MERGE: Matching Electronic Results with Genuine Evidence

for verifiable voting in person at remote locations

DRAFT V0.2

Arash Mirzaei, Vanessa Teague
{ arash.mirzaei, vanessa.teague}@anu.edu.au
CAC-Vote project*

March 14, 2024

Contents

*This project was funded by DARPA "Advanced Paper-based auditing techniques in deployed military settings," Agreement No.: HR00112290093

1 Introduction

Slow mail is one of the main barriers to electoral participation for many overseas military personnel. Some jurisdictions send blank ballots by paper mail and require ballot return by the same channel—this can be very slow, often resulting in ballots that are not returned by the deadline. Other jurisdictions allow for paperless voting by Internet, including email or pdf upload—this is very fast, but introduces unacceptable risks to integrity and privacy. One compromise is electronic delivery and paper returns, in which voters download a blank ballot and mail back a printed paper vote, having filled it in either manually or electronically. This has similar integrity and privacy properties to vote-by-mail, and takes only half the time that a fully paper solution would take. Nevertheless, the return mail delivery is often still too slow.

This paper describes the MERGE protocol, which enhances the basic scheme of electronic delivery and paper returns in a way that increases speed without significantly detracting from privacy or integrity. The main idea is to send an untrusted electronic record back quickly, which can be confirmed if the paper ballot arrives in time for the audit, or distrusted if it does not. This extends the time available for the paper ballot to arrive.

Assume a public bulletin board (BB) which is an authenticated broadcast channel with memory. We will extensively use the assumption that people in different locations can rely on seeing the same data on the BB.

- In the polling place, the voter makes both a verifiable paper record, which is placed inside an envelope and mailed, and an untrusted electronic record, which is encrypted and posted to the BB.

- The voter is issued a digital signature, presented on a sticker, which serves as verifiable evidence of participation. Envelopes are considered acceptable only if they bear stickers with valid signatures upon receipt.

- The untrusted electronic record can be incorporated into preliminary results and used as the cast vote record in a Risk Limiting Audit (RLA).

- If the paper ballot arrives in time for the RLA, it is used as the ballot, just like any other RLA.

- If no paper ballot arrives that corresponds to a given electronic record, we use the phantoms-to-zombies approach of Banuelos and Stark [BS12] to ensure that the risk limit is met even though the electronic record is not trusted.

Two assumptions are unavoidable: first, there must be an accurate count of people who legitimately participated; second, the paper ballot that the voter places into the envelope must accurately reflect their intentions. Our protocol also requires someone to check the data (specifically, the digital signature) on every envelope. These all need to be supported by human verification processes.

The paper and electronic records could be produced by scanning a hand marked paper ballot, or printing a paper ballot from a ballot marking device.[1] See subsection 1.5 for a detailed discussion of the trust assumptions.

MERGE is designed to carry a relatively small fraction of votes that are incorporated into a larger electoral process with an RLA. Compared to plain electronic delivery and paper returns, its main advantage is that, rather than needing ballot papers to arrive in time to be scanned for preliminary results, a vote can be safely counted if it arrives before the RLA. (Of course, an electronic commitment to the vote must still be made before the normal close of polls.) In practice, for most jurisdictions, this allows a few more weeks for mail to arrive. Its main disadvantage is that—if a large number of ballot papers do not arrive in time—the consequent large discrepancies may cause an RLA to fall back to a manual recount, when it would not have if those votes had simply been excluded from the beginning.

This document provides two different versions for the MERGE protocol, where each version is suitable for a specific adversarial model. The first version (basic-MERGE, section 2) aims only for *Software Independence* [Riv08]—we assume that the attacker can control any electronic components, but not alter ballots in the mail. We also present an end-to-end verifiable version (E2E-MERGE, section 5) and explain the implications for the RLA.

All the cryptographic operations are performed using Microsoft's ElectionGuard library, or some other cryptographic library openly available to the general public.

1.1 Locations

The election occurs in two classes of locations, each of which might have several instances.

Remote Voting Centers are controlled polling places in remote areas such as overseas military bases.

Local Counting Centers are state- or county-based electoral authorities, where the ordinary votes from most citizens are counted. These will also receive paper ballots from Remote Voting Centers and process them.

[1] Scientific opinion is divided on whether human verification of BMDs is sufficiently accurate to justify the assumption that the BMD printout matches the voter's intention. See [ADS20] and [KBW21] for examples of opposing views. At a minimum, this requires careful design to ensure that voters are motivated and encouraged to verify, and that there is something they can do if a misprint occurs.

There are likely to be many Remote Voting Centers and many Local Counting Centers, but we assume only one of each in this discussion for simplicity.

1.2 Authentication using CAC cards

Voters in remote areas own a smart card called a Common Access Card (CAC) card. It provides a Public Key Infrastructure (PKI) to enable secure authentication and digital signatures. Observers and officials in each Local Counting Center know the public keys of the voters they should include. Digital signatures are used on both the electronic and the paper records.

- Signatures are used to authenticate encrypted votes on the BB.
- Signatures on encrypted votes are printed as a QR code onto a sticker that is applied to the outside of the voting envelope.

Only votes with a valid CAC signature are accepted via either the electronic or paper channels.

This allows officials at the Local Counting Center to notify the voter when a properly-signed ballot from them has arrived. This is expected to be part of the practical implementation. It does not, however, form an important part of the cryptographic protocol because it does not include evidence that the paper ballot matched the electronic one.[2]

1.3 Process

The process will consist of the usual risk-limiting audit at the Local Counting Center. We assume that most of the votes are cast locally near the Local Counting Center, with extras incorporated from Remote Voting Centers as needed.

The ballot manifest—available to observers at the Local Counting Center—includes both the ordinary local ballots and all the ballots from Remote Voting Centers. So do the collections of preliminary Cast Vote Records (CVRs). This data is returned to the Local Counting Center electronically via the BB, while the ballot papers are mailed to the Local Counting Center.

The Risk Limiting Audit proceeds exactly as it would if all the votes had been cast locally: observers see the ballot manifest and (a commitment to) preliminary CVRs, then they watch a transparent process for seeding the PRNG that will be used to generate ballot samples, then they check the sequence of sampled ballots. For local ballots, preliminary CVRs are made in the usual way; for MERGE ballots, preliminary CVRs come from the BB in encrypted form. If the sampled ballot is local, officials can retrieve it in the usual way and compute its effect on the

[2] The protocol could be extended, with some extra mixing, to notify each voter of whether their specific ballot matched. The current version simply produces a collective tally of how many of the received ballots matched.

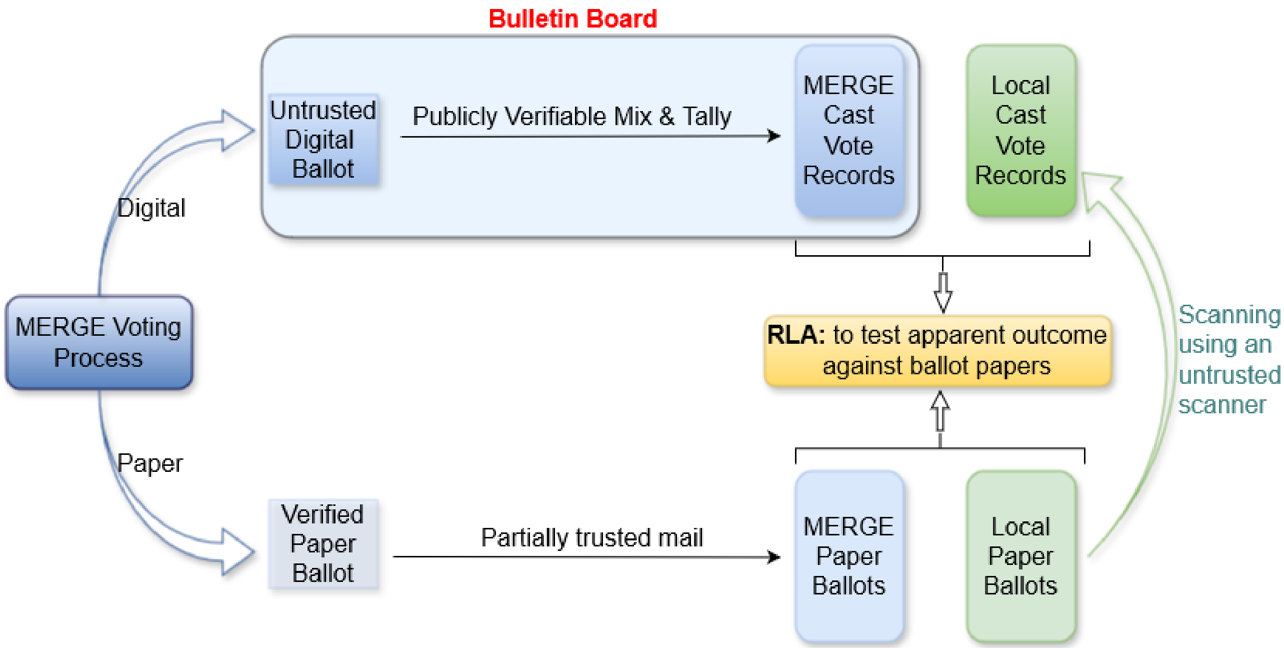


Figure 1: A high level overview of MERGE.

audit statistics. If the ballot was cast remotely, it will require some extra work to locate the mailed ballot and compare it with its electronic counterpart—this is described below. It is also more likely to be unavailable because it was delayed in the mail—we describe below how to deal conservatively with this situation.

The BB is the communication mechanism between the observers at the Local Counting Center and the Remote Voting Center. Assuming that the BB shows them all a consistent transcript, they need only trust each other (and possibly the paper ballot transport, depending on our threat model), not any of the other processes.

Figure 1 demonstrates a high level overview of MERGE.

1.4 Security goals

We desire an election system to satisfy the following properties:

- **Authentication:** Only the eligible voters should be able to cast their vote.

- **Privacy:** The value of a voter's cast ballot should be remain undisclosed.[3]
- **Receipt Freeness:** It should be impossible for a voter to convince anyone of the value of their vote, even if they actively collude with the coercer.[4]
- **Paper-based (individual) cast-as-intended verification:** Each voter can see that their paper vote accurately reflects their intention and refrain from casting it if it does not.[5]
- **(Public) tally verification:** anyone can verify whether the electronic election tally correctly reflects the recorded votes.
- **(Collective, probabilistic) recorded-as-cast verification:** if the apparent election outcome (incorporating the electronic tally) does not accurately reflect the paper evidence (including both MERGE and ordinary ballots), the RLA will fall back to a full manual count except with probability at most the risk limit.

The last property is the main contribution of this paper: we show how our cryptographic protocol can interface with an existing *Risk Limiting Audit* procedure to guarantee that the *overall* risk limit is still met, even in the presence of MERGE ballots. As far as we know, this is the first rigorous proof of an RLA guarantee from a cryptographic protocol. Of course, these guarantees are entirely dependent on the trustworthiness of the paper trail, which is outside the control of the cryptographic protocol, and may fail for a variety of reasons, such as security problems in the mail channel and difficulties verifying printouts. These need to be addressed by human procedures outside the MERGE protocol. Other assumptions are described below.

The end-to-end verifiable version (section 5) adds the following security goal:

- **End-to-end verifiability:** Each voter should be able to verify that their vote was cast as intended, recorded as cast and tallied as recorded.

1.5 Threat model and trust assumptions

Here, we define our trust assumptions as follows:

1. Cryptographic primitives used in our protocols meet the necessary security criteria.
2. The local electoral authorities maintain a list of all eligible voters within their county.

[3]There are significant assumptions behind the privacy guarantees of MERGE, which depend on which version is run. These are described below.

[4]MERGE reveals who participated, so it is possible to coerce someone to refrain from participating altogether.

[5]Our cryptographic protocol is agnostic about whether this paper vote is filled in by hand by the voter or printed by a BMD. It obviously needs to be trustworthy in order for the RLA to be valid.

3. The local electoral authorities maintain a list of CAC IDs assigned to registered eligible voters within their county.
    - Each voter's CAC card securely stores the corresponding private key.
    - The CAC Certificate Authority issues trustworthy certificates linking each CAC ID to its public key(s).
    - The local electoral authorities validate the CAC certificate for each signed ballot posted to the BB.[6]
4. Each voter verifies that the paper ballot accurately reflects their intention.
5. Each printed signature on the sticker is verified before sending, either by the voter or by some other trustworthy assistant at the Remote Voting Center.
6. The list of CAC IDs for MERGE ballots that have been validly signed on the BB aligns with the identifier list of voters who attempted to vote using the MERGE system.
7. The voter remains hidden from others within the confines of the voting booth. The adversary's visibility is restricted to solely observing
    - the BB,
    - other pieces of evidence (remembered or captured by the voter), which might be susceptible to forgery.
8. A pre-determined fraction of tallying authorities are trusted (relevant for privacy but not integrity),
9. Voters' eligibility is properly enforced at the remote voting center.

The end-to-end verifiable version (E2E-MERGE, section 5) adds the following.

10. If the voting system provides the voter audit capability and a voter chooses to audit their ballot, they diligently adhere to the provided audit instructions. At least a portion of them diligently perform cast-as-intended verification and engage in auditing their votes.
11. The adversary may observe the receipt (printed out by the voting machine) that the voter retrieves from the booth.

[6] In principle, the certificates could be posted on the BB with the signatures. However, in practice CAC certificates contain significant personal information that precludes public distribution. We therefore need to assume that they are verified by local authorities but not the public.

1.5.1 Attacker model

We define three distinct adversarial models, each characterized by the attacker's capabilities over the paper channel. In all three models, the adversary may control all computers and do any polynomial-time computation. The variations among these models lie in the extent of control the adversary possesses over the paper channel.

- *Electronic-only*: The adversary has no control over the paper channel. Every envelope arrives at the Local Counting Center unaltered within a specified timeframe and stuffing new envelopes is impossible.
- *Electronic&Drop*: Any envelope, mailed by a legitimate sender, may be selected by the adversary to experience extended delays or even go missing. However, if delivered, the envelope remains unaltered.
- *Electronic&Stuff*: Every envelope arrives at the Local Counting Center unaltered within a specified timeframe. Nonetheless, the adversary can stuff new envelopes on behalf of any legitimate voter.

The adversary in these models is unable to read the contents of a mailed envelope. Table 1 summarizes these models.

If ballots are collected by a responsible authority (for example, in a ballot box) and either transported to the Local Counting Center under supervision or audited on the spot with the Remote Voting Center acting as its own Local Counting Center, this corresponds to the Electronic-only model.

The Electronic&Drop model probably corresponds most closely to intuitive assumptions about postal mail voting (whether those assumptions are valid in practice or not). Jurisdictions vary greatly in how they prevent fraudulent ballots from being accepted. Our protocol's use of digital signature stickers makes it harder for an *external* attacker to generate apparently-valid fake votes, but since we assume that the attacker controls all the computers, it also controls the computers that get access to the voter's CAC signing key. We therefore do not make it harder for this *internal* attacker to stuff a valid-looking paper ballot into the paper mail than it would have been with traditional postal mail. If the jurisdiction was diligently checking each voter's handwritten signature, or if some other form of registered or secure mail is being used, our protocol does not undermine it, but nor do we add to the defences. So we simply assume that *something* prevents the internal attacker from adding fraudulent mail ballots.

The Electronic&Stuff model covers the cases where envelopes are mailed individually, but there is a reliable postal system which guarantees on-time mail delivery. Envelope stuffing is not prevented in this model though.

Table 1: Capabilities of the adversary in different models.

Adv. Model	Electronic Control	Env. Stuffing	Env. Drop	Content View
Electronic-only	✓	-	-	-
Electronic&Stuff	✓	✓	-	-
Electronic&Drop	✓	-	✓	-

Note that basic-MERGE does not claim to defend against an attacker who can both drop and stuff paper ballots in the same election. In section 5 we introduce a strong attacker model that assumes the adversary possesses all capabilities of both Electronic&Stuff and Electronic&Drop adversaries. However, this imposes significant verification requirements on voters.

1.5.2 Implications

Our protocol strives to detect any errors that may occur during the voting process. However, complete recovery from these errors cannot be guaranteed. It is important to acknowledge that if any assumptions fail, the errors or attacks in the given adversarial model may become undetectable or it might be impossible to differentiate one type of failure from another. For instance, if the voters do not carefully check that their paper matches their intention, or do not verify that the digital signature they are about to place in the mail is valid (and theirs), then manipulation may be undetectable.

1.5.3 Counting the number of votes and creating a trustworthy ballot manifest

As voters cast their ballots using their CAC cards, the digital signatures associated with each vote on the BB offer a dependable method to authenticate the voter. This robust authentication process ensures that only authorized votes are recorded on the BB, effectively preventing any unauthorized or fraudulent ballots from being included in the tally.

However, it does not guarantee protection against dropping ballots from the electronic record. This could be disastrous because, if the paper records are also delayed or dropped, there is no way to know how many ballots have been dropped, and consequently what the implications for the accuracy of the election result might be.

Moreover, even if dropping digital ballots is prevented and we have an accurate number of votes, the RLA process still requires that the available paper ballots reflect their actual voters' intention.

Table 2: Verification steps required by the voter under various adversarial models.

Adv. Model	Intended Vote Alignment	Sticker Sig. Validation†	BB Inclusion Check†
Electronic-only	✓	-	-
Electronic&Stuff	✓	✓	-
Electronic&Drop	✓	✓	✓

†: Also feasible for an official at the Remote Voting Center

Assumptions 5 and 6 are arguably enough to achieve both stated criteria in any aforementioned attacker models. However, we will demonstrate that within each attacker model, it is possible to relax these assumptions.

In the Electronic-only model, given each voter checks if their ballot align their intentions, the paper manifest reflects the legitimate voters' intentions. The situation is almost the same in the Electronic&Drop model. The envelope cast by any legitimate voter might get lost or experience extended delay. However, if delivered, the envelope remains unaltered and its ballot reflects its sender's intention. Therefore, as a requirement of all RLAs: there needs to be a record of how many ballot papers there are, derived independently of the scanners that are being audited. This assumption needs to be supported by practical work:

- officials at the Remote Voting Center could keep a record of who voted there and verify that it (eventually) matches the set of valid signatures on votes on the BB, or
- every voter (or their delegate) would need to verify the inclusion of their vote on the BB. The sticker on the envelope can be used as the participation evidence.

However, the situation for Electronic&Stuff model is different. Since stuffing new envelopes on behalf of the legitimate voter is possible, we need a way to guarantee that the envelope that is received at the Local Counting Center is accepted by the officials. Otherwise, the adversary could insert an envelope that is eventually accepted in place of the voter's actual envelope, which is rejected due to being identified as invalid. So, in this model, validity of the voting machine's signature on the envelope must be checked either by the voter or a trusted authority. The inclusion of the corresponding ballot on the BB might be skipped because ballot dropping will be detected once the envelope with a valid voting machine's signature is received at the Local Counting Center.

Table 2 summarizes the verification steps required by the voter under various adversarial models.

1.6 Notations and building blocks

The selections of voter i, choosing from m candidates, are represented as a binary vector $\mathbf{b_i}$ of length m, consisting of zeros for unselected options and ones for selections. In cases where it is clear and doesn't cause confusion, the index i can be omitted for simplicity. The binary vector $\mathbf{b}$ corresponds with a *selection option* vector $\mathbf{o}$ where each vector entry represents a selection option. Then, $\{\mathbf{o}_j | \mathbf{b}_j = 1\}$ is another way to represent the voter's vote. For example, Alice is another way to represent the vote vector $\mathbf{b} = [1\ 0\ 0]$ in a contest with the following candidate list [Alice Bob Charlie]. To simplify the notations, we often employ the vote vector representation to refer to the voter's selections. However, depending on the context, particularly when the selections are displayed on the screen or printed on the paper, the selection option representation might be a more user-friendly method.

The encryption of the vector $\mathbf{b}$ is denoted as $\mathbf{e} = E(\mathbf{b})$, wherein each entry $\mathbf{e}_j$ for $j = \{1, \cdots, m\}$ represents the encryption of the corresponding vote entry $\mathbf{b}_j$.

Encryption scheme Similar to ElectionGuard, encryption of votes in this work is performed using a variant exponential form of the ElGamal cryptosystem. The cryptosystem parameters (p, q, g) are defined as follows: p is a prime number equal to $2kq + 1$, where q is also a prime number, and g is a generator of the order q subgroup of $\mathbb{Z}_p^*$. To generate a public-private key pair, a private key $s \in \mathbb{Z}_q$ is randomly selected. The corresponding public key, K, is then computed as $K = g^s \mod p$ and made public. To encrypt a vote b, a nonce ξ is selected randomly such that $0 \leq \xi < q$. Then, encryption of b, denoted by $E(b)$, is computed as

$$E(b) = (g^\xi \mod p, K^b \cdot K^\xi \mod p) = (g^\xi \mod p, K^{b+\xi} \mod p).$$

The entity possessing the corresponding secret key s can decrypt (α, β) as $\beta / \alpha^s \mod p = K^b \mod p$.

This encryption scheme is additively homomorphic. Namely,

$$E(b_1, \xi_1) \cdot E(b_2, \xi_2) = E(b_1 + b_2, \xi_1 + \xi_2)$$

We can use the following equation to re-encrypt the ballot b

$$E(b, \xi_1) \cdot E(0, \xi_2) = E(b, \xi_1 + \xi_2)$$

Mixnet To maintain the confidentiality of each voter's ballot, a mixnet is used to mix that ballot with those of others. It takes a set of encrypted ballots and outputs a shuffled set of re-encrypted ballots, making it challenging to link any specific encrypted ballot in its output to any of its input encrypted ballots. The mixnet operation on the input encrypted ballot set $\{E(b_i)\}_i$ is denoted by $Mix(\{E(b_i)\}_i)$. We might also apply the mixnet to a batch of encrypted values to process multiple elements together. For instance, it can be applied to the set of encrypted ballot vectors $\{E(\mathbf{b_i})\}_i$. Then, its output is denoted by $Mix(\{E(\mathbf{b_i})\}_i)$.

Non-interactive zero-knowledge (NIZK) proofs Similar to ElectionGuard, we use NIZK proofs to demonstrate that an encrypted vote vector $\mathbf{e}$ is the encryption of a properly formed vote vector $\mathbf{b}$. Namely:

- The encryption linked to each option (e.g. $\mathbf{e}_j$) typically represents a valid value and is commonly either an encryption of zero or an encryption of one (i.e. $\mathbf{b}_j = 0$ or $\mathbf{b}_j = 1$).
- The sum of all encrypted options within each contest falls within a specific range.

NIZK proofs for an encrypted vote vector $\mathbf{e} = E(\mathbf{b})$ is denoted by $ZKP_{\text{VALID}}(\mathbf{b})$.

Similar to ElectionGuard, we also use NIZK proofs to demonstrate that an encrypted value (e.g. $e = E(b)$) is decrypted to a specific value (e.g. b), while keeping the encryption nonce and secret decryption keys undisclosed. Such a proof is denoted by $ZKP_{\text{DECRYPT}}(b, e)$.

Sometimes a plaintext, with a proof of proper decryption, will be supplied to some participants but not published on the BB. We call this *private decryption*.

Furthermore, we also use NIZK proofs to demonstrate that the Mixnet input $\{\mathcal{I}_i\}_i$ is correctly and honestly shuffled and re-encrypted to $\{\mathcal{O}_i\}_i$ without revealing any information about the order and correlation between input and output. Such a proof is denoted by $ZKP_{\text{MIX}}(\{\mathcal{I}_i\}_i, \{\mathcal{O}_i\}_i)$

Digital signature The digital signature on message m by voter i is represented as $sig_i(m)$.

Hash function The hash function, denoted as $H(m)$, generates the hash value for the message m.

Serial Number Each ballot includes a unique serial number sn which is randomly generated. The role of the serial number is to allow one-to-one matching between the paper ballot and its electronic counterpart.

1.7 Background on RLAs and ONEAudit

A *Risk Limiting Audit* (RLA) is a procedure for testing whether a trustworthy set of paper ballots implies that the announced election winner(s) won, and correcting the result by a fully hand count if not. It is parameterized by a *risk limit* α and has the following property:

> If the announced election result is wrong, the RLA will progress to a full hand count (and hence correct the result) with probability at least $1 - \alpha$.

RLAs proceed by taking a random sample of ballots and updating a P-value until it either falls below the risk limit (hence accepting the result), or the audit administrator decides to run a full hand count.

RLAs were developed by Stark and others [Sta08a, Sta08b, Sta10, LSY12, LS12, OSLM18, Sta20]. RLAs are known for a wide variety of election types and audit situations, including single- and multi-winner plurality contests, supermajority elections, party-list proportional elections and Instant Runoff Voting.

The *apparent outcome* is (a set of) announced winner(s). The *actual outcome* is the outcome that would be found if all the ballot papers were correctly counted. In this work we mostly concentrate on *ballot-level comparison audits* in which each individually sampled ballot paper is compared with its corresponding electronic record. (Using MERGE with other audit styles is discussed in section 3.) The apparent outcome is supported by a set of *cast vote records* (CVRs), which may or may not accurately reflect the voter's intentions. When the RLA samples a particular ballot, the paper ballot and its CVR are compared. If the CVR overstates a winner's tally compared with the paper record, it is an *overstatement* (for example, if the CVR is a vote for the winner and the paper ballot is a vote for the loser, it is a two-vote overstatement). If the CVR understates a winner's tally compared with the paper ballot, it is an *understatement*.

Two techniques are particularly useful in our setting. The first is the phantoms-to-zombies approach devised by Bañuelos and Stark [BS12]. This is a way of dealing with ballots that appear in the manifest but cannot be found on paper, a problem that may occur frequently when MERGE ballots are delayed or dropped in the mail. The idea is to "Pretend that the audit actually finds a ballot, an evil zombie ballot that shows whatever would increase the P-value the most." Bañuelos and Stark prove this to be conservative, in the sense that the RLA property still holds, assuming that it would have held had the correct ballot paper been located. The correct evil zombie ballot might be slightly different depending on the kind of RLA, but is generally a vote for the highest loser, or possibly an imaginary vote for all the losers, or (in the case of separate comparisons between the winner and each loser) a vote for whichever loser is being compared.

The second technique we use extensively is ONEAudit [Sta23].

Definition 1. *Two sets with the same number of CVRs are* overstatement net equivalent *if they produce the same totals.*

Proposition 3 and our use of ONEAudit in our protocol rely on the following proposition, which is proven in [Sta23] (though not explicitly stated as a proposition).

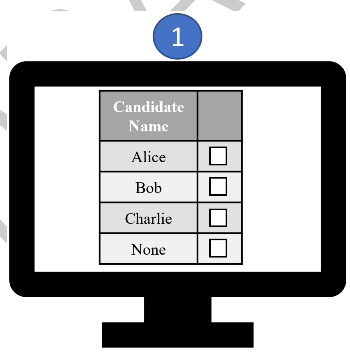
Proposition 2. *If we have an RLA procedure to test whether the net overstatement is greater than the margin using the real CVRs produced by the voting system, the same procedure can test whether the outcome is correct—with the same risk limit—if it is applied to* overstatement net equivalent *CVRs instead. (Audit sample sizes might be quite different.)*

2 Basic version: basic-MERGE

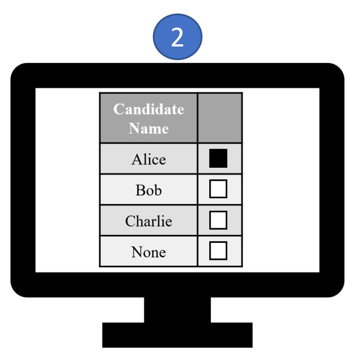
The process is as follows. The example given here uses a BMD to print a paper record, but the protocol would work just as well with a hand-marked paper ballot interpreted by a scanner. In that case, the "voting computer" would be a scanner with the capacity to produce an encrypted record and upload it to the BB. We will use the term "voting computer" to mean either a BMD or a scanner with computation and communication capacity.

2.1 Voting process

1. The voting machine displays the options to the voter.



2. The voter interacts with the machine and makes their selection.



Let their vote vector be $\mathbf{b}$.

3. The voting machine assigns a unique serial number sn_i to each ballot paper. The voting machine produces:

   - a signed, encrypted electronic record and serial number, with a proof of ballot validity, which goes on the BB:

   $$BB \hookleftarrow \quad sig_i(E(sn_i), E(\mathbf{b_i}), \mathit{ZKP}_{\mathsf{VALID}}(\mathbf{b_i})),$$

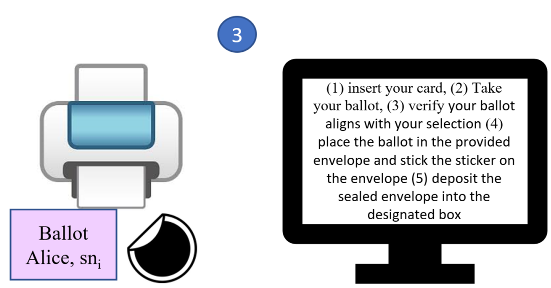
   - a plain paper record of the ballot $(sn_i, \mathbf{b_i})$, which is placed in a ballot box or in an envelope for mailing, and
   - a sticker that includes an address, a traditional mail tracking number, a space for a pen-and-ink signature (if required by regulation—not relevant to our cryptographic protocol) and a digital signature

   $$sig_{\text{Voting Computer}}(Id_i, Id_e, H_i),$$

   where Id_i and Id_e are the voter i's unique identifier and the election identifier, respectively. Also, H_i is the hash value created from the digital record that is stored for the voter i, i.e. $H_i = H(sig_i(E(sn_i), E(\mathbf{b_i}), \mathit{ZKP}_{\mathsf{VALID}}(\mathbf{b_i})))$

The voter must verify that her plaintext printout matches her intention, then stick the sticker on her envelope, then mail it. Someone also needs to verify that the digital signature on the sticker is valid, and that a matching encrypted vote eventually arrives on the BB, but this does not impact privacy and may be done by voters, officials or bystanders.

The purpose of the digital signatures is to complicate ballot stuffing—only ballots (either electronic or paper) with a valid accompanying digital signature will be accepted. The Voting computer's digital signature needs to be explicit on the mail sticker, but is implicit on the BB because that channel is already authenticated.



This entire flow could work just as well with a more standard ElectionGuard setup, with a hand-marked paper ballot input into a scanner. The serial numbers could either be generated by the scanner, or printed in advance.

After the voting period ends, electronic records undergo a series of processing steps. Additionally, paper ballots are sent to the Local Counting Center for auditing purposes. A diagram illustrating the whole process is in Figure 2. Details of all processing steps will be provided in the following sections.

2.2 Digital path

We explain digital processing of electronic ballots for one Local Counting Center's votes. Every other Local Counting Center's data is processed similarly—they do not interact because their results need to be delivered to the appropriate Local Counting Center, even for statewide contests. At the conclusion of the voting process, the following digital record containing the information of all n voters becomes publicly available on the BB, where the record with index i denotes the ballot from voter i, where $i \in \{1, \cdots, n\}$.

$$BB \hookleftarrow \quad \mathcal{D}_1 := \{sig_i(E(sn_i), E(\mathbf{b_i}), \mathit{ZKP}_{\textsf{VALID}}(\mathbf{b_i}))) : i \in \{1, \cdots, n\}\}$$

Subsequently, this digital record undergoes a series of processing steps as follows.

1. The validity of the signature and ZKP for each electronic ballot is verified.
2. The encryption of the total tally is calculated and published on the BB

$$BB \hookleftarrow \quad E(\mathbf{Tally}) := \prod_i E(\mathbf{b_i}).$$

3. $E(\mathbf{Tally})$ is decrypted and published on the BB along with the proof of the correct decryption.

$$BB \hookleftarrow \quad E(\mathbf{Tally}), \mathbf{Tally}, \mathit{ZKP}_{\textsf{DECRYPT}}(\mathbf{Tally}, E(\mathbf{Tally}))$$

(In some jurisdictions, this value may be sensitive because of a small set of MERGE voters. See section 3 for variations that allow public verifiability without publication of the separate tally.)

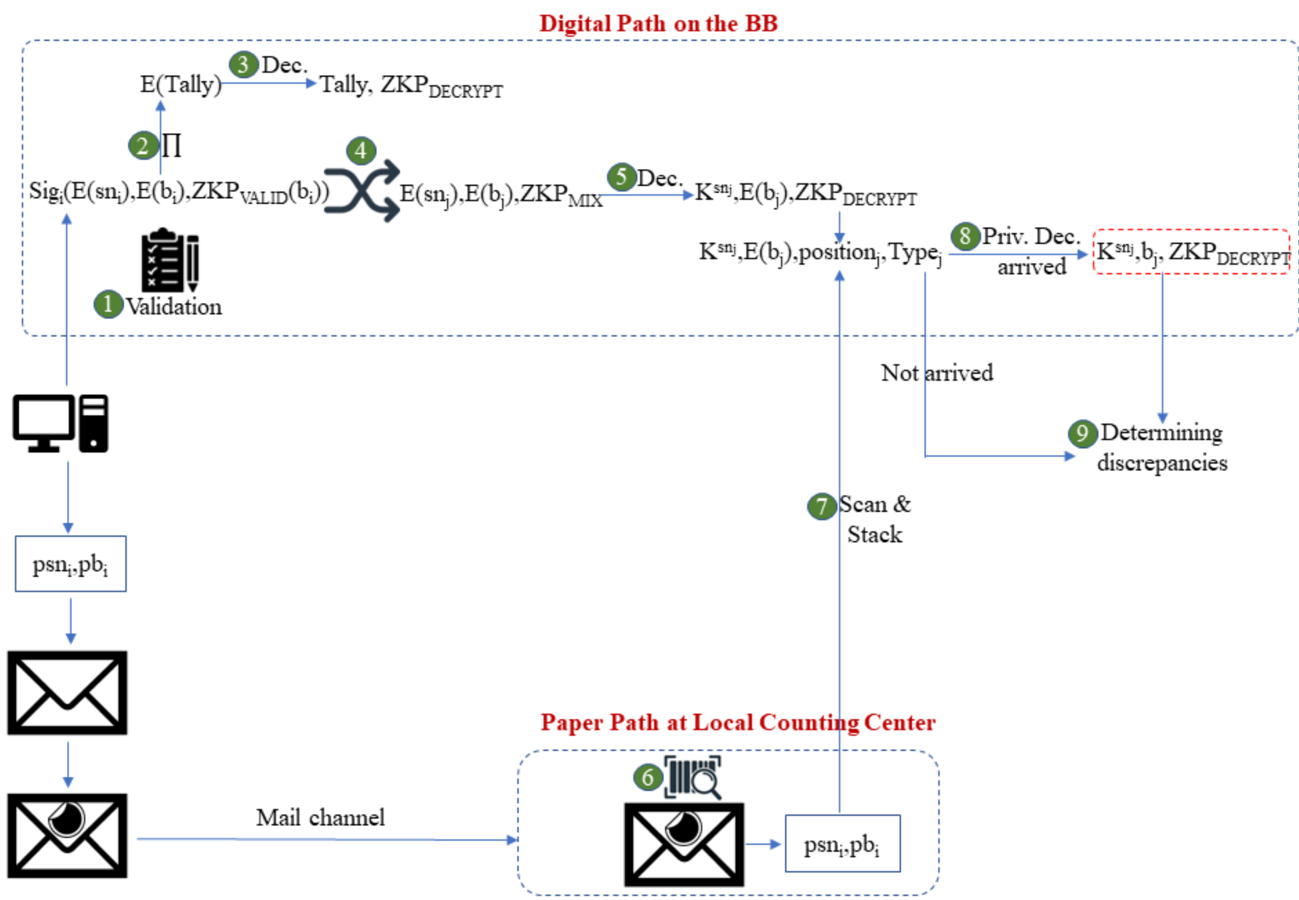


Figure 2: The whole process for basic-MERGE based on ballot comparison. Each Local Counting Center's votes are dealt with separately—the diagram shows the process for only one Local Counting Center.

4. The pairs $\mathcal{D}_2 := \{(E(sn_i), E(\mathbf{b_i}) : i \in \{1, \cdots, n\}\}$ are extracted from $\mathcal{D}_1$ and then mixed. The mixed pairs along with the proof of the correct mix operation are published on the BB. So, at the end of this step, we have the following data on the BB:

$$BB \hookleftarrow \quad \mathcal{D}_3 := \{(E(sn_j), E(\mathbf{b_j})) : j = \pi(i), \mathit{ZKP}_{\textsf{Mix}}(\mathcal{D}_2, \mathcal{D}_3)\}$$

where π reflects the permutation operation performed in the Mixnet operation.

5. Serial numbers for the mixed, arrived stack are decrypted and then the proof of the correct decryption is published. Therefore, at the end of this step, we have the following data on the BB.

$$BB \hookleftarrow \quad \mathcal{D}_4 := \{(K^{sn_j}, E(\mathbf{b_j}), \mathit{ZKP}_{\textsf{Decrypt}}(K^{sn_j}, E(sn_j))) : j = \pi(i)\}$$

It is also checked that there are no duplicate serial numbers. Any occurrence of duplicate serial numbers indicates significant problems with a voting computer—if there are any, the process should stop.

2.3 Paper path

At the conclusion of the voting process, the ballot papers are conveyed to the Local Counting Center, either individually in separate envelopes, or collectively in some organized way. Subsequently, this paper record undergoes a series of processing steps as follows.

6. If the jurisdiction has an existing process of scanning traditional handwritten signatures, this would apply at this point. Envelopes with invalid signatures must be set aside and handled properly.

   Each incoming envelope's sticker is scanned and its corresponding digital signature is verified. It is necessary to check that the data being signed matches the corresponding data the BB for this voter and the BMD's signature is valid. If any of these verifications fails, the process continues but the envelopes are set aside in a stack called "Rejected Envelopes" for further investigations. If the signature is valid but another validly signed envelope has already been received from the same voter, it is set aside in a stack called the "Eligibility Problem" stack.

   For all validly signed envelopes (excluding the ones inside the "Eligibility Problem" stack), the envelope contents are removed and physically shuffled, similar to standard postal voting procedures.

7. Each incoming ballot's serial number psn is scanned. For each serial number, K^{psn} is computed and it is checked if there is a digital ballot with a matching K^{sn} on the BB.[7] Then, the location of the paper ballot is appended to the corresponding digital ballot on the BB.[8] Each digital ballot is accompanied by a parameter called $Type$ where $Type =$ "not matching", $Type =$ "one-to-one" and $Type =$ "duplicated" respectively indicate if there exist zero, one or multiple paper ballots with the matching serial number. Therefore, we have the following data on the BB:

$$BB \hookleftarrow \quad \mathcal{D}_5 := \{(K^{sn_j}, E(\mathbf{b_j}), \{position_{j'}\}_{j'}, Type_j) : j = \pi(i), j' = \pi'(i)\}$$

The mapping π' represents the fact that paper ballots have been physically shuffled and $position_{j'}$ denotes the location of the associated paper ballots in paper storage. For digital ballots with multiple corresponding paper ballots (i.e., ballots with $Type =$ "duplicated"), the positions of all matching ballots are recorded on the BB. For digital ballots with $Type =$ "not matching", the parameter $position$ is null.

Any paper ballot with no matching K^{sn} on the BB is set aside. Call this the "problem" stack. These need to be dealt with outside the cryptographic protocol.

2.4 RLA

8. At the Local Counting Center the usual local Cast Vote Records (CVRs) are joined with the MERGE CVRs for auditing purposes, and the ballot manifest is updated to include both kinds of electronic records. When a local ballot is sampled, it is retrieved and dealt with as usual. When a MERGE ballot is sampled, if its type is either "one-to-one" or "duplicated", its encrypted vote $E(\mathbf{b})$ is privately decrypted to $\mathbf{b}$ and, alongside the proof of correct decryption, is supplied to all auditors and observers in the Local Counting Center. Therefore, they have access to the following data for the selected ballot:

Observers: $(K^{sn_j}, E(\mathbf{b_j}), \mathbf{b_j}, \mathit{ZKP}_{\text{DECRYPT}}(\mathbf{b_j}, E(\mathbf{b_j})), \{position_{j'}\}_{j'}, Type_j \neq \text{"not matching"})$

Then, the discrepancy is determined according to the guidelines outlined in Section 2.4.1 and then RLA statistics are updated accordingly, in exactly the same way for remote ballots

[7] Because of the way ElectionGuard encryption works, the number that naturally drops out after decryption is K^{sn}, not sn. Because we care only about exact matches, it does not matter whether we work in the exponential or plain form. We do not assume that K^{sn} hides sn, and do not need to hide the values of sn, because these are not publicly associated with the voter.

[8] This is its physical location in paper ballot storage, as in a ballot manifest. For example, "23rd ballot in batch 7, cabinet 12."

as they would be for local ones. The discrepancies can be made public in whatever way the Local Counting Center usually publishes its RLA calculations, or they can be appended to the relevant record on the BB.

If the selected ballot's type is "not matching", its discrepancy can be determined according to the process described in Section 2.4.1.

Alternatively, rather than private decryption of the ballot that is selected for RLA, we can privately decrypt all digital ballots before initiating the RLA process.

Local officials make local decisions about whether to accept the result or escalate to a larger sample, according to whatever RLA calculations they usually conduct.

2.4.1 Determining discrepancies

Assigning discrepancies to digital ballots is primarily determined by the stack to which the ballot belongs.

9.1. Let's start with the ballots with $Type =$ "one-to-one". Assume a digital ballot with the following data is selected for RLA:

$$(K^{sn_j}, \mathbf{b_j}, position_{j'}, Type_j = \text{“one-to-one”})$$

Then, we follow the below guidelines:

- Identify the paper ballot situated at position $position_{j'}$. Given that its serial number and content are represented by psn and $\mathbf{pb}$ respectively, verify whether K^{psn} matches K^{sn_j}. If not, output "serial number error" and terminate.
- Compare $\mathbf{pb}$ to $\mathbf{b_j}$ and record the discrepancy, exactly like any othe RLA.

Any occurrences of the "serial number error" indicate a significant problem with the scanning device. Therefore, encountering either of these errors necessitates restarting the entire process from step 7 onward using another scanning device.

9.2. Let's continue with ballots with $Type =$ "duplicated". Assume a digital ballot with the following data is selected for RLA:

$$(K^{sn_j}, \mathbf{b_j}, \{position_{j'}\}_{j'}, Type_j = \text{“duplicated”})$$

Then, the 2-step process described in step 9-1 is performed for all j' values and the final disrepancy is set to the *minimum* discrepancy among different j' values for the digital ballot.

Any occurrences of the “serial number error” indicate a significant problem with the scanning device.

9.3. The following options are available for determining discrepancies for ballots with $Type =$ “not matching”:

one-audit All encrypted ballots with $Type =$ “not matching” are homomorphically added and the tally privately decrypted for the auditors and observers in the Local Counting Center.[9] Let this sub-tally be denoted by $\mathbf{Tally_{n-m}}$.

For each audit statistic being computed, each individual CVR is then taken to be the average *overstatement net equivalent* CVR for $\#\{j | Type_j =$ “not matching”$\}$ number of votes. The (absent) paper ballot is taken to be the worst-case ballot (generally the loser in the case of a two-candidate comparison).

max-possible Regardless of the value of the given ballot, its discrepancy is set to the maximum value. This will usually be a +2 overstatement for plurality voting, but may be something different for other social choice functions.

9.4. Envelopes in the “Eligibility Problem” stack require further investigations as outlined later. Furthermore, based on the paper channel security model, validly signed envelopes in the “Eligibility Problem” might also impact the RLA process as follows: For any validly signed envelope in the “Eligibility Problem” with a distinct voter’s identifier, a maximum discrepancy (typically +2) is added to their county votes.

2.5 Verification Summary

There are three separate verification stages, which can be verified by different people: cast-as-intended verification, which is mostly done by the voter, universal (BB) verification, which can be done by any member of the public, verification by observers at the Local Counting Center that the data on the BB matches the ballot papers and the data included in the tally and the RLA matches the BB.

Each of these is detailed below.

Note that this version does *not* allow for individual recorded-as-cast verification: individual voters do not get evidence that their electronic record matches their intention. This evidence—or rather, evidence about whether the discrepancy is large enough to alter the result—is provided collectively through the matching and subsequent audit processes.

[9] This could also be published on the BB if the anonymity set is sufficiently large. Not clear what the advantage would be.

2.5.1 Cast-as-intended verification

By the voter

1- Verify that the ballot paper matches the intended vote.

By the voter or anyone else at the Remote Voting Center

2- Verify the digital signature on the sticker stuck onto the envelope.
3- Verify that a properly-signed vote from this voter is (eventually) on the BB.

2.5.2 BB transcript verification (public)

1. For each vote on the BB:
   (a) Verify digital signatures by voter and voting computer
   (b) Verify ZKP of proper ballot construction
2. Verify mixing ZKP
3. Verify decryption of all serial numbers sn_i.
4. Verify that each sn_i is unique.
5. Verify aggregation of $E(\mathbf{Tally})$.
6. Verify the decryption $\mathbf{Tally}$ of $E(\mathbf{Tally})$.

2.5.3 Verification at the Local Counting Center

1. Verify digital signatures on the stickers, and verify that they're signing the same data with appropriate domain separator.
2. Verify that the *Eligibility Problem Stack* includes all (and only) duplicate envelopes from the same voter.
3. Verify that the *Rejected Envelopes Stack* includes all (and only) envelopes with the invalid signatures.

4. Verify that $\mathbf{Tally}$ is properly added to the local CVRs or tallies.
5. Verify $ZKP_{\text{DECRYPT}}(,)$ for each ballot that is privately decrypted during the RLA process.
6. Verify aggregation of $E(\mathbf{Tally_{n-m}})$ and its decryption.
7. Verify that discrepancies are properly determined and then correctly incorporated into the RLA.
8. Verify there are no "serial number error"s.
9. For the case where ballots are collected by a responsible authority, verify if the number of ballots in the ballot box for each ballot style does not exceed the corresponding quantities on the BB.

2.6 Failure handling

In this section, we clarify the procedure that must be adopted if each verification, aforementioned above, fails.

2.6.1 Cast-as-intended verification

Item 1: If the paper ballot does not match the voter's selections, this could be attributed to a voting machine failure or a mistake by the voter. In such cases, the voter has the option to either retry the voting process or abandon it.

In either case, the following steps are taken:

- An authority records the voter's identity and the associated sticker, which is considered invalid from that point onward. If an envelope with an invalidated sticker arrives at the Local Counting Center, it is rejected. This recorded information serves as a reference in case the voter later claims possession of a sticker whose corresponding ballot is not found on the BB.
- The invalidated digital ballot is marked on the BB.

Furthermore, if the voter chooses to retry voting, the voting machine captures the new digital ballot as their vote and generates a corresponding paper ballot and sticker.

Item 2: If the digital signature on the sticker is not verified, it is the voting machine's fault.

Item 3: If a properly-signed vote, corresponding to the sticker's signature, from this voter does not (eventually) appear on the BB, it is the voting machine's fault.

2.6.2 BB transcript verification (public)

Items 1a and 1b: If for any digital ballot on the BB, digital signature or $ZKP_{\text{VALID}}()$ is not successfully verified, it is the voting machine's fault.

Item 2: If mixing ZKP is not successfully verified, it is the back-end server's fault. The RLA process must be performed after resolving the issue.

Item 3: If decryption of each encrypted serial number is not successfully verified, it is the back-end server's fault. The RLA process must be performed after resolving the issue.

Item 4: Given that it is of negligible probability that any two randomly generated serial numbers are equal, if each serial number is not unique, it is highly likely that it is the voting machine's fault.

Item 5: If computation of $E(\mathbf{Tally})$ is not successfully verified, it is the back-end server's fault. Decryption of $E(\mathbf{Tally})$ and then adding the decryption result to tallies from other systems must be performed after resolving the issue.

Item 6: If decryption of encrypted tally is not successfully verified, it is the back-end server's fault. Decryption of $E(\mathbf{Tally})$ and then adding the decryption result to tallies from other systems must be performed after resolving the issue.

2.6.3 Verification at the Local Counting Center

Item 1:

- If the sticker on any envelope carries a valid digital signature from the voting machine, but the corresponding signed data does not appear on the BB or it is associated with an incorrect domain separator, it is the voting machine's fault (unless the sticker has already been invalidated as part of the Cast-as-intended verification in item 1).

- If two stickers carry different valid signatures from a voting machine on the same message, that is the voting machine's fault.

- If two stickers carry valid signatures from different voting machines on the same voter' data, that is the voter's fault.

- If any two stickers contain a same valid signature by a voting machine, that could be considered a fault of anyone with access to the sticker (e.g., the voting machine, the voter or even someone working in the postal system).

Items 2 and 3: Any fault in the envelope stacking process is either due to the fault of the involved authorities or the fault of the device used for the signature verification.

Item 4: Any fault is the involved authorities' fault.

Items 5 and 6: If decryption of any ballot or aggregation of any sets of ballots and then its decryption are not successfully verified, it is the back-end server's fault.

Item 7: If discrepancies are not correctly determined and then integrated into the RLA computations, it is the involved authorities' fault.

Item 8: Any "serial number error" is the scanning device's fault.

Item 9: Any fault in counting the number of ballots in the ballot box is the involved authorities' fault.

2.7 A simpler design for Electronic-only model

If, instead of mailing envelopes individually, ballots are collected in a ballot box and conveyed in an organized manner, the voting process would differ, as no sticker is provided to the voter and each ballot is not placed in an envelope. Consequently, voters solely verify their ballots and place them in a ballot box. Furthermore, in the ballot box scenario, step 6, which is primarily relevant to sticker and envelope processing, will be omitted. However, it is crucial to verify that the number of ballots in the box for each ballot style does not exceed the corresponding quantities on the BB. This condition, for other adversarial models, is automatically met through step 6. This step in the paper path and the whole digital path remain consistent with the details outlined in Sections 2.2 and 2.3. The RLA process is also the same as what outlined in Section 2.4. The only distinction is that step 9.4, which is primarily relevant to "Eligibility Probelm" envelopes is not applicable in the ballot box scenario.

2.8 Security analysis: Verifiability

We will prove correctness assuming the one-audit method of accounting for "not matching" ballots. It is obvious that max-possible is more conservative.

In this section, we begin by outlining the audit process in an ideal scenario featuring trustworthy paper ballots and a one-to-one correspondence between each paper ballot and its electronic record. Although a specific subset of voters in the ideal world is controlled by the adversary, other voters follow the voting instructions. Subsequently, we gradually modify this ideal scenario step by step demonstrating that at each stage, we can establish the validity of at least one of the following statements:

- Any combination of digital and paper ballots in the newly defined world corresponds to a set of digital and paper ballots in the present world. This correspondence ensures identical (1) reported totals, (2) number of voters, and (3) honest voters. Furthermore, for any ballot selected for the RLA in the present world, the discrepancy value is equal to or smaller than the discrepancy value of its corresponding ballot in the newly defined world.

- For any combination of digital and paper ballots in the newly defined world, with total discrepancies of D, there exists a corresponding set of digital and paper ballots in the present world. This correspondence guarantees identical (1) reported totals, (2) number of voters, and (3) honest voters, with a total discrepancy equal to or smaller than D.

The first statement (which implies the second statement) proposes that, for any given risk limit, if the RLA procedure with any given sample set accepts a specific CVR in a newly defined world, it similarly accepts the corresponding sample set in the prior world.

The second statement implies that if we employ an RLA procedure to test whether the cumulative overstatements in the newly defined world are insufficient to alter the election outcome, the same procedure can be employed to test the correctness of the outcome in the current world. Proposition 3 proves this statement.

Since the last defined world in this section is the same as our protocol, the proof will show that our protocol has no greater likelihood of accepting a wrong result than an 'ordinary' RLA conducted on the trustworthy paper ballots and one-to-one correspondence between each paper ballot and its electronic record (Audit sample sizes might be quite different.)

Now we are ready to prove the following proposition.

Proposition 3. *Consider two digital ballot sets, $\mathcal{B}_1$ and $\mathcal{B}_2$, both with the same cardinality, represented as $|\mathcal{B}_1| = |\mathcal{B}_2|$ and the same totals. Let the total discrepancies for $\mathcal{B}_1$, denoted as D_1, be either equal to or less than the discrepancies for $\mathcal{B}_2$, denoted as D_2. If an RLA procedure, characterized by a specified risk limit, accepts $\mathcal{B}_2$, it also accepts $\mathcal{B}_1$.*

Proof. Reference [Sta23] has already addressed the scenario where $D_1 = D_2$. Now, consider the case where $D_2 > D_1$. Suppose we increase the discrepancy values of $k > 0$ ballots in $\mathcal{B}_1$ to form a set $\mathcal{B}'$ with total discrepancies equal to D_2. Then:

- As outlined in [Sta23], when an RLA procedure, configured with a specific risk level, accepts $\mathcal{B}_2$, it similarly accepts $\mathcal{B}'$, and
- As the discrepancy value for any ballot in $\mathcal{B}_1$ is at most as large as its counterpart in $\mathcal{B}'$, any specific sequence of ballots selected from $\mathcal{B}'$ for the RLA implies the existence of a corresponding sequence in $\mathcal{B}_1$ with a lower RLA risk. Consequently, if an RLA procedure, configured with a specific risk level, accepts $\mathcal{B}'$, it also accepts $\mathcal{B}_1$.

Based on the aforementioned considerations and utilizing the transitive property, we have demonstrated that if an RLA procedure, operating within a specified risk limit, accepts $\mathcal{B}_2$, it also accepts $\mathcal{B}_1$.

□

Now, we define an ideal world and step-by-step transform it to our protocol.

Ideal World Consider an Ideal World where every voter, physically visits a local polling place to verify their paper ballot and puts it in a ballot box. We assume a subset of voters is under the control of the adversary. Consequently, the adversary determines the participation status of each in the voting process. The adversary also has the potential to influence these voters, leading them to deviate from the prescribed voting process instructions. The voting machine records a digital ballot per voter where the validity of the digital ballot content is publicly verifiable. Digital ballots undergo tallying. The resulting tally is accurately incorporated into the local CVRs where authorities at the Local Counting Center, verify the correctness of this operation (i.e., Section 2.5.3, Verification at the Local Counting Center, item 4). In this ideal world, the voter's identifier is recorded on both the paper and its digital record. A precise one-to-one correspondence exists between each paper ballot and its electronic record. This correspondence is achieved relying on a trusted authority who verifies whether the digital ballot for the voter appears on the BB before allowing the voter to put their ballot in the box. In other words, the trusted authority ensures the fulfillment of two critical requirements: (1) the identifiers printed on the paper ballots form a subset of the identifiers recorded on the BB (ensuring inclusion), and (2) the quantity of votes in the ballot box is equivalent to the number of ballots recorded on the BB (ensuring exclusion). We assume the voting authority also has a way to ensure each voter's identity is not spoofed. The adversary also possesses control over all electronic devices, specifically the voting computer. Therefore, the voter's selection on each paper ballot might be different from its corresponding digital record. Therefore, after tallying the digital ballots and reporting the voting outcome, we need to verify if discrepancies between digital and paper ballots are not enough to change the voting outcome. So, we execute an RLA process by unsealing the ballot box and randomly sampling from the paper records and using the voter's identifier printed on the chosen paper to locate its corresponding digital record. Then, discrepancies are properly determined and then correctly incorporated into the RLA statistics where authorities at the Local Counting Center, verify the correctness of the operation (i.e., Section 2.5.3, Verification at the Local Counting Center, item 7). To guarantee that an incorrect voting outcome will lead to a manual tally with a predefined risk limit, each voter must check if the selections, printed on their paper ballot, is correct (i.e., Section 2.5.1, Cast-as-intended verification by the voter, item 1).

Ideal world Game The adversary $\mathcal{A}$ wins the ideal-world game if

- the apparent election result, according to the (MERGE and regular) CVRs, is different from the actual result, according to the (MERGE and regular) paper ballots, and

- the RLA procedure (including the MERGE cryptographic verification) confirms the result with probability greater than α, where the probability is taken over the random ballot selections of the RLA.

Intermediate World 1 Ballots in this world closely resemble those in the Ideal world. However, the RLA is carried out by randomly sampling from the digital records, identifying their corresponding paper ballots, and subsequently adjusting the RLA statistics in accordance with any discrepancies.

Proposition 4. *Any combination of digital and paper ballots in the Intermediate World 1 corresponds to a set of digital and paper ballots in the Ideal World. This correspondence ensures identical (1) reported totals, (2) number of voters, and (3) honest voters. Furthermore, for any ballot selected for the RLA in the Ideal World, the discrepancy value is equal to the discrepancy value of its corresponding ballot in the Intermediate World 1.*

Proof. Due to the one-to-one correspondence between digital records and paper ballots (resulting from inclusion and exclusion criteria), and the fact that RLA involves the random selection of ballots, any sequence of paper ballots in the Ideal World corresponds to a matching sequence of digital ballots in the Intermediate World 1 with the same discrepancy values and hence with the same RLA risk. □

Intermediate World 2 In this world, the ballots closely resemble those in Intermediate World 1. However, a unique serial number is recorded alongside the voter's unique identifier on digital ballots. Uniqueness of serial numbers published on the BB can be publicly verified. This serial number effectively replaces the voter's unique identifier on the corresponding paper ballot, serving as an alias for the voter. So, the authority in this world does not check if voters' identifiers printed on the paper ballots form a subset of the those recorded on the BB. The authority still checks if the quantity of votes in the ballot box is equivalent to the number of ballots recorded on the BB. The authority also ensures if the identities, published on the BB, are not spoofed. In this world, there is a potential for a malicious voting machine to deviate from the protocol by printing a permutation of serial numbers (which are published on the BB) on the paper ballots. In other words, although the set of all serial numbers published on the BB are the same as those printed on the paper ballots, the serial number that is assigned to voter i on the BB, might be printed on voter j's paper ballot where $i \neq j$. Consequently, the RLA process in this world unfolds as follows: A digital ballot is chosen at random for the RLA process, and its serial number is utilized to locate the corresponding paper ballot. Subsequently, discrepancies are accurately identified and appropriately integrated into the RLA statistics.

Proposition 5. *Given successful verification of the uniqueness of all serial numbers on the BB (i.e., Section 2.5.2, BB transcript verification (public), item 4), for any combination of digital and paper ballots in the Intermediate World 2, with total discrepancies of D, there exists a corresponding set of digital and paper ballots in the Intermediate World 1. This correspondence guarantees identical (1) reported totals, (2) number of voters, and (3) honest voters, with a total discrepancy equal to D.*

Proof. Let $\hat{\mathcal{B}}$ and $\hat{\mathcal{P}}$ represent the sets of all digital and paper ballots in the Intermediate World 2, denoted as $\hat{\mathcal{B}} = \{\hat{B}_i : i = \{1, \cdots, N\}\}$ and $\hat{\mathcal{P}} = \{\hat{P}_i : i = \{1, \cdots, N\}\}$, where $\hat{B}_i$ (and $\hat{P}_i$) represents the digital ballot recorded on the BB for the voter i (and the paper ballot provided to the voter i). Assume the permutation applied by the adversarial voting machine is denoted as π. Then, the digital ballot B_i in the RLA process is compared with the paper ballot $P_{\pi(i)}$. The total overstatement in this world is computed as follows:

$$\sum_i \hat{B}_i - \hat{P}_{\pi(i)} = \sum_i \hat{B}_i - \sum_i \hat{P}_{\pi(i)} = \sum_i \hat{B}_i - \sum_i \hat{P}_i \tag{1}$$

Let, $\mathcal{B}$ and $\mathcal{P}$, i.e., the sets of all digital and paper ballots for N voters in the Intermediate world 1 are established as follows. The digital ballot B_i is created from $\hat{B}_i$ by removing its voter's unique identifier. Also, P_i is created from $\hat{P}_i$ by replacing its serial number with the voter i's unique identifier. Then, the total overstatement in the Intermediate World 1 world is computed as follows:

$$\sum_i B_i - P_i = \sum_i B_i - \sum_i P_i \tag{2}$$

Since vote selections in each B_i are the same as those in $\hat{B}_i$, we have:

$$\sum_i \hat{B}_i = \sum_i B_i$$

Similarly, since vote selections in each P_i are the same as those in $\hat{P}_i$, we have:

$$\sum_i \hat{P}_i = \sum_i P_i$$

Therefore, the total overstatements in the Intermediate World 2 (computed in Equation 1) is the same as those in the Intermediate World 1 (computed in Equation 2).

□

Intermediate World 3 In this world, the ballots closely resemble those in Intermediate World 2. However, a malicious voting machine might deviate from the protocol by printing arbitrary, non-matching serial numbers on paper ballots. Consequently, the RLA process in this world unfolds as follows: A digital ballot is randomly selected for RLA and subsequently categorized into one of the following groups, each with its corresponding discrepancy specification, where authorities at the Local Counting Center verify this operation:

- 'not matching': When the serial number of a digital ballot fails to match any serial number in the paper ballot manifest, the discrepancy for the selected ballot is assigned according to the One-audit method (where is at most as large as the discrepancy obtained from max-possible method).
- 'one-to-one': When the serial number of a digital ballot matches a single entry in the paper ballot manifest, its discrepancy is computed accordingly and incorporated into the RLA statistics. It's important to note that there is a one-to-one correspondence between 'one-to-one' digital ballots and paper ballots.
- 'duplicated': When the serial number of a digital ballot matches multiple entries in the paper ballot manifest, the discrepancy between the digital record and each of those multiple entries is computed individually. Among these calculated discrepancies, the minimum discrepancy value is selected for inclusion in the RLA statistics. The paper ballot associated with this minimum discrepancy is identified as the corresponding paper ballot for the digital ballot. In cases where multiple paper ballots share the same minimum discrepancy, one of them is randomly chosen to serve as the corresponding paper ballot for the digital ballot.

Proposition 6. *For any combination of digital and paper ballots in the Intermediate World 3, with total discrepancies of D, there exists a corresponding set of digital and paper ballots in the Intermediate World 2. This correspondence guarantees identical (1) reported totals, (2) number of voters, and (3) honest voters, with a total discrepancy equal to or smaller than D.*

Proof. Let $\hat{\mathcal{B}}$ and $\hat{\mathcal{P}}$ represent the sets of all digital and paper ballots in the Intermediate World 3, denoted as $\hat{\mathcal{B}} = \{\hat{B}_i : i = \{1, \cdots, N\}\}$ and $\hat{\mathcal{P}} = \{\hat{P}_i : i = \{1, \cdots, N\}\}$. Each ballot in $\hat{\mathcal{B}}$ has been categorized into 'one-to-one', 'duplicated' or 'not matching' categories. Let $\hat{\mathcal{S}}$ denote the set of all digital ballots in $\hat{\mathcal{B}}$ that belong to 'one-to-one' or 'duplicated'. As outlined in the Intermediate World 3 specifications, each ballot in $\hat{\mathcal{S}}$ corresponds with a paper ballot in $\hat{\mathcal{P}}$. Let the set of these corresponding ballots in $\hat{\mathcal{P}}$ be denoted as $\hat{\mathcal{S}}'$.

Now, we establish a corresponding $\mathcal{B}$ and $\mathcal{P}$ with $|\mathcal{B}| = |\mathcal{P}| = N$ in the Intermediate World 2. Let $\mathcal{B} = \hat{\mathcal{B}}$. Furthermore, $\mathcal{S}' \subseteq \mathcal{P}$ is established s.t. $\mathcal{S}' = \hat{\mathcal{S}}'$. Furthermore, the set $\mathcal{P} \backslash \mathcal{S}'$ is established from $\hat{\mathcal{P}} \backslash \hat{\mathcal{S}}'$ such that the vote selections in each $P_i \in \mathcal{P} \backslash \mathcal{S}'$ is equal to the vote selections in $\hat{P}_i$. However, the set of serial numbers for ballots in $\mathcal{P}\backslash\mathcal{S}'$ are any arbitrary permutations of the set of serial numbers in $\mathcal{B}\backslash\mathcal{S}$. Consequently, there is a one-to-one correspondence between serial numbers in $\mathcal{B}$ and $\mathcal{P}$.

It is also important to note that the paper ballot for voter i in Intermediate World 2 mirrors the vote selections present on the paper ballot for the same voter in Intermediate World 3. Now let's compare the total discrepancies in these two worlds. If the discrepancy of the ballot B_i is denoted as d_i,since there is a one-to-one correspondence between serial numbers in $\mathcal{B}$ and $\mathcal{P}$, the total discrepancies in the Intermediate World 2 is computed as:

$$\begin{aligned}\sum_{\mathcal{B}} d_i = \sum_{\mathcal{B}} B_i - \sum_{\mathcal{P}} P_i =&[\sum_{\mathcal{S}} B_i + \sum_{\mathcal{B}\backslash\mathcal{S}} B_i] - [\sum_{\mathcal{S}'} P_i + \sum_{\mathcal{P}\backslash\mathcal{S}'} P_i] \\ =&[\sum_{\mathcal{S}} B_i - \sum_{\mathcal{S}'} P_i] + [\sum_{\mathcal{B}\backslash\mathcal{S}} B_i - \sum_{\mathcal{P}\backslash\mathcal{S}'} P_i]\end{aligned} \tag{3}$$

In the Intermediate World 3, digital ballots in $\hat{\mathcal{B}}\backslash\hat{\mathcal{S}}$ have no corresponding paper ballots and hence their discrepancies is set according to the One-audit. Therefore, if the discrepancy of the ballot $\hat{B}_i$ is denoted as $\hat{d}_i$, the total discrepancies in the Intermediate World 3 is computed as:

$$\sum_{\hat{\mathcal{B}}} \hat{d}_i =[\sum_{\hat{\mathcal{S}}} \hat{B}_i - \sum_{\hat{\mathcal{S}}'} \hat{P}_i] + [\sum_{\hat{\mathcal{B}}\backslash\hat{\mathcal{S}}} \hat{B}_i - |\hat{\mathcal{P}}\backslash\hat{\mathcal{S}}'| \cdot \min\{\hat{P}\}] \tag{4}$$

where $[\sum_{\hat{\mathcal{S}}} \hat{B}_i - \sum_{\hat{\mathcal{S}}'} \hat{P}_i]$ in 4 equals $[\sum_{\mathcal{S}} B_i - \sum_{\mathcal{S}'} P_i]$ in 3. However, since (1) vote selections in B_i are the same as those in $\hat{B}_i$, and (2) $|\hat{\mathcal{B}}\backslash\hat{\mathcal{S}}| = |\mathcal{B}\backslash\mathcal{S}|$, and (3) $|\hat{\mathcal{P}}\backslash\hat{\mathcal{S}}'| = |\mathcal{P}\backslash\mathcal{S}'|$, then $\sum_{\mathcal{P}\backslash\mathcal{S}'} P_i \geq |\hat{\mathcal{P}}\backslash\hat{\mathcal{S}}'| \cdot \min\{\hat{P}\}$ and hence $[\sum_{\hat{\mathcal{B}}\backslash\hat{\mathcal{S}}} \hat{B}_i - \min\{\sum_{\hat{\mathcal{P}}\backslash\hat{\mathcal{S}}'} \hat{P}_i\}]$ in 4 is at least as large as $[\sum_{\mathcal{B}\backslash\mathcal{S}} B_i - \sum_{\mathcal{P}\backslash\mathcal{S}'} P_i]$ in 3. So, the total overstatement in the Intermediate World 3 is at least as large as that in the Intermediate World 1. Note that since the total discrepancies obtained by One-audit method are at most as large as that from the max-possible method, the statement also holds if discrepancies for 'not matching' ballots are set according to the max-possible method.

□

Intermediate World 4 The only distinction between this world and the previous world is that voters controlled by the adversary might choose not to put their ballot in the ballot box or might be provided with multiple ballots to put in the ballot box. However, the authority checks if the quantity of ballots in the box does not exceed the number of ballots on the BB. Authority still ensures if voters identities, recorded on the BB, are not spoofed.

Proposition 7. *Given that the quantity of ballots in the box does not exceed the number of ballots on the BB (i.e., Section 2.5.3, Verification at the Local Counting Center, item 9), any combination of digital and paper ballots in the Intermediate World 4 corresponds to a set of digital and paper ballots in the Intermediate World 3. This correspondence ensures identical (1) reported totals, (2) number of voters, and (3) honest voters. Furthermore, for any ballot selected for the RLA in the Intermediate World 3, the discrepancy value is equal to or smaller than the discrepancy value of its corresponding ballot in the Intermediate World 4.*

Proof. Let $\tilde{\mathcal{B}}$ with $|\tilde{\mathcal{B}}| = N$ denote the set of all digital ballots on the BB in the Intermediate World 4. Also, let the number of honest voters be denoted by n. Since all honest voters verify their ballots and put them in the box, n paper records in the ballot box belong to the honest voters. Let those paper ballots be represented by $\mathcal{P}_h$. Also, due to the fact that the quantity of ballots in the box does not exceed the number of ballots on the BB, $m \leq N - n$ paper ballots in the box have been cast by the voters who are controlled by the adversary in the Intermediate World 4. Let paper ballots, cast by adversary's controlled voters, be denoted by $\mathcal{P}_m$.

Now let the set of digital records on the BB in the Intermediate World 3, denoted by $\hat{\mathcal{B}}$, be the same as $\tilde{\mathcal{B}}$. Consequently, any sequence of ballots chosen for auditing from $\tilde{\mathcal{B}}$ corresponds to a sequence of ballots from $\hat{\mathcal{B}}$. Also, let $\mathcal{P}_h$ and $\mathcal{P}_m$ be respectively provided to the honest voters and $|\mathcal{P}_m|$ members of the voters who are controlled by the adversary in the Intermediate World 3. Other adversary's controlled voters are provided with some arbitrary paper ballots selected by the adversary. Therefore, paper ballots in the Intermediate World 4, denoted by $\tilde{\mathcal{P}}$ form a subset of paper ballots in the Intermediate World 3, denoted by $\hat{\mathcal{P}}$.

Building upon this foundation, we are now ready to demonstrate that for any digital ballot in $\tilde{\mathcal{B}}$, its discrepancy is at least as large as the discrepancy of its corresponding digital ballot in $\hat{\mathcal{B}}$.

Let's consider a scenario where a digital ballot in $\tilde{\mathcal{B}}$ with serial number SN is randomly selected to be audit. Then, depending on the category of the corresponding digital ballot in $\hat{\mathcal{B}}$, we have the following cases:

- If the corresponding ballot in Intermediate World 3 falls into 'not matching', it will continue to belong to 'not matching' in Intermediate World 4. This arises from the fact that the paper ballots in $\tilde{\mathcal{P}}$ form a subset of those in $\hat{\mathcal{P}}$. Consequently, as there is no paper ballot with the serial number SN in $\hat{\mathcal{P}}$, there also exists no paper ballot with this serial number in $\tilde{\mathcal{P}}$. Therefore, the corresponding discrepancy will be set to the same value in both worlds.

- Suppose the corresponding digital record in Intermediate World 3 falls into 'one-to-one'. If its corresponding paper ballot in Intermediate World 3 remains a part of the paper manifest in Intermediate World 4, the discrepancy remains unchanged. Otherwise, that digital ballot in the Intermediate World 4 moves to the 'not matching' category, and its discrepancy is adjusted according to the One-audit method which is maximum discrepancy among all possible paper ballots for the selected digital ballot. This discrepancy is at least as large as the discrepancy in Intermediate World 3. The statement is clearly true if we adjust the discrepancy based on the max-possible method.

- Consider the corresponding digital record $\hat{B}$ in Intermediate World 3 falls into 'duplicated'. The corresponding paper ballots in $\hat{\mathcal{P}}$ with the serial number SN are represented by the set $\hat{\mathcal{S}'}_{SN}$. Discrepancy for $\hat{B}$ is determined based on the paper ballot within $\hat{\mathcal{S}'}_{SN}$ that exhibits the minimum discrepancy. As $\tilde{\mathcal{P}}$ is a subset of $\hat{\mathcal{P}}$, one of two scenarios occurs: either the paper ballot with the minimum discrepancy remains a part of $\tilde{\mathcal{P}}$, in which case the discrepancy in Intermediate World 4 remains unchanged, or it does not. In the latter situation, the discrepancy is calculated based on other paper ballots within $\hat{\mathcal{S}'}_{SN}$ that are part of $\tilde{\mathcal{P}}$, where the discrepancy is no less than that of the original non-received paper ballot. If none of the paper ballots in $\hat{\mathcal{S}'}_{SN}$ are included in the paper ballot manifest of Intermediate World 4, then the digital ballot in Intermediate World 4 transitions to 'not matching', resulting in its discrepancy being set to the maximum possible discrepancy among all possible paper ballots for the selected digital ballot. Thus, this discrepancy is at least as large as the discrepancy in Intermediate World 3. The statement is clearly true if we adjust the discrepancy based on the max-possible method.

Hence, in all possible scenarios, the discrepancy of a digital ballot in Intermediate World 4 will either match the corresponding discrepancy in Intermediate World 3 or be set to a larger value. $\square$

Intermediate World 5 In this world, the voting process and the paper path differ from the previous world as follows:

1. There is a trusted party who voters can rely on to guarantee the presence of a ballot corresponding to their unique identifier on the BB,
2. Voters place their ballot inside an envelope labeled with their ID and send it to the Local Counting Center.
3. Every paper ballot undergoes the same processing steps as those within basic-MERGE (steps 6 and 7 from the paper path explained in Section 2.3). However, rather than checking the signatures on the stickers, jurisdiction authorities accept any envelope labeled with IDs on the BB.

4. The adversarial model is according to Electronic&Stuff or Electronic&Drop models, defined in Section 1.5.1, and

5. In basic-MERGE, given that multiple validly signed envelopes from a given voter is received, the first one is accepted and the next ones are set aside. However, here we assume all of them are set aside, creating a stack called "Large Eligibility Problem" stack.

Proposition 8. *Given*

- *no reported errors from voters regarding missing votes on the BB (i.e., Section 2.5.1, Cast-as-intended verification, item 3),*
- *verification that the Rejected Envelopes Stack includes all envelopes with the invalid signatures (i.e., Section 2.5.3, Verification at the Local Counting Center, item 3),*
- *Verify that the Large Eligibility Problem Stack includes all duplicate envelopes from the same voter (including the first envelope from any given voter and the duplicate ones).*

any combination of digital and paper ballots in the Intermediate World 5 corresponds to a set of digital and paper ballots in the Intermediate World 4. This correspondence ensures identical (1) reported totals, (2) number of voters, and (3) honest voters. Furthermore, for any ballot selected for the RLA in the Intermediate World 4, the discrepancy value is equal to or smaller than the discrepancy value of its corresponding ballot in the Intermediate World 5.

Proof. Let N denote the total digital ballots on the BB. Assume the number of honest voters be denoted by n. Since no errors are reported by any honest voter, the n digital records on the BB correspond to these honest voters. So, we have $N \geq n$. Now assume an adversarial model with Electronic&Drop. In this model, although the envelopes mailed by honest voters might drop, the adversary is unable to stuff new envelopes on behalf of honest voters. Furthermore, the envelope that each honest voter mails, once received, remains unaltered and since the sticker on the envelope has already been checked by the voter or an authority, it is accepted at the Local Counting Center. Therefore, paper ballots inside envelopes received from honest voters at the Local Counting Center in the Intermediate World 5, form a subset of paper ballots viewed and mailed by the honest voters.

Similarly, in the Electronic&Stuff model, all envelopes transmitted by the voters are received at the Local Counting Center within a known timeframe. Also, since the presence of a vote with the correct identifier is verified by the voter, it is guaranteed that, once received at the Local Counting Center, envelopes will be identified as valid envelopes. Although some of these envelopes may be invalidated due to envelope stuffing, the remaining ones (i.e., those outside the Large Eligibility Problem Stack) remain consistent with the envelopes originally mailed by the voters. Since in this model, delivered envelopes remain unaltered, the paper ballots accepted from honest voters will be a subset of those viewed and mailed by honest voters.

Therefore, so far we have demonstrated that in the Intermediate World 5, for both Electronic&Stuff and Electronic&Drop models, paper ballots received and accepted from honest voters at the Local Counting Center represented by $\mathcal{P}'_h$ form a subset of paper ballots viewed and mailed by the honest voters, represented by $\mathcal{P}_h$. Moreover, since duplicate envelopes or envelopes with invalid signatures or envelopes with no corresponding ballot on the BB are stacked accordingly and their ballots are not used in the RLA process, the number of accepted paper ballots does not exceed the quantity of ballots on the BB. Consequently, for the papers included in envelopes received from voters who are controlled by the adversary, denoted by $\mathcal{P}_m$, we have: $0 \leq |\mathcal{P}_m| \leq N - n$. Now, consider the same set of digital ballots $\mathcal{B}$ recorded by the voting machine on the BB in the Intermediate World 4. Also, the voting machine provides the same paper ballots as $\mathcal{P}_h$ to their corresponding honest voters in the Intermediate World 8, and provides ballots in $\mathcal{P}_m$ to $|\mathcal{P}_m|$ members of voters controlled by the adversary, instructing them to place their ballots in the ballot box. The adversary instructs other voters not to place their ballots in the ballot box in the Intermediate World 4. Now digital ballots in both Intermediate World 4 and 5 are the same but paper ballots which are used as the input to the RLA process in the Intermediate World 5 are a subset of those in the Intermediate World 4. Building on the second paragraph of the proof of Proposition 7 onward, the discrepancy of any digital ballot in Intermediate World 5 will either match the corresponding discrepancy in Intermediate World 4 or be set to a larger value. This completes the proof.

□

The following world is gradually created from Intermediate World 4 or 5. For the Electronic-only model, Intermediate World 6 is directly established upon Intermediate World 4. But for the Electronic&Stuff or Electronic&Drop models, Intermediate World 6 is created by slightly changing the Intermediate World 5.

Intermediate World 6 The only distinction between this world and Intermediate World 5 (or Intermediate World 4) is that the scanning device might malfunction by recording incorrect serial numbers.

Proposition 9. *Given that no "serial number error" is observed during the RLA (i.e., Section 2.5.3, Verification at the Local Counting Center, item 8), any combination of digital and paper ballots in the Intermediate World 6 corresponds to a set of digital and paper ballots in the Intermediate World 5. This correspondence ensures identical (1) reported totals, (2) number of voters, and (3) honest voters. Furthermore, for any ballot selected for the RLA in the Intermediate World 5, the* *discrepancy value is equal to or smaller than the discrepancy value of its corresponding ballot in the Intermediate World 6.*

Proof. In Intermediate World 6, when a digital ballot is chosen for the RLA, its discrepancy is set to the maximum possible value for that ballot in the One-audit method (or +2 in the max-possible approach) unless a corresponding paper ballot with the same serial number and a lower discrepancy is identified in the paper record. If such a paper ballot exists in Intermediate World 6, it implies the coexistence of the same digital and paper ballot in Intermediate World 8 (or Intermediate World 5). Consequently, the discrepancy for a digital ballot in Intermediate World 8 (or Intermediate World 5), assuming a trustworthy scanning device, would not exceed its corresponding discrepancy value in the Intermediate World 6.

□

Intermediate World 7 In this world, given that multiple validly signed envelopes from a given voter is received, the first one is opened and the next ones are set aside to create a stack called Eligibility Problem stack (the same as basic-MERGE). Furthermore, the procedure mentioned in step 9.4 of Section 2.4.1, is accordingly used to adjust discrepancies. So, for every unique voter with a validly signed envelope in the "Eligibility Problem" stack, a maximum discrepancy (+2) is arbitrarily added to their county votes.

Proposition 10. *Given that Eligibility Problem Stack includes all (and only) duplicate envelopes from the same voter (i.e., Section 2.5.3, Verification at the Local Counting Center, item 2), for any combination of digital and paper ballots in the Intermediate World 7, with total discrepancies of D, there exists a corresponding set of digital and paper ballots in the Intermediate World 6. This correspondence guarantees identical (1) reported totals, (2) number of voters, and (3) honest voters, with a total discrepancy equal to or smaller than D.*

Proof. In Intermediate World 6, ballots outside the Large Eligibility Problem stack constitute a subset of the actual paper ballots mailed by voters (as discussed in the second paragraph of Proposition 8). Consequently, in Intermediate World 7, if a single validly signed envelope is received from a voter at the Local Counting Center, the contained ballot corresponds to the actual ballot mailed by that voter.

However, in this world, if multiple validly signed envelopes from the same voter are received at the Local Counting Center, only the first one is opened, while the rest are set aside.

Since in the Electronic&Stuff model, the first accepted envelope could potentially be a stuffed envelope, its contents may not necessarily represent the actual ballot sent by the voter. Consequently, the ballots outside the Eligibility Stack problem could contain some ballots that are not necessarily a subset of the ballots mailed by the voters. The count of validly signed envelopes with unique voter identifiers in the Eligibility Problem stack represents the number of potentially incorrect paper ballots. Each incorrect paper ballot could match an incorrect digital ballot recorded by the malicious voting machine in favor of the reported election winner. However, since in Intermediate World 7, a maximum discrepancy value (+2) is arbitrarily added to the county votes for each unique voter with a validly signed envelope in the 'Eligibility Problem' stack, the total discrepancies across all ballots in Intermediate World 7 are at least as great as those in Intermediate World 6.

In the Electronic&Drop model, since stuffing envelopes on behalf of the legitimate voter is prevented, the occurrence of receiving multiple valid envelopes from a single voter is only plausible if the voter mails all envelopes personally. Although this case requires further investigations, adding a maximum discrepancy value (+2) to the county votes for each unique voter with a validly signed envelope in the 'Eligibility Problem' stack guarantees that the number of discrepancies in Intermediate World 7 are at least as great as those in Intermediate World 6.

□

Intermediate World 8 Differences between this world and the Intermediate World 7 are as follows: The serial numbers and the vote contents of digital ballots are initially recorded by the voting machine in an encrypted format. Each digital ballot is also accompanied by a zero-knowledge proof demonstrating the validity of the (encrypted) vote. Subsequently, all digital ballots undergo the same processing steps as those within basic-MERGE (i.e., steps 1 to 5 from the digital path explained in Section 2.2) followed by private decryption in step 8.

Proposition 11. *Given that correctness of*

- *all $ZKP_{\textsc{Valid}}()$ proofs (i.e., Section 2.5.2, BB transcript verification (public), item 1b),*
- *all $ZKP_{\textsc{Decrypt}}(,)$ proofs including:*
  - *decryption of all serial numbers (i.e., Section 2.5.2, BB transcript verification (public), item 3),*
  - *decryption of vote selections for ballots that are selected for the RLA process (i.e., Section 2.5.3, Verification at the Local Counting Center, item 5), and*
  - *decryption of the encrypted tally obtained at step 3 of the digital path (i.e., Section 2.5.2, BB transcript verification (public), item 6)*
- *$ZKP_{\textsc{Mix}}(,)$ proof (i.e., Section 2.5.2, BB transcript verification (public), item 2), and*
- *aggregation of encrypted ballots at step 2 of the digital path in the Intermediate World 8 (i.e., Section 2.5.2, BB transcript verification (public), item 5)*

are successfully verified, any combination of digital and paper ballots in the Intermediate World 8 corresponds to a set of digital and paper ballots in the Intermediate World 7. This correspondence ensures identical (1) reported totals, (2) number of voters, and (3) honest voters. Furthermore, for any ballot selected for the RLA in the Intermediate World 7, the discrepancy value is equal to or smaller than the discrepancy value of its corresponding ballot in the Intermediate World 8.

Proof. Let $\mathcal{B}$ denote the list of paper ballots and digital ballots recorded on the BB for N voters before initiation of the step 1 in the Intermediate World 8:

$$\mathcal{B} = \{(E(sn_i), E(\mathbf{b_i}), \mathit{ZKP}_{\mathsf{VALID}}(\mathbf{b_i}), \mathit{Domain\_separator\_BB})) : i \in \{1, \cdots, N\}\}\}$$

By relying on the security guarantees provided by the $\mathit{ZKP}_{\mathsf{VALID}}()$ protocol, the validation of $\mathit{ZKP}_{\mathsf{VALID}}()$ for all ballots ensures that, for any $i \in \{1, \cdots, N\}$, $\mathbf{b_i}$ represents a valid vote. This crucial property will later enable us to use the same vote selections in the Intermediate World 7. After applying the mixnet, the list transforms to $\mathcal{B}'$:

$$\mathcal{B}' = \{(E(sn'_j), E(\mathbf{b'_j})) : j = \pi(i), i \in \{1, \cdots, N'\}\}\}$$

Due to validity of $\mathit{ZKP}_{\mathsf{MIX}}(,)$, we have $N' = N$, $sn'_j = sn_i$, $\mathbf{b'_j} = \mathbf{b_i}$ for any $i \in \{1, \cdots, N\}$ with π denoting a permutation from $\{1, \cdots, N\}$ to $\{1, \cdots, N\}$. Deviation from these conditions would have violated our security about the mixnet protocol. Digital ballots in $\mathcal{B}'$ are then privately decrypted and used for the RLA process. Digital ballots after private decryption, denoted by $\mathcal{B}''$, are as follows:

$$\mathcal{B}'' = \{(sn''_j, \mathbf{b''_j}) : j = \pi(i), i \in \{1, \cdots, N\}\}\}$$

The validity of $\mathit{ZKP}_{\mathsf{DECRYPT}}(,)$ ensures $sn''_j = sn'_j$ and $\mathbf{b''_j} = \mathbf{b'_j}$ (and hence $sn''_j = sn_i$ and $\mathbf{b''_j} = \mathbf{b_i}$), with $j = \pi(i)$. Otherwise, the security of the proof of decryption protocol would be violated.

Now let

$$\mathcal{B}''' = \{(sn_i, \mathbf{b_i}) : i \in \{1, \cdots, N\}\}\}$$

be the digital ballots recorded by the voting machine on the BB in Intermediate World 9. The tally in Intermediate World 7 ($\sum_{i=1}^{N} \mathbf{b_i}$) matches the result of decryption at step 2 of the protocol in the Intermediate World 8 ($D(\prod_{j=1}^{N} E(\mathbf{b'_j}))$). This equivalence is due to the homomorphic properties of the encryption algorithm. The private decryption at step 3 of the protocol in the Intermediate World 8 results in $\sum_{j=1}^{N} \mathbf{b'_j}$. Otherwise the security of the proof of decryption protocol would be violated. Now assume the paper manifest in both Intermediate Worlds 7 and 8 be the same. Each sequence of ballots in Intermediate World 8 corresponds to a sequence in Intermediate World 7 (established through the π^{-1} permutation function), maintaining identical discrepancy values. Consistency in both discrepancy values and tally across these intermediate worlds ensures an equivalent RLA risk. □

Intermediate World 9 In this world, instead of depending on a trusted third party, the voter's digital signature on their digital ballot serves as publicly verifiable evidence that the voter's identity is not spoofed.

Proposition 12. *Given that all digital signatures on the BB are successfully verified (i.e., Section 2.5.2, BB transcript verification (public), item 1a), any combination of digital and paper ballots in the Intermediate World 9 corresponds to a set of digital and paper ballots in the Intermediate World 8. This correspondence ensures identical (1) reported totals, (2) number of voters, and (3) honest voters. Furthermore, for any ballot selected for the RLA in the Intermediate World 8, the discrepancy value is equal to or smaller than the discrepancy value of its corresponding ballot in the Intermediate World 9.*

Proof. We first demonstrate that the voter's digital signature on their ballot proves that their identity is not spoofed. Consider the opposite in the Intermediate World 9: a scenario where a voter's identity on the BB is spoofed. The private key corresponding to each voter's digital signature is securely stored in their smart card, accessible only to the voter. If, however, the voting machine manages to generate a valid digital signature on the BB without the voter actively attempting to cast a vote and hence without access to their smart card, it would compromise the security of the digital signature system. Such a situation contradicts our initial assumption regarding the security of the cryptographic primitives employed in the system. Therefore, if all digital signatures on the BB are successfully verified, voters' identities are not spoofed. Moreover, since apart from the digital signature, the serial numbers and vote selections in paper and digital ballots in the Intermediate World 9 are the same as those in the Intermediate World 8, then for any sequence of ballots selected for auditing, the number of discrepancies detected by the audit in the Intermediate World 9 is equal to that in the Intermediate World 8 and hence RLA risks in both worlds are the same. □

Intermediate World 10 In this world, the voter with identifier Id_i who is voting in an election with the identifier Id_e, rather than depending on a trusted party to guarantee the presence of their digital ballots on the BB, is provided with a sticker containing the following voting machine's digital signature $sig_{\text{Voting Computer}}(Id_i, Id_e, H_i)$), where H_i is generated by the voting machine based on the digital ballot stored for this voter on the BB. Voters verify if (1) the digital signature on the sticker is successfully verified with correct voter's and election's identifiers and (2) if a ballot is ultimately published on the BB under their name. If either of these conditions are not met, the voter reports an error using the sticker data as evidence. Subsequently, they place the ballot in an envelope, affix the sticker, and dispatch the envelope to the Local Counting Center. Once received at the Local Counting Center, rather than verifying the ID on the envelope, the envelope is processed according to step 2.3 of the basic-MERGE.

Proposition 13. *Given that*

- *successful verification of each digital signature on a sticker by its corresponding voter (i.e., Section 2.5.1, Cast-as-intended verification, item 2),*
- *no reported errors from voters regarding missing votes on the BB (i.e., Section 2.5.1, Cast-as-intended verification, item 3),*
- *verification of the digital signatures on the stickers at the Local Counting Center (i.e., Section 2.5.3, Verification at the Local Counting Center, item 1),*

any combination of digital and paper ballots in the Intermediate World 10 corresponds to a set of digital and paper ballots in the Intermediate World 9. This correspondence ensures identical (1) reported totals, (2) number of voters, and (3) honest voters. Furthermore, for any ballot selected for the RLA in the Intermediate World 9, the discrepancy value is equal to or smaller than the discrepancy value of its corresponding ballot in the Intermediate World 10.

Proof. We first demonstrate that the voting machine's digital signature $sig_{\text{Voting Computer}}(Id_i, Id_e, H_i)$, printed on the sticker which is provided to a voter with unique identifier Id_i, proves the voting machine's commitment to publish a digital ballot for a voter with the same identifier Id_i in an election with the same election identifier Id_e on the BB. Consider the opposite in the Intermediate World 10: a scenario where $sig_{\text{Voting Computer}}(Id_i, Id_e, H_i)$ does not prove the voting machine's commitment to publish the corresponding digital ballot on the BB. The private key of the voting machine is securely held by the voting machine. If, however, anyone manages to generate a valid digital signature without the voting machine's involvement and hence without access to their private key, it would compromise the security of the digital signature system. Such a situation contradicts our initial assumption regarding the security of the cryptographic primitives employed in the system. Moreover, since apart from the digital signature, the serial numbers and vote selections in paper and digital ballots in the Intermediate World 10 are the same as those in the Intermediate World 9, then for any sequence of ballots selected for auditing, the number of discrepancies detected by the audit in the Intermediate World 20 is equal to that in the Intermediate World 9 and hence RLA risks in both worlds are the same.

□

Real world Game (adversarial model A) The adversary $\mathcal{A}$ wins the real-world game with adversarial model A if

- the apparent election result, according to the (MERGE and regular) CVRs, is different from the actual result, according to the (MERGE and regular) paper ballots, and
- the RLA procedure (including the MERGE cryptographic verification) confirms the result with probability negligibly greater than α, where the probability is taken over the random ballot selections of the RLA.

Proposition 14. *For any combination of digital and paper ballots in our protocol under the Electronic-only adversarial model and with total discrepancies of D, there exists a corresponding set of digital and paper ballots in the Ideal World. This correspondence guarantees identical (1) reported totals, (2) number of voters, and (3) honest voters, with a total discrepancy equal to or smaller than D.*

Proof. Our protocol with the Electronic-only adversarial model is the same as the Intermediate World 9 established upon the Intermediate World 8, which in turn stems from Intermediate World 6, where Intermediate World 6 is established upon Intermediate World 4. Therefore, the proof follows directly from Propositions 4 to 7, 9, 11 and 12. □

Corollary 15. *If the adversary wins the Real World Game in the Electronic-only adversarial model, then it wins the Ideal World Game.*

Proposition 16. *For any combination of digital and paper ballots in our protocol under the Electronic&Stuff or Electronic&Drop adversarial model and with total discrepancies of D, there exists a corresponding set of digital and paper ballots in the Ideal World. This correspondence guarantees identical (1) reported totals, (2) number of voters, and (3) honest voters, with a total discrepancy equal to or smaller than D.*

Proof. Since our protocol with the Electronic&Stuff or Electronic&Drop adversarial model is the same as the Intermediate World 7, the proof follows directly from Propositions 4 to 13. □

Corollary 17. *If the adversary wins the Real World Game in either the Electronic&Stuff or Electronic&Drop adversarial models, then it wins the Ideal World Game.*

2.9 Security analysis: Authentication

Only the eligible voters are able to cast their vote.

2.10 Security analysis: Privacy

In this section, we delve into the privacy aspects of our system, focusing on safeguarding the confidentiality of voter's intention. Privacy, in this context, involves preventing unauthorized

Table 3: Data accessible to various entities across different stages of the voting process.

Entity	ID	SN	ID&SN	Vote	Tally
General public	✓	✓	×	×	✓†
BMD machine	✓	✓	✓	✓	✓
RLA observers	×	✓	×	✓	✓†
Postal worker	✓	×	×	×	✓†

†: visible on the BB

access to information regarding individual voters' choices. The general public, i.e, anyone who views the BB, knows the list of identifiers and serial numbers for the participating voters. Due to the mixnet properties, they cannot link these two lists though. Also, the vote contents are encrypted on the BB, ensuring that no information about the voter's intention is revealed. Subsequently, after the mixnet, the decrypted vote contents are made accessible to the RLA auditor and observers. However, these decrypted ballots cannot be linked to the voters' identities, making each voter indistinguishable from others whose ballots contain the same voting options. Furthermore, the anonymity is maintained throughout the process at the Local Counting Center, as there are no unique markings on the paper ballot that would identify the voter. While the digital signature (and optionally the handwritten signature) on the envelope serves to identify the voter, the serial number and the ballot's vote contents remain hidden from postal workers. The BMD machine possesses complete information, including the voter's identifier, serial number, and vote contents. Table 3 provides an overview of the access levels granted to various stakeholders, aiding them in comprehending and analyzing individual voter choices.

According to this table, exclusive access to both the voter's identifier and their votes is limited to the BMD machine alone; no other single entity possesses this combination of information for any given voter.

Section 3.3 elaborates on the removal of serial numbers from our protocol, particularly in scenarios involving batch auditing. This ensures the preservation of voter privacy, even in the event of collusion between a coercer and auditors or observers at Local Counting Center.

However, there are still some privacy issues worth considering.

- **Small anonymity sets** may reveal individuals' preferences, for example if there are only a very small number of remote votes sent to one Local Counting Center, and they all make the same choices. Section 3.2 presents a solution to this problem. Our proposed solution involves merging our protocol's ballots with those from non-basic-MERGE voters, thereby enlarging the anonymity set.

Table 4: Data accessible to the coercer and the RLA observers.

Entity	ID	SN	ID&SN	Vote	Tally
Coercer	✓	✓	✓	×	✓†
RLA observers	×	✓	×	✓	✓†

†: visible on the BB

- **Italian attacks** may allow for coercion, if a person has many options on one ballot. However, our protocol resists this attack due to the fact that ballots on the BB are encrypted and contest-wise homomorphic tallying is performed on them. RLA auditors do have access to individual paper ballots, thereby presenting a coercion opportunity that appears inherent in any RLA process.
- **Group privacy** of the whole set of remote voters. This may cause concern even if the group is large, for example if the collective choices are strongly skewed relative to the rest of the population.

2.11 Security analysis: Receipt freeness

We assume the presence of a coercer outside the voting booth who is already aware of the target voter's identity and requests their serial number. Given that (1) serial numbers are ultimately disclosed on the BB and (2) serial numbers are sufficiently long, the voter is unable to provide false information about their serial number. Nevertheless, the coercer cannot rely on the voter's honesty regarding their cast vote. In other words, the voter cannot convince the coercer that the encrypted value $E(\mathbf{b_i})$ printed on the the sticker or published on the BB before the mixnet or the $E(\mathbf{b_i})$ in the Remote committed CVR (i.e. $(sn_i, E(\mathbf{b_i}))$ after the mixnet) is decrypted to the $\mathbf{b_i}$ that is claimed by the voter as her vote. However, as Table 4 shows, collusion between a coercer and auditors or observers at the Local Counting Center could enable the use of serial number data to connect voters' identities with their votes.

2.12 Security analysis: End-to-end verifiability

Not achieved. The electronic channel has no cast-as-intended verifiability, while the paper channel has no individual recorded-as-cast verifiability. The alignment between the electronic and paper channels relies on observers in the Local Counting Center to check the RLA.

3 Using MERGE with other RLA styles

The description above assumes a Local Counting Center using ballot-level comparison RLAs for all ballots, including those from MERGE. However, there are several other variants that work well with other RLA styles and may be more appropriate in some scenarios, particularly when the number of MERGE ballots is small.

3.1 VAULT-MERGE: incorporating MERGE in to an RLA that uses VAULT for all ballots

The VAULT RLA scheme [BST19] protects against pattern-based coercion attacks (often called "Italian attacks") by hiding individual ballots unless they are selected for RLA. MERGE ballots can easily be incorporated into a VAULT RLA. This protects against both the problem of small sub-tallies for MERGE ballots and also the individual pattern-based coercion attacks on all ballots (at least, all ballots that are not selected for audit).

There are two main steps. In the first step, all the votes are tabulated on a VAULT BB. This step is simple: the MERGE ballots that are *output* by the mix are in exactly the right form for VAULT. The ordinary ballots from the Local Counting Center can simply be appended—at this step, it is perfectly acceptable for the different types of votes to be distinguishable because they are all encrypted. The tallies are then computed over the whole set of votes. This obviates the need to publish the sub-tally of MERGE ballots in item 3 of the MERGE digital path.

In the second step, the RLA is conducted using the ballot-level comparison method, with samples taken at random from the VAULT-tabulated ballots. Discrepancies are calculated as follows:

- if the ciphertext is a MERGE ballot, it should be treated as described in subsubsection 2.4.1, using a ballot paper of matching serial number if possible, and a worst-case assumption if there is no ballot paper,
- if the ciphertext is an ordinary ballot, the paper ballot should be found and the discrepancy calculated as in a standard VAULT RLA.

The main restriction, compared with standard VAULT, is that for the MERGE ballots, the ciphertext must be decrypted rather than opened, because the authorities do not know the random factors used to generate it. The ordinary votes need to use the same encryption scheme as the MERGE ones, so that they can be aggregated together—they may be either decrypted like the MERGE ballots or opened (from the randomness used to generate them) like standard VAULT ballots.

Table 5: Data accessible to various entities in Sub-MERGE.

Entity	ID	SN	ID&SN	Vote	MERGE Tally
General public	✓	✓	×	×	×
BMD machine	✓	✓	✓	✓	✓
Coercer	✓	✓	✓	×	×
RLA observers	×	✓	×	✓	×
Postal worker	✓	×	×	×	×

This method allows for a very efficient audit (ballot-level comparison for all ballots) with good privacy properties, even if the MERGE ballots are only a small set.

3.2 Sub-MERGE: ballot-level comparison audits dealing with very small numbers of MERGE ballots

If a Local Counting Center has only a very small number of MERGE ballots, it may not be acceptable to publish their tally in item 3. By exposing the tallies for the subset of MERGE ballots, the protocol may reveal significant information about individual votes. They may be unanimous or near-unanimous, or there may be some contests in which only one MERGE voter was eligible.

When an attacker has direct access to the arriving mail ballot, we do not have a solution to this issue. However, if we consider a remote attacker who is looking at the sub-tallies on the BB, the problem can be addressed by including the MERGE tally into a larger tally of ballots. The best way to do this is to use VAULT (See subsection 3.1). This section describes a lightweight alternative that may be easier to implement for Local Counting Centers that do not already use VAULT.

The main idea is to make a VAULT-like batch that includes all the MERGE ballots and enough ordinary ballots to constitute a reasonable anonymity set. We tally the batch both electronically and on paper, then use one of several possible methods for incorporating the batch into an otherwise ballot-level comparison audit. Table 5 presents an overview of the access levels granted to various stakeholders in Sub-MERGE. Clearly, sub-tallies from MERGE ballots are not visible on the BB. This is explicitly depicted in the last column of the table.

Let's provide more details regarding this method. Call the (small) set of MERGE ballots on the BB M. When $|M|$ is small, mixing does not have much benefit. We could use the ballots output from the mix, but it is just as valid to use the signed, encrypted votes from *before* the mixing step. Either way, this gives us the encryption of the total tally, either the one published in item 2 of the digital path, or an equal one derived from the ballots output by the mix.

1. Gather a collection of local ballots L from those available in the Local Counting Center. They may have been cast in person, or they may be other (non-MERGE) absentee ballots. L should be chosen so that, when combined with the MERGE ballots, the anonymity set is large enough for the tallies to be published. Probably $|L \cup M| \approx 30$.

   We assume that ballots in L are already disassociated from the voter's name.

2. Encrypt the ballots in L using the encryption scheme from subsection 1.6, including validity proofs. Post them on the BB.

3. Use the homomorphic property to compute the combined aggregate of $L \cup M$. Decrypt it and publish the ciphertext with proof of correct decryption on the BB. This gives us, on the BB:

$$E(\mathbf{Tally_{L\cup M}}), \mathbf{Tally_{L\cup M}}, \mathit{ZKP}_{\text{DECRYPT}}(\mathbf{Tally_{L\cup M}}, E(\mathbf{Tally_{L\cup M}}))$$

4. Manually tally the combined batch comprising L and the MERGE ballots. Call the resulting tally $\mathbf{Tally}_{\text{MANUAL}}$.

5. Compute the overall discrepancy D between the electronic and manual tallies.

We now have the discrepancy between a plaintext electronic tally and a manual tally of paper ballots, each comprising both MERGE and ordinary ballots. Ideally, these would correspond perfectly. However, there may be fewer paper MERGE paper ballots than electronic ones, since mail may be dropped or delayed, or other discrepancies.

The resulting batch discrepancy can be incorporated in to a ballot-level comparison RLA, either by adopting the one-audit approach, or by assigning worst-case discrepancies like the strategy described in subsection 2.4.

If a MERGE ballot is selected for audit, but no corresponding paper ballot has arrived, it is important to ensure the right worst-case assumption in the case that overstatements of +2 or more are possible. We count the total number of MERGE electronic ballots, and subtract the number of MERGE ballots that have arrived, then explicitly add a "not arrived" zombie ballot to the collection of ballot papers. For the RLA, we then sample randomly from the collection of paper ballots, which includes ordinary paper ballots from the Local Counting Center, MERGE paper ballots that have arrived, and zombies representing those that have not arrived.

3.2.1 Using ONEAudit

A high-level description of ONEAudit is given below—for details see [Sta23].

- Create an *Overstatement Net Equivalent* CVR for each electronic record in $L \cup M$. This effectively spreads any discrepancy evenly across $L \cup M$. [10]
- If a ballot in $L \cup M$ is chosen for audit, use the discrepancy between its paper ballot and its *Overstatement Net Equivalent* CVR.

3.2.2 Assigning discrepancies arbitrarily

Another strategy is to assign the discrepancies arbitrarily, in advance of the audit, ensuring that the total overstatements are correct, and then use the assigned discrepancy if that ballot is selected for audit. This strategy is similar to that described in subsection 2.4.

3.3 Batch-MERGE: batch-level comparison audits

Everything described above would work just as well for batch-level comparison audits. It would make sense to define the batches according to some physical convenience function (for example, all the MERGE ballots that arrived on a certain date) because the corresponding electronic records should be easy to identify and need not be together on the BB.

This would make sense if the Local Counting Center already used batch-level comparison audits for their RLAs. If so, MERGE could easily be (slightly) modified to fit that auditing strategy.

It may even be possible to determine in advance which paper ballots will be in which batch—for example, if so few MERGE ballots are sent to one Local Counting Center that they are all treated as a single batch. In these cases, it may be possible to omit the serial numbers entirely, relying instead on the identity of the Local Counting Center to form the batch. The same sort of worst-case discrepancy calculation could be performed at a batch level.

Extending this idea further, if it was known in advance how many batches the MERGE ballots would comprise for a given Local Counting Center, it would not be necessary to put individual serial numbers on each ballot. It would suffice to label them with their intended batch, which would then need to be physically collated at the Local Counting Center. This would significantly improve privacy against an attacker who views the vote receiving process.

Note that it does not matter whether the attacker knows, or can change, which ballots are contained in the same batch—this worst-case attack assumption is already part of the attacker model of batch RLAs. As long as the attacker does not know which batches will be chosen, the risk limit is met.

Table 6 summarizes the access levels granted to various stakeholders in Batch-MERGE.

[10] The exact definition of a ONEAudit CVR depends on the audit method being used, but for a two-candidate contest, take the winner's tally minus the loser's and divide by the total number of ballots. This becomes a virtual CVR against which each sampled paper ballot is compared..

Table 6: Data accessible to various entities in Batch-MERGE.

Entity	ID	Batch No.	ID&Batch No.	Vote	Tally
General public	✓	✓	✓	×	✓[†]
BMD machine	✓	✓	✓	✓	✓
Coercer	✓	✓	✓	×	✓[†]
RLA observers	×	✓	×	✓	✓[†]
Postal worker	✓	✓	✓	×	✓[†]

[†]: visible on the BB

In the context of this table, we operate under the assumption that the batch number assigned to each participating voter is publicly disclosed on the BB. As indicated in the table, apart from the BMD machine, exclusive access to each ballot's vote content is granted only to the observers at the Local Counting Center. Nevertheless, the observers possess solely the batch number without any additional information to establish a connection between the vote contents and the respective voter's identity. Consequently, with a sufficiently large batch size, the privacy of the voter is preserved.

3.4 Summary and comparison of different MERGE variants

The main reason for choosing one variant or another is to fit with existing audit processes at the Local Counting Center. If VAULT is already being used, VAULT-MERGE fits easily into the existing process, retains the high efficiency of a ballot-level comparison audit, and inherits the coercion-resistant properties of VAULT. The main remaining disadvantage is that serial numbers are still required, and the protocol therefore has the same susceptibility to coercion as plain MERGE if the coercer colludes with RLA observers.

The other variants batch the MERGE ballots, obviating the need for serial numbers. They have larger anonymity sets but possibly less auditing efficiency. The Sub-MERGE approach has pros and cons compared with plain MERGE. One advantage is that there is no need for serial numbers on ballots. A disadvantage is that the resulting audit may be less efficient than ballot-level comparison audits, though this is unlikely to be a problem for small sets. This is worth considering for jurisdictions that currently do ballot-level comparison audits but are concerned about the privacy implications for their small sets of MERGE voters.

Batch-MERGE is ideal for jurisdictions that already perform batch-level comparison audits.

4 Optimizations for multiple contests

Up to now, we have assumed that the election consists of a single contest, resulting in the vote vector $\mathbf{b} = b_{n-1}\|\cdots\|b_0$ containing one bit per candidate within that contest. Therefore, the encrypted vote vector $\mathbf{e}$ is computed as:

$$\mathbf{e} = E(\mathbf{b}) = (E(b_{n-1}), \cdots, E(b_0))$$

To expand our design to accommodate multiple contests while maintaining a common ballot style, we can consider $\mathbf{b}$ as a concatenation of multiple vote vectors, with each vote vector corresponding to a distinct contest. For example, in an election featuring two contests, we express $\mathbf{b}$ as $\mathbf{b_1}||\mathbf{b_0}$, where $\mathbf{b_0}$ and $\mathbf{b_1}$ represent the vote vectors for contests 0 and 1, respectively. The encrypted vote vector $\mathbf{e}$ is accordingly computed as

$$\mathbf{e} = E(\mathbf{b}) = (E(\mathbf{b_1}), E(\mathbf{b_0}))$$

In this configuration, the overall process remains largely unchanged compared to a single-contest election, with the exception that NIZK proofs and discrepancy assignments must be conducted separately for each contest on a ballot.

Furthermore, to extend our design to accommodate elections with multiple ballot styles, it's required to store the ballot style in plaintext alongside each ballot. This is necessary for tallying the ballots within each contest. We must also ensure that the ballot style for each ballot undergoes the same mixnet process as the ballot itself and is presented in plaintext format before the RLA. Otherwise, if the digital ballot that is selected for the RLA lacks a corresponding paper ballot, it would be unclear which contest's RLA statistics should be updated.

Consider $\mathbf{b} = \mathbf{b_{k-1}}||\cdots||\mathbf{b_0}$, representing a set of k distinct contests, where each $\mathbf{b_i}$ is an n_i vote vector representing the i^{th} contest. The total number of options in total is denoted as n, where $n = \sum_i n_i$. In the typical scenario, we would require n encryption values to represent $\mathbf{e} = E(\mathbf{b})$. As mentioned earlier, it's possible to conduct the tally and the RLA process separately for each contest and for elections with different ballot styles. While this approach offers flexibility, it also entails a higher number of encryption values per ballot, namely n, which can potentially complicate and slow down the mixnet process.

Now, we provide some optimizations for elections with multiple contests.

Let $\mathbf{e}$ comprise n ElGamal encryption values corresponding to an n-bit vote vector. To accelerate the mixnet process, following the tally phase, we apply the following COMP transformation to $\mathbf{e} = (e_{n-1}, \cdots, e_0)$, with each $e_i = (\alpha_i, \beta_i)$, in order to compact it into a single ElGamal encryption value:

$$e' = \mathsf{COMP}(\mathbf{e}) = (\prod_i \alpha_i^{2^i} \mod p, \prod_i \beta_i^{2^i} \mod p)$$

Our assumption here is that n is smaller than the size of the group $|q|$ in bits. For larger vote vectors, we must partition the vector $\mathbf{e}$ and construct separate ciphertexts for each segment. While this approach is highly efficient, it lacks flexibility for RLA purposes, as outlined below:

- For digital ballots with non-matching serial numbers on paper ballots (i.e., digital ballots with $Type =$ "not matching"), one-audit would not be an alternative any more.

5 End-To-End Verifiable version (E2E-MERGE)

In the preceding sections, we explained how inconsistencies resulting from flaws in the voting machine can be identified during the audit process and reflected in the RLA results. However, basic-MERGE does not address the strong adversarial model where the adversary fully controls both the voting machine and the mail system. In such a case, the adversary could record an electronic ballot that aligns with their own interests and manipulate the corresponding paper ballot while it is in transit. This particular attack would not be detectable through the standard audit process. To mitigate this risk, we offer an end-to-end verifiable voting system—E2E-MERGE—that empowers voters to detect any discrepancies between their intended votes and the digital ballots recorded by the voting machines, with a probability of 0.5. The RLA parameters will be accordingly configured based on the (estimated) number of voters participating in the verification process. By introducing this end-to-end verifiable voting system, we aim to enhance transparency and ensure that voters can have confidence in the integrity of the electoral process, even in the face of potential adversarial control over both the voting machine and mail system.

5.1 Strong adversarial model

In the *strong adversarial* model, the adversary gains control over both ways of sending the vote: the voting machine and the mail system. The adversary may do *both of*:

- drop any envelope, mailed by a legitimate voter, and stuff new envelopes on behalf of any legitimate voter, and
- control any of the computers used for voting.

The adversary may also control the machines used to process votes at the Local Counting Center.

The adversary may *not* control the devices used for verification at the Remote Voting Center—we assume that each voter has access to at least one trustworthy device that can be used for verification. This device need not be trusted for privacy, because vote verification does not require any information about the vote, so voters can safely ask others to verify their receipts.

5.2 Setup

The format of ballots in this version mirrors that of pre-encrypted ballots in ElectionGuard V2.0. Our ballots use the same cryptographic construction as ElectionGuard V2.0, and use similar verification, but are not printed in advance—only the voted ballot will be printed.

A pre-encrypted ballot with m selection options for voter i, denoted as $\mathbf{E_i}$, is computed as encryption of the $m \times m$ binary identity matrix $\mathbf{I}$, wherein each entry $\mathbf{E}_{\mathbf{i}_{j,k}}$ for $j = \{1, \cdots, m\}$ and $k = \{1, \cdots, m\}$ represents the encryption of the corresponding entry $\mathbf{I}_{j,k}$. The j^{th} row of the matrix $\mathbf{E_i}$ is represented as $\mathbf{E}_{\mathbf{i}_j}$ and corresponds with the j^{th} selection option. The encryption vector $\mathbf{e_i}$ for voter i is then computed as the product of their vote vector $\mathbf{b_i}$ and the pre-encrypted ballot $\mathbf{E_i}$, as follows:

$$\mathbf{e_i} = \mathbf{b_i} \cdot \mathbf{E_i}$$

To enhance clarity, moving forward, we will omit the index i unless it is necessary for reference.

Each selection option is hashed to form a *selection hash*, building a selection hash vector $\mathbf{h}$, where $\mathbf{h}_j$ denotes the selection hash for the j^{th} row of $\mathbf{E}$. Additionally, a condensed representation called the *short code* $\mathbf{s}_j$ is derived from the selection hash $\mathbf{h}_j$ using a specialized *hash trimming function* Ω. For instance, Ω could extract the last byte of its input and represent it in a specified form. All selection hashes within a contest are sorted and then hashed to generate the *contest hash* χ specific to that contest. The *confirmation code* for a pre-encrypted ballot is computed as the hash of all the contest hashes present on the ballot in sequential order and is denoted as $\mathcal{C}$. Selection hashes, contest hashes and confirmation code are computed as specified by ElectionGuard.

The next section describes the voting process from the voter's perspective. The voting machine creates two pre-encrypted ballots $\mathbf{E}^1$ and $\mathbf{E}^2$ for each voter where $\mathbf{E}^1$'s short codes are green and $\mathbf{E}^2$'s codes are orange.

5.3 Voting

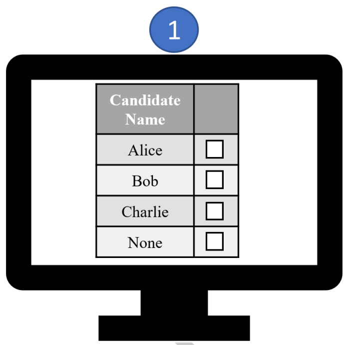
The voting process is as follows:

1. The voting machine creates two ballots $\mathbf{E}^1$ and $\mathbf{E}^2$ for voter i and prints the corresponding confirmation codes $\mathcal{C}^1$ and $\mathcal{C}^2$ on the receipt.

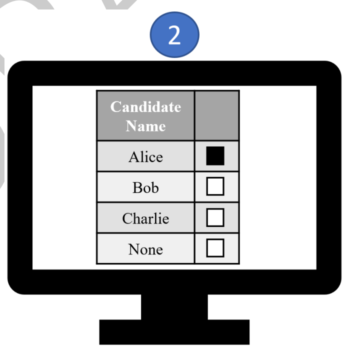
   (Optional) Verification step. The voter checks that two confirmation codes have been printed.

2. The following steps are are repeated for each contest.

   (a) The voting machine displays the options to the voter.



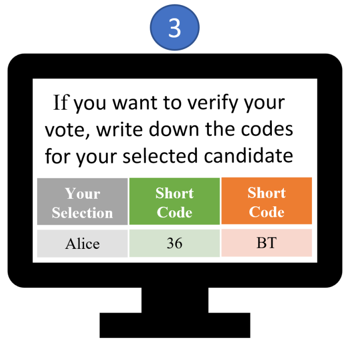
   (b) The voter interacts with the machine and makes their selection.[11]



   Let their vote vector be $\mathbf{b}$.

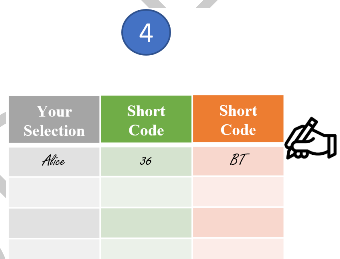
   (c) Then, the set $\{(\mathbf{o}_j, \mathbf{s}_j^1, \mathbf{s}_j^2)|\mathbf{b}_j = 1\}$ are displayed on the screen.

[11] ElectionGuard ballot construction and codes can incorporate multiple selections in the same contest, but not preferential voting or other scoring or ranking systems. We assume in this discussion that only one selection is allowed, and leave the details of multiple selections for later.



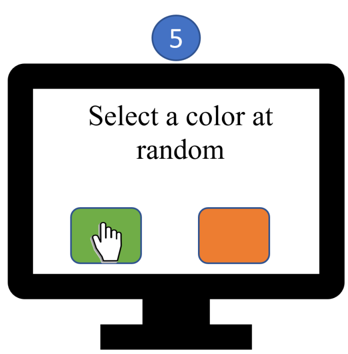
There needs to be an option here, which could be quite unobtrusive, to display all the short codes for the other options along with their corresponding plaintext ballot options (i.e. $\{(\mathbf{o}_j, \mathbf{s}_j^1, \mathbf{s}_j^2) | \mathbf{b}_j = 0\}$). The rationale behind this will be further explored in Section 5.9.

(d) (Optional) Verification step. The voter writes down the short codes corresponding with their intended selection options, for both colors.



We make some blank, colour coded notebooks freely available for everyone.

3. When all their selections are made, the voter randomly selects either $\mathbf{E}^1$ or $\mathbf{E}^2$ (green or orange) to cast.



The other will be opened for audit. Let the one selected by the voter be denoted by $\mathbf{E}^c$ with $c \in \{1, 2\}$.

4. The voting machine assigns a unique serial number sn_i to each ballot paper. The voting machine produces:

   - a signed, encrypted electronic record and serial number and a proof of ballot validity:

   $$sig_i(E(sn), \mathbf{E}^c, \mathit{ZKP}_{\text{VALID}}(\mathbf{E}^c))), \mathit{Domain\_separator\_BB}),$$

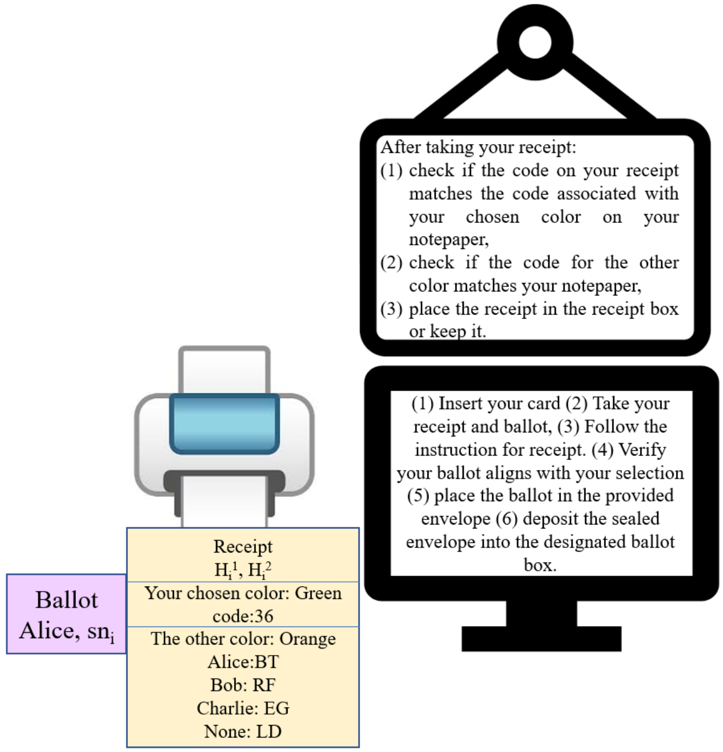
     which goes on the BB,
   - a plain paper record of the ballot including sn and $\mathbf{b}$, which is placed in a ballot box or in an envelope for mailing,
   - a sticker that includes an address, a traditional mail tracking number, a space for a pen-and-ink signature (if required by regulation—not relevant to our cryptographic protocol) and a digital signature

   $$sig_{\text{Voting Computer}} sig_i(E(sn_i), \mathbf{E}^c, \mathit{ZKP}_{\text{VALID}}(\mathbf{E}^c))), \mathit{Domain\_separator\_mail}).$$

   - a receipt record of $\mathcal{C}^1$ and $\mathcal{C}^2$ (previously printed in Step 1) along with $\{\mathbf{s}^c_j | \mathbf{b}_j = 1\}$ and $\mathbf{o}, \mathbf{s}^{3-c}$, that is, the selected codes for the voted ballot and all the codes for the opened ballot with their corresponding option name.

5. The voter must verify that her plaintext printout matches her intention.

6. (Optional) Verification step. For cast-as-intended verification of the encrypted vote, the voter should verify that the short codes for her chosen selection on the receipt match what the machine showed her in Step 2c and the voter documented in Step 2d.



5.4 Digital path

Digital path is mostly the same as that in the basic-MERGE. The only distinction is that for each voter, the following information is additionally presented on the BB:

- cast ballot additional data:
  - all the selection hashes on the ballot, sorted numerically within each contest, and
  - short codes associated with all selections on the cast ballot made by the voter
- uncast ballot additional data:
  - all the selection hashes on the ballot, sorted numerically within each contest,
  - all options on the ballot,
  - all encrypted values on the ballot, and
  - all short codes,

For example, given that the voter i's vote is the j^{th} selection option in a contest with m selection options, the following data is presented on the BB for voter i ($c' = 3 - c$):

$$
\begin{array}{c}
\mathbf{h}^c_{\sigma(1)} \\
\vdots \\
\mathbf{h}^c_{\sigma(j-1)} \\
\mathbf{h}^c_{\sigma(j)}, \mathbf{s}^c_{\sigma(j)} \\
\mathbf{h}^c_{\sigma(j+1)} \\
\vdots \\
\mathbf{h}^c_{\sigma(1)}
\end{array}
$$

and

$$
\begin{array}{c}
\mathbf{o}_{\sigma(1)}, \mathbf{E}^{c'}_{\sigma(1)}, \mathbf{h}^{c'}_{\sigma(1)}, \mathbf{s}^{c'}_{\sigma(1)} \\
\vdots \\
\mathbf{o}_{\sigma(m)}, \mathbf{E}^{c'}_{\sigma(m)}, \mathbf{h}^{c'}_{\sigma(m)}, \mathbf{s}^{c'}_{\sigma(m)}
\end{array}
$$

where σ is a permutation that represents the sorting of the selection hashes. This means that $\sigma(j) < \sigma(k)$ implies that $\mathbf{h}_{\sigma(j)} < \mathbf{h}_{\sigma(k)}$.

Figure 3 illustrates the data published on the BB for E2E-MERGE.

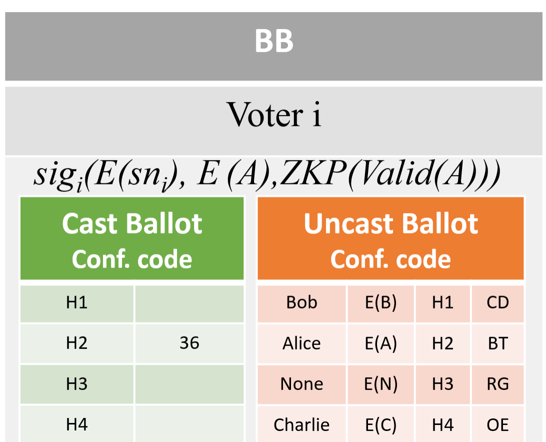


Figure 3: Bulletin Board data for E2E-MERGE.

5.5 Paper path and joining it to the digital path

Paper path and the way paper path's data is joined to the digital path's data similar to those in basic-MERGE.

5.6 The RLA

In the strong adversarial model, where the adversary gains control over both the paper channel and the voting machine, conventional RLA prove ineffective. The challenge lies in the adversary's ability to manipulate the voting machine to record their desired vote while simultaneously altering the paper ballot during transportation to match the falsified vote. Even Furthermore, in electronic-only or even a weak adversarial model, if a substantial number of paper ballots are missing, standard RLA or even a complete manual tally may fail to verify the accuracy of the election results. Consider a scenario where electronic records indicate a winning candidate with a lead of 5000 votes over their opponent. However, there are no corresponding paper ballots for at least 2500 electronic records, all of which purportedly indicate votes for the winning candidate. This discrepancy raises serious doubts about the integrity of the election outcome. In such scenarios, we might depend on voters' end-to-end verification to validate the election results.

In our analysis, we assume that the malicious voting machine can only manipulate votes based on the voters' actual intended choices. In other words, the adversary cannot use voters' identities or behavior as criteria for deciding whether to cheat or not. Let the electronic records reflect the victory of the reported winner over their opponent by a margin of m votes. In this scenario, for the actual loser to appear as the winning candidate, the voting machine would need to manipulate the results by at least $\frac{m}{2}$ votes. Assuming each voter diligently engages in end-to-end verification with a probability of r_{e2e}, each instance of fraud is detected and reported by the voter with a probability of $\frac{r_{e2e}}{2}$ where $\frac{1}{2}$ reflects this fact that each end-to-end verification conducted during a fraudulent event has a 0.5 chance of detection. Therefore, if there are at least $\frac{m}{2}$ errors, the probability of detecting $k \leq \frac{m}{2}$ errors can be expressed as:

$$\binom{\frac{m}{2}}{k}(\frac{r_{e2e}}{2})^k(1-\frac{r_{e2e}}{2})^{\frac{m}{2}-k}$$

Similar to RLA, there exists a risk that the end-to-end verification process may confirm an incorrect election result. Our goal is to limit this risk to an acceptable level, ensuring a high probability of rejecting the outcome if it's incorrect. Let α represent the acceptable risk. Hence, we seek an appropriate threshold, denoted by T, for the acceptable number of reported errors,

Table 7: Reported error threshold for various values of m, r_{e2e} and α.

	$r_{e2e} = 0.001$ and α			$r_{e2e} = 0.002$, α			$r_{e2e} = 0.005$, α			$r_{e2e} = 0.01$, α		
m	0.02	0.05	0.10	0.02	0.05	0.10	0.02	0.05	0.10	0.02	0.05	0.10
500	-	-	-	-	-	-	-	-	-	-	-	-
1000	-	-	-	-	-	-	-	-	-	-	-	0
2000	-	-	-	-	-	-	-	-	0	0	1	1
5000	-	-	-	-	-	0	1	1	2	5	6	7
10000	-	-	0	0	1	1	5	6	7	14	16	18
20000	0	1	1	3	4	5	14	16	18	35	38	40

where the associated risk is equal to or lower than α. The threshold T is calculated as:

$$\max_T\{\sum_{k=0}^{T}\binom{\frac{m}{2}}{k}(\frac{r_{e2e}}{2})^k(1-\frac{r_{e2e}}{2})^{\frac{m}{2}-k} \leq \alpha\}$$

Table 7 shows some sample figures.

5.7 Verification

compared with basic-MERGE, we have some extra verification steps.

5.7.1 Matching the Receipt with BB data

The following verification steps are performed using each voter's receipt. However, these steps reveal nothing about how the person voted and can therefore be delegated to any other interested party. For example, voters could be offered the chance to leave their receipts in a special tub where politically-motivated observers at the Remote Voting Center could gather them all and verify them as a set.

Therefore, the voter i or anyone holding the voter i's receipt checks if the information on the receipt match the data presented on the BB:

- the signed, encrypted vote appears on the BB (just like in basic-MERGE),
- short codes $\{\mathbf{s}_j^c|\mathbf{b}_j = 1\}$ on her receipt matches short codes from voter i's cast ballot additional data on the BB,
- short codes $\mathbf{s}^{3-c}$ on the receipt match the corresponding short codes of voter's i uncast ballot additional data on the BB.

Voter i completes the above checks by comparing the receipt with their own electronic data on the BB. Anyone holding the voter i's receipt can complete the above checks by comparing the receipt with any voter's electronic records on the BB with matching confirmation codes.

5.7.2 Verifying BB data for ballot construction

In addition to the verification steps performed in basic-MERGE, everyone (including voter i or election authorities) verifies if

- All confirmation codes on the BB are unique.
- The short codes $\{\mathbf{s}_j^c | \mathbf{b}_j = 1\}$ in voter i's cast ballot additional data is correctly computed from its corresponding selection hash which is, in turn, correctly computed from the encryption ballot $E(\mathbf{b})$ on the BB.
- All short codes $\mathbf{s}^{c'}$ with $c' = 3 - c$ in voter i's uncast ballot additional data are correctly computed from their corresponding selection hashes which are, in turn, correctly computed from their corresponding encryption values which are, in turn, correctly computed from their corresponding plaintext options.

5.8 Example of E2E-MERGE

To enhance comprehension of E2E-MERGE, we present a simplified illustration in the form of a simple election with two contests. The first contest contains 4 selection options as Alice, Bob, Charlie and None. The second contest contains 2 selection options as YES and NO. Each selection option within a contest corresponds with a vote vector as follows:

Alice:	$[1\ 0\ 0\ 0]$
Bob:	$[0\ 1\ 0\ 0]$
Charlie:	$[0\ 0\ 1\ 0]$
None:	$[0\ 0\ 0\ 1]$
YES:	$[1\ 0]$
NO:	$[0\ 1]$

Two pre-encrypted ballots are created for voter i. For the sake of simplicity assume that each encrypted or hash value is represented by a 4-character hexadecimal value (i.e., a value between 0x0000 and 0xFFFF) and each short code is represented by a combination of a digit and a capital letter (e.g., $4M$). Then, the first pre-encrypted ballot is created as follows:

$$
\begin{array}{ll}
\text{Alice:} & [1\ 0\ 0\ 0] \xrightarrow{E} \\
& [E(1)\ E(0)\ E(0)\ E(0)] = [\texttt{0x5148 0x3B68 0x9F42 0xA943}] \xrightarrow{H} \\
& H(\texttt{0x5148}, \texttt{0x3B68}, \texttt{0x9F42}, \texttt{0xA943}) = \texttt{0x8731} \xrightarrow{\Omega} 3M \\
\text{Bob:} & [0\ 1\ 0\ 0] \xrightarrow{E} [\texttt{0x128D 0x0642 0x7B44 0x91C4}] \xrightarrow{H} \texttt{0x309C} \xrightarrow{\Omega} 6H \\
\text{Charlie:} & [0\ 0\ 1\ 0] \xrightarrow{E} [\texttt{0x387F 0x13C6 0x8E45 0xDE40}] \xrightarrow{H} \texttt{0x9B31} \xrightarrow{\Omega} 9L \\
\text{None:} & [0\ 0\ 0\ 1] \xrightarrow{E} [\texttt{0x3927 0x168C 0xBB73 0x12E4}] \xrightarrow{H} \texttt{0x28E0} \xrightarrow{\Omega} 9U \\
\text{YES:} & [1\ 0] \xrightarrow{E} [\texttt{0xFA59 0xCB07}] \xrightarrow{H} \texttt{0x928F} \xrightarrow{\Omega} 1J \\
\text{NO:} & [0\ 1] \xrightarrow{E} [\texttt{0x66A3 0xF15D}] \xrightarrow{H} \texttt{0x5B28} \xrightarrow{\Omega} 7W
\end{array}
$$

where the contest hash value for the first and second contests are computed as follows:

$$
\begin{aligned}
\chi_1^1 &= H(\texttt{0x28E0}, \texttt{0x309C}, \texttt{0x8731}, \texttt{0x9B31}) = \texttt{0x7044} \\
\chi_2^1 &= H(\texttt{0x5B28}, \texttt{0x928F}) = \texttt{0x32FB}
\end{aligned}
$$

The confirmation code for this ballot is computed as:

$$
\mathcal{C}^1 = H(\texttt{0x7044}, \texttt{0x32FB}) = \texttt{0x1A75}
$$

Similarly, the second pre-encrypted ballot is created as follows:

$$
\begin{aligned}
&\text{Alice:} && [1\ 0\ 0\ 0] \xrightarrow{E} \\
& && [E(1)\ E(0)\ E(0)\ E(0)] = [\texttt{0x49D3 0xA6F9 0xBF34 0x10F4}] \xrightarrow{H} \\
& && H(\texttt{0x49D3}, \texttt{0xA6F9}, \texttt{0xBF34}, \texttt{0x10F4}) = \texttt{0x3450} \xrightarrow{\Omega} 7R \\
&\text{Bob:} && [0\ 1\ 0\ 0] \xrightarrow{E} [\texttt{0x9C3A 0x1F5A 0x999C 0x735A}] \xrightarrow{H} \texttt{0xB23C} \xrightarrow{\Omega} 1K \\
&\text{Charlie:} && [0\ 0\ 1\ 0] \xrightarrow{E} [\texttt{0x26A2 0x5A35 0x3A20 0x0500}] \xrightarrow{H} \texttt{0xA366} \xrightarrow{\Omega} 6V \\
&\text{None:} && [0\ 0\ 0\ 1] \xrightarrow{E} [\texttt{0x55D3 0x29A8 0x4F3A 0x9745}] \xrightarrow{H} \texttt{0xED4F} \xrightarrow{\Omega} 8H \\
&\text{YES:} && [1\ 0] \xrightarrow{E} [\texttt{0x84D7 0xCE48}] \xrightarrow{H} \texttt{0x185D} \xrightarrow{\Omega} 9N \\
&\text{NO:} && [0\ 1] \xrightarrow{E} [\texttt{0x4868 0x195D}] \xrightarrow{H} \texttt{0x3A8F} \xrightarrow{\Omega} 3Q
\end{aligned}
$$

where the contest hash value for the first and second contests are computed as follows:

$$
\begin{aligned}
\chi_1^2 &= H(\texttt{0x3450}, \texttt{0x9B31}, \texttt{0xA366}, \texttt{0xB23C}) = \texttt{0x55D3} \\
\chi_2^2 &= H(\texttt{0x185D}, \texttt{0x3A8F}) = \texttt{0xDA49}
\end{aligned}
$$

The confirmation code for this ballot is computed as:

$$
\mathcal{C}^2 = H(\texttt{0x55D3}, \texttt{0xDA49}) = \texttt{0x308A}
$$

The ballots are displayed to the voter as illustrated below:

Your Selection	Short Code	Short Code
☐ Alice	3M	7R
☐ Bob	6H	1K
☐ Charlie	9L	6V
☐ None	9U	8H
☐ YES	1J	9N
☐ NO	7W	3Q

Let the voter select Alice and YES. Then she writes down the corresponding short codes as illustrated below:

Your Selection	Short Code	Short Code
Alice	3M	7R
YES	1J	9N

Suppose she selects green to cast and orange to audit. Then the receipt for her vote would contain the following information:

Confirmation codes:	$\mathcal{C}^1 = \texttt{0x1A75}, \mathcal{C}^2 = \texttt{0x308A}$
Chosen color:	Green
Green codes for your selections:	$3M, 1J$
Other color:	Orange
Orange Codes:	Alice:$7R$, Bob:$1K$, Charlie:$6V$, None:$8H$
	YES:$9N$, NO:$3Q$

The voter checks if the selected color matches her intention and codes on the receipt match the ones written on her notepaper. The corresponding encrypted vote that is stored on the BB for the voter would be

$$[\texttt{0x5148 0x3B68 0x9F42 0xA943}], [\texttt{0xFA59 0xCB07}]$$

which corresponds with Alice and YES on the first pre-encrypted ballot. From this ballot, the confirmation code, all selection hashes, sorted numerically within each contest, along with the short codes corresponding with the voter's selection options are also published on the BB. So, for our sample ballots, the data below is published on the BB:

Conf. Code: 0x1A75

Contest 1

$\texttt{0x28E0}, \texttt{0x309C}, \texttt{0x8731} \xrightarrow{\Omega} 3M, \texttt{0x9B31}$

Contest 2

$\texttt{0x5B28}, \texttt{0x928F} \xrightarrow{\Omega} 1J$

For the second ballot, which is the audit ballot, the confirmation code, selection options, encrypted values, selection hashes and short codes are recorded on the BB as follows:

Conf. Code:	0x1A75
Contest 1	
Alice:	$[\texttt{0x49D3 0xA6F9 0xBF34 0x10F4}], \texttt{0x3450}, 7R$
Charlie:	$[\texttt{0x26A2 0x5A35 0x3A20 0x0500}], \texttt{0xA366}, 6V$
Bob:	$[\texttt{0x9C3A 0x1F5A 0x999C 0x735A}], \texttt{0xB23C}, 1K$
None:	$[\texttt{0x55D3 0x29A8 0x4F3A 0x9745}], \texttt{0xED4F}, 8H$
Contest 2	
NO:	$[\texttt{0x4868 0x195D}]\texttt{0x3A8F}, 3Q$
YES:	$[\texttt{0x84D7 0xCE48}], \texttt{0x185D}, 9N$

5.9 Security analysis

The correctness, authentication and privacy of this version remain consistent with those of basic-MERGE.

5.9.1 End-to-end verifiability

This version provides end-to-end verifiability because the voter is able to verify that their votes are cast as intended, collected as cast and tallied as collected. In more detail, each voter is able to challenge the voting machine to ensure that her vote is cast as intended. If the encrypted values on the paper ballots (e.g. $E(b_{i,j})$) do not match with their corresponding unencrypted values on the paper ballot (e.g. $b_{i,j}$), then there is a 50% probability that the voter detects it when the non-cast ballot set is decrypted on the BB. The voter also verifies that their ballot was collected as cast by comparing their receipt with the information published on the BB alongside their name and verifying that the ciphertexts were correctly constructed. Everyone, including the voter, will be able to verify that the votes are tallied as collected by verifying the correctness of all operations (i.e. mix and decryption) performed on the cast electronic ballots which is published on the BB.

5.9.2 Receipt Freeness

The coercibility will be discussed further. Assuming that the coercer requests the receipt and short code of the uncast ballot, as documented by the voter during the voting process, certain implications arise. If the voting machine solely displays the short codes for the voter's selected options, the remaining short codes on the uncast ballot would remain unknown to the voter. Consequently, the voter is faced with a dilemma of either providing the actual documented short code to the coercer, which would compromise the secrecy of their vote once uncast ballots are revealed on the BB, or providing a fake short code, which would likely be detected by the coercer as it is improbable for a fake short code to appear on the BB and align with the coercer's intention. To address this issue, the voting machine should present all short codes and their corresponding plaintext ballot options for the uncast ballot to the voter.

6 Protocol specification

6.1 Encryption scheme

The parameters of the ElGamal cryptosystem $\mathcal{G} = (p, q, g)$ are specified as follows: p is a prime number given by $2kq + 1$, where q is also a prime number, and g represents a generator of the order q subgroup of $\mathbb{Z}_p^*$. ElGamal cryptosystem includes four main algorithms as follows:

ElGamal key generation $\mathsf{KeyGen}(\mathcal{G})$

Input: ElGamal parameters $\mathcal{G} = (p, q, g)$
Output: Public-private key pair (K, s)

1. $s \xleftarrow{\$} \mathbb{Z}_q^*$
2. $K = g^s \mod p$
3. Output (K, s)

ElGamal encryption

Input: ElGamal parameters $\mathcal{G} = (p, q, g)$, message $\sigma \in \mathbb{Z}_q^*$, public key K
Output: $E(\sigma)$

1. $\xi \xleftarrow{\$} \mathbb{Z}_q^*$
2. $\alpha = g^\xi \mod p, \beta = K^\sigma K^\xi \mod p$
3. Output (α, β)

ElGamal encryption aggregation

Input: ElGamal parameters $\mathcal{G} = (p, q, g)$, encryption values $(\alpha_i, \beta_i) = E(\sigma_i)$
Output: $E(\Sigma_i \sigma_i \mod q)$

1. $A = \prod_i \alpha_i \mod p, B = \prod_i \beta_i \mod p$
2. Output (A, B)

ElGamal decryption

Input: ElGamal parameters $\mathcal{G} = (p, q, g)$, ciphertext $C = (A, B)$, private key s
Output: $D(C; s)$

1. $T = \frac{B}{A^s} \mod p$
2. Output T

6.2 Hash computation

The function H that is used in this whole section is the same as one specified in ElectionGuard. It is based on hashed message authentication code HMAC-SHA-256. The function H takes two inputs, B_0 and B_1, where B_0 is 256 bit long and corresponds to the key in HMAC and B_1, which is of arbitrary length, is the actual input to the HMAC. These inputs are separated by a semicolon. If B_1 is comprised of multiple elements (e.g. $B_1 = a||b||c$), then these elements are separated by commas(e.g. $H(B_0; a, b, c)$).

Hash function

Input: B_0 and $B_1 = B_{11}||B_{12}|| \cdots ||B_{1n}$
Output: $H(B_0; B_1)$

1. Output HMAC-SHA-256(B_0, B_1)

6.3 Distributed ElGamal

Distributed ElGamal is a variant of the traditional ElGamal encryption scheme that involves multiple parties collaboratively generating the encryption keys and decrypting the ciphertexts.

Key generation

Principals: n guardians G_i with $i = \{1, \cdots, n\}$
Input: ElGamal parameters $\mathcal{G} = (p, q, g)$
Output: Public key shares K_i for $i = \{1, \cdots, n\}$, aggregated public key K
Output (G_i): Private key share s_i

1. G_i: $(K_i, s_i) \leftarrow \mathsf{KeyGen}(\mathcal{G})$.
2. G_i: Send K_i and $\mathsf{KnowDlog}(g, K_i)$ to G_j for any $j \neq i$
3. G_i: Proceed if all proofs $\mathsf{KnowDlog}(g, K_i)$ for $i = \{1, \cdots, n\}$ are verified. Otherwise, stop.
4. G_i: Output private key share s_i
5. All guardians: Output public key shares K_i for $i = \{1, \cdots, n\}$ the aggregated public key $K = \prod_i K_i \mod p$

Decryption

Principals: n guardians G_i with $i = \{1, \cdots, n\}$
Public input: ElGamal parameters $\mathcal{G} = (p, q, g)$, ciphertext $C = (A, B)$, public key shares K_i for $i = \{1, \cdots, n\}$
Private input (G_i): Private key share s_i
Output: $\mathsf{DistDec}(C, \sum_{i=1}^{n} s_i)$

1. G_i: Publish share $M_i = A^{s_i} \mod p$ and proof EqDlogs(g, A, K_i, M_i).

2. Proceed if all proofs EqDlogs(g, A, K_i, M_i) for $i = \{1, \cdots, n\}$ are verified. Otherwise, stop.

3. Compute $M = \prod_i M_i \mod p$

4. Output $T = B/M \mod p$

Required operations per guardian: 1 exp., 1 EqDlogs, $n-1$ EqDlogs verification: in total 3 exp, $2(n-1)$ multi-exp

Required operations for verification: n EqDlogs verification: $2n$ multi-exp

Note that in the protocol presented above, anyone with access to the public input and the proofs EqDlogs(g, A, K_i, M_i) for $i = \{1, \cdots, n\}$ can verify the correctness of the distributed decryption operation by following steps 2-4 of the protocol. With the following protocol, guardians can collaboratively provide a proof of decryption correctness that can be verified by anyone who has access to the aggregated public key K rather than each individual guardian's public key share.

Proof of decryption correctness

Principals: n guardians G_i with $i = \{1, \cdots, n\}$ and the verifier V
Public input: ElGamal parameters $\mathcal{G} = (p, q, g)$, ciphertext $C = (A, B)$, public key shares K_i for $i = \{1, \cdots, n\}$, aggregated public key K and two deployment-dependant constants $cons_0$ and $cons_1$
Private input (G_i): Private key share s_i
Output (guardians): Plaintext $T :=$ DistDec$(C; K)$ and proof *ZKP*$_{\text{DECRYPT}}(T, C)$

1. G_i: Compute share $M_i = A^{s_i} \mod p$ and send M_i to G_j for any $j \neq i$

2. G_i: Compute $M = \prod_j M_j \mod p$, $T = \frac{B}{M} \mod p$, $u_i \xleftarrow{\$} \mathbb{Z}_q^*$, $a_i = g^{u_i} \mod p$ and $b_i = A^{u_i} \mod p$ and send (a_i, b_i) to G_j for any $j \neq i$

3. G_i: Compute $a = \prod_j a_j$ and $b = \prod_j b_j$ and $c = H(cons_0; cons_1, K, A, B, a, b, M)$

4. G_i: Send $v_i = u_i - cs_i$ to G_j for any $j \neq i$

5. G_i: Compute $a'_j = g^{v_j} K_j^c$ and $b'_j = A^{v_j} M_j^c$ and verify if $a_j = a'_j$ and $b_j = b'_j$ for any $j \neq i$. Otherwise, stop.

6. All guardians: Compute $v = \sum_j v_j$ and send the plaintext T and the proof (c, v) to V

7. V: Compute $M = \frac{B}{T} \mod p$, $a = g^v K^c \mod p$ and $b = A^v M^c \mod p$ and verify if $v \in \mathbb{Z}_q$ and $c = H(cons_0; cons_1, K, A, B, a, b, M)$. Otherwise, reject.

Required operations per guardian: 3 exp., $2(n-1)$ multi-exp
Required operation for verification: 2 multi-exp

6.4 Zero-knowledge proofs

KnowDlog(g, K)

Principals: The prover P and the verifier V
Public input: ElGamal parameters $\mathcal{G} = (p, q, g)$, K and two deployment-dependant constants $cons_0$ and $cons_1$
Private input (P): s s.t. $K = g^s \mod p$

1. P: computes
   - $u \xleftarrow{\$} \mathbb{Z}_q$
   - $h = g^u \mod p$
   - $c = H(cons_0; cons_1, K, h)$
   - $v = u - cs \mod q$
2. $P \xrightarrow{c,v} V$
3. V: Compute $h = g^v K^c \mod p$ and then verify if $c = H(cons_0; cons_1, K, h)$. Otherwise, reject.

Required operations: 1 exp
Required operation for verification: 1 multi-exp

EqDlogs(x, y, X, Y)

Principals: The prover P and the verifier V
Public input: ElGamal parameters $\mathcal{G} = (p, q, g)$, x, y, X, Y and two deployment-dependant constants $cons_0$ and $cons_1$
Private input (P): s s.t. $X = x^s \mod p$ and $Y = y^s \mod p$

1. P: compute

- $u \xleftarrow{\$} \mathbb{Z}_q$
- $h = x^u \mod p$
- $h' = y^u \mod p$
- $c = H(cons_0; cons_1, x, y, X, Y, h, h')$
- $v = u - cs \mod q$

2. $P \xrightarrow{h,h',c,v} V$

3. V: Compute $h = x^v X^c \mod p$ and $h' = y^v Y^c \mod p$ and verify if $c = H(cons_0; cons_1, x, y, X, Y, h, h')$. Otherwise, reject.

Required operations: 2 exp

Required operation for verification: 2 multi-exp

Proof that (a, b) is an encryption of an integer in the range $0, \cdots, L$

Principals: The prover P and the verifier V
Public input: ElGamal parameters $\mathcal{G} = (p, q, g)$, ciphertext (α, β), public key K and two deployment-dependant constants $cons_0$ and $cons_1$
Private input (P): ξ, l s.t. $(\alpha, \beta) = (g^\xi, K^l K^\xi)$

1. P:
   - $u_j \xleftarrow{\$} \mathbb{Z}_q$ for $j = \{0 \cdots, L\}$
   - $c_j \xleftarrow{\$} \mathbb{Z}_q$ for $j = \{0 \cdots, L\}, j \neq l$
   - $t_j := (u_j + (l - j)c_j) \mod q$, $(a_j, b_j) := (g^{u_j} \mod p, K^{t_j} \mod p$ for $j = \{0 \cdots, L\}, j \neq l$,
   - $(a_l, b_l) := (g^{u_l} \mod p, K^{u_l} \mod p$
   - $c = H(cons_0; cons_1, K, \alpha, \beta, a_0, b_0, a_1, b_1, \cdots, a_L, b_L)$
   - $c_l := (c - \sum_{j \neq l} c_j) \mod q$
   - $v_j = (u_j - c_j \xi) \mod q$
   - Send (c_j, v_j) for $j = \{0, \cdots, L\}$ to V

2. V:

   (a) $(a_j, b_j) := (g^{v_j} \alpha^{c_j} \mod p, K^{v_j - jc_j} \beta^{c_j} \mod p)$

(b) Verify if $\sum_j c_j = H(cons_0; cons_1, K, \alpha, \beta, a_0, b_0, a_1, b_1, \cdots, a_L, b_L)$. Otherwise, reject.

Required operations: $2L$ exp

Required operation for verification: $2L$ multi-exp

Proof that $(a, b) = (g^\xi, K^\xi)$ is an encryption of zero or one

---

Principals: The prover P and the verifier V
Public input: ElGamal parameters $\mathcal{G} = (p, q, g)$, ciphertext $(\alpha, \beta) = (g^\xi, K^\xi)$, public key K and two deployment-dependant constants $cons_0$ and $cons_1$
Private input (P): ξ s.t. $(\alpha, \beta) = (g^\xi, K^\xi)$

1. P:
   - $u_0, u_1, c_1 \xleftarrow{\$} \mathbb{Z}_q$
   - $(a_0, b_0) = (g^{u_0} \mod p, K^{u_0} \mod p)$, $(a_1, b_1) = (g^{u_1} \mod p, K^{u_1 - c_1} \mod p)$
   - $c = H(cons_0; cons_1, K, \alpha, \beta, a_0, b_0, a_1, b_1)$
   - $c_0 = c - c_1$, $v_0 = u_0 - c_0\xi \mod q$, $v_1 = u_1 - c_1\xi \mod q$
   - Send (c_0, c_1, v_0, v_1) to V
2. V
   (a) $(a_0, b_0) := (g^{v_0}\alpha^{c_0} \mod p, K^{v_0}\beta^{c_0} \mod p)$
   (b) $(a_1, b_1) := (g^{v_1}\alpha^{c_1} \mod p, K^{v_1 - c_1}\beta^{c_1} \mod p)$
   (c) Verify if $c_0 + c_1 = H(cons_0; cons_1, K, \alpha, \beta, a_0, b_0, a_1, b_1)$. Otherwise, reject.

Required operations: 4 exp

Required operation for verification: 4 multi-exp

Proof that $(\alpha, \beta) = (g^\xi, KK^\xi)$ is an encryption of zero or one

---

Principals: The prover P and the verifier V
Public input: ElGamal parameters $\mathcal{G} = (p, q, g)$, ciphertext $(\alpha, \beta) = (g^\xi, KK^\xi)$, public key K, two deployment-dependant constants $cons_0$ and $cons_1$
Private input (P): ξ s.t. $(\alpha, \beta) = (g^\xi, KK^\xi)$

1. P:

- $u_0, u_1, c_0 \xleftarrow{\$} \mathbb{Z}_q$
- $(a_0, b_0) = (g^{u_0} \mod p, K^{u_0+c_0} \mod p)$, $(a_1, b_1) = (g^{u_1} \mod p, K^{u_1} \mod p)$
- $c = H(cons_0; cons_1, K, \alpha, \beta, a_0, b_0, a_1, b_1)$
- $c_1 = c - c_0$, $v_0 = u_0 - c_0\xi \mod q$, $v_1 = u_1 - c_1\xi \mod q$
- Send (c_0, c_1, v_0, v_1) to V

2. V

   (a) $(a_0, b_0) := (g^{v_0}\alpha^{c_0} \mod p, K^{v_0}\beta^{c_0} \mod p)$

   (b) $(a_1, b_1) := (g^{v_1}\alpha^{c_1} \mod p, K^{v_1-c_1}\beta^{c_1} \mod p)$

   (c) Verify if $c_0 + c_1 = H(cons_0; cons_1, K, \alpha, \beta, a_0, b_0, a_1, b_1)$. Otherwise, reject.

Required operations: 4 exp

Required operation for verification: 4 multi-exp

References

[ADS20] Andrew W Appel, Richard A DeMillo, and Philip B Stark. Ballot-marking devices cannot ensure the will of the voters. *Election Law Journal: Rules, Politics, and Policy*, 19(3):432–450, 2020.

[BS12] Jorge H Banuelos and Philip B Stark. Limiting risk by turning manifest phantoms into evil zombies. *arXiv preprint arXiv:1207.3413*, 2012. https://arxiv.org/abs/1207.3413.

[BST19] Josh Benaloh, Philip B Stark, and Vanessa Teague. Vault: Verifiable audits using limited transparency. *Proceedings of E-Vote ID*, 2019. https://www.stat.berkeley.edu/~stark/Preprints/vault19.pdf.

[KBW21] Philip Kortum, Michael D Byrne, and Julie Whitmore. Voter verification of ballot marking device ballots is a two-part question: Can they? mostly, they can. do they? mostly, they don't. *Election Law Journal: Rules, Politics, and Policy*, 20(3):243–253, 2021.

[LS12] Mark Lindeman and Philip B Stark. A gentle introduction to risk-limiting audits. *IEEE Security & Privacy*, 10(5):42–49, 2012.

[LSY12] Mark Lindeman, Philip B Stark, and Vincent S Yates. Bravo: Ballot-polling risk-limiting audits to verify outcomes. In *EVT/WOTE*, 2012.

[OSLM18] Kellie Ottoboni, Philip B Stark, Mark Lindeman, and Neal McBurnett. Risk-limiting audits by stratified union-intersection tests of elections (suite). In *International Joint Conference on Electronic Voting*, pages 174–188. Springer, 2018. ArXiv: https://arxiv.org/pdf/1809.04235.pdf.

[Riv08] Ronald L Rivest. On the notion of 'software independence' in voting systems. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 366(1881):3759–3767, 2008. https://people.csail.mit.edu/rivest/pubs/RW06.pdf.

[Sta08a] Philip B Stark. Conservative statistical post-election audits. 2008. https://projecteuclid.org/journals/annals-of-applied-statistics/volume-2/issue-2/Conservative-statistical-post-election-audits/10.1214/08-AOAS161.pdf.

[Sta08b] Philip B Stark. A sharper discrepancy measure for post-election audits. 2008. https://projecteuclid.org/journals/annals-of-applied-statistics/volume-2/issue-3/A-sharper-discrepancy-measure-for-post-election-audits/10.1214/08-AOAS171.pdf.

[Sta10] Philip B Stark. {Super-Simple} simultaneous {Single-Ballot}{Risk-Limiting} audits. In *2010 Electronic Voting Technology Workshop/Workshop on Trustworthy Elections (EVT/WOTE 10)*, 2010.

[Sta20] Philip B Stark. Sets of half-average nulls generate risk-limiting audits: Shangrla. In *International Conference on Financial Cryptography and Data Security*, pages 319–336. Springer, 2020. ArXiv: https://arxiv.org/abs/1911.10035.

[Sta23] Philip B Stark. Overstatement-net-equivalent risk-limiting audit: Oneaudit. In *Financial Cryptography and Data Security - Voting Workshop*. Springer LNCS 13953, 2023. https://arxiv.org/abs/2303.03335.