Lecture 8

Kinds of graphs. Data structures for graph representation. Connectivity: The naíve algorithm and Warshall algorithm

23 November 2015

Remember that:

Graph = mathematical structure $G = (V, E)$ where

- V : set of nodes (or vertices)
- E : set of edges incident to 2 nodes, or 1 node

Depending on the kind of edges $e \in E$, graphs are of two kinds:

- **Undirected**: Every edge has one or two endpoints

  Graphical representation: $a \overset{e}{\text{——}} b$ or $\overset{e}{\frown} a$ (loop)

- **Directed** (or digraphs): Every edge $e \in E$ has a source (or start) end a destination (or end)

  Graphical representation: $a \xrightarrow{e} b$ or $\overset{e}{\circlearrowright} a$ (loop)

  The directed edges are called arcs.

Simple graph: graph (directed or not) with at most one arc between any pair of nodes, and no loop.

$$G = (V, E)$$

$e_1, e_2 \in E$ are parallel edges if they are incident to the same nodes, and

- If G is directed, then $start(e_1) = start(e_2)$ and $end(e_1) = end(e_2)$

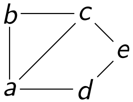
- **Multigraph** (directed or not):no loops, and if the graph is
  - undirected: it can have parallel edges
  - directed: it can have parallel arcs
- **Pseudograph**: undirected graph that can have loops and parallel edges.
- **Weighted graph**: every edge $e \in E$ has a weight $w(E)$; usually $w(e) \in \mathbb{R}$.

Graphical representations of graphs

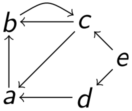
Illustrative examples

- Simple graphs: draw lines or arcs between the connected nodes

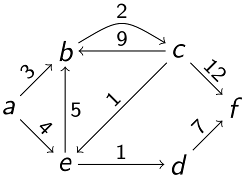
  Simple **undirected** graph:



  Simple **directed** graph:

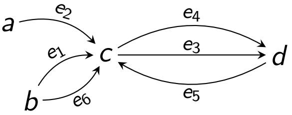


- Simple weighted graphs: we indicate the weights along the corresponding connections

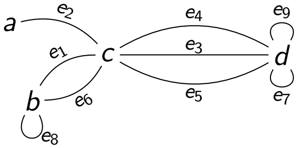


Multigraphs or pseudographs: if we want to distinguish parallel edges, we can label them:

multigraph:



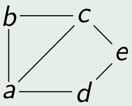
pseudograph:



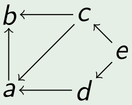
Concrete representations of graphs

1. List of nodes + list of edges
2. Adjacency lists
3. Adjacency matrix
4. Incidence matrix
5. Weight matrix

Example



List of nodes $V = [a, b, c, d, e]$
List of edges $E = [\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, e\}, \{d, e\}]$
Remarks: $\{a, b\} = \{b, a\}$, $\{a, c\} = \{c, a\}$, a.s.o.
edge $\leftrightarrow$ set of nodes adjacent to the edge



List of nodes $V = [a, b, c, d, e]$
List of arcs $E = [(a, b), (c, a), (c, b), (d, a), (e, c), (e, d)]$
Remarks: $(a, b) \neq (b, a)$, $(a, c) \neq (c, a)$, a.s.o.
edge $\leftrightarrow$ pair (start,end)

Remark

If the graph has no isolated nodes (with 0 neighbors), there is no need to store the list of nodes V:

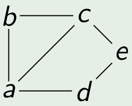
- V can be computed from E

Simple graphs
The representation with adjacency lists

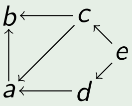
For every node $u \in V$ we keep the list of nodes that are adjacent to u

- If G is **undirected**, v is adjacent to u of there is an edge with endpoints u and v.
  - In undirected graphs, the relation of adjacency is symmetric.
- If G is **directed**, v is adjacent to u if there is an arc $e \in E$ from u to v, i.e. $start(e) = u$ and $end(e) = v$.

Example



$a \mapsto [b, c, d]$ $\quad d \mapsto [a, e]$
$b \mapsto [a, c]$ $\quad e \mapsto [c, d]$
$c \mapsto [a, b, e]$



$a \mapsto [b]$ $\quad d \mapsto [a]$
$b \mapsto []$ $\quad e \mapsto [c, d]$
$c \mapsto [a, b]$

If G has n nodes, $A_G = (m_{ij})$ has dimension $n \times n$ and

$$m_{ij} := \text{number of edges from the } i\text{-th node to the } j\text{-th node.}$$

Remarks

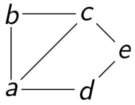
1. Before computing M_G from G, be must fix an enumeration of its nodes: $[v_1, v_2, \ldots, v_n]$
2. If G is undirected, A_G is a symmetric matrix
3. If G is a simple graph, A_G contains just 0s and 1s

Undirected graphs

The adjacency matrix A_G of an undirected graph G

The adjacency matrix the undirected graph

G :



for the node enumeration $[a, b, c, d, e]$ is

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

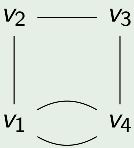
Remark: the matrix A_G is symmetric.

If A is a symmetric matrix of size $n \times n$ with $a_{ij} \in \mathbb{N}$ for all i, j, an undirected graph G whose adjacency matrix is A can be built as follows:

1. Draw n points $v_1, \ldots, v_n$ in plane
2. For every $i, j \in \{1, \ldots, n\}$, draw a_{ij} distinct edges between v_i and v_j

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix} \Rightarrow G :$$



Digraphs

The adjacency matrix A_G of a digraph G

The adjacency matrix of the digraph

$$G: \begin{tikzcd} b & c \arrow[l] \arrow[dd, "" ', start anchor=south west, end anchor=north east, shift left=0, to path={-- (\tikztotarget)}] \arrow[ddl] & \\ & & e \arrow[ul] \arrow[dl] \\ a \arrow[uu] & d \arrow[l] & \end{tikzcd}$$

for the enumeration $[a, b, c, d, e]$ of nodes is

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Digraphs

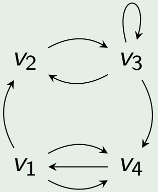
The digraph corresponding to an adjacency matrix

If A is an $n \times n$ matrix with $a_{ij} \in \mathbb{N}$ for all i, j, a digraph G whose adjacency matrix is A can be constructed as follows:

1. Draw n points $v_1, \ldots, v_n$ in the plane
2. For every $i, j \in \{1, \ldots, n\}$, draw a_{ij} distinct arcs from v_i to v_j

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow G :$$



We assume given two lists (or enumerations):

- $V = [v_1, \ldots, v_n]$ of the nodes of G
- $L = [e_1, \ldots, e_p]$ of the labels of edges of G

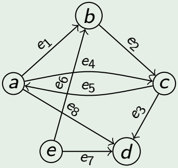
The incidence matrix $M_G = (m_{ij})$ has dimension $n \times p$ and

$$m_{ij} = \begin{cases} -1 & \text{if } \mathrm{start}(e_j) = v_i \\ 1 & \text{if } \mathrm{end}(e_j) = v_i \\ 0 & \text{otherwise.} \end{cases}$$

Example

If $V = [a, b, c, d, e]$, $L = [e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8]$ and

$$M_G = \begin{pmatrix} -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix} \Rightarrow$$



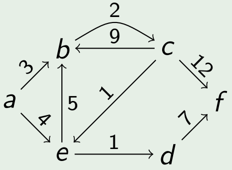
Simple weighted graphs
The representation with weight matrix

The weight matrix $W_G = (w_{ij})$ of a simple weighted graph G with n nodes $[v_1, \dots, v_n]$ has dimension $n \times n$ and

- $w_{ii} = 0$ for all $1 \le i \le n$.
- $w_{ij} = w(\{v_i, v_j\})$ for every edge $\{v_i, v_j\} \in E$, if G is undirected.
- $w_{ij} = w((v_i, v_j))$ for every arc $(v_i, v_j) \in E$, if G is directed.
- $w_{ij} = \infty$ otherwise.

Example (Weight digraph with node enumeration $[a, b, c, d, e, f]$)

$G :$



$$\Rightarrow W_G = \begin{pmatrix} 0 & 3 & \infty & \infty & 4 & \infty \\ \infty & 0 & 2 & \infty & \infty & \infty \\ \infty & 9 & 0 & \infty & 1 & 12 \\ \infty & \infty & \infty & 0 & \infty & 7 \\ \infty & 5 & \infty & 1 & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & 0 \end{pmatrix}$$

- The representation with list of edges
  - Adequate for the representation of simple graphs without isolated nodes, and with $|E| \ll |V|$
  - Space complexity: $O(|E|)$
- The representation with lists of adjacencies
  - Allows the fast enumeration of the neighbors of a node
  - Space complexity: $O(|V| + |E|)$
- The representation with adjacency matrix $A_G = (a_{ij})$ or with weight matrix $W_G = (w_{ij})$
  - Fast test to check direct connection between 2 nodes: $O(1)$
    - $\nexists (v_i, v_j) \in E$ if $a_{ij} = 0$ or if $w_{ij} = \infty$
  - Space complexity: $O(|V|^2)$
    - inadequate representation when $|E| \ll |V|^2$
- The representation with incidence matrix M_G
  - Time complexity: $O(|V| \cdot |E|)$

We assume that G is a simple (di)graph with n nodes and with the adjacency matrix $A_G = (a_{ij})$

- $a_{ij} = 0$ or `false`) if $\nexists$ arc $v_i \to v_j$
- $a_{ij} = 1$ (or `true`) if $\exists$ arc $v_i \to v_j$

We define:

- I_n : the identity matrix of size $n \times n$
- The Boolean operations $\odot$ (conjunction) and $\oplus$ (disjunction):

$\odot$	0	1
0	0	0
1	0	1

$\oplus$	0	1
0	0	1
1	1	1

Remarks:
$a \odot b = \min(a, b)$
$a \oplus b = \max(a, b)$

- If $U = (u_{ij})$, $V = (v_{ij})$ are $n \times n$ matrices with elements 0 or 1, we define
  - $U \oplus V = (c_{ij})$ if $c_{ij} = u_{ij} \oplus v_{ij}$ for all i, j
  - $U \odot V = (d_{ij})$ if $d_{ij} = (u_{i1} \odot v_{1j}) \oplus \ldots \oplus (u_{in} \odot v_{nj})$
  - $U^k = \underbrace{U \odot \ldots \odot U}_{k \text{ times}}$ for every $k > 0$

Properties

1. If $A_G^k = (a_{ij}^{(k)})$ for $k \geq 1$ then $a_{ij}^{(k)} = 1$ if and only if there is a path with length k from node v_i to v_j.
2. Let $A_G^* = I_n \oplus A_G \oplus A_G^2 \oplus \ldots \oplus A_G^{n-1} = (\overline{a}_{ij})$. Then
   - $\overline{a}_{ij} = 1$ if and only if there is a path with length $j \in \{1, \ldots, n-1\}$ from node v_i to v_j.
   - A_G^* can be computed in $O(n^4)$.
   - A_G^* is called reflexive and transitive closure of A_G.
3. v_i and v_j are connected $\Leftrightarrow \exists$ a simple path $v_i \rightsquigarrow v_j \Leftrightarrow \overline{a}_{ij} = 1$.
4. $I_n \oplus A_G \oplus A_G^2 \oplus \ldots \oplus A_G^{k+1} = I_n \oplus (I_n \oplus A_G \oplus A_G^2 \oplus \ldots \oplus A_G^k) \odot A_G$
   $\Rightarrow A^*$ can be computed in $O(n^4)$.

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   $\Rightarrow A^*$ can be computed in $O(n^4)$.

Corollary

The connectivity between all pairs of nodes in a simple digraph can be checked in $O(n^4)$.

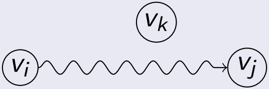
Simple digraphs
Better ways to compute A_G^*

Warshall algorithm computes A_G^* in $O(n^3)$.

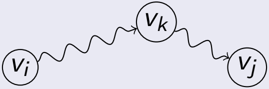
Warshall's main idea

If $V = [v_1, \ldots, v_n]$ is an enumeration of the nodes of G and $v_k \in V$, then every simple path $\pi : v_i \rightsquigarrow v_j$ is of one of the following two kinds:

1. v_k does not appear along π between v_i and v_j



2. v_k appears exactly once in π, between v_i and v_j



Assumption: $A_G = (a_{ij})$ has size $n \times n$

- Compute recursively $C^{[n]} = (c_{ij}^{[n]})$ where

$$c_{ij}^{[k]} := \begin{cases} a_{ij} & \text{if } k = 0 \\ c_{ij}^{[k-1]} \oplus (c_{ik}^{[k-1]} \odot c_{kj}^{[k-1]}) & \text{if } k \geq 1 \end{cases}$$

Properties

1. $C^{[0]} = A_G$
2. $c_{ij}^{[k]} = 1$ if and only if there is a path $\pi : v_i \rightsquigarrow v_j$ where all intermediary nodes are from the subset $\{v_1, \ldots, v_k\}$
3. $C^{[n]} = A_G^*$
4. $C^{[n]}$ can be computed in $O(n^3)$.

Consider the simple digraph $G = (V, E)$ with $V = \{1, \dots, n\}$ and the weight function $w : E \to \mathbb{R}^+$

- In G, the weight of a path $\pi : v_1 \to v_2 \to \dots \to v_p$ is
  - $w(\pi) = \sum_{i=1}^{p-1} w((v_i, v_{i+1}))$
  - (we add up the weights of all arcs of π)
- For every pair of nodes (i, j) din V, we wish to find
  - a lightest path from node i to node j
    (there can be more than one lightest path)
  - the weight of a lightest path

Remember that the weight matrix of G is $W_G = (w_{ij})$ where

$$w_{ij} = \begin{cases} 0 & \text{if } j = i, \\ w((i,j)) & \text{if } (i,j) \in E, \\ \infty & \text{if } (i,j) \notin E \end{cases}$$

Let $k \in V = \{1, \ldots, n\}$, and $\pi : i \rightsquigarrow j$ be a lightest path from i to j. We distinguish two cases:

1. k is not an intermediary node of π
2. k is an intermediary node of π. Then $\pi = i \overset{\pi_1}{\rightsquigarrow} k \overset{\pi_2}{\rightsquigarrow} j$ and $w(\pi) = w(\pi_1) + w(\pi_2)$.

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An useful auxiliary data structure:

- A matrix of paths $P^{[k]} = (p_{ij}^{[k]})$ where

$$p_{ij}^{[k]} := \begin{cases} \bullet & \text{special value: } \nexists \text{ path } i \rightsquigarrow j \text{ through} \\ & \text{intermediary nodes from } \{1, \ldots, k\} \\ \pi & \text{a lightest path from } i \text{ to } j \text{ through} \\ & \text{intermediary nodes from } \{1, \ldots, k\} \end{cases}$$

We assume that $W_G = (w_{ij})$

$$p_{ij}^{[k]} := \begin{cases} \bullet & \text{if } k = 0 \text{ and } w_{ij} = \infty \\ [i, j] & \text{if } k = 0 \text{ and } w_{ij} < \infty \\ p\text{-}min(p_{ij}^{[k-1]}, p_{ik}^{[k-1]} \asymp p_{kj}^{[k-1]}) & \text{if } k \geq 1 \end{cases}$$

where $p_{ik}^{[k-1]} \asymp p_{kj}^{[k-1]}$ denotes the concatenation of the paths $p_{ik}^{[k-1]}$ and $p_{kj}^{[k-1]}$. If both exist (i.e., they are not $\bullet$), and $\bullet$ otherwise.

Remark

If $p_{ij}^{[n]} = \bullet$, there is no path from node i to j. Otherwise, $p_{ij}^{[n]}$ is a lightest path from node i to j.

Compute recursively and simultaneously two matrices:

$$W^{[k]} = \left(w_{ij}^{[k]}\right) \quad \text{and} \quad P^{[k]} = \left(p_{ij}^{[k]}\right) \text{ for } 0 \le k \le n$$

$$w_{ij}^{[k]} = \begin{cases} w_{ij} & \text{if } k = 0, \\ \max\left(w_{ij}^{[k-1]}, \min\{w_{ik}^{[k-1]}, w_{kj}^{[k-1]}\}\right) & \text{otherwise.} \end{cases}$$

$$p_{ij}^{[k]} = \begin{cases} \bullet & \text{if } k = 0 \text{ and } w_{ij}^{[0]} = \infty, \\ [i,j] & \text{if } k = 0 \text{ and } w_{ij}^{[0]} \ne \infty, \\ w_{ij}^{[k-1]} & \text{if } k > 0 \text{ and } w_{ij}^{[k-1]} \le w_{ik}^{[k-1]} + w_{kj}^{[k-1]}, \\ w_{ik}^{[k-1]} \asymp w_{kj}^{[k-1]} & \text{otherwise.} \end{cases}$$

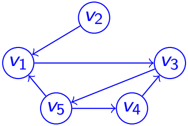
Remark

$W^{[n]}$ and $P^{[n]}$ can be computed in $O(n^3)$.

A special case of Warshall algorithm

Finding the shortest paths in a simple digraph

All arcs are assumed to have weight 1



$$P_G^{[0]} = \begin{pmatrix} \bullet & \bullet & [v_1, v_3] & \bullet & \bullet \\ [v_2, v_1] & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & [v_3, v_5] \\ \bullet & \bullet & [v_4, v_3] & \bullet & \bullet \\ [v_5, v_1] & \bullet & \bullet & [v_5, v_4] & \bullet \end{pmatrix},$$

$$P_G^{[1]} = \begin{pmatrix} \bullet & \bullet & [v_1, v_3] & \bullet & \bullet \\ [v_2, v_1] & \bullet & [v_2, v_1, v_3] & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & [v_3, v_5] \\ \bullet & \bullet & [v_4, v_3] & \bullet & \bullet \\ [v_5, v_1] & \bullet & [v_5, v_1, v_3] & [v_5, v_4] & \bullet \end{pmatrix},$$

$$P_G^{[2]} = P_G^{[1]}, \; P_G^{[3]} = \begin{pmatrix} \bullet & \bullet & [v_1, v_3] & \bullet & [v_1, v_3, v_5] \\ [v_2, v_1] & \bullet & [v_2, v_1, v_3] & \bullet & [v_2, v_1, v_3, v_5] \\ \bullet & \bullet & \bullet & \bullet & [v_3, v_5] \\ \bullet & \bullet & [v_4, v_3] & \bullet & [v_4, v_3, v_5] \\ [v_5, v_1] & \bullet & [v_5, v_1, v_3] & [v_5, v_4] & \bullet \end{pmatrix}, \; P_G^{[4]} = P_G^{(3)},$$

$$P_G^{[5]} = \begin{pmatrix} [v_1, v_3, v_5, v_1] & \bullet & [v_1, v_3] & [v_1, v_3, v_5, v_4] & [v_1, v_3, v_5] \\ [v_2, v_1] & \bullet & [v_2, v_1, v_3] & [v_2, v_1, v_3, v_5, v_4] & [v_2, v_1, v_3, v_5] \\ [v_3, v_5, v_1] & [v_3, v_5, v_1, v_3] & [v_3, v_5, v_1, v_3] & \bullet & [v_3, v_5] \\ [v_4, v_3, v_5, v_1] & \bullet & [v_4, v_3] & [v_4, v_3, v_5, v_4] & [v_4, v_3, v_5] \\ [v_5, v_1] & \bullet & [v_5, v_1, v_3] & [v_5, v_4] & [v_5, v_1, v_3, v_5] \end{pmatrix}$$