

Evaluation of Certain Euler-Style Sums

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1 Preliminaries

The Harmonic numbers are defined as follows:

$$H_n = \sum_{k=1}^n \frac{1}{k}$$

We note the following generating function for the Harmonic numbers:

$$\begin{aligned} -\frac{\ln(1-x)}{1-x} &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^{k+n+1}}{k+1} \\ &= \sum_{m=0}^{\infty} x^{m+1} \sum_{k=0}^m \frac{1}{k+1} \\ &= \sum_{m=0}^{\infty} x^{m+1} H_{m+1} \\ &= \sum_{m=1}^{\infty} x^m H_m \end{aligned}$$

The following Euler sum is well-known.

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3)$$

A solution method is as follows. We first note that:

$$\begin{aligned}
\int_0^1 \frac{\ln^2(1-x)}{x} dx &= 2 \int_0^1 \frac{1}{x} \cdot \sum_{k=1}^{\infty} \frac{x^{k+1} H_k}{k+1} dx \\
&= 2 \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} \\
&= 2 \sum_{k=1}^{\infty} \frac{H_{k+1}}{(k+1)^2} - 2 \sum_{k=1}^{\infty} \frac{1}{(k+1)^3} \\
&= 2 \sum_{k=1}^{\infty} \frac{H_k}{k^2} - 2\zeta(3)
\end{aligned}$$

We further note that:

$$\begin{aligned}
\int_0^1 \frac{\ln^2(1-x)}{x} dx &= \int_0^1 \frac{\ln^2(x)}{1-x} dx \\
&= \sum_{k=0}^{\infty} \int_0^1 x^k \ln^2(x) dx \\
&= \sum_{k=0}^{\infty} \frac{2}{(k+1)^3} \\
&= 2\zeta(3)
\end{aligned}$$

Comparing our answers, we ascertain that:

$$\boxed{\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3)}$$

2 Euler Sum of Order 2

Let us begin by defining the Euler sum of order n as follows (this is non-standard notation, but I shall stick with it for the purposes of this paper):

$$\boxed{E(n) = \sum_{k=1}^{\infty} \frac{H_{nk}}{k^2}}$$

Earlier we found that $E(1) = 2\zeta(3)$. Now, we shall attempt to evaluate $E(2)$.

2.1 A Lemma

We now note that:

$$\begin{aligned}
-\ln(1-x)\ln(1+x) &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \\
&= \sum_{m=0}^{\infty} x^{m+2} \cdot \sum_{n=0}^m \frac{(-1)^n}{(n+1)(m-n+1)} \\
&= \sum_{m=0}^{\infty} \frac{x^{m+2}}{m+2} \cdot \sum_{n=0}^m (-1)^n \left(\frac{1}{n+1} + \frac{1}{m-n+1} \right) \\
&= \sum_{m=0}^{\infty} \frac{x^{m+2}}{m+2} \cdot \sum_{n=0}^m \frac{(-1)^n + (-1)^{m-n}}{n+1} \\
&= \sum_{m=0}^{\infty} \frac{x^{2m+2}}{m+1} \cdot \sum_{n=0}^{2m} \frac{(-1)^n}{n+1} \\
&= \sum_{m=0}^{\infty} \frac{x^{2m+2}}{m+1} \cdot (H_{2m+1} - H_m) \\
&= \sum_{m=1}^{\infty} \frac{x^{2m}}{m} \cdot (H_{2m-1} - H_{m-1}) \\
&= \sum_{m=1}^{\infty} \frac{x^{2m}}{m} \cdot \left(H_{2m} - \frac{1}{2m} - H_m + \frac{1}{m} \right) \\
&= \boxed{\sum_{m=1}^{\infty} x^{2m} \cdot \frac{H_{2m} - H_m}{m} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{x^{2m}}{m^2}}
\end{aligned}$$

As a clear consequence of the above lemma:

$$\begin{aligned}
-\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x} dx &= \frac{1}{2} \sum_{m=1}^{\infty} \frac{H_{2m}}{m^2} - \frac{1}{2} \sum_{m=1}^{\infty} \frac{H_m}{m^2} + \frac{1}{4} \sum_{m=1}^{\infty} \frac{1}{m^3} \\
&= \frac{1}{2} E(2) - \frac{3}{4} \zeta(3)
\end{aligned}$$

2.2 Alternate Approach to the Integral

Let us now define and evaluate three relatively straightforward integrals - I_1 , I_2 , and I_3 :

$$\begin{aligned}
I_1 &= \int_0^1 \frac{\ln^2(1-x^2)}{x} dx \\
&= \frac{1}{2} \cdot \int_0^1 \frac{\ln^2(1-x)}{x} dx \\
&= \zeta(3) \quad (\text{From above.})
\end{aligned}$$

I_2 is practically the same thing:

$$\begin{aligned} I_2 &= \int_0^1 \frac{\ln^2(1-x)}{x} dx \\ &= 2\zeta(3) \quad (\text{From above.}) \end{aligned}$$

Whereas for I_3 , we proceed as follows:

$$\begin{aligned} I_3 &= \int_0^1 \frac{\ln^2(1+x)}{x} dx \\ &= \int_1^2 \frac{\ln^2(u)}{u-1} du \\ &= \int_1^2 \frac{u^{-1} \ln^2(u)}{1-u^{-1}} du \\ &= \sum_{k=0}^{\infty} \int_1^2 u^{-(n+1)} \ln^2(u) du \\ &= \frac{\ln^3(2)}{3} + \sum_{k=1}^{\infty} \left(\left[-\frac{u^{-n}}{n} \cdot \ln^2(u) \right]_1^2 + \frac{2}{n} \int_1^2 u^{-(n+1)} \ln(u) du \right) \\ &= \frac{\ln^3(2)}{3} + \sum_{k=1}^{\infty} \left(-\frac{\ln^2(2)}{n \cdot 2^n} + \frac{2}{n} \left(\left[-\frac{u^{-n}}{n} \cdot \ln(u) \right]_1^2 + \frac{1}{n} \int_1^2 u^{-(n+1)} du \right) \right) \\ &= \frac{\ln^3(2)}{3} + \sum_{k=1}^{\infty} \left(-\frac{\ln^2(2)}{n \cdot 2^n} + \frac{2}{n} \left(-\frac{\ln(2)}{n \cdot 2^n} - \frac{1}{n^2 \cdot 2^n} + \frac{1}{n^2} \right) \right) \\ &= \frac{\ln^3(2)}{3} - \ln^3(2) - 2\ln(2)\text{Li}_2\left(\frac{1}{2}\right) - 2\text{Li}_3\left(\frac{1}{2}\right) + 2\zeta(3) \\ &= \frac{1}{4}\zeta(3) \end{aligned}$$

(Where in the last line of the evaluation of I_3 , we have quoted literature values for the dilogarithm and trilogarithm of half.)

2.3 Conclusion

We now realize that:

$$-\ln(1-x)\ln(1+x) = \frac{1}{2}\ln^2(1-x) + \frac{1}{2}\ln^2(1+x) - \frac{1}{2}\ln^2(1-x^2)$$

As a result, we have:

$$-\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x} dx = \frac{1}{2}(I_2 + I_3 - I_1) = \frac{5}{8}\zeta(3)$$

Now, comparing both our answers (to the integral), we obtain:

$$\boxed{E(2) = \frac{11}{4}\zeta(3)}$$

3 Euler Sum of Order 4

We first note an important relationship:

$$\begin{aligned}\int_0^{\pi/2} x \ln(\tan x) dx &= \int_0^\infty \frac{\arctan(x) \ln(x)}{1+x^2} dx \\&= \int_0^1 \frac{\arctan(x) \ln(x)}{1+x^2} dx + \int_1^\infty \frac{\arctan(x) \ln(x)}{1+x^2} dx \\&= \int_0^1 \frac{\arctan(x) \ln(x)}{1+x^2} dx + \int_0^1 \left(\arctan(x) - \frac{\pi}{2} \right) \cdot \frac{\ln(x)}{1+x^2} dx \\&= 2 \int_0^1 \frac{\arctan(x) \ln(x)}{1+x^2} dx - \frac{\pi}{2} \cdot \int_0^1 \frac{\ln(x)}{1+x^2} dx \\&= 2 \int_0^1 \frac{\arctan(x) \ln(x)}{1+x^2} dx + \frac{\pi G}{2} \\&= \frac{\pi G}{2} - \int_0^1 \frac{\arctan^2(x)}{x} dx\end{aligned}$$

Now, we relate the resulting integral to the Euler sum of order 4 as follows:

$$\begin{aligned}
\int_0^1 \frac{\arctan^2(x)}{x} dx &= \int_0^1 \frac{1}{x} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} dx \\
&= \int_0^1 \frac{1}{x} \cdot \sum_{m=0}^{\infty} (-1)^m x^{2m+2} \cdot \sum_{k=0}^m \frac{1}{(2k+1)(2m-2k+1)} dx \\
&= \int_0^1 \frac{1}{x} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2}}{2m+2} \cdot \sum_{k=0}^m \left(\frac{1}{2k+1} + \frac{1}{2m-2k+1} \right) dx \\
&= \int_0^1 \frac{1}{x} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2}}{m+1} \cdot \sum_{k=0}^m \frac{1}{2k+1} dx \\
&= \int_0^1 \frac{1}{x} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2}}{m+1} \cdot \left(H_{2m+1} - \frac{1}{2} H_m \right) dx \\
&= \frac{1}{2} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^2} \cdot \left(H_{2m+1} - \frac{1}{2} H_m \right) \\
&= \frac{1}{2} \cdot \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^2} \cdot \left(H_{2m-1} - \frac{1}{2} H_{m-1} \right) \\
&= \frac{1}{2} \cdot \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^2} \cdot \left(H_{2m} - \frac{1}{2m} - \frac{1}{2} H_m + \frac{1}{2m} \right) \\
&= \frac{1}{2} \cdot \sum_{m=1}^{\infty} \frac{(-1)^{m-1} H_{2m}}{m^2} - \frac{1}{4} \cdot \sum_{m=1}^{\infty} \frac{(-1)^{m-1} H_m}{m^2} \\
&= 2 \left(\frac{1}{4} E(2) - \frac{1}{8} E(4) \right) - \frac{1}{4} \left(E(1) - \frac{1}{2} E(2) \right) \\
&= \frac{39}{32} \zeta(3) - \frac{1}{4} E(4)
\end{aligned}$$

Now, we realize that:

$$\int_0^{\pi/2} x \ln(\tan x) dx = 2 \int_0^{\pi/2} x \ln(\sin x) dx - \int_0^{\pi/2} x \ln\left(\frac{\sin(2x)}{2}\right) dx$$

For the latter integral, we may observe that:

$$\int_0^{\pi/2} x \ln(\sin x) dx = \frac{\pi}{2} \int_0^{\pi/2} \ln(\sin x) dx - \int_0^{\pi/2} x \ln(\cos x) dx$$

Hence:

$$\int_0^{\pi/2} x \ln\left(\frac{\sin(2x)}{2}\right) dx = \frac{\pi}{2} \int_0^{\pi/2} \ln(\sin x) dx = -\frac{\pi^2}{4} \ln(2)$$

(Where we have quoted the literature value for Euler's log-sine integral.)

The former integral has been extensively studied in the literature as well. Two notable approaches are through usage of the imaginary number i (and series expansions) and through usage of the Fourier series of $\ln(\sin x)$. We shall also quote its literature value, and hence obtain:

$$\int_0^{\pi/2} x \ln(\tan x) dx = \frac{7}{8}\zeta(3)$$

From earlier, we have:

$$\int_0^{\pi/2} x \ln(\tan x) dx = \frac{\pi G}{2} - \frac{39}{32}\zeta(3) + \frac{1}{4}E(4)$$

Comparing, we arrive at the rather beautiful:

$$E(4) = \frac{67}{8}\zeta(3) - 2\pi G$$