Propositional Logic

Outline

- What is a Proposition
- Logical Operations
- Truth Tables
- Examples using Truth Tables
- Tautology
- Contradiction
- Contingent Proposition
- Tautology/Contradiction Examples

Propositions

- A proposition is a statement which is either true or false but not both.

Examples:

i) p: Colombo is in Sri Lanka

ii) q: 17 is a prime number

iii) r: $x=3$ is a solution of $x^2=4$

Propositions p, q are true and r is false.

- Are the following statements propositions?

i) What is your age?

ii) Do your studies well

iii) $x + 2 = 2x$

Composite / compound Propositions

- Some propositions are composite, that is they are composed of sub statements

e.g. 17 is a prime number and 10 is an even number

- A fundamental property of a composite statement is that its truth value is completely determined by the truth value of each of its sub-statements and the way they are connected to form the composite statement

Primitive Propositions

A proposition which cannot be broken down into simpler propositions is called a primitive proposition.

p_1: 3 + 8 = 11

p_2: Elephants can fly

A primitive statement has a truth value either **True** or **False**

Truth Table

- A truth table is a computational device by which we can determine the truth value of a proposition once we know the truth value of each of its components

- We can combine primitive propositions using the basic logical operations.

Negation

- When any proposition is given, another proposition called Negation of p can be formed by inserting the word “not” before p. Negation of p is denoted by ∼p or “not p” or ¬p.

p	∼p
F	T
T	F

- T denotes true and F denotes false. Another way is to replace T with 1 and F with 0.

Conjunction

- Any two given propositions can be combined with the word "and" to form a compound proposition called the conjunction of the two given propositions.

- If p and q are propositions then, $\mathbf{p \land q}$ (read as "p and q") denotes the conjunction of the propositions p, q.

Conjunction

- Consider the two propositions p, q. If p is true and q is true, then p ∧ q is true. Otherwise, p ∧ q is false.

p	***q***	**p ∧ q**
F	F	F
F	T	F
T	F	F
T	T	T

Disjunction

- Any two given propositions can be combined with the word “or” to form a compound proposition called the disjunction of the two given propositions.

- If p and q are propositions then, $p \vee q$ (read as “p or q”) denotes the disjunction of the propositions p, q.

Disjunction

- Consider the two propositions p, q. If p is true or q is true, then p $\vee$ q is true. Otherwise, p $\vee$ q is false.
- T denotes true and F denotes false. Another way is to replace T with 1 and F with 0.

p	**q**	**p $\vee$ q**
F	F	F
F	T	T
T	F	T
T	T	T

Construct the following truth tables

1. Construct the truth table of

i. $(\sim p) \land q$

ii. $(\sim p) \lor (\sim q)$

iii. $\sim(p \land \sim q)$

2. Verify the following:

i. $\sim (p \land q) = (\sim p) \lor (\sim q)$

ii. $\sim (p \lor q) = (\sim p) \land (\sim q)$

iii. $\sim ((p \lor q) \land r) = (\sim p \land \sim q) \lor \sim r$

Construct the following truth tables

3. Prove using tables that

i. $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

ii. $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Implication

- An implication is of the form "if p is true, then q follows" or take the more shorter form "if p, then q"
- An implication is denoted by $p \Rightarrow q$

p	q	**p ⇒q**
F	F	T
F	T	T
T	F	F
T	T	T

If p is false it doesn't matter what will be the truth value of q,

$p \Rightarrow q$ is always TRUE

Equivalence

- An equivalence $p \Leftrightarrow q$ is true *if and only if* p and q have the same truth value. An equivalence is denoted by $\Leftrightarrow$

p	q	**p ⇔ q**
F	F	T
F	T	F
T	F	F
T	T	T

Construct the following truth tables

1. Construct the truth table of

i. $(p \Rightarrow q) \wedge (q \Rightarrow p)$

ii. $p \Leftrightarrow q \wedge (q \Rightarrow p)$

2. Construct the truth tables of :

i. $\sim(p \Rightarrow \sim q)$

ii. $\sim (p \wedge q) \vee \sim(q \Leftrightarrow p)$

Precedence rules

Propositional logic uses precedence rules

$\neg$		highest
$\wedge$		
$\vee$		
$\Rightarrow$		
$\Leftrightarrow$		lowest

Brackets override
B,N,C,D,I,E

Exclusive OR

Let p and q be two propositions. The exclusive OR of p and q (denoted by $p \oplus q$) is a proposition that simply means exactly one of p and q will be true but both cannot be true

Example :

When you buy a car from XYZ company, you get either Rs. 50000 cashback or accessories worth Rs. 50000

Tautology

- Some compound propositions contain only **T** in the last column of their truth tables.
- A compound proposition which is true under all possible assignments of truth values to its prime propositions is called a tautology or a valid proposition.

Consider the statements

e.g. 1. The newborn baby is either male or female

2. The train will either arrive in time or will not arrive in time

The above propositions are tautologies because they are always true. The propositions $p \lor \sim p$ is a tautology.

p	$\neg p$	$p \lor \neg p$
T	F	T
F	T	T

Tautology

Q. Show that [(p=>q) ∧ (q=>r)] => (p=>r) is a tautology.

Contradiction

- Similarly, some compound propositions contain only **F** in the last column of their truth tables.
- A compound proposition which is false under all possible assignments of truth values to its prime propositions is called a contradiction or an inconsistent proposition.

Example: $p \land \neg p$

p	$\neg p$	$p \land \neg p$
T	F	F
F	T	F

Contingent Proposition

- A compound proposition which is neither a tautology nor a contradiction is called a contingent proposition.

Example: $p \Rightarrow \neg p$

p	$\neg p$	$p \Rightarrow \neg p$
T	F	F
F	T	T

Laws in Propositional Logic

1. Commutative laws

$$p \land q = q \land p$$

$$p \lor q = q \lor p$$

2. Associative Laws

$$p \land (q \land r) = (p \land q) \land r$$

$$p \lor (q \lor r) = (p \lor q) \lor r$$

3. Distributive Laws

$$p \lor (q \land r) = (p \lor q) \land (p \lor r)$$

$$p \land (q \lor r) = (p \land q) \lor (p \land r)$$

Laws in Propositional Logic

4. De Morgan's Laws

$$\sim(p \land q) = \sim p \lor \sim q$$

$$\sim(p \lor q) = \sim p \land \sim q$$

5. Law of Negation

$$\sim(\sim p) = p$$

6. Law of Excluded Middle

$$p \lor \sim p = \text{true}$$

7. Law of Contradiction

$$p \land \sim p = \text{false}$$

Laws in Propositional Logic

8. Law of Implication

$$p \Rightarrow q = \sim p \vee q$$

9. Contrapositive Law

$$p \Rightarrow q = (\sim q \Rightarrow \sim p)$$

10. Law of Equivalence

$$p <=> q = (p \Rightarrow q) \wedge (q \Rightarrow p)$$

11. Idempotence

$$p \vee p = p$$

$$p \wedge p = p$$

Laws in Propositional Logic

12. Laws of Simplification-1

$$p \wedge \text{true} = p$$

$$p \vee \text{true} = \text{true}$$

$$p \wedge \text{false} = \text{false}$$

$$p \vee \text{false} = p$$

13. Laws of Simplification-2

$$p \vee (p \wedge q) = p$$

$$p \wedge (p \vee q) = p$$

Laws in Propositional Logic

14. Distributive Laws

$$a \lor (b \land c \land d) = (a \lor b) \land (a \lor c) \land (a \lor d)$$

$$a \land (b \lor c \lor d) = (a \land b) \lor (a \land c) \lor (a \land d)$$