Killed diffusion – On Monte Carlo methods to price some path-dependent payoffs

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Contents

In this study, we implement two numerical schemes for stochastic differential equations where the diffusion stops if it escapes from a defined domain. Theoretical results are given and sketches of the proofs are highlighted. We apply these results to a quantitative finance problem: the pricing of path-dependent option, in particular we play with an up-and-out call and a double no-touch call. The code is made public and accessible in a GitHub repository.

1 Killed diffusion, simulation schemes, what are we talking about?

We are investigating a paper from Emmanuel Gobet published in 2000 [Gob00]. We give a short finance introduction on some exotic options before providing proofs and delving into numerical schemes to approximate such payoffs.

1.1 Getting started – why?

We are interested in a path-dependent payoff on an asset with diffusion $(X_t)_{0\leq t\leq T}$. In particular, we kill the diffusion when it leaves the open set D.

With $\tau \coloneqq \inf\{t > 0 : X_t \notin D\}$, this leads to the computation of $\mathbb{E}\left[\mathbb{1}_{T<\tau} f(X_T)\right]$. The direct application in finance is with barrier and digital options that pay respectively the underlying asset, one unit of a specified currency, conditional on the barrier levels not having been broken.

Barrier options are either up, down, and in, out: an up-and-out barrier option with barrier B leads to the boundary condition $V(B,t) = 0 \quad \forall t \leq T$. With $D = (-\infty, B]$, the risk-neutral evaluation of the up-and-out call gives a price at time t:

$$C_{up-out} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}(S_T - K)^+ \mathbb{1}_{T<\tau} | \mathcal{F}_t\right]$$

where $\tau = \inf\{t > 0 : S_t \notin D\} = \inf\{t > 0 : S_t > B\}$ if we make the assumption that $S_0 \leq B$.

Similarly, we can define an up-and-in call option using $D = [B, +\infty)$, usually with $S_0 \geq B$. Note that if $K \geq B$, the option simply becomes a vanilla call option of strike K and maturity T.

For simplicity, we write $\mathbb{E}(.)$ as the expectation taken under risk-neutral measure $\mathbb{Q}$.

Overall, being able to study such payoffs opens the **realm of barrier options** as well as no-touch and one-touch options that are some path-dependent **binary options** – the payoff is either $\$1M$ or $\$0$ based on a condition (*e.g.* the EURUSD price remained under 1.087 up until maturity of the option) being filled: these are high leverage speculative instruments that can lead to significant gains for those who master their use.

The paper implements two discretisation schemes to price such payoffs using Monte Carlo methods: a continuous Euler scheme and a discrete Euler scheme. We will be implementing those algorithms and running a set of experiments to evaluate the claims of the paper: do we actually observe a $o(N^{-1})$ convergence for the continuous scheme while only a $o(N^{-1/2})$ for the discrete scheme? How do the simulation prices compare to the theoretical closed-form formulas under Black-Scholes dynamics?

1.2 Deriving the path-dependent options cost under Black-Scholes

Let's assume a geometric brownian motion dynamic for the underlying asset:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

For all barrier options, an arbitrage argument gives the following parity formula between the prices of barrier options and vanilla call and put options:

$$\begin{aligned}
&C_{\text{up-in}}(t) + C_{\text{up-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T - K)^+ \mid \mathcal{F}_t\right] \\
&C_{\text{down-in}}(t) + C_{\text{down-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T - K)^+ \mid \mathcal{F}_t\right] \\
&P_{\text{up-in}}(t) + P_{\text{up-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(K - S_T)^+ \mid \mathcal{F}_t\right] \\
&P_{\text{down-in}}(t) + P_{\text{down-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(K - S_T)^+ \mid \mathcal{F}_t\right]
\end{aligned}$$

where the price of the European call, resp. put option with strike price K are obtained from the Black-Scholes formula. Consequently, in what follows we will only compute the prices of the up-and-out barrier call.

Let's compute the price of the up-and-out barrier call option.

Introducing the maximum of the geometric brownian motion:

$$M_0^T = \max_{0 \le u \le T} S_u$$

We have the following proposition:

Proposition 1.0.1. *When $K \le B$, the price of the up-and-out barrier call option with maturity T, strike price K and (upper) barrier level B is given by:*

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T - K)^+ \mathbb{1}_{\{M_0^T < B\}} \mid \mathcal{F}_t\right] = \\
&\mathbb{1}_{\{M_0^t < B\}} \mathrm{Bl}(S_t, K, r, T-t, \sigma) - S_t \mathbb{1}_{\{M_0^t < B\}} \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right) \\
&- B\left(\frac{B}{S_t}\right)^{2r/\sigma^2} \mathbb{1}_{\{M_0^t < B\}}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right) \\
&+ \mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{M_0^t < B\}} \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right) \\
&+ \mathrm{e}^{-(T-t)r} K \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \mathbb{1}_{\{M_0^t < B\}}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right).
\end{aligned}$$

where

$$\delta_\pm^\tau(s) = \frac{1}{\sigma\sqrt{\tau}}\left(\log s + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right), \quad s > 0$$

Note that the price of the up-and-out barrier call option vanishes when $B \le K$.

Note also that as $B \to \infty$, the price of the up-and-out barrier call option converges to the price of the European call option:

$$\begin{aligned}\lim_{B\to\infty} \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T - K)^+ \mathbb{1}_{\left\{M_0^T < B\right\}} \mid \mathcal{F}_t\right] &= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T - K)^+ \mid \mathcal{F}_t\right] \\ &= S_t \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) - K\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)\end{aligned}$$

Proof. See A □

Proposition 1.0.2. *Price of a double no-touch option with lower and upper barriers S_{min}, S_{max}:*

$$V(S,t) = 2\pi \left(\frac{S}{S_{min}}\right)^{\alpha} e^{\beta(T-t)} \sum_{n=1}^{\infty} n \left[\frac{(1-(-1)^m e^{-\alpha H})}{\alpha^2 H^2 + n^2\pi^2}\right] \exp\left(-\frac{1}{2}\sigma^2 \frac{n^2\pi^2}{H^2} t\right) \sin\left(\frac{n\pi}{H} \ln \frac{S}{S_{min}}\right),$$

with $H = \ln(S_{max}/S_{min})$, $\alpha = -\tilde{r}/\sigma^2$ and $\beta = -\tilde{r}/2\sigma^2 - r$, $\tilde{r} = r - \frac{1}{2}\sigma^2$.

Proof. See B □

2 Monte Carlo methods

Our objective is to estimate $\mathbb{E}\left[\mathbb{1}_{T<\tau} f(X_T)\right]$, as seen before. Let us introduce a more general diffusion than the Black-Scholes model, the Itô diffusion, as used in the article of Emmanuel Gobet:

Let $(X_t)_{t\geq 0}$ be the Itô diffusion taking its values in $\mathbb{R}^d$ defined by

$$X_t = x + \int_0^t B\left(X_s\right) ds + \int_0^t \sigma\left(X_s\right) dW_s$$

where $(W_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^{d'}$. Let $\tau := \inf\left\{t > 0 : X_t \notin D\right\}$ be its first exit time from the open set $D \subset \mathbb{R}^d$.

We need to make assumptions on the regularity of the coefficients B and σ in order to ensure the existence and uniqueness of the solution to the SDE. We also need to make assumptions on the boundary of the open set D in order to ensure the existence of the first exit time τ.

In the case of our barrier option, we can take $D = (-\infty, b)$, where b is the barrier level. We can also take $D = (a, +\infty)$, or $D = (a, b)$, where a and b are the lower and upper barriers, respectively, in the case of a double no-touch barrier option.

Using the theory of Monte Carlo numerical methods for the Stochastic Differential Equation (SDE) with a first exit time, we can estimate the price of a barrier option.

We can use the Euler scheme, which is a simple and efficient method to simulate the diffusion process.

The question remains: shall we use a discrete or continuous Euler scheme? The continuous Euler scheme is more accurate, but it is more difficult to simulate the first exit time from the open set D. Indeed between two time steps the diffusion can still unfortunately exit the domain.

2.1 Methods

1. Discrete Euler scheme

   To evaluate the expectation of the functional of the diffusion, the simplest way to approximate the process is to use its discrete Euler scheme $\left(\tilde{X}_{t_i}\right)_{0\leq i\leq N}$ with discretization step T/N, defined if $t_i = iT/N$ is the i-th discretization time by

$$\begin{aligned}
\tilde{X}_0 &= x \\
\tilde{X}_{t_{i+1}} &= \tilde{X}_{t_i} + B\left(\tilde{X}_{t_i}\right) T/N + \sigma\left(\tilde{X}_{t_i}\right)\left(W_{t_{i+1}} - W_{t_i}\right)
\end{aligned}$$

The simulation involves simulation N independent Gaussian variables for the increments $\left(W_{t_{i+1}} - W_{t_i}\right)_{0 \le i \le N-1}$.

2. Continuous Euler scheme

   Introducing a more complex scheme, our goal is to refine the discrete Euler scheme by considering a modified stopping time $\tilde{\tau}_c$, which is the first exit time from the open set D for the continuous Euler scheme.

   $\left(\tilde{X}_t\right)_{t_i \le t \le t_{i+1}}$ shall be defined as:

$$\tilde{X}_t = \tilde{X}_{t_i} + B\left(\tilde{X}_{t_i}\right)(t - t_i) + \sigma\left(\tilde{X}_{t_i}\right)(W_t - W_{t_i}), \text{ for } t_i \le t < t_{i+1},$$ [1]

   Between two time steps, the diffusion can still exit the domain. From a simulation standpoint, Gobet suggests drawing N additional independent Bernoulli variables to simulate if $\left(\tilde{X}_t\right)_{0 \le t \le T}$ has left D between two discretization times or not. Each parameter involved for the simulation of the Bernoulli variables is related to the quantity

$$\mathbb{P}\left(\forall t \in [t_i, t_{i+1}]\, \tilde{X}_t \in D | \tilde{X}_{t_i} = z_1, \tilde{X}_{t_{i+1}} = z_2\right) := p\left(z_1, z_2, T/N\right)$$

   In the 1-dimensional case, $p\left(z_1, z_2, \Delta\right)$ is the cumulative of the infimum and supremum of a linear Brownian bridge and has a simple expression[2]:

   1. if $D = (-\infty, b)$, we have $p\left(z_1, z_2, \Delta\right) = \mathbb{1}_{b > z_1} \mathbb{1}_{b > z_2}\left(1 - \exp\left(-2\frac{(b-z_1)(b-z_2)}{\sigma^2(z_1)\Delta}\right)\right)$;
   2. if $D = (a, +\infty)$, we have $p\left(z_1, z_2, \Delta\right) = \mathbb{1}_{z_1 > a} \mathbb{1}_{z_2 > a}\left(1 - \exp\left(-2\frac{(a-z_1)(a-z_2)}{\sigma^2(z_1)\Delta}\right)\right)$;
   3. if $D = (a, b)$, we have

$$\begin{aligned} p\left(z_1, z_2, \Delta\right) = &\mathbb{1}_{b > z_1 > a} \mathbb{1}_{b > z_2 > a}\left(1 - \sum_{k=-\infty}^{+\infty}\left[\exp\left(-2\frac{k(b-a)\left(k(b-a) + z_2 - z_1\right)}{\sigma^2\left(z_1\right)\Delta}\right)\right.\right. \\ &\left.\left. - \exp\left(-2\frac{\left(k(b-a) + z_1 - b\right)\left(k(b-a) + z_2 - b\right)}{\sigma^2\left(z_1\right)\Delta}\right)\right]\right). \end{aligned}$$

[2] see [RY13], p. 105, for instance

2.2 Theoretical results

Let us introduce the following (weak) convergence errors for the two respective schemes:

- Continuous Euler Scheme

$$\mathcal{E}_{\mathrm{c}}(f,T,x,N) := \mathbb{E}_x\left[\mathbb{1}_{T<\tilde{\tau}_c} f\left(\tilde{X}_T\right)\right] - \mathbb{E}_x\left[\mathbb{1}_{T<\tau} f\left(X_T\right)\right]$$

- Discrete Euler Scheme

$$\mathcal{E}_{\mathrm{d}}(f,T,x,N) := \mathbb{E}_x\left[\mathbb{1}_{T<\tilde{\tau}_d} f\left(\tilde{X}_T\right)\right] - \mathbb{E}_x\left[\mathbb{1}_{T<\tau} f\left(X_T\right)\right]$$

We are interested in evaluating the errors $\mathcal{E}_{\mathrm{c}}(f,T,x,N)$ and $\mathcal{E}_{\mathrm{d}}(f,T,x,N)$ as a function of N, the number of discretization steps. We first state an easy result, which shows that both errors tend to 0 when N goes to infinity under mild assumptions. We then provide a more precise result, which gives the rate of convergence of the errors.

2.2.1 Limit behavior of the weak convergence

Assume that B and σ are globally Lipschitz functions, that D is defined by $D = \left\{x \in \mathbb{R}^d : F(x) > 0\right\}, \partial D = \left\{x \in \mathbb{R}^d : F(x) = 0\right\}$ for some globally Lipschitz function F. Provided that the condition (C) below is satisfied

$$(\mathbf{C}) : \mathbb{P}_x\left(\exists t \in [0,T] \quad X_t \notin D; \forall t \in [0,T] \quad X_t \in \bar{D}\right) = 0$$

for all function $f \in C_b^0(\bar{D}, \mathbb{R})$, we have

$$\lim_{N\to+\infty} \mathcal{E}_{\mathrm{c}}(f,T,x,N) = \lim_{N\to+\infty} \mathcal{E}_{\mathrm{d}}(f,T,x,N) = 0.$$

2.2.2 Order of convergence of the weak convergence error

Under regularity assumptions on B, σ, D and an uniform ellipticity condition, one has

- for the continuous Euler scheme:

  $$\mathcal{E}_{\mathrm{c}}(f,T,x,N) = CN^{-1} + o\left(N^{-1}\right)$$

  provided that f is a measurable function with support strictly included in D (Theorem 2.1, see below). The support condition can be weakened if f is smooth enough (Theorem 2.2, see below).

- for the discrete Euler scheme:

$$\mathcal{E}_{\mathrm{d}}(f,T,x,N) = O\left(N^{-1/2}\right)$$

2.2.3 Theorems

We make some reasonable assumptions on the regularity of the coefficients of the diffusion process, the smoothness of the survival domain, and the payoff function.

These allow us to bound the error functions, at the rate N^{-1} for the continuous scheme and the rate $N^{-1/2}$ for the discrete scheme.

2.3 Proof sketches

Introducing $v(t,x) := \mathbb{E}_x\left[\mathbb{1}_{T-t<\tau} f(X_{T-t})\right]$, we break the error into two parts to make appear the process and stopping time from the scheme: $\tilde{X}$ and $\tilde{\tau}$. The difference between the continuous and discrete errors essentially stems from the fact that Itô's formula is applicable on v in the former scheme but not the latter as discontinuities of v on the boundary of the domain D would lead to jumping spatial derivatives for.

For instance:

$$\begin{aligned}
\mathcal{E}_{\mathrm{c}}(f,T,x,N) =& \mathbb{E}_x\left[\mathbb{1}_{T<\tilde{\tau}_c} f\left(\tilde{X}_T\right)\right] - \mathbb{E}_x\left[v\left((T-T/N)\wedge\tilde{\tau}_c, \tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c}\right)\right] \\
&+ \mathbb{E}_x\left[v\left((T-T/N)\wedge\tilde{\tau}_c, \tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c}\right)\right] - \mathbb{E}_x\left[v\left(0,\tilde{X}_0\right)\right] \\
:=& C_1(N) + C_2(N)
\end{aligned}$$

and C_1 will yield a contribution of the order $O(N^{-3/2})$ while C_2 will be broken down between two other error terms, one having a dominating contribution of the order $O(N^{-1})$ – this involves applying Itô's formula and the Bernstein inequality for martingales.

3 Simulations

3.1 Programming choices

To implement the different schemes and run the Monte Carlo pricers, we mainly need to do a lot of additions and multiplications, and a way to generate realisations from a standard normal distribution, thus a random number generator. For those reasons, we decided to implement the code in C++, to scale the number of simulations without triggering impatience issues... and to expose the main functions in Python so they can be used in a notebook for instance, and produce visualisations quickly and easily.

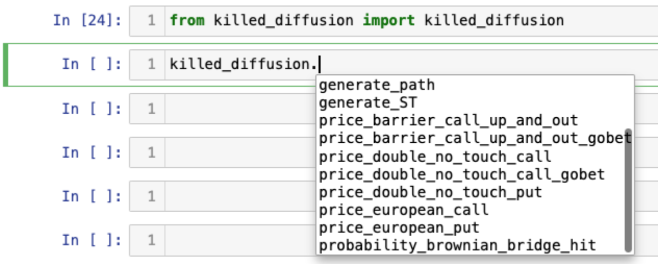

Figure 1: The killed diffusion module and available functions exposed in Python

The corresponding C++ files of interest lie in `src/closed_formula.cpp` and `src/montecarlo.cpp`.

For the simulations, we will consider the following set of parameters:

- $S_0 = 100$
- $K = 100$
- $r = 5\%$
- $T = 1$
- $\sigma = 0.22$
- (number of discretization steps) $N = 500$
- (number of Monte Carlo simulations) $M = 15000$
- upper barrier $B = 120$
- lower barrier $L = 80$

Throughout the analysis of the results, we call "MC" the discrete naive scheme while we use the name "MC - Gobet" for the continuous scheme.

3.2 Payoffs

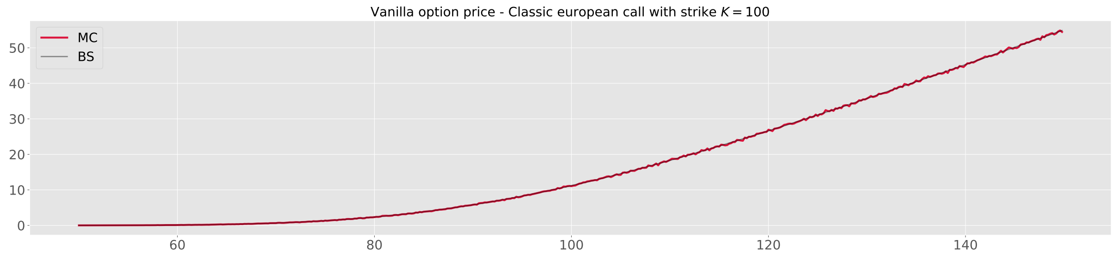

Figure 2: Evolution of the vanilla call option price against S_0

The Monte Carlo price from a discrete scheme matches very closely the theoretical price for this vanilla call.

Now, let's visualize the difference between classic Monte Carlo estimators using the discrete Euler scheme and the refinement introduced by Emmanuel Gobet using the law of some Brownian bridge to model the evolution between two increments.

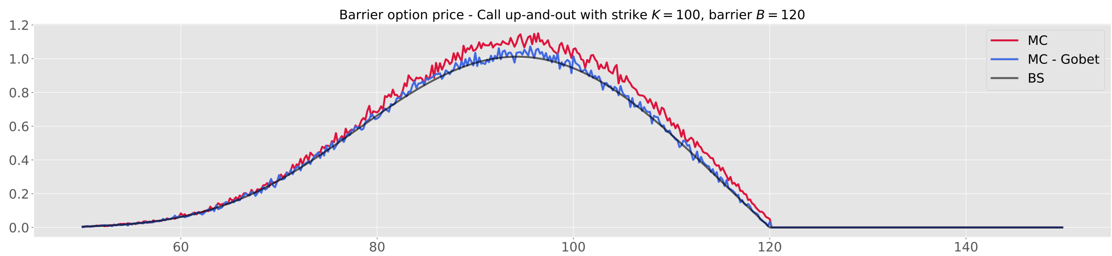

Figure 3: Evolution of the up-and-out barrier call option price against S_0

The Gobet **continuous scheme appears more accurate** than a "naive" Monte Carlo pricer: indeed, the option is path-dependent and the discrete scheme leaves out some possibilities to get the diffusion killed, overestimating the survival probability of the diffusion leads to an inflated price.

We observe that the **price is lower than for a vanilla call option**: of course if the barrier is hit the option expires worthless. This product is interesting as it gives a different exposure than the vanilla call: there is limited upside so a trader might also express a more detailed view on volatility when buying this cheaper option.

Still, we observe that despite a high number of simulations and a reasonable discretization stepsize, the Monte Carlo price **lacks regularity** (the MC price gravitates around the theoretical price): if we were to be the seller of such an option, we would build a confidence interval around a given price by running the pricer several times (one instance of the Gobet method takes a little less than half a second on our personal machines) and price this uncertainty in the spread we would charge to the potential buyer – and we certainly would not choose a geometric brownian motion dynamic for the underlying asset.

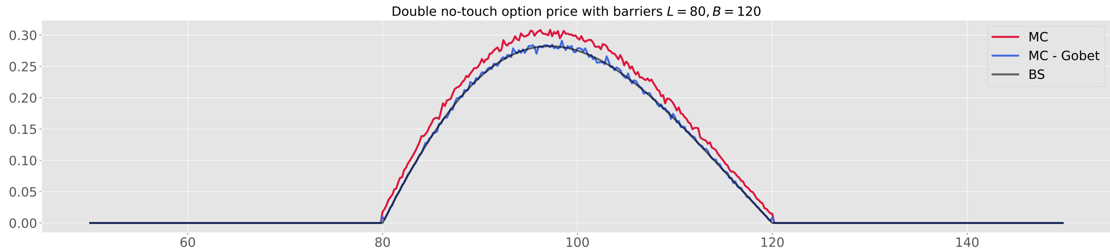

Figure 4: Evolution of the double no-touch option that pays \$1 against S_0

Here we can make the same remarks as above: the continuous scheme appears to match better the theoretical price, while the discrete scheme does not account for potential death of the diffusion in-between two time-steps.
Here, we see that the ATM double no-touch option would roughly pay 4-to-1, provided that the underlying asset remains between \$80 and \$120 during the whole year, knowing the asset has an annualized volatility of 22%.

3.3 Killed diffusion

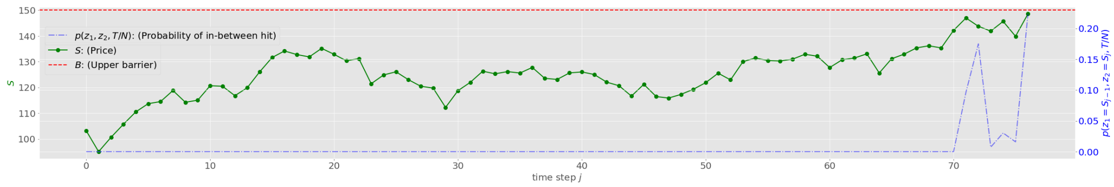

Figure 5: Killed diffusion of the underlying asset S - Up-and-out barrier

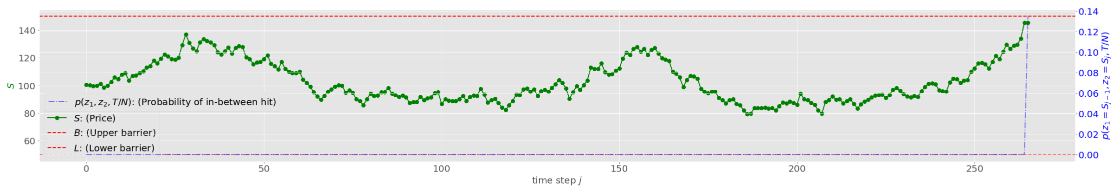

Figure 6: Killed diffusion of the underlying asset S - double no-touch

As we can see, the probability of hitting the barrier between two consecutive time steps rapidly grows, as the diffusion comes closer to the barrier. This contribution to the killing likelihood of the diffusion is what drives the price a bit down compared to the discrete scheme. We plot only up to the point of stopping time, *i.e.* exactly where we kill the diffusion when the barrier is hit.

3.4 Convergence

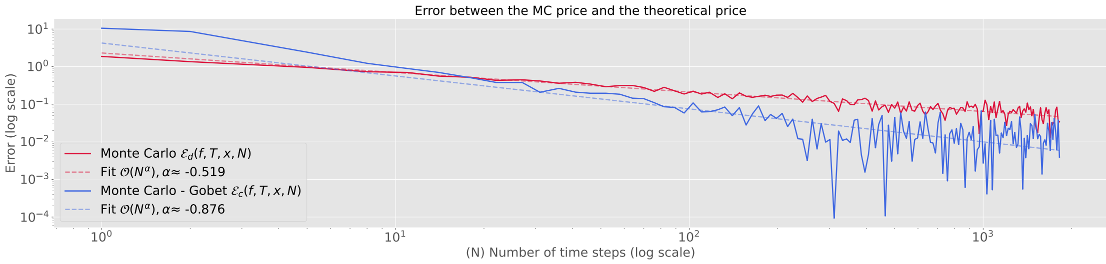

Figure 7: Convergence of the errors - Discrete Euler scheme (red) vs Continuous Euler scheme (blue) with linear fits in the log-space to check for rates of convergence

To achieve this view, we sampled the two different errors by running our Monte Carlo methods using logarithmic scaling with base 1.5. This allowed to obtain estimates with varying values of N ranging from 1 to roughly 2000 discretization time steps.

Note that we retrieve the convergence results of section 2. for both discrete and continuous Euler schemes. Using a linear regression on a log scale we found **empirical rates of convergence** that roughly match the one from the paper for the respective errors:

1. for the continuous Euler scheme: $\mathcal{E}_{\mathrm{c}}(f, T, x, N) = CN^{-1} + o\left(N^{-1}\right)$ provided that f is a measurable function with support strictly included in D (Theorem 2.1). The support condition can be weakened if f is smooth enough (Theorem 2.2). Empirically we got 0.88 for 1 theoretically.

2. for the discrete Euler scheme: $\mathcal{E}_{\mathrm{d}}(f, T, x, N) = O\left(N^{-1/2}\right)$ for functions f satisfying analogous hypotheses as before (Theorem 2.3). The rate $N^{-1/2}$ is optimal and intrinsic to the choice of a discrete killing time (Theorem 2.4). Empirically we got 0.52 for 0.5 theoretically.

In the case of our up-and-out barrier call option, f is defined as $x \mapsto (x - K)_+$, for the double no-touch option it simply is $x \mapsto 1$, or any amount that the counterparty agrees to pay upon meeting the staying in-between the barriers during the lifetime of the option condition.

Appendices

A Call up-and-out price derivation

Proof. [3]

Let's take a deep breath and expand the barrier replication cost:

$$\mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T-K)^+\,\mathbb{1}_{\left\{M_0^T<B\right\}}\mid\mathcal{F}_t\right]$$

$$\begin{aligned}
&= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T-K)^+\,\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{1}_{\left\{M_t^T<B\right\}}\mid\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}\left[(S_T-K)^+\,\mathbb{1}\left\{\mathrm{Max}_{t\le r\le T}S_r<B\right\}\mid\mathcal{F}_t\right]
\end{aligned}$$

Since, $\mathbb{1}_{\left\{M_0^t<B\right\}}$ is $\mathcal{F}_t$-measurable, we can take it out of the expectation. We then have:

$$= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}\left[\left(x\frac{S_T}{S_t}-K\right)^+\mathbb{1}_{\left\{x\,\mathrm{Max}_{t\le r\le T}\,\frac{S_r}{S_t}>B\right\}}\right]_{x=S_t}$$

In the case of the Black-Scholes model, we have an explicit expression for the diffusion: $\frac{S_r}{S_t}=\mathrm{e}^{(r-\sigma^2/2)(r-t)+\sigma(W_r-W_t)}$

$$\begin{aligned}
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}\left[\left(x\frac{S_{T-t}}{S_0}-K\right)^+\mathbb{1}_{\left\{x\,\mathrm{Max}_{0\le r\le T-t}\,\frac{S_r}{S_0}<B\right\}}\right].\\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}\left[\left(x\mathrm{e}^{\sigma\widetilde{W}_{T-t}}-K\right)^+\mathbb{1}\left\{{}_{x\,\mathrm{Max}_{0\le r\le T-t}}\mathrm{e}^{\sigma\widetilde{W}_r}<B\right\}\right]_{x=S_t}.
\end{aligned}$$

where $\widetilde{W}_t=W_t+\mu t$ and $\mu:=-\frac{2r-\sigma^2}{2\sigma}$.

Therefore, introducing the following process:

$$\widehat{X}_0^T=\mathrm{Max}_{t\in[0,T]}\,\widetilde{W}=\mathrm{Max}_{t\in[0,T]}\,(W_t+\mu t)$$

We have:

[3] This proof is inspired by what is highlighted in https://personal.ntu.edu.sg/nprivault/MA5182/barrier-options.pdf

$$
\begin{aligned}
\mathbb{E}\left[(S_T-K)^+ \mathbb{1}_{\left\{M_0^T<B\right\}}\right] &= \mathbb{E}\left[\left(S_0\mathrm{e}^{\sigma\widetilde{W}_T}-K\right)\mathbb{1}\left\{S_0\mathrm{e}^{\sigma\widetilde{W}_{T>K}}\right\}\mathbb{1}\left\{S_0\mathrm{e}^{\sigma\widehat{X}_0^T}<B\right\}\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\mathrm{e}^{\sigma y}-K\right)\mathbb{1}_{\{S_0\mathrm{e}^{\sigma y}>K\}}\mathbb{1}_{\{S_0\mathrm{e}^{\sigma x}<B\}}\mathrm{d}\mathbb{P}\left(\widehat{X}_0^T\leq x,\widetilde{W}_T\leq y\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\mathrm{e}^{\sigma y}-K\right)\mathbb{1}_{\{\sigma y>\log(K/S_0)\}}\mathbb{1}_{\{\sigma x<\log(B/S_0)\}}\varphi_{\widehat{X}_T,\widetilde{W}_T}(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\mathrm{e}^{\sigma y}-K\right)\mathbb{1}_{\{\sigma y>\log(K/S_0)\}}\mathbb{1}_{\{\sigma x<\log(B/S_0)\}}\mathbb{1}_{\{y\vee 0<x\}} \\
&\times \varphi_{\widehat{X}_T,\widetilde{W}_T}(x,y)dxdy \\
&= \frac{1}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{\sigma^{-1}\log(K/S_0)}^{\sigma^{-1}\log(B/S_0)}\int_{y\vee 0}^{\sigma^{-1}\log(B/S_0)}\left(S_0\mathrm{e}^{\sigma y}-K\right)(2x-y) \\
&\times \mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy \\
&= \frac{\mathrm{e}^{-\mu^2T/2}}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{\sigma^{-1}\log(K/S_0)}^{\sigma^{-1}\log(B/S_0)}\left(S_0\mathrm{e}^{\sigma y}-K\right)\mathrm{e}^{\mu y-y^2/(2T)} \\
&\times \int_{y\vee 0}^{\sigma^{-1}\log(B/S_0)}(2x-y)\mathrm{e}^{2x(y-x)/T}dxdy
\end{aligned}
$$

where $\varphi_{M_0^T,S_T}$ is the joint probability density function of $\left(M_0^T,S_T\right)$, which satisfies

$$
\mathbb{P}\left(M_0^T\leq x \text{ and } S_T\leq y\right)=\int_0^x\int_0^y\varphi_{M_0^T,S_T}(u,v)dudv,\quad x,y\geq 0.
$$

If $B\geq K$ and $B\geq S_0$ (otherwise the option price is 0), with $\mu:=r/\sigma-\sigma/2$ and $y\vee 0=\operatorname{Max}(y,0)$. Letting $a:=y\vee 0$ and $b:=\sigma^{-1}\log\left(B/S_0\right)$, we have

$$
\begin{aligned}
\int_a^b(2x-y)\mathrm{e}^{2x(y-x)/T}dx &= \int_a^b(2x-y)\mathrm{e}^{2x(y-x)/T}dx \\
&= -\frac{T}{2}\left[\mathrm{e}^{2x(y-x)/T}\right]_{x=a}^{x=b} \\
&= \frac{T}{2}\left(\mathrm{e}^{2a(y-a)/T}-\mathrm{e}^{2b(y-b)/T}\right) \\
&= \frac{T}{2}\left(\mathrm{e}^{2(y\vee 0)(y-y\vee 0)/T}-\mathrm{e}^{2b(y-b)/T}\right) \\
&= \frac{T}{2}\left(1-\mathrm{e}^{2b(y-b)/T}\right)
\end{aligned}
$$

hence, letting $c:=\sigma^{-1}\log\left(K/S_0\right)$, we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[(S_T-K)^+\,\mathbb{1}_{\{M_0^T<B\}}\Big] = \\
&\frac{\mathrm{e}^{-\mu^2T/2}}{\sqrt{2\pi T}}\int_c^b (S_0\mathrm{e}^{\sigma y}-K)\,\mathrm{e}^{\mu y-y^2/(2T)}\left(1-\mathrm{e}^{2b(y-b)/T}\right)dy = \\
&S_0\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{(\sigma+\mu)y-y^2/(2T)}\left(1-\mathrm{e}^{2b(y-b)/T}\right)dy \\
&-K\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{\mu y-y^2/(2T)}\left(1-\mathrm{e}^{2b(y-b)/T}\right)dy = \\
&S_0\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{(\sigma+\mu)y-y^2/(2T)}dy \\
&-S_0\mathrm{e}^{-\mu^2T/2-2b^2/T}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{(\sigma+\mu+2b/T)y-y^2/(2T)}dy \\
&-K\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{\mu y-y^2/(2T)}dy \\
&+K\mathrm{e}^{-\mu^2T/2-2b^2/T}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{(\mu+2b/T)y-y^2/(2T)}dy.
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{e}^{-rT}\mathbb{E}\Big[(S_T-K)^+\,\mathbb{1}_{\{M_0^T<B\}}\Big] = \\
&\quad S_0\mathrm{e}^{-\left(r+\mu^2/2\right)T+(\sigma+\mu)^2T/2}\left(\Phi\left(\frac{-c+(\sigma+\mu)T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+(\sigma+\mu)T}{\sqrt{T}}\right)\right) \\
&\quad -S_0\mathrm{e}^{-\left(r+\mu^2/2\right)T-2b^2/T+(\sigma+\mu+2b/T)^2T/2)} \\
&\qquad \times\left(\Phi\left(\frac{-c+(\sigma+\mu+2b/T)T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+(\sigma+\mu+2b/T)T}{\sqrt{T}}\right)\right) \\
&\quad -K\mathrm{e}^{-rT}\left(\Phi\left(\frac{-c+\mu T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+\mu T}{\sqrt{T}}\right)\right) \\
&\quad +K\mathrm{e}^{-\left(r+\mu^2/2\right)T-2b^2/T+(\mu+2b/T)^2T/2} \\
&\quad \times\left(\Phi\left(\frac{-c+(\mu+2b/T)T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+(\mu+2b/T)T}{\sqrt{T}}\right)\right) \\
&\quad = S_0\left(\Phi\left(\delta_+^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_+^T\left(\frac{S_0}{B}\right)\right)\right) \\
&\quad -S_0\mathrm{e}^{-\left(r+\mu^2/2\right)T-2b^2/T+(\sigma+\mu+2b/T)^2T/2}\left(\Phi\left(\delta_+^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_+^T\left(\frac{B}{S_0}\right)\right)\right) \\
&\quad -K\mathrm{e}^{-rT}\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right) \\
&\quad +K\mathrm{e}^{-\left(r+\mu^2/2\right)T-2b^2T+(\mu+2b/T)^2T/2}\left(\Phi\left(\delta_-\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_-\left(\frac{B}{S_0}\right)\right)\right),
\end{aligned}
$$

$0 \leq x \leq B$, where $\delta_\pm^T(s)$ is defined above.

Using the fact that:

$$-T\left(r+\frac{\mu^2}{2}\right)-2\frac{b^2}{T}+\frac{T}{2}\left(\sigma+\mu+\frac{2b}{T}\right)^2=2b\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)=\left(1+\frac{2r}{\sigma^2}\right)\log\frac{B}{S_0},$$

and

$$-T\left(r+\frac{\mu^2}{2}\right)-2\frac{b^2}{T}+\frac{T}{2}\left(\mu+\frac{2b}{T}\right)^2=-rT+2\mu b=-rT+\left(-1+\frac{2r}{\sigma^2}\right)\log\frac{B}{S_0}$$

this yields:

$$\begin{aligned}
\mathrm{e}^{-rT}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}\left\{M_0^T<B\right\}\right] &= S_0\left(\Phi\left(\delta_+^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_+^T\left(\frac{S_0}{B}\right)\right)\right)\\
&-\mathrm{e}^{-rT}K\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right)\\
&-B\left(\frac{B}{S_0}\right)^{2r/\sigma^2}\left(\Phi\left(\delta_+^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_+^T\left(\frac{B}{S_0}\right)\right)\right)\\
&+\mathrm{e}^{-rT}K\left(\frac{S_0}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_-^T\left(\frac{B}{S_0}\right)\right)\right)\\
&=S_0\left(\Phi\left(\delta_+^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_+^T\left(\frac{S_0}{B}\right)\right)\right)\\
&-S_0\left(\frac{B}{S_0}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_+^T\left(\frac{B}{S_0}\right)\right)\right)\\
&-\mathrm{e}^{-rT}K\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right)\\
&+\mathrm{e}^{-rT}K\left(\frac{S_0}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_-^T\left(\frac{B}{S_0}\right)\right)\right)
\end{aligned}$$

□

B Double no-touch option derivation

Proof. We have terminal condition $V(S_T)=1$ for $L<S_T<B$ and boundary conditions $V(L,t)=V(B,t)=0$ for $0\leq t\leq T$.
The Black-Scholes PDE for the contract value is written:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

In log-space with $X_t=\log(S_t/S_0)$, with the transformation $V(X,t)=e^{\alpha X+\beta(T-t)}U(X,t)$, the convenient $\alpha=-\tilde{r}/\sigma^2$ and $\beta=-\tilde{r}/2\sigma^2-r$, $\tilde{r}=r-\frac{1}{2}\sigma^2$, we get the heat equation:

$$\frac{\partial U}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2U}{\partial X^2}=0.$$

The conditions stemming from the double no-touch option payoff now are: $U(0,t) = U(H,t) = 0$, with $H = \ln \frac{B}{L}$ for $0 \leq t \leq T$, and terminal condition $U(X,T) = e^{-\alpha X}$.

We use a Fourier series ansatz to solve such an equation:

$$U(X,t) = \sum_{n=1}^{\infty} A_n(t) \sin\left(\frac{n\pi X}{H}\right).$$

Plugging this into the heat equation solved by U gives a first order differential equation verified by the coefficients A_n, thus $A_n(t) = a_n \exp\left(\sigma^2 \frac{n^2\pi^2}{2H^2} t\right)$

To find an expression for the little a_n's, we use the identity:

$\int_0^H \sin\left(\frac{n\pi}{H} \ln \frac{S}{S_{min}}\right) \sin\left(\frac{m\pi}{H} \ln \frac{S}{S_{min}}\right) = \frac{1}{2} H \delta_{n,m}$, with $\delta_{n,m}$ the Kronecker delta. We find out that for each $m \in \mathbb{N}^*$,

$$\frac{1}{2} a_m \exp\left(\sigma^2 \frac{n^2\pi^2}{2H^2} t\right) = \frac{m\pi H}{\alpha^2 H^2 + m^2\pi^2}(1 - (-1)^m e^{-\alpha H}).$$

Plugging everything back in, we get the theoretical price for a double no-touch barrier under Black-Scholes dynamics:

$$V(S,t) = 2\pi \left(\frac{S}{S_{min}}\right)^{\alpha} e^{\beta(T-t)} \sum_{n=1}^{\infty} n \left[\frac{(1-(-1)^m e^{-\alpha H})}{\alpha^2 H^2 + n^2\pi^2}\right] \exp\left(-\frac{1}{2}\sigma^2 \frac{n^2\pi^2}{H^2} t\right) \sin\left(\frac{n\pi}{H} \ln \frac{S}{S_{min}}\right),$$

with $H = \ln(S_{max}/S_{min})$, $\alpha = -\tilde{r}/\sigma^2$ and $\beta = -\tilde{r}/2\sigma^2 - r$, $\tilde{r} = r - \frac{1}{2}\sigma^2$.

Here, we are happy as this is (almost, except to notations choices) what is given in [EMT02].

$\square$

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