GPU programming

Monte Carlo simulation of Heston model

Vivien Tisserand, Meriem Hafid

ENSAE Paris

May 2, 2024

Motivation

Heston model introduced in [Hes93] is a stochastic volatility model that assumes the instantaneous variance dynamics follows a CIR (or square-root) process. It has dynamics:

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{V_t}S_t\,dW_t \\ dV_t = \kappa(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dB_t, \end{cases} \tag{Heston}$$

with the two Brownians having correlation ρ: $\langle dW, dB\rangle_t = \rho dt$. In the variance process, parameters are the long term variance level θ, the mean-reversion speed κ and the vol-of-vol σ.

Motivation

Alternatively, we can use equivalent dynamics:

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{V_t}S_t\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right) \\ dV_t = \kappa(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dW_t^1, \end{cases} \qquad \text{(Heston V2)}$$

with two uncorrelated Brownians.

Under integral form, we have

$$\begin{cases} S_t = S_u \exp\left\{ r(t-u) - \frac{1}{2}\int_u^t V_s ds + \rho\int_u^t \sqrt{V_s}dW_1^s + (1-\rho^2)\int_u^t \sqrt{V_s}dW_s^2 \right\} \\ V_t = V_u + \kappa\theta(t-u) - \kappa\int_u^t V_s ds + \sigma\int_u^t \sqrt{V_s}dW_s^1, \end{cases} \text{(Heston integral)}$$

Setup

Throughout the simulation, we work with with the parameters set $\Theta = \{S_0 = 1, K = 1, V_0 = 0.1, r = 0.0\%, \kappa = 2.0, \theta = 0.1, \sigma = 0.02, \rho = -0.30, T = 5\}$.

We compile and execute the code on a Tesla T4 GPU.

Our implementation is available on GitHub.

Outline

1. Discretization scheme
2. Exact sampling
3. Performance

Table of Contents

Euler naive scheme

Choosing a discretization grid $\pi = \{t_0 = 0 < t_1 < \cdots < t_N = T\}$, we have:

$$\begin{cases} S^{\pi}_{t_{i+1}} = S^{\pi}_{t_i} + rS^{\pi}_{t_i}(t_{i+1} - t_i) + \sqrt{V_{t_i}} S^{\pi}_{t_i}(W_{t_{i+1}} - W_{t_i}) \\ V^{\pi}_{t_{i+1}} = V^{\pi}_{t_i} + \kappa(\theta - V^{\pi}_{t_i})(t_{i+1} - t_i) + \sigma\sqrt{V_{t_i}}(B_{t_{i+1}} - B_{t_i}), \end{cases}$$

Adaptations

Although the coefficients in Heston dynamics are not Lipschitz, it can be shown that there exist a strong solution to Heston. The Feller condition $2\kappa\theta \geq \sigma^2$ assures the upward drift is large enough to make 0 unattainable in the CIR process.
However we have to be careful with the square-root: in the above scheme nothing prevent the instantaneous variance process to go negative. An adaptation could be:

$$V^{\pi}_{t_{i+1}} = V^{\pi}_{t_i} + \kappa(\theta - V^{\pi}_{t_i})(t_{i+1} - t_i) + \sigma\sqrt{V^{+}_{t_i}}(B_{t_{i+1}} - B_{t_i})$$

Numerical tools

To execute such a scheme, we need:

- A (pseudo-) random number generator, *e.g.* Mersenne Twister [MN00] or Marsaglia's XORWOW which is `cuRAND`'s default,
- Normal distribution sampling, *e.g.* Box-Muller transform (available in CUDA's `cuRAND` library)[1].

The algorithm runs on 2^{10} threads per block and 2^{10} blocks (then `__syncthreads` and reduction; atomic operation to copy back to the host memory).

[1] see documentation.

Milstein scheme for a higher convergence order

We can achieve any order of convergence for discretization schemes, but at a complexity cost, most of them are covered in [Alf05]. By the way, the Milstein scheme alleviates considerations on the posititivity of the instantaneous variance process.

Table of Contents

Apparition of a χ^2 distribution

In their seminal paper, Cox, Ingersoll and Ross showed that the transition law of a CIR-following quantity can be expressed as a *noncentral chi-square* distribution [CIR05; Fel51][2]. It can be applied to exact simulation of some (affine?) stochastic volatility models, in particular Heston [BK06].

$$V_t|V_u \sim \frac{\sigma^2(1-e^{-\kappa(t-u)})}{4\kappa}\chi_\delta'^2\left(\frac{4\kappa e^{-\kappa(t-u)}}{\sigma^2(1-e^{-\kappa(t-u)})}\right), \quad u < t,$$

$\chi_\delta'^2(\lambda)$ the noncentral chi-squared random variable with d degrees of freedom, and noncentrality parameter λ and $\delta = 4\theta\kappa/\sigma^2$.

[2] The proof relies on the Fokker-Planck equation to make the link with parabolic PDEs and Laplace transformation to solve for the probability density.

The Broadie-Kaya simulation procedure

Exact sampling from the joint distribution (S_t, V_t) is not trivial, Broadie and Kaya break the task into the following steps [BK06]:

- Sample from the conditional distribution of V_t using chi-squared properties,
- Sampling from the distribution of $\int_u^t V_s ds | V_u, V_t$,
- Using the dynamics Heston integral, recover samples for $\int_u^t V_s dW_1^s$,

Simulating V

A nice result relates non-central and ordinary chi-squared distributions:

$$\chi_\delta'^2(\lambda) = \chi_1'^2(\lambda) + \chi^2_{\delta-1},$$

then – provided $\delta > 1$ – with $\varepsilon \sim \mathcal{N}(0,1)$, we can sample

$$\chi_\delta'^2(\lambda) = (\varepsilon + \sqrt{\lambda})^2 + \chi^2_{\delta-1}.$$

For any $d > 0$, there exists a method sampling from an ordinary chi-squared with a random degrees of freedom ($\chi^2_{\delta+2N} \stackrel{(d)}{=} \chi_\delta'^2(\lambda)$ with $N \sim \mathcal{P}(\lambda/2)$).

GS* and GKMs sampling algorithms

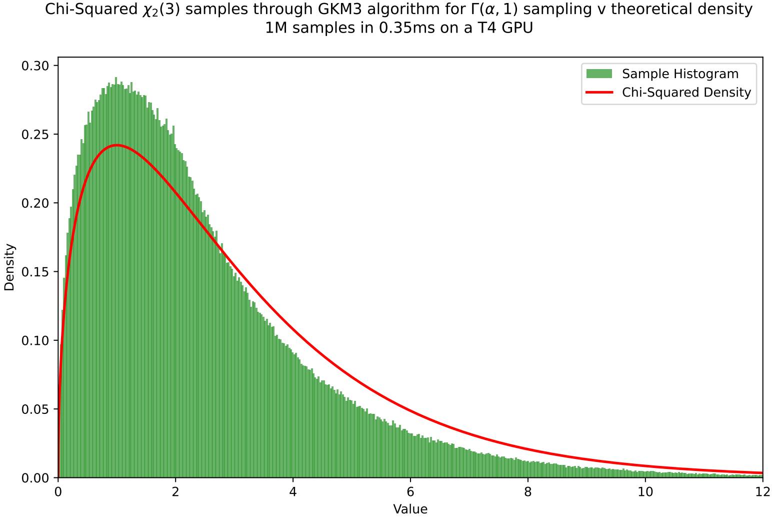
Fishman highlights algorithms to sample from a gamma distribution [Fis13]. Fortunately, chi-squared is a special case of the gamma distribution.

A gamma $\Gamma(\alpha, \beta)$ distribution has density: $f(z) = \dfrac{z^{\alpha-1} e^{-z/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$ (under the shape-scale parametrization) thus $\Gamma(k/2, 2) \overset{(d)}{=} \chi_k^2$.

If $X \sim \Gamma(\alpha, \beta)$, then $cX \sim \Gamma(\alpha, c\beta)$, thus knowing how to sample from $\Gamma(\alpha, 1)$ is enough.

We have several algorithms to do so: GS* (for $0 < \alpha < 1$), GKM1, GKM2 and GKM3 (for $\alpha > 1$), all based on rejection sampling.

χ^2 samples



Towards sampling S...

We now know how to simulate V. Still, what matters is the spot price. Its simulation involves the computation of $\int_u^t V_s ds$ (given V_u, V_t), possibly though Fourier inversion or quadrature. We approximate this integral with a sum given the instantaneous variance sample paths. We use the dynamics from Heston integral:

$$\int_u^t \sqrt{V_s} dW_s^1 = \frac{1}{\sigma}\left(V_t - V_u - \kappa\theta(t-u) + \kappa \int_u^t V_s ds\right).$$

Simulating S

The conditional distribution of $\log S_t$ is normal because the two Brownians are uncorrelated ($V_t \perp\!\!\!\perp W_t^2$ so we essentially have a Wiener integral). We derive mean and variance:

- $m(u,t) = \log S_u + r(t-u) - \frac{1}{2}\int_u^t V_s ds + \rho \int_u^t \sqrt{V_s} dW_s^1$,
- $\sigma^2(u,t) = (1-\rho^2)\int_u^t V_s ds$.

We now know how to sample from the distribution of (S_t, V_t)!

Table of Contents

Heston prices

There exists quasi closed-form formulae to get vanilla option prices under Heston dynamics (they involve solving Riccati equations for the characteristic function and FFT computations).

Using `cuFFT`?

Exploring quasi Monte Carlo alternatives to speed up estimations (Sobol is available in `cuRAND`)? As well as classic variance reduction techniques?

Performance comparison

The literature tends to say that although biased, the Euler-Maruyama scheme is much cheaper than the exact simulation through non-central chi-squared sampling.

References I

[Fel51] William Feller. "Two singular diffusion problems". In: *Annals of mathematics* (1951), pp. 173–182.

[Hes93] Steven L Heston. "A closed-form solution for options with stochastic volatility with applications to bond and currency options". In: *The review of financial studies* 6.2 (1993), pp. 327–343.

[MN00] Makoto Matsumoto and Takuji Nishimura. "Dynamic creation of pseudorandom number generators". In: *Monte-Carlo and Quasi-Monte Carlo Methods 1998: Proceedings of a Conference held at the Claremont Graduate University, Claremont, California, USA, June 22–26, 1998*. Springer. 2000, pp. 56–69.

[Alf05] Aurélien Alfonsi. "On the discretization schemes for the CIR (and Bessel squared) processes". In: *Monte Carlo Methods Appl.* 11.4 (2005), pp. 355–384.

References II

[CIR05] John C Cox, Jonathan E Ingersoll Jr, and Stephen A Ross. "A theory of the term structure of interest rates". In: *Theory of valuation*. World Scientific, 2005, pp. 129–164.

[BK06] Mark Broadie and Özgür Kaya. "Exact simulation of stochastic volatility and other affine jump diffusion processes". In: *Operations research* 54.2 (2006), pp. 217–231.

[Fis13] George Fishman. *Monte Carlo: concepts, algorithms, and applications*. Springer Science & Business Media, 2013.