Homework 10

1. Let $\prec$ be a binary predicate symbol.

   (a) We know the following:

   $$\underset{x,y,z}{\forall}(x \prec y \wedge y \prec z \Rightarrow x \prec z), \qquad (transitivity)$$

   $$\underset{x}{\forall}\neg(x \prec x). \qquad (irreflexivity)$$

   Prove[1] (under the assumptions above):

   $$\underset{x,y}{\forall}(x \prec y \Rightarrow \neg(y \prec x)). \qquad (asymmetry)$$

   Proof[2]:
   To prove $(asymmetry)$, take x_0 arbitrary but fixed, and prove:

   $$\underset{y}{\forall}(x_0 \prec y \Rightarrow \neg(y \prec x_0)). \qquad (G1)$$

   `[Comment] The goal's outermost construct was ∀, so the "arbitrary but fixed" rule was applied. The new goal's outermost construct is still ∀, so apply the rule again. Note that the knowledge base does not change by the application of the new rule (think proof situations).`

   To prove $(G1)$, take y_0 arbitrary but fixed and prove:

   $$x_0 \prec y_0 \Rightarrow \neg(y_0 \prec x_0). \qquad (G2)$$

   `[Comment] Now (G2)'s outermost symbol is ⇒. Apply the "deduction rule":`

   To prove $(G2)$, assume

   $$x_0 \prec y_0 \qquad (A1)$$

   and show

   $$\neg(y_0 \prec x_0). \qquad (G3)$$

   `[Comment] To prove the negation (G3), apply the rule presented in the lecture.`
   We prove $(G3)$ by contradiction:

[1] This is an example and is proved by A.Crăciun. The rest of the exercises are homework.
[2] Some comments are included, `indicated with these fonts`. These comments are about choosing the inference rules, applying them. You need not include in your proofs these comments, but are expected to explain what you are doing and why, when proving.

Assume

$$y_0 \prec x_0. \qquad (A2)$$

Since we know $(transitivity)$, in particular, we know:

$$x_0 \prec y_0 \wedge y_0 \prec x_0 \Rightarrow x_0 \prec x_0, \qquad (A3)$$

From $(A1), (A2)$, and $(A3)$, by "modus ponens" $x_0 \prec x_0$. But this is in contradiction with $(irreflexivity)$. The proof is complete (we derived a contradiction).

(b) Now, assuming $(transitivity)$ and $(assymetry)$, prove $(irreflexivity)$ (all formulae as defined above).

(c) Give some examples (at least one) of predicates (relations) that satisfy $(transitivity)$, $(assymetry)$ and $(irreflexivity)$.

2. Let $\approx$ be a binary predicate.

(a) Assume:

$$\forall_{x,y,z}(x \approx y \wedge y \approx z \Rightarrow x \approx z), \quad (transitivity)$$

$$\forall_{x,y}(x \approx y \Rightarrow y \approx x), \qquad (symmetry)$$

$$\forall_{x}(x \approx x). \qquad (reflexivity)$$

Prove

$$\forall_{x,y,z,u}((x \approx y \wedge x \approx z \wedge y \approx u) \Rightarrow z \approx u). \quad (double\ transitivity)$$

(b) Give examples (at least one) of predicates (relations) that satisfy the properties of $\approx$, as defined above.

3. The *theory of groups* $\mathcal{GR}$ is described by:

- the *symbols in the language*:
  - function symbols, $\mathcal{F}_{\mathcal{GR}} = \{\circ, ^{-1}\}$, with $\circ$ binary, and $^{-1}$, the inverse function, unary,
  - predicate symbols,$\mathcal{P}_{\mathcal{GR}} = \{=\}$,$=$ binary,
  - constants, $\mathcal{C}_{\mathcal{GR}} = \{e\}$, e is the neutral element.
- the *knowledge base* $\mathcal{KB}_{GR}$:
  - axioms of groups:

$$\forall_{x}(x \circ e = x), \qquad (right\ identity)$$

$$\forall_{x}(x \circ x^{-1} = e), \qquad (right\ inverse)$$

$$\forall_{x,y,z}((x \circ y) \circ z = x \circ (y \circ z)). \quad (associativity)$$

- the *inference mechanism* $\mathcal{IR}_{GR)}$ consists of the inference rules of predicate logic, and rules for equality.

(a) Prove

$$\underset{x,y,z}{\forall}(x \circ z = y \circ z \Rightarrow x = y), \quad (\textit{right cancellation})$$

$$\underset{x}{\forall}(e \circ x = x), \quad (\textit{left identity})$$

$$\underset{x}{\forall}(x^{-1} \circ x = e), \quad (\textit{left inverse})$$

$$\underset{x,y,z}{\forall}(z \circ x = z \circ y \Rightarrow x = y) \quad (\textit{left cancellation})$$

$$\underset{x}{\forall}(x \circ x = x \Rightarrow x = e). \quad (\textit{nonidempotence})$$

(b) Give 3 examples of groups.