

## Homework 9

1. Determine whether:

- (a)  $y + 1$  is substitutable for  $x$  in  $\exists y(x = 2y)$ ,
- (b)  $y + 1$  is substitutable for  $y$  in  $\forall y(x = 2y)$ ,
- (c)  $vw$  is substitutable for  $x$  in  $\exists y(x < vx \Rightarrow (\exists w(w < v)))$ ,
- (d)  $vw$  is substitutable for  $v$  in  $\exists y(x < vx \Rightarrow (\exists w(w < v)))$ ,
- (e)  $vw$  is substitutable for  $w$  in  $\exists y(x < vx \Rightarrow (\exists w(w < v)))$ .

In the situation when the substitution can be performed, do it.

2. In the lecture we discussed “syntactic sugar” (i.e. shortcut notations for expressions – terms and formulas). For the following expressions<sup>1</sup>, written using syntactic sugar, give the translation into “pure” expressions (i.e. according to the definitions of terms, formulae given in the lecture). Note that for this exercise, all variables are bound. If any symbol appears free, it is considered a constant. Which are the functions and predicates occurring in the expressions?

(a)  $\forall_{x,y,z \in \mathbb{R}} \exists_{k,l \in \mathbb{N}} (x, y, z \leq k \Rightarrow x + y + z \leq l),$

(b)

$$\forall_{x < y < z < 1} \exists_{-1 \leq \delta \leq 1} \forall_{-|\delta| < \varepsilon_1, \varepsilon_2 < |\delta|} (z - y < \varepsilon_1 \Rightarrow y - x < \varepsilon_2 \Rightarrow z - x \geq \varepsilon_1 + \varepsilon_2 + |\delta|)$$

(c)

$$\exists!_{x \in \mathbb{N}} (x^2 = 7).$$

3. Find 3 examples of syntactic sugar used in mathematics (based on your experience), and show their translation in the pure predicate logic language presented in the lecture.

4. Consider the following story:

“If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent.”

Based on this, prove Superman does not exist.

Hint: use the proof rules introduced in the lecture.

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<sup>1</sup>This is a purely syntactic exercise, the meaning of these expressions is not relevant here.