# Lecture 5

## Pólya's Theory of Counting

# Motivating problems

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2. How many different necklaces with $n$ beads can be formed using $m$ different kinds of beads?

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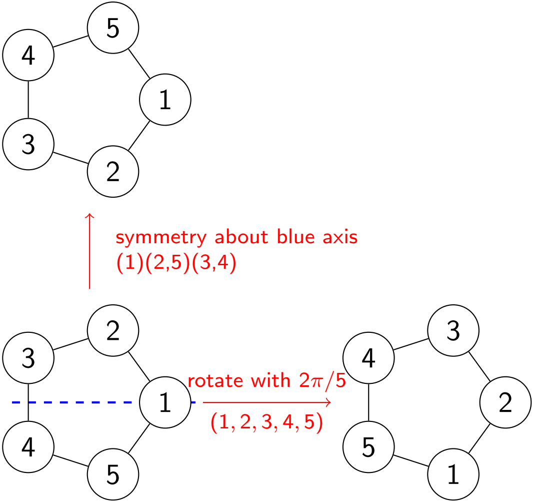
  2. Similarly, two necklaces should be considered identical if we can transform one into the other by rotating the necklace or by turning it over.
- We can rephrase these problems in the language of group theory.

# Examples of symmetries

## Necklace with 5 beads



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- Pólya noticed that the symmetries of an object form a group.

# Groups

A group is a set $G$ together with a binary operator $\circ$ defined on $G$ which satisfies 4 properties:

**Closure:** $a \circ b \in G$ for all $a, b \in G$.

**Associativity:** $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$

**Identity:** There exists $e \in G$ such that $e \circ a = a \circ e = a$ for all $a \in G$. The element $e$ is called the identity or neutral element of $G$.

**Inverses:** For every $a \in G$ there exists $b \in G$ such that $a \circ b = b \circ a = e$. The element $b$ is called the inverse of $a$.

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   - The neutral element is 0, the reverse of $r \in \mathbb{R}$ is $-r$.

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## Example

1. The set $\mathbb{R}$ of reals with addition $(+)$ is a group:
   - The neutral element is 0, the reverse of $r \in \mathbb{R}$ is $-r$.
2. The set $S_n$ of all permutations $\langle a_1, \ldots, a_n \rangle$ of $\{1, \ldots, n\}$ with function composition $\circ$ is a group.

# Subgroups. Permutation groups

- A subgroup of a group $(G, \circ)$ with neutral element $e$ is a subset $H$ of $G$ such that
  - $a \circ b \in H$ for all $a, b \in H$ (closure)
  - $e \in H$, and
  - For all $a \in H$ there is $b \in H$ such that $a \circ b = b \circ a = e$.
- A permutation group is a subgroup of the set $S_n$ of the permutations of $\{1, \ldots, n\}$.
- For a permutation $\pi \in S_n$ we define the powers
  - $\pi^0 = \langle 1, 2.3, \ldots, n\rangle = (1)(2)(3)\ldots(n)$,
  - $\pi^1 = \pi$, and
  - $\pi^n = \pi \circ \pi^{n-1}$ if $n > 1$.

**Example**

If $\pi = (2,3)(1,4,5,6)$ then

▷ $\pi^0 = (1)(2)(3)(4)(5)(6)$, $\pi^1 = \pi = (2,3)(1,4,5,6)$,

▷ $\pi^2 = \pi \circ \pi = (1,5)(2)(3)(4,6)$, $\pi^3 = \pi \circ \pi^2 = (1,6,5,4)(2,3)$,

▷ $\pi^4 = \pi \circ \pi^3 = (1)(2)(3)(4)(5)(6) = \pi^0$.

# Ciclic groups of permutations. Reflections

For a permutation $\pi \in S_n$, we define the set $\langle \pi \rangle = \{\pi^m \mid m \geq 0\}$. Note that $\langle \pi \rangle$ is a subgroup of $S_n$. $\langle \pi \rangle$ is the cyclic subgroup generated by $\pi$ in $S_n$.

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**Example**

$C_n = \langle (1, 2, \ldots, n) \rangle$ is the cyclic group generated by the cycle $(1, 2, \ldots, n) \in S_n$. $C_n$ consists of $n$ elements:

$$
\begin{aligned}
(1, 2, \ldots, n)^0 &= \langle 1, 2, \ldots, n \rangle = (1)(2)\ldots(n) \\
(1, 2, \ldots, n)^1 &= \langle 2, 3, \ldots, n, 1 \rangle \\
(1, 2, \ldots, n)^2 &= \langle 3, 4, \ldots, 1, 2 \rangle \\
&\ldots \\
(1, 2, \ldots, n)^{n-1} &= \langle n, 1, \ldots, n-1 \rangle \\
(1, 2, \ldots, n)^n &= \langle 1, 2, \ldots, n \rangle
\end{aligned}
$$

The reflection of a permutation $\langle a_1, a_2, \ldots, a_n \rangle$ is the permutation $\langle a_n, \ldots, a_2, a_1 \rangle$.

$C_n$ is the group of rotational symmetries of a regular polygon with $n$ vertices.

**Example**

$C_4 = \{(1)(2)(3)(4), (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$ corresponds to the permutation group of the nodes from the figure below, produced by rotations with $90^\circ$, $180^\circ$, or $270^\circ$.

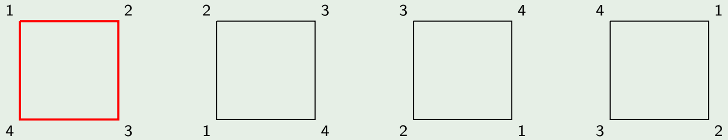



Figure: $C_4$ as a group of rotations of a square.

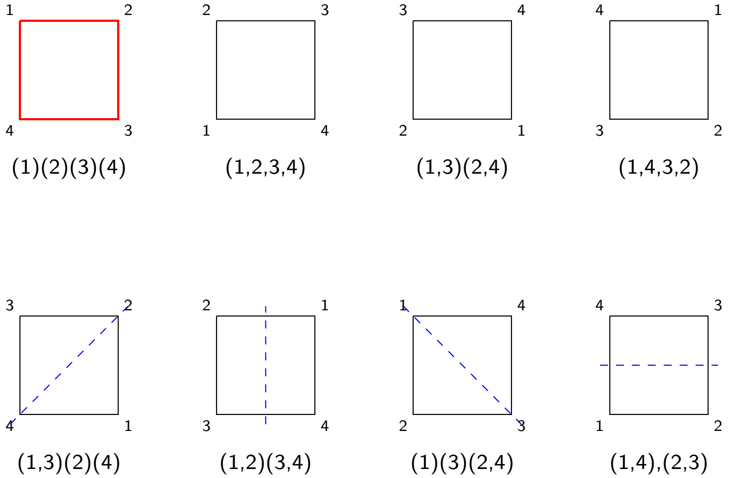
# Dihedral group

- The dihedral group $D_n$ consists of the elements of $C_n$ and their reflections. For instance:

$$
\begin{aligned}
D_4 =&\{\langle 1,2,3,4\rangle, \langle 2,3,4,1\rangle, \langle 3,4,1,2\rangle, \langle 4,1,2,3\rangle\}\cup \\
&\{\langle 4,3,2,1\rangle, \langle 1,4,3,2\rangle, \langle 2,1,4,3\rangle, \langle 3,2,1,4\rangle\} \\
=&\{(1)(2)(3)(4), (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}\cup \\
&\{(1,4)(2,3), (1)(2,4)(3), (1,2)(3,4), (1,3)(2)(4)\}.
\end{aligned}
$$

- $D_n$ has $2 \cdot n$ elements.
- $D_n$ can be identified with the group of rotational symmetries and reflections of a regular polygon with $n$ elements.

# $D_4$ as group of symmetries of a square



# Alternating group

Preliminary remarks

- A transposition is a cycle of length 2.
- Every cycle $(a_1, a_2, \dots, a_p)$ of an $n$-permutation can be written as a composition of transpositions:
  $(a_1, a_2, \dots, a_p) = (a_1, a_2)(a_2, a_3) \dots (a_{p-1}, a_p)$.
- Every $n$-permutation is a composition of cycles (cf. Lecture 4) $\Rightarrow$ every $n$-permutation is a composition of transpositions.
- An $n$-permutation is even if it is the composition of an even number of transpositions, and odd otherwise.
- It can be shown that an $n$-permutation can not be simultaneously even and odd.

The alternating group $A_n$ consists of the even permutations of $S_n$.

A coloring of $n$ objects $\{1, 2, \ldots, n\}$ is a map $c : \{1, 2, \ldots, n\} \rightarrow K$ where $K = \{k_1, \ldots, k_m\}$ contains $m$ colors.

- Every coloring $c$ can be represented as a permutation with repetition $\langle c(1), \ldots, c(n) \rangle$.
- There are $m^n$ possible colorings.

**Example**

The colouring $c : \{1, 2, 3, 4\}$ which maps $1 \mapsto r, 2 \mapsto g, 3 \mapsto r, 4 \mapsto r$ is represented by $\langle r, g, r, r \rangle$.

- Let $C$ be the set of all colourings $c : \{1, \ldots, n\} \rightarrow K$. If $\pi$ is a permutation and $c = \langle c(1), \ldots, c(n) \rangle$ is a colouring, we define the map $\pi^* : C \rightarrow C$ as follows: $\pi^*(\langle c(1), \ldots, c(n) \rangle) := \langle c(\pi(1)), \ldots, c(\pi(n)) \rangle$.

**Example**

If $\pi = (1, 2, 3, 4)$, then $\pi^*(\langle r, g, r, r \rangle) = \langle g, r, r, r \rangle$.

# Groups of colourings. Equivalent colourings

$G$: group of $n$-permutations

$c_1, c_2 : \{1, 2, \ldots, n\} \to K$ colourings

- $c_1$ and $c_2$ are equivalent with respect to $G$, and we write $c_1 \sim_G c_2$ if there is $\pi \in G$ such that $c_2 = \pi^*(c_1)$.

## Example

$G = C_4$, $c = \langle g, g, g, r \rangle$

- $C_4 = \langle \pi \rangle = \{\pi^n \mid n \geq 0\}$ where $\pi = (1, 2, 3, 4)$.
- $C_4 = \{\langle 1, 2, 3, 4 \rangle, \langle 2, 3, 4, 1 \rangle, \langle 3, 4, 1, 2 \rangle, \langle 4, 1, 2, 3 \rangle\}$

$\Rightarrow$ the colourings equivalent with $c$ are:

$\langle c(1), c(2), c(3), c(4) \rangle = \langle g, g, g, r \rangle$
$\langle c(2), c(3), c(4), c(1) \rangle = \langle g, g, r, g \rangle$
$\langle c(3), c(4), c(1), c(2) \rangle = \langle g, r, g, g \rangle$
$\langle c(4), c(1), c(2), c(3) \rangle = \langle r, g, g, g \rangle$

# Groups of colourings. Equivalent colourings

$G$: group of $n$-permutations
$C$ : set of colourings $c : \{1, 2, \ldots, n\} \to \{k_1, \ldots, k_m\}$ such that $\pi^*(c) \in C$ for every colouring $c \in C$

REMARKS:

1. $\sim_G$ is an equivalence relation (reflexive/symmetric/transitive) $\Rightarrow C$ can be partitioned in equivalence classes

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How many equivalence classes has $C$?

**Example (Square colourings with 2 colors)**

$S = \{1, 2, 3, 4\}$ is the set of nodes of a square, and $C$ the set of all possible colourings of the nodes with red (r) and green (g):

$$\begin{aligned} C = \{&\langle g,g,g,g\rangle, \langle g,g,g,r\rangle, \langle g,g,r,g\rangle, \langle g,g,r,r\rangle,\\ &\langle g,r,g,g\rangle, \langle g,r,g,r\rangle, \langle g,r,r,g\rangle, \langle g,r,r,r\rangle,\\ &\langle r,g,g,g\rangle, \langle r,g,g,r\rangle, \langle r,g,r,g\rangle, \langle r,g,r,r\rangle,\\ &\langle r,r,g,g\rangle, \langle r,r,g,r\rangle, \langle r,r,r,g\rangle, \langle r,r,r,r\rangle\} \end{aligned}$$

Two colourings are considered equivalent if one can be obtained from the other by rotating the square $\Rightarrow$ we consider $G = C_4 \Rightarrow$ 6 equivalence classes of $C$:

$$\begin{aligned} \overline{\langle g,g,g,g\rangle} =& \{\langle g,g,g,g\rangle\},\\ \overline{\langle g,g,g,r\rangle} =& \{\langle g,g,g,r\rangle, \langle g,g,r,g\rangle, \langle g,r,g,g\rangle, \langle r,g,g,g\rangle\}\\ \overline{\langle g,g,r,r\rangle} =& \{\langle g,g,r,r\rangle, \langle g,r,r,g\rangle, \langle r,g,g,r\rangle, \langle r,r,g,g\rangle\},\\ \overline{\langle g,r,g,r\rangle} =& \{\langle g,r,g,r\rangle, \langle r,g,r,g\rangle\},\\ \overline{\langle g,r,r,r\rangle} =& \{\langle g,r,r,r\rangle, \langle r,g,r,r\rangle, \langle r,r,g,r\rangle, \langle r,r,r,g\rangle\},\\ \overline{\langle r,r,r,r\rangle} =& \{\langle r,r,r,r\rangle\} \end{aligned}$$

- **The round table problem:** $S$ is the set of $n$ places at the table, $G$ is $C_n$, and $C$ is the collection of $n!$ seating assignments.
- **The necklace problem:** $S$ is the set of $n$ bead positions, $G$ is $D_n$, and $C$ is the collection of the $m^n$ possible arrangements of the $m$ kinds of beads on the necklace.

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- We want a general method to solve problems like these.

$G$: group of permtuations

$C$ : set of colourings $c : \{1, 2, \ldots, n\} \to \{k_1, \ldots, k_m\}$ such that $\pi^*(c) \in C$ for all $\pi \in G$

- The invariant set of a permutation $\pi \in Gi$ in $C$ is

$$C_\pi = \{c \in C \mid \pi^*(c) = c\}.$$

- The stabilizer of a coloring $c \in C$ is the set of permutations

$$G_c = \{\pi \in G \mid \pi^*(c) = c\}.$$

  $G_c$ is always a subgroup of $G$.

- The set of colorings in $C$ that are equivalent to $c$ under the action of the group $G$ is

$$\overline{c} = \{\pi^*(c) \mid \pi \in G\}.$$

  Thus $\overline{c}$ is the equivalence class of $c$ under the relation $\sim$ . $\overline{c}$ is also called the orbit of $c$ under the action of $G$.

**Example (Coloring the nodes of a square with 2 colors)**

$$\begin{aligned}C = \{&\langle g,g,g,g\rangle, \langle g,g,g,r\rangle, \langle g,g,r,g\rangle, \langle g,g,r,r\rangle,\\ &\langle g,r,g,g\rangle, \langle g,r,g,r\rangle, \langle g,r,r,g\rangle, \langle g,r,r,r\rangle,\\ &\langle r,g,g,g\rangle, \langle r,g,g,r\rangle, \langle r,g,r,g\rangle, \langle r,g,r,r\rangle,\\ &\langle r,r,g,g\rangle, \langle r,r,g,r\rangle, \langle r,r,r,g\rangle, \langle r,r,r,r\rangle\}.\end{aligned}$$

and the dihedral group $G = D_4$. Then

$$\begin{aligned}\overline{\langle g,g,g,r\rangle} &= \{\langle g,g,g,r\rangle, \langle g,g,r,g\rangle, \langle g,r,g,g\rangle, \langle r,g,g,g\rangle\}\\ G_{\langle g,g,g,r\rangle} &= \{(1)(2)(3)(4), (1,3)(2)(4)\}\\ \overline{\langle g,r,g,r\rangle} &= \{\langle g,r,g,r\rangle, \langle r,g,r,g\rangle\}\\ G_{\langle g,r,g,r\rangle} &= \{(1)(2)(3)(4), (1,3)(2,4), (1,3)(2)(4), (1)(2,4)(3)\}\end{aligned}$$

Observation. $|G_{\langle g,g,g,r\rangle}| \cdot |\overline{\langle g,g,g,r\rangle}| = 2 \cdot 4 = 8 = |G|$ and $|G_{\langle g,r,g,r\rangle}| \cdot |\overline{\langle g,r,g,r\rangle}| = 4 \cdot 2 = 8 = |G|$.

**Lemma**

Suppose a group $G$ acts on a set of colorings $C$. For any coloring $c \in C$ we have $|G_c| \cdot |\overline{c}| = |G|$.

**Burnside's Lemma**

The number $N$ of equivalence classes of the set $C$ in the presence of symmetries $G$ is given by

$$N = \frac{1}{|G|} \sum_{\pi \in G} |C_\pi|.$$

$$\begin{aligned}
\frac{1}{|G|}\sum_{\pi\in G}|C_\pi| &= \frac{1}{|G|}\sum_{\pi\in G}\sum_{c\in C}[\pi^*(c)=c] \\
&= \frac{1}{|G|}\sum_{c\in C}\sum_{\pi\in G}[\pi^*(c)=c] \\
&= \frac{1}{|G|}\sum_{c\in C}|G_c| \\
&= \sum_{c\in C}\frac{1}{\overline{c}} \\
&= \sum_{\overline{c}}\sum_{c\in\overline{c}}\frac{1}{\overline{c}} = \sum_{\overline{c}} 1 = N.
\end{aligned}$$

where $[\pi^*(c)=c] := \begin{cases} 1 & \text{if } \pi^*(c)=c \\ 0 & \text{otherwise} \end{cases}$

- To use Burnside's Lemma to count the number of equivalence classes of a set of colorings $C$, we must compute the size of the invariant set $C_\pi$ associated with every permutation $\pi \in G$.
- How can we count the size of $C_\pi$?
  - If $c$ is invariant under the action of $\pi$ then all objects permuted by the same cycle of $\pi$ must have the same color.
  - If $\pi$ has $k$ disjoint cycles, the number of colorings invariant under the action of $\pi$ is $|C_\pi| = m^k$, where $m$ is the number of colors.

**Example**

If $S$ is the set of nodes of a square and $G = D_4$ then $|C_{(1,2,3,4)}| = m$, $|C_{(1,2)(3,4)}| = m^2$, $|C_{(1,2)(2)(4)}| = m^3$, and $|C_{(1)(2)(3)(4)}| = m^4$.

# References


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