Aquaculture

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Abstract

As wild fish catches have plateaued, aquaculture has expanded, producing nearly half of fish consumed in 2009. To grow in a sustainable way, aquaculture will need to produce more fish per unit of land and water and reduce its reliance on wild-caught fish for feed. In order to do that, we need to determine the best time for harvesting the fish.

Introduction

Aquaculture, also known as aquafarming, is the farming of aquatic organisms such as fish, crustaceans, molluscs and aquatic plants. Aquaculture involves cultivating freshwater and saltwater populations under controlled conditions, and can be contrasted with commercial fishing, which is the harvesting of wild fish. According to the FAO, aquaculture "is understood to mean the farming of aquatic organisms including fish, molluscs, crustaceans and aquatic plants".

Problem

U.N. reports that 32% of global fish stocks are over exploited or depleted and as much as 90% of large species like tuna and marlin have been fished out in the past half-century. In order to encourage aquaculture over natural fish catches, we want to minimize the costs as much as possible by finding the optimal harvesting time.

Defining our model

A differential equation describing the growth of fish may be expressed as

$$\frac{dW}{dt} = KW^{\alpha} \tag{1}$$

where W(t) is the weight of the fish at time t and K and α are empirically determined growth constants. The functional form of this relationship is similar to that of the growth models for other species. Modeling the growth rate or metabolic rate by a term like W a is a common assumption. Biologists often refer to equation (1) as the **allometric equation**.It can be supported by plausibility arguments such as growth rate depending on the surface area of the gut (which varies like $W^{\frac{2}{3}}$) or depending on the volume of the animal (which varies like W).

Implicit solution

We can solve the equation for $\alpha \neq 1$

$$\frac{dW}{dt} = KW^{\alpha} \text{ is a separable equation}$$

$$\frac{dW}{W^{\alpha}} = Kdt \text{ we integrate in both sides}$$

$$\int W^{\alpha}\, dW = \int K\, dt$$

$$\frac{W^{1-\alpha}}{1-\alpha} = Kt + C$$

$$W^{1-\alpha} = \ (1-\alpha)(Kt + C)$$

$$W = [\ (1-\alpha)\ Kt + C)]^{\frac{1}{1-\alpha}}$$

More real model

The solution obtained in the first part grows large without bound, but in practice there is some limiting maximum weight W_{max} for the fish. This limiting weight may be included in the differential equation describing growth by inserting a dimensionless variable S that can range between 0 and 1 and involves an empirically determined parameter μ.

Namely, we now assume that

$$\frac{dW}{dt} = KW^{\alpha}S \tag{2}$$

where $S := 1 - (\frac{W}{W_{max}})^{\mu}$. When $\mu = 1 - \alpha$, equation (2) has a closed form solution. We can find W(t) using some constant given for t measured in months ($K = 10$, $\alpha = \frac{3}{4}$, $\mu = \frac{1}{4}$, $W_{max} = 81$ ounces and $W(0) = 1$ ounce).

$$\frac{dW}{dt} = KW^{\alpha}S, \text{ where } K = 10, \alpha = \frac{3}{4},\ W_{max} = 81$$

$$S = \left(1 - \frac{W}{W_{max}}\right)^{\mu} \text{, where } \mu = 1 - \alpha$$

$$\frac{dW}{dt} = 10W^{\alpha}\left(\left(1 - \frac{W}{W_{max}}\right)^{1-\alpha}\right)$$

$$\frac{dW}{dt} = 10W^{\alpha} - \left(\frac{10W^{\alpha}W^{1-\alpha}}{W_{max}^{1-\alpha}}\right)$$

$$\frac{dW}{dt} = 10W^{\alpha} - \left(\frac{10W}{W_{max}{}^{\frac{1}{4}}}\right)$$

$$\frac{dW}{dt} = 10W^{\alpha} - \left(\frac{10W}{3}\right)$$

The Bernoulli equation:

$$\frac{dW}{dt} + \left(\frac{10W}{3}\right) = 10W^{\alpha}$$

$$W^{-\alpha}\frac{dW}{dt} + \left(\frac{10W^{1-\alpha}}{3}\right) = 10$$

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We make a substitution of weight with $v = W^{1-\alpha}$ and solve the linear equation obtained this way:

$$\frac{dv}{dt} = (1-\alpha)W^{-\alpha}\frac{dW}{dt}$$

$$W^{-\alpha}\frac{dW}{dt} = \frac{1}{1-\alpha}\frac{dv}{dt}$$

$$\frac{1}{1-\alpha}\frac{dv}{dt} + \frac{10}{3}v = 10$$

$$a_1 = \frac{1}{1-\alpha}$$

$$a_0 = \frac{10}{3}$$

$$P(t) = \frac{a_0}{a_1} = \frac{10(1-\alpha)}{3} = \frac{10}{3 \cdot 4}$$

$$Q(t) = 10(1-\alpha) = \frac{10}{4}$$

$$\frac{dv}{dt} + P(t)v = Q(t)$$

$$\frac{dv}{dt} - \frac{10}{3 \cdot 4}v = \frac{10}{4}$$

$$\mu = e^{\int \frac{-10}{3}(1-\alpha)\,\mathrm{d}t} = e^{\frac{-10}{12}t}$$

$$v = e^{\frac{-10}{12}t}\left(\frac{10}{4}\int e^{\frac{-10}{12}t}\,\mathrm{d}t + c\right)$$

$$v = e^{\frac{-10}{12}t}\left(\frac{10}{4} \cdot \frac{6}{5} \cdot e^{\frac{5t}{6}} + c\right)$$

$$v = e^{\frac{-10}{12}t}\left(3e^{\frac{5t}{6}} + c\right)$$

$$v = W^{\frac{1}{4}} => W = v^4$$

$$W = e^{\frac{-10}{3}t}\left(3e^{\frac{5t}{6}} + c\right)^4$$

We substitute with the initial value condition W(0) = 1 to obtain the weight function.

$$W(0) = e^0\left(3e^0 + c\right)^4$$

$$(3+e)^4 = 1$$

The roots must be real, so the two possible solutions are -2 and -4. The weight function must be positive so the only constant value that makes sense it's **-2**. Thus, the weight function is:

$$W(t) = e^{\frac{-10}{3}t}\left(3e^{\frac{5t}{6}} - 2\right)^4$$

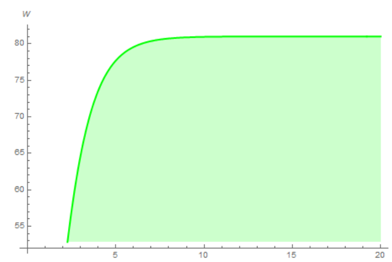

Figure 1: Weight function.

Cost function

The differential equation describing the total cost in dollars $C(t)$ of raising a fish for t months has one constant term K_1 that specifies the cost per month (due to costs such as interest, depreciation, and labor) and a second constant K_2 that multiplies the growth rate (because the amount of food consumed by the fish is approximately proportional to the growth rate). That is,

$$\frac{dC}{dt} = K_1 + K_2\frac{dW}{dt} \tag{3}$$

We can find the function $C(t)$ using some empiric constants: $K_1 = 0.4$, $K_2 = 0.1$, $C(0) = 1.1$ (dollars), and $W(t)$ we determined already.

$$\frac{dC}{dt} = K_1 + K_2\frac{dW}{dt}$$ We integrate the equation:

$$C(t) = K_1 t + K_2 W(t) + K_3$$

$$C(t) = 0.4t + 0.1[e^{\frac{-10}{3}t}\left(3e^{\frac{5t}{6}} - 2\right)] + K_3$$

$$C(0) = 1.1 \rightarrow \text{ initial conditions}$$

$$C(0) = 0 + 0.1[1(3-2)] + K_3 = 1.1$$

$$0.1 + K_3 = 1.1 \rightarrow K_3 = 1$$

$$C(t) = 0.4t + 0.1[e^{\frac{-10}{3}t}\left(3e^{\frac{5t}{6}} - 2\right)] + 1$$

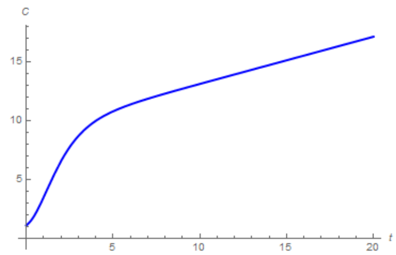

Figure 2: Cost function.

Optimal time for harvesting

To determine the optimal time for harvesting the fish, we can sketch the ratio $\frac{C(t)}{W(t)}$. This ratio represents the total cost per ounce as a function of time. When this ratio reaches its minimum (that is, when the total cost per ounce is at its lowest) it is the optimal time to harvest the fish.

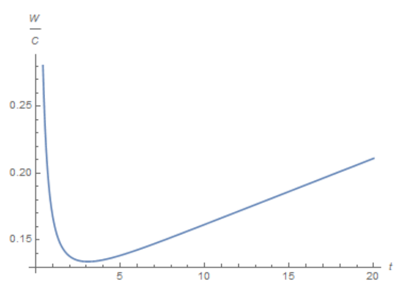

Figure 3: Cost / Weight ratio.

From this graphic, we can see that the optimal time to harvest the fish is around **3 months** (3.0853) with a cost ratio of **0.13**.

Conclusion

Wild fish stock populations fluctuate and have limitations on availability throughout the year. Most commercial fish species are severely overfished and supplies cannot meet the demands of the world market. Aquaculture can provide consistently stable supplies of finish and shellfish into the marketplace to meet this demand.

Aquaculture can be a means to enhance and restore commercial, recreational, and ecologically important species and habitats.

References

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