Seminar 6

Eulerian and Hamiltonian graphs

Exercises

1. For each of the following conditions, draw an Eulerian graph which fulfils them (if possible), or indicate why the condition can not be fulfilled:

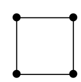
   (a) An even number of nodes and even number of edges.

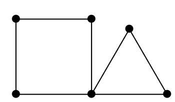
   (b) An even number of nodes and odd number of edges.

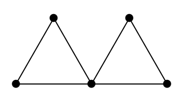
   (c) An odd number of nodes and even number of edges.

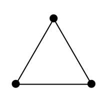
   (d) An odd number of nodes and odd number of edges.

   Answer:

   (a) even number of nodes and even number of edges: C_4: 

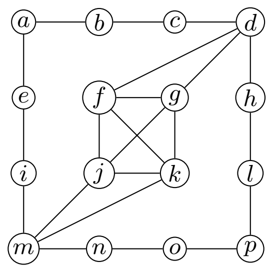
   (b) even number of nodes and odd number of edges: 

   (c) odd number of nodes and even number of edges: 

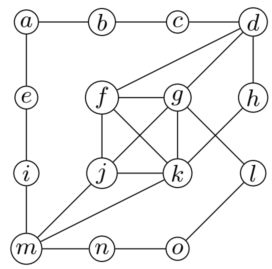
   (d) odd number of nodes and odd number of edges: C_3: 

   Note that all these graphs are Eulerian because they are connected and all their nodes have even degree.

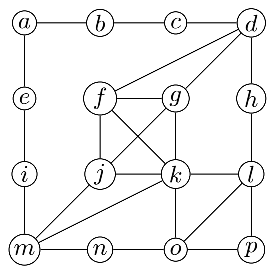
2. Which of the following graphs is Eulerian, and which is not? Indicate a reason for every answer you give. For the Eulerian graphs, indicate an Eulerian circuit.



G_1



G_2



G_3

ANSWER: G_1 is Eulerian because it is connected and all its nodes have even degree. G_2 is not Eulerian because $\deg(g) = 5$ is odd. G_3 is Eulerian because all its nodes have even degree.

Eulerian circuits for G_1 and G_3 can be found with Hierholzer algorithm:

- G_1 has the disjoint cycles $Q_1 = (a, b, c, d, h, l, p, o, n, m, i, e, a)$, $Q_2 = (d, f, g, d)$, $Q_3 = (m, j, k, m)$, and $Q_4 = (j, f, k, g, j)$. By patching them, we obtain the Eulerian circuit

$$(a, b, c, d, f, g, d, h, l, p, o, n, m, j, f, k, g, j, k, m, i, e, a)$$

- G_3 has the disjoint cycles $Q_1 = (a, b, c, d, h, l, p, o, n, m, i, e, a)$, $Q_2 = (d, f, g, d)$, $Q_3 = (m, j, k, m)$, $Q_4 = (j, f, k, g, j)$, and $Q_5 = (k, o, l, k)$. By patching them we get the Eulerian circuit

$$(a, b, c, d, f, g, d, h, l, p, o, n, m, j, f, k, g, j, k, o, l, k, m, i, e, a)$$

3. Let $G = K_{m,n}$.

   (a) For what values of m and n does G have an Eulerian trail?

   (b) For what values of m and n is G an Eulerian graph?

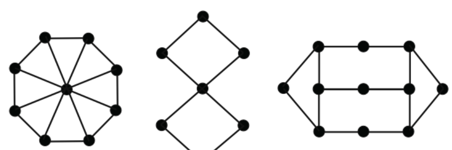
   ANSWER: Note that some nodes in $K_{m,n}$ have degree m, and all the other have degree n. Therefore:

   (a) G has an Eulerian trail if and only if at most two nodes have odd degree. This can happen only in the following situations:

- m and n are both even; in this case, all nodes have even degree.
- $m = n = 1$; in this case, exactly 2 nodes have odd degree.
- m is odd, and n is 2; in this case, exactly 2 nodes have odd degree.

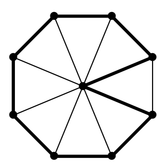
(b) G is Eulerian if and only if all nodes have even degree. This can happen if and only if both m and n are even numbers.

4. Which of the following graphs is traceable, Hamiltonian, or none of these?

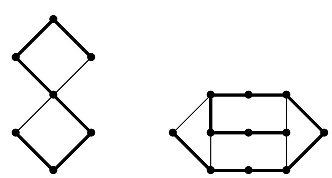


G_1 G_2 G_3

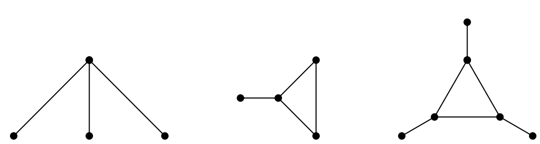
ANSWER: G_1 is Hamiltonian, thus traceable too. A Hamiltonian cycle of G_1 is depicted with thick lines below:



G_2 and G_3 are not Hamiltonian, but is traceable. Traces of G_2 and G_3 are shown below with thick lines:

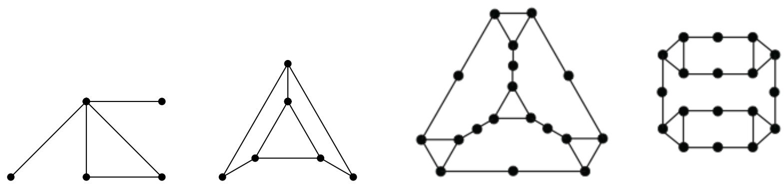


5. Consider the following graphs:



$K_{1,3}$ Z_1 N

Which of the following graphs are $K_{1,3}$-free? Which are Z_1-free? Which are N-free?



Answer:

- The first graph is not $K_{1,3}$-free, not Z_1-free, and N-free.
- The second graph is not $K_{1,3}$-free, not Z_1-free, and N-free.
- The third graph is $K_{1,3}$-free, not Z_1-free, and not N-free.
- The last graph is $K_{1,3}$-free, not Z_1-free, and not N-free.