Graph Theory

Introduction. Distance in Graphs. Trees

November 2014

What is Graph Theory?

- The study of graphs as mathematical structures $G = (V, E)$ used to model pairwise relations (*a.k.a.* edges) between objects of a collection V.
  - The objects are modeled as nodes (or vertices) of a set V
  - The pairwise relations are modeled as edges, which are elements of a set E.
- Graphs differ mainly by the types of edges between nodes. Most common types of graphs are:
  - **Undirected:** there is no distinction between the nodes associated with each edge.
  - **Directed:** edges are arcs from one node to another.
  - **Weighted:** every edge has a weight which is typically a real number.
  - **Labeled:** every edge has its own label.

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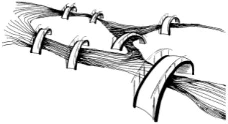
Graphs are among the most frequently used models in problem solving.

History of graph theory

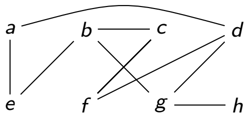
- 1736: L. Euler publishes "Seven Bridges of Königsberg" – first paper on graph theory.



  - Later: Euler's formula relating the number of edges, vertices, and faces of a convex polyhedron ⇒ generalizations by Cauchy and L'Huillier ⇒ study of topology and special classes of graphs.
- 1852: De Morgan introduces the "Four Color Map Conjecture": four is the minimum number of colors required to color any map where bordering regions are colored differently.
  - 1969: Heesch publishes a solving method
  - 1976: computer-generated proof of K. Appel and W. Haken.
- 1878: The term "graph" was first used by Sylvester in a publication in *Nature*.
- 1936: D. König publishes first textbook on graph theory.

ASSUMPTION: $G = (V, E)$ is a simple graph or digraph.

- The order of G is $|V|$, the number of its nodes.
- The size of G is $|E|$, the number of its edges.
- The neighborhood of $v \in V$ is $N(v) = \{x \in V \mid (v, x) \in E\}$.
- The closed neighborhood of $v \in V$ is $N[v] = \{v\} \cup N(v)$.
- The degree of $v \in V$ is the number of edges incident with V: $\deg(v) = |\{e \in E \mid e = (u, x) \text{ or } e = (x, u) \text{ for some } x \in V\}|$
- The maximum degree of G is $\Delta(G) = \max\{\deg(v) \mid v \in V\}$.
- The minimum degree of G is $\delta(G) = \min\{\deg(v) \mid v \in V\}$.
- The degree sequence of G with order n is the n-term sequence (usually written in descending order) of the vertex degrees of its nodes.



$G = (V, E)$ where $V = \{a, b, c, d, e, f\}$, $E = \{\{a, d\}, \{a, e\}, \{b, c\}, \{b, e\}, \{b, g\}, \{c, f\}, \{d, f\}, \{d, g\}, \{g, h\}\}$

- $N(d) = \{a, f, g\}$, $N[d] = \{a, d, f, g\}$,
- $\Delta(G) = \deg(b) = 3$
- $\delta(G) = \deg(h) = 1$,
- The degree sequence is $3, 3, 3, 2, 2, 2, 2, 1$

First Theorem of Graph Theory

Theorem

In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges. Consequently, the number of vertices with odd degree is even.

COMBINATORIAL PROOF.

Let $S = \sum_{v \in V} \deg(v)$. Notice that in counting S, we count each edge exactly twice. Thus, $S = 2|E|$ (the sum of the degrees is twice the number of edges). Since

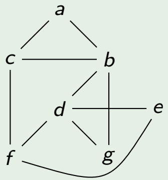
$$S = \sum_{\substack{v \in V \\ deg(v) \text{ even}}} deg(v) + \sum_{\substack{v \in V \\ deg(v) \text{ odd}}} deg(v)$$

and S is even, the second sum must be even, thus the number of vertices with odd degree is even.

ASSUMPTION: $G = (V, E)$ is a simple graph or digraph.

- A walk or path in G is a sequence of (not necessarily distinct) nodes $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \ldots, k-1$. Such a walk is sometimes called a $v_1 - v_k$ walk.
  - v_1 and v_k are the end vertices of the walk.
  - If the vertices in a walk are distinct, then the walk is called a simple path.
  - If the edges in a walk are distinct, then the walk is called a trail.
- A cycle is a simple path $v_1, \ldots, v_k$ (where $k \geq 3$) together with the edge (v_k, v_1).
- A circuit or closed trail is a trail that begins and ends at the same node.
- The length of a walk (or simple path, trail, cycle, circuit) is its number of edges, counting repetitions.

Example



- a, c, f, c, b, d is a walk of length 5.
- b, a, c, b, d is a trail of length 4.
- d, g, b, a, c, f, e is a simple path of length 6.
- g, d, b, c, a, b, g is a circuit.
- e, d, b, a, c, f, e is a cycle.

Note that walks, trails and simple paths can have length 0. The minimum length of a cycle or circuit is 3.

Theorem

In a graph G with vertices u and v, every $u-v$ walk contains a $u-v$ simple path.

PROOF. Let W be a $u-v$ walk in G. We prove this theorem by induction on the length of the walk W.

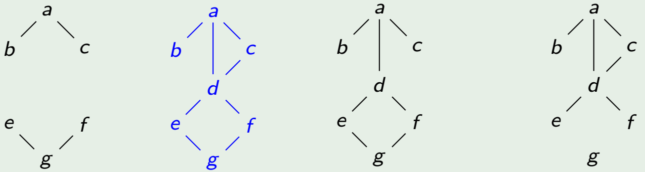
- If W has length 1 or 2, then it is easy to see that W must be a simple path.
- For the induction hypothesis, suppose the result is true for all walks of length $< k$ and suppose W has length k. Say that W is $u = w_0, w_1, \ldots, w_{k-1}, w_k = v$. If the nodes are distinct, then W itself is the desired $u-v$ simple path. If not, then let j be the smallest integer such that $w_j = w_r$ for some $r > j$. Let W_1 be the walk $u = w_0, \ldots, w_j, w_{r+1} \ldots, w_k = v$. This walk has length strictly less than k, and thus W_1 contains a $u-v$ simple path by induction hypothesis. Thus W contains a simple $u-v$ path.

ASSUMPTIONS: $G = (V, E)$ is a simple graph,
$v \in V$, $S \subseteq V$, $e \in E$, $T \subseteq E$

- Vertex deletion:
  - $G - v$ is the graph obtained by removing v and all edges incident with v from G.
  - $G - S$ is the graph obtained by removing each node of S and each edge incident with a node of S from G.
- Edge deletion:
  - $G - e$ is the graph obtained by removing only the edge e from G (its end nodes stay).
  - $G - T$ is the graph obtained by removing each edge of T from G.
- G is connected if every pair of nodes can be joined by a path. Otherwise, G is disconnected.
- A component of G is a maximal connected piece of G.
- v is a cut vertex if $G - v$ has more components than G.
- e is a bridge if $G - e$ has more components than G.

Operations on graphs and properties related to connectivity

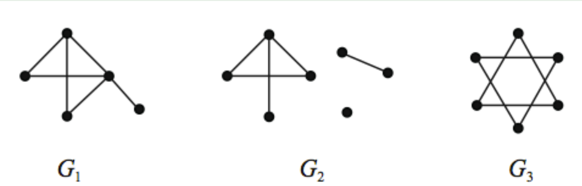
Example (Deletion operations)



$G - d$ $\quad$ G $\quad$ $G - (c, d)$ $\quad$ $G - \{(e, g), (f, g)\}$

d is a cut node in G. (a, b) is a bridge in G.

Example (Connected and disconnected graphs)



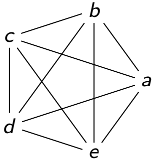
ASSUMPTION: $G = (V, E)$ is a graph.

- $\emptyset \neq S \subsetneq V$ is a node cut set of G if $G - S$ is disconnected.
- G is complete if every node is adjacent to every other node. We write K_n for the complete graph with n nodes.
  - The complete graphs K_n have no node cut sets because $K_n - S$ is connected for all proper subsets S of the set of nodes.
- If G is not complete then the connectivity of G, denoted by $\kappa(G)$, is the minimum size of a node cut set of G.
  - If G as a connected and incomplete graph of order n, then $1 \leq \kappa(G) \leq n - 2$.
  - If G is disconnected, then $\kappa(G) = 0$.
  - If $G = K_n$ then we say that $\kappa(G) = n - 1$.
- If $k > 0$, we say that G is k-connected if $k \leq \kappa(G)$.

Consequences of the definitions

1. A graph is connected if and only if $\kappa(G) \geq 1$.
2. $\kappa(G) \geq 2$ if and only if G is connected and has no cycles.
3. Every 2-connected graph contains at least one cycle.
4. For every graph G, $\kappa(G) \leq \delta(G)$.

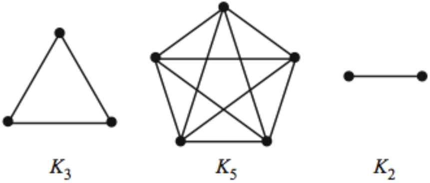
1. If G is a graph of order n, what is the maximum number of edges in G?
2. Prove that for any graph G of order at least 2, the degree sequence has at least one pair of repeated entries.
3. Consider the complete graph K_5 shown in the following figure.



a. How many different simple paths have c as an end vertex?
b. How many different simple paths avoid vertex c altogether?
c. What is the maximum length of a circuit in this graph? Give an example of such a circuit.

4. Let G be a graph where $\delta(G) \geq k$.
   a. Prove that G has a simple path of length at least k.
   b. If $k \geq 2$, prove that G has a cycle of length at least $k + 1$.
5. Prove that every closed odd walk in a graph contains an odd cycle.
6. Let P_1 and P_2 be two paths of maximum length in a connected graph G. Prove that P_1 and P_2 have a common vertex.
7. Prove that every 2-connected graph contains at least one cycle.

1. The complete graphs K_n. The graph K_n has order n and a connection between every two nodes. Examples:

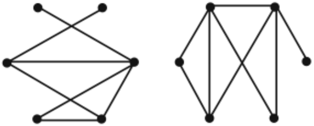


2. Empty graphs E_n. The graph E_n has order n and no edges. Example:

Special types of graphs
Complements and regular graphs

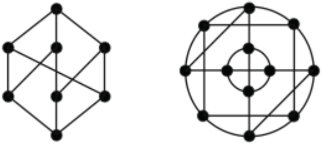
ASSUMPTION: $G = (V, E)$ is a graph.

- The complement of G is the graph $\overline{G}$ whose node set is the same as that of G and whose
  edge set consists of all the edges that are not in E. For example

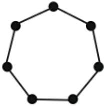


G $\qquad$ $\overline{G}$

- G is regular if all its nodes have the same degree. G is r-regular if $\deg(v) = r$ for all nodes v in G.
  K_n are $(n-1)$-regular graphs; E_n are 0-regular graphs.



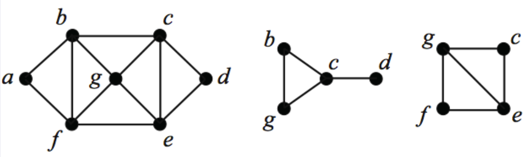
- The cycle C_n is simply a cycle on n vertices. Example: The graph C_7 looks as follows:



- The graph P_n is a simple path on n vertices. For example, the graph P_6 looks as follows:

- Given a graph $G = (V, E)$ and a subset $S \subseteq V$, the subgraph of G induced by S, denoted $\langle S \rangle_G$, is the subgraph with vertex set S and with edge set $\{(u, v) | u, v \in S \text{ and } u, v \in E\}$. So, $\langle S \rangle_G$ contains all vertices of S and all edges of G whose end vertices are both in S.

Example: a graph and two of its induced subgraphs

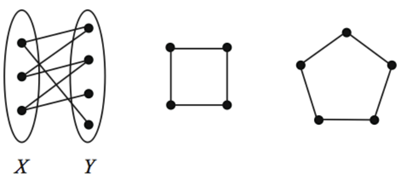


- A graph $G = (V, E)$ is bipartite if V can be partitioned into two sets X and Y such that every edge of G has one end vertex in X and the other in Y.
  - In this case, X and Y are called the partite sets.
- A complete bipartite graph is a bipartite graph with partite sets X and Y such that its edge set is $E = \{(x, y) \mid x \in X, y \in Y\}$.
  - Such a graph is complete bipartite graph denoted by $K_{|X|,|Y|}$.

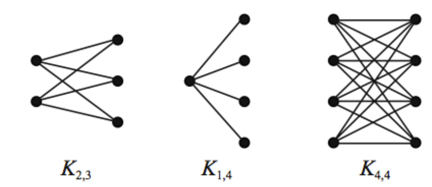
Special types of graphs: Examples

Bipartite graphs

The first two graphs in the following figure are bipartite, whereas the third graph is not bipartite.



Complete bipartite graphs



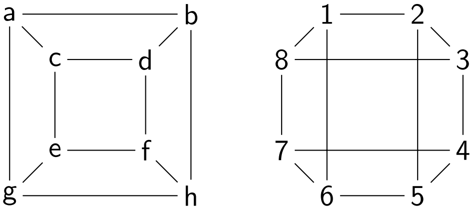
Theorem

A graph with at least 2 nodes is bipartite if and only if it contains no odd cycles.

PROOF.
"$\Rightarrow$:" Let $G = (V, E)$ be a bipartite graph with partite sets X and Y, and let $C = v_1, \ldots, v_k, v_1$ be a cycle in G. We can assume $v_1 \in X$ w.l.o.g. Then $v_i \in X$ for all even i and $v_i \in Y$ for all odd i. Since $(v_k, v_1) \in E$, we must have k even $\Rightarrow$ we can not have an odd cycle in G.
"$\Leftarrow$:" We can assume w.l.o.g. that G is connected, for otherwise we could treat each of its components separately.
Let $v \in V$ and define
$X = \{x \in V \mid \text{the shortest path from } x \text{ to } v \text{ has even length}\}$,
$Y = V \setminus X$.
It is easy to verify that G is a bipartite graph with partite sets X and Y.

Isomorphic graphs

Note that the following graphs are the same:



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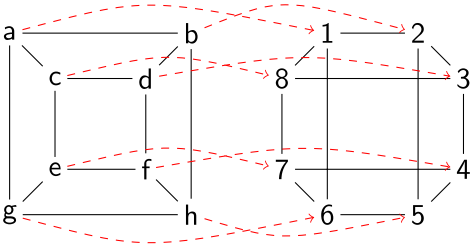

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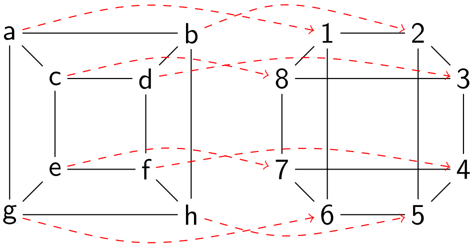

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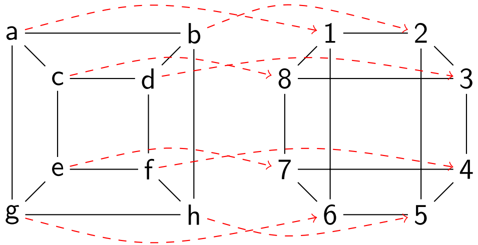


This is so because one graph can be redrawn to look like the other. The idea of isomorphism formalizes this phenomenon.

Isomorphic graphs

Two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ are isomorphic if there is a bijective mapping $f : V_1 \to V_2$ such that $(x, y) \in E_1$ if and only if $(f(x), f(y)) \in E_2$.

Isomorphic graphs

- When two graphs G and H are isomorphic, it is not uncommon to simply say that "$G = H$" or that "G is H."
- If G and H are isomorphic then they have the same order and size. The converse of this statement is not true, as seen in Figure 1 below.

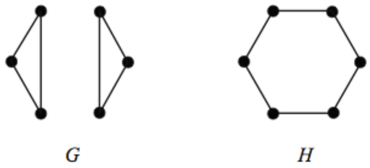


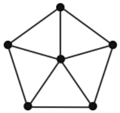
Figure : Two graphs G and H with same order and size, which are not isomorphic.

- If G and H are isomorphic then their degree sequences coincide. The converse of this statement is not true.

Exercises

1. For $n \geq 2$ prove that K_n has $n(n-1)/2$ edges.
2. Determine whether K_4 is a subgraph of $K_{4,4}$. If yes, then exhibit it. If no, then explain why not.
3. The line graph $L(G)$ of a graph G is defined in the following way:
   - the vertices of $L(G)$ are the edges of G, $V(L(G)) = E(G)$, and
   - two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex.

   a. Find $L(G)$ for the graph

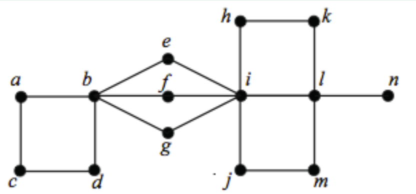
   

   b. Find the complement of $L(K_5)$.

ASSUMPTION: $G = (V, E)$ is a connected graph.

- The distance $d(u, v)$ from node u to node v in G is the length of the shortest path $u{-}v$ from u to v in G.
- The eccentricity $ecc(v)$ of v in G is the greatest distance from v to any other node.

Example

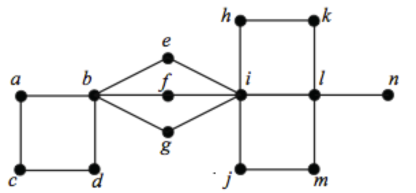


$d(b, k) = 4$, $d(c, m) = 6$.
$ecc(a) = 5$ since the farthest nodes from a are k, m, n, and they are a distance 5 from a.

ASSUMPTION: $G = (V, E)$ is a connected graph.

- The radius $rad(G)$ of G is the value of the smallest eccentricity.
- The diameter $diam(G)$ of G is the value of the greatest eccentricity.
- The center of G is the set of nodes v such that $ecc(v) = rad(G)$.
- The periphery of G is the set of nodes v such that $ecc(v) = diam(G)$.

Example



$rad(G) = 3$ and $diam(G) = 6$. The center of G is $\{e, f, g\}$.
The periphery of G is $\{c, k, m, n\}$.

Theorem 1

For any connected graph G, $rad(G) \leq diam(G) \leq 2\,rad(G)$.

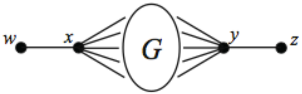
PROOF. By definition, $rad(G) \leq diam(G)$, so we just need to prove the second inequality. Let u, v be nodes in G such that $d(u,v) = diam(G)$. Let c be a node in the center of G. Then

$$diam(G) = d(u,v) \leq d(u,c) + d(c,v) \leq 2\,ecc(c) = 2\,rad(G).$$

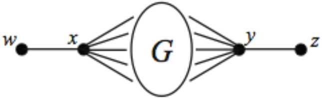
Theorem 2

Every graph $G = (V, E)$ is isomorphic to the center of some graph.

PROOF. We construct a new graph H by adding 4 nodes w, x, y, z to G along with the following edges:
$\{(w,x),(y,z)\} \cup \{(x,a) \mid a \in V\} \cup \{(b,y) \mid b \in V\}$.
The newly constructed graph H looks as shown in the figure below.



Proof of Theorem 2 continued.

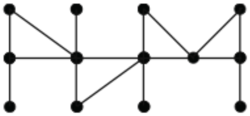


- $ecc(w) = ecc(z) = 4$, $ecc(y) = ecc(x) = 3$, and
- for any node v of G: $ecc(v) = 2$.

Therefore G is the center of H.

Exercises

1. Find the radius, diameter and center of the following graph:



2. Find the radius and diameter of each of the following graphs: P_{2k}, P_{2k+1}, C_{2k}, C_{2k+1}, K_n, and $K_{m,n}$.
3. Given a connected graph $G = (V, E)$ and a positive integer k, the *k-th power of G*, denoted G^k, is the graph whose set of nodes os V and where vertices u and v are adjacent in G_k if and only if $d(u, v) \leq k$ in G.
   a. Draw the 2-nd and 3-rd powers of P_8 and C_{10}.
   b. For a graph G of order n, what is $G^{diam(G)}$?

Trees

A tree is a connected graph which contains no cycles.

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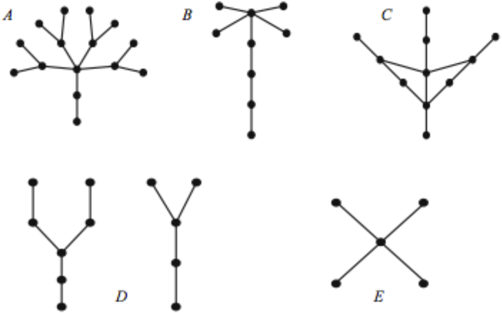
- **Quiz:** Which ones are trees?



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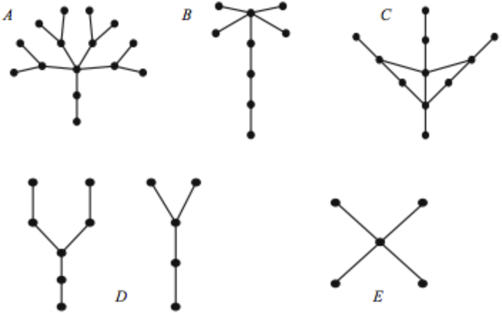

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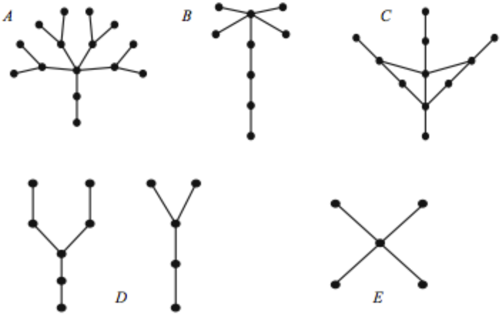

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- A leaf in a tree is a node with degree 1.

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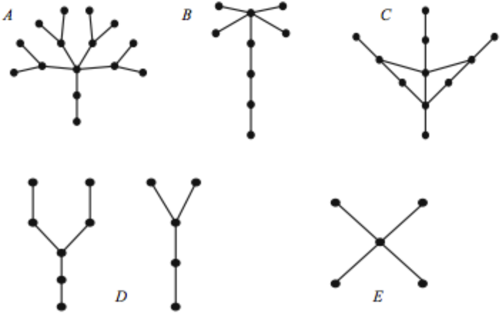

- A forest is a graph whose connected components are trees. E.g., the graph D is a forest.
- A leaf in a tree is a node with degree 1.
- Note that K_1 and K_2 are the only trees of order 1 and 2, respectively. P_3 is the only tree of order 3.

- If T is a tree of order n, then T has $n-1$edges.
- If F is a forest of order n containing k connected components, then F contains $n-k$ edges.
- A graph of order n is a tree if and only if it is connected and contains $n-1$ edges.
- A graph of order n is a tree if and only if it is acyclic and contains $n-1$ edges.
- Let T be the tree of order $n \geq 2$. Then T has at least two leaves.
- In any tree, the center is either a single vertex or a pair of adjacent vertices.

1. Show that every edge in a tree is a bridge.
2. Show that every nonleaf in a tree is a cut vertex.

References

J. M. Harris, J. L. Hirst, M. J. Mossinghoff. *Combinatorics and Graph Theory, Second Edition.* Springer 2008.

Chapter 1: Graph Theory. Sections §1.1, §1.2 and §1.4.