HOMEWORK #3: Network Thinking and Linear Algebra and All That ID number and Affiliation: 20225640 | VML Name: Vanessa Tan	Score /

1. **Networks Aplenty! (25 pts)**
   1.1. Name three different types of ties between you and people you know, each belonging to the social, the biological, and the physical types? (15 pts)

   (a) Social tie: Friendship

   (b) Biological tie: Family

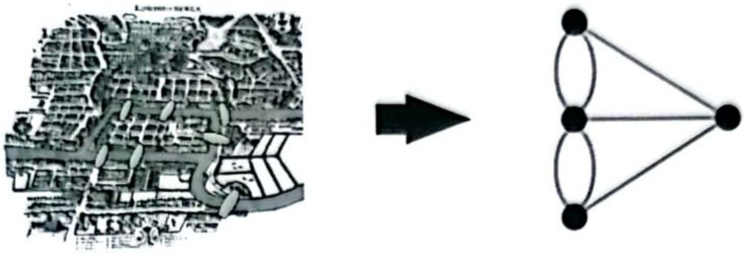
   (c) Physical tie: Roommates

   1.2. A network with multiple edge types is called a *multiplex network*. With the three types of edges you just named, how many different types of connections can exist between two people? (10 pts)

There can be 6 types of connections between two people.
These connections are as follows:
- social → friendship
- biological → family
- physical → roommates
- social - biological → sibling friends
- social - physical → your friends are your roommates
- biological - physical → your siblings are your roommates

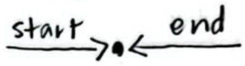
2. **Let's Take a Walk! (50 pts)** Königsberg, now a Russian city named Kaliningrad, was a beautiful Prussian (German) city where a celebrated mathematician Leonhardt Euler lived. In Euler's **Königsberg bridge problem**, the goal is to find a path that crosses each of the seven bridges of Königsberg once. Euler made this into a network problem by replacing each land mass with a vertex, and each bridge with an edge. Your goal is to find out if such a path exists. Remember that since you're allowed to cross a bridge only one, when you enter and exit a land mass, you must take separate

bridges. Now solve this problem by answering the following short questions in order.



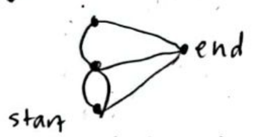
2.1. Assume that such a path exists. If the starting node and the ending node are the same node, is the degree of that node an odd number or an even number? (10 pts)

If the starting node and the ending node are the same node, the degree of that node should be even.



2.2. Assume that such a path exists. If the starting node and the ending node are different nodes, are the degrees of those nodes odd or even? (10 pts)

If the starting and ending node are different nodes, the degree of those nodes are odd.



Start → degree 3

End → degree 3

2.3. Assume that a solution exists. Is it possible for a node that is neither the starting nor the ending node to have an odd degree? (10 pts)

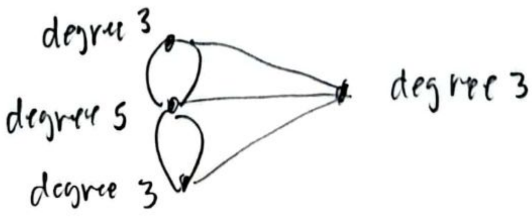
It is not possible for a node that is neither the starting nor the ending node to have an odd degree.

2.4. What is the maximum number of odd-degree nodes if an Eulerian path may exist? (10 pts)

The maximum number of odd-degree nodes for an Eulerian path is 2, the starting and ending nodes.

2.5. Why is there no solution in the Königsberg bridge problem? (10 pts)

The Königsberg bridge problem has no solution because it is not Eulerian. All of the nodes are odd degrees.

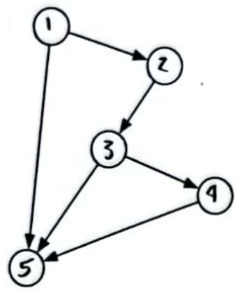


3. Adjacency Matrix and Directed Networks (25 pts). Consider a simple directed network whose adjacency matrix is $\mathbf{A}$.

3.1. What are the diagonal elements of $\mathbf{A}^2$? What would it mean if all those elements were zero? (10 pts)

the diagonal elements of A are the self-edges. If all elements are zero, there are no self-edges.

3.2. Consider the following network:



3.2.1. What is the longest path in the network? (5 pts)

The longest path is $1 \to 2 \to 3 \to 4 \to 5$.

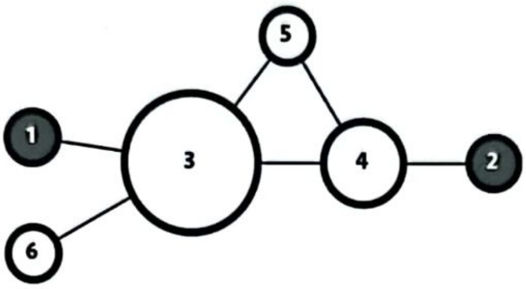
3.2.2. How do you show that this network is *acyclic*, meaning that there is no path leading from one node to itself, using the adjacency matrix? (10 pts)

(1,2) (1,5) (2,3) (3,4) (3,5) (4,5)

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The network is acyclic when the adjacency matrix is an upper triangular matrix and have zero diagonal elements

3. Lifting Yourself Up By Your Own Bootstrap (40 pts). The definition of eigenvector centrality appears quite circular — "you are only as important as your friend, but your friend is only as important as you!". What happens if one tries to give some people initial preferential treatment? We will now see that whatever the origin of centrality, the network (i.e. connection patterns) takes care of mixing them up to equilibrium in the end. Consider the following network:



3.1. What is the adjacency matrix $\mathbf{A}$? (5 pts)

(1,3)(3,1)(6,3)(3,6)(3,5)(5,3)(3,4)(4,3)(5,4)(4,5)(4,2)(2,4)

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

3.2. Assume that we 'start' with a initial original centrality $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. What is the first transformation via the matrix, i.e. $\vec{x}(1) = \mathbf{A} \cdot \vec{x}(0)$? (5 pts)

$$\vec{x}(1) = \underset{\sim}{A} \cdot \vec{x}(0) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

3.3. In the next table, fill in the values of the next iterations, i.e. $\vec{x}(2) = \mathbf{A} \cdot \vec{x}(1)$, $\vec{x}(3) = \mathbf{A} \cdot \vec{x}(2)$, ... (10 pts)

iteration step (t)	*node1*	*node2*	*node3*	*node4*	*node5*	*node6*
0	*1*	*1*	*0*	*0*	*0*	*0*

iteration step (t)	node1	node2	node3	node4	node5	node6
1	0	0	1	1	0	0
2	1	1	1	1	2	1
3	1	1	5	4	2	1
4	5	4	8	8	9	5
5	8	8	27	21	16	8
6	27	21	53	51	48	27
7	53	51	153	122	104	53
8	153	122	332	308	275	153
9	332	308	889	729	640	332
10	889	729	2033	1837	1618	889

3.4. Start from $\vec{x}(0) = (4,4,1,2,3,4)$. In the next table, fill in the values. (10 pts)

iteration step (t)	node1	node2	node3	node4	node5	node6
0	4	4	1	2	3	4
1	1	2	13	8	3	1
2	13	8	13	18	21	13
3	13	18	65	42	31	13
4	65	42	99	114	107	65
5	99	114	351	248	213	99
6	351	248	659	678	599	351
7	659	678	1979	1506	1337	659
8	1979	1506	4161	3994	3485	1979
9	4161	3994	11437	9152	8155	4161
10	11437	9152	25629	23586	20589	11437

3.5. We see that the elements of the vector $\vec{x}(t)$ will keep growing *t* increases. However, we learned that due to the definition of the eigenvector of a matrix $\mathbf{A}\vec{e} = \lambda\vec{e}$, the eigenvector multiplied by a constant number γ is also an eigenvector, as $\mathbf{A} \cdot (\gamma\vec{e}) = \gamma(\mathbf{A} \cdot \vec{e}) = \gamma(\lambda\vec{e}) = \lambda(\gamma\vec{e})$. A popular way to choose γ is such that the length of the vector is 1 (a vector whose length is 1 is called a *unit vector*). The most common definition of the length of a vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ is $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$. Therefore,

$$\vec{x}' = \frac{\vec{x}}{|\vec{x}|} = \frac{1}{\sqrt{\sum_{i=1}^{n} x_i^2}}(x_1, x_2, \cdots, x_n)$$

is a unit vector. Find the unit vectors of $\vec{x}(0)$ and $\vec{x}(10)$'s from (c) and (d), up to three significant digits (5 pts):

	node1	**node2**	**node3**	**node4**	**node5**	**node6**
$\vec{x}(0)$ (c)	0.707	0.707	0	0	0	0
$\vec{x}(0)$ (d)	0.508	0.508	0.127	0.254	0.381	0.508
$\vec{x}(10)$ (c)	0.254	0.208	0.581	0.525	0.462	0.254
$\vec{x}(10)$ (d)	0.257	0.206	0.576	0.530	0.462	0.257

3.6. The actual unit eigenvector of $\mathbf{A}$ is as follows, and indeed the node 1 has a higher centrality than node 2:

	node1	**node2**	**node3**	**node4**	**node5**	**node6**
	0.244778	0.211823	0.598675	0.518075	0.456601	0.244778

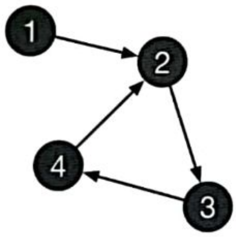
People often compare two vectors $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ by way of the **Pearson Correlation Coefficient (P.C.C.)**

$$\rho(\vec{u}, \vec{v}) \equiv \frac{\sum_{i=1}^{n}(u_i - \bar{u})(v_i - \bar{v})}{\sqrt{\sum_{i=1}^{n}(u_i - \bar{u})}\sqrt{\sum_{i=1}^{n}(v_i - \bar{v})}},$$ where $\bar{x} \equiv \frac{1}{n}\sum_{i=1}^{n} x_i$ is the average

of the elements of the vector. $-1 \le \rho(\vec{u},\vec{v}) \le 1$, and a $\rho(\vec{u},\vec{v})$ that is closer to 1 means that $\vec{u}$ and $\vec{v}$ are more similar. Show that regardless of the initial trial vectors $\vec{x}(0)$, the iterative scheme does converge to the eigenvector by filling in the following table (5 pts):

	initial correlation	***last correlation***
initial vector from (a)	−0.704	0.998
initial vector from (b)	−0.996	0.997

4. **Eigenvalue vs Katz Centralities. (50 pts)** The difference between the Katz centrality and the Eigenvalue centrality is the existence of a one-time "free" centrality for each node that lets us avert the pesky problem of a node with no in-component. Consider the following network:



a) Write down the adjacency matrix $\mathbf{A}$ for this network (10 pts):

(1,2) (2,3) (3,4) (4,2)

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

b) What is the eigenvector centrality of node 1? (5 pts)

$Ax = \lambda x$ node1 : 0.282

c) What are the eigenvector centralities of nodes 2, 3, and 4? (10 pts)

node 2 : 0.612

node 3 : 0.523

node 4 : 0.523

d) The simplest Katz scheme gives a free centrality of 1 to each node, so that the Katz centrality $\vec{x}$ (not to be confused with the Eigenvector centrality) is defined as $\vec{x} = \alpha \mathbf{A} \vec{x} + \vec{1}$. It can be further manipulated to solve for $\vec{x}$ as follows:

$$\mathbf{I} \cdot \vec{x} - \alpha \mathbf{A} \cdot \vec{x} = \vec{1} \quad \cdots\cdots \quad (1)$$

$$(\mathbf{I} - \alpha \mathbf{A}) \cdot \vec{x} = \vec{1} \quad \cdots\cdots \quad (2)$$

$$\therefore \vec{x} = (\mathbf{I} - \alpha \mathbf{A})^{-1} \cdot \vec{1}.$$

We see here that α is no longer the eigenvalue of $\mathbf{A}$ — rather, a *free* parameter that whose value is up to us. Complete the following table: (10 pts)

Legend: unnormalized
normalized *

α	Katz Centrality			
	1	2	3	4
0	1 0.5 *	1 0.5 *	1 0.5 *	1 0.5 *
0.5	-6 0.263 *	-14 0.614 *	-12 0.526 *	-12 0.526 *
0.85	-0.198 0.084 *	-1.409 0.601 *	-1.318 0.562 *	-1.318 0.562 *
1	0 0 *	-1 0.577 *	-1 0.577 *	-1 0.577 *
2	0.6 -0.567 *	-0.2 0.189 *	-0.6 0.567 *	-0.6 0.567 *

e) You must have seen that there are α values that appear to make no sense. Why do you think that is, given the meaning of α? (5 pts)

when $\alpha = 0$: $x = (I - \alpha A)^{-1} \cdot \vec{1}$

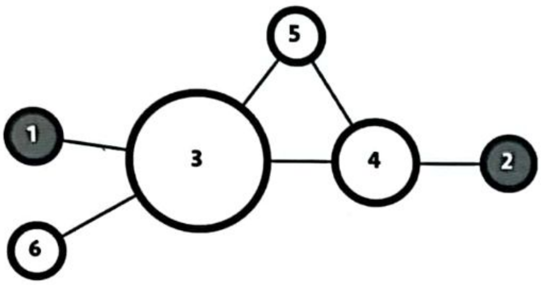
$x = I \implies$ all centralities are 1

$\alpha \geq \frac{1}{\lambda}$: note λ is the largest eigenvalue

the centralities diverge and will not be comparable to the eigenvector centrality

(e.g. $\alpha \geq 1$)

4. The Three Centralities (40 pts). Consider the following network:



$$\underset{\sim}{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

4.1. Give the degree centralities $\vec{k}$.

node 1 = 1
node 2 = 1
node 3 = 4
node 4 = 3
node 5 = 2
node 6 = 1

$$\vec{k} = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

4.2. Calculate the eigenvector centralities $\vec{e}$. Remember, the eigenvector centralities are positive (even if the computer gives you a negative number), and correspond to the leading (largest) eigenvalue.

$$\vec{e} = \begin{bmatrix} 0.245 \\ 0.212 \\ 0.599 \\ 0.518 \\ 0.457 \\ 0.245 \end{bmatrix}$$

4.3. Calculate the PageRank™ centralities $\vec{p}$ (the undirected version). Use the original Google $\alpha = 0.85$. Remember $\vec{p} = (\mathbf{I} - \alpha \mathbf{A} \mathbf{K}^{-1})^{-1} \cdot \vec{1}$, where $\mathbf{K}^{-1}$ is the diagonal matrix $1/k_i$ on the diagonal and the rest of the elements are 0.

$$\vec{p} = \begin{bmatrix} 0.093 \\ 0.093 \\ 0.320 \\ 0.241 \\ 0.161 \\ 0.093 \end{bmatrix}$$

4.4. What are the pairwise Pearson correlations between $\vec{k}$, $\vec{e}$, and $\vec{p}$?

$\vec{k}$ 와 $\vec{e}$ = 0.966 $\quad$ $\vec{k}$ 와 $\vec{p}$ = 0.999

$\vec{e}$ 와 $\vec{p}$ = 0.958