

Outline

- 1 Conceptual Exercises
- 2 Summary of Naive Bayes
- 3 Examples of Naive Bayes
- 4 Handling Missing Data
- 5 Naive Bayes on the Heart Disease Data

More Naive Bayes

Data Mining
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Estimating Conditional Probabilities

- Say the joint counts table for the random variables Z and W in a data set is:

	$W = a$	$W = b$	$W = c$
$Z = 1$	25	13	31
$Z = 2$	73	41	102

Using these data, estimate the conditional distribution of W given that $Z = 2$. Estimate the (marginal) probability distribution of W . Estimate the joint probability that $Z = 1$ and $W = c$. Are Z and W approximately independent?

- A. Yes
- B. No

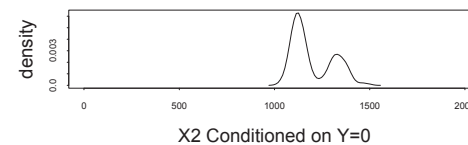
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Classification

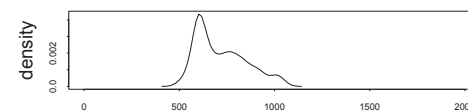
Exploration of some training data gives you the following information: The probability that $Y = 0$ is approximately 52%, and otherwise $Y = 1$. The distribution of X_1 conditional on Y is approximately:

	$X_1 = 1$	$X_1 = 2$
$Y = 0$	0.27	0.73
$Y = 1$	0.65	0.35

The distribution of X_2 conditional on Y is approximately:



X_2 Conditioned on $Y=0$



X_2 Conditioned on $Y=1$

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Classification

Say that we want to create a “classification rule” that predicts Y accurately based only on a single predictor variable (either X_1 or X_2). Which predictor variable would you use, to get the best accuracy?

- A. X_1
- B. X_2

What would be your classification rule? E.g., X_2 is the predictor variable, and the rule is to predict $Y = 1$ when $X_2 \leq 500$ and $Y = 0$ when $X_2 > 500$.

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Naive Bayes Training / Prediction

There are 2 steps to a supervised learning method:

- 1 **TRAINING:** Learn a decision rule (rule for prediction) using a training data set, in which both the predictors and outcome are observed
 -
- 2 **PREDICTION:** For new observations, predict the value of the outcome variable given the predictors.

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Example

Naive Bayes Training:

- For each predictor k estimate the conditional probability table $\Pr(X_k|Y)$ to be equal to the conditional frequency table from the training data, called $\hat{\Pr}(X_k|Y)$
- Estimate $\Pr(Y)$ to be equal to the frequencies from the training data, called $\hat{\Pr}(Y)$

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Example

Naive Bayes Prediction:

- To predict the probability that $Y = y$, given that $X_1 = x_1, \dots, X_K = x_K$ we use the formula:

$$\begin{aligned} \Pr(Y = y | X_1 = x_1, \dots, X_K = x_K) \\ = \frac{\hat{\Pr}(Y = y) \left[\prod_{k=1}^K \hat{\Pr}(X_k = x_k | Y = y) \right]}{\sum_{y'} \hat{\Pr}(Y = y') \left[\prod_{k=1}^K \hat{\Pr}(X_k = x_k | Y = y') \right]} \end{aligned}$$

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Example

For the CAD example, say we calculate

$$Pr(CAD = \text{Yes}) \left[\prod_{k=1}^K Pr(X_k = x_k | CAD = \text{Yes}) \right] = 0.005$$

and

$$Pr(CAD = \text{No}) \left[\prod_{k=1}^K Pr(X_k = x_k | CAD = \text{No}) \right] = 0.01$$

What is the conditional probability that CAD = Yes, given the predictor values?

- A. In the interval [0, .25)
- B. In the interval [.25, .5)
- C. In the interval [.5, .75)
- D. In the interval [.75, 1]

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Example

- Example: Y is CAD, predictors X_1 is Sex and X_2 is Exercise Induced Angina

- Estimate $Pr(X_1 | Y)$ to be equal to the table from the data:

	Sex = Fem	Sex = Mal
CAD = N	0.45	0.55
CAD = Y	0.17	0.83

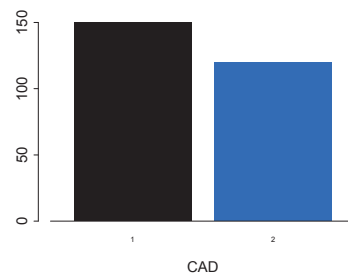
- Estimate $Pr(X_2 | Y)$ to be equal to the table from the data:

	EIA = N	EIA = Y
CAD = N	0.85	0.15
CAD = Y	0.45	0.55

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Example

- Estimate $Pr(Y)$ to be equal to the frequencies from the data:



- $Pr(Y = \text{Yes}) = 0.56$ and $Pr(Y = \text{No}) = 0.44$

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Example

- What is the chance that a female with EIA has CAD?

- A. In the interval [0, .25)
- B. In the interval [.25, .5)
- C. In the interval [.5, .75)
- D. In the interval [.75, 1]

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Example

- We need to predict CAD or no CAD for such a patient. What do we pick?
- Typically we predict CAD if $Pr(Y = \text{Yes} | X_1, \dots, X_K) > 0.5$ and no CAD otherwise
- I.e. we use the **threshold 0.5**. This is the default choice.
- But we know that not sending a CAD patient to angiography is a more costly mistake than sending a non-CAD patient to angiography
- So maybe we want to use a different threshold (lower or higher?)
- We will discuss this next time

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Summary

Summary

Naive Bayes is a model-based method. What is the statistical model (assumptions)? What are the parameters of the model? How many parameters are there for the simple CAD example on the previous slides?

- 1-4
- 5-8
- 9-12
- 13-16

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Naive Bayes

For a different problem our training data look like: What are the estimates of $Pr(Y)$, $Pr(X_1|Y)$, and $Pr(X_2|Y)$ for naive Bayes using these data?

X_1	X_2	Y
a	0	1
a	0	0
b	0	1
b	0	1
a	1	1
b	0	0
a	1	1
a	0	0
b	1	0
b	1	0
b	0	1

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Naive Bayes

Estimate the probability that $Y = 0$, given that $X_1 = b$ and $X_2 = 0$.

Predict Y , given that $X_1 = b$ and $X_2 = 0$.

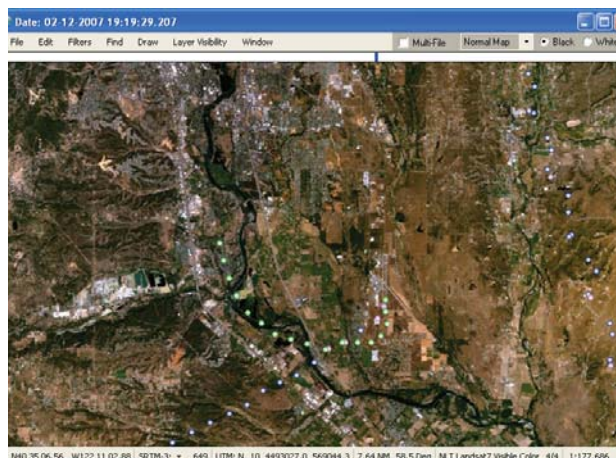
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Naive Bayes

- Give an example where the naive Bayes assumption is reasonable, i.e., a prediction problem where this assumption would approximately hold.
- Give an example where the naive Bayes assumption definitely would not hold.

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Naive Bayes



This assumption does not hold when you are trying to predict the type of aircraft based on its location at a series of time points.

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Naive Bayes

In naive Bayes the outcome Y does not have to be a binary variable, as in the CAD example. It can take any finite # of possible values. Examples:

-
-

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Missing Data

Missing Data in Naive Bayes

- How do we do prediction using naive Bayes if we are missing a value for one of the predictors?

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Missing Data

- What do we do if one of the records (rows) in the **test** data is missing a value for one of the predictors?
- When doing classification for that record we simply do not condition on that predictor.
- If we are missing the first predictor X_1 , instead of calculating

$$Pr(Y|X_1, \dots, X_K)$$

to do classification, we calculate

$$Pr(Y|X_2, \dots, X_K)$$

- How do we calculate this?

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Missing Data

Calculation:

For any possible y, x_2, \dots, x_K ,

$$Pr(Y = y | X_2 = x_2, \dots, X_K = x_K)$$

=

$$= \frac{Pr(Y = y) \prod_{k=2}^K Pr(X_k = x_k | Y = y)}{\sum_{y'} Pr(Y = y') \prod_{k=2}^K Pr(X_k = x_k | Y = y')}$$

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Missing Data

- How do we do prediction if there is more than one predictor missing?
- Same: Just leave those predictors out of the product term when calculating the conditional prob. that $Y = y$.

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Missing Data

- What do we do if one of the records in the **training** data has a missing value for one of the predictors (say, X_1)?

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Missing Data

- When estimating $Pr(X_1|Y)$, we just calculate as if that data point was not in the data set
- How do we change our estimation of $Pr(X_k|Y)$ for $k \neq 1$?

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Missing Data

- How do we change our estimation of $Pr(X_k|Y)$ for $k \neq 1$?
- It does not change. Use the whole data set as usual.

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Missing Data

Say our training dataset looks like:

X_1	X_2	Y
a	1	1
a	0	0
b	0	1
b	0	1
a	1	1
b	1	0
?	1	1
?	0	0
b	1	0
b	1	0
b	0	1

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Missing Data

What are the (frequentist) estimates of $Pr(Y)$, $Pr(X_1|Y)$, and $Pr(X_2|Y)$ for naive Bayes?

X_1	X_2	Y
a	1	1
a	0	0
b	0	1
b	0	1
a	1	1
b	1	0
?	1	1
?	0	0
b	1	0
b	1	0
b	0	1

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Missing Data

- Estimate the probability that $Y = 1$, given that $X_1 = b$ and $X_2 = 0$.
- Using threshold 0.5, predict the value of Y , knowing that $X_1 = b$ and $X_2 = 0$.

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Missing Data

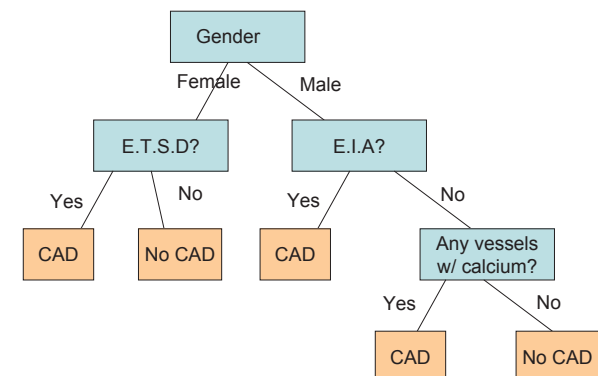
- Estimate the probability that $Y = 1$, given that $X_2 = 0$ and X_1 is missing.

- Using threshold 0.4, predict the value of Y , knowing that $X_2 = 0$ and X_1 is missing.

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Naive Bayes on the Heart Disease Data

- Recall that we constructed a classification tree for heart disease using exploratory data analysis:



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Naive Bayes on the Heart Disease Data

- This tree had 80% accuracy on the training data
- I never told you the accuracy on the test data

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Naive Bayes on the Heart Disease Data

The training data (from Cleveland):

```
63.0,1.0,1.0,145.0,233.0,1.0,2.0,150.0,0.0,2.3,3.0,0.0,6.0,0
67.0,1.0,4.0,160.0,286.0,0.0,2.0,108.0,1.0,1.5,2.0,3.0,3.0,2
67.0,1.0,4.0,120.0,229.0,0.0,2.0,129.0,1.0,2.6,2.0,2.0,7.0,1
37.0,1.0,3.0,130.0,250.0,0.0,0.0,187.0,0.0,3.5,3.0,0.0,3.0,0
41.0,0.0,2.0,130.0,204.0,0.0,2.0,172.0,0.0,1.4,1.0,0.0,3.0,0
56.0,1.0,2.0,120.0,236.0,0.0,0.0,178.0,0.0,0.8,1.0,0.0,3.0,0
62.0,0.0,4.0,140.0,268.0,0.0,2.0,160.0,0.0,3.6,3.0,2.0,3.0,3
57.0,0.0,4.0,120.0,354.0,0.0,0.0,163.0,1.0,0.6,1.0,0.0,3.0,0
63.0,1.0,4.0,130.0,254.0,0.0,2.0,147.0,0.0,1.4,2.0,1.0,7.0,2
53.0,1.0,4.0,140.0,203.0,1.0,2.0,155.0,1.0,3.1,3.0,0.0,7.0,1
57.0,1.0,4.0,140.0,192.0,0.0,0.0,148.0,0.0,0.4,2.0,0.0,6.0,0
56.0,0.0,2.0,140.0,294.0,0.0,2.0,153.0,0.0,1.3,2.0,0.0,3.0,0
56.0,1.0,3.0,130.0,256.0,1.0,2.0,142.0,1.0,0.6,2.0,1.0,6.0,2
44.0,1.0,2.0,120.0,263.0,0.0,0.0,173.0,0.0,0.0,1.0,0.0,7.0,0
52.0,1.0,3.0,172.0,188.0,1.0,0.0,162.0,0.0,0.5,1.0,0.0,7.0,0
```

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Naive Bayes on the Heart Disease Data

The last column is heart disease status (0-3)

```
63.0,1.0,1.0,145.0,233.0,1.0,2.0,150.0,0.0,2.3,3.0,0.0,6.0,0
67.0,1.0,4.0,160.0,286.0,0.0,2.0,108.0,1.0,1.5,2.0,3.0,3.0,2
67.0,1.0,4.0,120.0,229.0,0.0,2.0,129.0,1.0,2.6,2.0,2.0,7.0,1
37.0,1.0,3.0,130.0,250.0,0.0,0.0,187.0,0.0,3.5,3.0,0.0,3.0,0
41.0,0.0,2.0,130.0,204.0,0.0,2.0,172.0,0.0,1.4,1.0,0.0,3.0,0
56.0,1.0,2.0,120.0,236.0,0.0,0.0,178.0,0.0,0.8,1.0,0.0,3.0,0
62.0,0.0,4.0,140.0,268.0,0.0,2.0,160.0,0.0,3.6,3.0,2.0,3.0,3
57.0,0.0,4.0,120.0,354.0,0.0,0.0,163.0,1.0,0.6,1.0,0.0,3.0,0
63.0,1.0,4.0,130.0,254.0,0.0,2.0,147.0,0.0,1.4,2.0,1.0,7.0,2
53.0,1.0,4.0,140.0,203.0,1.0,2.0,155.0,1.0,3.1,3.0,0.0,7.0,1
57.0,1.0,4.0,140.0,192.0,0.0,0.0,148.0,0.0,0.4,2.0,0.0,6.0,0
56.0,0.0,2.0,140.0,294.0,0.0,2.0,153.0,0.0,1.3,2.0,0.0,3.0,0
56.0,1.0,3.0,130.0,256.0,1.0,2.0,142.0,1.0,0.6,2.0,1.0,6.0,2
44.0,1.0,2.0,120.0,263.0,0.0,0.0,173.0,0.0,0.0,1.0,0.0,7.0,0
52.0,1.0,3.0,172.0,188.0,1.0,0.0,162.0,0.0,0.5,1.0,0.0,7.0,0
```

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Naive Bayes on the Heart Disease Data

We just predicted heart disease present (1-3) or absent (0)

```
63.0,1.0,1.0,145.0,233.0,1.0,2.0,150.0,0.0,2.3,3.0,0.0,6.0,0
67.0,1.0,4.0,160.0,286.0,0.0,2.0,108.0,1.0,1.5,2.0,3.0,3.0,2
67.0,1.0,4.0,120.0,229.0,0.0,2.0,129.0,1.0,2.6,2.0,2.0,7.0,1
37.0,1.0,3.0,130.0,250.0,0.0,0.0,187.0,0.0,3.5,3.0,0.0,3.0,0
41.0,0.0,2.0,130.0,204.0,0.0,2.0,172.0,0.0,1.4,1.0,0.0,3.0,0
56.0,1.0,2.0,120.0,236.0,0.0,0.0,178.0,0.0,0.8,1.0,0.0,3.0,0
62.0,0.0,4.0,140.0,268.0,0.0,2.0,160.0,0.0,3.6,3.0,2.0,3.0,3
57.0,0.0,4.0,120.0,354.0,0.0,0.0,163.0,1.0,0.6,1.0,0.0,3.0,0
63.0,1.0,4.0,130.0,254.0,0.0,2.0,147.0,0.0,1.4,2.0,1.0,7.0,2
53.0,1.0,4.0,140.0,203.0,1.0,2.0,155.0,1.0,3.1,3.0,0.0,7.0,1
57.0,1.0,4.0,140.0,192.0,0.0,0.0,148.0,0.0,0.4,2.0,0.0,6.0,0
56.0,0.0,2.0,140.0,294.0,0.0,2.0,153.0,0.0,1.3,2.0,0.0,3.0,0
56.0,1.0,3.0,130.0,256.0,1.0,2.0,142.0,1.0,0.6,2.0,1.0,6.0,2
44.0,1.0,2.0,120.0,263.0,0.0,0.0,173.0,0.0,0.0,1.0,0.0,7.0,0
52.0,1.0,3.0,172.0,188.0,1.0,0.0,162.0,0.0,0.5,1.0,0.0,7.0,0
```

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Naive Bayes on the Heart Disease Data

The other columns are age, gender (0/1), E.I.A. (0/1), etc.

```
63.0,1.0,1.0,145.0,233.0,1.0,2.0,150.0,0.0,2.3,3.0,0.0,6.0,0
67.0,1.0,4.0,160.0,286.0,0.0,2.0,108.0,1.0,1.5,2.0,3.0,3.0,2
67.0,1.0,4.0,120.0,229.0,0.0,2.0,129.0,1.0,2.6,2.0,2.0,7.0,1
37.0,1.0,3.0,130.0,250.0,0.0,0.0,187.0,0.0,3.5,3.0,0.0,3.0,0
41.0,0.0,2.0,130.0,204.0,0.0,2.0,172.0,0.0,1.4,1.0,0.0,3.0,0
56.0,1.0,2.0,120.0,236.0,0.0,0.0,178.0,0.0,0.8,1.0,0.0,3.0,0
62.0,0.0,4.0,140.0,268.0,0.0,2.0,160.0,0.0,3.6,3.0,2.0,3.0,3
57.0,0.0,4.0,120.0,354.0,0.0,0.0,163.0,1.0,0.6,1.0,0.0,3.0,0
63.0,1.0,4.0,130.0,254.0,0.0,2.0,147.0,0.0,1.4,2.0,1.0,7.0,2
53.0,1.0,4.0,140.0,203.0,1.0,2.0,155.0,1.0,3.1,3.0,0.0,7.0,1
57.0,1.0,4.0,140.0,192.0,0.0,0.0,148.0,0.0,0.4,2.0,0.0,6.0,0
56.0,0.0,2.0,140.0,294.0,0.0,2.0,153.0,0.0,1.3,2.0,0.0,3.0,0
56.0,1.0,3.0,130.0,256.0,1.0,2.0,142.0,1.0,0.6,2.0,1.0,6.0,2
44.0,1.0,2.0,120.0,263.0,0.0,0.0,173.0,0.0,0.0,1.0,0.0,7.0,0
52.0,1.0,3.0,172.0,188.0,1.0,0.0,162.0,0.0,0.5,1.0,0.0,7.0,0
```

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Naive Bayes on the Heart Disease Data

The test data (from California):

```
63,1,4,140,260,0,1,112,1,3,2,?,?,2
44,1,4,130,209,0,1,127,0,0,?,?,?,0
60,1,4,132,218,0,1,140,1,1.5,3,?,?,2
55,1,4,142,228,0,1,149,1,2.5,1,?,?,1
66,1,3,110,213,1,2,99,1,1.3,2,?,?,0
66,1,3,120,0,0,1,120,0,-0.5,1,?,?,0
65,1,4,150,236,1,1,105,1,0,?,?,?,3
60,1,3,180,0,0,1,140,1,1.5,2,?,?,0
60,1,3,120,0,?,0,141,1,2,1,?,?,3
60,1,2,160,267,1,1,157,0,0.5,2,?,?,1
56,1,2,126,166,0,1,140,0,0,?,?,?,0
59,1,4,140,0,0,1,117,1,1,2,?,?,1
62,1,4,110,0,0,0,120,1,0.5,2,?,?,3,1
63,1,3,?,0,0,2,?,?,?,?,?,1
57,1,4,138,0,1,1,148,1,1,2,?,?,1
```

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Naive Bayes on the Heart Disease Data

The last column is the heart disease status (0-3)

```
63,1,4,140,260,0,1,112,1,3,2,?,?,2
44,1,4,130,209,0,1,127,0,0,?,?,?,0
60,1,4,132,218,0,1,140,1,1.5,3,?,?,2
55,1,4,142,228,0,1,149,1,2.5,1,?,?,1
66,1,3,110,213,1,2,99,1,1.3,2,?,?,0
66,1,3,120,0,0,1,120,0,-0.5,1,?,?,0
65,1,4,150,236,1,1,105,1,0,?,?,?,3
60,1,3,180,0,0,1,140,1,1.5,2,?,?,0
60,1,3,120,0,?,0,141,1,2,1,?,?,3
60,1,2,160,267,1,1,157,0,0.5,2,?,?,1
56,1,2,126,166,0,1,140,0,0,?,?,?,0
59,1,4,140,0,0,1,117,1,1,2,?,?,1
62,1,4,110,0,0,0,120,1,0.5,2,?,?,3,1
63,1,3,?,0,0,2,?,?,?,?,?,1
57,1,4,138,0,1,1,148,1,1,2,?,?,1
```

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Naive Bayes on the Heart Disease Data

The test data has a lot of missing values for the predictors!

```
63,1,4,140,260,0,1,112,1,3,2,?,?,2
44,1,4,130,209,0,1,127,0,0,?,?,?,0
60,1,4,132,218,0,1,140,1,1.5,3,?,?,2
55,1,4,142,228,0,1,149,1,2.5,1,?,?,1
66,1,3,110,213,1,2,99,1,1.3,2,?,?,0
66,1,3,120,0,0,1,120,0,-0.5,1,?,?,0
65,1,4,150,236,1,1,105,1,0,?,?,?,3
60,1,3,180,0,0,1,140,1,1.5,2,?,?,0
60,1,3,120,0,?,0,141,1,2,1,?,?,3
60,1,2,160,267,1,1,157,0,0.5,2,?,?,1
56,1,2,126,166,0,1,140,0,0,?,?,?,0
59,1,4,140,0,0,1,117,1,1,2,?,?,1
62,1,4,110,0,0,0,120,1,0.5,2,?,?,3,1
63,1,3,?,0,0,2,?,?,?,?,?,1
57,1,4,138,0,1,1,148,1,1,2,?,?,1
```

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Naive Bayes on the Heart Disease Data

- We can't use our classification tree when we are missing the value of one of the predictors that is used in the tree.
- In fact, the original heart disease training data (303 patients) had missing data as well
- The missing data had been removed in the data set that we used in Lecture 3, leaving 270 patients.
- But now we can fit a naive Bayes classifier to the full training data, and predict on the test data, regardless of the missing values!

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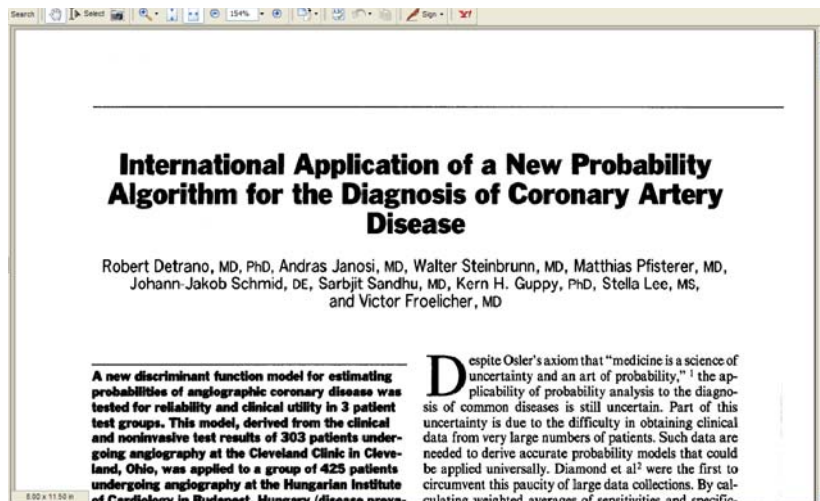
Naive Bayes on the Heart Disease Data

- Applying naive Bayes to the heart disease data gives a 75% accuracy on the test data.

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Naive Bayes on the Heart Disease Data

- Recall the article on heart disease prediction:



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Naive Bayes on the Heart Disease Data

- The authors report an accuracy of about 77% for their methods on the same test data.
- Our classifier, despite being "naive", does almost as well!

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