A. Differential Calculus of Two-Variable Function

▶ Def.

1. $X(x,y), A(x_0,y_0) \in \mathbb{R}^2$, Distance $\equiv \sqrt{(x-x_0)^2+(y-y_0)^2} := \|X-A\|$

2. Open circle plane centered at A radius r $= \{X \in \mathbb{R}^2 \mid \|X-A\| < r\} := B(A,r)$

   or, $B(A)$

3. $D \subseteq \mathbb{R}^2$, $A \in D \leftrightarrow B(A) \subseteq D$: $\forall A \in D \rightarrow$ Open Set

4. $D \subseteq \mathbb{R}^2$, $A \in \mathbb{R}^2 \longleftrightarrow \exists B(A) \subseteq D \wedge \exists B(A) \nsubseteq D$

   $\forall$(Bound. A) $\in D \rightarrow$ Closed set

4.1 There exists "Neither closed nor open set"

5. $D \subseteq \mathbb{R}^2$, $A \in \mathbb{R}^2$: A is $\overbrace{\text{a limit point}}^{A'}$ $\leftrightarrow \forall B(A)\left((B(A)-\{A\}) \cap D \neq \emptyset\right)$

   Including interior points. Sometimes boundary points.

   $A \notin D_1 \wedge D = D_1 \cup \{A\} \rightarrow$ A is a boundary but not limit point (outside)

6. $D \subseteq \mathbb{R}^2$, D is a bounded set $\leftrightarrow (X,Y) \in \mathbb{R}^2, (a \leq b, c \leq d) : (a \leq X \leq b \wedge c \leq Y \leq d)$

<u>Def. Limit</u> : $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2 \wedge (x_0,y_0) = D'$

$\rightarrow f(x,y) = L$ as $(x,y) \rightarrow (x_0,y_0)$; $L \in \mathbb{R}$

or $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

$\leftrightarrow (\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(|f(x,y)-L| < \varepsilon \ \forall (x,y) \in D;$

$0 < \|(x,y)-(x_0,y_0)\| < \delta$

$\equiv \forall (x,y) \in D \left(0 < \|(x,y)-(x_0,y_0)\| < \delta \rightarrow |f(x,y)-L| < \varepsilon\right)$

Example Show that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Given $(\forall \varepsilon \in \mathbb{R}^+)(\forall \delta \in \mathbb{R}^+)\left(\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon \ \forall (x,y) \in D\right)$,

$D = \{(x,y) \mid (x,y) \neq (0,0)\}$

Suppose $0 < \|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$

Check
$$\begin{aligned}
\left|\frac{3x^2y}{x^2+y^2} - 0\right| &= \frac{|3x^2y|}{|x^2+y^2|} \\
&= \frac{3x^2|y|}{x^2+y^2} \\
&\leq \frac{3(x^2+y^2)\sqrt{x^2+y^2}}{x^2+y^2} \\
&= 3\sqrt{x^2+y^2} \\
&< 3\delta
\end{aligned}$$

Choose $\delta = \frac{\varepsilon}{3}$; $\varepsilon \leq 3\delta$, $\varepsilon > 0$, $\delta > 0$

$$\begin{aligned}
\left|\frac{3x^2y}{x^2+y^2} - 0\right| &< 3\delta \\
&= 3\left(\frac{\varepsilon}{3}\right) \\
&= \varepsilon
\end{aligned}$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$ ■

<u>Def. Curve</u> $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2 \wedge (x_0, y_0) = D'$, C is a curve in $\mathbb{R}^2$ passes (x_0, y_0) : $f(x,y) = L$ as $(x,y) \to (x_0, y_0)$ along C

or $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ along C

$\leftrightarrow (\forall \varepsilon \in \mathbb{R}^+)(\forall \delta \in \mathbb{R}^+)(|f(x,y) - L| < \varepsilon \ \forall (x,y) \in C \cap D,$
$0 < \|(x,y) - (x_0, y_0)\| < \delta)$

or $(\forall (x,y) \in C \cap D)(0 < \|(x,y) - (x_0,y_0)\| < \delta \to |f(x,y) - L| < \varepsilon)$

<u>Cont'd Thm.</u>

1. $\forall C$ passes (x_0, y_0) : $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \overset{(\leftrightarrow)}{\to} \lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ along C

2. C_1, C_2 passes (x_0, y_0) : $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ along $C_1 \neq \lim_{(x,y)\to(x_0,y_0)} f(x,y)$ along C_2
$\to \lim_{(x,y)\to(x_0,y_0)} f(x,y)$ D.N.E. $(\notin \mathbb{R})$

3. C passes (x_0, y_0) : $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ along C D.N.E. $\to \lim_{(x,y)\to(x_0,y_0)} f(x,y)$ D.N.E.

- $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$; $D \subseteq \mathbb{R}^2 \wedge (x_0, y_0) = D' \wedge c, A, B \in \mathbb{R}$,

1. $\forall (x,y) \in D \ (f(x,y) = c \to \lim_{(x,y)\to(x_0,y_0)} f(x,y) = c)$
2. $\forall (x,y) \in D \ (f(x,y) = x \to \lim_{(x,y)\to(x_0,y_0)} f(x,y) = x_0)$
3. $\forall (x,y) \in D \ (f(x,y) = y \to \lim_{(x,y)\to(x_0,y_0)} f(x,y) = y_0)$

A. $\forall (x,y) \in D \ (\lim_{(x,y)\to(x_0,y_0)} f(x,y) = A \wedge \lim_{(x,y)\to(x_0,y_0)} g(x,y) = B)$ cont'd:

4.1 $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) + g(x,y)] = A + B$

4.2 $\lim_{(x,y)\to(x_0,y_0)} f(x,y) g(x,y) = AB$

4.3 $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{A}{B}$; $B \neq 0$

4.4 $\lim_{(x,y)\to(x_0,y_0)} |f(x,y)| = |A|$

4.5 $\lim_{(x,y)\to(x_0,y_0)} f^{\frac{1}{m}}(x,y) = A^{\frac{1}{m}}$; $m > 1, m \in \mathbb{Z}^+ \wedge A^{\frac{1}{m}} \in \mathbb{R}$

To prove $\lim_{(x,y)\to(x_0,y_0)} f(x,y)g(x,y) = 0$, consider

<u>Thm.</u> $f: D \to \mathbb{R}$, $g: D' \to \mathbb{R}$

$$\Big[(\exists M \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in \overset{D'}{D})(0 < \|(x,y)-(x_0,y_0)\| < \delta \to |f(x,y)| \leq M)$$
$$\wedge \lim_{(x,y)\to(x_0,y_0)} g(x,y) = 0\Big] \to \lim_{(x,y)\to(x_0,y_0)} f(x,y)g(x,y) = 0$$

<u>Pf.</u> $f(x,y), g(x,y)$ are functions,

$(\exists M \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \forall y \in \overset{D'}{D} \times \mathbb{R})(0 < \|(x,y)-(x_0,y_0)\| < \delta) \wedge \lim_{(x,y)\to(x_0,y_0)} g(x,y) = 0$

Given $\varepsilon > 0 \to \delta' > 0$: $|g(x,y) - 0| < \frac{\varepsilon}{M+1}$ $\forall x \forall y \in D' \times \mathbb{R}$,

$0 < \|(x,y)-(x_0,y_0)\| < \delta'$, $\delta'' = \min(\delta, \delta')$

$\exists x \exists y \in (D' \cap D) \times \mathbb{R}$: $0 < \|(x,y)-(x_0,y_0)\| < \delta''$: $|f(x,y)| \leq M$

and $|g(x,y)| < \frac{\varepsilon}{M+1}$

$$\begin{aligned} \to |f(x,y)g(x,y) - 0| &= |f(x,y)g(x,y)| \\ &= |f(x,y)||g(x,y)| \\ &< M\left(\frac{\varepsilon}{M+1}\right) \\ &= \frac{M}{M+1}\varepsilon \\ &< (1)\varepsilon \\ &= \varepsilon \end{aligned}$$

$\therefore \lim_{(x,y)\to(x_0,y_0)} f(x,y)g(x,y) = 0$ ∎

Continuity **Thm.** $f: D \to \mathbb{R}$; $D \subseteq \mathbb{R}^2$, $(x_0, y_0) \in \mathbb{R}^2$

f is cont. $\leftrightarrow$ 1. $f(x_0, y_0)$ exists

2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

For $S \subseteq D$: f is cont. on S $\leftrightarrow$ $\forall f(x,y)$ is cont. on S

Thm. f, g is cont. at $(x_0,y_0) \to$ 1. $f+g$, $f-g$, fg, cf, $|f|$ is cont. at (x_0,y_0)

2. $g(x_0,y_0) \neq 0 \to \frac{f}{g}$ is cont. at (x_0,y_0)

Thm. $f,g: \mathbb{R}^2 \to \mathbb{R}$; $f(x,y) = x$, $g(x,y) = y$ $\to$ f,g is cont. on $\mathbb{R}^2$

f,g is a projection function

$\to$ Polynomial function $f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} x^i y^j$ is cont. on $\mathbb{R}^2$

Composite Function

Def. $f: D_1 \to \mathbb{R}$; $D_1 \subseteq \mathbb{R}^2$ and $g: D_2 \to \mathbb{R}$; $R_f \subseteq D_2$

Composite function $g \circ f : D_1 \to \mathbb{R}$; $(g\circ f)(x,y) = g(f(x,y))$ $\forall (x,y) \in D_1$

Thm. $f: D_1 \to \mathbb{R}$; $D_1 \subseteq \mathbb{R}^2$ and $g: D_2 \to \mathbb{R}$; $R_f \subseteq D_2$, $(x_0,y_0) \in D_1$

f and g is cont. at (x_0,y_0) $\to$ $g\circ f$ is also cont. at (x_0,y_0)

▶ Partial Derivative

$z = f(x,y)$ at (x_0,y_0) $\to$ $\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(x_0,y_0)}$ or $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$

$f_x(x_0,y_0)$

Def. $$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \equiv \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \equiv \lim_{k\to 0} \frac{f(x_0,y_0+k) - f(x_0,y_0)}{k}$$

Chain Rule Thm.

$z = f(x,y)$, $x = x(t)$, $y = y(t)$.

x, y have deriv. at t_0 and z has deriv. at $(x(t_0), y(t_0))$:

$$\left.\frac{dz}{dt}\right|_{t=t_0} = \frac{\partial}{\partial x} f(x(t_0), y(t_0)) \cdot \frac{d}{dt} x(t_0) + \frac{\partial}{\partial y} f(x(t_0), y(t_0)) \cdot \frac{d}{dt} y(t_0)$$

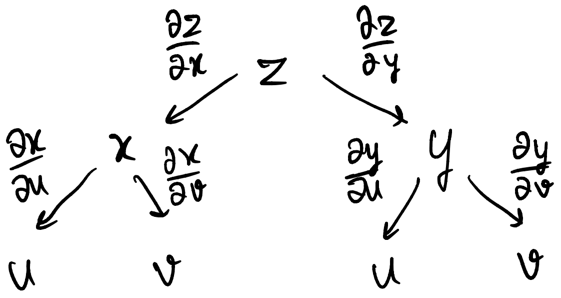
$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

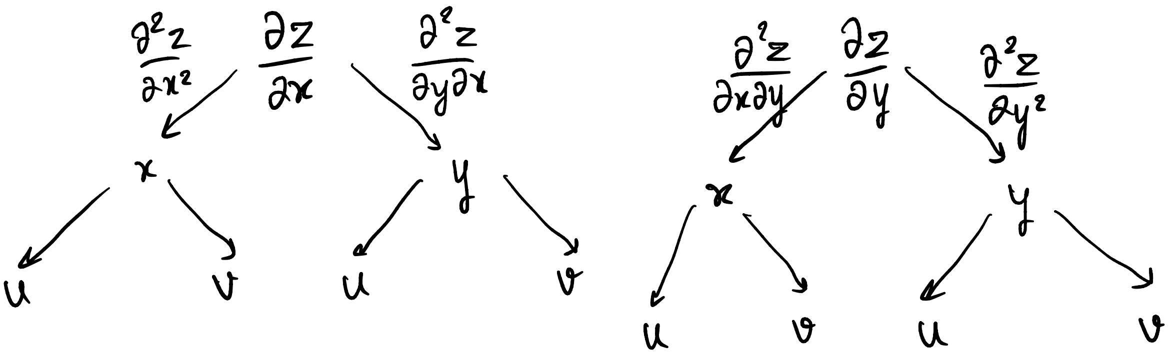
$z = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$,

x, y have deriv. at (u_0, v_0) and z has deriv. at $(x(u_0,v_0), y(u_0,v_0))$

$$\begin{cases} \dfrac{\partial z}{\partial u} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial z}{\partial v} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial v} \end{cases}$$

Diagram $\dfrac{\partial z}{\partial t}$





By Part $e^{ax}\cos(bx)$:

	D	I	
$+$	e^{ax}	$\cos(bx)$	$\sin(bx)$
$-$	ae^{ax}	$\frac{1}{b}\sin(bx)$	$-\frac{1}{b}\cos(bx)$
$+$	a^2e^{ax}	$-\frac{1}{b^2}\cos(bx)$	$-\frac{1}{b^2}\sin(bx)$

$$\int e^{ax}\cos(bx)\,dx = \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}e^{ax}\cos(bx) - \frac{a^2}{b^2}\int e^{ax}\cos(bx)\,dx$$

$$= \frac{e^{ax}}{\left(1+\frac{a^2}{b^2}\right)}\left(\frac{1}{b}\sin(bx) + \frac{a}{b^2}\cos(bx)\right)$$

$$\boxed{\begin{aligned}\int e^{ax}\cos(bx)\,dx &= \frac{e^{ax}}{a^2+b^2}\left(a\cos(bx) + b\sin(bx)\right)\\ \int e^{ax}\sin(bx)\,dx &= \frac{e^{ax}}{a^2+b^2}\left(a\sin(bx) - b\cos(bx)\right)\end{aligned}}$$

Pt. 6 : First Order Differential Equation

- Order : $f^{(n)}(x)$
- Degree : $[f^{(max)}(x)]^n$

— General Term : $F(x,y) = \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$: $\boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$

I. Separable Eqn.

* $F(x)dx + G(y)dy = 0$ → Sol'n $\int F(x)\,dx + \int F(y)\,dy = C$

II. Homogeneous Eqn. (เอกพันธ์)

* $F(kx, ky) = k^n F(x,y)$, $k \in \mathbb{R}^+$

* ข้อสังเกต $M(x,y)dx + N(x,y)dx = 0$ เป็น homogeneous ถ้า $x^m y^n$: $m+n$ เท่ากันหมด

Sol'n ให้ $\boxed{y = vx}$ → Integrate dx, dv (separable) → $v = \frac{y}{x}$

III. Exact Eqn.

* $dF(x,y) = \underbrace{M(x,y)}_{\frac{\partial F}{\partial x}}dx + \underbrace{N(x,y)}_{\frac{\partial F}{\partial y}}dy$

* $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Sol'n $F(x,y) = C$
$dF = 0$

- A, B } หา $\frac{\partial F}{\partial \xi_1}$ → $\frac{\partial F}{\partial \xi_2}$ จับเท่ากับ C
- C จัดรูป $d(f(x,y) \cdot g(x,y))$

– Integrating Factor

$M(x,y)dx + N(x,y)dy = 0$ is not Exact → $\underbrace{\mu(x,y)}_{\text{I.F.}}[M\,dx + N\,dy] = 0$ is Exact.

Use

1. $f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

   $\mu = \exp\left(\int f(x)\,dx\right)$

2. $g(y) = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

   $\mu = \exp\left(\int g(y)\,dy\right)$

สังเกต ถ้าเป็น Exact แต่แรก :
$f(x) = 0$, $g(x) = 0$ → $\mu = 1$

Jum $\frac{1}{\text{other}}\left(\Delta_{\text{other}}\right)$

IV. Linear Eqn.

* $\frac{dy}{dx} + P(x)y = Q(x) \rightarrow [P(x)y - Q(x)]dx + dy = 0$

* $P(x)$ เป็น $f(x) \rightarrow \mu = \exp\left(\int P(x)\,dx\right)$

Sol'n $\boxed{\mu y = \int \mu Q(x)\,dx + C}$

- Bernoulli Eqn.

  * $\frac{dy}{dx} + P(x)y = Q(x)y^n$

  Sol'n ให้ $\boxed{z = y^{1-n}}$ $\xrightarrow{\text{จัดรูป}}$ $(\text{degree}_{n-1})\frac{dy}{dx} - (\text{degree}_n)$

  $$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

Applications

$P = P(t)$: $\frac{dx}{dt} = kx$: $x = Ae^{kt}$

$x(t) = x(0)e^{kt}$

$$\frac{dx}{dt} = kAB$$