540 = 0 \qquad ...(i)$$

To remove the x^2 term, put $y = x - (-3/3)$ or $x = y - 1$ in (*i*)

so that
$$(y - 1)^3 + 3(y - 1)^2 - 144(y - 1) + 540 = 0$$

or
$$y^3 - 147y + 686 = 0 \qquad ...(ii)$$

To solve (*ii*), let $y = u + v$, so that

$$y^3 - 3uvy - (u^3 + v^3) = 0 \qquad ...(iii)$$

Comparing (*ii*) and (*iii*), we get

$$uv = 49, \quad u^3 + v^3 = -686. \text{ so that } u^3v^3 = (343)^2.$$

$\therefore u^3, v^3$ are the roots of the quadratic

$$t^2 + 686t + (343)^2 = 0 \text{ or } (t + 343)^2 = 0$$

$\therefore$
$$t = -343 \quad i.e., u^3 = v^3 = -343 \quad \text{or } u = v = -7.$$

Thus
$$y = u + v = -14 \text{ and } x = y - 1 = -15.$$

Dividing L.H.S. of (*i*) by $x + 15$, we get

$$(x - 6)^2 = 0 \text{ or } x = 6, 6.$$

$\therefore$ The roots of (*i*) are $-15, 6, 6$.

Hence the roots of the given equation are $-5, 2, 2$.

Example 1·18. *Solve $x^3 - 3x^2 + 3 = 0$.* (*S.V.T.U., 2006*)

Sol. Given equation is $x^3 - 3x^2 + 3 = 0$ ...(*i*)

To remove the x^2 term, put $y = x - 3/3$ or $x = y + 1$,

so that (*i*) becomes $(y + 1)^3 - 3(y + 1)^2 + 3 = 0$

or
$$y^3 - 3y + 1 = 0 \qquad ...(ii)$$

To solve it, put $y = u + v$

so that
$$y^3 - 3uvy - (u^3 + v^3) = 0 \qquad ...(iii)$$

Comparing (*ii*) and (*iii*), we get $uv = 1$, $u^3 + v^3 = -1$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + t + 1 = 0$

Hence $$u^3 = \frac{-1 + i\sqrt{3}}{2} \text{ and } v^3 = \frac{-1 - i\sqrt{3}}{2}$$

$$\therefore \quad u = \left(\frac{-1 + i\sqrt{3}}{2}\right)^{1/3}$$

| put $-\frac{1}{2} = r\cos\theta$ and $\sqrt{3}/2 = r\sin\theta$ so that $r = 1$, $\theta = 2\pi/3$

$$= [r(\cos\theta + i\sin\theta)]^{1/3}$$

$= [\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]^{1/3}$, where n is any integer or zero. Using De Moivre's theorem (p ...).

$$u = \cos\frac{\theta + 2n\pi}{3} + i\sin\frac{\theta + 2n\pi}{3}$$

Giving n the values 0, 1, 2 successively, we get the three values of u to be

$$\cos\frac{\theta}{3} + i\sin\frac{\theta}{3},\ \cos\frac{\theta + 2\pi}{3} + i\sin\frac{\theta + 2\pi}{3},\ \cos\frac{\theta + 4\pi}{3} + i\sin\frac{\theta + 4\pi}{3}$$

i.e. $$\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9},\ \cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9},\ \cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}.$$

The corresponding values of v are

$$\cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9},\ \cos\frac{8\pi}{9} - i\sin\frac{8\pi}{9},\ \cos\frac{14\pi}{9} - i\sin\frac{14\pi}{9}.$$

$\therefore$ The three values of $y = u + v$ are

$$2\cos 2\pi/9,\ 2\cos 8\pi/9,\ 2\cos 14\pi/9.$$

Hence the roots of (*i*) are found from $x = 1 + y$ to be

$$1 + 2\cos 2\pi/9,\ 1 + 2\cos 8\pi/9,\ 1 + 2\cos 14\pi/9.$$

Problems 1.3

Solve the following equations by Cardan's method :

1. $x^3 - 27x + 54 = 0$ *(U.P.T.U., 2003)*
2. $x^3 - 18x + 35 = 0$
3. $x^3 - 15x = 126$ *(S.V.T.U., 2007)*
4. $2x^3 + 5x^2 + x - 2 = 0$ *(Osmania, 2003)*
5. $9x^3 + 6x^2 - 1 = 0$
6. $x^3 - 6x^2 + 6x - 5 = 0$ *(U.P.T.U, 2003)*
7. $x^3 - 3x + 1 = 0$
8. $27x^3 + 54x^2 + 198x - 73 = 0$

1.7. SOLUTION OF BIQUADRATIC EQUATIONS

(1) Ferrari's method

This method of solving a biquadratic equation is illustrated by the following examples :

Example 1.19. *Solve the equation $x^4 - 12x^3 + 41x^2 - 18x - 72 = 0$ by Ferrari's method.* *(S.V.T.U., 2007)*

Sol. Combining x^4 and x^3 terms into a perfect square, the given equation can be written as

$$(x^2 - 6x + \lambda)^2 + (5 - 2\lambda)x^2 + (12\lambda - 18)x - (\lambda^2 + 72) = 0$$

or $$(x^2 - 6x + \lambda)^2 = [(2\lambda - 5)x^2 + (18 - 12\lambda)x + (\lambda^2 + 72)] \quad ...(i)$$

This equation can be factorised if R.H.S. is a perfect square

i.e. if $$(18 - 12\lambda)^2 = 4(2\lambda - 5)(\lambda^2 + 72)$$

i.e. if $$2\lambda^3 - 41\lambda^2 + 252\lambda - 441 = 0 \quad \text{which gives} \quad \lambda = 3.$$ $[b^2 - 4ac$

$\therefore$ (i) reduces to $(x^2 - 6x + 3)^2 = (\lambda - 9)^2$

i.e. $$(x^2 - 5x - 6)(x^2 - 7x + 12) = 0.$$

Hence the roots of the given equation are $-1, 3, 4$, and 6.

Example 1·20. *Solve the equation $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$ by Ferrari's method*

Sol. Combining x^4 and x^3 terms into a perfect square, the given equation can be written as $(x^2 - x + \lambda)^2 = (2\lambda + 6)x^2 - (2\lambda + 10)x + (\lambda^2 + 3)$. This equation can be factorised, if R.H.S is a perfect square *i.e.* if $(2\lambda + 10)^2 = 4(2\lambda + 6)(\lambda^2 + 3)$ $[b^2 = 4ac]$

or $$2\lambda^3 + 5\lambda^2 - 4\lambda - 7 = 0 \text{ which gives } \lambda = -1.$$

$\therefore$ (i) reduces to $(x^2 - x - 1)^2 = 4x^2 - 8x + 4$

or $$(x^2 - x - 1)^2 - (2x - 2)^2 = 0 \text{ or } (x^2 + x - 3)(x^2 - 3x + 1) = 0$$

$\therefore$ $$x = \frac{-1 \pm \sqrt{1 + 12}}{2} \text{ or } \frac{3 \pm \sqrt{9 - 4}}{2}$$

Hence the roots are $\dfrac{-1 \pm \sqrt{13}}{2}$, $\dfrac{3 \pm \sqrt{5}}{2}$.

(2) Descarte's method

This method of solving a biquadratic equations consists in removing the term in x^3 and then expressing the new equation as product of two quadratics. It is best illustrated by the following examples :

Example 1·21. *Solve the equation $x^4 - 8x^2 - 24x + 7 = 0$ by Descarte's method.*

(U.P.T.U., 2001)

Sol. In the given equation, the term in x^3 is already absent so we assume that

$$x^4 - 8x^2 - 24x + 7 = (x^2 + px + q)(x^2 - px + q') \quad ...(i)$$

Equating coefficients of the like powers of x in (i), we get

$$-8 = q + q' - p^2, \; -24 = p(q' - q); \; 7 = qq'$$

$\therefore$ $$q + q' = p^2 - 8, \; q - q' = 24/p$$

$\therefore$ $$(p^2 - 8)^2 - (24/p)^2 = 4 \times 7$$

$$p^6 - 16p^4 + 36p^2 - 576 = 0 \text{ or } t^3 - 16t^2 + 36t - 576 = 0 \text{ where } t = p^2$$

Now $t = 16$ satisfies this cubic so that $p = 4$.

$$q + q' = 8, \; q - q' = 6. \; \therefore \; q = 7, q' = 1$$

$\therefore$ (i) takes the form $(x^2 + 4x + 7)(x^2 - 4x + 1) = 0$

Whence $$x = \frac{-4 \pm \sqrt{(16 - 28)}}{2} \text{ or } x = \frac{4 \pm \sqrt{(16 - 4)}}{2}$$

Hence $x = -2 \pm \sqrt{3}i, \; 2 \pm \sqrt{3}$.

Example 1·22. *Solve the equation $x^4 - 6x^3 - 3x^2 + 22x - 6 = 0$ by Desarte's method.*

Sol. Here sum of roots = 6 and number of roots = 4

$\therefore$ To remove the second term, we have to diminish the roots by 6/4 (= 3/2) which will be a fraction. Therefore, we first multiply the roots by 2 $\therefore$ $y^4 - 12y^3 + 12y^2 + 176y - 96 = 0$ where $y =$

Now diminishing the roots by 3, we obtain $z^4 - 42z^2 + 32z + 297 = 0$ where $z = y - 3$.

Assuming that $z^4 - 42z^2 + 32z + 297 = (z^2 + pz + q)(z^2 - pz + q')$ ...(i) and comparing coefficients, we get

$$-42 = q + q' - p^2, \; 32 = p(q' - q); \; 297 = q q'$$

$\therefore$ $$q + q' = p^2 - 42; \; q - q' = -32/p, \; q q' = 297$$

$\therefore$ $$(p^2 - 42)^2 - (-32/p)^2 = 4 \times 297$$

or $$t^3 - 84t^2 + 576t - 1024 = 0 \text{ where } t = p^2$$

Now $t = 4$ satisfies this cubic so that $p = 2$.

$\therefore q + q' = -38, \; q - q' = -16, \; \therefore q = -27, q' = -11.$

Thus (i) takes the form $(z^2 + 2z - 27)(z^2 - 2z - 11) = 0$

Whence $$z = \frac{-2 \pm \sqrt{(4 + 108)}}{2} \text{ or } z = \frac{2 \pm \sqrt{(4 + 44)}}{2}$$

or $$x = \frac{1}{2}y = \frac{1}{2}(z + 3) = \frac{1}{2}(2 \pm \sqrt{28}), \; \frac{1}{2}(4 \pm \sqrt{12})$$

Hence $x = 1 \pm \sqrt{7}, \; 2 \pm \sqrt{3}$.

Problems 1.4

Solve by Ferrari's method, the equations :

1. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$
2. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ (U.P.T.U., 2003)
3. $x^4 - 10x^2 - 20x - 16 = 0$ (U.P.T.U., 2002)
4. $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0$ (U.P.T.U., 2005)

Solve the following equations by Descartes method :

5. $x^4 - 6x^3 + 3x^2 + 22x - 6 = 0$
6. $x^4 + 12x - 5 = 0$
7. $x^4 - 8x^3 - 24x + 7 = 0$ (U.P.T.U., 2001)
8. $x^4 - 10x^3 + 44x^2 - 104x + 96 = 0$

Obs. We have obtained algebraic solutions of cubic and biquadratic equations. But the need often arises to solve higher degree or transcendental equations for which no algebraic methods are available in general. Such equations can be best solved by *graphical methods* (explained below) or by *numerical methods* (§28.2).

1.8. GRAPHICAL SOLUTION OF EQUATIONS

Let the equation be $f(x) = 0$.

(i) Find the interval (a, b) in which a root of $f(x) = 0$ lies.

[At least one root of $f(x) = 0$ lies in (a, b) if $f(a)$ and $f(b)$ are of opposite signs —§1.2(III) p. 2].

(ii) Write the equation $f(x) = 0$ as $\phi(x) = \psi(x)$ where $\psi(x)$ contains only terms in x and the constants.

(iii) Draw the graphs of $y = \phi(x)$ and $y = \psi(x)$ on the same scale and with respect to the same axes.

(iv) Read the abscissae of the points of intersection of the curves $y = \phi(x)$ and $y = \psi(x)$. These are required real roots of $f(x) = 0$.

Sometimes it may not be convenient to write the given equation $f(x) = 0$ in the form $\phi(x) = \psi(x)$. In such cases, we proceed as follows :

(i) Form a table for the value of x and $y = f(x)$ directly.

(ii) Plot these points and pass a smooth curve through them.

(iii) Read the abscissae of the points where this curve cuts the x-axis. These are the required roots of $f(x) = 0$.

Obs. The roots, thus located graphically are approximate and to improve their accuracy, the curves are replotted on the larger scale in the immediate vicinity of each point of intersection. This gives a better approximation to the value of desired root. The above graphical operation may be repeated until the root is obtained correct upto required number of decimal places. But this method of repeatedly drawing graphs is very tedious. It is, therefore, advisable to improve upon the accuracy of an approximate root by numerical methods of §28.2.

Example 1.23. *Find graphically an approximate value of the root of the equation*

$$3 - x = e^{x-1}$$

Sol. Let

$$f(x) = e^{x-1} + x - 3 = 0 \quad \ldots(i)$$

$$f(1) = 1 + 1 - 3 = -\text{ve}$$

and

$$f(2) = e + 2 - 3 = 2.718 - 1 = +\text{ve}$$

$\therefore$ A root of (i), lies between $x = 1$ and $x = 2$.

Let us write (i) as $e^{x-1} = 3 - x$.

The abscissa of the point of intersection of the curves

$$y = e^{x-1} \quad \ldots(ii)$$

and

$$y = 3 - x \quad \ldots(iii)$$

will give the required root.

To plot (ii), we form the following table of values :

$x =$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$y = e^{x-1}$	1.11	1.22	1.35	1.49	1.65	1.82	2.01	2.23	2.46	2.72

Taking the origin at (1, 1) and 1 small unit along either axis = 0.02, we plot these points and pass a smooth curve through them as shown in Fig. 1.1.

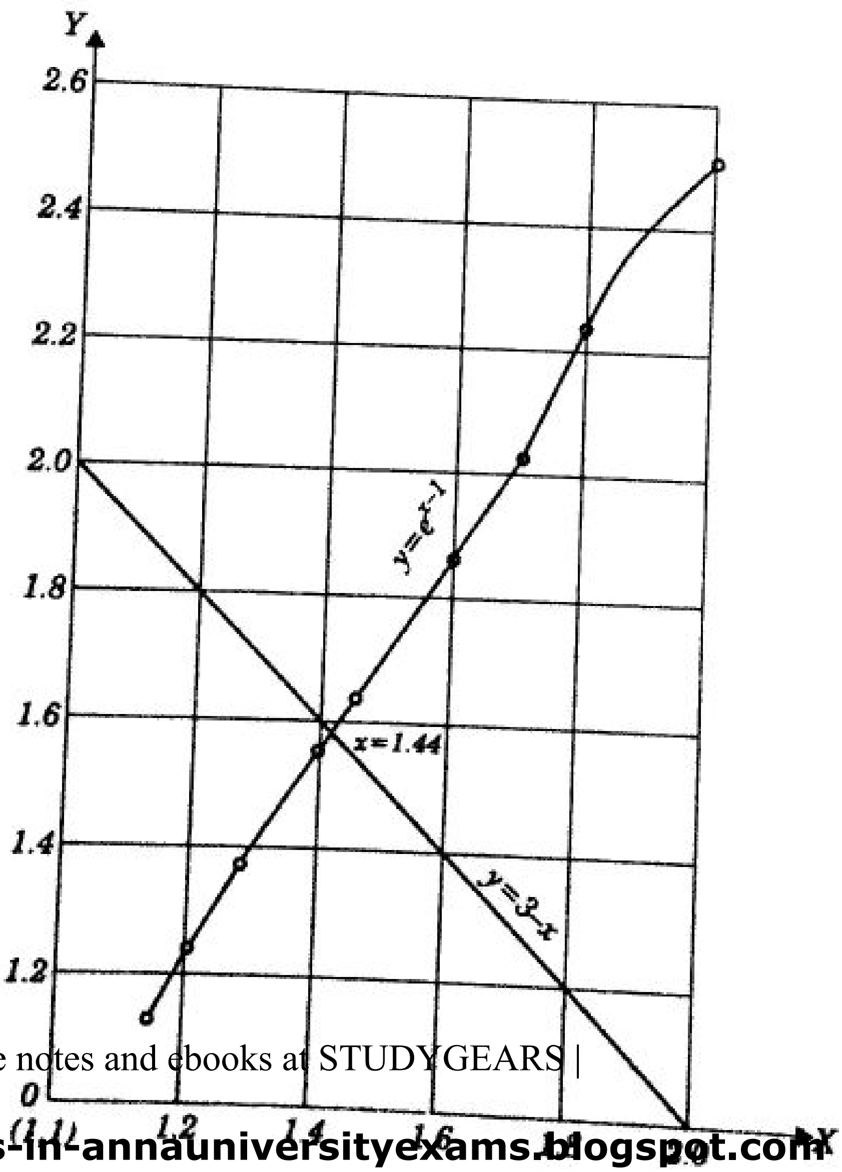

Fig. 1.1.

To draw the line (*iii*), we join the points (1, 2) and (2, 1) on the same scale and with the same axes.

From the figure, we get the required root to be $x = 1{\cdot}44$ nearly.

Example 1·24. *Obtain graphically an approximate value of the root of* $x = \sin x + \pi/2$.

Sol. Let us write the given equation as $\sin x = x - \pi/2$

The abscissa of the point of intersection of the curve $y = \sin x$ and the line $y = x - \pi/2$ will give a rough estimate of the root.

To draw a curve $y = \sin x$, we form the following table :

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
y	0	0·71	1	0·71	0

Taking 1 unit along either axis $= \pi/4 = 0{\cdot}8$ nearly, we plot the curve as shown in Fig 1·2.

Also we draw the line $y = x - \pi/2$ to the same scale and with the same axis.

From the graph, we get $x = 2{\cdot}3$ radians approximately.

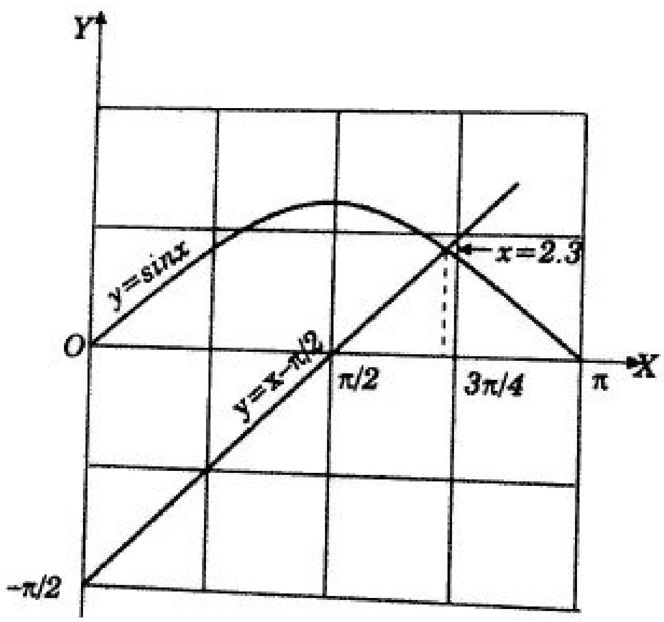

Fig. 1·2.

Example 1·25. *Obtain graphically the lowest root of* $\cos x \cosh x = -1$.

Sol. Let $\quad f(x) = \cos x \cosh x + 1 = 0$ ...(*i*)

$\because\ f(0) = +\text{ve},\ f(\pi/2) = +\text{ve}$ and $f(\pi) = -\text{ve}$.

$\therefore$ The lowest root of (*i*) lies between $x = \pi/2$ and $x = \pi$.

Let us write (*i*) as $\cos x = -\operatorname{sech} x$.

The abscissa of the point of intersection of the curves

$$y = \cos x \quad ...(ii)$$

and

$$y = -\operatorname{sech} x \quad ...(iii)$$

will give the required root.

To draw (*ii*), we form the following table :

$x =$	$\pi/2 = 1{\cdot}57$	$3\pi/4 = 2{\cdot}36$	$\pi = 3{\cdot}14$
$y = \cos x$	0	$-0{\cdot}71$	-1

Taking the origin at (1·57, 0) and 1 unit along either axis $= \pi/8 = 0{\cdot}4$ nearly, we plot the cosine curve as shown in Fig. 1·3.

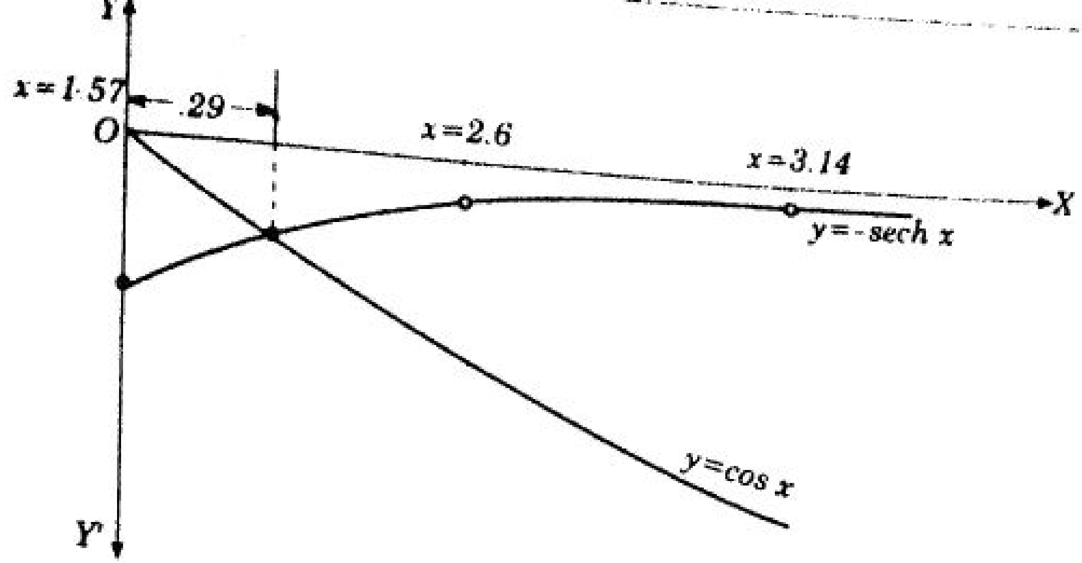

Fig. 1.3.

To draw (*iii*), we form the following table :

$x =$	1.57	2.36	3.14
$\cosh x =$	2.58	5.56	11.12
$y = -\text{sech } x$	−0.39	−0.18	−0.09

Then we plot the curve (*iii*) to the same scale with the same axes.

From the figure we get the lowest root to be approximately $x = 1.57 + 0.29 = 1.86$.

Problems 1.5

Solve the following equations graphically :

1. $x^3 - x - 1 = 0$ (*Madras, 2000 S*)
2. $x^3 - 3x - 5 = 0$
3. $x^3 - 6x^2 + 9x - 3 = 0$.
4. $\tan x = 1.2x$
5. $x = 3 \cos (x - \pi/4)$
6. $e^x = 5x$ which is near $x = 0.2$.

1.9. OBJECTIVE TYPE OF QUESTIONS

Problems 1.6

Choose the correct answer or fill up the blanks in the following problems :

1. If for the equation $x^3 - 3x^2 + kx + 3 = 0$, one root is the negative of another, then the value of k is
(*a*) 3 (*b*) −3 (*c*) 1 (*d*) −1.
2. If the roots of the equation $x^n - 1 = 0$ are $1, \alpha_1, \alpha_2, \ldots\ldots, \alpha_{n-1}$, then
$(1 - \alpha_1)(1 - \alpha_2) \ldots\ldots (1 - \alpha_{n-1})$ is equal to
(*a*) 0 (*b*) 1 (*c*) n (*d*) $n + 1$
3. If α, β, γ are the roots of $2x^3 - 3x^2 + 6x + 1 = 0$, then $\alpha^2 + \beta^2 + \gamma^2$ is
(*a*) 15/4 (*b*) −3 (*c*) −15/4 (*d*) 33/4.
4. $x + 2$ is a factor of
(*a*) $x^4 + 2$ (*b*) $x^4 - x^2 + 12$
(*c*) $x^4 - 2x^3 - x + 2$ (*d*) $x^4 + 2x^3 - x - 2$.
5. If $\alpha + \beta + \gamma = 5$; $\alpha\beta + \beta\gamma + \gamma\alpha = 7$; $\alpha\beta\gamma = 3$, then the equation whose roots are α, β and γ is
(*a*) x^3 [illegible] $= 0$ (*b*) $x^3 - 7x^2 + 3 = 0$
(*c*) $x^3 - 5x^2 + 7x - 3 = 0$ (*d*) $x^3 + 7x^2 - 3 = 0$.

6. If one of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$ is 2, then the other two roots are
   (a) 1 and 3 (b) 0 and 4
   (c) -1 and 5 (d) -2 and 6.
7. The equation whose roots are the reciprocals of the roots of $x^3 + px^2 + r = 0$ is
   (a) $x^3 + 1/p \ x^2 + 1/r = 0$ (b) $1/r \cdot x^3 + 1/p \cdot x + 1 = 0$
   (c) $rx^3 + px^2 + 1 = 0$ (d) $rx^3 + px + 1 = 0$.
8. If 1 and 2 are two roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, then the remaining two roots are
   (a) -3 and 5 (b) 3 and -5 (c) -6 and 5 (d) 6 and -5.
9. If the roots of $x^3 - 3x^2 + px + 1 = 0$, are in arithmetic progression then the sum of squares of the largest and the smallest roots is
   (a) 3 (b) 5 (c) 6 (d) 10.
10. A root of $x^3 - 8x^2 + px + q = 0$ where p and q are real numbers is $3 + i\sqrt{3}$. The real root is
    (a) 2 (b) 6 (c) 9 (d) 12.
11. One of the roots of the equation $f(x) = x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1x + a_0 = 0$ where $a_0, a_1, \ldots\ldots a_{n-1}$ are real, is given to be $2 - 3i$. Of the remaining, the next $n - 2$ roots are given to be $1, 2, 3, \ldots\ldots, n - 2$. The nth root is
    (a) n (b) $n - 1$ (c) $2 + 3i$ (d) $-2 + 3i$.
12. If a real root of $f(x) = 0$ lies in $[a, b]$, then the sign of $f(a) \cdot f(b)$ is ......
13. Descartes rule of signs states that ....
14. If α, β, γ are the roots of the equation $x^3 - px + q = 0$, then $\Sigma\, 1/\alpha =$ ....
15. If α, β, γ are the roots of $x^3 = 7$, then $\Sigma\, \alpha^3$ is ......
16. A root of $x^3 - 3x^2 + 2{\cdot}5 = 0$ lies between $1{\cdot}1$ and $1{\cdot}2$. (True or False)
17. In an equation with real coefficients, imaginary roots must occur in ......
18. If $f(\alpha)$ and $f(\beta)$ are of opposite signs, then $f(x) = 0$ has at least one root between α and β provided ....
19. If α, β, γ are the roots of the equation $x^3 + 2x + 3 = 0$, then $\alpha + 3, \beta + 3, \gamma + 3$ are the roots of the equation ....
20. If one root is double of another in $x^3 - 7x^2 + 36 = 0$, then its roots are ....
21. The equation whose roots are 10 times those of $x^3 - 2x - 7 = 0$, is ....
22. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then $\Sigma\,(1/\alpha\beta) =$ ....
23. $\sqrt{3}$ and $-1 + i$ are the roots of the biquadratic equation ....
24. If α, β, γ are the roots of $x^3 - 3x + 2 = 0$, then the value of $\alpha^2 + \beta^2 + \gamma^2$ is ....
25. If there is a root of $f(x) = 0$ in the interval $[a, b]$, then sign of $f(a)/f(b)$ is ....
26. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then the condition for $\alpha + \beta = 0$ is ....
27. The three roots of $x^3 = 1$ are ....
28. One real root of the equation $x^3 + x - 5 = 0$ lies in the interval
    (i) (2, 3), (ii) (3, 4), (iii) (1, 2), (iv) $(-3, -2)$
29. If two roots of $x^3 - 3x^2 + 2 = 0$ are equal, then its roots are ....
30. The cubic equation whose two roots are 5 and $1 - i$ is ....
31. The sum and product of the roots of the equation $x^5 = 2$ are .... and ....
32. The equation $x^6 - x^5 - 10x + 7 = 0$ has four imaginary roots. (True or False)
33. One real root of the equation $x^3 + 2x^2 + 5 = 0$ lies between ....
34. If the roots of the equation $x^4 + 2x^3 - ax^2 - 22x + 40 = 0$ are $-5, -2, 1$ and 4 then $a =$ ......

2

Linear Algebra : Determinants, Matrices

1. Introduction. 2. Determinants, Cofactors, Laplace's expansion. 3. Properties of determinants. 4. Multiplication of determinants. 5. Matrices, Special matrices. 6. Matrix operations. 7. Related matrices. 8. Rank of a matrix, Elementary transformations, Elementary matrices, Inverse from elementary matrices, Normal form of a matrix. 9. Partition method. 10. Solution of linear system of equations. 11. Consistency of linear system of equations. 12. Linear and orthogonal transformations. 13. Vectors ; Linear dependence. 14. Eigen values and eigen vectors. 15. Properties of eigen values. 16. Cayley-Hamilton theorem. 17. Reduction to diagonal form. 18. Reduction of quadratic form to canonical form. 19. Nature of quadratic form. 20. Complex matrices. 21. Objective Type of Questions.

2.1. INTRODUCTION

Linear algebra comprises of the theory and applications of linear system of equations, linear transformations and eigen value problems. In linear algebra, we make a systematic use of matrices and to a lesser extent determinants and their properties.

Determinants were first introduced for solving linear systems and have important engineering applications in systems of differential equations, electrical networks, eigen-value problems and so on. Many complicated expressions occurring in electrical and mechanical systems can be elegently simplified by expressing them in the form of determinants.

Cayley* discovered matrices in the year 1860. But it was not until the twentieth century was well-advanced that engineers heard of them. These days, however, matrices have been found to be of great utility in many branches of applied mathematics such as algebraic and differential equations, mechanics, theory of electrical circuits, nuclear physics, aerodynamics and astronomy. With the advent of computers, the usage of matrix methods has been greatly facilitated.

2.2. DETERMINANTS

(1) Definition. The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a *determinant of the second order* and stands for '$a_1b_2 - a_2b_1$'. It contains 4 numbers a_1, b_1, a_2, b_2 (called *elements*) which are arranged along two horizontal lines (called *rows*) and two vertical lines (called *columns*).

Similarly $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a *determinant of the third order*. It consists of 9 *elements* which are arranged in 3 *rows* and 3 *columns*.

In general, *a determinant of the nth order* is denoted by $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & l_1 \\ a_2 & b_2 & c_2 & d_2 & l_2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & d_n & l_n \end{vmatrix}$

*Arthur Cayley (1821–1895) was a professor at Cambridge and is known for his important contributions to algebra, matrices and differential equations.

which is a block of n^2 *elements* arranged in the form of a square along n rows and n columns. The diagonal through the left hand top corner which contains the elements $a_1, b_2, c_3, \ldots$ is called the *leading or principal diagonal*.

(2) Cofactors

The cofactor *of any element in a determinant is obtained by deleting the row and column which intersect in that element with the proper sign. The sign of an element in the ith row and jth column is* $(-1)^{i+j}$. The cofactor of an element is usually denoted by the corresponding capital letter.

For instance, in $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the cofactor of b_3 *i.e.*, $B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ and $C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$.

(3) Laplace's expansion.* *A determinant can be expanded in terms of any row (or column) as follows :*

Multiply each element of the row (or column) in terms of which we intend expanding the determinant, by its cofactor and then add up all these terms.

$\therefore$ Expanding by R_1 (*i.e.* 1st row),

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1 = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Similarly expanding by C_2 (*i.e.* 2nd column)

$$\Delta = b_1B_1 + b_2B_2 + b_3B_3 = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$
$$= -b_1(a_2c_3 - a_3c_2) + b_2(a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1)$$

and expanding by R_3 (*i.e.* 3rd row), $\Delta = a_3A_3 + b_3B_3 + c_3C_3$.

Thus Δ is the sum of the products of the elements of any row (or column) by the corresponding cofactors.

If, however, the sum of the products of the elements of any row (or column) by the cofactors of another row (or column) be taken, the result is zero.

e.g., in Δ, $a_3A_2 + b_3B_2 + c_3C_2 = -a_3 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_3 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

$$= -a_3(b_1c_3 - b_3c_1) + b_3(a_1c_3 - a_3c_1) - c_3(a_1b_3 - a_3b_1) = 0$$

In general, $a_i A_j + b_i B_j + c_i C_j = \Delta$ when $i = j$
$= 0$ when $i \neq j$

Example 2.1. *Expand* $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Sol. Expanding by R_1, $\Delta = a \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \begin{vmatrix} h & f \\ g & c \end{vmatrix} + g \begin{vmatrix} h & b \\ g & f \end{vmatrix}$

$$= a(bc - f^2) - h(hc - gf) + g(hf - gb) = abc + 2fgh - af^2 - bg^2 - ch^2$$

* Named after a great French mathematician Pierre Simon Marquis De Laplace (1749–1827). He made important contributions in probability theory, special functions, potential theory and astronomy. While a professor in Paris, he taught Napolean Bonapart for a year.

Example 2.2. *Find the value of* $\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

Sol. Since there are two zeros in the second row, therefore, expanding by R_2, we get

$$\Delta = -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 0 - 3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} + 0$$

(Expand by C_1) (Expand by R_1)

$$= -[1(0 \times 2 - 1 \times 1) - 3(2 \times 2 - 1 \times 3) + 0] - 3[0 - (2 \times 2 - 3 \times 1) + 3(2 \times 0 - 3 \times 3)]$$

$$= -(-1 - 3) - 3(-1 - 27) = 4 + 84 = 88.$$

2.3. PROPERTIES OF DETERMINANTS

The following properties, are proved for determinants of the third order, but these hold good for determinants of any order. These properties enable us to simplify a given determinant and evaluate it without expanding the given determinant.

I. *A determinant remains unaltered by changing its rows into columns and columns into rows.*

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ [Expand by R_1]

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Then $\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ [Expand by R_1]

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = \Delta.$$

Obs. 1. Any theorem concerning the rows of a determinant, therefore, applies equally to its columns and *vice-versa*.

2. When a *row* or a *column* is referred to in a general manner, it is called *a line*.

II. *If two parallel lines of a determinant are interchanged, the determinant retains its numerical value but changes in sign.*

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ [Expand by R_1]

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Interchanging C_2 and C_3, we have

$$\Delta' = \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} \quad \text{[Expand by } R_1]$$

$$= a_1(c_2b_3 - c_3b_2) - c_1(a_2b_3 - a_3b_2) + b_1(a_2c_3 - a_3c_2)$$

$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] = -\Delta.$$

Cor. *If a line of Δ be passed over two parallel lines, i.e., if the resulting determinant is like*

$$\Delta' = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} \quad \text{then} \quad \Delta' = (-1)^2 \Delta$$

In general, *if any line of a determinant be passed over m parallel lines, the resulting determinant*

$$\Delta' = (-1)^m \Delta.$$

III. *A determinant vanishes if two parallel lines are identical.*

Consider a determinant Δ in which two parallel lines are identical.

Interchange of the identical lines leaves the determinant unaltered yet by the previous property, the interchanges of two parallel lines changes the sign of the determinant.

Hence $\qquad \Delta = \Delta' = -\Delta \quad \text{or} \quad 2\Delta = 0, \text{ or } \Delta = 0.$

IV. *If each element of a line be multiplied by the same factor, the whole determinant is multiplied by that factor.*

i.e.
$$\begin{vmatrix} a_1 & pb_1 & c_1 \\ a_2 & pb_2 & c_2 \\ a_3 & pb_3 & c_3 \end{vmatrix} = p \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

For on expanding by C_2,

$$\text{L.H.S.} = -pb_1(a_2c_3 - a_3c_2) + pb_2(a_1c_3 - a_3c_1) - pb_3(a_1c_2 - a_2c_1)$$
$$= p\,(-b_1B_1 + b_2B_2 - b_3B_3) = \text{R.H.S.}$$

Similarly, $\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Cor. *If two parallel lines be such that the elements of one are equi-multiples of the elements of the other, the determinant vanishes.*

i.e.
$$\begin{vmatrix} a_1 & b_1 & pb_1 \\ a_2 & b_2 & pb_2 \\ a_3 & b_3 & pb_3 \end{vmatrix} = p \begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = p(0) = 0$$

V. *If each element of a line consists of m terms, the determinant can be expressed as the sum of m determinants.*

Consider the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 + d_1 - e_1 \\ a_2 & b_2 & c_2 + d_2 - e_2 \\ a_3 & b_3 & c_3 + d_3 - e_3 \end{vmatrix}$

each of whose third column elements consists of three terms.

Expanding Δ by C_3, we have

$$\Delta = (c_1 + d_1 - e_1)(a_2b_3 - a_3b_2) - (c_2 + d_2 - e_2)(a_1b_3 - a_3b_1) + (c_3 + d_3 - e_3)(a_1b_2 - a_2b_1)$$
$$= [c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)]$$
$$+ [d_1(a_2b_3 - a_3b_2) - d_2(a_1b_3 - a_3b_1) + d_3(a_1b_2 - a_2b_1)]$$
$$- [e_1(a_2b_3 - a_3b_2) - e_2(a_1b_3 - a_3b_1) + e_3(a_1b_2 - a_2b_1)]$$
$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix}$$

Further, *if the elements of three parallel lines consist of m, n and p terms respectively, the determinants can be expressed as the sum of mnp determinants.*

Example 2·3. *If* $\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0$ *in which a, b, c are different, show that abc = 1.*

Sol. As each term of C_3 in the given determinant consists of two terms, we express it as a n of two determinants.

$$\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & -1 \\ b & b^2 & -1 \\ c & c^2 & -1 \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

[Taking common a, b, c from R_1, R_2, R_3 respectively of the first determinant and -1 from of the second determinant.]

$$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

[Passing C_3 over C_2 and C_1 in the second determinant]

$\therefore \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (abc - 1) = 0$ Hence $abc = 1$, since $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$ as a, b, c are all different.

VI. *If to each elements of a line be added equi-multiples of the corresponding elements of one r more parallel lines, the determinants remains unaltered.*

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $\Delta' = \begin{vmatrix} a_1+pb_1-qc_1 & b_1 & c_1 \\ a_2+pb_2-qc_2 & b_2 & c_2 \\ a_3+pb_3-qc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} -qc_1 & b_1 & c_1 \\ -qc_2 & b_2 & c_2 \\ -qc_3 & b_3 & c_3 \end{vmatrix}$

$= \Delta + 0 + 0 = \Delta.$ [by *IV-Cor.*]

Obs. *This property is very useful for simplifying determinants.* To add equi-multiples of parallel lines. we shall employ the following **notation :**

Suppose to the elements of the second row, we add p times the elements of the first row and q times he element of the third row ; then we say :

Operate $R_2 + pR_1 + qR_3$.

Similarly Operate '$C_3 + mC_1 - nC_2$'

means that to the elements of the third column add m times the elements of the first column and $-n$ times the elements of the second column.

Example 2·4. *Evaluate* $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 6 & 7 & 1 & 2 \end{vmatrix}$

Sol. Operating $R_1 - R_2 - R_4$, $R_2 - 3R_3$, $R_3 - 2R_4$, the given determinant

$$\Delta = \begin{vmatrix} -8 & -12 & 0 & -2 \\ 6 & -2 & 0 & 1 \\ -4 & -6 & 0 & -1 \\ 6 & 7 & 1 & 2 \end{vmatrix}$$ [Expand by C_1]

$$\begin{vmatrix} -8 & 12 & 2 \\ 6 & 2 & 1 \\ -4 & 6 & -1 \end{vmatrix} = 0 \qquad | \ R_1 \ 2R_2$$

Example 2 5. *Solve the equation* $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0.$

Sol. Operating $R_3 - (R_1 + R_2)$, we get

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0 \qquad \text{(Operate } R_2 - R_1 \text{ and } R_1 - R_2\text{)}$$

or $$\begin{vmatrix} x+2 & 2x+4 & 6x+12 \\ x+1 & x+1 & x+1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0 \quad \text{or} \quad (x+1)(x+2)\begin{vmatrix} 1 & 2 & 6 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

To bring one more zero in C_1, operate $R_1 - R_2$.

$$(x+1)(x+2)\begin{vmatrix} 0 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

Now expand by C_1 $\therefore -(x+1)(x+2)(3x+8-5) = 0$ or $-3(x+1)(x+2)(x+1) = 0$

Thus $x = -1, -1, -2.$

Example 2 6. *Prove that* $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$

Sol. Let Δ be the given determinant. Taking a, b, c, d common from R_1, R_2, R_3, R_4 respectively, we get

$$\Delta = abcd\begin{vmatrix} a^{-1}+1 & a^{-1} & a^{-1} & a^{-1} \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

[Operate $R_1 + (R_2 + R_3 + R_4)$ and take out the common factor from R_1]

$$= abcd\,(1 + a^{-1} + b^{-1} + c^{-1} + d^{-1})\begin{vmatrix} 1 & 1 & 1 & 1 \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

[Operate $C_2 - C_1, C_3 - C_1, C_4 - C_1$]

$$= abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ b^{-1} & 1 & 0 & 0 \\ c^{-1} & 0 & 1 & 0 \\ d^{-1} & 0 & 0 & 1 \end{vmatrix} = abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

Obs. If all elements on one side of the leading diagonal are zero, then the determinant is equal to the product of leading diagonal elements and such a determinant is called a *triangular determinant*.

Example 2 7. *Show that* $\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix} = \lambda^3(a^2 + b^2 + c^2 + d^2 + \lambda)$

Let Δ be the given determinant.

tiply C_1, C_2, C_3, C_4 by a, b, c, d respectively and divide by $abcd$. Then

$$\Delta = \frac{1}{abcd}\begin{vmatrix} a^3 + a\lambda & ab^2 & ac^2 & ad^2 \\ a^2b & b^3 + b\lambda & bc^2 & bd^2 \\ a^2c & b^2c & c^3 + c\lambda & cd^2 \\ a^2d & b^2d & c^2d & d^3 + d\lambda \end{vmatrix}$$

king a, b, c, d common from R_1, R_2, R_3 and R_4 respectively, we get

$$\Delta = \frac{abcd}{abcd}\begin{vmatrix} a^2 + \lambda & b^2 & c^2 & d^2 \\ a^2 & b^2 + \lambda & c^2 & d^2 \\ a^2 & b^2 & c^2 + \lambda & d^2 \\ a^2 & b^2 & c^2 & d^2 + \lambda \end{vmatrix}$$

perate $C_1 + (C_2 + C_3 + C_1)$ and take out the common factor from C_1]

$$= (a^2 + b^2 + c^2 + d^2 + \lambda)\begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 1 & b^2 + \lambda & c^2 & d^2 \\ 1 & b^2 & c^2 + \lambda & d^2 \\ 1 & b^2 & c^2 & d^2 + \lambda \end{vmatrix} \text{[Operate } R_2 - R_1, R_3 - R_1, R_4 - R_1]$$

$$= (a^2 + b^2 + c^2 + d^2 + \lambda)\begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^3(a^2 + b^2 + c^2 + d^2 + \lambda)$$

VII. Factor Theorem. *If the elements of a determinant Δ are functions of x and two parallel become identical when $x = a$, then $x - a$ is a factor of Δ.*

Let $\qquad \Delta = f(x)$

Since $\qquad \Delta = 0$ when $x = a$, $\qquad \therefore\ f(a) = 0.$

$(x - a)$ is a factor of $f(x)$.

Hence $x - a$ is a factor of Δ.

Obs. *If k parallel lines of a determinant Δ become identical when $x = a$, then $(x - a)^{k-1}$ is a factor of Δ*

Example 2·8. *Factorize* $\Delta = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$

Sol. Putting $a = b$, $R_1 \equiv R_2$ and hence $\Delta = 0$. $\therefore\ a - b$ is a factor of Δ.

Similarly, $a - c$ and $a - d$ are also factors of Δ.

Again putting $b = c$, $R_2 \equiv R_3$ and hence $\Delta = 0$. $\therefore\ b - c$ is a factor of Δ.

Similarly $b - d$ and $c - d$ are also factors of Δ.

Also Δ is of the sixth degree in a, b, c, d and therefore, there cannot be any other algebraic tor of Δ.

Suppose $\Delta = k(a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$, where k is a numerical constant

The leading term in $\Delta = a^3b^2c$. The corresponding term on R.H.S. $= ka^3b^2c$.

$\therefore \qquad k = 1.$

Hence $\qquad \Delta = (a - b)(a - c)(a - d)(b - c)(b - d)(c - d).$

Example 2.9. *Factorize* $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

Sol. Putting $a = b$, $C_1 \equiv C_2$ and hence $\Delta = 0$.

$\therefore$ $a - b$ is a factor of Δ. Similarly $b - c$ and $c - a$ are also factors of Δ.

$\therefore$ $(a - b)(b - c)(c - a)$ is a third degree factor of Δ which itself is of the fifth degree as is judged from the leading term b^2c^3.

$\therefore$ The remaining factor must be of the second degree. As Δ is symmetrical in a, b, c, the remaining factor must, therefore, be of the form $k\,(a^2 + b^2 + c^2) + l(bc + ca + ab)$.

$\therefore$ $\Delta = (a - b)(b - c)(c - a)\,\{k(a^2 + b^2 + c^2) + l(bc + ca + ab)\}$.

If $k \neq 0$, we shall get terms like a^4b, b^4c etc. which do not occur in Δ. Hence k must be zero

$\therefore$ $\Delta = l(a - b)(b - c)(c - a)(bc + ca + ab)$. The leading term in $\Delta = b^2c^3$.

The corresponding term on R.H.S. $= l\,b^2c^3$. $\therefore$ $l = 1$

Hence $\Delta = (a - b)(b - c)(c - a)(bc + ca + ab)$.

Example 2.10. *Prove that* $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a + b + c)^3$ *(J.N.T.U., 1998)*

Sol. Let the given determinant be Δ. If we put $a = 0$,

$$\Delta = \begin{vmatrix} (b+c)^2 & 0 & 0 \\ 0 & c^2 & b^2 \\ c^2 & c^2 & b^2 \end{vmatrix} = 0$$

$\therefore$ a is a factor of Δ. Similarly b and c are its factors.

Again if we put $a + b + c = 0$.

$$\Delta = \begin{vmatrix} (-a)^2 & a^2 & a^2 \\ b^2 & (-b^2) & b^2 \\ c^2 & c^2 & (-c)^2 \end{vmatrix} = 0$$

In this, three columns being identical, $(a + b + c)^2$ is a factor of Δ.

As Δ is of the sixth degree and is symmetrical in a, b, c the remaining factor must therefore, be of the first degree and of the form $k\,(a + b + c)$.

Thus $\Delta = kabc\,(a + b + c)^3$

To determine k, put $a = b = c = 1$, then

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 27k \quad \text{or } 54 = 27k \qquad i.e. \quad k = 2$$

Hence $\Delta = 2abc\,(a + b + c)^3$.

Otherwise : Operating $C_1 - C_3$ and $C_2 - C_3$, we have

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix} \quad [\text{Take } (a + b + c) \text{ common from } C_1 \text{ and } C_2]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \quad [\text{Operate } R_3 - R_1 - R_2]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad \left[\text{Operate } C_1 + \frac{1}{a} C_3,\ C_2 + \frac{1}{b} C_3\right]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c & a^2/b & a^2 \\ b^2/a & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \quad [\text{Expand by } R_3]$$

$$= 2ab\,(a+b+c)^2\,[(b+c)(c+a) - ab] = 2abc\,(a+b+c)^3.$$

4. MULTIPLICATION OF DETERMINANTS

The product of two determinants of the same order is itself a determinant of that order.

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$

en their product is defined as

$$\Delta_1 \Delta_2 = \begin{vmatrix} a_1l_1 + b_1m_1 + c_1n_1, & a_1l_2 + b_1m_2 + c_1n_2, & a_1l_3 + b_1m_3 + c_1n_3 \\ a_2l_1 + b_2m_1 + c_2n_1, & a_2l_2 + b_2m_2 + c_2n_2, & a_2l_3 + b_2m_3 + c_2n_3 \\ a_3l_1 + b_3m_1 + c_3n_1, & a_3l_2 + b_3m_2 + c_3n_2, & a_3l_3 + b_3m_3 + c_3n_3 \end{vmatrix}$$

milarly the product of two determinants of the nth order is a determinant of the nth order.

Example 2·11. *Evaluate* $\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$

Sol. By the rule of multiplication of determinants, the resulting determinant

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

here $d_{11} = (a^2+\lambda^2)\lambda + (ab+c\lambda)c + (ca-b\lambda)(-b) = \lambda(a^2+b^2+c^2+\lambda^2)$

$d_{12} = (a^2+\lambda^2)(-c) + (ab+c\lambda)\lambda + (ca-b\lambda)a = 0$

$d_{13} = 0,$

$d_{21} = 0,\ d_{22} = \lambda(a^2+b^2+c^2+\lambda^2),\ d_{23} = 0.$

$d_{31} = 0,\ d_{32} = 0,\ d_{33} = \lambda(a^2+b^2+c^2+\lambda^2).$

Hence $\Delta = \begin{vmatrix} \lambda(a^2+b^2+c^2+\lambda^2) & 0 & 0 \\ 0 & \lambda(a^2+b^2+c^2+\lambda^2) & 0 \\ 0 & 0 & \lambda(a^2+b^2+c^2+\lambda^2) \end{vmatrix}$

$= \lambda^3(a^2+b^2+c^2+\lambda^2)^3.$

Example 2·12. *Show that* $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$ *where A, B etc. are the co-factors* *of a, b, etc. respectively in the determinant* $(a_1b_2c_3)$.

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Then

$$\Delta\Delta' = \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1, & a_1A_2 + b_1B_2 + c_1C_2, & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1, & a_2A_2 + b_2B_2 + c_2C_2, & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1, & a_3A_2 + b_3B_2 + c_3C_2, & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

Hence $\Delta' = \Delta^2$.

Obs. Δ' is called the **reciprocal** *or* **adjugate determinant** *of* Δ.

Example 2·13. *Express* $\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$

as the square of a determinant, and hence find its value.

Sol. Given determinant

$$= \begin{vmatrix} a.(-a) + b.c + c.b, & a.(-b) + b.a + c.c, & a.(-c) + b.b + c.a \\ b.(-a) + c.c + a.b, & b.(-b) + c.a + a.c, & b.(-c) + c.b + a.a \\ c.(-a) + a.c + b.b, & c.(-b) + a.a + b.c, & c.(-c) + a.b + b.a \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

[Take out (–1) common from C_1 and interchange C_2, C_3]

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1)^2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \Delta^2$$

where $\quad \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

Hence the given determinant $= \Delta^2 = (a^3 + b^3 + c^3 - 3abc)^2$.

Problems 2·1

1. Prove, without expanding, that

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} \text{ vanishes.}$$

2. If $\begin{vmatrix} x & x^2 & 1 + x^3 \\ y & y^2 & 1 + y^3 \\ z & z^2 & 1 + z^3 \end{vmatrix} = 0$, then prove, without expansion, that $xyz = -1$ where x, y, z are unequal.

(Andhra, 1999 ; Assam, 1999)

3. Show that $\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x - \alpha)(x - \beta)(x - \gamma)$.

4. Show that $\begin{vmatrix} a & b & c \\ b + c & c + a & a + b \\ a^2 & b^2 & c^2 \end{vmatrix} = -(a - b)(b - c)(c - a)(a + b + c)$.

5. If a, b, c are all different and $\begin{vmatrix} a & a^3 & a^4 - 1 \\ b & b^3 & b^4 - 1 \\ c & c^3 & c^4 - 1 \end{vmatrix} = 0$, then show that $abc\,(bc + ca + ab) = a + b + c$

6. Evaluate (i) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$ (ii) $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$

Prove the following results : (7 to 12)

7. $\begin{vmatrix} a+b & b+c & c+a \\ l+m & m+n & n+l \\ p+q & q+r & r+p \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix}$

8. $\begin{vmatrix} a-b-c & 2b & 2c \\ 2a & b-c-a & 2c \\ 2a & 2b & c-a-b \end{vmatrix} = (a+b+c)^3$

9. $\begin{vmatrix} 1+a^2-b^2 & 2b & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a^2 & 1-a^2-b^2 \end{vmatrix}$ is a perfect cube

10. $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$ is a perfect square.

11. $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$ vanishes

12. $\begin{vmatrix} 1 & \cos A & \sin A \\ 1 & \cos B & \sin B \\ 1 & \cos C & \sin C \end{vmatrix} = 4 \sin \dfrac{B-C}{2} \sin \dfrac{C-A}{2} \sin \dfrac{A-B}{2}$

Factorize each of the following determinants : (13 to 15)

13. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ *(Andhra, 1998)*

14. $\begin{vmatrix} b^2c^2+a^2 & bc+a & 1 \\ c^2a^2+b^2 & ca+b & 1 \\ a^2b^2+c^2 & ab+c & 1 \end{vmatrix}$

15. $\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ a & b & c & d \\ 1 & 1 & 1 & 1 \\ bcd & cda & dab & abc \end{vmatrix}$

16. If $a+b+c=0$, solve $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ *(Andhra, 19*

17. Solve the equation $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$

18. Show that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = 4a^2b^2c^2$

19. Prove that the determinant $\begin{vmatrix} 2b_1+c_1 & c_1+3a_1 & 2a_1+3b_1 \\ 2b_2+c_2 & c_2+3a_2 & 2a_2+3b_2 \\ 2b_3+c_3 & c_3+3a_3 & 2a_3+3b_3 \end{vmatrix}$ is a multiple of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

2·5. MATRICES

(1) Definition. *A system of mn numbers arranged in a rectangular formation along m r... and n columns and bounded by the brackets [ ] is called an m by n* **matrix** *; which is writte...*

... A matrix is also denoted by a single capital letter.

$$\text{Thus } A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1j} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2j} & \dots a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots a_{ij} & \dots a_{in} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{mj} & \dots a_{mn} \end{bmatrix}$$

is a matrix of *order mn*. It has *m rows* and *n columns*. Each of the *mn* numbers is called element of the matrix.

To locate any particular element of a matrix, the elements are denoted by a letter followed two suffixes which respectively specify the rows and the columns. Thus a_{ij} is the element in the i-th row and j-th column of A. In this notation, the matrix A is denoted by $[a_{ij}]$.

A matrix should be treated as a single entity with a number of components, rather than a collection numbers. For example, the co-ordinates of a point in solid geometry, are given by a set of three numbers which can be represented by the matrix $[x, y, z]$. Unlike a determinant, a matrix cannot reduce to a single number and the question of finding the value of a matrix never arises. The difference between a determinant and a matrix is brought out by the fact that an interchange of rows and columns does not alter the determinant but gives an entirely different matrix.

(2) Special matrices

Row and column matrices. *A matrix having a single row is called a row matrix e.g.* $[1 \quad 3 \quad 5 \quad 7]$.

A matrix having a single column is called a column matrix, e.g., $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Row and column matrices are sometimes called *row vectors* and *column vectors.*

Square matrix. *A matrix having n rows and n columns is called a square matrix of order n.*

The determinant having the same elements as the square matrix A is called the *determinant of the matrix* and is denoted by the symbol $|A|$. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

The diagonal of this matrix containing the elements 1, 3, 5 is called the *leading or principal diagonal. The sum of the diagonal elements of a square matrix A is called the* **trace** *of A.*

A square matrix is said to be **singular** *if its determinant is zero otherwise* **non-singular.**

Diagonal matrix. *A square matrix all of whose elements except those in the leading diagonal are zero is called a diagonal matrix.*

A diagonal matrix whose all the leading diagonal elements are equal is called a *scalar matrix*

For example,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are the diagonal and scalar matrices respectively.

Unit matrix. *A diagonal matrix of order n which has unity for all its diagonal elements, is called a unit matrix or an identity matrix of order n* and is denoted by I_n. For example, unit matrix of order 3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null matrix. *If all the elements of a matrix are zero, it is called a null or zero matrix and is* **denoted by '0'** ; **e.g.,**

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a null matrix.}$$

Symmetric and skew-symmetric matrices. *A square matrix $A = [a_{ij}]$ is said to be symmetric when $a_{ij} = a_{ji}$ for all i and j.*

If $a_{ij} = -a_{ji}$ for all i and j so that all the leading diagonal elements are zero, then the matrix is called a skew-symmetric matrix. Examples of symmetric and skew-symmetric matrices are respectively.

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$$

Triangular matrix. A square matrix all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix*. A square matrix all of whose elements above the leading diagonal are zero, is called a *lower triangular matrix*. Thus

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$$

are upper and lower triangular matrices respectively.

2·6. MATRIX OPERATIONS

(1) Equality of Matrices

Two matrices A and B are said to equal if and only if

(i) they are of the same order

and (ii) each element of A is equal to the corresponding element of B.

(2) Addition and subtraction of matrices. *If A, B be two matrices of the same order, then their sum A + B is defined as the matrix each element of which is the sum of the corresponding elements of A and B.*

Thus $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} + \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \\ a_3 + c_3 & b_3 + d_3 \end{bmatrix}$$

Similarly *$A - B$ is defined as a matrix whose elements are obtained by subtracting the elements of B from the corresponding elements of A.*

Thus $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} - \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 - c_1 & b_1 - d_1 \\ a_2 - c_2 & b_2 - d_2 \end{bmatrix}$$

Obs. 1. Only matrices of the same order can be added or subtracted.

2. Addition of matrices is *commutative,*

i.e. $A + B = B + A$

3. Addition and subtraction of matrices is *associative*

i.e. $(A + B) - C = A + (B - C) = B + (A - C).$

(3) Multiplication of matrix by a scalar. *The product of a matrix A by a scalar k is a matrix whose each element is k times the corresponding elements of A.*

Thus $$k \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$$

The distributive law holds for such products, *i.e.* $k(A + B) = kA + kB$.

Obs. All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.

Example 2.14. *Find x, y, z and w given that*

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 5 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 6 & x+y \\ z+w & 5 \end{bmatrix}$$

Sol. We have $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+6 & 5+x+y \\ -1+z+w & 2w+5 \end{bmatrix}$

Equating the corresponding elements, we get

$$3x = x+6,\ 3y = 5+x+y,\ 3z = -1+z+w,\ 3w = 2w+5.$$

or $$2x = 6,\ 2y = 5+x,\ 2z = w-1,\ w = 5$$

Hence $x = 3, y = 4, z = 2, w = 5$

Example 2.15. *Express* $\begin{bmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{bmatrix}$ *as the sum of a lower triangular matrix wi… leading diagonal and an upper triangular matrix.*

Sol. Let $L = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix}$ be the lower triangular matrix with zero leading diagonal

and $U = \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$ be the upper traingular matrix.

Then $\begin{bmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix} + \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$

Equating corresponding elements from both sides, we obtain $3 = l, 5 = m, -7 = n, -8$… $11 = p, 4 = q, 13 = b, -14 = c, 6 = r$.

Hence $L = \begin{bmatrix} 0 & 0 & 0 \\ -8 & 0 & 0 \\ 13 & -14 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & 5 & -7 \\ 0 & 11 & 4 \\ 0 & 0 & 6 \end{bmatrix}$

(4) Multiplication of matrices. *Two matrices can be multiplied only when the numb… columns in the first is equal to the number of rows in the second. Such matrices are said t…* **conformable.**

For instance, the product $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix}$

is defined as the matrix $\begin{bmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 \\ a_4l_1 + b_4m_1 + c_4n_1 & a_4l_2 + b_4m_2 + c_4n_2 \end{bmatrix}$

In general, if $A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$

… and $n \times p$ conformable matrices, then their product is defined as the $m \times p$ matr…

$$AB = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mp} \end{bmatrix}$$

where $c_{ij} = a_{i1} b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$, *i.e.*, the element in the ith row and the jth column of the matrix AB is obtained by weaving the ith row of A with jth column of B. The expression for c_{ij} is known as the *inner product* of the ith row with the jth column.

Post-multiplication and Pre-multiplication. In the product AB, the matrix A is said to be *post-multiplied* by the matrix B. Whereas in BA, the matrix A is said to be *pre-multiplied* by B. In one case the product may exist and in the other case it may not. Also the products in both cases may exist yet may or may not be equal.

Obs. 1. *Multiplication of matrices is associative. i.e.* $(AB)C = A(BC)$ provided A, B are conformable for the product AB and B, C are conformable for the product BC. (Ex. 2.16)

Obs. 2. *Multiplication of matrices is distributive. i.e.* $A(B + C) = AB + AC$. provided A, B are conformable for the product AB and A, C are conformable for the product AC.

Obs. 3. Power of a matrix. If A be a square matrix, then the product AA is defined as A^2. Similarly we define higher powers of A. *i.e.* $A A^2 = A^3$, $A^2 . A^2 = A^4$ etc.

If $A^2 = A$, *then the matrix A is called idempotent.*

Example 2.16. *If* $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$ *form the product of AB. Is BA defined ?*

Sol. Since the number of columns of A = the number of rows of B (each being = 3).

$\therefore$ The product AB is defined and

$$= \begin{bmatrix} 0.1 + 1.-1 + 2.2 & 0.-2 + 1.0 + 2.-1 \\ 1.1 + 2.-1 + 3.2 & 1.-2 + 2.0 + 3.-1 \\ 2.1 + 3.-1 + 4.2 & 2.-2 + 3.0 + 4.-1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

Again since the number of columns of $B \neq$ the number of rows of A.

$\therefore$ The product BA is not possible.

Example 2.17. *If* $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, *compute AB and BA and show that* $AB \neq BA$.

Sol. Considering rows of A and columns of B, we have

$$AB = \begin{bmatrix} 1.2 + 3.1 + 0.-1, & 1.3 + 3.2 + 0.1, & 1.4 + 3.3 + 0.2 \\ -1.2 + 2.1 + 1.-1, & -1.3 + 2.1 + 1.1, & -1.4 + 2.3 + 1.2 \\ 0.2 + 0.1 + 2.-1, & 0.3 + 0.2 + 2.1, & 0.4 + 0.3 + 2.2 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$$

Again considering the rows of B and columns of A, we have

$$BA = \begin{bmatrix} 2.1 + 3.-1 + 4.0, & 2.3 + 3.2 + 4.0, & 2.0 + 3.1 + 4.2 \\ 1.1 + 2.-1 + 3.0, & 1.3 + 2.2 + 3.0, & 1.0 + 2.1 + 3.2 \\ -1.1 + 1.-1 + 2.0, & -1.3 + 1.2 + 2.0, & -1.0 + 1.1 + 2.2 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$$

Evidently $AB \neq BA$.

Example 2.18. *If* $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$, *find the matrix B such that* $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ \text{[illegible]} & 6 & 4 \end{bmatrix}$

(Bombay, 2005)

Sol. Let $AB = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} l & m & n \\ p & q & r \\ u & v & w \end{bmatrix}$

$$= \begin{bmatrix} 3l + 2p + 2u & 3m + 2q + 2v & 3n + 2r + 2w \\ l + 3p + u & m + 3q + v & n + 3r + w \\ 5l + 3p + 4u & 5m + 3q + 4v & 5n + 3r + 4w \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix} \quad \text{(given)}$$

Equating corresponding elements, we get

$3l + 2p + 2u = 3, \quad l + 3p + u = 1, \quad 5l + 3p + 4u = 5$ ...(i)

$3m + 2q + 2v = 4, \quad m + 3q + v = 6, \quad 5m + 3q + 4v = 6$ ...(ii)

$3n + 2r + 2w = 2, \quad n + 3r + w = 1, \quad 5n + 3r + 4w = 4$ ...(iii)

Solving the equations (*i*), we get $l = 1,\ p = 0,\ u = 0$

Similarly equations (*ii*) give $m = 0, q = 2, v = 0$

and equations (*iii*) give $n = 0,\ r = 0,\ w = 1$

Thus $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 2.19. *Prove that* $A^3 - 4A^2 - 3A + 11I = 0$, *where* $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

Sol. $A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$

$A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$

$A^3 - 4A^2 - 3A + 11\,I$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11, & 37-28-9+0, & 26-20-6+0 \\ 10-4-6-0, & 5-16+0+11, & 1-4+3+0 \\ 35-32-3+0, & 42-36-6+0, & 34-36-9+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Example 2.20. *By mathematical induction, prove that if*

$$A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \textit{ then } A^n = \begin{bmatrix} 1 + 10n & -25n \\ 4n & 1 - 10n \end{bmatrix}$$

Sol. When $n = 1$, A^n gives $A^1 = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$

Let us assume that the result is true for any positive integer k, so that

$$A^k = \begin{bmatrix} 1 + 10k & -25k \\ 4k & 1 - 10k \end{bmatrix}$$

$$A^{k+1} = A^k \cdot A^1 = \begin{bmatrix} 1 + 10k & -25k \\ 4k & 1 - 10k \end{bmatrix} \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 11(1+10k) - 100k & -25(1+10k)+225k \\ 44k + 4(1-10k) & -100k - 9(1-10k) \end{bmatrix}$$

$$= \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}$$

This is true for $n = k+1$ ...(*ii*)

We have seen in (*i*) that the result is true for $n = 1$.

$\therefore$ It is true for $n = 1 + 1 = 2$ [by (*ii*)]

Similarly it is true for $n = 2 + 1 = 3$ and so on.

Hence by mathematical induction, the result is true for all positive integers n.

Example 2.21. *Prove that $(AB)C = A(BC)$, where A, B, C are matrices conformable for the products.* (*J.N.T.U., 2002 S*)

Sol. Let $A = [a_{ij}]$ be of order $m \times n$, $B = [b_{ij}]$ be of order $n \times p$ and $C = [c_{ij}]$ be of order of $p \times q$.

Then $AB = [a_{ik}]\,[b_{kj}] = \sum_{k=1}^{n} a_{ik}\, b_{kj}$

$$\therefore \quad (AB)C = \left[\sum_{k=1}^{n} a_{ik}\, b_{kl}\right][c_{lj}] = \left[\sum_{l=1}^{p}\left(\sum_{k=1}^{n} a_{ik}\, b_{kl}\right)c_{lj}\right] = \left[\sum_{k=1}^{n}\sum_{l=1}^{p} a_{ik}\, b_{kl}\, c_{lj}\right]$$

Similarly, $\quad BC = [b_{kl}]\;[c_{lj}] = \sum_{l=1}^{p} b_{kl}\, c_{lj}$

$$\therefore \quad A(BC) = [a_{ik}]\left[\sum_{l=1}^{p} b_{kl}\, c_{lj}\right] = \left[\sum_{k=1}^{n} a_{ik}\left(\sum_{l=1}^{p} b_{kl}\, c_{lj}\right)\right] = \left[\sum_{k=1}^{n}\left(\sum_{l=1}^{p} a_{ik}\, b_{kl}\, c_{lj}\right)\right]$$

Hence $\quad (AB)C = A(BC)$.

Problems 2.2

1. For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular ?

2. Find the values of x, y, z and a which satisfy the matrix equation $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.

3. Matrix A has x rows and $x+5$ columns. Matrix B has y rows and $11-y$ columns. Both AB and BA exist. Find x and y.

4. If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$, calculate the product AB.

5. If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, find AB or BA, whichever exists.

6. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$, verify that $(AB)C = A(BC)$ and $A(B+C) = AB + AC$.

7. Evaluate (*i*) $[x \;\; y \;\; z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 1 & -1 \\ \text{[illegible]} & 5 & 6 \\ -3 & 7 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -6 & 4 \\ -2 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix}$.

(iii) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \times [4 \quad 5 \quad 2] \times \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \times [3 \quad 2]$

8. Prove that the product of two matrices

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is a null matrix when θ and ϕ differ by an odd multiple of $\pi/2$

9. If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$, show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

10. If $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, find the value of $A^2 - 6A + 8I$, where I is a unit matrix of second order (B P T U

11. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$, and I is the unit matrix of order 3, evaluate $A^2 - 3A + 9I$.

12. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$. Verify the result $(A + B)^2 = A^2 + BA + AB + B^2$.

13. If $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

calculate the products EF and FE and show that $E^2F + FE^2 = E$.

14. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, when n is a positive integer

15. Factorize the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into the form LU, where L is lower triangular and U is up[per]

triangular matrix.

2·7. RELATED MATRICES

(1) Transpose of a matrix. *The matrix obtained from any given matrix A, by interchangi[ng] rows and columns is called the* **transpose** *of A and is denoted by A′.*

Thus the transposed matrix of $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$ is $A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$

Clearly the transpose of an $m \times n$ matrix is an $n \times m$ matrix. Also the transpose of t[he] **transpose of a matrix coincides with itself**, *i.e.* $(A')' = A$

For a symmetric matrix, $A' = A$ and for a skew-symmetric matrix, $A' = -A$.

Obs. 1. *The transpose of the product of the two matrices is the product of their transposes taken in t[he] **reverse order** i.e.* $(\mathbf{AB})' = \mathbf{B'A'}$.

For, the element in the ith row and jth col. of $(AB)'$

= element in the jth row and ith col. of AB = inner product of jth row of A with ith col. of B

= inner product of jth col. of A' with ith row of B' = element in the ith row and jth col. of $B'A'$

Hence $(AB)' = B'A'$.

Obs. 2. *Every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symm[etric] matrix.*

Let A be the given square matrix, then $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

Let $B = \frac{1}{2}(A + A')$ and $C = \frac{1}{2}(A - A')$

or $\quad A\dfrac{\text{Adj } A}{|A|} = I \quad [\,|A| \neq 0]$ or $\dfrac{\text{Adj } A}{|A|}$ is the inverse of A

Obs. 1. *Inverse of a matrix is unique.*

For if possible, let the two inverses of the matrix A be B and C,

then $\quad AB = BA = I \quad$ and $\quad AC = CA = I$

$CAB = (CA)B = IB = B \quad$ and $\quad CAB = C(AB) = CI = C$

Thus $\quad B = C$

Obs. 2. *The reciprocal of the product of two matrices is the product of their reciprocals taken in the reverse order i.e.* $(AB)^{-1} = B^{-1}A^{-1}$

If A, B be two matrices, then the reciprocal of their product is $(AB)^{-1}$

Clearly $\quad (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ (by Associative law)

$= AIA^{-1} = AA^{-1} = I$

Similarly, $\quad (B^{-1}A^{-1})(AB) = I$

Hence $B^{-1}A^{-1}$ is the reciprocal of AB

Obs. 3. Multiplication by an inverse matrix plays the same role in matrix algebra that division plays in ordinary algebra

i.e. if $\quad |A||B| = |C||D| \quad$ then $\quad |A|^{-1}|A||B| = |A^{-1}||C||D|$

or $\quad B = A^{-1}|C||D| \quad$ i.e. $\quad \dfrac{|C||D|}{|A|} = A^{-1}|C||D|$

Example 2.23. *Find the inverse of* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Sol. The determinant of the given matrix A is

$$\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ (say)}$$

If $A_1, A_2, \ldots$ be the co-factors of $a_1, a_2, \ldots$ in Δ, then $A_1 = -24$, $A_2 = -8$, $A_3 = -12$, $B_1 = 10$, $B_2 = 2$, $B_3 = 6$, $C_1 = 2$, $C_2 = 2$, $C_3 = 2$.

Thus $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = -8$.

$$\text{and } adj\, A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}.$$

Hence the inverse of the given matrix A

$$= \frac{adj\, A}{\Delta} = \frac{1}{-8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Note. For other methods see Examples 2.25, 2.28 and 2.46

Problems 2.3

1. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $AA' = I = A'A$, where I is the unit matrix

2. Express each of the following matrices as the sum of a symmetric and a skew-symmetric matrix

(i) $\begin{bmatrix} 2 & 7 & 1 \\ 5 & 4 & 0 \end{bmatrix}$ ... (ii) $\begin{vmatrix} a & a & b \\ c & b & b \\ & a & c \end{vmatrix}$

or $\quad A \dfrac{Adj\,A}{|A|} = I \quad [\,|A| \neq 0]$ or $\dfrac{Adj\,A}{|A|}$ is the inverse of A

Obs. 1. *Inverse of a matrix is unique.*

For if possible, let the two inverses of the matrix A be B and C.

then $\quad AB = BA = I \quad$ and $\quad AC = CA = I$

$CAB = (CA)B = IB = B \quad$ and $\quad CAB = C(AB) = CI = C$

Thus $\quad B = C$

Obs. 2. *The reciprocal of the product of two matrices is the product of their reciprocals taken in the reverse order i.e.* $(AB)^{-1} = B^{-1}A^{-1}$

If A, B be two matrices, then the reciprocal of their product is $(AB)^{-1}$

Clearly $\quad (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ [By Associativity]

$= AIA^{-1} = AA^{-1} = I$

Similarly, $\quad (B^{-1}A^{-1})(AB) = I$

Hence $B^{-1}A^{-1}$ is the reciprocal of AB

Obs. 3. Multiplication by an inverse matrix plays the same role in matrix algebra that division plays in ordinary algebra

i.e. if $\quad |A||B| = |C||D| \quad$ then $\quad |A|^{-1}|A||B| = |A^{-1}||C||D|$

or $\quad B = A^{-1}|C||D| \quad$ i.e. $\quad \dfrac{|C||D|}{|A|} = A^{-1}|C||D|$

Example 2.23. *Find the inverse of* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Sol. The determinant of the given matrix A is

$$\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ (say)}$$

If $A_1, A_2, \ldots$ be the co-factors of $a_1, a_2, \ldots$ in Δ, then $A_1 = -24$, $A_2 = -8$, $A_3 = -12$, $B_1 = 10$, $B_2 = 2$, $B_3 = 6$, $C_1 = 2$, $C_2 = 2$, $C_3 = 2$.

Thus $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = -8$.

$$\text{and } adj\,A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}.$$

Hence the inverse of the given matrix A

$$= \frac{adj\,A}{\Delta} = \frac{1}{-8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Note. For other methods see Examples 2.25, 2.28 and 2.46.

Problems 2.3

1. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $AA' = I = A'A$, where I is the unit matrix.

2. Express each of the following matrices as the sum of a symmetric and a skew-symmetric matrix:

(i) $\begin{bmatrix} 2 & 7 & 1 \\ 5 & 4 & 0 \\ \ldots \end{bmatrix}$ $\quad$ (ii) $\begin{bmatrix} a & a & b \\ c & b & b \\ \ldots & a & c \end{bmatrix}$

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, compute $adj\ A$ and A^{-1}. Also verify that $AA^{-1} = I$.

4. If A is a non-singular matrix of order n, prove that $A\ adj\ A = |A|\ I$ *(Bombay 2006)*

Verify that $A\ (adj\ A) = (adj\ A)\ A = |A|\ I$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

5. Find the inverse of the matrix (i) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$ *(B.P.T.U. 2005)*

6. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

(i) find A^{-1}; (ii) show that $A^3 = A^{-1}$

7. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, prove that $A^{-1} = A'$.

8. Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$

9. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)' = B'A'$, where A' is the transpose of A.

10. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

11. If A is a square matrix, show that (i) $A + A'$ is symmetric, and (ii) $A - A'$ is skew-symmetric. *(P.T.U., 1999)*

12. If $D = \text{diag}\ [d_1, d_2, d_3]$, $d_1, d_2, d_3 \neq 0$, prove that $D^{-1} = \text{diag}\ [d_1^{-1}, d_2^{-1}, d_3^{-1}]$

13. If A and B are square matrices of the same order and A is symmetrical, show that $B'AB$ is also symmetrical.

[**Hint.** Show that $(B'AB)' = B'AB$]

14. If a non-singular matrix A is symmetric, show that A^{-1} is also symmetric.

2·8. (1) RANK OF A MATRIX

If we select any r rows and r columns from any matrix A, deleting all the other rows and columns, then the determinant formed by these $r \times r$ elements is called the *minor of A of order r*. Clearly there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

Def. *A matrix is said to be of rank r when*

(i) it has at least one non-zero minor of order r,

and (ii) every minor of order higher than r vanishes.

Briefly, *the rank of a matrix is the largest order of any non-vanishing minor of the matrix.*

If a matrix has a non-zero minor of order r, its rank is $\geq r$.

If all minors of a matrix of order $r + 1$ are zero, its rank is $\leq r$.

The rank of a matrix A shall be denoted by $\rho(A)$.

(2) Elementary transformation of a matrix. The following operations, three of which refer to rows and three to columns are known as *elementary transformations* :

I. The interchange of any two rows (columns).

II. The multiplication of any row (column) by a non-zero number.

III. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Notation. The *elementary row transformations* will be denoted by the following symbols:

(*i*) R_{ij} for the interchange of the *i*th and *j*th rows.

(*ii*) kR_i for multiplication of the *i*th row by *k*.

(*iii*) $R_i + pR_j$ for addition to the *i*th row, *p* times the *j*th row.

The corresponding column transformation will be denoted by writing *C* in place of *R*.

Elementary transformations do not change either the order or rank of a matrix. While the value of minors may get changed by the transformation I and II, their zero or non-zero character remains unaffected.

(3) Equivalent matrix. *Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations.* Two equivalent matrices have the same order and *the same rank*. The symbol ~ is used for equivalence.

Example 2·24. *Determine the rank of the following matrices :*

$$(i)\begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 2\\ 2 & 6 & 5\end{bmatrix} \qquad (ii)\begin{bmatrix}0 & 1 & -3 & -1\\ 1 & 0 & 1 & 1\\ 3 & 1 & 0 & 2\\ 1 & 1 & -2 & 0\end{bmatrix}$$

Sol. (*i*) Operate $R_2 - R_1$ and $R_3 - 2R_1$ so that the given matrix

$$\sim \begin{bmatrix}1 & 2 & 3\\ 0 & 2 & -1\\ 0 & 2 & -1\end{bmatrix} = A \text{ (say)}$$

Obviously, the 3rd order minor of *A* vanishes. Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is $\begin{vmatrix}1 & 3\\ 0 & -1\end{vmatrix} = -1 \neq 0$.

$\therefore$ $\rho(A) = 2$. Hence the rank of the given matrix is 2.

(*ii*) Given matrix

$$\sim \begin{bmatrix}0 & 1 & -3 & -1\\ 1 & 0 & 0 & 0\\ 3 & 1 & -3 & -1\\ 1 & 1 & -3 & -1\end{bmatrix} \qquad \sim \begin{bmatrix}0 & 1 & -3 & -1\\ 1 & 0 & 0 & 0\\ 3 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

[Operating $C_3 - C_1, C_4 - C_1$] [Operating $R_3 - R_1, R_4 - R_1$]

$$\sim \begin{bmatrix}0 & 1 & -3 & -1\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad \sim \begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} = A \text{ (say)}$$

[Operating $R_3 - 3R_2, R_4 - R_2$] [Operating $C_3 + 3C_2, C_4 + C_2$]

Obviously, the 4th order minor of *A* is zero. Also every 3rd order minor of *A* is zero. But, of all the 2nd order minors, only $\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix} = -1 \neq 0$. $\therefore$ $\rho(A) = 2$

Hence the rank of the given matrix is 2.

(4) Elementary matrices. *An elementary matrix is that, which is obtained from a unit matrix, by subjecting it to any of the elementary transformations.*

Examples of elementary matrices obtained from

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ are } R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C_{23}\,;\ kR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix},\ R_1 + pR_2 = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) Theorem. *Elementary row (column) transformations of a matrix A can be obtained by pre-multiplying (post-multiplying) A by the corresponding elementary matrices.*

Consider the matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Then $R_{23} \times A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{bmatrix}$

So a pre-multiplication by R_{23} has interchanged the 2nd and 3rd rows of A. Similarly pre-multiplication by kR_2 will multiply the 2nd row of A by k and pre-multiplication by $R_1 + pR_2$ will result in the addition of p times the 2nd row of A to its 1st row.

Thus the pre-multiplication of A by elementary matrices results in the corresponding elementary row transformation of A. It can easily be seen that post-multiplication will perform the elementary column transformations.

(6) Gauss-Jordan method of finding the inverse*. *Those elementary row transformations which reduce a given square matrix A to the unit matrix, when applied to unit matrix I give the inverse of A.*

Let the successive row transformations which reduce A to I result from pre-multiplication by the elementary matrices $R_1, R_2, \ldots R_i$ so that

$$R_i R_{i-1} \ldots\ldots R_2 R_1 A = I$$

$\therefore$ $$R_i R_{i-1} \ldots\ldots R_2 R_1 A A^{-1} = I A^{-1}$$

or $$R_i R_{i-1} \ldots\ldots R_2 R_1 I = A^{-1} \qquad [\because AA^{-1} = I]$$

Hence the result.

Working rule to evaluate A^{-1}. *Write the two matrices A and I side by side. Then perform the same row transformations on both. As soon as A is reduced to I, the other matrix represents* A^{-1}.

Example 2·25. *Using the Gauss-Jordan method, find the inverse of the matrix*

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \qquad \text{(Kurukshetra, 2006)}$$

Sol. Writing the same matrix side by side with the unit matrix of order 3, we have

$$\left[\begin{array}{rrr:rrr} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right] \qquad \text{(Operate } R_2 - R_1 \text{ and } R_3 + 2R_1)$$

* Named after the great German mathematician *Carl Friedrich Gauss* (1777–1855) who made his first great discovery as a student at Gottingen. His important contributions are to algebra, number theory, mechanics, complex analysis, differential equations, differential geometry, non-Euclidean geometry, numerical analysis, astronomy and electromagnetism. He became director of the observatory at Gottingen in 1807.

* Named after the German mathematician and geodesist *Wilhelm Jordan* (1842–1899).

$$\sim \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 2 & -6 & : & -1 & 1 & 0 \\ 0 & -2 & 2 & : & 2 & 0 & 1 \end{bmatrix} \quad \left(\text{Operate } \frac{1}{2}R_2 \text{ and } \frac{1}{2}R_3\right)$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & : & 1 & 0 & \frac{1}{2} \end{bmatrix} \quad (\text{Operate } R_1 - R_2 \text{ and } R_3 + R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & : & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & : & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \left[\text{Operate } R_1 + 3R_3, R_2 - \frac{3}{2}R_3 \text{ and } \left(-\frac{1}{2}\right)\right.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & : & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & : & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Hence the inverse of the given matrix is $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$ [cf. Example 2·20

(7) Normal form of a matrix. *Every non-zero matrix A of rank r, can be reduced b sequence of elementary transformations, to the form*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ called the } \textbf{normal form} \text{ of } A.$$

Cor. 1. *The rank of a matrix A is r if and only if it can be reduced to the normal form (i)*

Cor. 2. *Since each elementary transformation can be affected by pre-multiplication or p multiplication with a suitable elementary matrix and each elementary matrix is non-singular, therefore have the following result :*

Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that P equals (i).

If A be a $m \times n$ matrix, then P and Q are square matrices of orders m and n respectively.

Example 2·26. *Reduce the following matrix into its normal form and hence find its rank*

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$ (U.P.T.U., 200

Sol.

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$ [by R_{12}

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad [\text{by } R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad [\text{by } C_2 + C_1, C_3 + 2C_1, C_4 + 4C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{by } R_4 - R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{by } R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{by } R_3 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{by } C_3 + 6C_2, C_4 + 3C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{by } \frac{1}{33} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{by } C_4 - 22\, C_3$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = 3$.

Example 2·27. *For the matrix* $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$,

find non-singular matrices P and Q such that PAQ is in the normal form.

Hence find the rank of A. *(Kurukshetra, 2005)*

Sol. We write $A = IAI$, *i.e.* $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We shall affect every elementary row (column) transformation of the product by subjecting the prefactor (post-factor) of A to the same.

Operate $C_2 - C_1, C_3 - 2C_1$ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $R_2 - R_1$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Operate $C_3 - C_2$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $R_3 + R_2$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

which is of the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Hence $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\rho(A) = 2$.

Problems 2.4

Determine the rank of the following matrices (1–4) :

1. $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ (*P.T.U., 2005*)

2. $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ (*W.B.T.U., 2005*)

4. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ (*Kottayam, 2005*)

5. $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$

6. Use Gauss-Jordan method to find the inverse of the following matrices:

(*i*) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(*iii*) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ (*B.P.T.U., 2006*) (*iv*) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (*Kurukshetra, 2006*)

7. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ find A^{-1}. Also find two non-singular matrices P and Q such that $PAQ = I$, where I is the unit matrix and verify that $A^{-1} = QP$.

8. Reduce each of the following matrices to normal form and hence find their ranks

(*i*) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ (*Kurukshetra, 2005*) (*ii*) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ (*U.P.T.U., 2005*)

$$\text{(iii)} \begin{bmatrix} 1 & -1 & -2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$ (Bombay, 2006)

9. Find non-singular matrices P and Q such that PAQ is in the normal form for the matrices

(i) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (J.N.T.U., 2002) (ii) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

2.9. PARTITION METHOD OF FINDING THE INVERSE

According to this method of finding the inverse, if the inverse of a matrix A_n of order n is known, then the inverse of the matrix A_{n+1} can easily be obtained by adding $(n+1)$th row and $(n+1)$th column to A_n.

Let $$A = \begin{bmatrix} A_1 & : & A_2 \\ \cdots & : & \cdots \\ A_3' & : & \alpha \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \cdots & : & \cdots \\ X_3' & : & x \end{bmatrix}$$

where A_2, X_2 are column vectors and A_3', X_3' are row vectors (being transposes of column vectors A_3, X_3) and α, x are ordinary numbers. We also assume that A_1^{-1} is known.

Then $AA^{-1} = I_{n+1}$ *i.e.* $$\begin{bmatrix} A_1 & : & A_2 \\ \cdots & : & \cdots \\ A_3' & : & \alpha \end{bmatrix} \begin{bmatrix} X_1 & : & X_2 \\ \cdots & : & \cdots \\ X_3' & : & x \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$$

gives
$$A_1X_1 + A_2X_3' = I_n \quad \ldots(i)$$
$$A_1X_2 + A_2x = 0 \quad \ldots(ii)$$
$$A_3'X_1 + \alpha X_3' = 0 \quad \ldots(iii)$$
$$A_3'X_2 + \alpha x = 1 \quad \ldots(iv)$$

From (ii), $X_2 = -A_1^{-1}A_2x$ and using this, (iv) gives $x = (\alpha - A_3'A_1^{-1}A_2)^{-1}$

Hence x and then X_2 are given.

Also from (i), $X_1 = A_1^{-1}(I_n - A_2X_3')$

and using this, (iii) gives $X_3' = -A_3'A_1^{-1}(\alpha - A_3'A_1^{-1}A_2)^{-1} = -A_3'A_1^{-1}x$

Then X_1 is determined and hence A^{-1} is computed.

Obs. This is also known as the '*Escalator method*'. For evaluation of A^{-1} we only need to determine two inverse matrices A_1^{-1} and $(\alpha - A_3'A_1^{-1}A_2)^{-1}$.

Example 2.28. *Using the partition method, find the inverse of* $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$.

Sol. Let $$A = \begin{bmatrix} 1 & 1 & : & 1 \\ 4 & 3 & : & -1 \\ \cdots & \cdots & : & \cdots \\ 3 & 5 & : & 3 \end{bmatrix} = \begin{bmatrix} A_1 & : & A_2 \\ \cdots & : & \cdots \\ A_3' & : & \alpha \end{bmatrix}$$

So that $$A_1^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}^{-1} = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$$

Let $$A^{-1} = \begin{bmatrix} X_1 & \vdots & X_2 \\ \dots & \vdots & \dots \\ X_3' & \vdots & x \end{bmatrix} \text{ so that } AA^{-1} = I.$$

$$\alpha - A_3' A_1^{-1} A_2 = 3 + [3 \quad 5] \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -10$$

$$\therefore \quad x = (\alpha - A_3' A_1^{-1} A_2)^{-1} = -\frac{1}{10}$$

Also $$X_2 = -A_1^{-1} A_2 x = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10} \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Then $$X_3' = -A_3' A_1^{-1} x = [3 \quad 5] \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10} [-11 \quad 2]$$

Finally $$X_1 = A_1^{-1} (I - A_2 X_3') = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} [-11 \quad 2]$$

$$= \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} -44 & 8 \\ 55 & -10 \end{bmatrix} = \begin{bmatrix} 1\cdot4 & 0\cdot2 \\ -1\cdot5 & 0 \end{bmatrix}$$

Hence $$A^{-1} = \begin{bmatrix} 1\cdot4 & 0\cdot2 & -0\cdot4 \\ -1\cdot5 & 0 & 0\cdot5 \\ 1\cdot1 & -0\cdot2 & -0\cdot1 \end{bmatrix}$$

Example 2·29. *If A and C are non-singular matrices, then show that*

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$$

Hence find inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$ *(Bombay, 200*

Sol. Let the given matrix be $M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ and its inverse be $M^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ both in th partioned form where A, B, C, P, Q, R, S are all matrices.

$$\therefore \quad MM^{-1} = \begin{bmatrix} A & O \\ B & C \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = I$$

or $$\begin{bmatrix} AP + OR & AQ + OS \\ BP + CR & BQ + CS \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$\therefore$ **Equating** corresponding elements, we have

$$AP + OR = I, \; AQ + OS = 0, \; BP + CR = 0, \; BQ + CS = I.$$

Second relation gives $AQ = 0$ *i.e.* $Q = 0$ as A is non-singular.

First relation gives $AP = I$ *i.e.* $P = A^{-1}$.

From third equation, $BP + CR = 0$ *i.e.* $CR = -BP = -BA^{-1}$

$\therefore C^{-1}CR = -C^{-1}BA^{-1}$ or $IR = -C^{-1}BA^{-1}$ or $R = -C^{-1}BA^{-1}$

From fourth equation, $BQ + CS = I$, or $CS = I$ or $S = C^{-1}$

Hence $$M^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$$

(ii) Let $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} A & O \\ B & C \end{bmatrix}$

Whence $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $C^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

$\therefore \quad -C^{-1}(BA^{-1}) = -\frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\left\{\frac{1}{2}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

$$= -\frac{1}{24}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = -\frac{1}{24}\begin{bmatrix} 18 & 0 \\ 0 & 4 \end{bmatrix}$$

Hence $M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ -3/4 & 0 & 1/4 & 0 \\ 0 & -1/6 & 0 & 1/3 \end{bmatrix}$

Problems 2·5

Find the inverse of each of the following matrices using the partition method.

1. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (*Nagpur, 1997*)

2. $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -1 & 10 & 2 \\ 5 & 1 & 20 & 3 \\ 9 & 7 & 39 & 4 \\ 1 & -2 & 2 & 1 \end{bmatrix}$

5. If A and C are non-singular square matrices of order n, show that $\begin{bmatrix} A & O \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & O \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$

Hence find $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 4 & 0 & 2 \end{bmatrix}$ (*Bombay, 2002 S*)

2·10. SOLUTION OF LINEAR SYSTEM OF EQUATIONS

(1) Method of determinants — Cramer's* rule

Consider the equations
$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \ldots(i)$$

If the determinant of coefficients be $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

*Gabriel Cramer (1704–1752), a Swiss mathematician.

then $$x\Delta = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix} \quad [\text{Operate } C_1 + yC_2 + zC_3]$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Thus $$x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ provided } \Delta \neq 0$$

Similarly, $$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $$z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Equations (ii), (iii) and (iv) giving the values of x, y, z constitute the **Cramer's rule**, which reduces the solution of the linear equations (i) to a problem in evaluation of determinants.

(2) Matrix inversion method

If $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then the equations (i) are equivalent to the matrix equation $AX = D$ where A is the *coefficient matrix*.

Multiplying both sides of (v) by the reciprocal matrix A^{-1}, we get

$$A^{-1}AX = A^{-1}D \quad \text{or} \quad IX = A^{-1}D \qquad [\because A^{-1}A = I]$$

or $$X = A^{-1}D \quad i.e., \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

where A_1, B_1 etc. are the cofactors of a_1, b_1 etc. in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} (\Delta \neq 0)$

Hence equating the values of x, y, z to the corresponding elements in the product on the right side of (vi), we get the desired solutions.

Obs. *When A is a singular matrix, i.e., Δ = 0, the above methods fail. These also fail when the number of equations and the number of unknowns are unequal. Matrices can, however, be usefully applied to solve such equations as will be seen in § 2.11(2).*

Example 2.30. *Solve the equations* $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$ *by (i) determinants (ii) matrices.*

(i) **Sol.** *by determinants.*

Here $\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$ [Expanding by C_1]

$$\therefore \quad x = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} \quad \text{[Expand by } C_1\text{]}$$

$$= \frac{1}{8}[3(-3+2) + 3(1-4) + 4(-1+6)] = 1$$

Similarly, $y = \frac{1}{\Delta}\begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 2$ and $z = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1$

Hence $x = 1, y = 2, z = -1$.

Note. The use of Cramer's rule involves a lot of labour when the number of equations exceeds four. In such and other cases, the numerical methods given in § 28.4 to 28.6 are preferable.

***(ii)* Sol.** *by Matrices :*

Here $\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (say).

Then $A_1 = -1, A_2 = 3, A_3 = 5$; $B_1 = -3, B_2 = 1, B_3 = 7$; $C_1 = 7, C_2 = -5, C_3 = -11$

Also $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = 8$.

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence $x = 1, y = 2, z = -1$.

Example 2·31. *Solve the equations* $x_1 - x_2 + x_3 + x_4 = 2$; $x_1 + x_2 - x_3 + x_4 = -4$; $x_1 + x_2 + x_3 - x_4 = 4$; $x_1 + x_2 + x_3 + x_4 = 0$, *by finding the inverse by elementary row operations*

Sol. Given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

To find A^{-1}, we write

$$[A : I] = \begin{bmatrix} 1 & -1 & 1 & 1 & : & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & : & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & : & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & : & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} [R_2 - R_1 \\ [R_3 + R_1 \\ [R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 & : & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & : & -1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & : & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 2 & : & 1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} [\frac{1}{2}R_2 \\ [\frac{1}{2}R_3 \\ [\frac{1}{2}R_4 \end{matrix}$$

$$= \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 1 & 0 & 1 & 1 & 1/2 & 0 & 0 & 1/2 \end{array}\right] \quad \begin{array}{l} [R_1 + R_2 \\ \\ \\ [R_4 - R_3 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array}\right] \quad \begin{array}{l} [R_1 - R_4 \\ [R_2 + R_3 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & -1/2 \\ 1 & 1 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array}\right] \quad \begin{array}{l} [R_2 - R_1 \\ [R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & -1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array}\right]$$

Thus, $$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}$$

Hence, $$X = A^{-1}B = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$$

i.e. $x_1 = 1,\ x_2 = -1,\ x_3 = 2,\ x_4 = -2.$

Problems 2.6

Solve the following equations with the help of determinants (1 to 4)

1. $x + y + z = 4$; $x - y + z = 0$, $2x + y + z = 5$.
2. $x + 3y + 6z = 2$, $3x - y + 4z = 9$; $x - 4y + 2z = 7$ *(Osmania* [illegible]
3. $x + y + z = 6.6$, $x - y + z = 2.2$; $x + 2y + 3z = 15.2$
4. $x^2z^3/y = e^8$, $y^2z/x = e^4$, $x^3y/z^4 = 1$
5. $2vw - wu + uv = 3uvw$, $3vw + 2wu + 4uv = 19uvw$, $6vw + 7wu - uv = 17uvw$.

Solve the following system of equations by matrix method (6 to 8)

6. $x_1 + x_2 + x_3 = 1$, $x_1 + 2x_2 + 3x_3 = 6$, $x_1 + 3x_2 + 4x_3 = 6$
7. $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$. *(P.T.U.* [illegible]
8. $2x - 3y + 4z = -4$, $x + z = 0$, $-y + 4z = 2$ *(Bhopal* [illegible]
9. $2x - y + 3z = 8$, $x - 2y - z = -4$, $3x + y - 4z = 0$ *(W.B.T.U.* [illegible]
10. $2x_1 + x_2 + 2x_3 + x_4 = 6$, $4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$, $6x_1 + 6x_2 + 6x_3 + 12x_4 = 36$, $2x_1 + 2x_2$ [illegible] *(Raj*[illegible]

(U.P.T.U. [illegible]

11. By finding A^{-1} solve the linear equation $AX = B$, where $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B =$ [illegible]

12. In a given electrical network, the equations for the currents i_1, i_2, i_3 are [illegible]

Calculate i_1 and i_3 by Cramer's rule.

13. Using the loop current method on a circuit, the following equations are obtained

$$7i_1 - 4i_2 = 12, -4i_1 + 12i_2 - 6i_3 = 0, -6i_2 + 14i_3 = 0.$$

By matrix method, solve for i_1, i_2 and i_3.

14. Solve the following equations by calculating the inverse by elementary row operations :

$$2x_1 + 2x_2 + 2x_3 - 3x_4 = 2 \; ; \; 3x_1 + 6x_2 - 2x_3 + x_4 = 8 \; ; \; x_1 + x_2 - 3x_3 - 4x_4 = -1 \; ; \; 2x_1 + x_2 + 5x_3 + x_4 = 5$$

2·11. (1) CONSISTENCY OF LINEAR SYSTEM OF EQUATIONS

Consider the system of m linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= k_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= k_m \end{aligned}\right\}$$

containing the n unknowns $x_1, x_2, \ldots, x_n$. To determine whether the equations (*i*) are consistent (*i.e.* possess a solution) or not, we consider the ranks of the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & k_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & k_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & k_m \end{bmatrix}$$

A is the *co-efficient matrix* and K is called the *augmented matrix* of the equations (*i*).

(2) Rouche's theorem. *The system of equations (i) is consistent if and only if the coefficient matrix A and the augmented matrix K are of the same rank otherwise the system is inconsistent.*

Proof. We consider the following two possible cases :

I. Rank of A = rank of K = r ($r \leq$ the smaller of the numbers m and n). The equations (*i*) can, by suitable row operations, be reduced to

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + \ldots\ldots + b_{1n}x_n &= l_1 \\ 0.x_1 + b_{22}x_2 + \ldots\ldots + b_{2n}x_n &= l_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0.x_1 + 0.x_2 + \ldots\ldots + b_{rn}x_n &= l_r \end{aligned}\right\} \qquad \ldots(ii)$$

and the remaining $m - r$ equations being all of the form $0.x_1 + 0.x_2 + \ldots\ldots + 0.x_n = 0$.

The equations (*ii*) will have a solution, though $n - r$ of the unknowns may be chosen arbitrarily. The solution will be unique only when $r = n$. Hence the equations (*i*) are consistent.

II. Rank of A (i.e. r) < rank of K. In particular, let the rank of K be $r + 1$. In this case, the equations (*i*) will reduce, by suitable row operations, to

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \ldots\ldots + b_{1n}x_n &= l_1, \\ 0.x_1 + b_{22}x_2 + \ldots\ldots + b_{2n}x_n &= l_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0.x_1 + 0.x_2 + \ldots\ldots + b_{rn}x_n &= l_r, \\ 0.x_1 + 0.x_2 + \ldots\ldots + 0.x_n &= l_{r+1} \end{aligned}$$

and the remaining $m - (r + 1)$ equations are of the form $0.x_1 + 0.x_2 + \ldots\ldots + 0.x_n = 0$.

Clearly, the $(r + 1)$th equation cannot be satisfied by any set of values for the unknowns. Hence the equations (*i*) are inconsistent.

[**Procedure to test the consistency of a system of equations in n unknowns :**

Find the ranks of the coefficient matrix A and the augmented matrix K, by reducing A to the triangular form by elementary row operations. Let the rank of A be r and that of K be r'.

(*i*) *If $r \neq r'$, the equations are inconsistent, i.e. there is no solution.*

(*ii*) *If $r = r' = n$, the equations are consistent and there is a unique solution.*

(*iii*) *If $r = r' < n$, the equations are consistent and there are infinite number of solutions.* Giving arbitrary values to $n - r$ of the unknowns, we may express the other r unknowns in terms of these.]

Example 2·32. *Test for consistency and solve*

$$5x + 3y + 7z = 4,\ 3x + 26y + 2z = 9,\ 7x + 2y + 10z = 5.$$

(*J.N.T.U., 2005 ; P.T.U., 2005 ; V.T.U., 2004*)

Sol. We have $\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

Operate $3R_1, 5R_2$, $\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix}$

Operate $R_2 - R_1$, $\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix}$

Operate $\frac{7}{3}R_1, 5R_3, \frac{1}{11}R_2$, $\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix}$

Operate $R_3 - R_1 + R_2, \frac{1}{7}R_1$, $\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

The ranks of coefficient matrix and augmented matrix for the last set of equations, are both 2. Hence the equations are consistent. Also the given system is equivalent to

$$5x + 3y + 7z = 4, \quad 11y - z = 3, \qquad \therefore \qquad y = \frac{3}{11} + \frac{z}{11} \text{ and } x = \frac{7}{11} - \frac{16}{11}z$$

where z is a parameter.

Hence $x = \frac{7}{11}$, $y = \frac{3}{11}$ and $z = 0$, is a particular solution.

Obs. In the above solution, the coefficient matrix is reduced to an upper triangular matrix by row-transformations.

Example 2·33. *Investigate the values of* λ *and* μ *so that the equations*

$$2x + 3y + 5z = 9,\ 7x + 3y - 2z = 8,\ 2x + 3y + \lambda z = \mu,$$

have (i) no solution, (ii) a unique solution and (iii) an infinite number of solutions.

(*Ranchi, 2000*)

Sol. We have $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$

The system admits of unique solution if, and only if, the coefficient matrix is of rank 3. T... requires that

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5 - \lambda) \neq 0$$

Thus for a unique solution $\lambda \neq 5$ and μ may have any value. If $\lambda = 5$, the system will have ... solution for those values of μ for which the matrices

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \text{ and } K = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

are not of the same rank. But A is of rank 2 and K is not of rank 2 unless $\mu = 9$. Thus if $\lambda = 5$ a... $\mu \neq 9$, the system will have no solution.

If $\lambda = 5$ and $\mu = 9$, the system will have an infinite number of solutions.

(3) System of linear homogeneous equations. Consider the homogeneous line... equations

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = 0 \end{array}\right\} \quad \ldots(iii)$$

Find the rank r of the coefficient matrix A by reducing it to the triangular form by elementar... row operations.

I. *If $r = n$, the equations (iii) have only a trivial zero solution*

$$x_1 = x_2 = \ldots\ldots = x_n = 0$$

If $r < n$, the equations (iii) have $(n - r)$ linearly independent solutions.

The number of linearly independent solutions is $(n - r)$ means, if arbitrary values ar... assigned to $(n - r)$ of the variables, the values of the remaining variables can be uniquely found. Thus the equations (iii) will have an infinite number of solutions.

II. ***When*** $m < n$ (*i.e.* the number of equations is less than the number of variables), *th... solution is always other than $x_1 = x_2 = \ldots\ldots = x_n = 0$.* The number of solutions is infinite.

III. ***When*** $m = n$ (*i.e.* the number of equations = the number of variables), *the necessary and sufficient condition for solutions other than $x_1 = x_2 = \ldots\ldots = x_n = 0$, is that the determinant of the coefficient matrix is zero.* In this case the equations are said to be *consistent* and such a solution ... *non-trivial solution. The determinant is called the* **eliminant** *of the equations.*

Example 2·34. *Solve the equations*

(i) ... $2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 12z = 0$

(ii) ... $+ 2y + z + 3w = 0$, $6x + 3y + 4z + 7w = 0$, $2x + y + w = 0$. (Osmania, 1999)

Sol. (i) Rank of the coefficient matrix

$$\begin{bmatrix} \ldots & 2 & 3 \\ \ldots & 4 & 4 \\ \ldots & 10 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix} \qquad [\text{Operating } R_2 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\text{Operating } R_3 - 7R_1 - 2R_2]$$

which = the number ... (n)

$\therefore$ **The equations have only a trivial solution :** $x = y = z = 0$.

(ii) Rank of the coefficient matrix

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix} \qquad \left[\text{Operating } R_2 - \frac{3}{2}R_1,\ R_3 - \frac{1}{2}R_1\right.$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left[\text{Operating } R_3 + \frac{1}{5}R_2\right.$$

is 2 which < the number of variable (*i.e.* $r < n$)

$\therefore$ Number of independent solutions $= 4 - 2 = 2$. Given system is equivalent to

$$4x + 2y + z + 3w = 0, \quad z + w = 0.$$

$\therefore$ We have $z = -w$ and $y = -2x - w$

which give an infinite number of non-trivial solutions, x and w being the parameters.

Example 2·35. *Find the values of k for which the system of equations $(3k - 8)x + 3y + 3z = 0,\ 3x + (3k - 8)y + 3z = 0,\ 3x + 3y + (3k - 8)z = 0$ has a non-trivial solution.* (*U.P.T.U., 2006*)

Sol. For the given system of equations to have a non-trivial solution, the determinant of the coefficient matrix should be zero.

i.e.
$$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix} = 0$$
[Operating $C_1 + (C_2 + C_3)$]

or
$$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0 \text{ or } (3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3k-11 & 0 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0$$
[Operating $R_2 - R_1,\ R_3 - R_1$]

or $$(3k-2)(3k-11)^2 = 0 \text{ whence } k = 2/3,\ 11/3,\ 11/3.$$

Example 2·36. *If the following system has non-trivial solution, prove that $a + b + c = 0$ or $a = b = c$: $ax + by + cz = 0,\ bx + cy + az = 0,\ cx + ay + bz = 0$.* (*Bombay, 2006*)

Sol. For the given system of equations to have non-trivial solution, the determinant of the coefficient matrix is zero.

i.e.
$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \text{ or } \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$
[Operating $R_1 + R_2 + R_3$]

or
$$(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \text{ or } (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = 0$$
[Operating $C_2 - C_1,\ C_3 - C_1$]

or $$(a+b+c)\ [(c-b)(b-c) - (a-c)(a-b)] = 0$$

or $$(a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca) = 0$$

i.e. $$a + b + c = 0 \quad \text{or} \quad a^2 + b^2 + c^2 - ab - bc - ca = 0$$

or $$a + b + c = 0 \text{ or } \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

or $$a + b + c = 0\ ;\ a = b,\quad b = c,\quad c = a.$$

... given system has a non-trivial solution if $a + b + c = 0$ or $a = b = c$.

Example 2.37 *Find the values of λ for which the equations*

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$
$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$
$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

are consistent, and find the ratios of x : y : z when λ has the smallest of these values. What happens when λ has the greater of these values. (Kurukshetra 2006, Delhi 2002)

Sol. The given equations will be consistent, if

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0 \qquad \text{(Operate } R_2 - R_1\text{)}$$

or if,
$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ 0 & \lambda - 3 & 3 - \lambda \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0 \qquad \text{(Operate } C_3 + C_2\text{)}$$

or if,
$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 5\lambda + 1 \\ 0 & \lambda - 3 & 0 \\ 2 & 3\lambda + 1 & 6\lambda - 2 \end{vmatrix} = 0 \qquad \text{(Expand by } R_2\text{)}$$

or if,
$$(\lambda - 3)\begin{vmatrix} \lambda - 1 & 5\lambda + 1 \\ 2 & 2(3\lambda + 1) \end{vmatrix} = 0 \text{ or if, } 2(\lambda - 3)[(\lambda - 1)(3\lambda + 1) - (5\lambda + 1)] = 0$$

or if,
$$6\lambda(\lambda - 3)^2 = 0 \text{ or if, } \lambda = 0 \text{ or } 3$$

(a) When $\lambda = 0$, the equations become $-x + y = 0$

$$-x - 2y + 3z = 0$$
$$2x + y - 3z = 0$$

Solving (*ii*) and (*iii*), we get $\dfrac{x}{6 - 3} = \dfrac{y}{6 - 3} = \dfrac{z}{-1 + 4}$. Hence $x = y = z$.

(b) When $\lambda = 3$, equations becomes identical.

Problems 2.7

1. **Investigate** for consistency of the following equations and if possible find the solutions: $4x - 2y + 6z = 8$, $x + y - 3z = -1$, $15x - 3y + 9z = 21$. (Osmania 1999)
2. **For what values** of k the equations $x + y + z = 1$, $2x + y + 4z = k$, $4x + y + 10z = k^2$ have a solution and solve them completely in each case. (V.T.U. 2006)
3. **Investigate** for what values of λ and μ the simultaneous equations
   $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$
   have (*i*) no solution, (*ii*) a unique solution, (*iii*) an infinite number of solutions.
   (U.P.T.U. 2006, P.T.U. 2002, Sambalpur 2002)
4. **Test for consistency and solve**
   (*i*) $2x - 3y + 7z = 5$, $3x + y - 3z = 13$, $2x + 19y - 47z = 32$ (Kurukshetra 2005, Raipur 2005)
   (*ii*) $x + 2y + z = 3$, $2x + 3y + 2z = 5$, $3x - 5y + 5z = 2$, $3x + 9y - z = 4$ (Bhilai 2005, Madras 2003)
   (*iii*) $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$. (Rajasthan 2005)
5. **Find the values** of a and b for which the equations
   $x + ay + z = 3$, $x + 2y + 2z = b$, $x + 5y + 3z = 9$
   are consistent. When will these equations have a unique solution? (Kurukshetra 2005, Madras 2003)
6. **Show that** if $\lambda \neq -5$, the system of equations
   have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

7. Show that the equations

$$3x + 4y + 5z = a,\ 4x + 5y + 6z = b,\ 5x + 6y + 7z = c$$

do not have a solution unless $a + c = 2b$. (*Raipur, 2004 ; Nagpur 200*)

8. Prove that the equations $5x + 3y + 2z = 12$, $2x + 4y + 5z = 2$, $39x + 43y + 45z = c$ are incompatible unless $c = 74$; and in that case the equations are satisfied by $x = 2 + t$, $y = 2 - 3t$, $z = -2 + 2t$, where t is an arbitrary quantity.

9. Find the values of λ for which the equations $(2 - \lambda)x + 2y + 3 = 0$, $2x + (4 - \lambda)y + 7 = 0$, $2x + 5y + (6 - \lambda) = 0$ are consistent and find the values of x and y corresponding to each of these values of λ.

10. Show that there are three real values of λ for which the equations $(a - \lambda)x + by + cz = 0$, $bx + (c - \lambda)y + az = 0$, $cx + ay + (b - \lambda)z = 0$ are simultaneously true and that the product of these values of λ is $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

11. Determine the values of k for which the following system of equations has non-trivial solutions and find them :

$$(k-1)x + (4k-2)y + (k+3)z = 0,\ (k-1)x + (3k+1)y + 2kz = 0,\ 2x + (3k+1)y + 3(k-1)z = 0$$

(*Bombay, 2005*)

12. Show that the system of equations $2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$ can possess a non-trivial solution only if $\lambda = 1$, $\lambda = -3$. Obtain the general solution in each case.

13. Determine the values of λ for which the following set of equations may possess non-trivial solution

$$3x_1 + x_2 - \lambda x_3 = 0,\ 4x_1 - 2x_2 - 3x_3 = 0,\ 2\lambda x_1 + 4x_2 + \lambda x_3 = 0.$$

For each permissible value of λ, determine the general solution (*Kurukshetra, 2006*)

14. Solve completely the system of equations

(*i*) $x + y - 2z + 3w = 0$; $x - 2y + z - w = 0$; $4x + y - 5z + 8w = 0$; $5x - 7y + 2z - w = 0$.

(*ii*) $3x + 4y - z - 6w = 0$; $2x + 3y + 2z - 3w = 0$; $2x + y - 14z - 9w = 0$; $x + 3y + 13z + 3w = 0$

(*J.N.T.U., 2002 S*)

2·12. (1) LINEAR TRANSFORMATIONS

Let (x, y) be the co-ordinates of a point P referred to set of rectangular axes OX, OY. Then its co-ordinates (x', y') referred to OX', OY', obtained by rotating the former axes through an angle θ are given by

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta, \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\} \qquad ...(i)$$

A more general transformation than (*i*) is

$$\left.\begin{aligned} x' &= a_1x + b_1y \\ y' &= a_2x + b_2y \end{aligned}\right\} \qquad ...(ii)$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Such transformations as (*i*) and (*ii*), are called *linear transformations* in two dimensions.

Similarly the relations of the type

$$\left.\begin{aligned} x' &= l_1x + m_1y + n_1z \\ y' &= l_2x + m_2y + n_2z \\ z' &= l_3x + m_3y + n_3z \end{aligned}\right\} \qquad ...(iii)$$

give a *linear transformation* from (x, y, z) to (x', y', z') in three dimensional problems.

In general, the relation $Y = AX$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $A = \begin{bmatrix} a_1 & b_1 & c_1 & \dots & k_1 \\ a_2 & b_2 & c_2 & \dots & k_2 \\ \dots & & & & \dots \\ a_n & b_n & c_n & \dots & k_n \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ ...(iv) give *linear transformation* from n variables $x_1, x_2, \ldots, x_n$ to the variables $y_1, y_2, \ldots, y_n$ i.e. the transformation of the vector X to the vector Y.

This transformation is called linear because the linear relations $A(X_1 + X_2) = AX_1 + AX_2$ and $A(bX) = bAX$, hold for this transformation.

If the transformation matrix A is singular, the transformation also is said to be singular otherwise non-singular. For a non-singular transformation $Y = AX$, we can also write the inverse transformation $X = A^{-1} Y$. A non-singular transformation is also called a *regular* transformation.

Cor. If a transformation from (x_1, x_2, x_3) to (y_1, y_2, y_3) is given by $Y = AX$ and another transformation of (y_1, y_2, y_3) to (z_1, z_2, z_3) is given by $Z = BY$, then the transformation from (x_1, x_2, x_3) to (z_1, z_2, z_3) is given by
$$Z = BY = B(AX) = (BA)X$$

(2) Orthogonal transformation. *The linear transformation (iv), i.e. $Y = AX$, is said to be* ***orthogonal*** *if it transforms*
$$y_1^2 + y_2^2 + \ldots + y_n^2 \text{ into } x_1^2 + x_2^2 + \ldots + x_n^2.$$
The matrix of an orthogonal transformation is called an **orthogonal matrix.**

We have $X'X = [x_1 x_2 \ldots x_n] \times \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \ldots + x_n^2$

and similarly $\quad Y'Y = y_1^2 + y_2^2 + \ldots + y_n^2$

$\therefore$ If $\quad Y = AX$ is an orthogonal transformation, then

$X'X = Y'Y = (AX)'(AX) = X'A'AX$ which is possible only if $A'A = I$

But $A^{-1}A = I$, therefore, $A' = A^{-1}$ for an orthogonal transformation.

Hence a *square matrix A is said to be orthogonal if $AA' = A'A = I$*

Obs. 1. *If A is orthogonal, A' and A^{-1} are also orthogonal.* (Assam, 1999)

Since A is orthogonal, $A' = A^{-1}$

$\therefore \quad (A')' = (A^{-1})' = (A')^{-1} \quad$ i.e. $\quad B' = B^{-1}\quad$ where $B = A'$

Hence B (*i.e.* A') is orthogonal. As $A' = A^{-1}$, A^{-1} is also orthogonal.

Obs. 2. *If A is orthogonal, then $|A| = \pm 1$* (Bombay, 2006)

Since $\quad AA' = A'A = I \quad \therefore \quad |A|\,|A'| = |I|$

But $\quad |A'| = |A|, \quad \therefore \quad |A|\,|A| = 1$

or $\quad |A|^2 = 1 \quad$ i.e. $\quad |A| = \pm 1.$

Example 2.38. *Show that the transformation*
$$y_1 = 2x_1 + x_2 + x_3,\ y_2 = x_1 + x_2 + 2x_3,\ y_3 = x_1 - 2x_3$$
is regular. Write down the inverse transformation.

Sol. The given transformation may be written as
$$Y = AX$$

where $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

Now $\quad |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -1$

Thus the matrix A is non-singular and hence the transformation is regular.

$\therefore$ The inverse transformation is given by

$$X = A^{-1} Y$$

where $$A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Thus $x_1 = 2y_1 - 2y_2 - y_3$; $x_2 = -4y_1 + 5y_2 + 3y_3$; $x_3 = y_1 - y_2 - y_3$ is the inverse transformation.

Example 2·39. *Prove that the following matrix is orthogonal :*

$$\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$ *(Kurukshetra,*

Sol. We have $AA' = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \times \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$

$$= \begin{bmatrix} 4/9 + 1/9 + 4/9 & -4/9 + 2/9 + 2/9 & -2/9 - 2/9 + 4/9 \\ -4/9 + 2/9 + 2/9 & 4/9 + 4/9 + 1/9 & 2/9 - 4/9 + 2/9 \\ -2/9 - 2/9 + 4/9 & 2/9 - 4/9 + 2/9 & 1/9 + 4/9 + 4/9 \end{bmatrix} = I$$

Hence the matrix is orthogonal.

Example 2·40. *If* $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ *is orthogonal, find a, b, c and* A^{-1}. *(Bombay, 2*

Sol. As A is orthogonal, $AA' = I$

$$\therefore \quad \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} 1 + 4 + a^2 & 2 + 2 + ab & 2 - 4 + ac \\ 2 + 2 + ab & 4 + 1 + b^2 & 4 - 2 + bc \\ 2 - 4 + ac & 4 - 2 + bc & 4 + 4 + c^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\therefore \quad 5 + a^2 = 9,\ 5 + b^2 = 9,\ 8 + c^2 = 9 \text{ i.e. } a^2 = 4,\ b^2 = 4,\ c^2 = 1$$

Thus $a = 2$, $b = 2$, $c = 1$.

Since A is orthogonal, $A^{-1} = A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$.

2·13. (1) VECTORS

Any quantity having n-components is called a *vector of order n*. Therefore, the coefficients in a linear equation or the elements in a row or column matrix will form a vector. Thus any numbers $x_1, x_2, \ldots\ldots, x_n$ written in a particular order, constitute a vector **x**.

(2) Linear dependence. *The vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots\ldots, \mathbf{x}_n$ *are said to be* **linearly dependent**, *if there exist r numbers* $\lambda_1, \lambda_2, \ldots\ldots, \lambda_r$ *not all zero, such that*

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_r\mathbf{x}_r = \mathbf{0}.$$

If no such numbers, other than zero, exist, the vectors are said to be **linearly independent**.

If $\lambda_1 \neq 0$, transposing $\lambda_1\mathbf{x}_1$ to the other side and dividing by $-\lambda_1$, we write (i) in the form

$$\mathbf{x}_1 = \mu_2\mathbf{x}_2 + \mu_3\mathbf{x}_3 + \ldots\ldots + \mu_r\mathbf{x}_r.$$

Then the vector $\mathbf{x}_1$ is said to be a linear combination of the vectors $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_r$.

Example 2.41. *Are the vectors* $\mathbf{x}_1 = (1, 3, 4, 2)$, $\mathbf{x}_2 = (3, -5, 2, 2)$ *and* $\mathbf{x}_3 = (2, -1, 3, 2)$ *linearly dependent ? If so express one of these as a linear combination of the others.*

Sol. The relation $\qquad \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3 = \mathbf{0}$.

i.e., $\qquad \lambda_1(1, 3, 4, 2) + \lambda_2(3, -5, 2, 2) + \lambda_3(2, -1, 3, 2) = \mathbf{0}$

is equivalent to $\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$, $3\lambda_1 - 5\lambda_2 - \lambda_3 = 0$,

$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$, $2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$

As these are satisfied by the values $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -2$ which are not zero, the given vectors are linearly dependent. Also we have the relation,

$$\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 = \mathbf{0}$$

by means of which any of the given vectors can be expressed as a linear combination of the others.

Obs. Applying elementary row operations to the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, we see that the matrices

$$A = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 \end{bmatrix}$$

have the same rank. The rank of B being 2, the rank of A is also 2. Moreover $\mathbf{x}_1$, $\mathbf{x}_2$ are linearly independent and $\mathbf{x}_3$ can be expressed as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ $\left[\because\ \mathbf{x}_3 = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)\right]$. Similar results will hold for column operations and for any matrix. In general, we have the following results :

If a given matrix has r linearly independent vectors (rows or columns) and the remaining vectors are linear combinations of these r vectors, then rank of the matrix is r. Conversely, if a matrix is of rank r, it contains r linearly independent vectors and remaining vectors (if any) can be expressed as a linear combination of these vectors.

Problems 2.8

1. Represent each of the transformations
$$x_1 = 3y_1 + 2y_2,\ y_1 = z_1 + 2z_2 \text{ and } x_2 = -y_1 + 4y_2,\ y_2 = 3z_1$$
by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

2. If $\xi = x\cos\alpha - y\sin\alpha$, $\eta = x\sin\alpha + y\cos\alpha$, write the matrix A of transformation and prove that $A^{-1} = A'$. Hence write the inverse transformation.

3. A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $Y = AX$, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $Z = BY$, where
$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$. Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.

4. Find the inverse transformation of $y_1 = x_1 + 2x_2 + 5x_3$; $y_2 = 2x_1 + 4x_2 + 11x_3$; $y_3 = -x_2 + 2x_3$.

5. Verify that the following matrix is orthogonal :

(i) $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ (Hissar, 2005 S ; P.T.U., 2003)

(ii) $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ (Kurukshetra, 2005)

6. Find the values of a, b, c if $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal ? (Bombay, 2005 S)

7. Prove that $\begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix}$ is orthogonal when $l = 2/7$, $m = 3/7$, $n = 6/7$

8. If A and B are orthogonal matrices, prove that AB is also orthogonal. (Ann...

9. Are the following vectors linearly dependent. If so, find the relation between them.

(i) (3, 2, 7), (2, 4, 1), (1, – 2, 6).

(ii) (1, 1, 1, 3), (1, 2, 3, 4), (2, 3, 4, 9).

(iii) $\mathbf{x}_1 = (1, 2, 4)$, $\mathbf{x}_2 = (2, -1, 3)$, $\mathbf{x}_3 = (0, 1, 2)$, $\mathbf{x}_4 = (-3, 7, 2)$. (U.P.T.U., 2003, Nagpu...

2.14. (1) EIGEN VALUES

If A is any square matrix of order n, we can form the matrix $A - \lambda I$, where I is the nth unit matrix. The determinant of this matrix equated to zero, *i.e.*

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the *characteristic equation of A*. On expanding the determinant, the charact... equation takes the form

$$(-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \dots + k_n = 0,$$

where k's are expressible in terms of the elements a_{ij}. The roots of this equation are calle... *eigen-values* or *latent roots* or *characteristic roots* of the matrix A.

(2) Eigen vectors

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ then the linear transformation $Y = AX$

carries the column vector X into the column vector Y by means of the square-matrix ... practice, it is often required to find such vectors which transform into themselves or to a s... multiple of themselves.

Let X be such a vector which transforms into λX by means of the transformation (i).

Then $\lambda X = AX$ or $AX - \lambda IX = 0$ or $[A - \lambda I]X = 0$

This matrix equation represents n homogeneous linear equations

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right\}$$

which will have a non-trivial solution only if the coefficient matrix is singular, ... $|A - \lambda I| = 0$.

This is called the characteristic equation of the transformation and is same as ... characteristic equation of the matrix A. It has n roots and corresponding to each root, ... equation (ii) [or (iii)] will have a non-zero solution

$X = [x_1, x_2, \dots\dots, x_n]'$, which is known as the *eigen vector* or *latent vector*.

Obs. 1. Corresponding to n distinct eigen values, we get n independent eigen vectors. But when tw... more eigen values are equal, it may or may not be possible to get linearly independent ... corresponding to the repeated roots.

Obs. 2. If X_i is a solution for a eigen value λ_i then it follows from (ii) that cX_i is also a solution, where c is an arbitrary constant. Thus *the eigen vector corresponding to a eigen value is not unique* but may be any one of the vectors cX_i

Example 2·42. *Find the eigen values and eigen vectors of the matrix* $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ (Assam, 1999)

Sol. The characteristic equation is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$ or $\lambda^2 - 7\lambda + 6 = 0$

or $(\lambda - 6)(\lambda - 1) = 0$ $\therefore$ $\lambda = 6, 1.$

Thus the eigen values are 6 and 1.

If x, y be the components of an eigen vector corresponding to the eigen value λ, then

$$[A - \lambda I]\, X = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Corresponding to $\lambda = 6$, we have $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $-x + 4y = 0$

$\therefore$ $\dfrac{x}{4} = \dfrac{y}{1}$ giving the eigen vector $(4, 1)$.

Corresponding to $\lambda = 1$, we have $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $x + y = 0$

$\therefore$ $\dfrac{x}{1} = \dfrac{y}{-1}$ giving the eigen vector $(1, -1)$.

Example 2·43. *Find the eigen values and eigen vectors of the matrix* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(Raipur, 2005)

Sol. The characteristic equation is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$ *i.e.* $\lambda^3 - 7\lambda^2 + 36 = 0$

Since $\lambda = -2$ satisfies it, we can write this equation as

$$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0 \quad \text{or} \quad (\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$$

Thus the eigen values of A are $\lambda = -2, 3, 6$.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ, we have

$$[A - \lambda I]X = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \ldots(i)$$

Putting $\lambda = -2$, we have $3x + y + 3z = 0$, $x + 7y + z = 0$, $3x + y + 3z = 0$.

The first and third equations being the same, we have from the first two

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \quad \text{or} \quad \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

Hence the eigen vector is $(-1, 0, 1)$. Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = -2$.

Similarly the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 6$ are the arbitrary non-zero multiples of the vectors $(1, -1, 1)$ and $(1, 2, 1)$ which are obtained from (i).

Hence the three eigen vectors may be taken as $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$.

Example 2·44. *Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

(U.P.T.U.

Sol. The characteristic equation is

$$|A - \lambda I| = 0 \quad i.e. \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$$

or $$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

Thus the eigen values of A are 2, 3, 5.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ, we

$$[A - \lambda I] X = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Putting $\lambda = 2$, we have $x + y + 4z = 0, 6z = 0, 3z = 0$ *i.e.* $x + y = 0$ and $z = 0$.

$$\therefore \qquad \frac{x}{1} = \frac{y}{-1} = \frac{z}{0} = k_1 \text{ (say)}$$

Hence the eigen vector corresponding to $\lambda = 2$ is $k_1 (1, -1, 0)$.

Putting $\lambda = 3$, we have $y + 4z = 0, -y + 6z = 0, 2z = 0$ *i.e.* $y = 0, z = 0$

$$\therefore \qquad \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = k_2$$

Hence the eigen vector corresponding to $\lambda = 3$ is $k_2 (1, 0, 0)$.

Similary the eigen vector corresponding to $\lambda = 5$ is $k_3 (3, 2, 1)$.

2·15. PROPERTIES OF EIGEN VALUES

I. Any square matrix A and its transpose A′ have the same eigen values.

We have
$$(A - \lambda I)' = A' - \lambda I' = A' - \lambda I$$
$$|(A - \lambda I)'| = |A' - \lambda I|$$
$$|A - \lambda I| = |A' - \lambda I|$$

$\therefore$ $|A - \lambda I| = 0$ is and only if $|A' - \lambda I| = 0$ [$\because$ $|B'|$ ...

i.e. λ is an eigen value of A if and only if it is an eigen value of A'.

II. The eigen values of a triangular matrix are just the diagonal elements of the matrix

Let
$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix}$$ be a triangular matrix of order n.

Then $|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda).....(a_{nn} - \lambda)$.

$\therefore$ Roots of $|A - \lambda I| = 0$ are $\lambda = a_{11}, a_{22}, ..., a_{nn}$.

Hence the eigen values of A are the diagonal elements of A *i.e.* $a_{11}, a_{22}, ... a_{nn}$.

Cor. *The eigen values of a diagonal matrix are just the diagonal elements of the matrix*

III. The eigen values of an idempotent matrix are either zero or unity.

Let A be an idempotent matrix so that $A^2 = A$. If λ be an eigen value of A, then there exists a non-zero vector X such that $AX = \lambda X$

$A(AX) = A(\lambda X)$ i.e. $A^2X = \lambda(AX)$

i.e. $AX = \lambda(\lambda X)$

$AX = \lambda^2 X$

From (1) and (2), we get $\lambda^2 X = \lambda X$ or $(\lambda^2 - \lambda)X = 0$

or $\lambda^2 - \lambda = 0$ whence $\lambda = 0$ or 1

Hence the result.

IV. The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

(This property will be proved for a matrix of order 3 but the method will be capable of extension to matrices of any order.)

Consider the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

so that $|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$ (On expanding)

$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \ldots$$

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A, then $|A - \lambda I| = (-1)^3(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$

$$= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) + \ldots$$

Equating the right hand sides of (*ii*) and (*iii*) and comparing coefficients of λ^2, we get

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$. Hence the result.

V. The product of the eigen values of a matrix A is equal to its determinant.

Putting $\lambda = 0$ in (*iii*), we get the result.

VI. If λ is an eigen value of a matrix A, then $1/\lambda$ is the eigen value of A^{-1}.

If X be the eigen vector corresponding to λ, then $AX = \lambda X$

Premultiplying both sides by A^{-1}, we get $A^{-1}AX = A^{-1}\lambda X$

i.e. $IX = \lambda A^{-1}X$ or $X = \lambda(A^{-1}X)$ i.e. $A^{-1}X = (1/\lambda)X$

This being of the same form as (*i*), shows that $1/\lambda$ is an eigen value of the inverse matrix A^{-1}.

VII. If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen value.

We know that if λ is an eigen value of a matrix A, then $1/\lambda$ is an eigen value of A^{-1} [Property VI]. Since A is an orthogonal matrix, A^{-1} is same as its transpose A'.

$\therefore$ $1/\lambda$ is an eigen value of A'.

But the matrices A and A' have the same eigen values, since the determinants $|A - \lambda I|$ and $|A' - \lambda I|$ are the same.

Hence $1/\lambda$ is also an eigen value of A.

VIII. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (m being a positive integer).

Let λ_i be the eigen value of A and X_i the corresponding eigen vector. Then

$$AX_i = \lambda_i X_i$$

We have

$$A^2X_i = A(AX_i) = A(\lambda_i X_i) = \lambda_i(AX_i) = \lambda_i(\lambda_i X_i) = \lambda_i^2 X_i$$

Similarly, $A^3X_i = \lambda_i^3 X_i$. In general, $A^m X_i = \lambda_i^m X_i$ which is of the same form as (i).

Hence λ_i^m is an eigen value of A^m.

The corresponding eigen vector is the same X_i.

2.16. CAYLEY-HAMILTON THEOREM*

Every square matrix satisfies its own characteristic equation ; *i.e.* if the characteristic equation for the nth order square matrix A is

$$| A - \lambda I | = (-1)^n \lambda^n + k_1 \lambda^{n-1} + \ldots\ldots + k_n = 0$$

then

$$(-1)^n A^n + k_1 A^{n-1} + \ldots\ldots + k_n = 0.$$

Let the adjoint of the matrix $A - \lambda I$ be P. Clearly the elements of P will be polynomials of $(n-1)$th degree in λ, for the cofactors of the elements in $| A - \lambda I |$ will be such polynomials.

$\therefore$ P can be split up into a number of matrices, containing terms with the same powers of λ, such that

$$P = P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \ldots\ldots + P_{n-1}\lambda + P_n$$

where $P_1, P_2, \ldots\ldots, P_n$ are all the square matrices of order n whose elements are functions of the elements of A.

Since the product of a matrix by its adjoint = determinant of the matrix × unit matrix.

$\therefore$ $$[A - \lambda I]P = | A - \lambda I | \times I$$

$\therefore$ by (i) and (ii),

$$[A - \lambda I][P_1\lambda^{n-1} + P_2\lambda^{n-2} + \ldots\ldots + P_{n-1}\lambda + P_n]$$
$$= [(-1)^n\lambda^n + k_1\lambda^{n-1} + \ldots\ldots + k_{n-1}\lambda + k_n]I.$$

Equating the coefficients of various powers of λ, we get

$$-P_1 = (-1)^n I \qquad [\because IP_1 = P_1]$$
$$AP_1 - P_2 = k_1 I,$$
$$AP_2 - P_3 = k_2 I,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$AP_{n-1} - P_n = k_{n-1} I,$$
$$AP_n = k_n I.$$

Now pre-multiplying the equations by $A^n, A^{n-1}, \ldots\ldots, A, I$ respectively and adding, we get

$$(-1)^n A^n + k_1 A^{n-1} + \ldots\ldots + k_{n-1}A + k_n I = 0, \qquad (iii)$$

for the terms on the left cancel in pairs. This proves the theorem.

Cor. Another method of finding the inverse.

Multiplying (iii) **by** A^{-1}, **we get**

$$(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots\ldots + k_{n-1} I + k_n A^{-1} = 0$$

whence $$A^{-1} = -\frac{1}{k_n}[(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots\ldots + k_{n-1} I].$$

This result gives the inverse of A in terms of $n-1$ powers of A and is considered as a practical method for the computation of the inverse of large matrices. As a by-product of the computation, the characteristic **equation and the determinant of the matrix are also obtained**

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*See footnote on p. 19. William Rowan Hamilton (1805–1865) an Irish mathematician who is known for his work in dynamics

Example 2·45. *Verify Cayley-Hamilton theorem for the matrix* $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ *and find its inverse. Also express* $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ *as a linear polynomial in A*

Sol. The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 4\lambda - 5 = 0 \qquad \ldots(i)$$

By Cayley-Hamilton theorem, A must satisfy its characteristic equation (*i*), so that

$$A^2 - 4A - 5I = 0 \qquad \ldots(ii)$$

Now $A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

This verifies the theorem.

Multiplying (*ii*) by A^{-1}, we get $A - 4I - 5A^{-1} = 0$

or
$$A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now dividing the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I$ by the polynomial $\lambda^2 - 4\lambda - 5$, we obtain

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 - \lambda - 10I = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$$
$$= \lambda + 5 \qquad [\text{by } (i)$$

Hence $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5$ which is a linear polynomial in A.

Example 2·46. *Find the characteristic equation of the matrix* $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ *and hence find its inverse.*

Sol. The characteristic equation is $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$ *i.e.*, $\lambda^3 - 20\lambda + 8 = 0$.

By Cayley-Hamilton theorem, $A^3 - 20A + 8I = 0$, whence $A^{-1} = \frac{5}{2}I - \frac{1}{8}A^2$,

$$= \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} \qquad [\text{cf. Ex. 2.23}$$

Example 2·47. *Find the characteristic equation of the matrix,* $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ *and hence compute* A^{-1}. *Also find the matrix represented by*

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$ (*Rajasthan, 2005 ; U.P.T.U., 2003*)

Sol. The characteristic equation of the matrix A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \text{ or } \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$$

According to Cayley-Hamilton theorem, we have $A^3 - 5A^2 + 7A - 3I = 0 \qquad \ldots(i)$

Multiplying (i) by A^{-1}, we get

$$A^2 - 5A + 7I - 3A^{-1} = 0 \text{ or } A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

But $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$$A^2 - 5A + 7I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Hence from (ii), $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

$$= A^2 + A + I \qquad [\because A^3 - 5A^2 + 7A - 3I = 0]$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Problems 2.9

1. Find the sum and product of the eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ *(Madras 20..)*

2. Find the product of the eigen values of $\begin{bmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

3. Find the eigen values and eigen vectors of the matrices

(a) $\begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$ *(W.B.T.U. 2005)* (b) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ *(Bhopal 200..)*

4. Find the latent roots and the latent vectors of the matrices

(a) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ *(J.N.T.U. 2006; Bhillai 2005; V.T.U. 200..)*

(b) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ *(J.N.T.U. 2005; Kurukshetra 20..)*

(c) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ *(B.P.T.U. 2006; U.P.T.U. 200..)*

(d) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ *(Kurukshetra 2006)* (e) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ *(Madras 200..)*

5. If λ be an eigen value of a non-singular matrix A, show that $|A|/\lambda$ is an eigen value of the matrix adj A. *(U.P.T.U. 20..)*

6. Two eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are = 1 each. Find the eigen values of A^{-1}.

7. Show that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the latent roots of a matrix A, then A^2 has the latent roots $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$. *(P.T.U., 2005)*

8. For a symmetrical square matrix, show that the eigen vectors corresponding to two unequal eigen values are orthogonal.

9. Using Cayley-Hamilton theorem, find the inverse of

(i) $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ *(Osmania, 2000 S)*

(iii) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$ *(Bhopal, 2002 S)* (iv) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *(U.P.T.U., 2006)*

10. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ Show that the equation is satisfied by A and hence obtain the inverse of the given matrix. *(Anna, 2005 ; Kerala, 2005)*

11. Verify Cayley-Hamilton theorem for the matrix A and find its inverse.

(i) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ *(Madras, 2006 ; U.P.T.U., 2005)* (ii) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ *(Coimbatore, 2001)*

(iii) $\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$ *(P.T.U., 2006)*

12. Using Cayley-Hamilton theorem, find A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. *(Anna, 2003)*

13. If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$, evaluate A^{-1}, A^{-2} and A^{-3}.

14. If $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find A^4. *(Madras, 2006)*

2.17. (1) REDUCTION TO DIAGONAL FORM

If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

[This result will be proved for a square matrix of order 3 but the method will be capable of easy extension to matrices of any order.]

Let A be a square matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values and

$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding eigen vectors.

Denoting the square matrix $[X_1 X_2 X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ by P, we have

$$AP = A[X_1 X_2 X_3] = [AX_1, AX_2, AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = PD,$$ where D is the diagonal matrix

$\therefore$ $P^{-1}AP = P^{-1}PD = D$, which proves the theorem.

Obs. 1. *The matrix P which diagonalises A is called the* ***modal matrix*** *of A and the resulting diagonal matrix D is known as the* ***spectral matrix*** *of A.*

2. *The diagonal matrix has the eigen values of A as its diagonal elements.*

3. *The matrix P, which diagonalise A, consitutes the eigen vectors of A.*

(2) Similarity of matrices. *A square matrix $\hat{A}$ of order n is called* **similar** *to a square matrix A of order n if*

$$\hat{A} = P^{-1}AP \text{ for some non-singular } n \times n \text{ matrix } P.$$

This transformation of a matrix A by a non-singular matrix P to $\hat{A}$ is called a **similarity transformation.**

Obs. *If the matrix $\hat{A}$ is similar to the matrix A, then $\hat{A}$ has the same eigen values as A.*

If $\mathbf{x}$ *is an eigen vector of A, then $y = P^{-1}\mathbf{x}$ is an eigen vector of $\hat{A}$ corresponding to the same eigen value.*

(3) Powers of a matrix. *Diagonalisation of a matrix is quite useful for obtaining powers of a matrix.*

Let A be the square matrix. Then a non-singular matrix P can be found such that

$$D = P^{-1}AP \qquad \ldots(i)$$

$\therefore$ $$D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P \qquad [\because PP^{-1} = I]$$

Similarly $$D^3 = P^{-1}A^3P \text{ and in general, } D^n = P^{-1}A^nP$$

To obtain A^n, premultiply (i) by P and post-multiply by P^{-1}.

Then $$PD^nP^{-1} = PP^{-1}A^nPP^{-1} = A^n \quad \text{which gives } A^n.$$

Thus, $$A^n = PD^nP^{-1} \text{ where, } D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

Working procedure :

1. Find the eigen values of the square matrix A.
2. Find the corresponding eigen vectors and write the modal matrix P.
3. Find the diagonal matrix D from $D = P^{-1}AP$
4. Obtain A^n from $A^n = PD^nP^{-1}$.

Example 2·48. *Reduce the matrix* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ *to the diagonal form.*

(*U.P.T.U., 2006 ; Raipur, 2004*)

Sol. The characteristic equation of A is

$$\begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} = 0 \text{ or } \lambda^3 - \lambda^2 - 5\lambda + 5 = 0.$$

Solving, we get $\lambda_1 = 1$, $\lambda_2 = \sqrt{5}$, $\lambda_3 = -\sqrt{5}$ as the eigen values of A.

When $\lambda = 1$, the corresponding eigen vector is given by

$$-2x + 2y - 2z = 0, \; x + y + z = 0, \; -x - y - z = 0$$

Solving the first two equations, we get $\frac{x}{2} = \frac{y}{0} = \frac{z}{-2}$ giving the eigen vector $(1, 0, -1)$

When $\lambda = \sqrt{5}$, the corresponding eigen vector is given by

$$(-1-\sqrt{5})x + 2y - 2z = 0,\ x + (2-\sqrt{5})y + z = 0,\ x - y - (1+\sqrt{5})z = 0$$

Solving 2nd and 3rd equations, we get

$$\frac{x}{6-2\sqrt{5}} = \frac{y}{-1+\sqrt{5}} = \frac{z}{1-\sqrt{5}} \quad \text{or} \quad \frac{x}{\sqrt{5}-1} = \frac{y}{1} = \frac{z}{-1}$$

giving the eigen vector $(\sqrt{5}-1, 1, -1)$.

Similarly the eigen vector corresponding to $\lambda = -\sqrt{5}$ is $(\sqrt{5}+1, -1, 1)$.

Writing the three eigen vectors as the three columns, we get the transformation [illegible] matrix as $P = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Hence the diagonal matrix is

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

Example 2.49. *Find a matrix P which transforms the matrix* $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ *to the diagonal form. Hence calculate* A^4.

Sol. The eigen values of A (found in Ex. 2.36) are $-2, 3, 6$ and the eigen vectors are $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$. Writing these eigen vectors as the three columns, the required transformation matrix (*modal matrix*) is

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

To find P^{-1}. $\quad |P| = \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (say)

$$A_1 = -3,\ B_1 = 2,\ C_1 = 1,\ A_2 = 0,\ B_2 = -2,\ C_2 = 2,\ A_3 = 3,\ B_3 = 2,\ C_3 = 1$$

Also $\quad |P| = a_1A_1 + b_1B_1 + c_1C_1 = 6$

$$\therefore \quad P^{-1} = \frac{1}{|P|}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Thus

$$D = P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\therefore \quad D^4 = \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 6^4 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}$$

Hence $A^4 = PD^4P^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -8 & 0 & 8 \\ 27 & -27 & 27 \\ 216 & 512 & 216 \end{bmatrix} = \begin{bmatrix} 251 & 485 & 235 \\ 485 & 1051 & 485 \\ 235 & 485 & 251 \end{bmatrix}$$

2.18. REDUCTION OF QUADRATIC FORM TO CANONICAL FORM

A homogeneous expression of the **second degree** in any number of variables is called *quadratic form*.

For instance, if $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X' = [x \quad y \quad z]$, then

$$X'AX = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

which is a *quadratic form*.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of the matrix A and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

be its corresponding eigen vectors in the normalized form (*i.e.* each element is divided by square root of sum of the squares of all the three elements in the eigen vector).

Then by § 2·17(1), $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ where $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Hence the *quadratic form (i) is reduced to a* **canonical form (or sum of squares form or Principal axes form.)**

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$$

and P is the **matrix of transformation** which is an orthogonal matrix.

Note. Congruent (or orthogonal) transformation. The diagonal matrix D and the matrix A are called *congruent matrices* and *the above method of reduction is called* **congruent** (or **orthogonal**) **transformation**

Remember that the matrix A corresponding to the quadratic form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

is $\begin{bmatrix} \text{coeff. of } x^2 & \frac{1}{2}\text{coeff. of } yz & \frac{1}{2}\text{coeff. of } zx \\ \frac{1}{2}\text{coeff. of } yz & \text{coeff. of } y^2 & \frac{1}{2}\text{coeff. of } xy \\ \frac{1}{2}\text{coeff. of } zx & \frac{1}{2}\text{coeff. of } xy & \text{coeff. of } z^2 \end{bmatrix}$ *i.e.* $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Example 2·50. *Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form and specify the matrix of transformation.* (*Kurukshetra, 2006 ; Madras, 20..*)

Sol. The matrix of the given quadratic form is $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Its characteristic equation is $|A - \lambda I| = 0$ *i.e.* $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

which gives $\lambda = 2, 3, 6$ **as its eigen values. Hence** the given quadratic form reduces to the **canonical form**

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$$

To find the matrix of transformation

From $[A - \lambda I]X = 0$, we obtain the equations

$$(3-\lambda)x - y + z = 0,\ -x + (5-\lambda)y - z = 0;\ x - y + (3-\lambda)z = 0$$

Now corresponding to $\lambda = 2$, we get $x - y + z = 0$, $-x + 3y - z = 0$, and $x - y + z = 0$

whence
$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

∴ The eigen vector is $X_1(1, 0, -1)$ and its normalised form is $(1/\sqrt{2}, 0, -1/\sqrt{2})$

Similarly corresponding to $\lambda = 3$, the eigen vector is $X_2(1, 1, 1)$ and its normalised form is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Finally, corresponding to $\lambda = 6$, the eigen vector is $X_3(1, -2, 1)$ and its normalised form is $(1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6})$.

Hence the matrix of transformation is $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

2.19. NATURE OF A QUADRATIC FORM

Let $Q = X'AX$ be a quadratic form in n variables $x_1, x_2, \ldots x_n$

Index. *The number of positive terms in its canonical form is called the index of the quadratic form.*

Signature (S) *of the quadratic form is the difference of positive and negative terms in the canonical form. If the rank of the matrix A is r and the signature of the quadratic form Q is S then the quadratic form is said to be*

(*i*) *positive definite* if $r = n$ and $s = n$

(*ii*) *negative definite* if $r = n$ and $s = 0$

(*iii*) *positive semidefinite* if $r < n$ and and $s = r$

(*iv*) *negative semidefinite* if $r < n$ and $s = 0$

(*v*) *indefinite* in all other cases.

In other words *a real quadratic form X'AX in n variables is said to be*

(*i*) **positive definite** *if all the eigen values of A > 0.*

(*ii*) **negative definite** *if all the eigen values of A < 0.*

(*iii*) **positive semidefinite** *if all the eigen values of A ≥ 0 and at least one eigen value = 0*

(*iv*) **negative semidefinite** *if all the eigen values of A ≤ 0 and at least one eigen value = 0*

(*v*) **indefinite** *if some of the eigen values of A are positive and others negative.*

Example 2.51. *Reduce the quadratic form* $2x_1x_2 + 2x_1x_3 - 2x_2x_3$ *to a canonical form by an orthogonal reduction and discuss its nature.* (*Madras 2006*)

Also find the modal matrix

Sol. (*i*) The matrix of the given quadratic form is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

Its characteristic equation is $|A - \lambda I| = 0$ i.e. $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = 0$

which gives $\lambda^3 - 3\lambda + 2 = 0$

Solving, we get $\lambda = 1, 1, -2$ as the eigen values. Hence the given quadratic form reduces to the canonical form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0 \quad i.e. \quad x^2 + y^2 - 2z^2 = 0$$

(ii) Since some of the eigen values of A are positive and others are negative, the given quadratic form is *indefinite*.

(iii) *To find the matrix of transformation*

From $[A - \lambda I]\, X = 0$, we get the equations

$$-\lambda x + y + z = 0, \quad x - \lambda y + z = 0, \quad x - y - \lambda z = 0$$

When $\lambda = -2$, we get $2x + y + z = 0, x + 2y - z = 0, x - y + 2z = 0$.

Solving first and second equations, we get

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$$

$\therefore$ The corresponding eigen vector $X_1 = (-1, 1, 1)$ and its normalised form is $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

When $\lambda = 1$, we get $-x + y + z = 0, x - y - z = 0, x - y - z = 0$.

These equations are same. Take $y = 0$ so that $x = z$.

$\therefore$ The corresponding eigen vector $X_2 = (1, 0, 1)$ and its normalised form is $(1/\sqrt{2}, 0, 1/\sqrt{2})$

To find the eigen vector $X_3 = (l, m, n)$ (say)

Since X_3 is orthogonal to X_1, $\therefore -l + m + n = 0$

Since X_3 is orthogonal to X_2, $\therefore l + n = 0$

These equations give $\dfrac{l}{1} = \dfrac{m}{2} = \dfrac{n}{-1}$.

$\therefore$ The eigen vector $X_3 = (1, 2, -1)$ and normalised form is $(1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6})$.

Hence the modal matrix is

$$P = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

Problems 2.10

1. If $A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $P^{-1}AP$ is a diagonal matrix

2. Show that the linear transformation

$H = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where $\theta = \dfrac{1}{2}\tan^{-1}\dfrac{2h}{a - b}$, changes the matrix

$C = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ to the diagonal form $D = HCH'$.

3. Reduce the matrix $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$ to the diagonal form. (B.P.T.U. 2005)

4. By diagonalising the matrix $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$, find A^4 (Madras 2006)

5. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, calculate A^4. *(Coimbatore, 2001)*

6. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ find A^*

7. Find the index and signature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$. *(Madras, 2006)*

8. Find the eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and hence reduce

$6x^2 + 3y^2 + 3z^2 - 2yz + 4zx - 4xy$ to a 'sum of squares'. Also write the nature of the matrix *(Calicut, 2005)*

9. Reduce the quadratic form $2xy + 2yz + 2zx$ into canonical form.
(Kurukshetra, 2006, Bombay, 2003, Madras, 2002)

10. Find the eigen values, eigen vectors and the modal matrix of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ and hence reduce the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to a canonical form. *(Andhra, 2000)*

11. Reduce the following quadratic forms into a 'sum of squares' by an orthogonal transformation and give the matrix of transformation. Also state the nature of each of these.
(*i*) $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$.
(*ii*) $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4zx$ *(Anna, 2002 S)*

12. Show that the form $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x^1 + 6x_1x_2$ is a positive semi-definite and find a non-zero set of values of x_1, x_2, x_3 which make the form zero. *(P.T.U., 2003)*

2·20. COMPLEX MATRICES

So far, we have considered matrices whose elements were real numbers. The elements of a matrix can, however, be complex numbers also.

(1) Conjugate of a matrix. *If the elements of a matrix* $A = [a_{rs}]$ *are complex numbers* $\alpha_{rs} + i\,\beta_{rs}$, α_{rs} and β_{rs} *being real, then the matrix*

$\bar{A} = [\bar{a}_{rs}] = [\alpha_{rs} - i\beta_{rs}]$ *is called the* **conjugate matrix** *of A.*

The transpose of a conjugate of a matrix A is denoted by A^* or A^θ *i.e.* $(\bar{A})' = A^*$

(2) Hermitian matrix. *A square matrix A such that* $A' = \bar{A}$ *is said to be a* **Hermitian matrix.*** The elements of the leading diagonal of a Hermitian matrix are evidently real, while every other element is the complex conjugate of the element in the transposed position. For instance $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & -5 \end{bmatrix}$ is a Hermitian matrix, since $A' = \begin{bmatrix} 2 & 3-4i \\ 3+4i & -5 \end{bmatrix} = \bar{A}$

(3) Skew-Hermitian matrix. *A square matrix A such that* $A' = -\bar{A}$ *is said to be a* **skew-Hermitian matrix.** This implies that the leading diagonal elements of a skew-Hermitian matrix are either all zeros or all purely imaginary.

Obs. *A Hermitian matrix is a generalisation of a real symmetric matrix* as every real symmetric matrix is Hermitian. Similarly, *a skew-Hermitian matrix is a generalisation of a real skew-symmetric matrix*

Properties.

I. Any square matrix A can be written as the sum of a Hermitian and skew-Hermitian matrices.

Take $B = \frac{1}{2}(A + \bar{A}')$ and $C = \frac{1}{2}(A - \bar{A}')$

* Named after the French mathematician *Charles Hermite* (1822–1901), known for his contributions to algebra and number theory

Then $$B' = \frac{1}{2}(A + \bar{A}')' = \frac{1}{2}(A' + \bar{A})$$

and $$\bar{B} = \frac{1}{2}\overline{(A + \bar{A}')} = \frac{1}{2}(\bar{A} + A') = B'$$

i.e. B is a Hermitian matrix.

Again $$C' = \frac{1}{2}(A - \bar{A}')' = \frac{1}{2}(A' - \bar{A})$$

and $$\bar{C} = \frac{1}{2}\overline{(A - \bar{A}')} = \frac{1}{2}(\bar{A} - A') = -C'$$

$\therefore$ $C' = -\bar{C}$ *i.e.* C is a skew-Hermitian matrix.

Thus $$A = \frac{1}{2}(A + \bar{A}') + \frac{1}{2}(A - \bar{A}') = B + C$$

Hence the result.

II. If A is a Hermitian matrix, then (iA) is a skew-Hermitian matrix.

We have $$(\overline{iA})' = (\bar{i}\,\bar{A})' = (-i\bar{A})' = -i\bar{A}'$$
$$= -iA \qquad [\because \bar{A}' = A$$

Thus (iA) is a skew-Hermitian matrix.

Similarly *if A is a skew-Hermitian matrix then (iA) is a Hermitian matrix.*

III. The eigen values of a Hermitian matrix are real. (see Fig. 2.1)

Let λ be the eigen value and X the corresponding eigen vector of a Hermitian matrix A, so that

$$AX = \lambda X$$
$$\therefore \quad \bar{X}' AX = \bar{X}' \lambda X = \lambda \bar{X}'X$$
or $$\lambda = \bar{X}' AX / \bar{X}'X$$

Since $\bar{X}'X = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \ldots + \bar{x}_n x_n = |x_1|^2 + |x_2|^2 + \ldots + |x_n|^2$ is real and non-zero. Also $\bar{X}'AX$ is a Hermitian form which is always real.

$\therefore$ λ, the eigen value of a Hermitian matrix is real.

IV. The eigen values of a skew-Hermitian matrix are purely imaginary or zero.

Let λ be the eigen value and X the corresponding eigen vector of a skew-Hermitian matrix B so that $BX = \lambda X$.

$$\therefore \quad \bar{X}' BX = \bar{X}'\lambda X = \lambda \bar{X}'X \qquad \text{or} \qquad \lambda = \bar{X}' BX / \bar{X}' X$$

Since $\bar{X}'X$ is real and non-zero. Also $\bar{X}BX$ is a skew-Hermitian form which is purely imaginary or zero

$\therefore$ λ, the eigen value of a skew-Hermitian matrix is purely imaginary or zero.

(4) Unitary matrix. A square matrix U such that $\bar{U}' = U^{-1}$ is called a **unitary matrix.** For a unitary matrix U, $U \cdot U^* = U^* \cdot U = I$.

This is a generalisation of the orthogonal matrix in the complex field.

Properties

I. Inverse of a unitary matrix is unitary

If U is a unitary matrix, then

$$\bar{U}' = \overline{U'} = U^{-1}$$
or $$U' = \overline{U^{-1}}$$
$$\therefore \quad [(U^{-1})^{-1}]' = \overline{U^{-1}}$$

Writing $U^{-1} = V$, we have
$$[V^{-1}]' = \bar{V} \quad \text{or} \quad V^{-1} = \bar{V}'$$

Thus $V (= U^{-1})$ is also unitary

Cor. *Inverse of an orthogonal matrix is orthogonal.*

II. Transpose of a unitary matrix is unitary

If U is a unitary matrix, $\bar{U}' = U^{-1}$

or

or $$[(\overline{U'})]' = [U^{-1}]' = [U']^{-1}$$

Writing $U' = V$, we have $\overline{V}' = V^{-1}$

Thus V (*i.e.* U') is also unitary.

Cor. *Transpose of an orthogonal matrix is orthogonal.*

III. Product of two unitary matrices is a unitary matrix.

If U and V are unitary matrices then

$$U' = \overline{U}^{-1}, \quad V' = \overline{V}^{-1}$$

Now,
$$(\overline{UV})^{-1} = (\overline{U}\,\overline{V})^{-1} = \overline{V}^{-1}\,\overline{U}^{-1}$$
$$= V'U' \qquad [\because U, V \text{ are unitary.}$$
$$= (UV)'$$

Thus UV is a unitary matrix.

Cor. *Product of two orthogonal matrixes is an orthogonal matrix.*

IV. The eigen value of a unitary matrix has absolute value 1.

If U is a unitary matrix then $$UX = \lambda X \qquad ...(1)$$

Taking conjugate transpose of (1),

$$(\overline{UX})' = (\overline{U}\,\overline{X})' = \overline{X}'\,\overline{U}' = \overline{X}'\,\overline{U}^{-1}$$

Also $$(\overline{UX})' = (\bar{\lambda}\,\overline{X})' = \bar{\lambda}\,\overline{X}'$$

i.e. $$\overline{X}'\,U^{-1} = \bar{\lambda}\,\overline{X}' \qquad ...(2)$$

Post-multiplying (2) by (1), we get

$$(\overline{X}'\,U^{-1})\,(UX) = (\bar{\lambda}\,\overline{X}')\,(\lambda X)$$
$$\overline{X}'\,(U^{-1}\,U)\,X = (\bar{\lambda}\lambda)\,(\overline{X}'\,X)$$
$$\overline{X}'\,X = (\lambda\lambda')\,\overline{X}'\,X \qquad [\because U^{-1}\,U = I$$

Thus $$\lambda\lambda' = |\lambda|^2 = 1. \qquad [\because \overline{X}X \neq 0$$

Hence the result.

Cor. *The eigen value of an orthogonal matrix has absolute value 1.*

Example 2·52. *If* $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, *show that* AA^* *is a Hermitian matrix, where* A^* *is the conjugate transpose of A.* (*J.N.T.U., 2005 ; U.P.T.U., 2003*)

Sol. We have $A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$

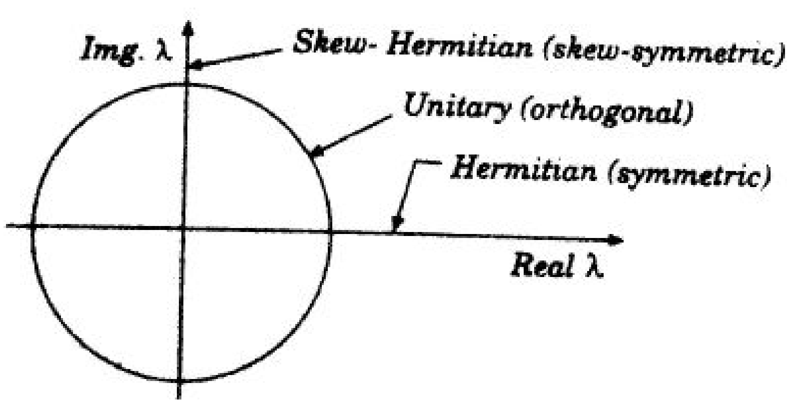


Fig. 2.1. Eigen values of various matrices.

and
$$A^* = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$\therefore$
$$AA^* = \begin{bmatrix} 2-i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}\begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$= \begin{bmatrix} 4-i^2+9+1-9i^2, & -10-5i-3i-10+10i \\ -10+5i+3i-10-10i, & 25-i^2+16-4i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -20+2i \\ -20-2i & 46 \end{bmatrix}, \text{ which is a Hermitian matrix.}$$

Example 2·53. *Prove that the matrix* $A = \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$ *is unitary and find* A^{-1}

(*Bombay, 2006*)

Sol. Conjugate of A *i.e.* $\overline{A} = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(-1-i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(1+i) \end{bmatrix}$

$\therefore$ Transpose of $\overline{A}$ *i.e.* $A^\theta = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}$

Now $A^\theta \cdot A = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}\begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{4}(1+1)+\frac{1}{4}(1+1) & -\frac{1}{4}(1-i)^2+\frac{1}{4}(1-i)^2 \\ -\frac{1}{4}(1+i)^2+\frac{1}{4}(1+i)^2 & \frac{1}{4}(1+1)+\frac{1}{4}(1+1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly $AA^\theta = I$.

Hence A is a unitary matrix.

Also $\quad A^{-1} = A^\theta = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{bmatrix}$.

Example 2·54. *Given that* $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, *show that* $(I-A)(I+A)^{-1}$ *is a unitary matrix.*

Sol. $\quad I+A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$, $|I+A| = 1-(-1-4) = 6.$

$$(I+A)^{-1} = \begin{bmatrix} 1 & 1-2i \\ 1-2i & 1 \end{bmatrix} \div 6 \quad \text{Also } I-A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\therefore \quad (I-A)(I+A)^{-1} = \begin{bmatrix} 1 & 1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6 = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \quad ...(i)$$

Its conjugate-transpose $= \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$ ...(ii)

$\therefore$ Product of (i) and (ii) $= \frac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I$

Hence the result.

Problems 2.11

1. Prove that every Hermitian matrix can be written as $A + iB$, where A is real and symmetric and B is real and skew-symmetric.
2. Show that every square matrix can be expressed as $P + iQ$, where P and Q are Hermitian matrices. (P.T.U. 1996)
3. Show that a Hermitian matrix remains Hermitian when transformed by an orthogonal matrix. (Bhopal 2002 S)
4. Show that the matrix $\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is a unitary matrix, if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$. (U.P.T.U. 2006)
5. Show that $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ is a Hermitian matrix.
6. If $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$, show that A is a Hermitian matrix and iA is a skew-Hermitian matrix. (Sambalpur 2002)
7. Show that the following matrix is unitary $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$. (U.P.T.U. 2002)
8. Express $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ as $P + iQ$ where P is real and skew symmetric and Q is real and symmetric. (Bombay 2006)
9. If $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$, where $a = e^{2i\pi/3}$, prove that $S^{-1} = \frac{1}{3}\bar{S}$. (Kurukshetra, 2006 ; J.N.T.U. 2001)

2.21. OBJECTIVE TYPE OF QUESTIONS

Problems 2.12

Choose the correct answer or fill up the blanks in the following problems

1. To multiply a matrix by scalar k, multiply
   (a) any row by k (b) every element by k (c) any column by k
2. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then A^n is
   (a) $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ (b) $\begin{bmatrix} 3^n & (-4)^n \\ 1 & (-1)^n \end{bmatrix}$ (c) $\begin{bmatrix} 1+3n & 1-4n \\ 1+n & 1-n \end{bmatrix}$ (d) $\begin{bmatrix} 1+2n & 4n \\ 1+n & 1-2n \end{bmatrix}$
3. The inverse of the matrix $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ then the determinant AB has the value

(a) 4 (b) 8 (c) 16 (d) 32

5. The system of equations $x + 2y + z = 9$, $2x + y + 3z = 7$ can be expressed as

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$ (d) none of the above

6. If $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, then X equals

(a) $\begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -14 \\ 4 & -17 \end{bmatrix}$

7. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ then A (adj A) equals

(a) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$ (d) none of the above

8. If $3x + 2y + z = 0$, $x + 4y + z = 0$, $2x + y + 4z = 0$, be a system of equations, then

(a) it is inconsistent

(b) it has only the trivial solution $x = 0, y = 0, z = 0$

(c) it can be reduced to a single equation and so a solution does not exist

(d) determinant of the matrix of coefficients is zero.

9. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then

(a) $C = A\cos\theta - B\sin\theta$ (b) $C = A\sin\theta + B\cos\theta$

(c) $C = A\sin\theta - B\cos\theta$ (d) $C = A\cos\theta + B\sin\theta$.

10. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$, then

(a) A is row equivalent to B only when $\alpha = 2$, $\beta = 3$, and $\gamma = 4$

(b) A is row equivalent to B only when $\alpha \neq 0$, $\beta \neq 0$ and $\gamma = 0$

(c) A is not row equivalent to B

(d) A is row equivalent to B for all value of α, β, γ.

11. If $A \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then A is

(a) $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 \\ -1/2 & -1/2 \end{bmatrix}$

12. Matrix has a value. This statement

(a) is always true (b) depends upon the matrices (c) is false

13. If A is a square matrix such that $AA' = I$, then value of $A'A$ is

(a) A^2 (b) I (c) A^{-1}

14. If every minor of order r of a matrix A is zero, then rank of A is
(a) greater than r (b) equal to r
(c) less than or equal to r (d) less than r

15. A square matrix A is called orthogonal if
(a) $A = A^2$ (b) $A' = A^{-1}$ (c) $AA^{-1} = I$

16. The rank of matrix $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$ is

17. The sum of the eigen values of a matrix is the ... of the elements of the principal diagonal

18. Rank of a unit (identity) matrix of order 4 is ...

19. Inverse of $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & k \\ 2 & 2 & 5 \end{bmatrix}$ then k is

20. The equations $x + 2y = 1$, $7x + 14y = 12$ are consistent (True or False)

21. If two eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are 3 and 15, then the third eigen value is

22. A quadratic form is positive semi-definite when

23. $A_{m \times n}$ and $B_{p \times q}$ are two matrices. When will
(a) $A \cdot B$ exist (b) $A + B$ exist ?

24. The product of the eigen values of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ is

25. The rank of the matrix $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \end{bmatrix}$ is

26. An example of a 3×3 matrix of rank one is

27. The quadratic form corresponding to the symmetric matrix $\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix}$ is

28. Solving the equations $x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 12z = 0$ $x = \ldots$, $y = \ldots$, $z = \ldots\ldots\ldots$

29. The eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are

30. A matrix A is *idempotent* if ...

31. The rank of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ is ...

32. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, then the eigen values of A^2 are ...

33. If rank $(A) = 2$, rank $(B) = 3$, then rank $(AB) = 6$ (True or False)

34. The maximum value of the rank of a 4×5 matrix is ...

35. Any set of vectors which includes the zero vector is linearly independent (True or False)

36. If λ is an eigen value of a symmetric matrix, then λ is real (True or False)

37. The eigen values of matrix $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ are ...

38. The eigen values of a triangular matrix are

39. If the product of two eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16, then the third eigen value is .....
40. If λ_i, $i = 1, 2, \ldots\ldots n$ are the eigen values of a square matrix A, then the eigen values of A^T are .....
41. By applying elementary transformations to a matrix, its rank
    (a) increases (b) decreases (c) does not change.
42. If λ is an eigen value of A, then it is an eigen value of B, only if $B =$ ......
43. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$, then eigen values of A^{-1} are ......
44. The characteristic equation of $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ is ....
45. Every square matrix does not satisfy its own characteristic equation. (True or False)
46. If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen value. (True or False)
47. Every Hermitian matrix can be written as $A + iB$, where A is real and ...... and B is real and .....
48. The sum and product of the eigen values of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are .... and ....
49. If the rank of a matrix A is 3, then the rank of $3A^T$ is 1. (True or False)
50. The product of the eigen values of a matrix is equal to ......
51. The eigen values of $A = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$ are the roots of the equation ......
52. A system of linear non-homogeneous equations is consistent, if and only if the rank of coefficient matrix is equal to rank of ......
53. The vectors [1, 1, – 1, 1], [1, – 1, 2, – 1], [3, 1, 0, 1] are linearly dependent. (True or False)
54. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of a matrix A, then A^3 has the eigen values ......
55. If λ is an eigen value of a non-singular matrix A, then the eigen value of A^{-1} is ....
56. The matrix corresponding to the quadratic form $x^2 + 2y^2 - 7z^2 - 4xy + 8xz + 5yz$ is ....
57. Cayley Hamilton theorem states that ....
58. If the rank of a matrix A is 2, then the rank of A' is ....
59. The eigen values of a skew-symmetric matrix are real. (True or False)
60. Inverse of a unitary matrix is a unitary matrix. (True or False)
61. A is a non-zero column matrix and B is a non-zero row matrix, then rank of AB is one. (True or False)
62. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then eigen values of A^{-1} are ....
63. Matrix $\begin{bmatrix} x & 2 \\ 1 & x-1 \end{bmatrix}$ is singular for $x =$ ....
64. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^3 =$ ....
65. If the product of two eigen values of $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ is -4, then the third eigen value is ....
66. The sum of the squares of the eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ is ....
67. The index and signature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$ are respectively ...... and ......
68. The matrix of the quadratic form ... $2y^2 + z^2 - 2xy + 6zx$ is ......

3

Vector Algebra & Solid Geometry

VECTOR ALGEBRA

3.1. (1) VECTORS

A quantity which is completely specified by its magnitude only is called *a scalar*. Length, time, mass, volume, temperature, work, electric charge and numerical data in Statistics are all examples of scalar quantities.

A quantity which is completely specified by its magnitude and direction is called a **vector**. Weight, displacement, velocity, acceleration and electric current density are all vector quantities for each involves magnitude and direction.

A vector is represented by a directed line segment. Thus $\vec{PQ}$ represents a vector whose magnitude is the length PQ and direction is from P (starting point) to Q (end point). We denote a vector by a single letter in capital bold type (or with an arrow on it) and its magnitude (length) by the corresponding small letter in italics type. Thus if **V** is the velocity vector, its magnitude is v, the speed.

A vector of unit magnitude is called a *unit vector*. The idea of unit vectors is often used to represent concisely the direction of any vector. Unit vector corresponding to the vector **A** is written as $\hat{\mathbf{A}}$.

Fig. 3.1

A vector of zero magnitude (which can have no direction associated with it) is called a *zero (or null) vector* and is denoted by **0**—a thick zero.

The vector $\vec{QP}$ represents the negative of $\vec{PQ}$ *i.e.* $-\mathbf{A}$.

Two vectors* A *and* B *having the same magnitude and the same (or parallel) directions are said to be equal and we write $\mathbf{A} = \mathbf{B}$. Clearly the vectors $\vec{AB}$, $\vec{LM}$ and $\vec{PQ}$ are all equal (Fig. 3.1).

(2) Addition of vectors. Vectors are added according to the *triangle law of addition*, which is a matter of common knowledge. Let **A** and **B** be represented by two vectors $\vec{OP}$ and $\vec{PQ}$ respectively then $\vec{OQ} = \mathbf{C}$ is called the sum or resultant of **A** and **B**. Symbolically, we write,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

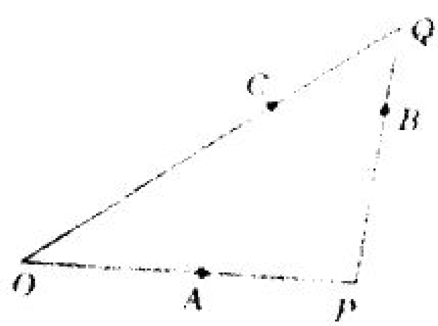


Fig. 3.2

(3) Subtraction of vectors. *The subtraction of a vector* **B** *from* **A** *is taken to be the addition of* –**B** *to* **A** *and we write*

$$\mathbf{A} + (-\mathbf{B}) = \mathbf{A} - \mathbf{B}$$

(4) Multiplication of vectors by scalars.

We have just seen that $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$

and $-\mathbf{A} + (-\mathbf{A}) = -2\,\mathbf{A}$

where both 2**A** and – 2**A** denote vectors of magnitude twice that of **A** ; the former having the same direction as **A** and the latter the opposite direction.

In general, *the product m***A** *of a vector* **A** *and a scalar m is a vector whose magnitude is m times that of* **A** *and direction is the same or opposite to* **A** *according as m is positive or negative.*

Thus $\mathbf{A} = a\,\hat{\mathbf{A}}$.

Example 3 1. *If* **A** *and* **B** *are the vectors determined by two adjacent sides of a regular hexagon. What are the vectors represented by the other sides taken in order ?*

Sol. Let *ABCDEF* be the given hexagon, such that

$$\overrightarrow{AB} = \mathbf{A} \text{ and } \overrightarrow{BC} = \mathbf{B}$$

$\therefore \quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{A} + \mathbf{B}$

Also $\quad \overrightarrow{AD} = 2\overrightarrow{BC} = 2\mathbf{B}$

$\therefore \quad \overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC} = 2\mathbf{B} - (\mathbf{A} + \mathbf{B}) = \mathbf{B} - \mathbf{A}$

Now $\quad \overrightarrow{DE} = -\overrightarrow{AB} = -\mathbf{A}$ $\quad [\because AB = \text{and} \parallel ED]$

$\overrightarrow{EF} = -\overrightarrow{BC} = -\mathbf{B}$ $\quad [\because BC = \text{and} \parallel FE]$

and $\quad \overrightarrow{FA} = -\overrightarrow{CD} = -(\mathbf{B} - \mathbf{A}) = \mathbf{A} - \mathbf{B}$ $\quad [\because CD = \text{and} \parallel AF]$

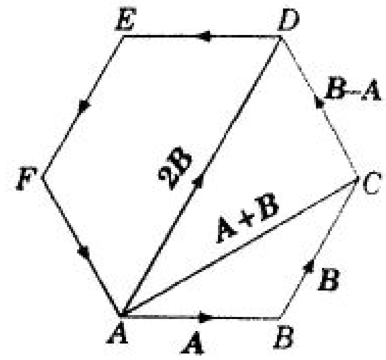


Fig. 3 3.

3·2. (1) Space co-ordinates. Let *X′OX* and *Y′OY*, *Z′OZ* be three mutually perpendicular lines which intersect at *O*. Then *O* is called the origin.

X′OX is called the **x-axis,** *Y′OY the* **y-axis,** *Z′OZ the* **z-axis** *and taken together these are called the* **co-ordinate axes.**

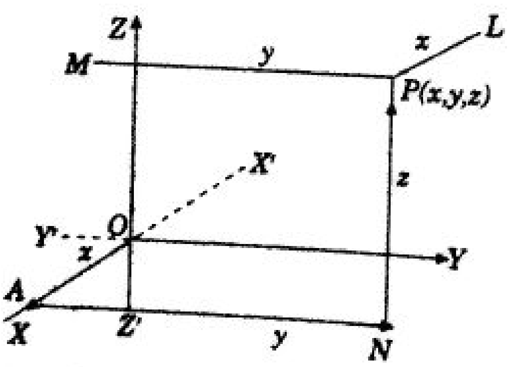


Fig. 3·4.

The plane YOZ is called the **yz-plane,** *the plane ZOX the* **zx-plane,** *the plane XOY the* **xy-plane** *and taken together these are called the* **co-ordinate planes.**

Let *P* be any point in space. Draw *PL*, *PM*, *PN* ⊥s to the *yz*, *zx* and *xy*-planes. Then *LP*, *MP*, *NP* are respectively called the coordinates of *P* (Fig. 3·4). *Thus the co-ordinates of any point in space are the perpendicular distances from the yz, zx and xy planes respectively. In practice, to find the co-ordinates of a point P draw PN* ⊥ to the *xy*-plane ; from *N* draw *NA* ⊥ to the *x*-axis (Fig. 3·4). If $OA = x$, $AN = y$, $NP = z$, then (x, y, z) are the co-ordinates of *P*.

The x, y, z co-ordinates are positive along OX, OY, OZ respectively and negative along OX′, OY′, OZ′.

The three co-ordinate planes divide the space into eight compartments called **octants.** The **octant** *OXYZ* in which all the co-ordinates are positive is called the **positive** or **first octant.**

Note. As a right-handed screw moving in the direction of OX rotates from OY towards OZ, therefore the above system of axes is called a **right-handed system.** The left-handed system will, however, be obtained by interchanging the axes of x and y.

In general, *three non-coplanar vectors* **A, B, C** *are said to form a* **right-handed** (*or a* **left-handed**) *system according as a right threaded screw rotated through an angle less than* 180° *from* **A** *to* **B** *will advance along (or opposite to)* **C** as shown in Fig. 3.5.

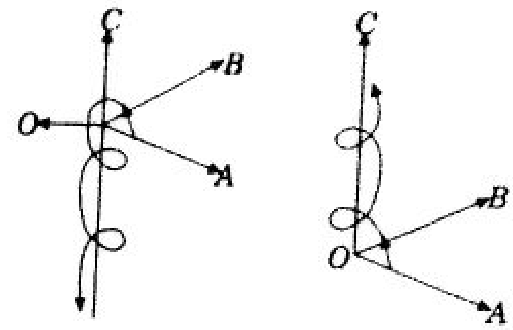


Fig. 3.5.

An area of a closed curve described in a given manner is represented by a vector whose magnitude is the given area and direction normal to the plane of the area.

Thus vector **A** representing the area is taken to be positive or negative according as the direction of description of the boundary of the curve and the sense of **A** correspond to a right-handed or a left-handed system.

We have explained the most commonly used system of co-ordinates namely the *Rectangular Cartesian Co-ordinates*. The other two systems of co-ordinates often used to locate a point in space are the *Polar Spherical co-ordinates* and *Cylindrical co-ordinates*, which are explained in § 8.21 and 8.20.

(2) Resolution of vectors. Let **I, J, K** denote unit vectors along OX, OY, OZ respectively. Let $P(x, y, z)$ be a point in space. On OP as diagonal, construct a rectangular parallelopiped with edges OA, OB, OC along the axes so that

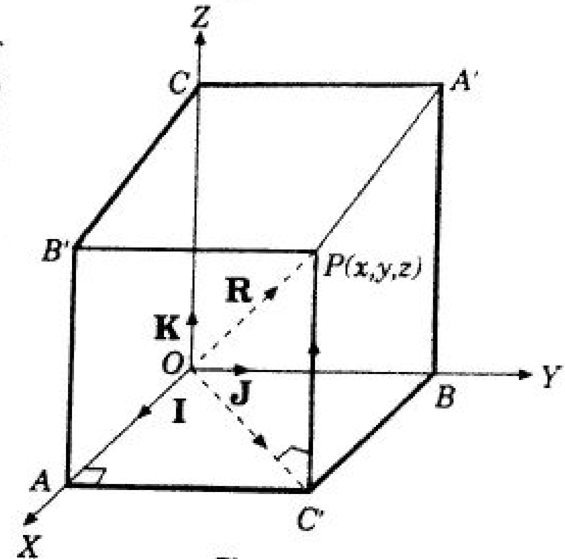


Fig. 3.6.

$$\vec{OA} = x\mathbf{I},\ \vec{OB} = y\mathbf{J},\ \vec{OC} = z\mathbf{K}$$

Then $$\mathbf{R} = \vec{OP} = \vec{OC'} + \vec{C'P}$$
$$= \vec{OA} + \vec{AC'} + \vec{OC} = \vec{OA} + \vec{OB} + \vec{OC}$$

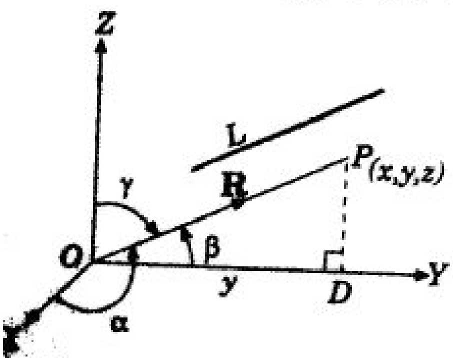


Fig. 3.7.

Hence $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ is called the *position vector* of P relative to origin O and

$$r = |\ \mathbf{R}\ | = \sqrt{(x^2 + y^2 + z^2)}$$

$$[\because r^2 = OP^2 = OC'^2 + C'P^2 = OA^2 + AC'^2 + C'P^2$$

(3) Direction cosines. Let any line L or its parallel OP, make angles α, β, γ with OX, OY, OZ respectively, then *$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of this line* which are usually denoted by l, m, n.

If l, m, n are direction cosines of a vector **R**, *then*

(i) $\hat{\mathbf{R}} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$, *(ii)* $\mathbf{l}^2 + \mathbf{m}^2 + \mathbf{n}^2 = 1$

Proof. Let D be the foot of the perpendicular from $P(x, y, z)$ on OY. Then

$$y = OD = r\cos\beta = mr. \text{ Similarly } z = nr \text{ and } x = lr.$$

$$\therefore \quad \mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = r(l\mathbf{I} + m\mathbf{J} + n\mathbf{K})$$

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{r} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$$

hich expresses a unit vector in terms of its direction cosines.

Also $$1 = |\ \hat{\mathbf{R}}\ | = \sqrt{(l^2 + m^2 + n^2)} \quad \text{Thus } \mathbf{l}^2 + \mathbf{m}^2 + \mathbf{n}^2 = 1$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Obs. Direction ratios. If the direction cosines of a line be proportional to a, b, c, then these are called proportional direction cosines or direction ratios of the line.

If the direction cosines be l, m, n, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \frac{1}{\sqrt{(\Sigma a^2)}}$$

$$\therefore \quad l = \frac{a}{\sqrt{(\Sigma a^2)}},\ m = \frac{b}{\sqrt{(\Sigma a^2)}},\ n = \frac{c}{\sqrt{(\Sigma a^2)}}$$

(4) Distance between two points $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *is*

$$\sqrt{[(\mathbf{x_2} - \mathbf{x_1})^2 + (\mathbf{y_2} - \mathbf{y_1})^2 + (\mathbf{z_2} - \mathbf{z_1})^2]}$$

and **direction ratios** *of* $\overrightarrow{PQ}$ *are* $\mathbf{x_2 - x_1, y_2 - y_1, z_2 - z_1}$

We have

$$\overrightarrow{OP} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}$$

and

$$\overrightarrow{OQ} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

$$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J} + (z_2 - z_1)\mathbf{K}$$

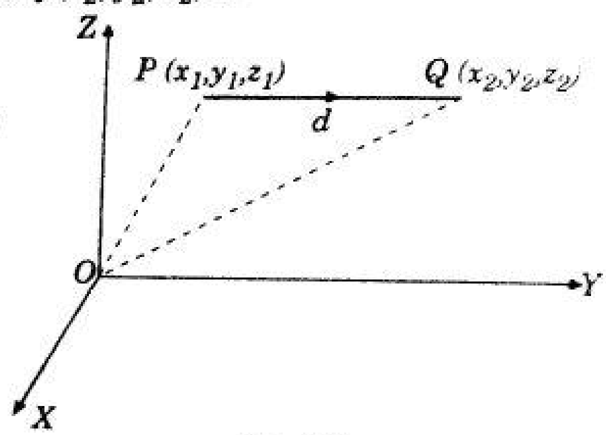


Fig. 3.8.

Thus $\quad d = |\overrightarrow{PQ}| = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

and direction cosines of $\overrightarrow{PQ}$ are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Example 3.2. *Show that the points $A(-4, 9, 6)$, $B(-1, 6, 6)$ and $C(0, 7, 10)$ form a right angled isosceles triangle. Also find the direction cosines of AB.*

Sol. We have

$$AB = \sqrt{[(-1+4)^2 + (6-9)^2 + (6-6)^2]} = 3\sqrt{2}$$

$$BC = \sqrt{[(0+1)^2 + (7-6)^2 + (10-6)^2]} = 3\sqrt{2}$$

and

$$CA = \sqrt{[(-4-0)^2 + (9-7)^2 + (6-10)^2]} = 6$$

Since $AB^2 + BC^2 = CA^2$ and $AB = BC$, it follows that ΔABC is a right-angled isosceles triangle. The direction ratios of $\overrightarrow{AB}$ are $-1+4, 6-9, 6-6$.

$\therefore$ Its direction cosines are $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0$.

3.3. SECTION FORMULAE

The point $\mathbf{R}(x, y, z)$ *dividing the join of the points* $\mathbf{A}(x_1, y_1, z_1)$ *and* $\mathbf{B}(x_2, y_2, z_2)$ *in the ratio* $m_1 : m_2$ *is*

$$\mathbf{R} = \frac{m_1\mathbf{B} + m_2\mathbf{A}}{m_1 + m_2} \quad i.e. \quad \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right) \quad ...(i)$$

Let $P(\mathbf{A})$ **and** $Q(\mathbf{B})$ be the given points referred to origin O. Let $R(\mathbf{R})$ be the point dividing **the line joining** P **and** Q in the ratio $m_1 : m_2$ so that

$$\frac{PR}{RQ} = \frac{m_1}{m_2} \quad i.e. \quad m_2 \ PR = m_1 \ RQ$$

$\therefore$ **We have** $\quad m_2\overrightarrow{PR} = m_1\overrightarrow{RQ}$

or $\quad m_2(\overrightarrow{OR} - \overrightarrow{OP}) = m_1(\overrightarrow{OQ} - \overrightarrow{OR})$

or $\qquad m_2(\mathbf{R}-\mathbf{A}) = m_1(\mathbf{B}-\mathbf{R})$

whence $$\mathbf{R} = \frac{m_1\mathbf{B} + m_2\mathbf{A}}{m_1 + m_2}$$

Since $\mathbf{A} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}$, $\mathbf{B} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$

and $$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$$

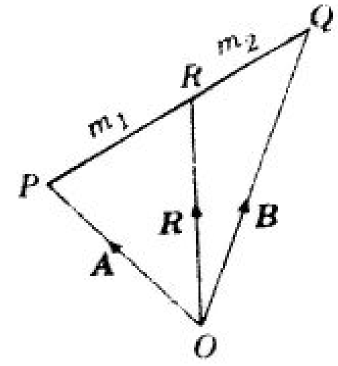


Fig. 3.9.

$$\therefore \quad x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = \frac{m_1(x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}) + m_1(x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K})}{m_1 + m_2}$$

Equating coefficients of **I, J, K,** we get the desired results (i).

Cor. 1. *Mid point of $P(\mathbf{A})$ and $Q(\mathbf{B})$ is $\frac{1}{2}(\mathbf{A}+\mathbf{B})$.*

2. Point R dividing the join of $P(\mathbf{A})$ and $Q(\mathbf{B})$ in the ratio $m_1 : m_2$ externally is $\mathbf{R} = \dfrac{m_1\mathbf{B} - m_2\mathbf{A}}{m_1 - m_2}$

Obs. Rewriting (i) as $m_2\mathbf{A} + m_1\mathbf{B} - (m_1 + m_2)\mathbf{R} = 0$, we note that the sum of the coefficients of **A, B** and **R** is zero.

Hence it follows that any *three points with position vectors* **A**, **B** *and* **C** *are colinear if* $\lambda\mathbf{A} + \mu\mathbf{B} + \gamma\mathbf{C} = \mathbf{0}$, where $\lambda + \mu + \gamma = 0$

Example 3.3. *In a trapezium, prove that the straight line joining the mid-points of the diagonals is parallel to the parallel sides and half their difference.*

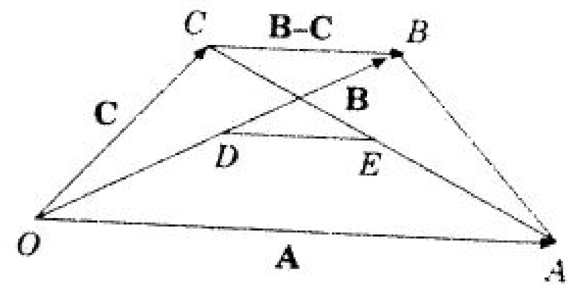


Fig. 3.10.

Sol. Consider a trapezium $OABC$ with parallel sides OA and BC. Take O as the origin and let the other vertices be $A(\mathbf{A})$, $B(\mathbf{B})$, $C(\mathbf{C})$.

Since CB is parallel to OA, therefore,

$$\mathbf{B} - \mathbf{C} = \overrightarrow{CB} = \lambda\overrightarrow{OA} = \lambda\mathbf{A}.$$

The mid-points of the diagonals OB and AC are $D(\mathbf{B}/2)$ and $E(\mathbf{A}+\mathbf{C})/2$.

$$\therefore \quad \overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = \frac{1}{2}(\mathbf{A}+\mathbf{C}) - \frac{1}{2}\mathbf{B} = \frac{1}{2}[\mathbf{A} - (\mathbf{B}-\mathbf{C})] \quad \ldots(i)$$

$$= \tfrac{1}{2}(1-\lambda)\mathbf{A} \quad \ldots(ii)$$

From (ii), it is clear that $\overrightarrow{DE}$ is parallel to $\overrightarrow{OA}$; from (i), it follows that $DE = \frac{1}{2}(OA - CB)$. Hence the result.

Example 3.4. *Show that the line joining one vertex of a parallelogram to the mid-point of an opposite side trisects the diagonal and is itself trisected there at.*

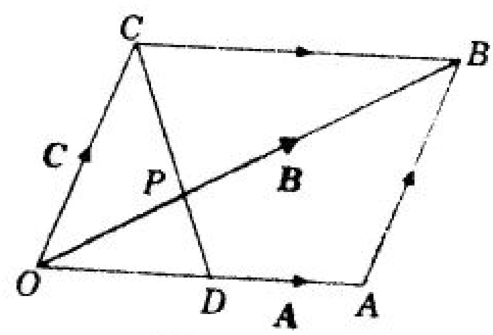


Fig. 3.11.

Sol. Consider a parallelogram $OABC$. Take O as the origin and let the other vertices be $A(\mathbf{A})$, $B(\mathbf{B})$ and $C(\mathbf{C})$.

The mid-point D of OA is $\mathbf{A}/2$.

Now since OA is equal to and parallel to CB.

$$\overrightarrow{OA} = \overrightarrow{CB}, \text{ i.e. } \mathbf{A} = \mathbf{B} - \mathbf{C}$$

which may be written as $\dfrac{2(\mathbf{A}/2) + 1\,.\,\mathbf{C}}{} \quad \mathbf{B}$

Problems 3.1

1. Given $\mathbf{R}_1 = 5\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$ and $\mathbf{R}_2 = \mathbf{I} + 3\mathbf{J} + 7\mathbf{K}$, find the magnitude and direction cosines of the vectors $\mathbf{R}_1 + \mathbf{R}_2$ and $2\mathbf{R}_1 - \mathbf{R}_2$.
2. Show that the points (0, 4, 1) ; (2, 3, – 1) ; (4, 5, 0) and (2, 6, 2) are the vertices of a square. *(Osmania, 1999 S)*
3. A straight line is inclined to the axes of x and y at angles of 30° and 60°. Find the inclination of the line to the z-axis. *(Madras, 2000)*
4. If a line makes angles α, β, γ with the axes, prove that *(V.T.U., 2000 ; Osmania, 1999)*
   (i) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.
   (ii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$.
5. If $\mathbf{A}$ and $\mathbf{B}$ are non-colinear vectors and $\mathbf{P} = (2x + 3y - 2)\mathbf{A} + (3x + 2y + 5)\mathbf{B}$ and $\mathbf{Q} = (-x + 4y - 2)\mathbf{A} + (3x - 4y + 7)\mathbf{B}$, find x, y such that $7\mathbf{P} = 3\mathbf{Q}$.
6. Prove that the line joining the mid-points of the two sides of a triangle is parallel to the third side and half of it.
7. Prove that (i) the diagonals of a parallelogram bisect each other ;
   (ii) a quadrilateral whose diagonals bisect each other is a parallelogram.
8. In a skew quadrilateral, prove that :
   (i) the figure formed by joining the mid-points of the adjacent sides is a parallelogram
   (ii) the joins of the mid-points of opposite sides bisect each other.
9. In a trapezium, prove that the straight line joining the mid-points of the non-parallel sides is parallel to the parallel sides and half their sum.
10. Prove that the vectors $\mathbf{A} = 3\mathbf{I} + \mathbf{J} - 2\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 3\mathbf{J} + 4\mathbf{K}$, $\mathbf{C} = 4\mathbf{I} - 2\mathbf{J} - 6\mathbf{K}$ can form the sides of a triangle. Also find the length of the median bisecting the vector $\mathbf{C}$. *(J.N.T.U., 1995 S)*
11. Find the ratio in which XOY plane divides the join of the points (– 3, 4, – 8) and (5, – 6, 4) and thus write the co-ordinates of the point of division.
12. Find the ratio in which the line joining (2, 4, 16) and (3, 5, – 4) is divided by the plane $2x - 3y + z + 6 = 0$. *(Mysore, 1995)*
13. Show that the three points $\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$, $2\mathbf{I} + 3\mathbf{J} - 4\mathbf{K}$, $-7\mathbf{J} + 10\mathbf{K}$ are colinear.
14. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be the position vectors of the vertices A, B, C of the triangle ABC, show that the three
    (i) medians concur at the point $\frac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C})$, called the *centroid*.
    (ii) internal bisectors of the angles concur at the point $\dfrac{a\mathbf{A} + b\mathbf{B} + c\mathbf{C}}{a + b + c}$, called the *incentre*.
15. Show that the co-ordinates of the *centroid of the triangle* whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are
$$\left[\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right]$$
16. Show that the co-ordinates of the *centroid of the tetrahedron* whose vertices are $(x_r, y_r, z_r) : r = 1, 2, 3, 4$, are
$$\left[\frac{1}{4}(x_1 + x_2 + x_3 + x_4), \frac{1}{4}(y_1 + y_2 + y_3 + y_4) ; \frac{1}{4}(z_1 + z_2 + z_3 + z_4)\right].$$

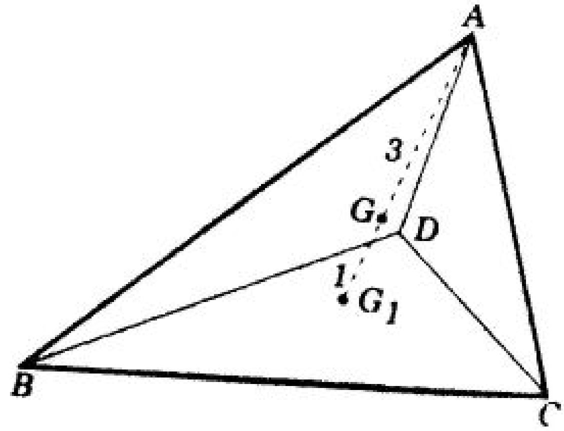


Fig. 3.12.

[**Def. A tetrahedron** *is a solid bounded by four triangular faces*. Thus the tetrahedron $ABCD$ has four faces—the Δs ABC, ACD, ADB, BCD. (Fig. 3.12).

It has four vertices A, B, C, D and three pairs of opposite edges AB, CD ; BC, AD ; CA, BD.

The centroid of the tetrahedron divides the join of each vertex to the centroid of the opposite triangular face in the ratio 3 : 1].

17. *M* and *N* are the mid-points of the diagonals AC and BD respectively of a quadrilateral $ABCD$. Show that the resultant of the vectors $\vec{AB}, \vec{AD}, \vec{CB}, \vec{CD}$ ...

3.4. PRODUCTS OF TWO VECTORS

Unlike the product of two scalars or that of a vector by a scalar, the product of two vectors is sometimes seen to result in a scalar quantity and sometimes in a vector. As such, we are led to define two types of such products, called the *scalar product* and the *vector product* respectively.

The scalar and vector products of two vectors **A** and **B** are usually written as **A . B** and **A × B** respectively and are read as **A** dot **B** and **A** cross **B**. In view of this notation, the former is sometimes called the *dot product* and the latter the *cross product*.

In vector algebra, the division of a vector by another vector is not defined.

3.5. SCALAR OR DOT PRODUCT

(1) **Definition.** *The scalar or dot product of two vectors* **A** *and* **B** *is defined as the scalar ab* cos θ, *where* θ *is the angle between* **A** and **B**.

Thus $\qquad \mathbf{A} \cdot \mathbf{B} = ab \cos \theta.$

(2) **Geometrical interpretation. A.B** *is the product of the length of one vector and the length of the projection of the other in the direction of the former.*

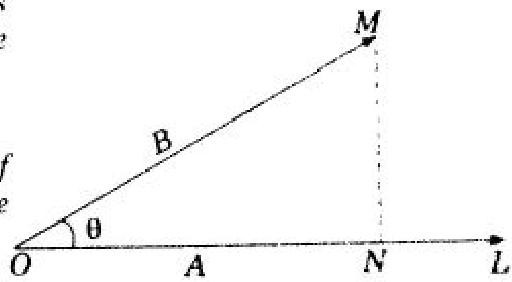


Fig. 3.13.

Let $\qquad \vec{OL} = \mathbf{A}, \ \vec{OM} = \mathbf{B} \qquad$ then

$$\mathbf{A} \cdot \mathbf{B} = ab \cos \theta = a(OM \cos \theta) = a(ON) = |\,A\,| \text{ Proj. of } |\,\mathbf{B}\,| \text{ in the direction of } \mathbf{A}$$

Similarly, **A . B** = | **B** | Proj. of | **A** | in the direction of **B**.

(3) **Properties and other results.**

I. Scalar product of two vectors is commutative

i.e. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ for $\mathbf{A} \cdot \mathbf{B} = ab \cos \theta = ba \cos(-\theta) = \mathbf{B} \cdot \mathbf{A}$

II. The necessary and sufficient condition for two vectors to be perpendicular is that their scalar product should be zero.

When the vectors **A** and **B** are perpendicular, $\mathbf{A} \cdot \mathbf{B} = ab \cos 90° = 0$.

Conversely when $\mathbf{A} \cdot \mathbf{B} = 0$, $ab \cos \theta = 0$ *i.e.* $\cos \theta = 0$. $\qquad (\because a \neq 0, b \neq 0)$, or $\theta = 90°$

III. $\mathbf{A} \cdot \mathbf{A} = a^2$ which is written as $\mathbf{A}^2$. Thus *the square of a vector is a scalar which stands for the square of its magnitude.*

IV. For the mutually perpendicular unit vectors **I, J, K**, we have the relations

$$\mathbf{I} \cdot \mathbf{J} = \mathbf{J} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{I} = 0$$

and $$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = 1$$

which are of great utility.

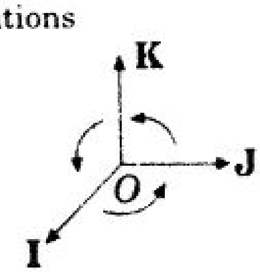


Fig. 3.14.

V. Scalar product of two vectors is distributive i.e.

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$$

*VI. Schwarz inequality** : $\qquad |\,\mathbf{A} \cdot \mathbf{B}\,| \leq |\,\mathbf{A}\,|\,|\,\mathbf{B}\,|$

$$|\,\mathbf{A} \cdot \mathbf{B}\,| = |\,\mathbf{A}\,|\,|\,\mathbf{B}\,|\,|\cos \theta\,| \leq |\,\mathbf{A}\,|\,|\,\mathbf{B}\,| \qquad [\because |\cos \theta| \leq 1]$$

VII. Scalar product of two vectors is equal to the sum of the products of their corresponding components.

* Named after the German mathematician *Hermann Amandus Schwarz* (1843–1921) who is known for his work in conformal mapping, calculus of variations and differential geometry. He succeeded *Weierstrass* in Berlin University.

For if $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, $\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$

then by the distributive law, $\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$

In particular, $\mathbf{A}^2 = a_1^2 + a_2^2 + a_3^3$.

VIII. **Angle between two lines** *whose direction cosines are l, m, n and l', m', n' is* $\cos^{-1}(\mathbf{ll' + mm' + nn'})$.

The unit vectors in the directions of the given lines are $\mathbf{U} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$ and $\mathbf{U}' = l'\mathbf{I} + m'\mathbf{J} + n'\mathbf{K}$.

If θ be the angle between the lines, then

$$\mathbf{U} \cdot \mathbf{U}' = (l\mathbf{I} + m\mathbf{J} + n\mathbf{K}) \quad (l'\mathbf{I} + m'\mathbf{J} + n'\mathbf{K})$$

or $$1 \cdot 1 \cdot \cos\theta = ll' + mm' + nn'$$

Hence $$\cos\theta = ll' + mm' + nn' \qquad ...(i)$$

Cor. 1. $\sin^2\theta = 1 - \cos^2\theta = 1 - (ll' + mm' + nn')^2$

$$= (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2$$

$$= (mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2$$

$\therefore \quad \sin\theta = \pm\sqrt{\Sigma(mn' - nm')^2}$. $\qquad ...(ii)$

Cor. 2. *The condition that the lines whose direction cosines are l, m, n and l', m', n' should be* **perpendicular** *is*

$$\mathbf{ll' + mm' + nn' = 0} \qquad ...(iii)$$

and **parallel** *is* $$\mathbf{l = l', m = m', n = n'} \qquad ...(iv)$$

These conditions easily follow from (*i*) and (*ii*).

Cor. 3. *The angle θ between two lines whose direction ratios are a, b, c and a', b', c' is given by*

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma a'^2)}}$$

or $$\sin\theta = \frac{\sqrt{[(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2}}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma a'^2)}}$$

These lines are (i) perpendicular if $aa' + bb' + cc' = 0$, (ii) parallel if $a/a' = b/b' = c/c'$.

IX. **Projection of the line joining two points** (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *on a line whose direction cosines are l, m, n is*

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

Let $$\vec{OP} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}, \quad \vec{OQ} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

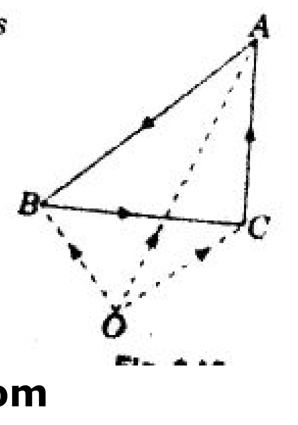
$\therefore$ $$\vec{PQ} = (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J} + (z_2 - z_1)\mathbf{K}$$

Also unit vector $\mathbf{U}$ along the given line is $l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$.

$\therefore$ Projection of PQ on the given line $= \vec{PQ} \cdot \mathbf{U}$.

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

Example 3·5. *Find the sides and angles of the triangle whose vertices are* $\mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$, $2\mathbf{I} + \mathbf{J} - \mathbf{K}$, *and* $3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$



Sol. Let $\vec{OA} = \mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$, $\vec{OB} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$, $\vec{OC} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$

Then $$\vec{BC} = \mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$$

$$\vec{CA} = -2\mathbf{I} - \mathbf{J}$$

and $$\vec{AB} = \mathbf{I} + 3\mathbf{J} - 3\mathbf{K}$$

$BC = \sqrt{14}$, $CA = \sqrt{5}$, $AB = \sqrt{19}$.

Now d.c.'s of AB and AC being

$1/\sqrt{19}, 3/\sqrt{19}, -3/\sqrt{19}$ and $2/\sqrt{5}, 1/\sqrt{5}, 0,$

We have $\cos A = \frac{1}{\sqrt{19}} \cdot \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{19}} \cdot \frac{1}{\sqrt{5}} + \frac{-3}{\sqrt{19}} \cdot 0 = \sqrt{(5/19)}$

i.e. $\angle A = \cos^{-1} \sqrt{(5/19)}$. Again d.c.'s of BC and BA being

$1/\sqrt{14}, -2/\sqrt{14}, 3/\sqrt{14}$ and $-1/\sqrt{19}, -3/\sqrt{19}, 3/\sqrt{19}$;

we have $\cos B = \frac{1}{\sqrt{14}} \cdot \frac{-1}{\sqrt{19}} + \frac{-2}{\sqrt{14}} \cdot \frac{-3}{\sqrt{19}} + \frac{3}{\sqrt{14}} \cdot \frac{3}{\sqrt{19}} = \sqrt{(14/19)}$ *i.e.* $\angle B = \cos^{-1} \sqrt{(14/19)}$

Finally d.c.'s of CA and CB being $-2/\sqrt{5}, -1/\sqrt{5}, 0$ and $-1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14}$.

we have $\cos C = \frac{-2}{\sqrt{5}} \cdot \frac{-1}{\sqrt{14}} + \frac{-1}{\sqrt{5}} \cdot \frac{2}{\sqrt{14}} + 0 \cdot \frac{-3}{\sqrt{14}} = 0$ *i.e.* $\angle C = 90°$.

Example 3·6. *Prove that the right-bisectors of the sides of a triangle concur at its circumcentre.*

Sol. Let $A(\mathbf{A})$, $B(\mathbf{B})$, $C(\mathbf{C})$ be the vertices of any triangle ABC. The mid-points of the sides BC, CA and AB are

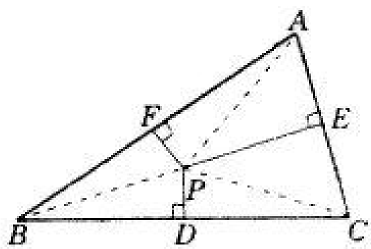


Fig. 3·16.

$$D\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right), E\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right), F\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)$$

Let the perpendiculars at D and E to BC and CA respectively intersect at the point $P(\mathbf{R})$. Then $\vec{DP} \cdot \vec{BC} = 0$

i.e. $$\left(\mathbf{R} - \frac{\mathbf{B}+\mathbf{C}}{2}\right) \cdot (\mathbf{C} - \mathbf{B}) = 0 \quad ...(i)$$

and $$\vec{EP} \cdot \vec{CA} = 0, \text{ i.e. } \left(\mathbf{R} - \frac{\mathbf{C}+\mathbf{A}}{2}\right) \cdot (\mathbf{A} - \mathbf{C}) = 0 \quad ...(ii)$$

Adding (i) and (ii), we get $\left(\mathbf{R} - \frac{\mathbf{A}+\mathbf{B}}{2}\right) \cdot (\mathbf{A} - \mathbf{B}) = 0$

which shows that FP is perpendicular to AB. Hence the result.

Further $PA = PB$ if $|\mathbf{A} - \mathbf{R}| = |\mathbf{B} - \mathbf{R}|$

or if, $(\mathbf{A} - \mathbf{R})^2 = (\mathbf{B} - \mathbf{R})^2$ or if, $\mathbf{A}^2 - 2\mathbf{A} \cdot \mathbf{R} = \mathbf{B}^2 - 2\mathbf{B} \cdot \mathbf{R}$

or if, $\left(\mathbf{R} - \frac{\mathbf{A}+\mathbf{B}}{2}\right) \cdot (\mathbf{A} - \mathbf{B}) = 0$, which is true.

Example 3·7. *If the distance between two points P and Q is d and the lengths of the projections of PQ on the coordinate planes are d_1, d_2, d_3, show that $2d^2 = d_1^2 + d_2^2 + d_3^2$.*

Sol. Let P be (x_1, y_1, z_1) and Q be (x_2, y_2, z_2), then

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

The feet of the perperdiculars drawn from P and Q on the XY-plane are the projections of P and Q on this plane. If these are L and M, then L is $(x_1, y_1, 0)$ and M is $(x_2, y_2, 0)$

$\therefore$ d_1 = projection of PQ on XY-plane *i.e.* LM

or $d_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

Similarly $d_2^2 = (y_1 - y_2)^2 + (z_1 - z_2)^2$ and $d_3^2 = (z_1 - z_2)^2 + (x_1 - x_2)^2$

$\therefore$ $d_1^2 + d_2^2 + d_3^2 = 2\,[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2] = 2d^2$

Example 3·8. *A line makes angles $\alpha, \beta, \gamma, \delta$ with diagonals of a cube, prove that*

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = 4/3.$$ (V.T.U., 2006 ; Osmania, 2000 S)

Sol. Take O, a corner of the cube as origin and OA, OB, OC the three edges through it, as the axes.

Let $OA = OB = OC = a$. Then the co-ordinates of the corners are as shown in Fig. 3.17.

The four diagonals are

OP, AA', BB' and CC'.

Clearly direction cosines of OP are

$$\frac{a-0}{\sqrt{(\Sigma a^2)}}, \frac{a-0}{\sqrt{(\Sigma a^2)}}, \frac{a-0}{\sqrt{(\Sigma a^2)}} \text{ i.e. } \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

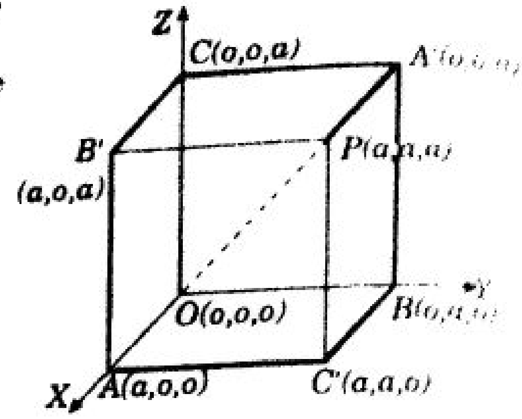


Fig. 3.17.

Similarly direction cosines of AA' are $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$;

" " " BB' are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$;

and " " " CC' are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$.

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with OP, AA', BB', CC' respectively. Then

$$\cos\alpha = \frac{1}{\sqrt{3}}(l+m+n)\,;\ \cos\beta = \frac{1}{\sqrt{3}}(-l+m+n)$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l-m+n)\,;\ \cos\delta = \frac{1}{\sqrt{3}}(l+m-n)$$

Squaring and adding, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3}[4(l^2+m^2+n^2)] = \frac{4}{3}. \qquad [\because l^2+m^2+n^2=1]$$

Example 3.9. *If the edges of a rectangular parallelopiped are a, b, c, show that the angle between the four diagonals are* $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$

Sol. Let $OA = a$, $OB = b$, $OC = c$ be the edges of the rectangular parallelopiped. Then the coordinates of the corners are as shown in Fig. 3.18. The four diagonals taken in pairs are (*i*) (OP, AA'), (*ii*) (OP, BB'), (*iii*) (OP, CC'), (*iv*) (AA', BB'), (*v*) (AA', CC') and (*vi*) (BB', CC').

Let the angles between these pairs of diagonals be $\theta_1, \theta_2, \ldots \theta_6$ respectively. Clearly d.r.'s OP are a, b, c ; d.r.'s, of AA' are $-a, b, c$, d.r.'s of BB' are $a, -b, c$ and d.r.'s of CC' are $a, b, -c$.

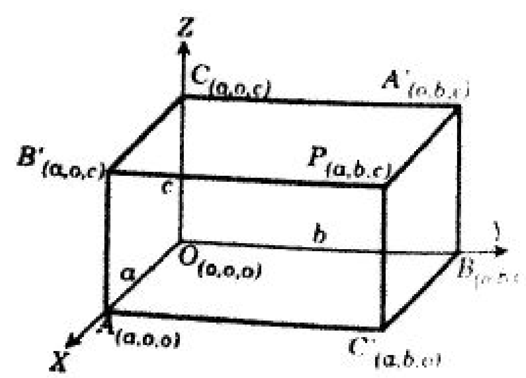


Fig. 3.18.

∴ For the pair (*i*) *i.e.* (OP, AA');

$$\cos\theta_1 = \frac{-a^2+b^2+c^2}{\sqrt{(a^2+b^2+c^2)}\sqrt{(a^2+b^2+c^2)}} = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

Similarly $\cos\theta_2 = \frac{a^2-b^2+c^2}{a^2+b^2+c^2}$, $\qquad \cos\theta_3 = \frac{a^2+b^2-c^2}{a^2+b^2+c^2}$;

$\cos\theta_4 = \frac{-a^2-b^2+c^2}{a^2+b^2+a^2}$, $\qquad \cos\theta_5 = \frac{-a^2+b^2-c^2}{a^2+b^2+c^2}$;

$$\cos\theta_6 = \frac{a^2 - b^2 - c^2}{a^2 + b^2 + c^2}$$

Thus, noting that at least one term in the numerator is negative, we have in general

$$\cos\theta = \frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}.$$

Example 3.10. *Prove that the lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $mn + nl + lm = 0$ are*

(i) perpendicular if $a^{-1} + b^{-1} + c^{-1} = 0$ (Burdwan, 2003)

(ii) parallel if $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$.

Sol. Eliminating n from the given relations, we have

$$(m + l)\left(-\frac{al + bm}{c}\right) + lm = 0 \quad \text{or } al^2 + (c - a - b)lm + bm^2 = 0$$

or $$a\,(l/m)^2 + (c - a - b)(l/m) + b = 0 \qquad ...(1)$$

If l_1, m_1, n_1 ; l_2, m_2, n_2, are the direction cosines of these lines then l_1/m_1, l_2/m_2 are the roots of the quadratic (1).

$$\therefore \quad \frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{b}{a} \qquad \text{or} \qquad \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \text{ (by symmetry)} = k \text{ (say)}.$$

The lines will be perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = k\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$

or if, $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

The lines will be parallel if $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$.

i.e. if, $l_1/m_1 = l_2/m_2$ *i.e.* if, $(c - a - b)^2 = 4ab$

or if, $c - a - b = \pm 2\sqrt{(ab)}$ or if, $c = a + b \pm 2\sqrt{(ab)} = (\sqrt{a} \pm \sqrt{b})^2$

or if, $\pm\sqrt{c} = \sqrt{a} \pm \sqrt{b}$ or if, $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$ [Taking necessary signs]

Example 3.11. *Find the angle between the lines whose direction cosines are given by the equation $l + 3m + 5n = 0$ and $5lm - 2mn + 6nl = 0$.*

Sol. Let us eliminate l from the given relations, by substituting $l = -3m - 5n$ in the second relation.

$$5m\,(-3m - 5n) - 2mn + 6n\,(-3m - 5n) = 0$$

i.e. $$15m^2 + 45mn + 30n^2 = 0 \quad \text{or} \quad m^2 + 3mn + 2n^2 = 0$$

or $$(m + n)(m + 2n) = 0 \quad i.e. \quad m + n = 0 \quad \text{or} \quad m + 2n = 0$$

Now let us first solve the equations $l + 3m + 5n = 0$ and $m + n = 0$

These give $m = -n$ and $l = -2n$ *i.e.* $\frac{l}{-2} = \frac{m}{-1} = \frac{n}{1}$ ...(i)

Similarly solving the equations $l + 3m + 5n = 0$ and $m + 2n = 0$,

We get $$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1} \qquad ...(ii)$$

(i) and (ii) give the direction ratios of the two lines.

If θ be the angle between these two lines, then

$$\cos\theta = \frac{(-2)\times 1 + (-1)\times(-2) + 1\times 1}{\sqrt{(2^2 + 1^2 + 1^2)}\,\sqrt{(1^2 + 2^2 + 1^2)}} = \frac{1}{6} \qquad i.e.\ \theta = \cos^{-1}\left(\frac{1}{6}\right)$$

Problems 3.2

1. If $\mathbf{A} = \mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 2\mathbf{J} + \mathbf{K}$ and $\mathbf{C} = 3\mathbf{I} + \mathbf{J}$, find t such that $\mathbf{A} + t\mathbf{B}$ is perpendicular to $\mathbf{C}$.
2. (i) Show that $\left(\frac{\mathbf{A}}{a^2} - \frac{\mathbf{B}}{b^2}\right)^2 = \left(\frac{\mathbf{A} - \mathbf{B}}{ab}\right)^2$.

   (ii) Interpret geometrically $(\mathbf{C} - \mathbf{A}) \cdot (\mathbf{B} - \mathbf{C}) = 0$.
3. If $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$, show that $\mathbf{A}$ and $\mathbf{B}$ are mutually perpendicular.
4. If $\mathbf{A} = \mathbf{I} + 2\mathbf{J} - 3\mathbf{K}$ and $\mathbf{B} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, show that $\mathbf{A} + \mathbf{B}$ is perpendicular to $\mathbf{A} - \mathbf{B}$. Also calculate the angle between $2\mathbf{A} + \mathbf{B}$ and $\mathbf{A} + 2\mathbf{B}$.
5. Show that the three concurrent lines with direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are coplanar if $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$.
6. Find the projection of the vector $\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ on $4\mathbf{I} - 4\mathbf{J} + 7\mathbf{K}$.
7. The projection of a line on the coordinate axes are 12, 4, 3. Find the length and direction cosines of the line. (*Rajasthan, 2006*)
8. Show (by vector methods) that the mid-point of the hypotenuse of a right-angled triangle is equidistant from its vertices.
9. Prove (by vector methods) that the angle in a semi-circle is a right angle.
10. Show (by vector methods) that the diagonals of a rhombus intersect at right angles.
11. Show that the altitudes of a triangle meet in a point (called the *ortho-centre*).
12. *ABCD* is a tetrahedron having the edges *BC* and *AC* at right angles to opposite edges *AD* and *BD* respectively. Show that the third pair of opposite edges *AB* and *CD* are also at right angles.
13. Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$. (*Rajasthan, 2005*)
14. Angle between the lines whose direction cosines are given by $3l + m + 5n = 0$ and $6mn - 2ln + 5lm = 0$.
15. Show that the lines whose direction cosines are given by the equations $4lm - 3mn - nl = 0$ and $3l + m + 2n = 0$ are perpendicular. (*Anna, 2005*)
16. Show that the lines whose direction cosines are given by the equations $l + m + n = 0$, $al^2 + bm^2 + cn^2 = 0$ are

   (i) perpendicular, if $a + b + c = 0$, (ii) parallel, if $a^{-1} + b^{-1} + c^{-1} = 0$.
17. Show that the straight lines whose direction cosines are given by the equations

   $al + bm + cn = 0$, $fmn + gnl + hlm = 0$ are (i) perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$. (*Osmania, 2003*)

   (ii) parallel if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$.
18. Show that the angle between any two diagonals of a cube is $\cos^{-1} 1/3$. (*V.T.U., 2006 ; Assam, 1999*)
19. (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are the direction cosines of three mutually perpendicular lines. Prove that the line whose d.c.'s are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ makes equal angles with the axes. (*V.T.U., 2003*)
20. *AB*, *BC* are the diagonals of adjacent faces of a rectangular box with its centre at the origin *O*, its edges are parallel to the axes. If the angles *BOC*, *COA* and *AOB* are equal to θ, φ, ψ respectively, prove that

$$\cos\theta + \cos\phi + \cos\psi = -1.$$

3.6. VECTOR, OR CROSS PRODUCT

(1) Definition. *The vector, or cross product of two vectors* **A** *and* **B** *is defined as a vector such that*

(i) its magnitude is ab sin θ, θ *being the angle between* **A** *and* **B**,

(ii) its direction is perpendicular to the plane of **A** *and* **B**,

and (iii) it forms with **A** *and* **B** *a right-handed system.*

If **N** be a unit vector normal to the plane of **A** and **B** (**A**, **B**, **N** forming a right handed system), then

$$\mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}.$$

(2) **Geometrical interpretation.** $\mathbf{A} \times \mathbf{B}$ *represents twice the vector area of the triangle having the vectors* **A** *and* **B** *as its adjacent sides.*

If **N** be a unit vector normal to the plane of the triangle OAB, then

$$\mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}$$

$$= 2\left(\frac{1}{2} ab \sin\theta\right)\mathbf{N} = 2\Delta OAB\, \mathbf{N} = 2\Delta\, \overrightarrow{OAB}$$

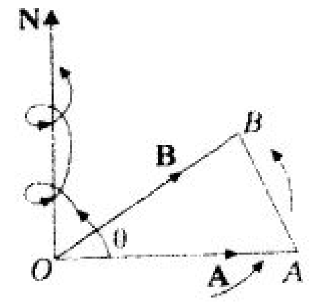


Fig. 3.19.

(3) **Properties and other results**

I. Vector product of two vectors is not commutative.

$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. In fact, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

for $\quad \mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}$ or $2\Delta\, \overrightarrow{OAB}$

and $\quad \mathbf{B} \times \mathbf{A} = ab \sin(-\theta)\, \mathbf{N} = -ab \sin\theta\, \mathbf{N}$ or $2\Delta\, \overrightarrow{OBA}$

II. The necessary and sufficient condition for two non-zero vectors to be parallel is that their vector product should be zero.

When the vectors **A** and **B** are parallel, the angle θ between them is 0 or 180° so that $\sin\theta = 0$, and as such $\mathbf{A} \times \mathbf{B} = \mathbf{0}$.

Conversely, when $\quad \mathbf{A} \times \mathbf{B} = \mathbf{0}\,;\ ab \sin\theta = 0$

i.e. $\quad \sin\theta = 0 \qquad (\because\ a \neq 0,\ b \neq 0)$

or $\quad \theta = 0$ or $180°$ In particular, $\mathbf{A} \times \mathbf{A} = \mathbf{0}$.

III. For the orthonormal vector triad **I, J, K,** we have the relations :

$$\mathbf{I} \times \mathbf{I} = \mathbf{J} \times \mathbf{J} = \mathbf{K} \times \mathbf{K} = \mathbf{0}$$
$$\mathbf{I} \times \mathbf{J} = \mathbf{K}, \quad \mathbf{J} \times \mathbf{I} = -\mathbf{K}$$
$$\mathbf{J} \times \mathbf{K} = \mathbf{I}, \quad \mathbf{K} \times \mathbf{J} = -\mathbf{I}$$
$$\mathbf{K} \times \mathbf{I} = \mathbf{J}, \quad \mathbf{I} \times \mathbf{K} = -\mathbf{J}.$$

IV. Relation between scalar and vector products.

We have $\quad (\mathbf{A} \cdot \mathbf{B})^2 = a^2b^2 \cos\theta = a^2b^2 - a^2b^2 \sin^2\theta = a^2b^2 - (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B})$

$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A} \cdot \mathbf{B})^2.$$

V. Vector product of two vectors is distributive

i.e. $\quad (\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}.$

VI. Analytical expression for the vector product.

If $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, $\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$ *then* $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

..., we get

$$\mathbf{A} \times \mathbf{B} = (a_2b_3 - a_3b_2)\,\mathbf{I} + (a_3b_1 - a_1b_3)\,\mathbf{J} + (a_1b_2 - a_2b_1)\,\mathbf{K}$$

... follows the required result.

Example 3·12. *If* $\mathbf{A} = 4\mathbf{I} + 3\mathbf{J} + \mathbf{K}$, $\mathbf{B} = 2\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, *find a unit vector* $\mathbf{N}$ *perpendicular to vectors* $\mathbf{A}$ *and* $\mathbf{B}$ *such that* $\mathbf{A}, \mathbf{B}, \mathbf{N}$ *form a right handed system. Also find the angle between the vectors* $\mathbf{A}$ *and* $\mathbf{B}$.

Sol. Since
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\mathbf{I} - 6\mathbf{J} - 10\mathbf{K}$$

and
$$|\mathbf{A} \times \mathbf{B}| = \sqrt{[(7)^2 + (-6)^2 + (-10)^2]} = \sqrt{185}$$

$\therefore$ Unit vector $\mathbf{N} \perp$ to $\mathbf{A}$ and $\mathbf{B} = \dfrac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = (7\mathbf{I} - 6\mathbf{J} - 10\mathbf{K})/\sqrt{185}$

Also $a = \sqrt{(4^2 + 3^2 + 1^2)} = \sqrt{26}$ and $b = 3$.

If θ be the angle between $\mathbf{A}$ and $\mathbf{B}$, then $|\mathbf{A} \times \mathbf{B}| = ab \sin\theta$, *i.e.* $\sin\theta = |\mathbf{A} \times \mathbf{B}|/ab$

Thus $\sin\theta = \sqrt{185}/3\sqrt{26}$ whence $\theta = 62° 40'$.

Example 3·13. (*i*) *Prove that the area of the triangle whose vertices are* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *is*
$$\frac{1}{2}\,|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}|$$

(*ii*) *Calculate the area of the triangle whose vertices are* A (*1, 0, – 1*), B (*2, 1, 5*) *and* C (*0, 1, 2*).

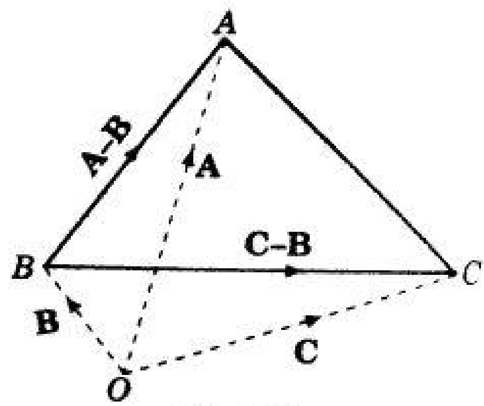


Fig. 3·20.

Sol. (*i*) Let $A(\mathbf{A})$, $B(\mathbf{B})$, $C(\mathbf{C})$ be the vertices of the triangle ABC (Fig. 3·20) and O, the origin so that
$$\vec{BC} = \vec{OC} - \vec{OB} = \mathbf{C} - \mathbf{B}$$
and
$$\vec{BA} = \vec{OA} - \vec{OB} = \mathbf{A} - \mathbf{B}$$

$\therefore$ Vector area of ΔABC
$$= \frac{1}{2}[\vec{BC} \times \vec{BA}] = \frac{1}{2}[(\mathbf{C} - \mathbf{B}) \times (\mathbf{A} - \mathbf{B})]$$
$$= \frac{1}{2}[\mathbf{C} \times \mathbf{A} - \mathbf{C} \times \mathbf{B} - \mathbf{B} \times \mathbf{A} + \mathbf{B} \times \mathbf{B}]$$
$$= \frac{1}{2}[\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}] \qquad [\because \mathbf{B} \times \mathbf{B} = 0]$$

Thus area of $\Delta ABC = \dfrac{1}{2}\,|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}|$.

(*ii*) Let O be the origin so that
$$\vec{OA} = \mathbf{I} - \mathbf{K},\ \vec{OB} = 2\mathbf{I} + \mathbf{J} + 5\mathbf{K} \text{ and } \vec{OC} = \mathbf{J} + 2\mathbf{K}$$
Then
$$\vec{BC} = \vec{OC} - \vec{OB} = -2\mathbf{I} - 3\mathbf{K}$$
and
$$\vec{BA} = \vec{OA} - \vec{OB} = -\mathbf{I} - \mathbf{J} - 6\mathbf{K}$$

$\therefore$ Vector area of $\Delta ABC = \dfrac{1}{2}(\vec{BC} \times \vec{BA}) = \dfrac{1}{2}\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -2 & 0 & -3 \\ -1 & -1 & -6 \end{vmatrix}$

Thus area of $\Delta ABC = \dfrac{1}{2}\,|-3\mathbf{I} - 9\mathbf{J} + 2\mathbf{K}| = \dfrac{1}{2}\sqrt{94}$.

Example 3·14. *In a triangle ABC ; D, E, F are the mid-points of the sides BC, CA, AB ; prove that*
$$\Delta DEF = \Delta FCE = \frac{1}{4}\Delta ABC.$$

Sol. Take B as the origin and let the position vectors of C and A be $\mathbf{C}$ and $\mathbf{A}$ (Fig. 3.21), so that the position vectors of D, E, F are

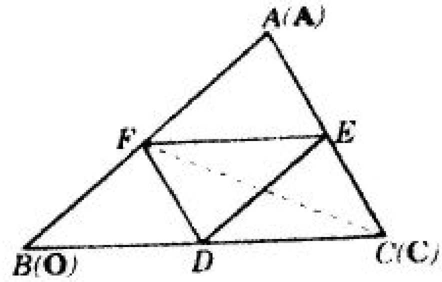


Fig. 3.21

$$\mathbf{C}/2,\ (\mathbf{C}+\mathbf{A})/2,\ \mathbf{A}/2.$$

$$\therefore \ \Delta\vec{DEF} = \frac{1}{2}(\vec{DE} \times \vec{DF}) = \frac{1}{2}\left(\frac{\mathbf{C}+\mathbf{A}}{2} - \frac{\mathbf{C}}{2}\right)\times\left(\frac{\mathbf{A}}{2} - \frac{\mathbf{C}}{2}\right)$$

$$= \frac{1}{8}[\mathbf{A} \times (\mathbf{A} - \mathbf{C})] = \frac{1}{8}\mathbf{C} \times \mathbf{A} = \frac{1}{4}\Delta\vec{ABC}$$

$$\Delta\vec{FCE} = \frac{1}{2}(\vec{FC} \times \vec{FE}) = \frac{1}{2}[(\mathbf{C} - \mathbf{A}/2) \times \mathbf{C}/2]$$

$$= \frac{1}{8}\mathbf{C} \times \mathbf{A} = \frac{1}{4}\Delta\vec{ABC}.$$ Hence the result.

Example 3.15. *Prove that*

(*i*) $\sin(A+B) = \sin A \cos B + \cos A \sin B.$

(*ii*) $\cos(A+B) = \cos A \cos B - \sin A \sin B.$

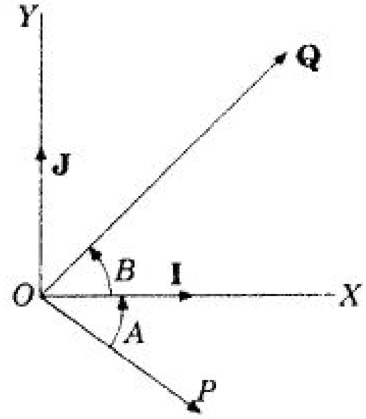


Fig. 3.22.

Sol. Let $\mathbf{I}, \mathbf{J}$ denote unit vectors along two perpendicular lines OX, OY so that

$$\mathbf{I}^2 = \mathbf{J}^2 = 1,\ \mathbf{I}\cdot\mathbf{J} = 0$$

and $$\mathbf{I}\times\mathbf{I} = \mathbf{J}\times\mathbf{J} = \mathbf{0}$$

Let $$\angle POX = A \text{ and } \angle XOQ = B,$$

so that $$\angle POQ = A + B.$$

If $OP = p$ and $OQ = q$, then the coordinates of P are $(p\cos A, -p\sin A)$ and those of Q are $(q\cos B, q\sin B)$ so that

$$\vec{OP} = (p\cos A)\mathbf{I} - (p\sin A)\mathbf{J}$$

$$\vec{OQ} = (q\cos B)\mathbf{I} + (q\sin B)\mathbf{J}$$

Then $$|\vec{OP} \times \vec{OQ}| = |[(p\cos A)\mathbf{I} - (p\sin A)\mathbf{J}] \times [(q\cos B)\mathbf{I} + (q\sin B)\mathbf{J}]|$$

$$= pq\,|\cos A\sin B\,(\mathbf{I}\times\mathbf{J}) - \sin A\cos B\,(\mathbf{J}\times\mathbf{I})|$$

$$= pq\,(\cos A\sin B + \sin A\cos B) \text{ for } |\mathbf{I}\times\mathbf{J}| = 1$$

Also $|\vec{OP} \times \vec{OQ}| = pq\sin(A+B)$. Equating the two expressions, we get (*i*).

Similarly, (*ii*) follows from $\vec{OP}\cdot\vec{OQ} = pq\cos(A+B)$.

Example 3.16. *In any triangle ABC, prove that*

(*i*) $a/\sin A = b/\sin B = c/\sin C.$ (*Sine formula*)

(*ii*) $a = b\cos C + c\cos B.$ (*Projection formula*)

(*iii*) $a^2 = b^2 + c^2 - 2bc\cos A.$ (*Cosine formula*)

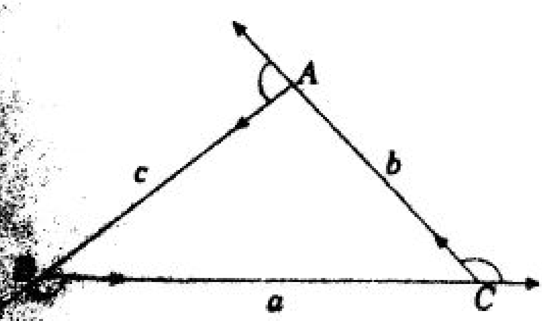


Fig. 3.23.

Sol. From ΔABC, we have $\vec{BC} + \vec{CA} + \vec{AB} = 0$

or $$\vec{CA} + \vec{AB} = -\vec{BC} \qquad (\lambda)$$

(*i*) Multiplying (λ) vectorially by $\vec{AB}$, we get

$$\vec{CA} \times \vec{AB} = -\vec{BC} \times \vec{AB}$$

or $$|\vec{CA} \times \vec{AB}| = |\vec{BC} \times \vec{AB}|$$

$\therefore$ $$bc\sin(\pi - A) = ac\sin(\pi - B)$$

or $$a/\sin A = b/\sin B.$$

Similarly, multiplying (λ) vectorially by $\vec{CA}$, we get

$a/\sin A = c/\sin C$, whence follows the result.

(*ii*) Multiplying (λ) scalarly by $\vec{BC}$, we get $\vec{CA} \cdot \vec{BC} + \vec{AB} \cdot \vec{BC} = -(\vec{BC})^2$

$\therefore \quad ba\cos(\pi - C) + ca\cos(\pi - B) = -a^2 \qquad$ or $\qquad a = b\cos C + c\cos B$

(*iii*) Squaring (λ), we get

$$(\vec{CA})^2 + (\vec{AB})^2 + 2\vec{CA} \cdot \vec{AB} = (\vec{BC})^2$$

i.e. $\quad b^2 + c^2 + 2bc\cos(\pi - A) = a^2 \quad$ or $\quad a^2 = b^2 + c^2 - 2bc\cos A$.

Problems 3.3

1. Given $\mathbf{A} = 2\mathbf{I} + 2\mathbf{J} - \mathbf{K}$, $\mathbf{B} = 6\mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$, find $\mathbf{A} \times \mathbf{B}$ and the unit vector perpendicular to both **A** and **B**. Also determine the sine of the angle between **A** and **B**.
2. If **A** and **B** are unit vectors and θ is the angle between them, show that $\sin\frac{\theta}{2} = \frac{1}{2}|\mathbf{A} - \mathbf{B}|$.
3. Find a unit vector normal to the plane of $\mathbf{A} = 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$ and $\mathbf{B} = \mathbf{I} + \mathbf{J} - 2\mathbf{K}$.
4. For any vector **A**, show that $|\mathbf{A} \times \mathbf{I}|^2 + |\mathbf{A} \times \mathbf{J}|^2 + |\mathbf{A} \times \mathbf{K}|^2 = 2|\mathbf{A}|^2$.
5. By vector method, find the area of the triangle whose vertices are (3, – 1, 2), (1, – 1, – 3) and (4, – 3, 1).
6. (*a*) Prove that the vector area of the quadrilateral *ABCD* is $\frac{1}{2}\vec{AC} \times \vec{BD}$.

   (*b*) If $3\mathbf{I} + \mathbf{J} - 2\mathbf{K}$ and $\mathbf{I} - 3\mathbf{J} - 4\mathbf{K}$ are the diagonals of a parallelogram. Find its area.
7. Given vectors $\mathbf{A} = \mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$ and $\mathbf{B} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$. Find the projection of $\mathbf{A} \times \mathbf{B}$ parallel to $5\mathbf{I} - \mathbf{K}$.
8. If $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$, prove that $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{C} = \mathbf{C} \times \mathbf{A}$, and interpret it geometrically.
9. Show that the perpendicular distance of the point **C** from the line joining **A** and **B** is $|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}| \div |\mathbf{B} - \mathbf{A}|$.
10. In *AC*, diagonal of the parallelogram *ABCD*, a point **P** is taken. Prove that $\Delta BAP = \Delta DAP$.
11. Prove by vector methods, that

    (*i*) $\sin(A - B) = \sin A\cos B - \cos A\sin B$; (*ii*) $\cos(A - B) = \cos A\cos B + \sin A\sin B$. (*Cochin, 1995*)
12. In any triangle *ABC*, prove by vector methods, that

    (*i*) $b = c\cos A + a\cos C$; (*ii*) $c^2 = a^2 + b^2 - 2ab\cos C$.

3.7. PHYSICAL APPLICATIONS

(1) Work done as a scalar product. If constant force **F** acting on a particle displaces it from the position *A* to position *B*, then

Work done = (resolved part of *F* in the direction of *AB*) . *AB*

$$= F\cos\theta \cdot AB = \mathbf{F} \cdot \vec{AB}$$

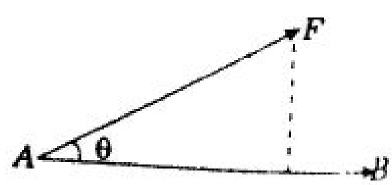
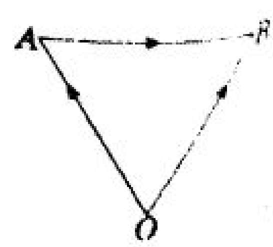


Fig. 3.24.

Thus, *the work done by a constant force is the scalar (or dot) product of the vectors representing the force and the displacement.*

Example 3.17. *Constant forces* $\mathbf{P} = 2\mathbf{I} - 5\mathbf{J} + 6\mathbf{K}$ *and* $\mathbf{Q} = -\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ *act on a particle. Determine the work done when the particle is displaced from A to B, the position vectors of A and B being* $4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}$ *and* $6\mathbf{I} + \mathbf{J} - 3\mathbf{K}$ *respectively.*



Sol. Resultant force $\quad \mathbf{F} = \mathbf{P} + \mathbf{Q} = \mathbf{I} - 3\mathbf{J} + 5\mathbf{K}$

and $\quad \vec{AB} = \vec{OB} - \vec{OA} = (6\mathbf{I} + \mathbf{J} - 3\mathbf{K}) - (4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}) = 2\mathbf{I} + 4\mathbf{J} - \mathbf{K}$

∴ *Work done* $= \mathbf{F} \cdot \overrightarrow{AB} = (\mathbf{I} - 3\mathbf{J} + 5\mathbf{K}) \cdot (2\mathbf{I} + 4\mathbf{J} - \mathbf{K})$

$= 1 \cdot 2 - 3 \cdot 4 + 5 \cdot (-1) = -15$ units.

(2) Normal flux. Consider the flow of a liquid through an element of area δs with a velocity **V** inclined at an angle θ to the outward unit normal **N** to the surface δs (Fig. 3·26).

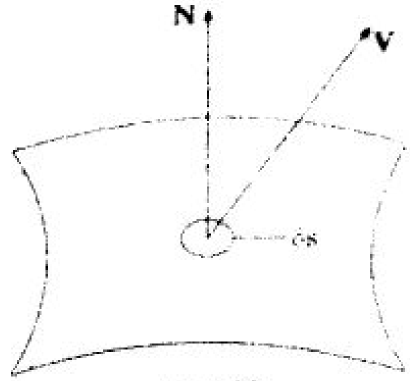


Fig. 3·26.

∴ Normal flux of the liquid through δs in unit time

$$V \cos\theta \cdot \delta s = \mathbf{V} \cdot \mathbf{N}\delta s.$$

Thus, *the rate of normal flux per unit area* $= \mathbf{V} \cdot \mathbf{N}$.

Obs. We can also apply this result to the case of electric or magnetic flux.

(3) Moment of a force about a point. Suppose the moment of the force **F** acting at the point P about the point A is required.

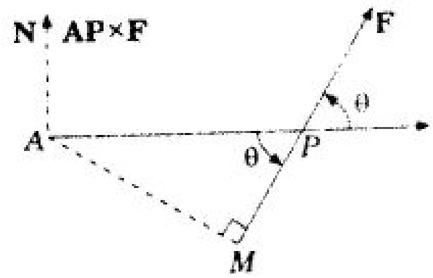


Fig. 3·27.

Draw $AM \perp$ the line of action of **F** (Fig. 3·27). If θ be the angle between $\overrightarrow{AP}$ and **F** and **N** be a unit vector $\perp$ to their plane, then

$$\overrightarrow{AP} \times \mathbf{F} = (AP \cdot F \sin\theta)\,\mathbf{N} = F(AP \sin\theta)\,\mathbf{N} = (F \cdot AM)\,\mathbf{N}$$

Clearly (*i*) the magnitude of $\overrightarrow{AP} \times \mathbf{F} = F \cdot AM$ which is the numerical measure of the moment of **F** about A.

and (*ii*) the direction of $\overrightarrow{AP} \times \mathbf{F}$ is the direction of the moment of **F** about A.

Hence the moment (or torque) of **F** about A is $\overrightarrow{AP} \times \mathbf{F}$.

Example 3·18. *Find the torque about the point* $2\mathbf{I} + \mathbf{J} - \mathbf{K}$ *of a force represented by* $4\mathbf{I} + \mathbf{K}$ *acting through the point* $\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.

Sol. Let O be the origin and P be the point, moment about which of the force $\overrightarrow{AB}$ through A, is required (Fig. 3·28).

$$\therefore \quad \overrightarrow{OP} = 2\mathbf{I} + \mathbf{J} - \mathbf{K},$$

$$\overrightarrow{OA} = \mathbf{I} - \mathbf{J} + 2\mathbf{K}, \quad \text{and} \quad \overrightarrow{AB} = 4\mathbf{I} + \mathbf{K}.$$

Then $\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = -\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$

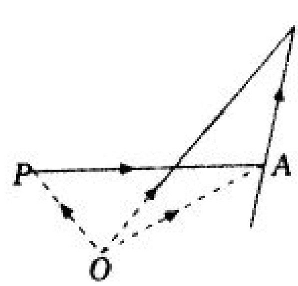


Fig. 3·28.

∴ Moment of the force $\overrightarrow{AB}$ about P

$$= \overrightarrow{PA} \times \overrightarrow{AB} = (-\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}) \times (4\mathbf{I} + \mathbf{K})$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -1 & -2 & 3 \\ 4 & 0 & 1 \end{vmatrix} = -2\mathbf{I} + 13\mathbf{J} + 8\mathbf{K}$$

∴ Magnitude of the moment $= \sqrt{(4 + 169 + 64)} = 15·4$.

(4) Moment of a force about a line.

Def. *The moment of a force* **F** *about a line* **D** *is the resolved part along* **D** *of the moment of* **F** *about any point on* **D**.

Example 3·19. *Find the moment about a line through the origin having direction of* $2\mathbf{I} + 2\mathbf{J} + \mathbf{K}$, *due to a 30 kg force acting at a point* $(-4, 2, 5)$ *in the direction of* $12\mathbf{I} - 4\mathbf{J} - 3\mathbf{K}$.

Sol. Let **D** be the given line through the origin O and **F** the force through $A(-4, 2, 5)$.

Clearly $\overrightarrow{OA} = -4\mathbf{I} + 2\mathbf{J} + 5\mathbf{K}$

and the force $\mathbf{F} = 30\left(\dfrac{12\mathbf{I} - 4\mathbf{J} - 3\mathbf{K}}{13}\right)$

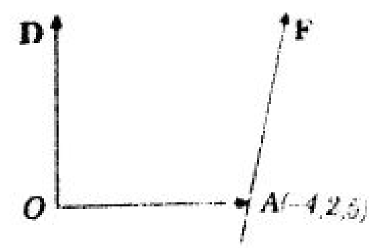


Fig. 3.29

$\therefore$ Moment of $\mathbf{F}$ about $O = \vec{OA} \times \mathbf{F}$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -4 & 2 & 5 \\ \dfrac{360}{13} & -\dfrac{120}{13} & -\dfrac{90}{13} \end{vmatrix} = \frac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K})$$

Thus the moment of $\mathbf{F}$ *about the line* $\mathbf{D}$

= resolved part of the moment of $\mathbf{F}$ about O along $\mathbf{D}$,

i.e. $\dfrac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K}) \cdot \hat{\mathbf{D}}$

$$= \frac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K}) \cdot \frac{2\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{(4 + 4 + 1)}} = \frac{20}{13}(7 \times 2 + 24 \times 2 - 4 \times 1) = 89 \ 23$$

(5) Angular velocity of a rigid body

Let a rigid body be rotating about the axis OM with angular velocity ω radians per second (Fig. 3·30). Let P be a point of the body such that $\vec{OP} = \mathbf{R}$ and $\angle MOP = \theta$. Draw $PM \perp OM$.

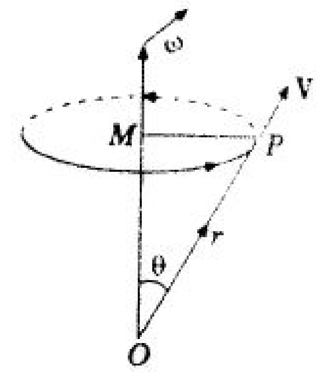


Fig. 3.30.

Now if $\mathbf{N}$ be a unit vector $\perp \vec{\omega}$ and $\mathbf{R}$ then

$$\vec{\omega} \times \mathbf{R} = \omega r \sin\theta \,.\, \mathbf{N} = \omega MP \,.\, \mathbf{N}$$
$$= (\text{speed of } \mathbf{P})\, \mathbf{N}$$
$$= \text{velocity } \mathbf{V} \text{ of } P \text{ in a direction} \perp \text{to the plane } MOP.$$

Hence $\mathbf{V} = \vec{\omega} \times \mathbf{R}$.

Example 3·20. *A rigid body is spinning with angular velocity 27 radians per second about an axis parallel to* $2\mathbf{I} + \mathbf{J} - 2\mathbf{K}$ *passing through the point* $\mathbf{I} + 3\mathbf{J} - \mathbf{K}$. *Find the velocity of the point of the body whose position vector is* $4\mathbf{I} + 8\mathbf{J} + \mathbf{K}$.

Sol. Unit vector along the direction of $\vec{\omega} = \dfrac{2\mathbf{I} + \mathbf{J} - 2\mathbf{K}}{\sqrt{(4 + 1 + 4)}} = \dfrac{1}{3}(2\mathbf{I} + \mathbf{J} - 2\mathbf{K})$

$\therefore$ Angular velocity $\vec{\omega} = \dfrac{27}{3}(2\mathbf{I} + \mathbf{J} - 2\mathbf{K}) = 9(2\mathbf{I} + \mathbf{J} - 2\mathbf{K})$

Let A be the point $\mathbf{I} + 3\mathbf{J} - \mathbf{K}$ and the point P of the body be $(4\mathbf{I} + 8\mathbf{J} + \mathbf{K})$ so that

$$\vec{AP} = (4\mathbf{I} + 8\mathbf{J} + \mathbf{K}) - (\mathbf{I} + 3\mathbf{J} - \mathbf{K}) = 3\mathbf{I} + 5\mathbf{J} + 2\mathbf{K}$$

$\therefore$ Velocity vector of $P = \mathbf{V} = \vec{\omega} \times \vec{AP} = 9(2\mathbf{I} + \mathbf{J} - 2\mathbf{K}) \times (3\mathbf{I} + 5\mathbf{J} + 2\mathbf{K})$

$$= 9\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 2 & 1 & -2 \\ 3 & 5 & 2 \end{vmatrix} = 9(12\mathbf{I} - 10\mathbf{J} + 7\mathbf{K})$$

and its magnitude $9\sqrt{(144 + 100 + 49)} = 9\sqrt{293}$.

Problems 3.4

1. A particle acted on by constant forces $4\mathbf{I} + \mathbf{J} - 3\mathbf{K}$ and $3\mathbf{I} + \mathbf{J} - \mathbf{K}$ is displaced from the point $\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$ to the point $5\mathbf{I} + 4\mathbf{J} + \mathbf{K}$. Find the total work done by the forces.
2. Forces $2\mathbf{I} - 5\mathbf{J} + 6\mathbf{K}$, $-\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ and $2\mathbf{I} + 7\mathbf{J}$ act on a particle P whose position vector is [illegible] $- 2\mathbf{K}$. Determine the work done by the forces in a displacement of the particle to the point [illegible], 1, − 3).

3. [illegible] the vector moment of the resultant of three forces acting at P about the point Q

3. Forces of magnitudes 5, 3, 1 units act in the directions $6\mathbf{I}+2\mathbf{J}+3\mathbf{K}$, $3\mathbf{I}-2\mathbf{J}+6\mathbf{K}$, $2\mathbf{I}-3\mathbf{J}-6\mathbf{K}$ respectively on a particle which is displaced from the point (2, 1, –3) to (5, –1, 1). Find the work done by the forces.
4. The point of application of the force (–2, 4, 7) is displaced from the point (3, –5, 1) to the point (5, 9, 7). But the force is suddenly halved when the point of application moves half the distance. Find the work done.
5. A force $\mathbf{F}=3\mathbf{I}+2\mathbf{J}-4\mathbf{K}$ is applied at the point (1, – 1, 2). Find the moment of the force about the point (2, – 1, 3) *(Assam, 1999)*
6. A force with components (5, – 4, 2) acts at a point P which is at a distance 3 units from the origin. If the moment of the force about origin has components (12, 8, – 14), find the co-ordinates of P.
7. Find the moment of the force $\mathbf{F}=2\mathbf{I}+2\mathbf{J}-\mathbf{K}$ acting at the point (1, – 2, 1) about z-axis.
8. A force of 10 kg acts in a direction equally inclined to the co-ordinate axes through the point (3, –2, 5). Find the magnitude of the moment of the force about a line through the origin and whose direction ratios are (2, –3, 6).
9. A rigid body is rotating at 2·5 radians per second about an axis OR, where R is the point $2\mathbf{I}-2\mathbf{J}+\mathbf{K}$ relative to O. Find the velocity of the particle of the body at the point $4\mathbf{I}+\mathbf{J}+\mathbf{K}$. (All lengths are in cm).

3·8. PRODUCTS OF THREE OR MORE VECTORS

With any three vectors **A, B, C,** we can form the products $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$, $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ and $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$. The first being the product of a scalar $\mathbf{A}\cdot\mathbf{B}$ and a vector **C**, represents a vector in the direction of **C**. The second being the scalar product of vectors $\mathbf{A}\times\mathbf{B}$ and **C**, represents a scalar and is usually called the *scalar product of three vectors.* The third being the vector product of the vectors $\mathbf{A}\times\mathbf{B}$ and **C**, represents a vector and is usually known as the *vector product of three vectors.*

The reader must, however, note that the products of the form $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$, $\mathbf{A}\times(\mathbf{B}\cdot\mathbf{C})$ and $\mathbf{A}(\mathbf{B}\times\mathbf{C})$ are meaningless.

In practical applications, we seldom come across products of more than three vectors. Such products if they occur can, in general, be reduced by using successively the expansion formula for vector triple products. As an illustration, we shall consider two products $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})$ and $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})$ of any four vectors, the former being a scalar and a latter a vector.

3·9. SCALAR PRODUCT OF THREE VECTORS

(1) Definition. *If* **A, B, C** *be any three vectors then the scalar or dot product of* $\mathbf{A}\times\mathbf{B}$ *with* **C** *is called the scalar product of the three vectors* **A, B, C** *and is written as* $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ *or* **[ABC]**.

No ambiguity can arise by omitting the brackets in $(A\times B)\cdot C$ as $A\times(B\cdot C)$ would be meaningless.

(2) Geometrical interpretation. *The product* $\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$ *represents numerically the volume of a parallelopiped having* **A, B, C** *as coterminous edges.*

Consider a parallelopiped with $\vec{OA}=\mathbf{A}$, $\vec{OB}=\mathbf{B}$, $\vec{OC}=\mathbf{C}$ as coterminous edges (Fig. 3·31).

Let V be its volume, α the area of each of the two faces parallel to the vectors **A** and **B** and p the perpendicular distance between these faces.

Then $|\mathbf{A}\times\mathbf{B}|=\alpha$ and $|\mathbf{C}|\cos\phi=p$ or $-p$ according as **A, B, C** form a right-handed or left-handed triad.

$\therefore\ \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}=|\mathbf{A}\times\mathbf{B}|\,.\,|\mathbf{C}|\cos\phi=\pm\alpha p=\pm V.$

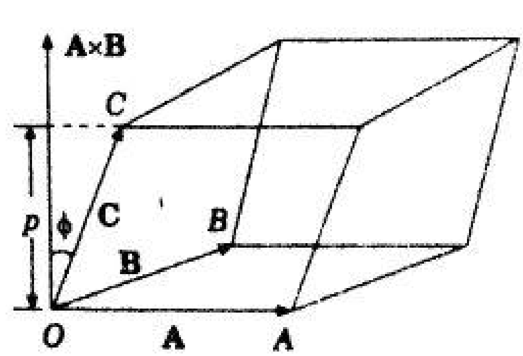


Fig. 3·31.