SPX-VIX joint calibration: extending the quintic OU model

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1 Introduction

The S&P500 is an equity index introduced to track the performance of 500 of the largest companies listed on stock exchanges in the United States. There are loads of exchange-traded products that embed this index as part of their behaviour.
In particular, the Chicago Board Options Exchange (CBOE) offers a vast range of option products with this index – referred to as the *SPX* – as an underlying.
On the other hand, a volatility index was introduced, now known as the *VIX*. It derives expected volatility levels for the trading month to come with current options and forward prices.

Investors are able to express their views or hedge themselves against moves in the VIX via VIX futures trading introduced in 2004, and further manage volatility positions via VIX options since 2006. However, a reliable pricing model for such options is yet to be found: VIX is a derivative of the SPX, hence any pricing model on the VIX needs to be consistent with SPX options prices. While there exists several methods for pricing VIX options the issue is rather in having a single model that can jointly calibrate both indices. This is referred to as the **joint calibration problem**.

We will be highlighting the relevant market behaviours such a model should embed, reviewing the current state-of-the-art in the joint calibration problem, before proposing a new stochastic dynamic to fit both the SPX and VIX surfaces as well as futures prices.

2 Stylized facts and goals

When designing a model for financial assets, we care mostly about the correctness according to the empirical market observations and the tractability of the model.

2.1 Empirical market behaviour

We retrieve popular stylized facts highlighted in the literature [Con01; Rat+23], namely:

1. **Absence of returns autocorrelation**: there is no predictive power to be found between current and future returns (exception for the high-frequency scale where market microstructure is involved).

2. **Heavy tails** in the distribution of returns: the normality assumption is not verified as rare events are more often observed than they should (one of the reasons for headlines with so-called "five sigma event" popping up so regularly...)

3. **Skewed return distribution** deviates from a Gaussian distribution: it exhibits an asymmetry which is in general negative.

4. **Aggregational Gaussianity**: the return distribution gets closer to a Gaussian distribution as the timescale over which returns are calculated increases.

5. **Volatility clustering:** shocks in prices tend to cluster in time, in the sense that large price variations are more likely to be followed by large price variations. Moreover, absolute returns remain correlated to past returns up to a significant time lag, with a slow decay that aligns well with a power law.

6. **Leverage effect** was first discussed by in [Bla76]: we observe a rise in volatility when prices drop – think of a crisis. This effect is also referred to as a negative volatility-return correlation.
   There also is time reversal asymmetry (TRA) in the sense that the predictive power is stronger in the past returns predict future volatilities than the other way around.

7. **Zumbach effect** refers to the predictive power of historical squared returns on future volatility – or integrated variance –, and it being greater than that

of historical volatility on future squared returns [CB14].
In 1963, Mandelbrot noticed that *large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes* [Man63].

8. **ATM skew**? (*still in discussion on whether or not to include this as a stylized facts or not*) $\mathcal{S}_T$ is defined as:

$$\mathcal{S}_T = \left.\frac{d\widehat{\sigma}(T,k)}{dk}\right|_{k=0},$$

and is believed to follow a – truncated? – power-law distribution. This involves a skew reconstruction step, to approximate the theoretical skew with a finite difference estimator [DMS23].

We design systematic tests to assess for the presence – or absence – of such facts in 1. observed assets trajectories and 2. simulated trajectories under different model dynamics. As we are concerned with fitting implied volatility smiles for options maturing several days in the future, we focus on daily returns instead of high-frequency returns as in [Rat+23].

1. Absence of return autocorrelation can be assessed by plotting the autocorrelogram of the returns [Rat+23]. We expect the autocorrelations to be within the confidence bands testing whether they are different from 0, up to a large confidence level (e.g., 95%) and for all lags larger than 0.

2. The heaviness of the tails of the return distribution can be determined by checking whether the empirical excess kurtosis of the daily returns is significantly larger than 0. We compute the empirical excess kurtosis using returns calculated over various timescales, e.g., one day, two days, ..., and then plot the kurtosis as a function of the timescale. We expect the returns to remain leptokurtic up to several days.

3. Skewed return distribution is examined in a similar way as the heaviness of the tails. We compute the empirical skewness using returns calculated over various timescales, and then plot the skewness as a function of the timescale. We expect the returns to be negatively skewed up to several days.

4. Aggregational Gaussianity can be observed by looking at different empirical moment measures of the return distribution. We check whether the empirical excess kurtosis and skewness of the returns converge towards 0 as the timescale increases.

5. Volatility clustering can be examined by plotting the autocorrelogram of the absolute returns. The expectation is for the autocorrelations to be significantly positive for a large confidence level and up to many lags, and to asymptotically go to zero, with a decay corresponding to a power law. Instead of checking whether the decay looks roughly linear on a log-log plot, as in [Rat+23], we will assess the fitting accuracy of various functions on the autocorrelations, like linear combinations of exponential and power law functions. We anticipate that a power law will give the best fit.

Note however that in the presence of heavy tails, estimators of the autocorrelation function may be unreliable [Con01].

6. Leverage effect is extensively studied in [BMP01] where the authors highlight it with exponential fits on individual stocks as well as on indices. The quantity of interest is $\mathcal{L}(\tau) = \langle |r_{t+\tau}|^2, r_t \rangle_t$, normalized by $\langle |r_{t+\tau}|^2 \rangle_t^2$, where r_t is the daily close-to-close log return.

7. Zumbach effect is investigated through the quantity:

$$\tilde{\mathcal{C}}^{(2)}(\tau) = \langle (\sigma_t^2 - \langle \sigma_t^2 \rangle)^2, r_{t-\tau}^2 \rangle$$

introduced in [CB14]. This quantifies (under stationarity assumptions) the covariance of integrated variance with past squared returns. Empirically, time-reversal asymmetry measured by $Z(\tau) = \tilde{\mathcal{C}}^{(2)}(\tau) - \tilde{\mathcal{C}}^{(2)}(-\tau)$ is found positive.
[El +20] continues the empirical study on the 31 stock indices in the Oxford-Man Institute dataset. They compute the Zumbach effect under the rough Heston model and indeed find it positive. [GJR20] further design a quadratic rough Heston which exhibits rough volatility and is consistent with the strong Zumbach effect.

2.2 Models

Over the years several models fulfilling different purposes have been introduced and considered "state-of-the-art". We briefly introduce the most popular ones before focusing on models designed to answer the joint calibration problem.

Black-Scholes provides a convenient framework for options (*e.g.* the definition of implied volatility stems from the this model) where the asset is assumed to follow a geometric brownian motion dynamic:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

however the assumption of a constant volatility is drastically bold, as **the volatility surface is not flat**.

Dupire worked on local volatility: going from a constant volatility function σ to a maturity and strike-dependent function to fit to the market data: $\sigma(T, K)$. This provides a better fit to the actual surfaces but fails to generalize to **smile dynamics**: volatility surfaces change in time, and local volatility does not account for such a fact – this is known as the time inhomogeneity problem.

Heston (and stochastic volatility models) eventually introduce a stochastic process for the instantaneous volatility σ_t. Heston [Hes93] introduces a CIR process for the instantaneous variance v_t:

$$\begin{cases} dS_t = \mu S_t \, dt + \sqrt{v_t} S_t \, dW_t \\ dv_t = \kappa(\theta - v_t) \, dt + \sigma \sqrt{v_t} \, dB_t, \end{cases}$$

with the two Brownians having correlation ρ: $\langle dW_t, dB_t \rangle = \rho dt$. In the variance process parameters are θ the long term level, κ the mean-reversion factor and σ the vol-of-vol.

We can simulate such dynamics following [And07; Gat22].

It is known that the Heston model misprices by construction the link between volatility and volatility of volatility, *"thereby leading to the mispricing of derivatives that are highly sensitive to the dynamics of implied volatility"* [GIP17] – indeed, the VIX and VVIX daily series exhibit a 60% correlation level in log-variations. The consequence lies in the failure at reproducing the at-the-money skew of the implied volatility observed in the market.

Rough models address the rough volatility phenomenom [GJR22]: it has been argued that volatility dynamics are better described by a fractional Brownian motion (fBM) than with a simple Brownian motion – however this remains a debated claim [GE22; DMS23; AL24]. This process B_t^H is a continuous zero-mean Gaussian with covariance function

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

where $H \in (0,1)$ is the Hurst exponent. It can also be built through the Mandelbrot-van Ness representation.

This paves the path to a subclass of non-Markovian stochastic volatility models [El 18]. Two extensions of the Heston model can be studied here: the rough Heston [El +20] and the quadratic rough Heston [GJR20].

There exist several ways of estimating H.

The quintic Ornstein-Uhlenbeck model where the dynamics of the stock price S, with no interest nor dividends, are given by [JI+22]:

$$\begin{cases} \dfrac{dS_t}{S_t} = \sigma_t dB_t, \\ \sigma_t = \sqrt{\xi_0(t)} \dfrac{p(X_t)}{\sqrt{\mathbb{E}\left[p(X_t)^2\right]}}, \quad p(x) = \alpha_0 + \alpha_1 x + \alpha_3 x^3 + \alpha_5 x^5, \\ X_t = \varepsilon^{H-1/2} \displaystyle\int_0^t e^{-(1/2-H)\varepsilon^{-1}(t-s)} dW_s, \end{cases}$$

This is close to an instance of a Volterra Bergomi model – with coefficients α_k fixed – with one-factor kernel in [AL24].

2.3 Results

We check the actual behaviour of equity indices trajectories from daily prices and five-minutes realized volatility estimates from the Oxford-Man Institute dataset as a proxy for the instantaneous volatility. This allows us to build a "checklist" from verified literature results and assess for the reproductibility of these features by different models.
We run simulation of trajectories according to a bunch of models fitted to market data so that calibrated parameters are consistent with reality.

Table 1: Different models and the stylized facts that they manage to capture.

	Black-Scholes	Heston	SABR	LSV	Rough Heston	Rough Bergomi	Quintic OU	new sota?
Absence of returns auto-correlation	✓		✓				✓	
Heavy tails	✗		✓				✓	
Gain/loss asymmetry	✗		✗				✗	
Volatility clustering	✗		✓				✓	
Volatility persistence	✗		✗					
Leverage effect	✗		✗				✓	
Zumbach effect	✓		✗					
Calibration time	?							

2.4 What about the VIX?

The VIX can be seen as the market expectation of the future realized volatility within the next trading month. It was introduced with the following formula:

$$\mathrm{VIX} = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right] \tag{VIX}$$

where: T is a time to expiration; F is the forward index level; K, is a strike price of the i^{th} out-of-the-money option a call if $K_i > F$ and a put if $K_i < F$; ΔK_i is an interval between strike prices: K_0 is a first strike below the forward index level, F: R is a risk-free interest rate to expiration; and $Q(K)$ is a midpoint of a bid-ask spread for each option with strike K,

In continuous time, it can also be written explicitly as:

$$\mathrm{VIX}_T^2 = -\frac{2}{\Delta}\mathbb{E}\left[\log(S_{T+\Delta}/S_T) \mid \mathcal{F}_T\right] \times 100^2 = \frac{100^2}{\Delta}\int_T^{T+\Delta} \xi_T(u)du,$$

with $\Delta = 30$ days and $\xi_T(u) := \mathbb{E}\left[\sigma_u^2 | \mathcal{F}_T\right]$ the forward variance process.

2.5 VX futures

3 Towards a new model?

3.1 Calibration

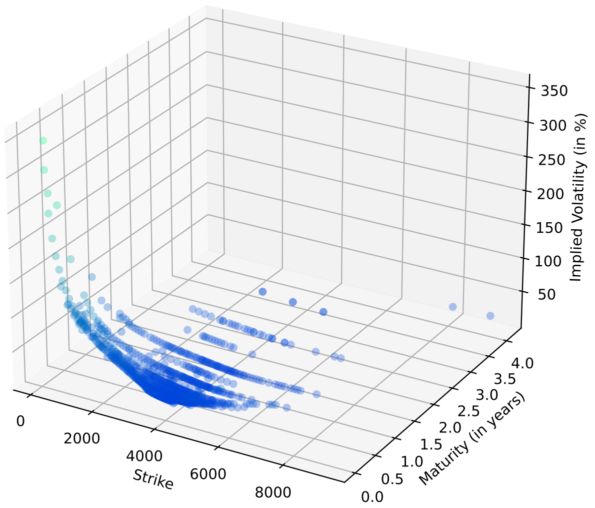

Figure 1: SPX surface

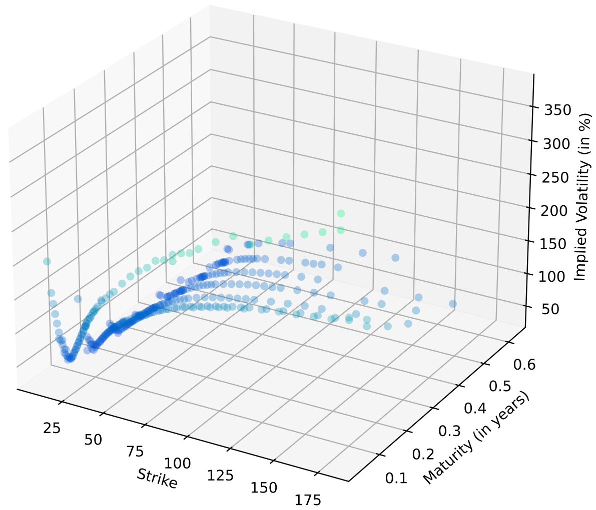

Figure 2: VIX surface

Figure 3: Implied Volatility quoted on November 4^{th} 2022, filtered on contracts with significant open interest

We observe on 3 that the VIX indeed does not behave like a random equity name:

Our model will be solving for the following problem:

$$\min_{\Theta} \left\{ c_1 \sqrt{\sum_{i,j} \left(\sigma^{\Theta}_{spx}(T_i, K_j) - \sigma^{mkt}_{spx}(T_i, K_j) \right)^2} + c_2 \sqrt{\sum_{i,j} \left(\sigma^{\Theta}_{vix}(T_i, K_j) - \sigma^{mkt}_{vix}(T_i, K_j) \right)^2} \right.$$
$$\left. + c_3 \sqrt{\sum_{i} \left(F^{\Theta}_{vix}(T_i) - F^{mkt}_{vix}(T_i) \right)^2} \right\}. \tag{1}$$

4 Roadmap

In the short-term we should be able to finalize the stylized facts pipeline with averaged results over several simulated paths and confidence bars;

We've listed stylized facts, implemented models (some still need some work, especially the fit method to get parameters from daily market data) and now are leaning towards developping a new model with some analytical derivation (e.g. zumbach effect as in the rough heston paper)

Leveraging options data is the next step where we will be fitting the SPX surface, the VIX surface as well as the VX futures term strucute;

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