Matrices

Outline

- What is a matrix?
- Different types of matrices
- Matrix Operations
- Solving a System of Linear Equations using Matrix Inversion

Matrices

- What is a matrix?

A matrix is a rectangular array of numbers. The numbers in the array are called the entries of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

Usually, we specify a position in the matrix as row, column. The size of a matrix A is written in terms of the number of its rows x the number of its columns.

a_{ij} is called ij-entry or ij element appears in row i and column j.

We also denote the above matrix as A= $[a_{ij}]_{m \times n}$

Q

- Identify the sizes of the following matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 2 \end{bmatrix}$$

$$F = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad G = \begin{bmatrix} 2 & 0 & 1 & 1 \end{bmatrix}$$

Square matrix

- A square matrix is a matrix with same number of rows as columns. An n x n square matrix is said to be of order n and is sometimes called an n-square matrix

- E.g.

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 3 \\ 1 & 4 & 3 & -1 \end{bmatrix}$$

- The operations of addition, multiplication, scalar multiplication, and transpose can be performed on any n x n matrices would result in an n x n matrix.

Identity Matrix

- The n-square identity or unit matrix, denoted by I_n, or simply I, is the n-square matrix with 1's on the diagonal and 0's elsewhere.
- For any square matrix A,

$$AI = IA = A$$

- Examples for identity matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Operations

Matrix addition

Let A=[a_{ij}] and B=[b_{ij}] be two matrices with the same size, say m x n matrices. The sum of A and B, written A+B, is the matrix obtained by adding corresponding elements from A and B. That is

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}_{m\times n}$$

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}_{m\times n}$$

Matrix Addition

- Example:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2+5 & 1+1 \\ 3+2 & 2+0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 5 & 2 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-5 & 1-1 \\ 3-2 & 2-0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 1 & 2 \end{bmatrix}$$

Matrix Operations

Scalar Multiplication

The product of the matrix A by a scalar k, written kA is the matrix obtained by multiplying each element of A by k. That is,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}_{m \times n}$$

Scalar Multiplication

- Example:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad 5A = 5 \times \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 15 & 10 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix} \qquad -2B = -2 \times \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 0 \\ -4 & -6 & 0 \\ -8 & -4 & -2 \end{bmatrix}$$

Multiplying Two Matrices

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} \qquad B=\begin{bmatrix}3 & -1\\ 2 & 0\\ 1 & 1\end{bmatrix}$$

If A*B =C

Steps:

1. Check whether the two matrices are compatible for multiplication.

   A is a 2 x 3 matrix. B is a 3 x 2 matrix.

   (No. of columns of A = No. of rows of B)

2. Identify the dimension of the resultant matrix. (2 x 2)

$$C=\begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}_{2\times 2}$$

Example: Multiplying Two Matrices

If A*B =C

Steps:

3. c_{ij} = dot product between i[th] row of A and j[th] column of B.

$$A = \begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad 5 \quad 6 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2 \times 2} \qquad C = \begin{bmatrix} \boxed{3+4+3} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2 \times 2}$$

$$C = \begin{bmatrix} 10 & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2 \times 2}$$

Example: Multiplying Two Matrices

If A*B =C

Steps:

3. c_{ij} = dot product between i^th row of A and j^th column of B.

$$A = \begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad 5 \quad 6 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2\times 2}$$

$$C = \begin{bmatrix} 10 & \boxed{-1+0+3} \\ c_{21} & c_{22} \end{bmatrix}_{2\times 2}$$

$$C = \begin{bmatrix} 10 & 2 \\ c_{21} & c_{22} \end{bmatrix}_{2\times 2}$$

Example: Multiplying Two Matrices

If A*B =C

Steps:

3. c_{ij} = dot product between i^th row of A and j^th column of B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2 \times 2} \qquad C = \begin{bmatrix} 10 & 2 \\ \boxed{12 + 10 + 6} & c_{22} \end{bmatrix}_{2 \times 2}$$

$$C = \begin{bmatrix} 10 & 2 \\ 28 & c_{22} \end{bmatrix}_{2 \times 2}$$

Example: Multiplying Two Matrices

If A*B =C

Steps:

3. c_{ij} = dot product between i[th] row of A and j[th] column of B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2 \times 2} \qquad C = \begin{bmatrix} 10 & 2 \\ 28 & \boxed{-4+0+6} \end{bmatrix}_{2 \times 2}$$

$$C = \begin{bmatrix} 10 & 2 \\ 28 & 2 \end{bmatrix}_{2 \times 2}$$

Example: Multiplying Two Matrices

If A*B =C

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \boxed{a_{i1} & a_{i2} & \cdots & a_{in}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}_{n \times p}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}_{m \times p}$$

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$$

Example: Multiply the following matrices

1. $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

4. $A = \begin{bmatrix} 3 & 2 & 1 \\ 8 & 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 0 & 1 \end{bmatrix}$

Determinant of a 2 x 2 matrix

- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

the determinant |A| is given by

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

Find the Determinant of the following matrices

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix}$$

Inverse of a 2 x 2 matrix

- A square matrix A is said to be **invertible** or **nonsingular** if the following relationship holds:

$$AA^{-1} = A^{-1}A = I$$

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- where $|A| = \mathrm{ad} - \mathrm{cb}$

  and $|A| \neq 0$

Solving simultaneous equations

A system of linear equations in matrix form.

e.g.
$$x + y = 10$$
$$x - 2y = 4$$

The two equations can be represented as follows:

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$$A\mathbf{x} = b \longrightarrow \mathbf{x} = A^{-1}b$$

Solve the following simultaneous equations

1. $x + y = 5$

   $x - y = 1$

2. $2x + y = 4$

   $x - 2y = -3$

3. $-x + 2y = -1$

   $x - 2y = 1$

4. $3x + 4y = 25$

   $4x + 3y = 24$

Minor

- The minor of an element is defined as a determinant obtained by deleting the row and column containing the element. For the matrix A,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

- Find the following minors: M_{13}, M_{21}, M_{22}, M_{23}, M_{31}, M_{32}, M_{33} of the matrix A.

Cofactor Matrix

- For the given matrix A,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

cofactors are obtained as follows:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$\mathbf{C_{ij}}$ is the elements of the cofactor matrix.

The cofactor matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = (-1)^{1+1} M_{11}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

Find all the cofactors .

Find the cofactor matrices of the following.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 2 & -3 \\ 1 & -2 & 2 \\ 3 & 2 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

Cofactor Matrix of A

$$C.M = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

$$C.M = \begin{bmatrix} (0-4) & -(0-6) & (0-3) \\ -(0-6) & (0-9) & -(2-6) \\ (4-3) & -(2-0) & (1-0) \end{bmatrix}$$

$$C.M = \begin{bmatrix} -4 & 6 & -3 \\ 6 & -9 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

Cofactor Matrix of B

$$B = \begin{bmatrix} -1 & 2 & -3 \\ 1 & -2 & 2 \\ 3 & 2 & -1 \end{bmatrix}$$

$$C.M = \begin{bmatrix} \begin{vmatrix} -2 & 2 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -3 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} -1 & -3 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} \end{bmatrix}$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

$$C.M = \begin{bmatrix} (2-4) & -(-1-6) & (2+6) \\ -(-2+6) & (1+9) & -(-2-6) \\ (4-6) & -(-2+3) & (2-2) \end{bmatrix}$$

$$C.M = \begin{bmatrix} -2 & 7 & 8 \\ -4 & 10 & 8 \\ -2 & -1 & 0 \end{bmatrix}$$

Transpose of a matrix

- A transpose of a matrix is obtained by replacing all elements a_{ij} with a_{ji}. The matrix transpose, most commonly written A^T.

DEFINITION If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

Interchange entries that are symmetrically positioned about the main diagonal.

Inverse-(Adjoint Method)

- If A is a n x n square matrix, its inverse is obtained by;

$$A^{-1} = \frac{Adj\ A}{Determinent\ of\ \ A}$$

$$A^{-1} = \frac{(Cofactor\ matrix)^T}{|A|}$$

Determinant of a n x n matrix

If

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n\times n}$$

the determinant |A| is given by

$$|A| = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

M_{ij} is the determinant of the matrix after deleting i^{th} row and j^{th} column

Determinant of a 3 x 3 matrix

- If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the determinant |A| is given by $a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Find the Determinant of the following matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 2 & -3 \\ 1 & -2 & 2 \\ 3 & 2 & -1 \end{bmatrix}$$

Find the Determinant of the following matrices

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix}$$

$$|A| = 1(0-4) - 2(0-6) + 3(0-3)$$

$$|A| = -4 + 12 - 9$$

$$|A| = -1$$

Find the Determinant of the following matrices

$$\mathrm{B} = \begin{bmatrix} -1 & 2 & -3 \\ 1 & -2 & 2 \\ 3 & 2 & -1 \end{bmatrix}$$

$$|B| = b_{11}M_{11} - b_{12}M_{12} + b_{13}M_{13}$$

$$|B| = -1\begin{vmatrix} -2 & 2 \\ 2 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix}$$

$$|B| = -1(2-4) - 2(-1-6) - 3(2--6)$$

$$|B| = 2 + 14 - 24$$

$$|B| = -8$$

Find the inverse of the following matrices

- We found the cofactor matrices and the determinants of A and B before. Now, use adjoint method to find the inverse of A and B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 2 & -3 \\ 1 & -2 & 2 \\ 3 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{(Cofactor\ matrix)^T}{|A|}$$

Inverse of A (A^{-1})

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}, \quad C.M = \begin{bmatrix} -4 & 6 & -3 \\ 6 & -9 & 4 \\ 1 & -2 & 1 \end{bmatrix}, \quad |A| = -1$$

$$A^{-1} = \frac{C\ M^T}{|A|}$$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -4 & 6 & 1 \\ 6 & -9 & -2 \\ -3 & 4 & 1 \end{bmatrix}$$

where $C.M^T = \begin{bmatrix} -4 & 6 & 1 \\ 6 & -9 & -2 \\ -3 & 4 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 4 & -6 & -1 \\ -6 & 9 & 2 \\ 3 & -4 & -1 \end{bmatrix}$$

$$B^{-1} = \frac{(Cofactor\ matrix)^T}{|B|}$$

Inverse of B $\quad (B^{-1})$

$$\text{B} = \begin{bmatrix} -1 & 2 & -3 \\ 1 & -2 & 2 \\ 3 & 2 & -1 \end{bmatrix}, \quad C.M = \begin{bmatrix} -2 & 7 & 8 \\ -4 & 10 & 8 \\ -2 & -1 & 0 \end{bmatrix}, \quad |B| = -8$$

$$B^{-1} = \frac{C\ M^T}{|B|}$$

$$B^{-1} = \frac{1}{-8}\begin{bmatrix} -2 & -4 & -2 \\ 7 & 10 & -1 \\ 8 & 8 & 0 \end{bmatrix}$$

where $C.M^T = \begin{bmatrix} -2 & -4 & -2 \\ 7 & 10 & -1 \\ 8 & 8 & 0 \end{bmatrix}$

$$B^{-1} = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ -0.875 & -1.25 & 0.125 \\ -1 & -1 & 0 \end{bmatrix}$$

System of Linear Equations

- When solving a system of linear equations adjoint method is used to find the inverse of the square matrix.

Solve the following linear systems using matrix inversion.

1.
$$x+y+z=6$$
$$x+2y-z=2$$
$$2x+y+2z=10$$

2.
$$2x+y+2z=5$$
$$x+2y-z=-3$$
$$2x-y-z=1$$

3.
$$x+2y+z=3$$
$$2x-y+2z=11$$
$$x-2y-z=1$$

System of Linear Equations

Q1 $x + y + z = 6$

$x + 2y - z = 2$

$2x + y + 2z = 10$

Represent the above system in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 10 \end{bmatrix}$$

$$A\mathbf{x} = b$$

$$\mathbf{x} = A^{-1}b$$

$$A^{-1} = \frac{C.M^T}{|A|}$$

System of Linear Equations

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 10 \end{bmatrix}$$

$$C.M = \begin{bmatrix} \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

$$C.M = \begin{bmatrix} (4-(-1)) & -(2-(-2)) & (1-4) \\ -(2-1) & (2-2) & -(1-2) \\ (-1-2) & -(-1-1) & (2-1) \end{bmatrix}$$

$$C.M = \begin{bmatrix} 5 & -4 & -3 \\ -1 & 0 & 1 \\ -3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 10 \end{bmatrix}$$

System of Linear Equations

$$C.M = \begin{bmatrix} 5 & -4 & -3 \\ -1 & 0 & 1 \\ -3 & 2 & 1 \end{bmatrix}$$

$$C.M^T = \begin{bmatrix} 5 & -1 & -3 \\ -4 & 0 & 2 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$|A| = 1\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$|A| = (4+1) - 1(2+2) + (1-4)$$

$$|A| = -2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 10 \end{bmatrix}$$

System of Linear Equations

$$A^{-1} = \frac{C.M^T}{|A|}$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -1 & -3 \\ -4 & 0 & 2 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\mathbf{x} = A^{-1}b$$

$$\mathbf{x} = \frac{1}{-2}\begin{bmatrix} 5 & -1 & -3 \\ -4 & 0 & 2 \\ -3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 10 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 30 - 2 - 30 \\ -24 + 0 + 20 \\ -18 + 2 + 10 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}}$$

System of Linear Equations

Q2 $2x + y + 2z = 5$

$x + 2y - z = -3$

$2x - y - z = 1$

Represent the above system in matrix form:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

$$A\mathbf{x} = b$$

$$\mathbf{x} = A^{-1}b$$

$$A^{-1} = \frac{C.M^T}{|A|}$$

System of Linear Equations

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

$$C.M = \begin{bmatrix} \begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

$$C.M = \begin{bmatrix} (-2-(1)) & -(-1-(-2)) & (-1-4) \\ -(-1-(-2)) & (-2-4) & -(-2-2) \\ (-1-4) & -(-2-2) & (4-1) \end{bmatrix}$$

$$C.M = \begin{bmatrix} -3 & -1 & -5 \\ -1 & -6 & 4 \\ -5 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

System of Linear Equations

$$C.M = \begin{bmatrix} -3 & -1 & -5 \\ -1 & -6 & 4 \\ -5 & 4 & 3 \end{bmatrix}$$

$$C.M^T = \begin{bmatrix} -3 & -1 & -5 \\ -1 & -6 & 4 \\ -5 & 4 & 3 \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$|A| = 2\begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$|A| = 2(-2-1) - 1(-1+2) + 2(-1-4)$$

$$|A| = -17$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

System of Linear Equations

$$A^{-1} = \frac{C.M^T}{|A|}$$

$$A^{-1} = \frac{1}{-17}\begin{bmatrix} -3 & -1 & -5 \\ -1 & -6 & 4 \\ -5 & 4 & 3 \end{bmatrix}$$

$$\mathbf{x} = A^{-1}b$$

$$\mathbf{x} = \frac{1}{-17}\begin{bmatrix} -3 & -1 & -5 \\ -1 & -6 & 4 \\ -5 & 4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \frac{1}{-17}\begin{bmatrix} -15+3-5 \\ -5+18+4 \\ -25-12+3 \end{bmatrix}$$

$$= \frac{1}{-17}\begin{bmatrix} -17 \\ 17 \\ -34 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\boxed{\begin{gathered} x = 1 \\ y = -1 \\ z = 2 \end{gathered}}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix}$$

System of Linear Equations

Q3 $x + 2y + z = 3$

$2x - y + 2z = 11$

$x - 2y - z = 1$

Represent the above system in matrix form:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix}$$

$$A\mathbf{x} = b$$

$$\mathbf{x} = A^{-1}b$$

$$A^{-1} = \frac{C.M^T}{|A|}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix}$$

System of Linear Equations

$$C.M = \begin{bmatrix} \begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \end{bmatrix}$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

$$C.M = \begin{bmatrix} (1 - 2(-2)) & -(-2 - 2) & (-4 - (-1)) \\ -(-2 - (-2)) & (-1 - 1) & -(-2 - 2) \\ (4 - (-1)) & -(2 - 2) & (-1 - 4) \end{bmatrix}$$

$$C.M = \begin{bmatrix} 5 & 4 & -3 \\ 0 & -2 & 4 \\ 5 & 0 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix}$$

System of Linear Equations

$$C.M = \begin{bmatrix} 5 & 4 & -3 \\ 0 & -2 & 4 \\ 5 & 0 & -5 \end{bmatrix}$$

$$C.M^T = \begin{bmatrix} 5 & 0 & 5 \\ 4 & -2 & 0 \\ -3 & 4 & -5 \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & -1 \end{bmatrix}$$

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$|A| = 1\begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$

$$|A| = 1(1 + 4) - 2(-2 - 2) + 1(-4 + 1)$$

$$|A| = 10$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix}$$

System of Linear Equations

$$A^{-1} = \frac{C.M^T}{|A|}$$

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 5 & 0 & 5 \\ 4 & -2 & 0 \\ -3 & 4 & -5 \end{bmatrix}$$

$$\mathbf{x} = A^{-1} b$$

$$\mathbf{x} = \frac{1}{10} \begin{bmatrix} 5 & 0 & 5 \\ 4 & -2 & 0 \\ -3 & 4 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 15 + 0 + 5 \\ 12 - 22 + 0 \\ -9 + 44 - 5 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 20 \\ -10 \\ 30 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 3 \end{aligned}}$$

Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(*a*) $A + B = B + A$ **(Commutative law for addition)**

(*b*) $A + (B + C) = (A + B) + C$ **(Associative law for addition)**

(*c*) $A(BC) = (AB)C$ **(Associative law for multiplication)**

(*d*) $A(B + C) = AB + AC$ **(Left distributive law)**

(*e*) $(B + C)A = BA + CA$ **(Right distributive law)**

(*f*) $A(B - C) = AB - AC$

(*g*) $(B - C)A = BA - CA$

(*h*) $a(B + C) = aB + aC$

(*i*) $a(B - C) = aB - aC$

(*j*) $(a + b)C = aC + bC$

(*k*) $(a - b)C = aC - bC$

(*l*) $a(bC) = (ab)C$

(*m*) $a(BC) = (aB)C = B(aC)$