Closures of Relations

Lecture/Week 12

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Introduction

Lecture/Week Outline & Learning Outcomes

1. **Lesson/Week Outline:**
   1.1 Different Types of Closures.
   1.2 Paths in Directed Graphs.
   1.3 Transitive Closures and Connectivity Relation.

2. **Learning Outcomes:**
   2.1 Find the Reflexive, Symmetric, and Transitive closures of a relation.
   2.2 Define the concepts of Paths and Circuits in Directed Graphs.

- **Closure of a Relation:** If R is a relation on a set A (i.e. $R \subseteq A \times A$), then the **Closure of** R with respect to a property, P, is:
  - the new relation, S (on the set A), having the property, P;
  - the new relation, S, containing the original relation, R; and
  - the new relation, S, is a subset of every relation in $A \times A$ which contains R as well as property, P.

- If A is a set and $\triangle = \{(a,a)|a \in A\}$. We call $\triangle$ the diagonal/identity relation on A.
  Thus, the **Reflexive Closure** on a relation, R, is the new relation formed via: $r(R) = R \cup \triangle$.
- Example: Consider the relation R on the set A, such that: $R = \{(1,2),(2,4),(3,3),(4,2)\}$ and $A = \{1,2,3,4\}$. Hence, compute $r(R)$.

  Solution: $r(R) = R \cup \triangle$

  $\triangle = \{(1,1),(2,2),(4,4)\}$

  $r(R) = \{(1,2),(2,4),(3,3),(4,2)\} \cup \{(1,1),(2,2),(4,4)\}$

  $r(R) = \{(1,2),(2,4),(3,3),(4,2),(1,1),(2,2),(4,4)\}$

Closures of Relations
Symmetric Closure

- The **Symmetric Closure** of a relation, R, is the new relation formed via: $s(R) = R \cup R^{-1}$; such that $R^{-1} = \{(b,a) | (a,b) \in R\}$ is the inverse of relation R.

- Example: Consider the relation $R = \{(1,2),(1,3),(2,2),(2,4),(4,3)\}$ on the set $A = \{1,2,3,4\}$. Hence, compute $s(R)$.

  Solution: $s(R) = R \cup R^{-1}$

  $$\begin{aligned} R &= \{(1,2),(1,3),(2,2),(2,4),(4,3)\} \\ R^{-1} &= \{(2,1),(3,1),(2,2),(4,2),(3,4)\} \\ s(R) &= \{(1,2),(1,3),(2,2),(2,4),(4,3),(2,1),(3,1), \\ &\qquad (4,2),(3,4)\} \end{aligned}$$

- Constructing the **Transitive Closure** of a relation is more complicated than constructing either the Reflexive Closure and Symmetric Closure.
- This is because the **Transitive Closure** of a relation can be found by adding new ordered pairs that must be present, and *then repeating this process until no new ordered pairs are needed*.
- Representing relations via *Directed Graphs (Digraphs)* aids in constructing Transitive Closure.

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Transitive Closure - Related Terminologies

- **Path:** A *path of length*, n, from a to b in a directed graph, G, is the sequence of edges: $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ with respect to G; where $n \geq 0$, $x_0 = a$, and $x_n = b$.
  - Also, this *path of length*, n, can be denoted via: $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$.
- **Cycle/Circuit:** This is a *path of length*, $n \geq 1$, that begins and ends at the same vertex (i.e. $x_0 = x_n$).
- NOTE: A *path* in a directed graph can pass through a vertex and/or edge more than once.

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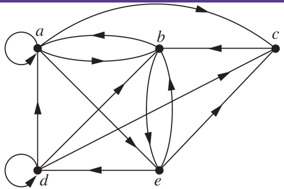
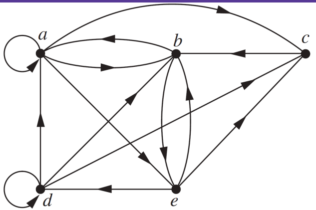
Transitive Closure - Related Terminologies



1. Which of the following are *valid paths* in the directed graph shown above?
   1.1 a, b, e, d
   1.2 a, e, c, d, b
   1.3 b, a, c, b, a, a, b
   1.4 d, c
   1.5 c, b, a

2. What are the lengths of the valid paths?

3. Which of the paths is/are *circuits*?



1. *valid/invalid paths*?
   1.1 valid path (len = 3). $a, b, e, d : (a, b), (b, e), (e, d)$.
   1.2 invalid path. $a, e, c, d, b : (a, e), (e, c), (c, d), (d, b)$. (c, d) is not an *edge*.
   1.3 valid path (len = 6). $b, a, c, b, a, a, b :$ $(b, a), (a, c), (c, b), (b, a), (a, a), (a, b)$.
   1.4 valid path (len = 1). $d, c : (d, c)$.
   1.5 valid path (len = 2). $c, b, a : (c, b), (b, a)$.

3. In (1.3) above: b, a, c, b, a, a, b is a *circuit/cycle* because it begins at vertex b and ends at vertex b.

- **Connectivity Relation,** R^***:** consists of the pairs, (a, b), such that there is a path of length, $n \geq 1$, from a to b in the relation R (*on a set* A).

$$R^* = \bigcup_{n=1}^{\infty} R^n \qquad (1)$$

- If the relation, R, is defined on a finite set, A, with the cardinality, $|A| = n$; then the **Connectivity Relation**, $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.
- The **Transitive Closure, t(R),** of a relation R is equivalent to the **Connectivity Relation**, R^*.

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Transitive Closure - Native Approach/Algorithm. Complexity = $O(n^4)$

- **Example 1:** Consider $R = \{(1,2),(2,2),(2,3),(3,3)\}$ on set $A = \{1,2,3\}$. Thus, compute Transitive Closure, $t(R)$, of R.
- **Solution:**

$$t(R) = R^* = \bigcup_{n=1}^{\infty} R^n \qquad (2)$$

$R^1 = \{(1,2),(2,2),(2,3),(3,3)\}$
$R^2 = R^1 \circ R^1 = \{(1,2),(1,3),(2,2),(2,3),(3,3)\}$
$R^3 = R^2 \circ R^1 = \{(1,2),(1,3),(2,2),(2,3),(3,3)\}$
$R^n \subseteq R^2$ where $n \geq 2$
$t(R) = R^1 \cup \ldots \cup R^n = \{(1,2),(1,3),(2,2),(2,3),(3,3)\}$

- **Example 2:** $R = \{(1,1),(1,3),(2,2),(3,1),(3,2)\}$ on set $A = \{1,2,3\}$. Thus, compute Transitive Closure, $t(R)$, of R.

- **Solution:**

$$t(R) = R^* = \bigcup_{n=1}^{\infty} R^n \tag{3}$$

$R^1 = \{(1,1),(1,3),(2,2),(3,1),(3,2)\}$

$R^2 = R^1 \circ R^1 = \{(1,1),(1,3),(1,2),(2,2),(3,1),(3,3),(3,2)\}$

$R^3 = R^2 \circ R^1 = \{(1,1),(1,3),(1,2),(2,2),(3,1),(3,3),(3,2)\}$

$R^n \subseteq R^2$ where $n \geq 2$

$t(R) = R^1 \cup \ldots \cup R^n = \{(1,1),(1,3),(1,2),(2,2),(3,1),(3,3),$
$(3,2)\}$

Closures of Relations

Transitive Closure - Warshall's Approach/Algorithm.

Complexity = $O(n^3)$

- **Example 3:** $R = \{(1,2),(2,1),(2,3),(3,4),(4,1)\}$ on set $A = \{1,2,3,4\}$. Thus, compute Transitive Closure, $t(R)$, of R using Warshall's algorithm.

- **Soln:** $W_{ij}^{[k]} = f(W^{[k-1]}) = \begin{cases} W_{ij}^{[k-1]} = 1 \\ W_{ik}^{[k-1]} \wedge W_{kj}^{[k-1]} = 1 \end{cases}$

$$W^{[0]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$W^{[1]} = f(W^{[0]}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

- **Soln:**

$$W^{[2]} = f(W^{[1]}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$W^{[3]} = f(W^{[2]}) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$W^{[4]} = f(W^{[3]}) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus, matrix, $W^{[4]}$, is the **Transitive Closure** of R.

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Transitive Closure - Warshall's Approach/Algorithm. Complexity = $O(n^3)$

- **Example 4:** $R = \{(1,2),(2,3),(3,4)\}$ on set $A = \{1,2,3,4\}$. Thus, compute Transitive Closure, $t(R)$, of R using Warshall's algorithm.

- **Soln:** $W_{ij}^{[k]} = f(W^{[k-1]}) = \begin{cases} W_{ij}^{[k-1]} = 1 \\ W_{ik}^{[k-1]} \wedge W_{kj}^{[k-1]} = 1 \end{cases}$

$$W^{[0]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W^{[1]} = f(W^{[0]}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Transitive Closure - Warshall's Approach/Algorithm. Complexity = $O(n^3)$

- **Soln:**

$$W^{[2]} = f(W^{[1]}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W^{[3]} = f(W^{[2]}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W^{[4]} = f(W^{[3]}) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, matrix, $W^{[4]}$, is the **Transitive Closure** of R.

Closures of Relations
Class/Game Activity

Navigate to: `www.kahoot.it`

Game PIN: available in-class

Questions?

&

Answers!