Graph Theory and Combinatorics

Exercises for Seminar 3

1. Write down the canonical cyclic structures of the following permutations:

   (a) $\langle 1,6,3,2,5,7,4,9,8\rangle$

   (b) $\langle 6,5,8,1,2,4,7,3,10,9\rangle$

   (c) $\langle 6,1,2,4,5,3,8,7\rangle$

   Answer:

   (a) $(1)(2,6,7,4)(3)(5)(8,9)$

   (b) $(1,6,4)(2,5)(3,8)(7)(9,10)$

   (c) $(1,6,3,2)(4)(5)(7,8)$

2. Write down the permutations with the following cyclic structures:

   (a) $(4,7,3)(8,5,1,2)(10,6,9)$

   (b) $(6)(1,2,3,4,5)$

   (c) $(2,4,6,8)(1,5,3,7)$

   Answer:

   (a) $\langle 1,6,3,2,5,7,4,9,8\rangle$

   (b) $\langle 6,5,8,1,2,4,7,3,10,9\rangle$

   (c) $\langle 6,1,2,4,5,3,8,7\rangle$

3. Write down the canonical cyclic structures of the cyclic structures from the previous exercise.

   Answer:

   (a) $(1,2,8,5)(3,4,7)(6,9,10)$

   (b) $(1,2,3,4,5)(6)$

(c) $(1,5,3,7)(2,4,6,8)$

4. Write down the types of the following permutations:

a) $\langle 1,2,3,4,5,6,7,8\rangle$	b) $\langle 8,1,2,3,4,5,6,7\rangle$
c) $\langle 4,5,6,7,1,2,3\rangle$	d) $\langle 3,4,5,6,7,1,2\rangle$
e) $\langle 1,2,3,4,6,5,8,7\rangle$	f) $\langle 7,8,5,6,4,3,2,1\rangle$

ANSWER:

(a) $\langle 1,2,3,4,5,6,7,8\rangle = (1)(2)(3)(4)(5)(6)(7)(8)$ are tipul [8,0,0,0,0,0,0,0]

(b) $\langle 8,1,2,3,4,5,6,7\rangle = (1,8,7,6,5,43,2)$ are tipul [0,0,0,0,0,0,0,1]

(c) $\langle 4,5,6,7,1,2,3\rangle = (1,4,7,3,6,2,5)$ are tipul [0,0,0,0,0,0,1]

(d) $\langle 3,4,5,6,7,1,2\rangle = (1,3,5,7,2,4,6)$ are tipul [0,0,0,0,0,0,1]

(e) $\langle 1,2,3,4,6,5,8,7\rangle = (1)(2)(3)(4)(5,6)(7,8)$ are tipul [4,2,0,0,0,0,0,0]

(f) $\langle 7,8,5,6,4,3,2,1\rangle = (1,7)(2,8)(3,5)(4,6)$ are tipul [0,4,0,0,0,0,0,0]

5. Which of the following lists is a valid permutation type:

(a) $[1,1,0,1]$	(b) $[1,1,0,0]$
(c) $[1,0,1,0,0]$	(d) $[0,0,0,0,1]$
(e) $[0,1,0,1,0,0]$	(f) $[1,0,1,0,0,0]$

ANSWER: (d), (e).

6. Write down all plausible types of the permutations of the set $\{1,2,3,4\}$. [SUGGESTION: Write down all integer partitions of 4, and use the relationship between integer partitions and permutation types described in Lecture 4.]

ANSWER: The following table indicates the 1-1 correspondence between the integer partitions of 4 and all permutation types of the set $\{1,2,3,4\}$:

integer partition of 4	permutation type of $\{1,2,3,4\}$
$\{4\}$	$[0,0,0,1]$
$\{1,3\}$	$[1,0,1,0]$
$\{2,2\}$	$[0,2,0,0]$
$\{1,1,2\}$	$[2,1,0,0]$
$\{1,1,1,1\}$	$[4,0,0,0]$

7. Write down all plausible types of the permutations of the set $\{1,2,3\}$.

ANSWER: The following table indicates the 1-1 correspondence between the integer partitions of 3 and all permutation types of the set $\{1,2,3\}$:

integer partition of 4	permutation type of $\{1,2,3,4\}$
$\{3\}$	$[0,0,1]$
$\{1,2\}$	$[1,1,0]$
$\{1,1,1,1\}$	$[3,0,0]$

8. How many permutations have the same type as the permutation $\langle 3,2,1,5,4\rangle$?

ANSWER: $\langle 3,2,1,5,4\rangle = (1,3)(2)(4,5)$ has type [1,2,0,0,0], and the number of permutations with type [1,2,0,0,0] is

$$\frac{5!}{1!\cdot 2!\cdot 0!\cdot 0!\cdot 0!\cdot 1^1\cdot 2^2\cdot 3^0\cdot 4^0\cdot 5^0} = \frac{5!}{2!\cdot 2^2} = \frac{2\cdot 3\cdot 4\cdot 5}{2\cdot 4} = 15.$$

9. Which of the following formulas is a solution of the recurrence relation $a_n = 8\,a_{n-1} - 16\,a_{n-2}$:

a) $a_n = 0$?	b) $a_n = 1$?
c) $a_n = 2^n$?	d) $a_n = 4^n$?
e) $a_n = n\cdot 4^n$?	f) $a_n = 2\cdot 4^n + 3\cdot n\cdot 4^n$?
g) $a_n = (-4)^n$?	h) $a_n = n^2\cdot 4^n$?

ANSWER: In every case, we shall check if the relation $a_n = 8\,a_{n-1} - 16\,a_{n-2}$ holds after replacing a_n with the corresponding expression:

(a) $8\,a_{n-1} - 16\,a_{n-2} = 8\cdot 0 - 16\cdot 0 = 0 = a_n$, thus $a_n = 0$ is a solution of this recurrence.

(b) $8\,a_{n-1} - 16\,a_{n-2} = 8\cdot 1 - 16\cdot 1 = -8 \neq 1 = a_n$, thus $a_n = 1$ is not a solution of this recurrence.

(c) $8\,a_{n-1} - 16\,a_{n-2} = 8\cdot 2^{n-1} - 16\cdot 2^{n-2} = 2^3\cdot 2^{n-1} - 2^4\cdot 2^{n-2} = 2^{n+2} - 2^{n+2} = 0 \neq 2^n$, thus $a_n = 2^n$ is not a solution of this recurrence.

(d) $8\,a_{n-1} - 16\,a_{n-2} = 8\cdot 4^{n-1} - 16\cdot 4^{n-2} = 2\cdot 4^n - 4^n = 4^n = a_n$, thus $a_n = 4^n$ is a solution of this recurrence.

(e) $8\,a_{n-1} - 16\,a_{n-2} = 8\,(n-1)\,4^{n-1} - 16\,(n-2)\,4^{n-2} = (2\,n-2)\,4^n - (n-2)\,4^n = n\cdot 4^n = a_n$, thus $a_n = 4^n$ is a solution of this recurrence.

Similarly, we can verify that (f) gives a solution to this recurrence relation, whereas (g) and (h) do not.

Another way to solve this exercise is to compute the general form of the solution of this recurrence relation. The characteristic equation is $r^2 - 8\,r + 16 = 0$, with the double root $r_1 = 4$ with multiplicity 2. This implies that the solution of the recurrence relation is $a_n = (a \cdot n + b) \cdot 4^n$ with $a, b \in \mathbb{R}$. It is easy to see that (a), (d), (e), (f) are solutions of this form, whereas (b), (c), (g), (h) do not.

10. Solve these recurrence relations together with the initial conditions given.
    (a) $a_n = 2\,a_{n-1}$ for $n \geq 1$, $a_0 = 3$.
    (b) $a_n = 5\,a_{n-1} - 6\,a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$.
    (c) $a_n = 4\,a_{n-1} - 4\,a_{n-2}$, $a_0 = 6$, $a_1 = 8$.
    (d) $a_n = 4\,a_{n-2}$, $a_0 = 0$, $a_1 = 4$.
    (e) $a_n = 7\,a_{n-1} - 10\,a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$.

11. Find the solution to $a_n = 2\,a_{n-1} + 5\,a_{n-2} - 6\,a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

    ANSWER: The characteristic equation of this relation is $r^3 - 2\,r^2 - 5\,r + 6 = 0$. As a first guess, we will check if some divisor of the free term 8 is root of this equation. It turns out that $r_1 = 1$ is solution, thus $r^3 - 2\,r^2 - 5\,r + 6$ is divisible with $r - 1$. If we divide $r^3 - 2\,r^2 - 5\,r + 6$ with $r - 1$, we get $r^3 - 2\,r^2 - 5\,r + 6 = (r^2 - r - 6)(r - 1) = (r + 2)(r - 3)(r - 1)$, thus the characteristic equation has three distinct roots: $r_1 = 1$, $r_2 = -2$, $r_3 = 3$. It follows that

$$a_n = a \cdot r_1^n + b \cdot r_2^n + c \cdot r_3^n = a + b \cdot (-2)^n + c \cdot 3^n$$

    From $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$ we obtain

$$\left\{ \begin{array}{ll} a + b + c & = 7 \\ a - 2\,b + 3\,c & = -4 \\ a + 4\,b + 9\,c & = 8 \end{array} \right. \Rightarrow \left\{ \begin{array}{ll} a & = 5 \\ b & = 3 \\ c & = -1 \end{array} \right. \Rightarrow a_n = 5 + 3 \cdot (-2)^n - 3^n.$$

12. Determine the values of the constants a and b such that $a_n = a\,n + b$ is a solution of the recurrence relation $a_n = 2\,a_{n-1} + n + 5$.
    ANSWER: If we replace a_n with $a\,n + b$ in this recurrence relation, we obtain

$$a\,n + b = 2\,(a\,(n-1) + b) + n + 5 = (2\,a + 1)\,n + 2\,b - 2\,a + 5$$

    for all $n \in \mathbb{N}$, which implies

$$\left\{ \begin{array}{ll} a & = 2\,a + 1 \\ b & = 2\,b - 2\,a + 5 \end{array} \right. \Rightarrow \left\{ \begin{array}{ll} a & = -1 \\ b & = -7 \end{array} \right.$$

13. What is the general form of a solution of the linear nonhomogeneous recursive relation $a_n = 6\,a_{n-1} - 12\,a_{n-2} + 8\,a_{n-3} + F(n)$, if

    (a) $F(n) = n^2$?

    (b) $F(n) = n\,2^n$?

    (c) $F(n) = n^2\,2^n$?

    (d) $F(n) = 2^n$?

    (e) $F(n) = (-2)^n$?

14. Find all solutions of the recurrence relation $a_n = 2\,a_{n-1} + 2\,n^2$.

15. Let

$$a_n = \sum_{k=1}^{n} \frac{k(k+1)}{2} \quad \text{for all } n \geq 1.$$

Show that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = a_{n-1} + \frac{n(n+1)}{2}$$

and the initial condition $a_1 = 1$. Then, solve this recurrence relation to determine a formula for a_n.

16. A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 12304056789 is valid, whereas 12098700567 is not. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

Answer: There are nine 1-digit codewords, namely 1,2,3,4,5,6,7,8,9. Thus $a_1 = 9$.

If $n > 1$, we distinguish 2 disjoint cases:

1. The codeword starts with a non-zero digit x, thus it is of the form xw where $1 \leq x \leq 9$ and w id a codeword with $n-1$ decimal digits. There are 9 choices for x, and a_{n-1} choices for w. By the rule of product, there are $9 \cdot a_{n-1}$ codewords in this case.
2. The codeword starts with 0, thus it is of the form $0w'$ where w' is a string with $n-1$ decimal digits which is not a codeword. There are 10^{n-1} strings with $n-1$ decimal digits. By the principle of inclusion and exclusion, there are $10^{n-1} - a_{n-1}$ possibilities to choose w'. Thus, there are $10^{n-1} - a_{n-1}$ codewords in this case.

By the rule of sum, $a_n = 9\,a_{n-1} + 10^{n-1} - a_{n-1} = 10^{n-1} + 8\,a_{n-1}$.

17. Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n+1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n$, to specify the order of multiplication. For example $C_3 = 5$ because there are five ways to parenthesize $x_0 \cdot x_1 \cdot x_2 \cdot x_3$ to determine the order of multiplication:

$$\begin{array}{lll} ((x_0 \cdot x_1) \cdot x_2) \cdot x_3 & (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3 & (x_0 \cdot x_1) \cdot (x_2 \cdot x_3) \\ x_0 \cdot ((x_1 \cdot x_2) \cdot x_3) & x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)) & \end{array}$$

Answer: Note that, however we insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n$, one '$\cdot$' operator remains outside all parentheses, namely, the operator for the final multiplication operation to be performed. For example, in $((x_0 \cdot x_1) \cdot x_2) \cdot x_3$ it is the final '$\cdot$', while in $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$ it is the second '$\cdot$'. This final operator appears between two of the $n+1$ numbers, say between x_k and x_{k+1}.

By the product rule, there are $C_k \cdot C_{n-k-1}$ ways to insert parentheses to determine the order of the $n+1$ numbers to be multiplied when the final operator appears between x_k and x_{k+1}, because:

- There are C_k ways to parenthesize the product $x_0 \cdot x_1 \cdot \ldots \cdot x_k$, and
- There are C_{n-k-1} ways to parenthesize the product $x_{k+1} \cdot \ldots \cdot x_n$.

Because the final operator can appear between any of the $n+1$ numbers, it follows from the rule of sum that

$$C_n = \sum_{k=0}^{n-1} C_k \cdot C_{n-k-1}.$$

Note that the initial conditions are $C_0 = 1$ and $C_1 = 1$.
From now we can compute the other values of C_n recursively:

$$\begin{array}{l} C_2 = C_0 \cdot C_1 + C_1 \cdot C_0 = 1 + 1 = 2 \\ C_3 = C_0 \cdot C_2 + C_1 \cdot C_1 + C_2 \cdot C_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5 \\ C_4 = C_0 \cdot C_3 + C_1 \cdot C_2 + C_2 \cdot C_1 + C_3 \cdot C_0 = 5 + 2 + 2 + 5 = 14 \\ \ldots \end{array}$$

18. A vending machine dispensing books of stamps accepts only bills of 1\$, 5\$, and 20\$. Let b_n be the number of ways to deposit n dollars in the vending machine, if the order in which the bills are inserted matters. Find a recursive formula for the computation of b_n.

Answer: We distinguish three distinct cases:

1. The first bill inserted in the vending machine is 1$ $\Rightarrow$ there are $n-1$ dollars left to insert in the machine, and this can be done in a_{n-1} ways.
2. The first bill inserted in the vending machine is 5$ $\Rightarrow$ there are $n-5$ dollars left to insert in the machine, and this can be done in a_{n-5} ways.
3. The first bill inserted in the vending machine is 20$ $\Rightarrow$ there are $n-20$ dollars left to insert in the machine, and this can be done in a_{n-20} ways.

By the rule of sum, $a_n = a_{n-1} + a_{n-5} + a_{n-20}$.

19. Let a_n be the number of n-bit strings that contain 01. Find a recurrence relation for the computation of a_n.
ANSWER: Let b_n be the number of all n-bit strings, and c_n be the number of all n-nit strings that do not contain 01. An n-bit string is of one of the following two kinds:

(a) It contains 01. There are a_n n-bit strings of this kind.
(b) It does not contain 01. There are c_n n-bit strings of this kind.

By the rule of sum, the total number of n-bit strings is $b_n = a_n + c_n$. By the rule of product, $b_n = 2^n$. Thus $a_n = b_n - c_n = 2^n - c_n$.

An n-bit string which does not contain 01 is of the form

$$\underbrace{1\ldots1}_{k \text{ times}}\ \underbrace{0\ldots0}_{n-k \text{ times}}$$

where $0 \le k \le n$. There are $n+1$ possible values for k, thus $c_n = n+1$.

We conclude that $a_n = 2^n - (n+1)$.

Exercises related to Lectures 2 and 3

20. Arrange the following permutations of $\{1,2,3,4,5\}$ in increasing lexicographic order: $\langle 4,3,1,2,5\rangle, \langle 2,1,5,3,4\rangle, \langle 3,4,5,2,1\rangle, \langle 3,4,1,5,2\rangle, \langle 4,3,1,5,2\rangle$.

ANSWER: $\langle 2,1,5,3,4\rangle, \langle 3,4,1,5,2\rangle, \langle 3,4,5,2,1\rangle, \langle 4,3,1,2,5\rangle, \langle 4,3,1,5,2\rangle$

21. What is the permutation that follows after $\langle 4,5,3,2,1\rangle$ in lexicographic order?

ANSWER: $\langle 5,1,2,3,4\rangle$.

22. What is the permutation that follows after $\langle 4,2,5,3,1\rangle$ in lexicographic order?

ANSWER: $\langle 4,3,1,2,5\rangle$.

23. What is the rank of the 3-permutation with repetition $\langle 2, 2, 5\rangle$ of the set $\{1, 2, 3, 4, 5\}$ in the lexicographic ordering?

    ANSWER: $1, 2, 3, 4, 5$ have ranks $0, 1, 2, 3, 4$ in the set $\{1, 2, 3, 4, 5\}$ (which has 5 elements), thus the rank of the 3-permutation with repetition $\langle 2, 2, 5\rangle$ is $114_{(5)} = 1 \cdot 5^2 + 1 \cdot 5 + 4 = 25 + 5 + 4 = 34$.

24. Compute the 6-permutation with repetition of the set $\{1, 2\}$ with rank 17 in the lexicographic ordering.

    ANSWER: The set $A = \{1, 2\}$ has 2 elements, thus we shall compute the representation in base 2 of 17, using 6 digits:

    $$17 = 16 + 1 = 2^4 + 1 = 010001_{(2)}$$

    and the 6-permutation with repetition of $\{1, 2\}$ corresponding to the binary number $010001_{(2)}$ is $\langle 121112\rangle$.

25. Compute the 5-permutation with repetition of the set $\{1, 2, 3\}$ with rank 17 in the lexicographic ordering.

    ANSWER: $17 = 00122_{(3)}$, thus the 5-permutation with repetition with rank 17 of $\{1, 2, 3\}$ is $\langle 1, 1, 2, 3, 3\rangle$.

26. Which is the fourth subset of $\{1, 2, 3, 4\}$ enumerated with the standard reflected Grey code?

27. Which is the subset of $\{1, 2, 3, 4\}$ that follows after $\{1, 3\}$ in the enumeration order via binary representations?

    ANSWER: $4 > 3 > 2 > 1$ and the binary representation of $\{1, 3\}$ is 0101. The nest binary encoding is 0110, which corresponds to the subset $\{2, 3\}$.

28. Which is the 3-combination of $\{1, 2, 3, 4, 5\}$ that follows after $\{1, 3, 4\}$ in the enumeration order via binary representations? ANSWER:

    $5 > 4 > 3 > 2 > 1$ and

3-combinaton	binary representation 54321
$\{1, 3, 4\}$	01101

    The next 5-bit number with 3 occurrences of 1, after 01101, is 01110.
    The 3-combination for 01110 is $\{2, 3, 4\}$.

29. What is the rank of the permutation $\langle 1, 2, 4, 5, 3\rangle$ in lexicographic order?

    ANSWER: $0 \cdot 4! + 0 \cdot 3! + 1 \cdot 2! + 1 \cdot 1! = 3$.

30. What is the rank of the permutation $\langle 2, 4, 1, 3\rangle$ in lexicographic order?

ANSWER: $1 \cdot 3! + 2 \cdot 2! + 0 \cdot 1! = 6 + 4 = 10$.