Graph Theory and Combinatorics

Lecture 1: Introduction.
Counting Principles. Permutations and Combinations.
Binomial and Multinomial Numbers

October 5, 2015

Purpose of this lecture

Become familiar with the basic notions from combinatorics and graph theory.

1. Counting principles, Arrangements, permutations, combinations.
2. Principle of inclusion and exclusion, enumeration techniques.
3. Combinations
4. The cyclic structure of permutations. Advanced counting techniques.
5. Polya's theory of counting
6. Graph theory: basic notions
7. Data structures for the representation of graphs
8. Transport networks, maximal flows, minimal cuts
9. Trees: definitions; generating trees; minimum cost trees
10. Paths, circuits, chains, and cycles
11. The traveling salesman problem. Planar graphs
12. Chromatic theory of graphs
13. Matchings

Organizatorial items

- Lecturer and TA: Isabela Drămnesc
- Course webpage: `http:/web.info.uvt.ro/~idramnesc`
  - Exercises
  - Seminar/Lab: working with *Combinatorica* in *Mathematica*
- Handouts: will be posted on the webpage of the lecture
- Grading:
  - 50% : weekly seminar assignments
  - 50% : 1 written exam at the end of the semester

Lecture outline

- Basic counting principles
  - The product rule
  - The sum rule
  - Combinatorial proofs; examples
- Counting techniques for
  - combinations - unordered selections of distinct elements of a finite set
  - permutations - ordered selections of distinct elements of a finite set
- Generalizations
  - permutations with repetition
  - combinations with repetition
  - permutations with indistinguishable elements
- Binomial and multinomial numbers

Product rule. If a procedure can be broken down into a sequence of two tasks, such that

- first task can be done in n_1 ways
- second task can be done in n_2 ways

then there are $n_1 \cdot n_2$ ways to do the procedure.

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Generalized product rule. If a procedure can be broken down into a sequence of m tasks, such that

- first task can be done in n_1 ways
- second task can be done in n_2 ways
- ...
- m-th task can be done in n_m ways

then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to do the procedure.

Applications of the product rule

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$\Rightarrow$ by the product rule, there are $12 \cdot 11 = 132$ ways.

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Answer

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- According to the product rule, there are $26 \cdot 25 = 650$ ways to do so.

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- Each of the 7 bits can be chosen in 2 ways, because each bit is either 0 or 1.

$\Rightarrow$ by the product rule, there are $2^7 = 128$ ways.

(4) How many functions are there from a set with m elements to a set with n elements?

Answer

- The procedure to define such a function can be broken down into a sequence of m subtasks, where each subtask fixes the output value for an input argument.
- Each subtask can be done in n ways (there are n possible output values)

$\Rightarrow$ by product rule, the number of functions is $\underbrace{n \cdot \ldots \cdot n}_{m \text{ times}} = n^m$

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$\Rightarrow$ By product rule, there are $n \cdot (n-1) \cdot \ldots \cdot (n-m+1)$ one-to-one functions.

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- For every subset B of A we define the bit string $b_1 b_2 \ldots b_n$ with

$$b_i = \begin{cases} 1 & \text{if } a_i \in B \\ 0 & \text{otherwise} \end{cases}$$

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$\Rightarrow$ there are 2^n subsets of S.

Sum rule. If a procedure can be done either in one of n_1 ways or in one of n_2 ways, and none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the procedure.

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Generalized sum rule. Suppose that a procedure can be done in one of n_1 ways, in one of n_2 ways, . . . , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + \ldots + n_m$.

Applications of the sum rule

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Answer

- The project can be chosen by selecting from the first list, the second list, or the third list.
- Because no project is in more than one list, we can apply the sum rule $\Rightarrow$ there are $9 + 8 + 12 = 29$ ways to choose a project.

More complex counting problems

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  - Let W_m be the number of strings of uppercase letters and digits of length m. By product rule, $W_m = (26 + 10)^m = 36^m$

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  - Let N_m be the number of strings of uppercase letters of length m. By product rule, $N_m = 26^m$.

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    - Let N_m be the number of strings of uppercase letters of length m. By product rule, $N_m = 26^m$.
- Note that $P_m = W_m - N_m$ (explain why).

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- By sum rule, we have $P = P_6 + P_7 + P_8$.
- To compute P_m for $m \in \{6, 7, 8\}$, we can proceed as follows:
  - Let W_m be the number of strings of uppercase letters and digits of length m. By product rule, $W_m = (26 + 10)^m = 36^m$
  - Let N_m be the number of strings of uppercase letters of length m. By product rule, $N_m = 26^m$.
- Note that $P_m = W_m - N_m$ (explain why).

$\Rightarrow$ $P = W_6 - N_6 + W_7 - N_7 + W_8 - N_8 = 36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8$.

More complex counting examples

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Answer

$$\begin{aligned} \text{R\&E} &= 5 \times \ \ 9 = 45 && \text{by product rule} \\ \text{R\&G} &= 5 \times 10 = 50 && \text{by product rule} \\ \text{E\&G} &= 9 \times 10 = 90 && \text{by product rule} \end{aligned}$$

$\Rightarrow 45 + 50 + 90 = 185$ ways (by sum rule).

Combinatorial proofs

- A combinatorial proof is a proof that uses counting arguments, such as the sum rule and product rule to prove something.
- The proofs illustrated in the previous examples are combinatorial proofs.

Assumption: A is a finite set with n elements.

- An r-permutation is an ordered sequence $\langle a_1, a_2, \ldots, a_r \rangle$ of r elements of A.
- A permutation of A is an ordered sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of all elements of A.
- An r-combination of A is an unordered selection $\{a_1, a_2, \ldots, a_r\}$ of r elements of A.

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Example

$\langle 3, 1, 2 \rangle$ and $\langle 1, 3, 2 \rangle$ are permutations of $\{1, 2, 3\}$.
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- $P(n, r) :=$ the number of r-permutations of a set with n elements.
- $C(n, r) :=$ the number of r-combinations of a set with n elements. Alternative notation: $\binom{n}{r}$.

Theorem

$P(n, r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$.

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PROOF

$A = \{a_1, \ldots, a_n\}$

r-permutation $= p_1, p_2, ..., p_r$

	choice tasks			
	$p_1 \in A$	$p_2 \in A - \{p_1\}$	$\ldots$	$p_r \in A - \{p_1, \ldots, p_{r-1}\}$
# of choices	n	$n-1$	$\ldots$	$n-r+1$

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$$\Rightarrow P(n, r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$$

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$$\Rightarrow P(n, r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$$

REMARK. $n!$ denotes the product $1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n$.

Theorem

$P(n, r) = C(n, r) \times P(r, r).$

COMBINATORIAL PROOF

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$\Rightarrow$ by product rule, we obtain $P(n, r) = C(n, r) \times P(r, r)$.

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$$\Rightarrow C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!} \cdot \frac{0!}{r!} = \frac{n!}{r!(n-r)!}$$

Properties of combinations

Theorem

$C(n, r) = C(n-1, r-1) + C(n-1, r)$ *for all* $n > r > 0$.

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- Let $S = \{a_1, a_2, \ldots, a_n\}$. There are $C(n,r)$ ways to choose r elements from S. We distinguish 2 distinct possibilities:

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  By sum rule, $C(n, r) = N_1 + N_2$. But:

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  By sum rule, $C(n, r) = N_1 + N_2$. But:
  - $N_1 = C(n-1, r-1)$ because we have to choose $r-1$ elements from $\{a_2, \ldots, a_n\}$
  - $N_2 = C(n-1, r)$ because we have to choose r elements from $\{a_2, \ldots, a_n\}$

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    - $N_2 = C(n-1,r)$ because we have to choose r elements from $\{a_2, \ldots, a_n\}$

$\Rightarrow C(n,r) = C(n-1,r-1) + C(n-1,r)$.

Quizzes

1. Give an algebraic proof, using the formulas for $C(n, r)$, of the fact that $C(n, r) = C(n-1, r-1) + C(n-1, r)$.
2. Give a combinatorial proof of the fact that $C(n, r) = C(n, n-r)$.
3. How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?
4. In how many ways can n people stand to form a ring?
5. How many permutations of the letters $ABCDEFGH$ contain the string ABC?
6. How many bit strings of length n contain exactly r 1s?

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Example

How many strings of length n can be formed with the lowercase and uppercase letters of the English alphabet?
Answer: $|Alphabet_{English}| = 52 \Rightarrow 52^n$ strings (by product rule)

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How many strings of length n can be formed with the lowercase and uppercase letters of the English alphabet?
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Theorem

The number of r-permutations of a set of n elements with repetition is n^r.

- An *r-combination with repetition* of a set of n elements is a choice of r elements from a bag of elements of n kinds, where we can choose the same kind of element any number of times.

Q: How many r-combinations with repetition of a set of n elements are there?

Example

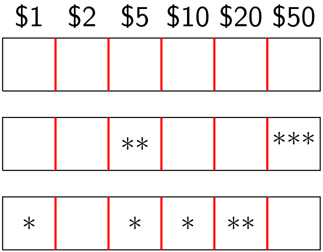
How many ways are there to select 5 bills from a cash box containing bills of \$1, \$2, \$5, \$10, \$20, \$50. Assume that: the order in which the bills are chosen does not matter; the bills are indistinguishable; there are at least 5 bills of each type.

Combinations with repetition
Example – continued

Five not necessarily distinct bills = a 5-combination with repetition from the set $\{\$1, \$2, \$5, \$10, \$20, \$50\}$ of bill kinds = a placement of five * in the slots of the cash box depicted below:

- The number of * in a slot represents the number of bills taken from that place.

$\Rightarrow$ The number of 5-combinations with repetition of a set with 6 elements = the number of ways to place 5 stars in 6 slots.



cash box with 6 types of bills

$||{**}|||{***}$

$*||*|*|{**}|$

...

Combinations with repetition

NOTE THAT

- Every placement of 5 stars in 6 possible slots is uniquely described by a string of 5 stars and 5 red bars.
- In general, the number of r-combinations with repetition of a set with n elements = the number of strings with r stars and $n - 1$ red bars.

Q: In how many ways can we arrange $n - 1$ bars and r stars ?

Combinations with repetition

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- Every placement of 5 stars in 6 possible slots is uniquely described by a string of 5 stars and 5 red bars.
- In general, the number of r-combinations with repetition of a set with n elements $=$ the number of strings with r stars and $n-1$ red bars.

Q: In how many ways can we arrange $n-1$ bars and r stars ?

A: The sequence has length $n+r-1$

$\Rightarrow$ there are $n+r-1$ positions in the sequence

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There are $C(n+r-1, r)$ such choices.

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Theorem

The number of r-combinations with repetition of n elements is $C(r+n-1, r)$.

Type	Repetition allowed?	Formula
r-permutations	No	$P(n,r) = \dfrac{n!}{(n-r)!}$
r-combinations	No	$C(n,r) = \dfrac{n!}{r!(n-r)!}$
r-permutations with repetition	Yes	n^r
r-combinations with repetition	Yes	$C(n+r-1,r) = \dfrac{(n+r-1)!}{r!(n-1)!}$

Permutation with indistinguishable objects

Problem

How many strings can be made by reordering the string SUCCESS?

Permutation with indistinguishable objects

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How many strings can be made by reordering the string SUCCESS?

- SUCCESS contains 3 Ss, 2 Cs, 1U, 1 E.
- placements of 3 Ss among 7 places: $C(7,3) \Rightarrow$ 4 places left.
- placements of 2 Cs among 4 places: $C(4,2) \Rightarrow$ 2 places left.
- placements of 1 U among 2 places: $C(2,1) \Rightarrow$ 1 place left.
- placements of 1 E among 1 place: $C(1,1)$.

$\Rightarrow$ by product rule, the number is

$$C(7,3) \times C(4,2) \times C(2,1) \times C(1,1) = \frac{7!}{3!2!1!1!}$$

Permutations with indistinguishable objects

Theorem

The number of different permutations of n objects, where there are

- n_1 indistinguishable elements of type 1
- n_2 indistinguishable elements of type 2

  ...
- n_k indistinguishable elements of type n_k

is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Binomial and multinomial numbers

- The binomial numbers are the coefficients $c_{n,k}$ in the formula

$$(x+y)^n = \sum_{k=0}^{n} c_{n,k} \cdot x^{n-k} y^k$$

- The multinomial numbers are the coefficients $c_{n,k_1,\ldots,k_r}$ in the formula

$$(x_1 + \ldots + x_r)^n = \sum_{k_1+\ldots+k_r=n} c_{n,k_1,\ldots,k_r} \cdot x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r}$$

Example

$$\begin{aligned}(x+y)^3 &= 1 \cdot x^3 + 3 \cdot x^2 y + 3 \cdot x\, y^2 + 1 \cdot y^3 \\ (x_1+x_2+x_3)^2 &= 1 \cdot x_1^2 + 1 \cdot x_2^2 + 1 \cdot x_3^2 + \\ &\quad 2 \cdot x_1 x_2 + 2 \cdot x_1 x_3 + 2 \cdot x_2 x_3\end{aligned}$$

$$(x_1 + \ldots + x_r)^n = \sum_{k_1+\ldots+k_r=n}^{n} \frac{n!}{k_1! \ldots k_r!} \cdot x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r}$$

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COMBINATORIAL PROOF

$$(x_1 + \ldots + x_r)^n = \overbrace{(x_1 + \ldots + x_r) \cdot \ldots \cdot (x_1 + \ldots + x_r)}^{n \text{ parenthesized expressions}}$$

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In how many ways can we produce the monomial $x_1^{k_1} \cdot \ldots \cdot x_r^{k_r}$?

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- Choose k_1 parentheses from where x_1 originates $\Rightarrow \binom{n}{k_1}$ choices.
- Choose k_2 parentheses from where x_2 originates $\Rightarrow \binom{n-k_1}{k_2}$ choices.

  ...
- Choose k_r parentheses from where x_r originates $\Rightarrow \binom{n-\sum_{i=1}^{r-1} k_i}{k_r}$ choices.

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- Choose k_r parentheses from where x_r originates $\Rightarrow \binom{n-\sum_{i=1}^{r-1} k_i}{k_r}$ choices.

$\Rightarrow$ by the product rule, the number of occurrences of $x_1^{k_1} \cdot \ldots \cdot x_r^{k_r}$ in the right hand side is $\binom{n}{k_1}\binom{n-k_1}{k_2} \cdot \ldots \cdot \binom{n-\sum_{i=1}^{r-1} k_i}{k_r} = \frac{n!}{k_1! \ldots k_r!}$

- For the formula $\frac{n!}{k_1!\ldots k_r!}$ with $k_1+\ldots+k_r=n$ we often use the notation $\binom{n}{k_1,\ldots,k_r}$.
- The binomial formula is

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

- The multinomial formulas

$$(x_1+\ldots+x_r)^n=\sum_{k_1+\ldots+k_r=n}\binom{n}{k_1,\ldots,k_r}x_1^{k_1}\cdot\ldots\cdot x_r^{k_r}$$

Remark. $\binom{n}{k}=\binom{n}{k,n-k}$ and

$$(x_1+x_2)^n=\sum_{k=0}^{n}\binom{n}{k}x_1^kx_2^{n-k}=\sum_{k_1+k_2=n}\binom{n}{k_1,k_2}x_1^{k_1}x_2^{k_2}.$$