Random slopes and the **slopes-as-outcomes** model

A note on nomenclature

- Multilevel models go by a number of different names:
  - Random coefficient models
  - Mixed models
  - Hierarchical linear models
- Additionally, the same model is often referred to in multiple different ways
  - e.g., a slopes-as-outcomes model vs. a model with a cross-level interaction
  - We will try to be as general as possible but if ever you are confused, **please just ask**!

Models we have learned

- A model with an effect of a variable at **Level 1**, which is allowed to vary over **Level 2** units
  - The **random intercept** model
- A model with an effect of a variable at **Level 1**, which is allowed to vary over **Level 2** units, along with an effect of a variable at **Level 2**
  - The **intercepts-as-outcomes** model
  - Note that the **Level 2** variable can be a group average of the **Level 1** variable, which we have centered

Random intercept model

Level 1

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij} \qquad r_{ij} \sim N\left(0, \sigma^2\right)$$

Level 2

$$\beta_{0j} = \gamma_{00} + u_{0j} \qquad u_{0j} \sim N\left(0, \tau_{00}\right)$$

$$\beta_{1j} = \gamma_{10}$$

β_{0j} is the predicted value of y_{ij} for a subject with a value of 0 on x_{ij}, given that they are a member of cluster j.

β_{1j} is the predicted increase in y_{ij} associated with a one-unit shift in x_{ij}, given that they are a member of cluster j.

Random intercept model

Reduced-form equation

$$y_{ij} = \underbrace{\gamma_{00} + \gamma_{10} x_{ij}}_{\text{fixed}} + \underbrace{u_{0j} + r_{ij}}_{\text{random}}$$

$$u_{0j} \sim N\left(0, \tau_{00}\right)$$

$$r_{ij} \sim N\left(0, \sigma^2\right)$$

Toy example: TV and math scores in the ECLS-K dataset

- We are have children nested within schools, and we are interested in the effect of **TV watching** on students' **math performance**. We also want to control for **school-level poverty**, which is indexed by the percentage of students qualifying for free and reduced lunch.
- What is Level 1 here? Which variables are at Level 1?
- What is Level 2 here? Which variables are at Level 2?

Random intercept model

Level 1

$$MathScore_{ij} = \beta_{0j} + \beta_{1j} HoursTV_{ij} + r_{ij} \quad r_{ij} \sim N\left(0, \sigma^2\right)$$

Level 2

$$\beta_{0j} = \gamma_{00} + u_{0j} \qquad u_{0j} \sim N\left(0, \tau_{00}\right)$$

$$\beta_{1j} = \gamma_{10}$$

β_{0j} is the predicted math score for a child who watches no TV, given that they are a student at school j.

β_{1j} is the effect of hours of TV watched on math score for school j. **Note that it is the same for all schools here.**

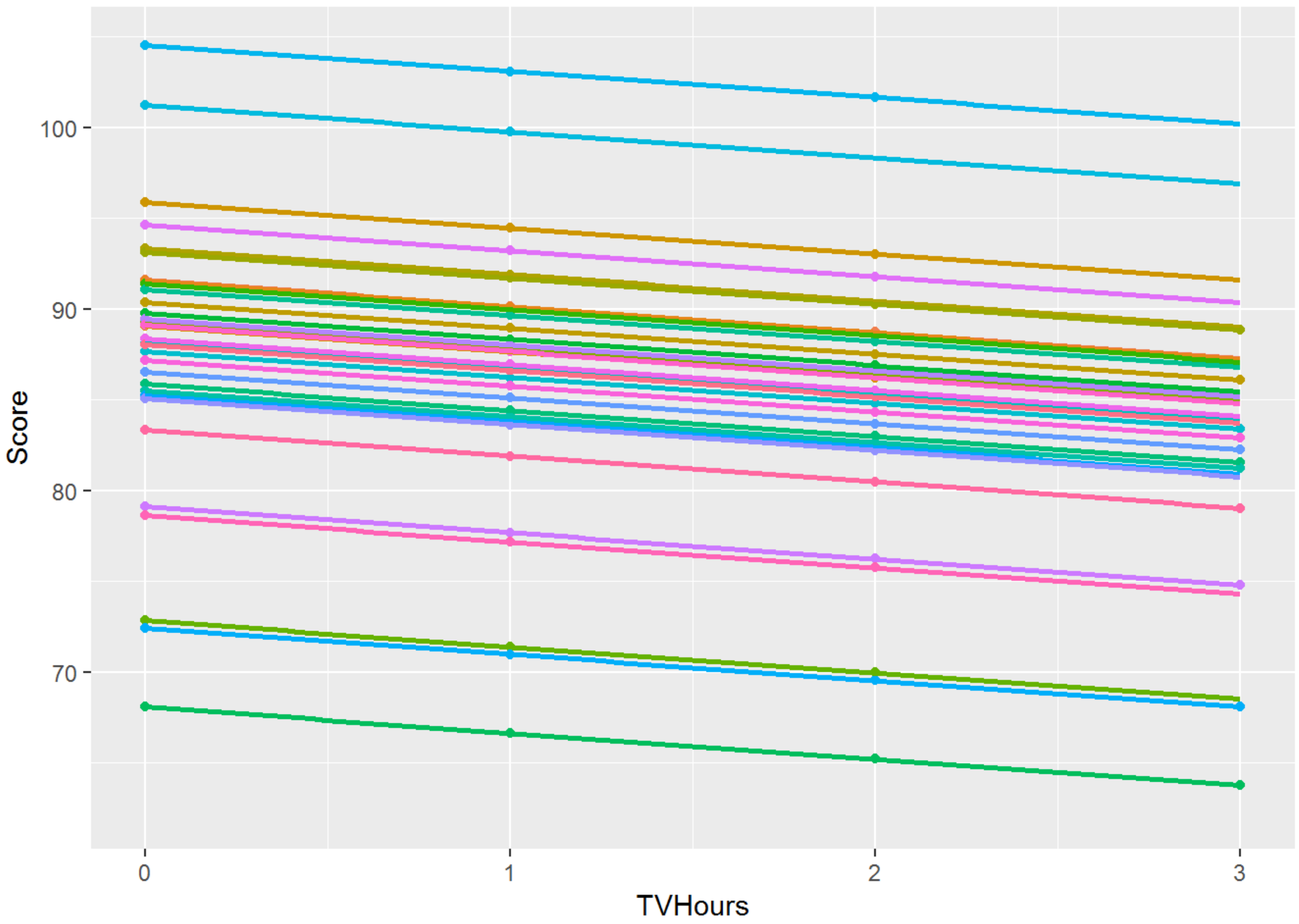
Random intercept model

Reduced-form equation

$$MathScore_{ij} = \underbrace{\gamma_{00} + \gamma_{10} HoursTV_{ij}}_{\text{fixed}} + \underbrace{u_{0j} + r_{ij}}_{\text{random}}$$

$$u_{0j} \sim N\left(0, \tau_{00}\right)$$

$$r_{ij} \sim N\left(0, \sigma^2\right)$$



100
90
80
70
Score
0
1
2
3
TVHours

Intercepts-as-outcomes model

Level 1

$$MathScore_{ij} = \beta_{0j} + \beta_{1j}HoursTV_{ij} + r_{ij} \quad r_{ij} \sim N\left(0, \sigma^2\right)$$

Level 2

$$\beta_{0j} = \gamma_{00} + \gamma_{01}PctFRL_j + u_{0j} \qquad u_{0j} \sim N\left(0, \tau_{00}\right)$$

$$\beta_{1j} = \gamma_{10}$$

Here γ_{01} conveys the effect of $PctFRL_j$ (the percentage of students qualifying for free or reduced lunch at school j) on the overall predicted math score for school j.

Intercepts-as-outcomes model

Reduced-form equation

$$MathScore_{ij} = \underbrace{\gamma_{00} + \gamma_{01} PctFRL_j + \gamma_{10} HoursTV_{ij}}_{\text{fixed}} + \underbrace{u_{0j} + r_{ij}}_{\text{random}}$$

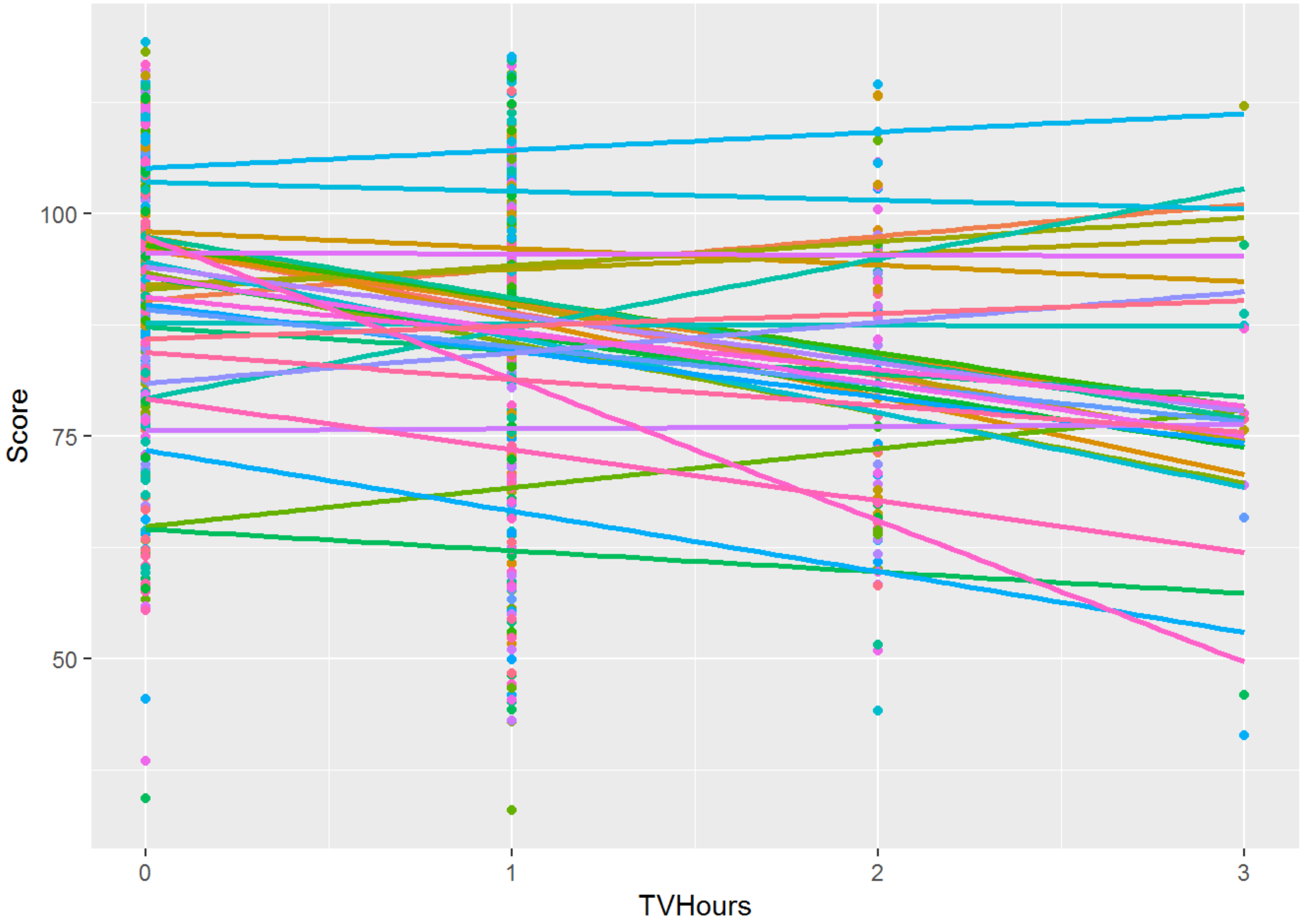
Note that even though $PctFRL_j$ is a school-level variable and $HoursTV_i$ is a child-level variable, both are fixed effects.

$$u_{0j} \sim N\left(0, \tau_{00}\right)$$

$$r_{ij} \sim N\left(0, \sigma^2\right)$$

Is this model sufficient?

- Depending on our questions, it may not be substantively interesting.
  - **Right now we can ask:**
    - Does the percentage of students with free and reduced lunch at a student's school influence their predicted math score?
  - **What if we wanted to ask**:
    - Is there variation among the schools in terms of how strongly TV watching impacts math performance?
    - Does TV watching differentially impact students based on the percentage of students at their school receiving free and reduced lunch?
- It may also violate the assumption of **independence of errors**.



Score
100
75
50
TVHours
0
1
2
3

Random slopes model

Level 1

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij} \qquad r_{ij} \sim N\left(0, \sigma^2\right)$$

Level 2

$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$

$$\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N \begin{bmatrix} \tau_{00} & \\ \tau_{01} & \tau_{11} \end{bmatrix}$$

Note that we have a **random effect** on β_{ij}, which has a variance τ_{11} and a covariance with β_{0j}.

Random slopes model

Level 1

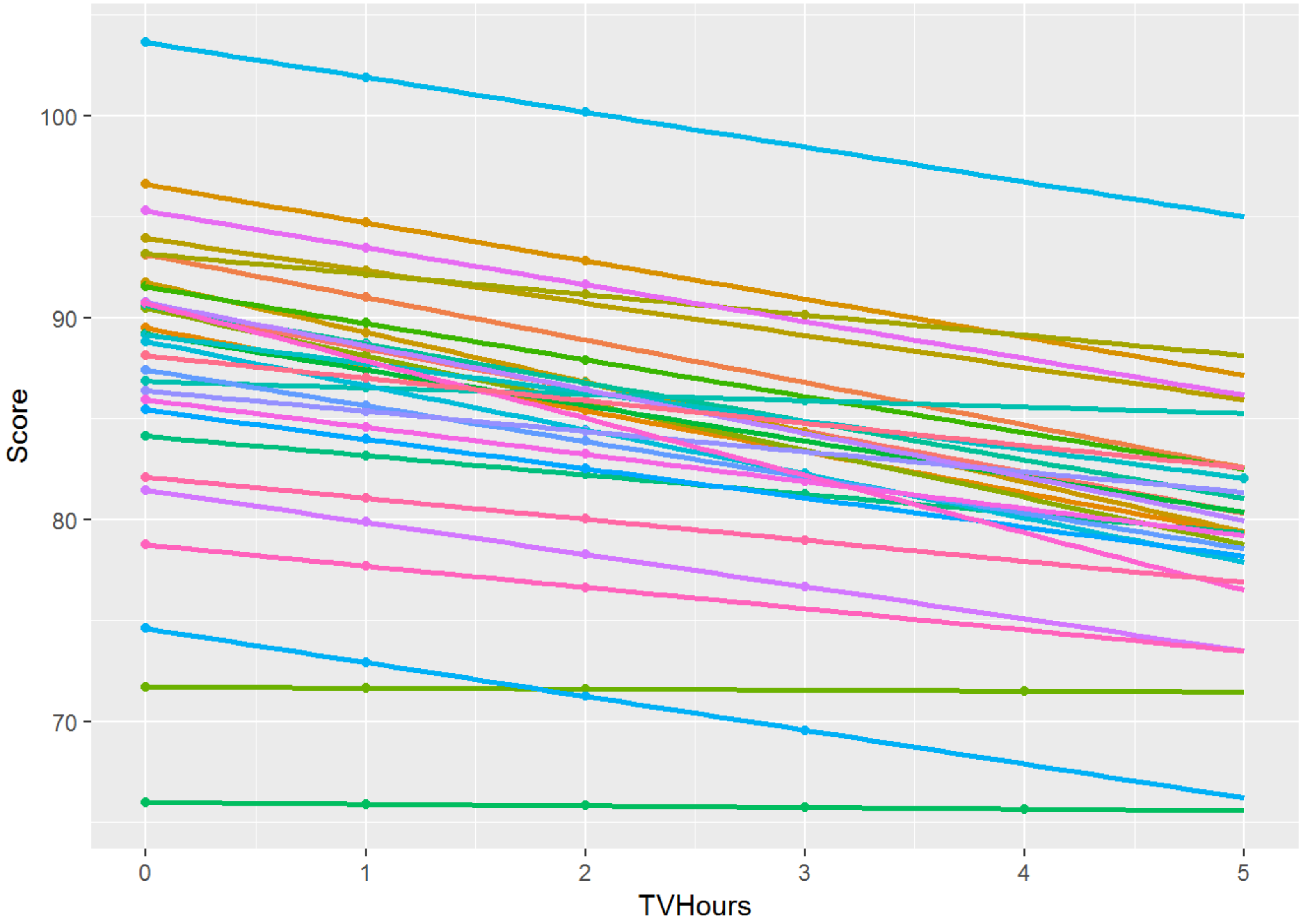
$$MathScore_{ij} = \beta_{0j} + \beta_{1j}HoursTV_{ij} + r_{ij} \quad r_{ij} \sim N\left(0, \sigma^2\right)$$

Level 2

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + u_{1j}$$

$$\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N \begin{bmatrix} \tau_{00} & \\ \tau_{01} & \tau_{11} \end{bmatrix}$$


Score
100
90
80
70
0
1
2
3
4
5
TVHours

Slopes-as-outcomes model

Level 1

$$MathScore_{ij} = \beta_{0j} + \beta_{1j} HoursTV_{ij} + r_{ij} \quad r_{ij} \sim N\left(0, \sigma^2\right)$$

Level 2

$$\beta_{0j} = \gamma_{00} + \gamma_{01} PctFRL_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11} PctFRL_j + u_{1j}$$

$$\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N \begin{bmatrix} \tau_{00} & \\ \tau_{01} & \tau_{11} \end{bmatrix}$$

Here γ_{01} conveys the effect of $PctFRL_j$ (the percentage of students qualifying for free or reduced lunch at school j) on the overall predicted math score for school j, and γ_{01} conveys the effect of $PctFRL_j$ on the effect of $HoursTV_i$.

Slopes-as-outcomes model

Reduced-form equation

$$MathScore_{ij} = \underbrace{\gamma_{00} + \gamma_{01} PctFRL_j}_{\text{fixed}} +$$

$$\underbrace{\left(\gamma_{10} + \gamma_{11} PctFRL_j\right) HoursTV_{ij}}_{\text{fixed}} +$$

$$\underbrace{u_{0j} + u_{1j} HoursTV_{ij} + r_{ij}}_{\text{random}}$$

$$r_{ij} \sim N\left(0, \sigma^2\right)$$

$$\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N \begin{bmatrix} \tau_{00} & \\ \tau_{01} & \tau_{11} \end{bmatrix}$$

A much more interesting example

- "motivation.sas7bdat"
- Students in grades 7 through 9, nested within 47 classrooms
  - **classid** (unique classroom identifier)
  - **studentid** (unique student identifier)
  - **goalstrct** (classroom-level variable indicating the extent to which the classroom emphasizes performance and demonstrating ability)
  - **sex** (0=female; 1=male)
  - **relperf** (student-level variable indicating the extent to which a student is focused on his or her relative performance)
  - **intrinsic** (a measure of the student's intrinsic motivation)

The Joint Influence of Personal Achievement Goals and Classroom Goal Structures on Achievement-Relevant Outcomes

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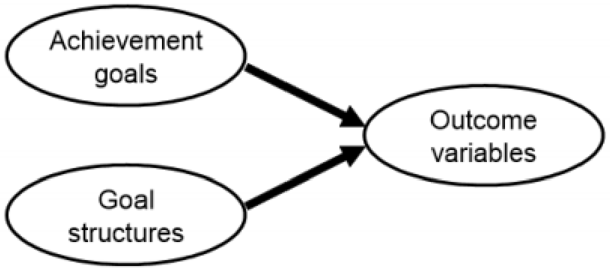
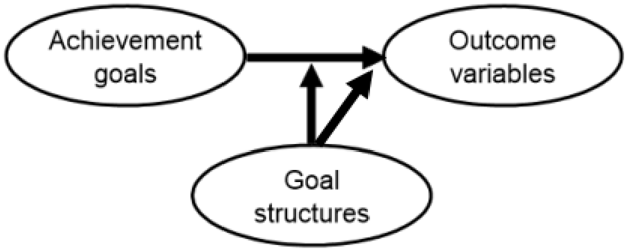
A much more interesting example

Current Model: Main Effects Only



Alternative Model: Moderation Effect



<u>Unique effects hypothesis</u>:
Student achievement goals and classroom goal structure are independently associated with students' intrinsic motivation

<u>Goal match hypothesis</u>:
Students' intrinsic motivation is strongest when classroom goal structure <u>matches</u> their own performance goals

A much more interesting example

- We will do a number of things to test these hypotheses during this class…
  - Make centering decisions about independent variables
  - Use exploratory data analyses to understand the relationships between relative performance and intrinsic motivation
  - Fit a model with random intercepts
  - Fit a model with random slopes, testing the effect of goal structure on the relationship between relative performance and intrinsic motivation
- …and during next week's class
  - Probe interactions between goal structure and relative performance

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