Set Theory

Outline

- Sets-
- Describing sets, notation, examples
- Set equality
- Membership relationships
- Set Operations-

  Union, Intersection, Complement and Difference
- Venn Diagrams
- DeMorgan’s Laws for sets
- Set partition
- Cardinality, Power set ,Relations, Cartesian product, Functions

Set Theory

- Set: A *Set* is any well-defined collection of "objects."
- $A = \{a_1, a_2, a_3, \ldots, a_n\}$ "A contains…"
- Denote sets with upper-case letters, elements with lower-case letters.

The following notation is used to show set membership

- $a_1 \in A$ "a_1 is an element of A"
  "a_1 is a member of A"
- $b_1 \notin A$ "b_1 is not an element of A"

Set Theory

- If B is the set of letters in the English Alphabet, then we have,
  $B = \{a, b, c, \dots\dots\dots, x, y, z\}$
- We can write $a \in B$, $b \in B$, $c \in B$,…… , $z \in B$
- $n(B) = 26$
- Note: **Number of elements of a given set A is denoted by n(A).** In the above example since the number of letters in the Alphabet is 26, $n(B)=26$.

Ways of Describing Sets

- List the elements

$$A=\{1,2,3,4,5,6\}$$

- Give a verbal description
  - "A is the set of all positive integers from 1 to 6, inclusive"

- Denote using the set builder notation / mathematical notation

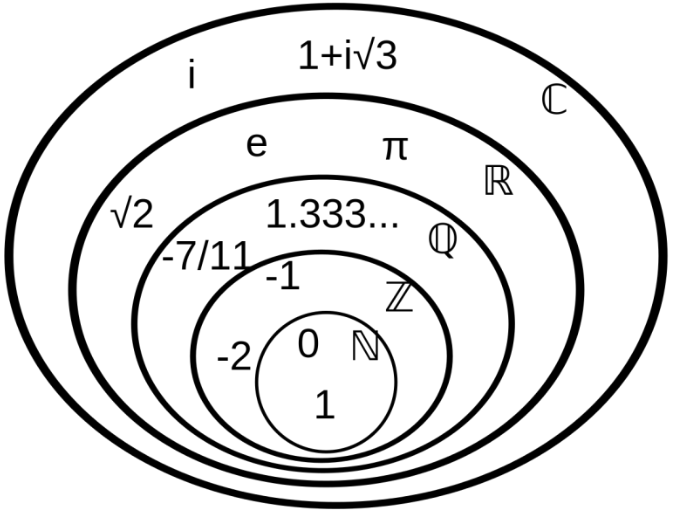
$$A= \{x \mid x \in Z^{+} , x<7\}$$

- Read as
  "A is the set of x such that x is a positive integer and less than 7"

Examples for Sets

"Standard" Sets:

- Natural numbers N = {1, 2, 3, …}
- Integers Z = {…, -2, -1, 0, 1, 2, …}
- Positive Integers Z^+ = {1, 2, 3, 4, …}
- Real Numbers R = {47.3, -12, π, …}
- Rational Numbers Q = {1.5, 2.6, -3.8, 15, …}



Q denotes the set of rational numbers, then

Q= $\{x \mid x=p/q \text{ where } p \in Z, q \in Z, q \neq 0\}$

Set Equality

- Sets A and B are equal if and only if they contain exactly the same elements.

Example

- A = {9, 2, 7, -3}, B = {7, 9, -3, 2} :

  A = B

  Order of elements is meaningless

Q) Identify which sets are equal

A= {x | x is an integer where x>0 and x<5 }

B= {x | x is a positive integer whose square is >0 and <25}

C= {x | x is an integer whose square is >0 and <25}

Some Special Sets

- The Null Set or Empty Set { }. This is a set with no elements, often symbolized by

$$\emptyset$$

- The Universal Set. This is the set of all elements currently under consideration, and is often symbolized by

$$\Omega$$

- Also symbolized by $\bigcup$

Membership Relationships

Subset.

$A \subseteq B$ "A is a subset of B"

We say "A is a subset of B" if $x \in A \Rightarrow x \in B$,

i.e., all the members of A are also members of B. The notation for subset means, in terms of the sets, "included in or equal to."

- B = {9, 2, 7, -3},

{9}, {9,2} ,{2,7,-3} are subsets of B and {9,2,7,-3} is also a subset of B

Membership Relationships

Proper Subset

$$A \subset B$$ “A is a proper subset of B”

We say “A is a proper subset of B” if all the members of A are also members of B, but in addition there exists at least one element c such that $c \in B$ but $c \notin A$.

The notation for proper subset means, in terms of the sets, “included in but not equal to.”

- {9, 2, 7, -3} is a subset of the set {9, 2, 7, -3} but not a proper subset of the set {9, 2, 7, -3} .

Questions

Fill in the blanks with $\in$, $\notin$, $\subseteq$, $=$ or $\neq$
Recall that Z is the set of all integers and ϕ is the empty set, {}.

a. 2	----------	{2, 4, 6}
b. {2}	----------	{2, 4, 6}
c. 1.5	----------	Z
d. -1.5	----------	Z
e. 15	----------	Z
f. {-15}	----------	Z
g. Ø	----------	Z
h. 54	----------	{6, 12, 18, …}
i. 54	----------	{6, 12, 18}
j. {1, 3, 3, 5}	----------	{1, 3, 5}
k. {-3, 1, 5}	----------	{1, 3, 5}
l. {3, 1, 5}	----------	{1, 3, 5}

Combining Sets – Set Union

$$A \cup B$$

- "A union B" is the set of all elements that are in A, or B, or both.

$$\Omega = \{1,2,3,4,5,6\}$$

$$A = \{1,2,3\}$$

$$B = \{3,4,5,6\}$$

$$A \cup B = \{1,2,3,4,5,6\}$$

Combining Sets – Set Intersection

$$A \cap B$$

- “A intersect B” is the set of all elements that is in both A and B.

$$\Omega = \{1,2,3,4,5,6\}$$

$$A = \{1,2,3\}$$
$$B = \{3,4,5,6\}$$

$$A \cap B = \{3\}$$

Set Complement A'

This is also denoted by $\overline{A}$

- "A complement," or "not A" is the set of all elements not in A.

$$\overline{\overline{A}} = A$$

$$\Omega = \{1,2,3,4,5,6\} \qquad B = \{3,4,5,6\}$$

$$\overline{B} = \{1,2\}$$

Set Difference

$$A - B$$

- The set difference "A minus B" is the set of elements that are in A, with those that are in B subtracted out. Another way of putting it is, it is the set of elements that are in A, and not in B, so

$$A - B = A \cap \overline{B}$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{3,4,5,6\}$$
$$B = \{1,2,3\}$$

$$A - B = \{4,5,6\}$$

Questions

Answer the following questions using the information given.

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

A = {2, 4, 6, 8, 10}

B = {1, 3, 7}

C = {3, 7}

Required:

a) Subsets of C

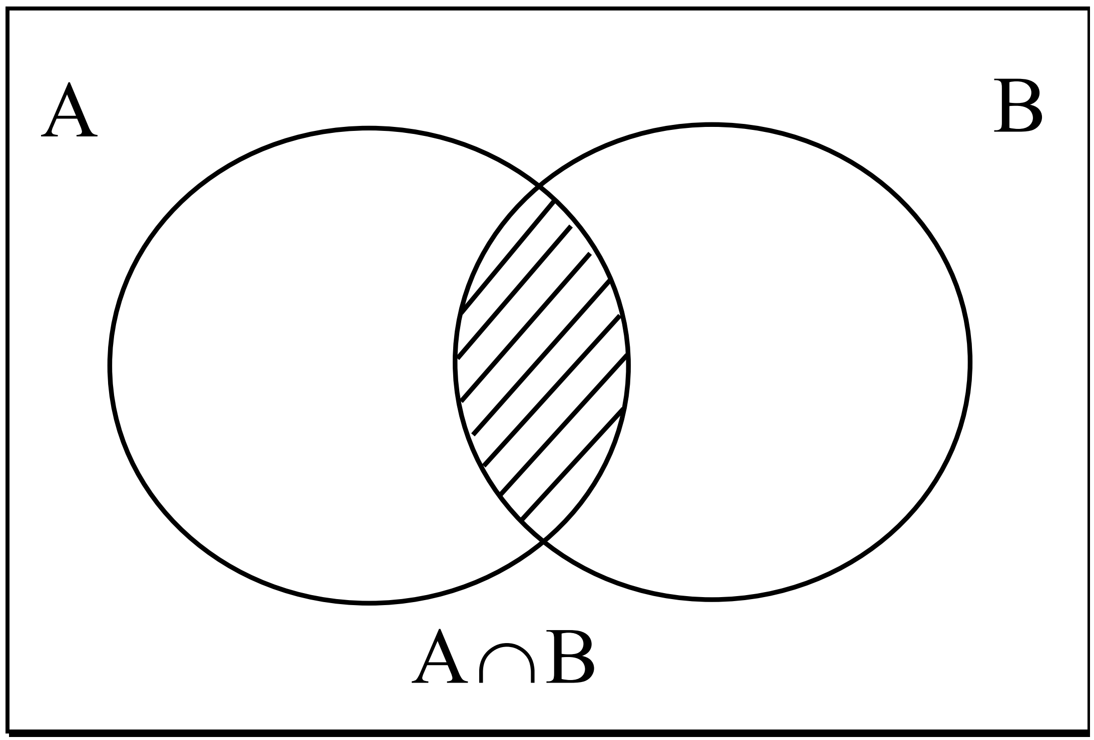
b) Proper subsets of B

c) $A \cap B$, $A \cup C$, A' , B' ,

$B \cap A'$, $B \cap C'$, $A - B$

Venn Diagrams

- Venn Diagrams use topological areas to stand for sets. I've done this one for you.



Exercise - Venn Diagrams

Identify the regions in separate Venn diagrams for following:

a. $A \cup B$

b. A’ (also symbolized by $\overline{\mathrm{A}}$)

c. B’ (also symbolized by $\overline{\mathrm{B}}$)

d. A - B

Venn Diagrams

- Shade the identified region in the following expressions

$$\overline{A} \cap \overline{B}$$

$$\overline{A \cup B}$$

$$\overline{A \cap B}$$

$$\overline{A} \cup \overline{B}$$

De Morgan's Law for Sets

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Let's use Venn Diagrams to verify DeMorgan's law

De Morgan's Law for Sets

$$\overline{(A \cup B) \cap C} = (\overline{A} \cap \overline{B}) \cup \overline{C}$$

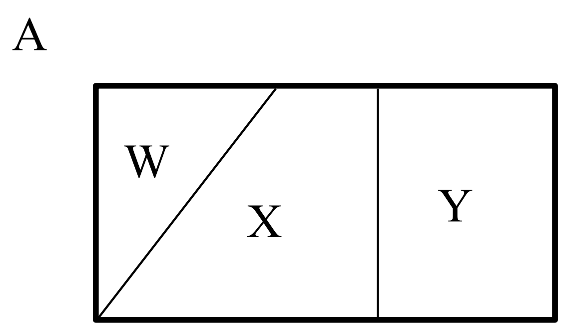
Mutually Exclusive and Exhaustive Sets

- Definition. We say that a group of sets is exhaustive of another set if their union is equal to that set. For example, if $A \cup B = C$ we say that A and B are exhaustive with respect to C.

- Definition. We say that two sets A and B are mutually exclusive if $A \cap B = \varnothing$,that is, the sets have no elements in common.

Set Partition

- Definition. We say that a group of sets partitions another set if they are mutually exclusive and exhaustive with respect to that set. When we "partition a set," we break it down into mutually exclusive and exhaustive regions, i.e., regions with no overlap. The Venn diagram below should help you get the picture. In this diagram, the set A (the rectangle) is partitioned into sets W,X, and Y.



Some Test Questions

$$\mathbf{A} \cup \varnothing = ?$$

Some Test Questions

$$A \cup \bar{A} = ?$$

Some Test Questions

$$A \cap \bar{A} = ?$$

Some Test Questions

$$\mathrm{A} \cup \Omega = ?$$

Some Test Questions

$$\mathrm{A} \cap \Omega = ?$$

Some Test Questions

$$\text{If } A \subset B \text{ then}$$

$$A \cap B = ?$$

Some Test Questions

If $A \subset B$ then

$$A \cup B = ?$$

Cardinality and Finiteness

- $|A|$ (read "the cardinality of set A") is a measure of how many different elements A has. (**Same as n(A)**)
- E.g.,

$$|\varnothing| = 0,$$

$$|\{1,2,3\}| = 3,$$

$$|\{a,b\}| = 2,$$

$$|\{\{1,2,3\},\{4,5\}\}| = 2$$

- We say A is infinite if it is not finite.
- What are some infinite sets we've seen?

$$\mathbb{N}\ \mathbb{Z}\ \mathbb{R}$$

The Power Set Operation

- The power set P(A) of set A is the set of all subsets of A.

  $P(A) = \{x \mid x \subseteq A\}$.

- E.g. If A={a,b } then $P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.

- **Cardinality of power set -P(A) is given by:**

  $$|P(A)| = 2^{|A|} \quad \text{(for finite A)}$$

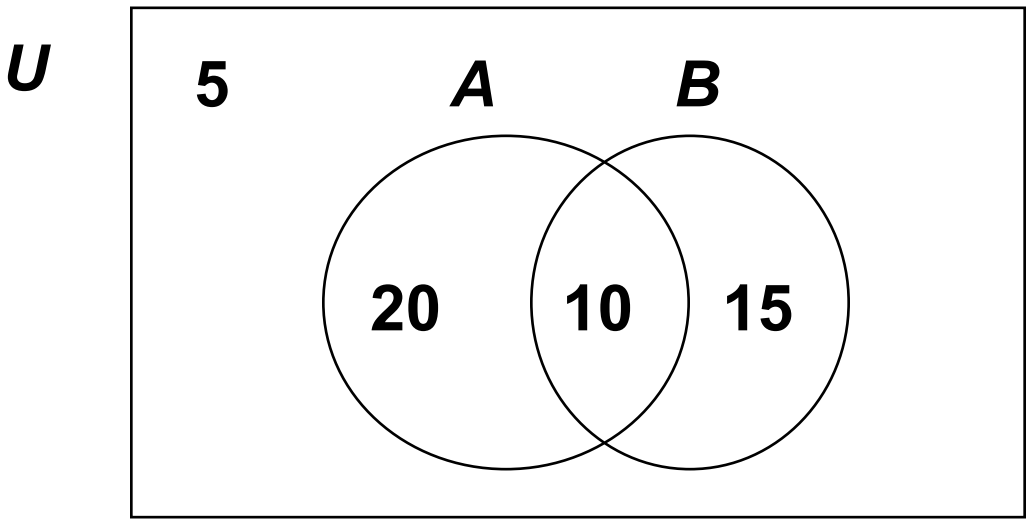
- It turns out that $|P(A)| > |A|$.

Question

1. If A={a,b,c}, how many subsets can be formed ? What are they ?
2. How many proper subsets exist? What are they?
3. Identify the P(A) (or power set A)?
4. What is the cardinality of A?
5. Identify the cardinality of P(A) (or power set A)

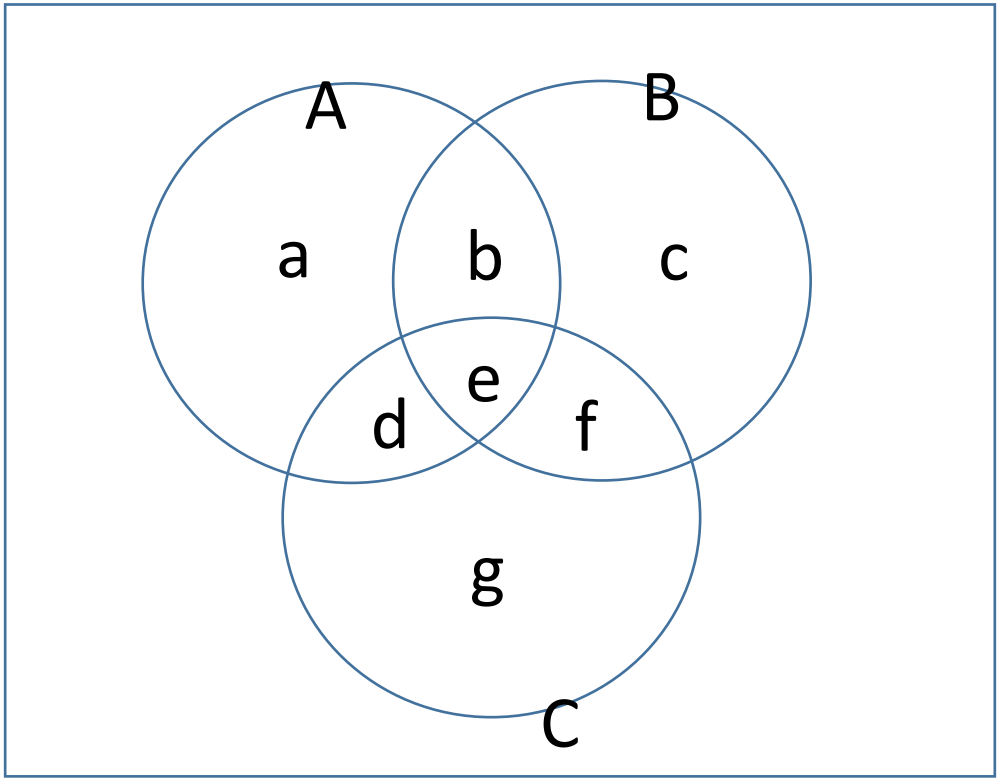
Counting and Venn Diagrams

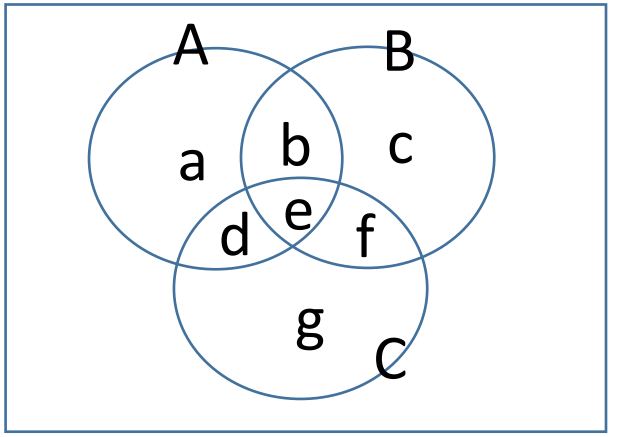
Example 1: In a class of 50 college freshmen, 30 are studying BASIC, 25 are studying PASCAL, and 10 are studying both. How many freshmen are studying either computer language?



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting and Venn Diagrams (Con.)





- $|A| = a + b + d + e$
- $|B| = b + c + e + f$
- $|C| = d + e + f + g$

$$|A| + |B| + |C| = a + c + g + 2b + 2d + 2f + 3e$$

- $|A \cap B| = b + e$
- $|B \cap C| = e + f$
- $|A \cap C| = d + e$

$$-$$

$$|A \cap B| + |B \cap C| + |\mathrm{A} \cap C| = \mathrm{b} + \mathrm{d} + \mathrm{f} + 3\mathrm{e}$$

$$+$$

- $|A \cap B \cap C| = \mathrm{e}$

$$|A \cap B \cap C| = \mathrm{e}$$

$$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |\mathrm{A} \cap C| + |A \cap B \cap C| = a + b + c + d + e + f + g$$
$$= |A \cup B \cup C|$$

Example 2: Defect types of a product:

D1: Level 1 Defect

D2: Level 2 Defect

D3: Level 3 Defect

Given 100 samples

set A: with D1

set B: with D2

set C: with D3

with |A|=23, |B|=26, |C|=30,

$|A \cap B| = 7, |A \cap C| = 8, |B \cap C| = 10,$

$|A \cap B \cap C| = 3$, how many samples have defects?

Counting and Venn Diagrams (Con.)

Given 100 samples

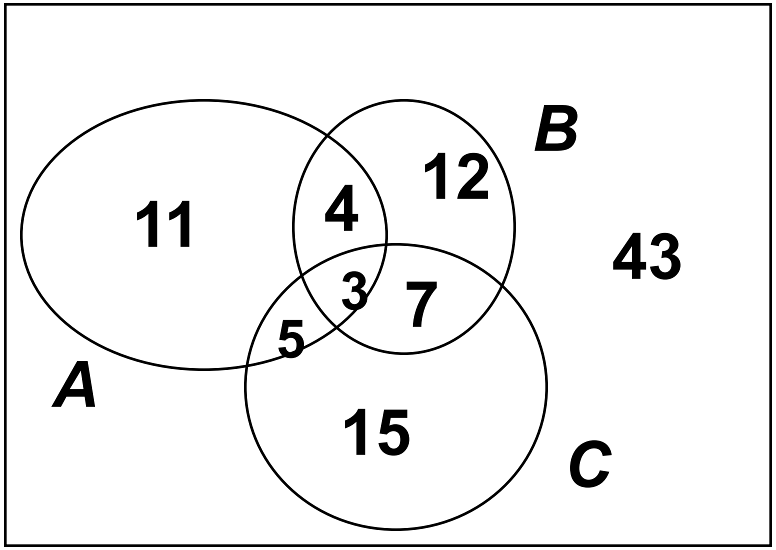
set A: with D1

set B: with D2

set C: with D3

with |A|=23, |B|=26, |C|=30,

$$|A \cap B| = 7, |A \cap C| = 8, |B \cap C| = 10,$$
$$|A \cap B \cap C| = 3$$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$$
$$- |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

how many samples have defects?

Ans:57

Questions on Venn Diagrams

1.1 In a class there are 30 students. 21 students like Mathematics, 16 like English, 6 students do not like Mathematics or English. How many Students like both Mathematics and English?

1.2 In a class there are;

8 students who play football and hockey, 7 students who do not play football or hockey, 13 students who play hockey, 19 students who play football.

How many student are there in the class?

Question

1.3

The following data refers to random sample of 200 attending a seminar; they are categorized according to their education qualifications.

- There are 115 BSc Engineers, 100 CIMA qualified people and 70 AAT qualified people. 40 are both BSc and CIMA qualified. 50 are both CIMA and AAT qualified. 20 BSC Engineers are AAT qualified. 10 have no qualification.
- Required: Find the no of people who have all three qualifications?

Laws of the Algebra of Sets

- Idempotent laws

  $A \cup A = A$

  $A \cap A = A$

- Associative laws

  $(A \cup B) \cup C = A \cup (B \cup C)$

  $(A \cap B) \cap C = A \cap (B \cap C)$

- Commutative laws

  $A \cup B = B \cup A$

  $A \cap B = B \cap A$

Laws of the Algebra of Sets

- Distributive laws

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- Identity laws

  $A \cup \emptyset = A \qquad A \cap \Omega = A$

  $A \cup \Omega = \Omega \qquad A \cap \emptyset = \emptyset$

- Involution law

  $\bar{\bar{A}} = A$ or $A'' = A$

Laws of the Algebra of Sets

- Complement laws

  $A \cup A' = \Omega$ $\qquad$ $A \cup A' = \emptyset$

  $\Omega' = \emptyset$ $\qquad$ $\emptyset' = \Omega$

- De Morgan's laws

  $(A \cup B)' = A' \cap B'$

  $(A \cap B)' = A' \cup B'$

Relations & Functions

Outline:

- Cartesian Product
- Ordered pairs
- Arrow Diagrams
- Directed Graphs
- Functions

Cartesian Product

Definition : Given two non-empty sets A and B, the set of all ordered pairs (x, y), where $x \in A$ and $y \in B$ is called Cartesian product of A and B

symbolically, we write $A \times B$

If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$

then $A \times B = \{(1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5)\}$

and $B \times A = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$

Cartesian Products of Sets

- For sets A, B, their Cartesian product
  $A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}$.

- E.g. $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$

Find the following Cartesian Products of sets

Q1) Let A={1,2} and B={a,b,c}

i. Find A x B

ii. B x A?

iii. A x A?

Note:

-The Cartesian product deals with ordered pairs, so the order in which the sets are considered is important

- For finite A,B n(A x B) =n(A).n(B)

Find the following Cartesian Products of sets

Q2) If $A=\{a,b\}$ and $B=\{1,2,3\}$, Find $A \times B$

Q3) If $A=\{a,b\}$, $B=\{1,2\}$, $C=\{2,3\}$ Find

i. $A \times (B \cup C)$

ii. $(A \times B) \cup (A \times C)$

iii. $A \times (B \cap C)$

iv. $(A \times B) \cap (A \times C)$

Relations

- A “relation" is just a relationship between sets of information.

- Think of all the students in your class and think of their heights. The pairing of names and heights is a relation.

- In relations, the pairs of names and heights are "ordered", which means one comes first and the other comes second.

Relations - Definition

A Relation **R** from a non-empty set A to a non-empty set B is a **subset of the Cartesian product set A × B**.

The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B.

The set of all first elements in a relation R, is called the **domain** of the relation R, and the set of all second elements, is called the **range** of R.

Definition (Cont.)

- Let A and B be sets. A binary relation or simply a relation from A to B is a subset of A x B.

Suppose R is a relation from A to B. Then R is the set of ordered pairs where each first element comes from A and each second element comes from B.

Relations

Ordered Pairs

Relations will be defined in terms of ordered pairs (a,b) of elements, where a is designated as the first element and b as the second element.

$$(a,b) = (c,d) \text{ if and only if } a = c \text{ and } b = d$$

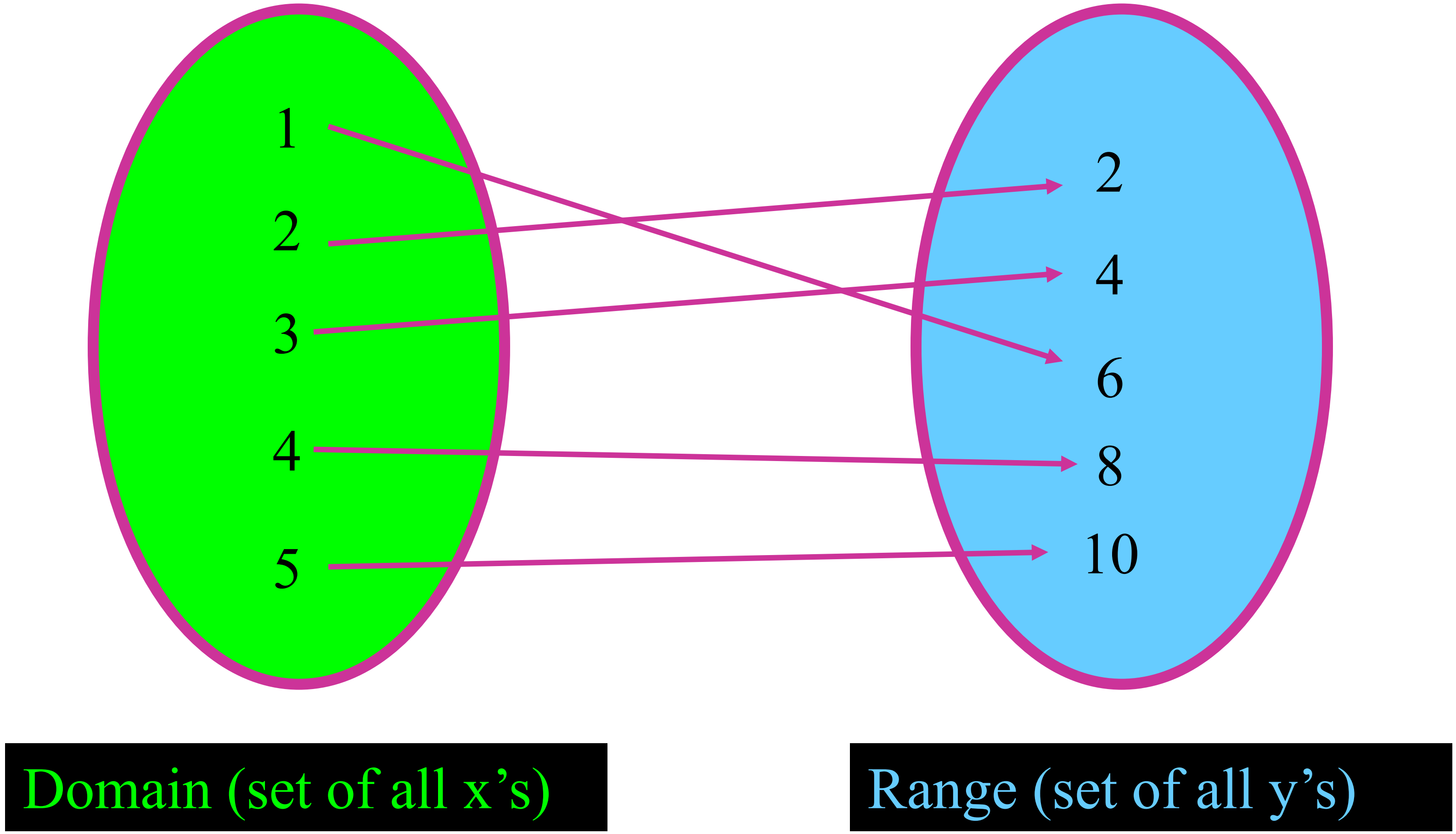
Pictorial representations of relations

- Now we look at different ways of picturing and representing binary relations.

1. Arrow Diagram:

Suppose A and B are two finite sets. Write down the elements of A and the elements of B in two disjoint disks. Then draw an arrow from $a \in A$, to $b \in B$ whenever a is related to b. This picture is called the arrow diagram of the relation

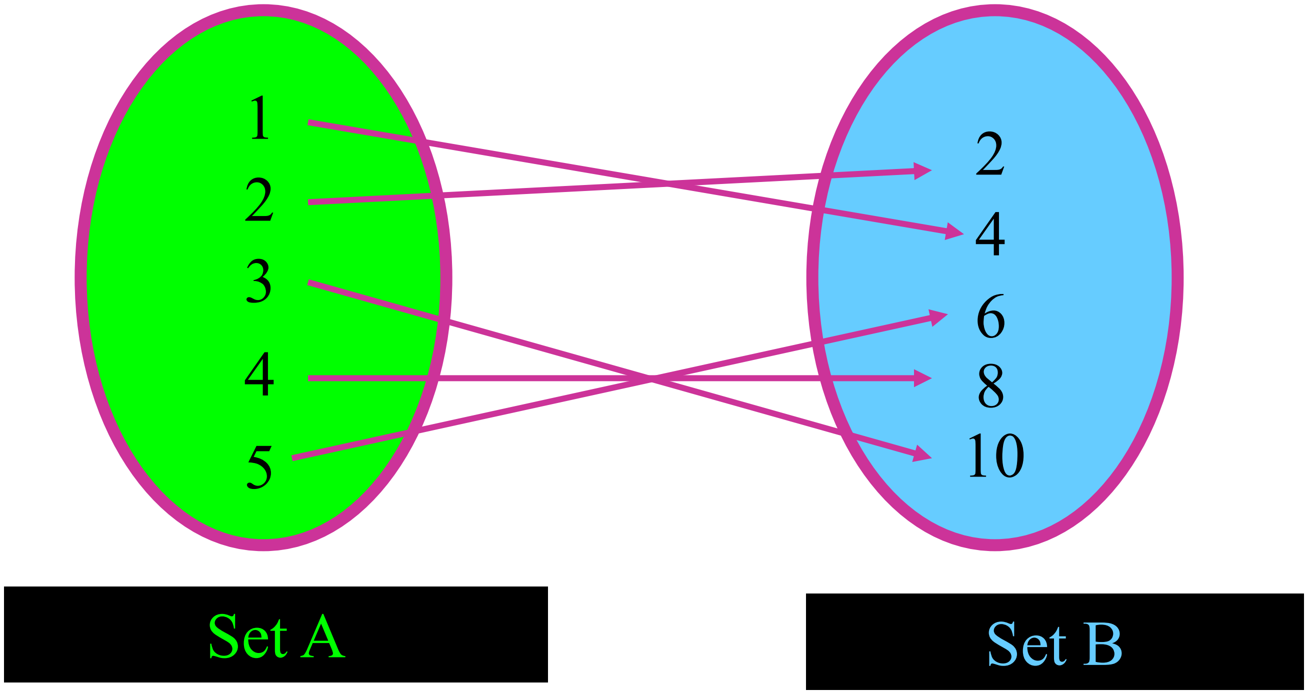
Pictorial representations of relations



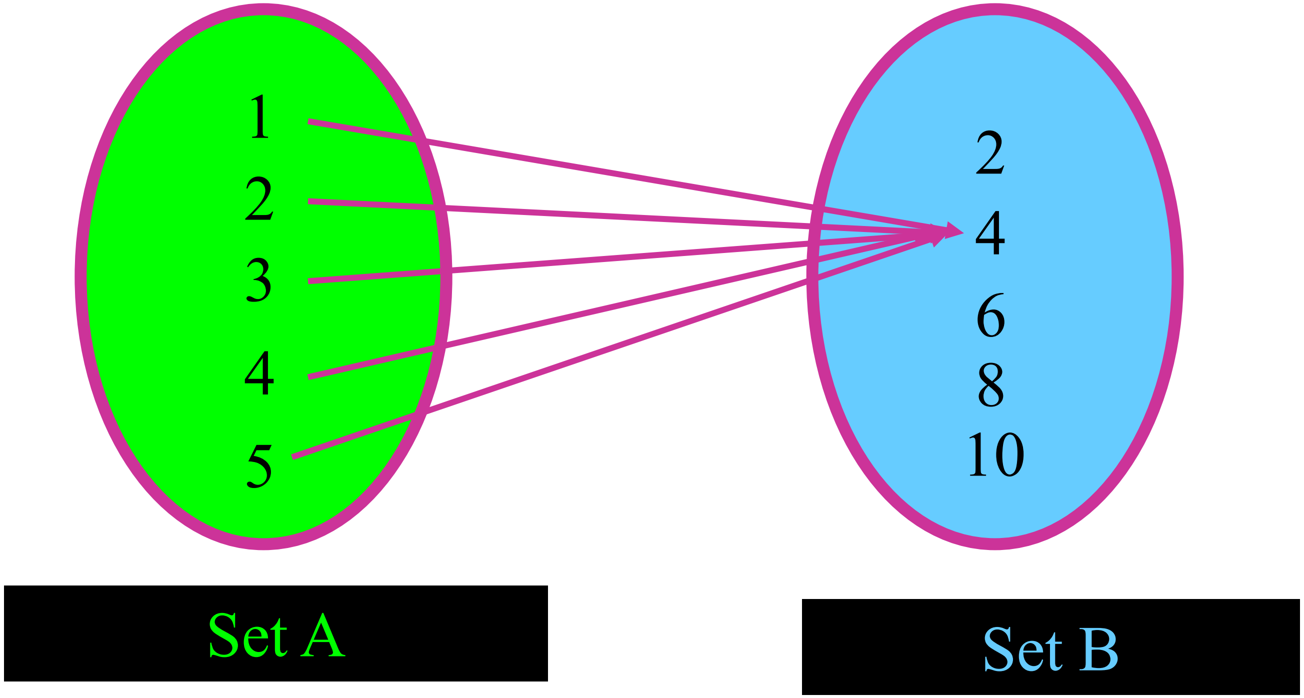
This relation can be written {(1,6), (2,2), (3,4), (4,8), (5,10)}

Q) Write the following relations in ordered pairs

1.



2.



Pictorial representations of relations

2. Directed Graphs

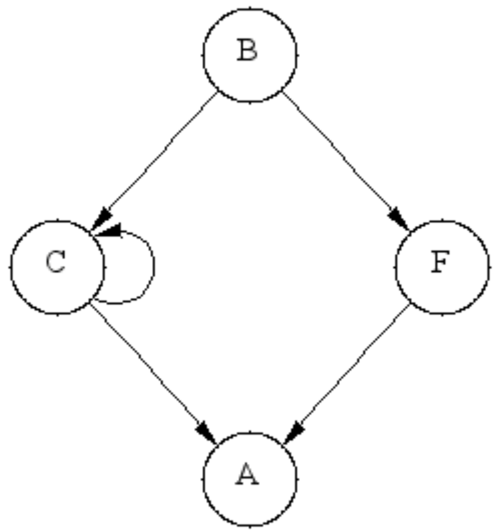
These graphs are used to picture a relation R, which is a relation from a finite set to itself.

Draw the directed graph of the relation R on the set
A ={1, 2, 3, 4},

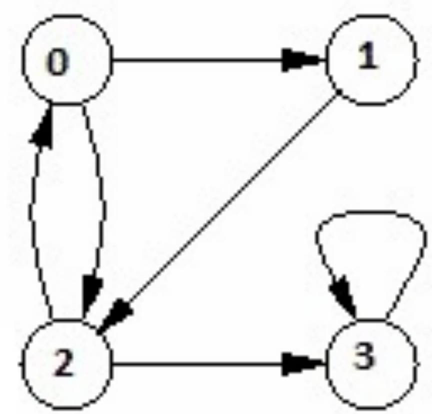
Let R={(1,2), (2,2), (2,4), (3,2), (3,4), (4,1), (4,3)}

Find the relations of the following directed graphs

1.

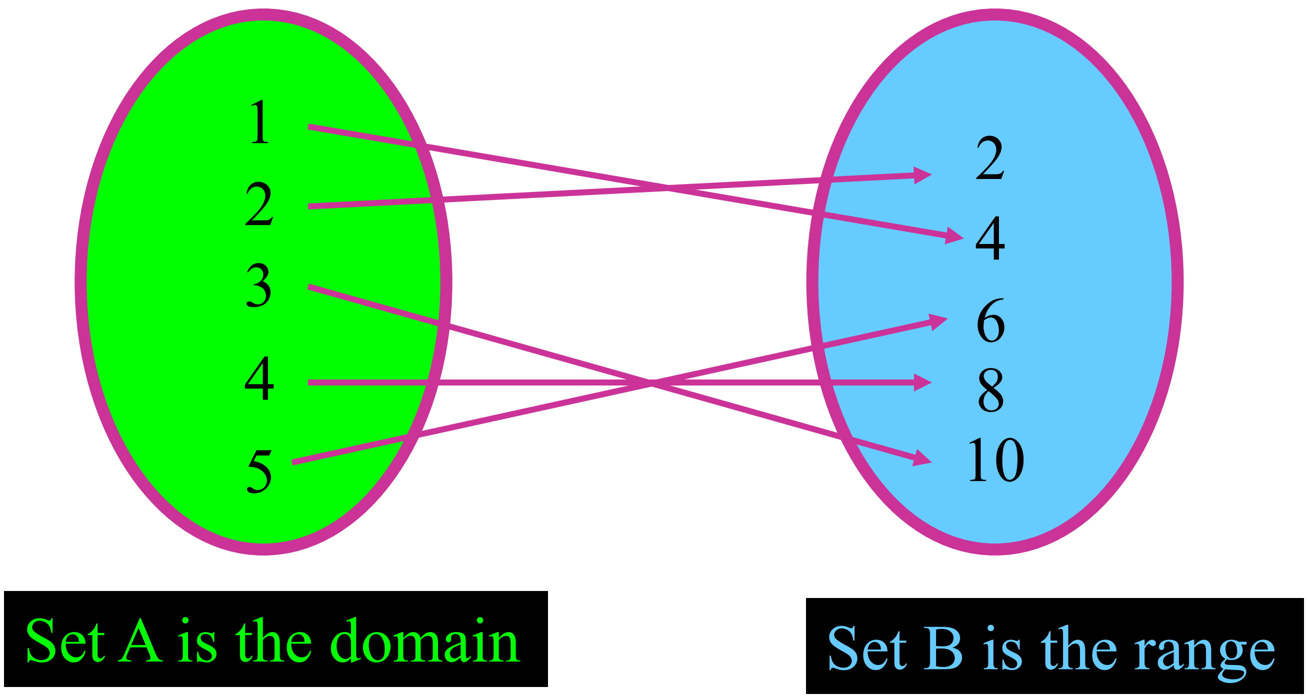


2.



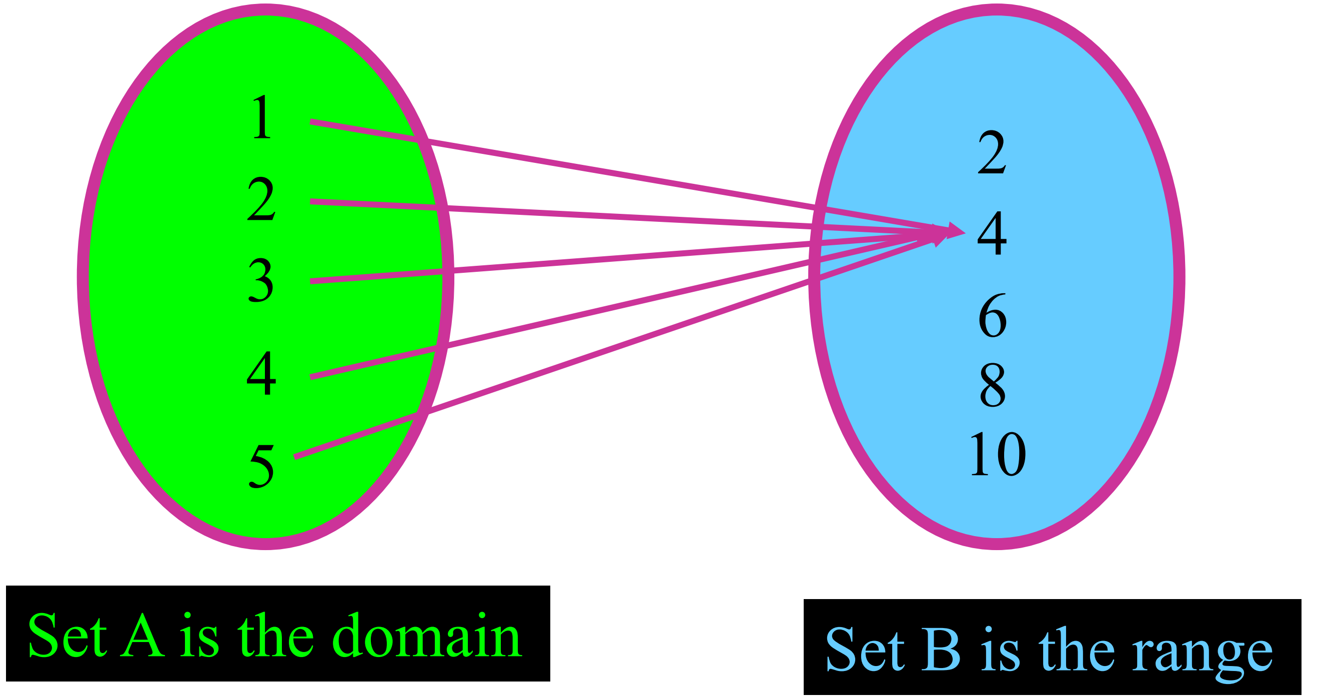
Functions

- A function f is a **relation** in which each element of the domain is paired with exactly one element of the range.



Functions

Is the following relation a function?

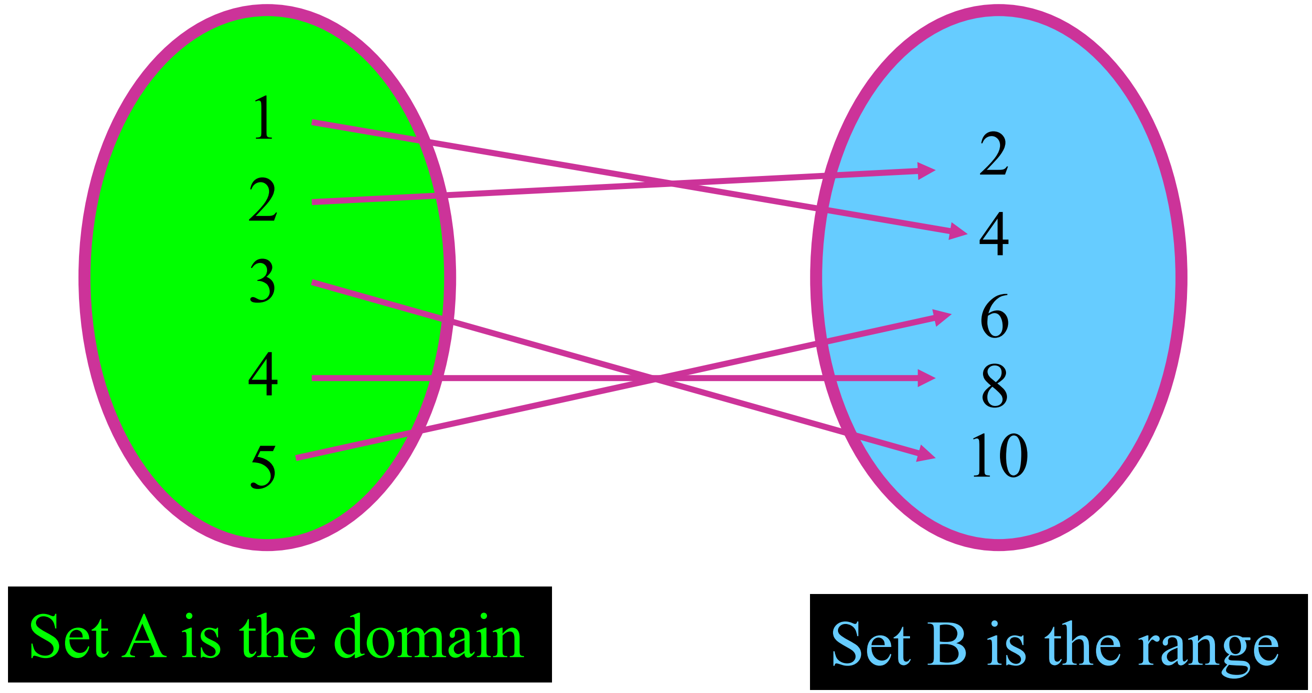


Must use all the x's

The x value can only be assigned to one y

Functions

Is the following relation a function?



Functions

Is the following relation a function?