ค่าสูงสุด-ต่ำสุด , จุดวิกฤต

จุดวิกฤตของ $f(x,y)$ ที่ (x_0,y_0) เมื่อ $f_x(x_0,y_0)$ ไม่มีค่า หรือ $f_y(x_0,y_0)$

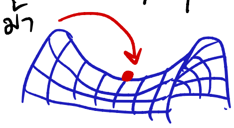
หรือ อนุพันธ์ที่ (x_0,y_0) มีค่า และ $\underbrace{f_x(x_0,y_0)=0 \text{ และ } f_y(x_0,y_0)=0}_{\vec{\nabla} f = \vec{0}}$

✱ หาจุดวิกฤตจาก $\begin{cases} f_x \\ f_y \end{cases}$ สัมพัทธ์

จุดสุดขีด (Extrema : Minima, Maxima) → สัมบูรณ์

→ สัมพัทธ์ (Local) → หาจากจุดวิกฤต (แต่ บางฟังก์ชันจุดวิกฤต ไม่เป็น จุดสุดขีด)

เช่น จุดอานม้า



✱ ทดสอบจุดวิกฤต สัมพัทธ์

Thm ให้ (a,b) เป็นจุดวิกฤตของ $f(x,y)$ ให้ $f_{xy} = f_{yx}$;

$$\boxed{D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)}$$

$D(a,b) = 0 \longrightarrow$ บอกไม่ได้

$D(a,b) > 0 \longrightarrow f_{xx}(a,b) > 0$: ต่ำสุดสัมพัทธ์

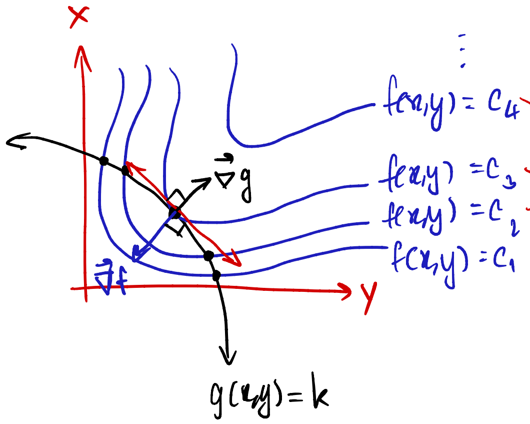
$\quad\quad\quad\quad\quad\quad\ \ \hookrightarrow f_{xx}(a,b) < 0$: สูงสุดสัมพัทธ์

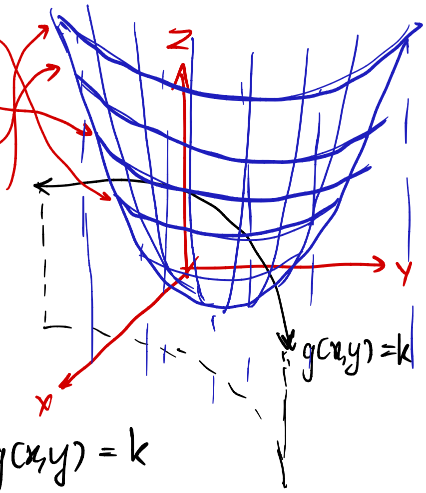
$D(a,b) < 0 \longrightarrow$ เป็นจุดอานม้า

Lagrange Multiplier

$$\begin{cases} f(x,y) \\ g(x,y) = k \end{cases}$$

⊛ หาสูงสุด / ต่ำสุดสัมบูรณ์





Thm. หากจุด (x_0,y_0) ให้ค่าสุดขีดกับ $f(x,y)$ ที่สอดคล้องกับ $g(x,y)=k$

แล้ว $\vec\nabla f(x_0,y_0)$ กับ $\vec\nabla g(x_0,y_0)$ ขนานกัน $(\vec A = k\vec B \leftrightarrow \vec A \parallel \vec B)$

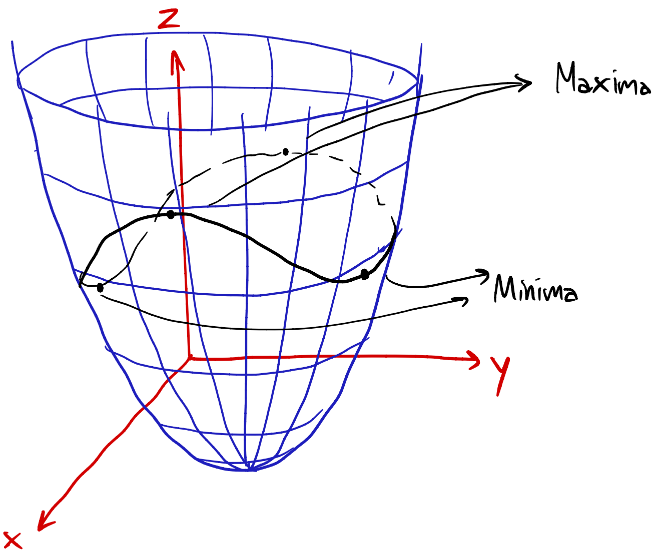
$$\left\{\boxed{\begin{aligned} \vec\nabla f_0 &= \lambda \vec\nabla g_0 \\ g(x,y) &= k \end{aligned}}\right. \quad ; \quad \begin{aligned} \vec\nabla f_0 &= \vec\nabla f(x_0,y_0,z_0,\ldots) \\ \vec\nabla g_0 &= \vec\nabla g(x_0,y_0,z_0,\ldots) \end{aligned}$$

เงื่อนไขบังคับ (Constraint)

จาก $\vec\nabla f(x_0,y_0) = \lambda \vec\nabla g(x_0,y_0)$ จะได้

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases}$$

$$\begin{cases} \langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle \\ g(x,y,z) = k \end{cases}$$



e.g. $\begin{cases} f(x,y,z) = 2x+2y+z \\ x^2+y^2+z^2 = 9 \end{cases} \longrightarrow g(x,y,z) = x^2+y^2+z^2$

$$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle$$

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

$$\begin{bmatrix} 1/\lambda \\ 1/\lambda \\ 1/2\lambda \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{---(1)}$$

constraint: $\frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 9 \longrightarrow \frac{9}{4\lambda^2} = 9 \longrightarrow \lambda^2 = \frac{1}{4}$

$\therefore \quad \lambda = \pm \frac{1}{2}$

on (1); $\langle x, y, z \rangle = \langle 2, 2, 1 \rangle, \langle -2, -2, -1 \rangle$

$\left.\begin{array}{l} f(2,2,1) = 9 \\ f(-2,-2,-1) = -9 \end{array}\right\} \quad \therefore \text{Max} = 9, \quad \text{Min} = -9$

eg. $\begin{cases} f(x,y,z) = xyz \\ x^2+2y^2+3z^2 = 6 \end{cases}$

$$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle$$

$$\begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 4y \\ 6z \end{bmatrix}$$

$$\begin{aligned} \therefore \quad 2\lambda x &= yz \rightarrow 2\lambda x^2 = xyz \quad \text{—(1)} \\ 4\lambda y &= xz \rightarrow 4\lambda y^2 = xyz \quad \text{—(2)} \\ 6\lambda z &= xy \rightarrow 6\lambda z^2 = xyz \quad \text{—(3)} \end{aligned}$$

(1)+(2)+(3) : $3xyz = 2\lambda(x^2+2y^2+3z^2)$

$3xyz = 12\lambda$

$\therefore xyz = 4\lambda$

$$\left.\begin{aligned} &(1) : 2\lambda x^2 = 4\lambda \ ; \ \lambda = 0 \\ &\qquad \lambda \neq 0 : x^2 = 2 \\ &\qquad\qquad x = \pm\sqrt{2} \\ &(2) : 4\lambda y^2 = 4\lambda \ ; \ \lambda = 0 \\ &\qquad \lambda \neq 0 : y^2 = 1 \\ &\qquad\qquad y = \pm 1 \\ &(3) : 6\lambda z^2 = 4\lambda \ ; \ \lambda = 0 \\ &\qquad \lambda \neq 0 : z^2 = \tfrac{2}{3} \\ &\qquad\qquad z = \pm\sqrt{\tfrac{2}{3}} \end{aligned}\right\}$$

8 จุด $\left(\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}}\right)$

$\therefore$ Max $= f(\sqrt{2}, 1, \sqrt{\frac{2}{3}}) = \frac{2}{\sqrt{3}}$

Min $= f(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}) = -\frac{2}{\sqrt{3}}$

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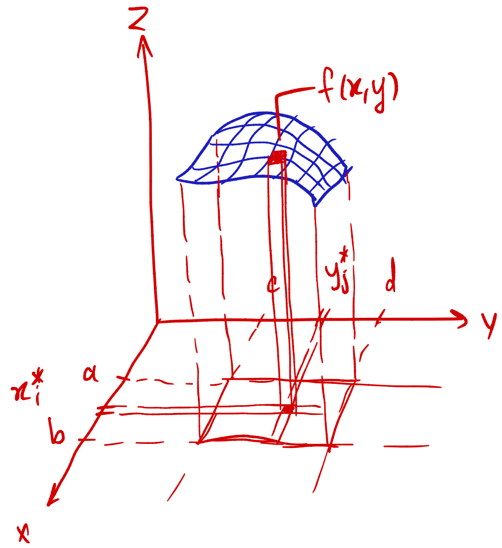
MULTIPLE INTEGRALS

$$\iint_D f(x,y)\,dA \quad , \quad \iiint_D f(x,y,z)\,dV$$



$$\iint_R f(x,y)\,dA \;;\; R = [a,b]\times[c,d]$$

$$= \lim_{(m,n)\to(\infty,\infty)} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i^*, y_j^*)\,\Delta x\,\Delta y$$

" Integral ซ้อน (Iterated Integral) "

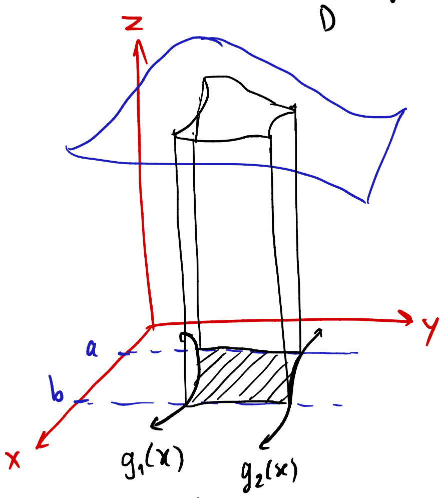
Thm. Fubini's Theorem

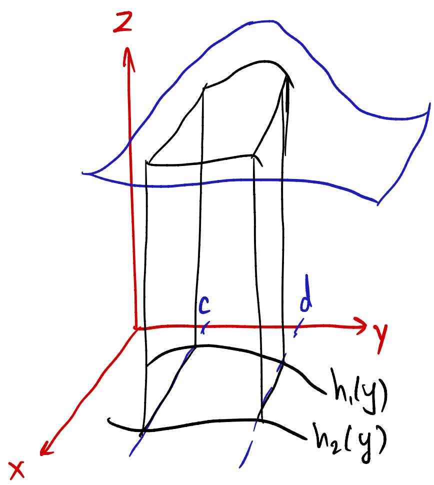
ให้ $R=[a,b]\times[c,d]$:

$$\iint_R f(x,y)\,dA = \underbrace{\int_a^b\int_c^d f(x,y)\,dy\,dx}_{\text{Treat เหมือน } \frac{\partial f}{\partial y}} = \underbrace{\int_c^d\int_a^b f(x,y)\,dx\,dy}_{\text{Treat เหมือน } \frac{\partial f}{\partial x}}$$

หมายถึง $dA = dx\,dy = dy\,dx$

<u>โดเมนใดๆ</u> D ที่ $\iint_D f(x,y)\,dA$





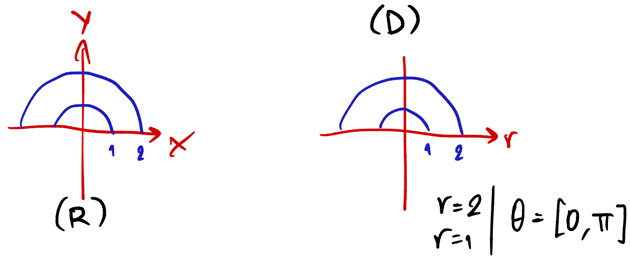
$$\iint_D f(x,y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dx\,dy \quad , \quad \iint_D f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dy\,dx$$

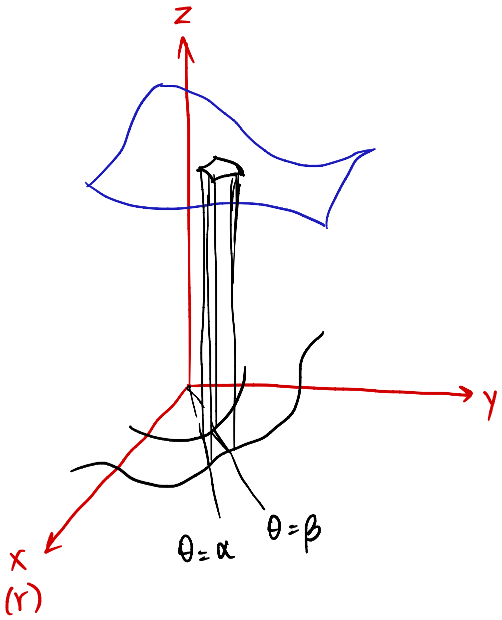
① ②

(ไม่ออกสอบ.)
CALC 2.

โดเมนใดๆ R ที่เป็นเชิงขั้ว (D)

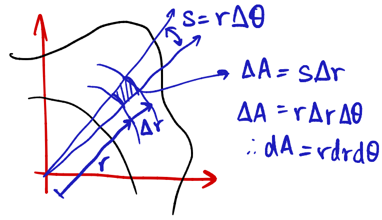
เช่น



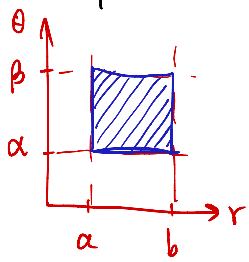


$$\iint_R f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

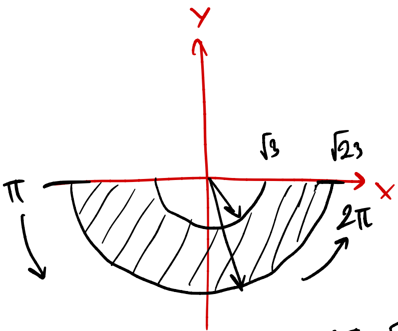
$(dA = r\,dr\,d\theta)$



D สี่เหลี่ยมมุมฉากเชิงขั้ว.



e.g. $D = \{(x,y) \mid y<0,\ 3 \le x^2+y^2 \le 23\}$ หา $\iint_D \frac{x^2}{x^2+y^2}\,dA$



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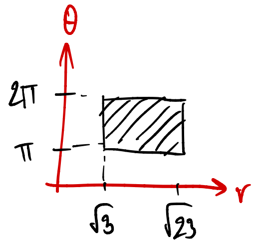


$$\iint_D \frac{x^2}{x^2+y^2}\,dA = \int_{\pi}^{2\pi}\int_{\sqrt{3}}^{\sqrt{23}} \frac{\cancel{r^2}\cos^2\theta}{\cancel{r^2}\cos^2\theta + \cancel{r^2}\sin^2\theta}\, r\,dr\,d\theta$$

$$= \int_{\pi}^{2\pi}\int_{\sqrt{3}}^{\sqrt{23}} r\cos^2\theta\,dr\,d\theta$$

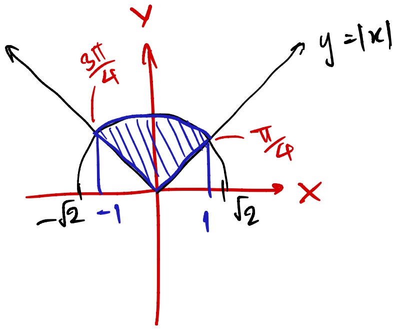
$$= \int_{\pi}^{2\pi} \cos^2\theta\,d\theta \int_{\sqrt{3}}^{\sqrt{23}} r\,dr$$

$\cos^2\theta \rightarrow \frac{1+\cos 2\theta}{2}$

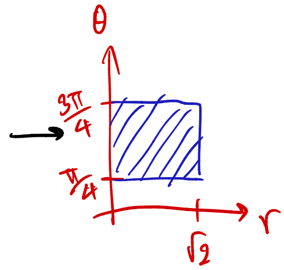
$= 5\pi$ ※

e.g. ให้ $D = \{(x,y) \mid -1 < x \le 1,\ |x| \le y \le \sqrt{2-x^2}\}$

$$\left(\text{หา} \int_{-1}^{1}\int_{|x|}^{\sqrt{2-x^2}} \sqrt{x^2+y^2}\, dy\, dx \right)$$



$\rightarrow$



$\therefore D = \{(r,\theta) \mid 0 \le r \le \sqrt{2},\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\}$

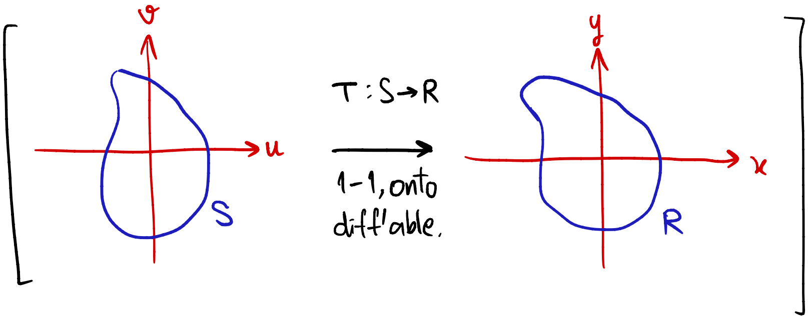
$$\text{ได้} \int_{\pi/4}^{3\pi/4}\int_{0}^{\sqrt{2}} r \cdot r\, dr\, d\theta \quad \left(\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r \right)$$

u,v substitution

Calc I. $\int_a^b f(x)\,dx$; Let $x = g(u)$, $dx = g'(u)\,du$ $\rightarrow$ $\int_{\alpha = g^{-1}(a)}^{\beta = g^{-1}(b)} \underbrace{f(g(u))}_{f(x)}\, \underbrace{g'(u)}_{\frac{dx}{du}}\, du$

$\iint_R f(x,y)\,dA$; Map, bijection $(x,y) \mapsto (u,v)$

1. เปลี่ยน f
2. เปลี่ยน dA
3. เปลี่ยน Domain



ให้ $T: \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

Thm. Jacobian $J_T = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$

$\rightsquigarrow |J_T| = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

General Form :

$$J_f = \vec{\nabla}^T f$$

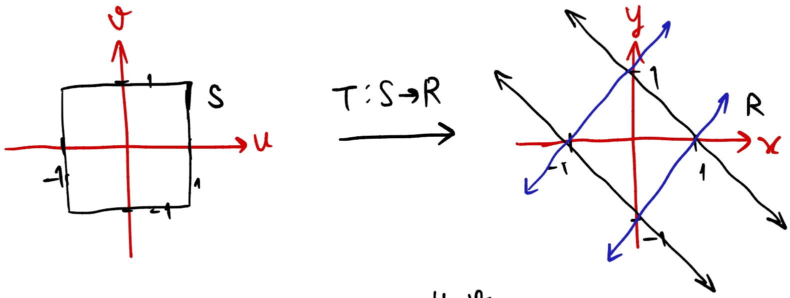
e.g. Jacobian of : $x = uv$, $y = \frac{u}{v}$: $|J| = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v}$

***

New $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}$ และ $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|}$; จาก $\det(A^{-1}) = \dfrac{1}{\det(A)}$

e.g. $S = \{(u,v) \mid u \in [-1,1],\ v \in [-1,1]\}$

$R = \{(x,y) \mid -1 \le x+y \le 1,\ -1 \le x-y \le 1\}$



$T: \quad u = x+y, \quad v = x-y \quad \Big| \quad x = \dfrac{u+v}{2}, \quad y = \dfrac{u-v}{2}$

$$dA = \left\| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right\| \cdot du \cdot dv = (x_u y_v - y_u x_v)\, du\, dv$$

Thm.

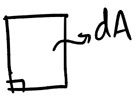
$$\iint_R f(x,y)\, dA = \iint_S f(u,v) \underbrace{\left\| \frac{\partial(x,y)}{\partial(u,v)} \right\|}_{dA} du\, dv$$

$\vec{du} = du \cdot \hat{du}$, $\vec{dv} = dv \cdot \hat{dv}$ (differentials)

Proof. $x = x(u,v) \longrightarrow \vec{dx} = (x_u \vec{du} + x_v \vec{dv})$

$y = y(u,v) \longrightarrow \vec{dy} = (y_u \vec{du} + y_v \vec{dv})$

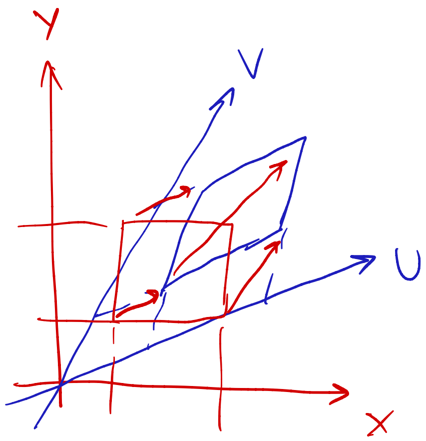
$\vec{dy} = dy\, \hat{j}$



$\vec{dx} = dx\, \hat{i}$

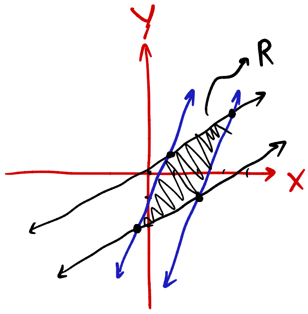
$dA = \|\vec{dA}\| = \|\vec{dx} \times \vec{dy}\|$

$$\begin{aligned}
\vec{dx} \times \vec{dy} &= (x_u \vec{du} + x_v \vec{dv}) \times (y_u \vec{du} + y_v \vec{dv}) \\
&= x_u y_u (\vec{0}) + x_v d_v (\vec{0}) + x_u y_v (\vec{du} \times \vec{dv}) + x_v y_u (\vec{dv} \times \vec{du}) \\
&= (x_u y_v - x_v y_u)\, du\, dv \quad \square
\end{aligned}$$



หาพื้นที่ตาม Domain

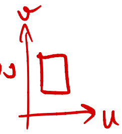
e.g. $\displaystyle\iint_R \frac{x-2y}{3x-y}\,dA$ โดย $R: \left\{\begin{array}{lll} x-2y=0 & ; & x-2y=4 \\ 3x-y=1 & ; & 3x-y=8 \end{array}\right\}$ ปิดล้อม.

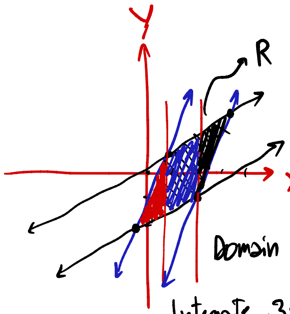


Let $u = x-2y$
$v = 3x-y$

$S: \begin{cases} u\in[0,4] \\ v\in[1,8] \end{cases}$

หาพื้นที่ในกล่อง





Domain แบบที่ 1 : Integrate 3 รอบ. (HARD!)

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}^{-1} = 5^{-1} = \frac{1}{5}$$

$$\iint_R \frac{x-2y}{3x-y}\,dA = \iint_S \frac{u}{v}\cdot\left|\frac{1}{5}\right| du\,dv = \frac{1}{5}\int_0^4 \underbrace{u\,du}_{\frac{u^2}{2}} \cdot \int_1^8 \frac{1}{v}\,dv$$

$$= \frac{1}{5}\cdot\frac{16}{2}\cdot\ln 8 = \frac{8}{5}\ln 8 \quad \square$$

e.g. $\displaystyle\iint_R (2y^2+x)e^{y^2-x}\,dA$; R ปิดล้อมด้วย $xy=1$, $xy=3$
$x=y^2+1$, $x=y^2+4$

Let $u = xy$
$v = x-y^2$

$S: u\in[1,3]$
$v\in[1,4]$

$u_x = y \quad v_x = 1$
$u_y = x \quad v_y = -2y$

$(2y^2+x>0$ ใน $R)$

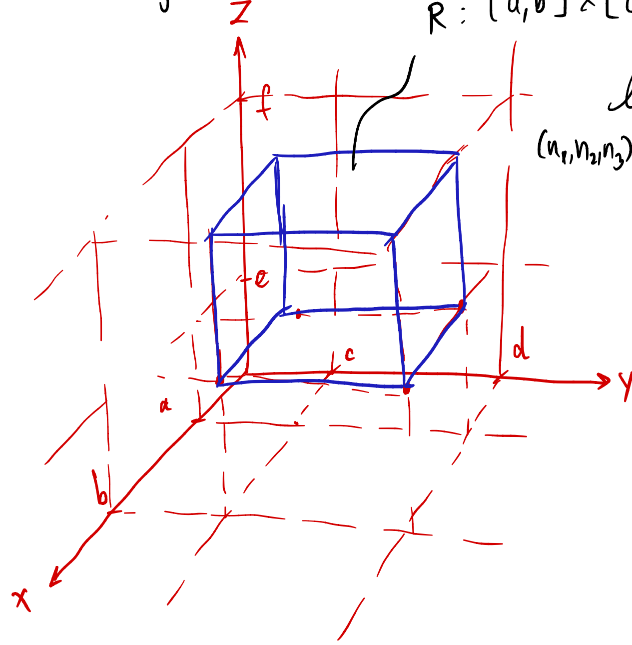
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} y & x \\ 1 & -2y \end{vmatrix}^{-1} = (-2y^2-x)^{-1} = -\frac{1}{2y^2+x} \rightarrow |J| = \left|-\frac{1}{2y^2+x}\right| = \left|\frac{1}{2y^2+x}\right|$$

$$= \int_1^4\int_1^3 e^{-v}\cdot(\cancel{2y^2+x})\cdot\frac{1}{\cancel{2y^2+x}}\,du\,dv$$

$$= \int_1^4 e^{-v}\,dv\cdot\int_1^3 du$$

$$= 2(e^{-1}-e^{-4}) \quad \square$$

Triple Integral



$$R^3 : [a,b]\times[c,d]\times[e,f]$$

$$\lim_{(n_1,n_2,n_3)\to(\infty,\infty,\infty)} \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} f(x_i^*, y_j^*, z_k^*)\,\Delta x\,\Delta y\,\Delta z$$

$$= \iiint_R f(x,y,z)\,\underbrace{dx\,dy\,dz}_{dV}$$

dV สลับได้ 6 แบบ.

e.g. $\iiint_R xy^2\sin(xyz)\,dV$; $R = [\frac{\pi}{2},\pi]\times[1,2]\times[0,1]$

$$= \iint_{[\frac{\pi}{2},\pi]\times[1,2]} \int_0^1 xy^2\sin(xyz)\,dz\,dA$$

$$= \int \frac{\cancel{xy^2}}{\cancel{xy}}\cdot(-\cos(xyz))\Big|_{z=0}^{1}\,dA$$

$$= -\int y\cos(xy) - y\,dA$$

$$= -\int_1^2\int_{\pi/2}^{\pi} y\cos(xy) - y\,dx\,dy$$

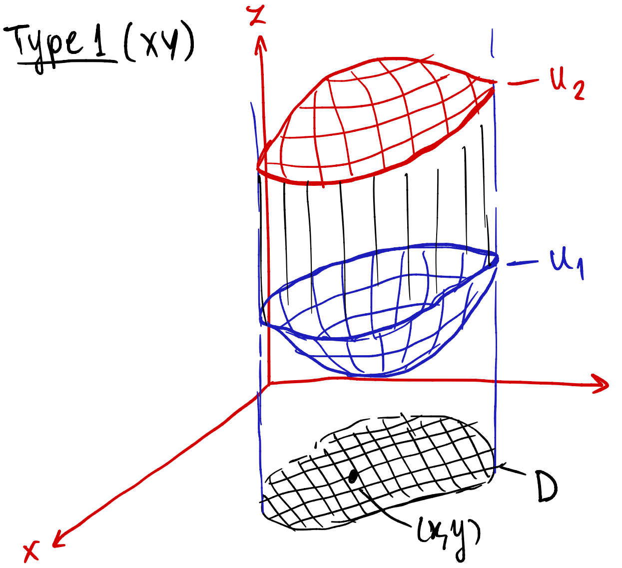
$$= -\int_1^2 \left[\frac{\cancel{y}\sin xy}{\cancel{y}} - xy\right]_{\pi/2}^{\pi} dy$$

$$= -\int_1^2 \sin\pi y + \sin\frac{\pi}{2}y - \pi y + \frac{\pi y}{2}\,dy = \frac{3\pi}{4} \qquad \square$$

Triple Integral on General Domains

โดยที่ $T = \{(x,y,z) \mid (x,y) \in D\ ,\ u_1(x,y) \le z \le u_2(x,y)\}$

Type 1 (XY)

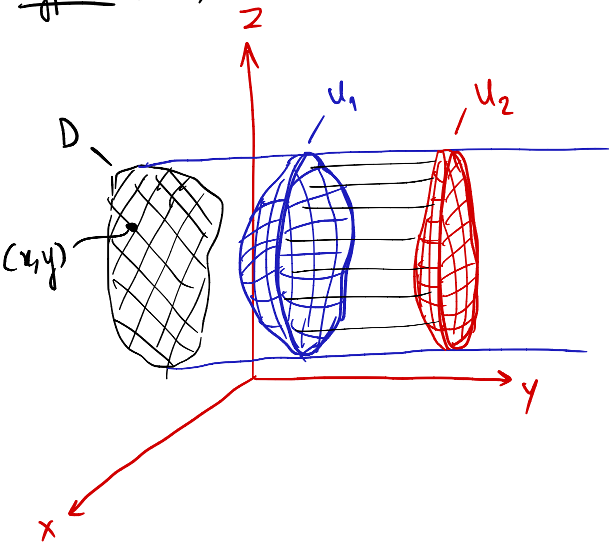


$$I = \iiint_T f(x,y,z)\,dV$$

$$I = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

$dA \rightarrow dxdy\ ,\ dydx$

Type 2 (XZ)



$$I = \iiint_T f(x,y,z)\,dV$$

$$I = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right) dA$$

$dA \rightarrow dxdz\ ,\ dzdx$

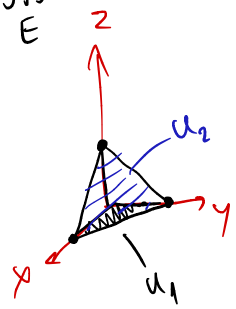
Type 3 (YZ) $\longrightarrow$ เช่นกัน.

$$I = \iiint_T f(x,y,z)\,dV$$

$$I = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right) dA$$

$dA \rightarrow dydz\ ,\ dzdy$

e.g. $\iiint_E z\,dV$; E ปิดล้อม $x=0,\ y=0,\ z=0,\ x+y+z=1$



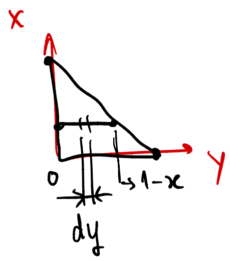
$u_2(x,y) = 1-x-y$

$u_1(x,y) = 0$

$1-x-y=0$

$y = 1-x$

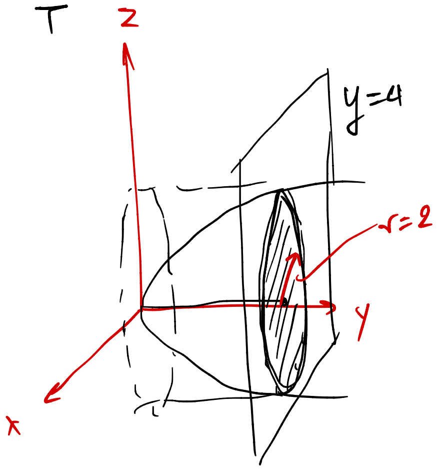
$$I = \iiint_E z\,dV = \iint_D \left(\int_0^{1-x-y} z\,dz \right) dA$$



$$I = \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy\,dx$$

$$I = \ldots \quad \square$$

e.g. $\iiint_T \sqrt{x^2+z^2}\,dV$, T ปิดล้อมด้วย $y=x^2+z^2,\ y=4$

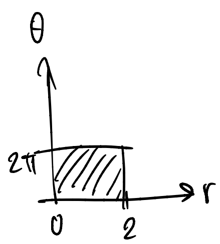


$$\iiint_T \sqrt{x^2+y^2}\,dV$$

$$= \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy \right) dA$$

$$= \iint_D \sqrt{x^2+z^2}\,(4-(x^2+z^2))\,dA$$

ใช้ (r,θ) :

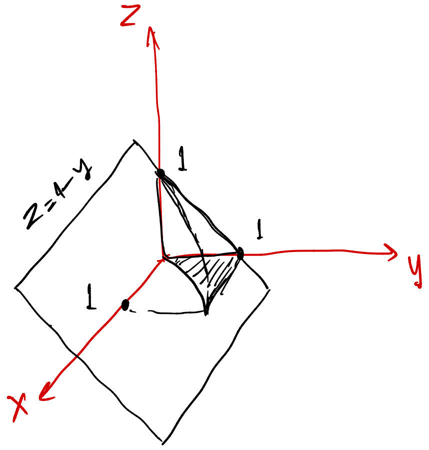


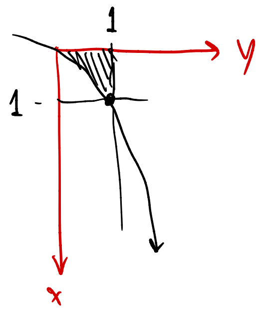
$$= \int_0^{2\pi} \int_0^2 r(4-r^2)\,r\,dr\,d\theta$$

$$= 2\pi \int_0^2 4r^2 - r^4\,dr$$

$$= 2\pi \left[\frac{4}{3}r^3 - \frac{r^5}{5} \right]_0^2 = \ldots \quad \square$$

e.g. $\displaystyle\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,\underbrace{dy\,dx}_{dA} \longrightarrow dx\,\underbrace{dz\,dy}_{dA}$



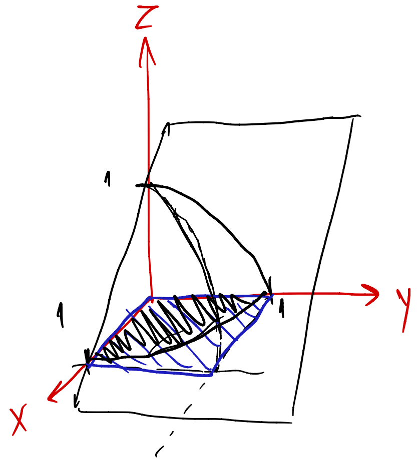


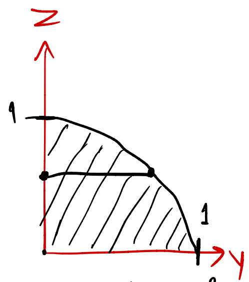
$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

e.g. $\displaystyle\int_0^1\int_0^{1-z^2}\int_0^{1-z} f(x,y,z)\,dx\,\underbrace{dy\,dz}_{dA}$

$dy\,\underline{dz\,dx}$ (1)

$dz\,\underline{dy\,dx}$ (2)





$y = 1-z^2$

$$(1) = \int_0^1\int_0^{1-x}\int_0^{1-z^2} f(x,y,z)\,dy\,dz\,dx \qquad \square$$

$$(2) = \int_0^1\int_0^{1-y^2}\int_0^{1-x} f(x,y,z)\,dz\,dy\,dx \qquad \square$$

Check

$$\int_0^1\int_0^{1-z^2}\int_0^{1-z} dx\,dy\,dz = \int_0^1 (1-z)(1-z^2)\,dz$$

$$= \int_0^1 1-z-z^2+z^3\,dz$$

$$= z-\frac{z^2}{2}-\frac{z^3}{3}+\frac{z^4}{4}\Big|_0^1$$

$$= 1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4} \quad /$$

$$(1) = \int_0^1\int_0^{1-x}(1-z^2)\,dz\,dx$$

$$= \int_0^1 (1-x)-\frac{(1-x)^3}{3}\,dx$$

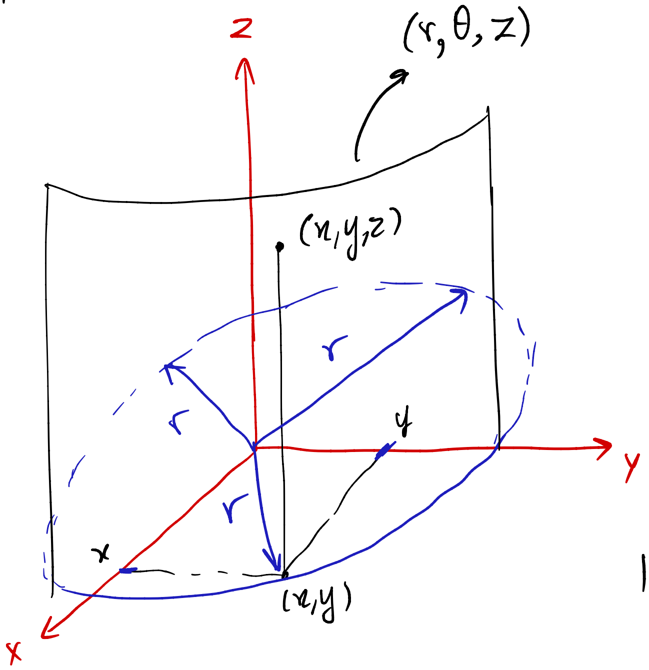
$$= \int_1^0 \frac{(x-1)^3}{3}+x-1\,dx$$

$$= \frac{(x-1)^4}{12}+\frac{x^2}{2}-x\Big|_1^0$$

$$= 1-\frac{1}{2}+\frac{1}{12} \quad \checkmark$$

$$(2) = \int_0^1\int_0^1 (1-x)\,dy\,dx =$$

Cylindrical Coordinates



Jacobian

2D: $x = x(u, v)$

$y = y(u, v)$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

3D: $x = x(u, v, w)$

$y = y(u, v, w)$

$z = z(u, v, w)$

$$|J| = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

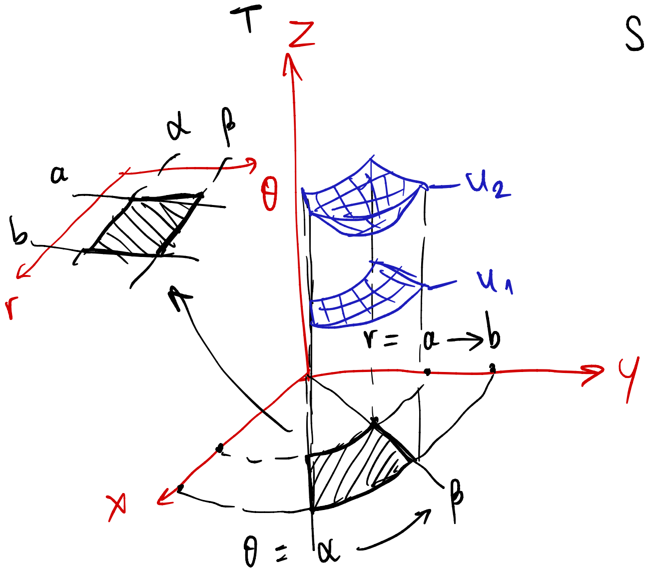
พิสูจน์ทรงกระบอก : Jacobian $= r$; จาก $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta)$$

$$|J| = r \quad \square$$

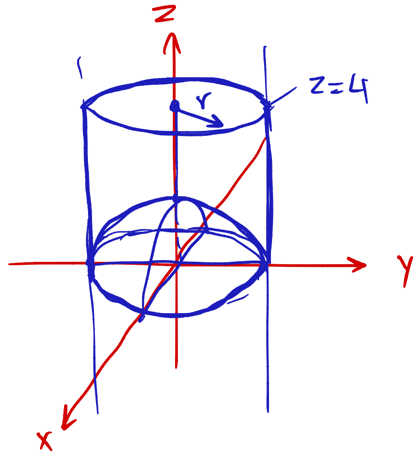
Transforming Cartesian → Cylindrical

$$\iiint_T f(x,y,z)\,dV = \iiint_S f(r\cos\theta, r\sin\theta, z)\,\overset{*}{r}\,dr\,d\theta\,dz$$



$$= \int_{u_1}^{u_2}\int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz.$$

e.g. E เป็นทรงตันล้อมรอบด้วย $x^2+y^2=1$, $z=4$, $z=1-x^2-y^2$ หา $\iiint_E \sqrt{x^2+y^2}\,dV$



$x=r\cos\theta$
$y=r\sin\theta$

$\{(r,\theta),\ 0\le r\le 1,\ 0\le\theta\le 2\pi\}$

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^{1}\int_{1-r^2}^{4} \underline{r}\cdot\underline{r\,dz\,dr\,d\theta}$$

$$= \int_0^{2\pi} d\theta \int_0^{1} r^2 \int_{1-r^2}^{4} dz\,dr$$

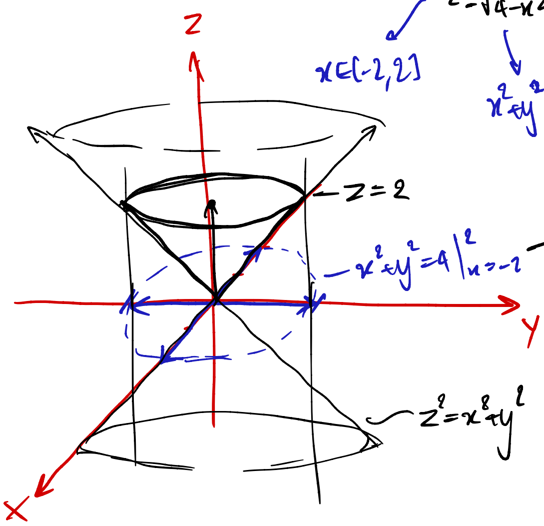
$$= 2\pi\int_0^1 r^2(4-1+r^2)\,dr$$

$$= 2\pi\left[\frac{\cancel{3}r^3}{\cancel{3}}+\frac{r^5}{5}\right]_0^1$$

$$= \frac{12\pi}{5} \quad \square$$

e.g. แปลงให้เป็น Cylindrical coord. : $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$

$x\in[-2,2]$ $\quad x^2+y^2=4$ $\quad z^2=x^2+y^2$ $(z=r)$



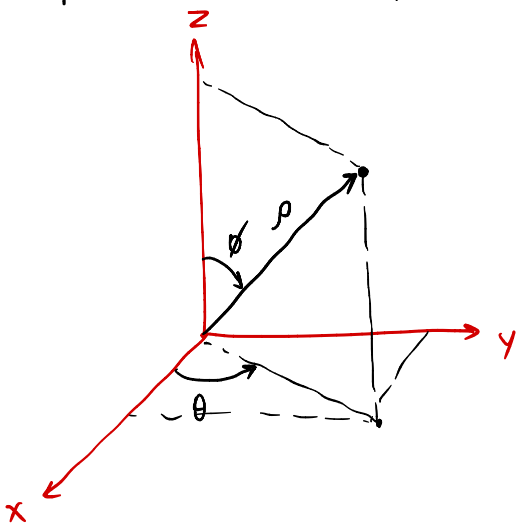
$D=\{(r,\theta)\,|\,0\le r\le 2,\ 0\le\theta\le 2\pi\}$

$$I = \int_0^{2\pi}\int_0^{2}\int_r^{2} \overset{r^2\cdot r}{r^3}\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^2 (2r^3-r^4)\,dr = 2\pi\left(\frac{2r^4}{4}-\frac{r^5}{5}\right)\Big|_0^2$$

$$= 2\pi\left(8-\frac{32}{5}\right) = \frac{16\pi}{5} \quad \square$$

Spherical Coordinates (ρ, θ, ϕ)



$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases} \quad \rho = x^2+y^2+z^2$$

$$|J| = \left\| \begin{matrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{matrix} \right\|$$

$$|J| = \rho^2 \sin\phi$$

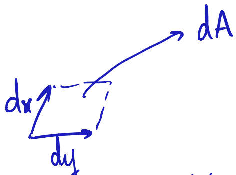
Special cases : $\rho = c$: sphere
$\theta = c$: half plane
$\phi = c$: cone part

จะได้ $$\iiint_T f(x,y,z)\,dV = \iiint_S f \cdot \rho^2 \sin\phi \, dV \; ;$$

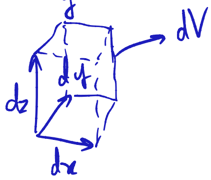
$$f = f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

Applications

* พื้นที่ : $A = \iint_D dA$



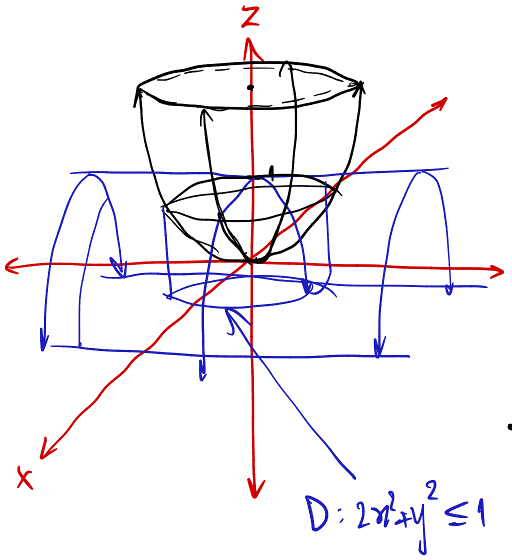
* ปริมาตร : $V = \iiint_T dV$

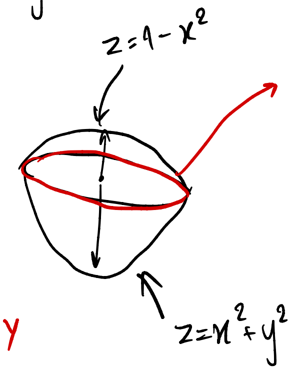


* มวล , โมเมนต์ความเฉื่อย (I_x, I_y, I_z) , จุดศูนย์กลางมวล $(\bar{x}, \bar{y}, \bar{z})$

$$m = \iiint_T \rho(x,y,z)\,dV \quad (dm = \rho(x,y,z)\,dV)$$

e.g. ปริมาตรทรงตันปิดล้อมด้วย $z = x^2+y^2,\ 1-x^2$





$x^2+y^2 = 1-x^2$

$2x^2+y^2 = 1 \rightarrow D.$

$$V = \iint_D \int_{x^2+y^2}^{1-x^2} dz\, dA = \iint_D 1-2x^2-y^2\, dA$$

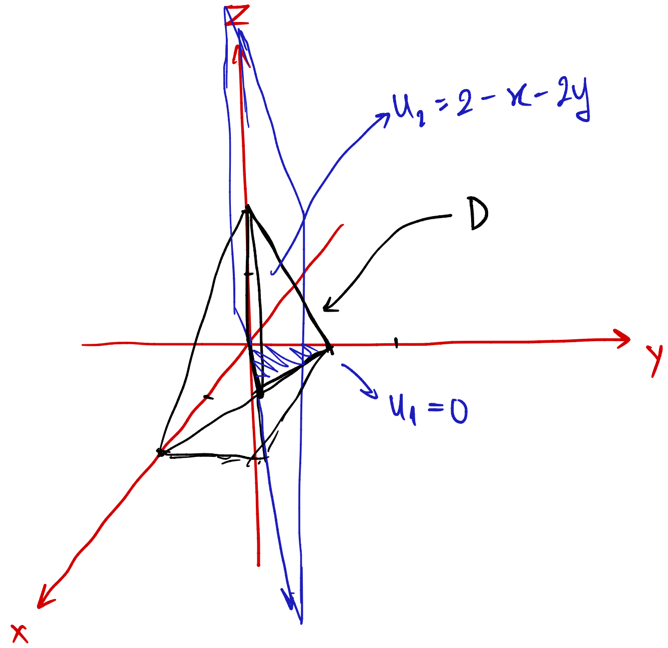
Using: $u=\sqrt{2}x$, $v=y$ $\longrightarrow$ $S:\ u^2+v^2 \le 1$ $\longrightarrow$ $R: 0\le r \le 1,\ 0 \le \theta \le 2\pi$

$$|J| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{2}}$$

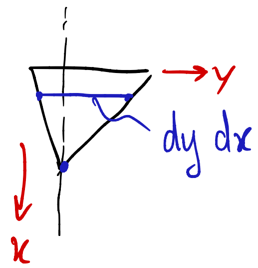
$$V = \iint_S \left[1-(u^2+v^2)\right]\frac{1}{\sqrt{2}}dA = \int_0^{2\pi}\int_0^{1} (1-r^2)\cdot\frac{1}{\sqrt{2}}\cdot r\,dr\,d\theta$$

$$= \sqrt{2}\pi \int_0^1 \overbrace{r-r^3\,dr}^{\frac{1}{2}-\frac{1}{4}=\frac{1}{4}} = \frac{\sqrt{2}}{4}\pi \quad \square$$

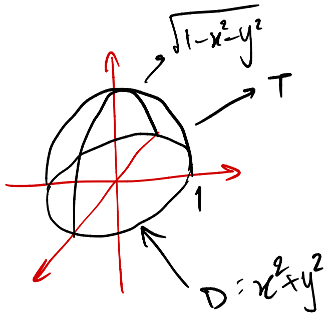
e.g. $x+2y+z=2, \quad x=2y, \quad x=0, \quad z=0$



$$V = \iint_D \int_0^{2-x-2y} dz\, dA = \dots$$



e.g. มวลของทรงตันทรงกลมครึ่งบน $x^2+y^2+z^2 \le 1$, $z \ge 0$, $\rho(x,y,z) = \sqrt{x^2+y^2+z^2} = r$



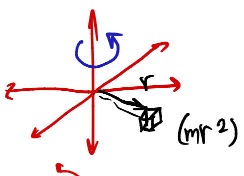
$u_1 = 0$

$T \to S$: $$m = \iiint_T \sqrt{x^2+y^2+z^2}\, dV$$

$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1} r \cdot r^2 \sin\phi \, dr\, d\phi\, d\theta$$

□

T : $$m = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$$



Center of Mass, Moment of Inertia [$dm = \rho dV$]

$\frac{1}{m}\int x \, dm$

$\int r^2 dm$

$$\bar{x} = \frac{1}{m}\iiint_S x \underbrace{\rho dV}_{dm} \qquad I_x = \iiint_S (y^2+z^2)\rho\, dV$$

$$\bar{y} = \frac{1}{m}\iiint_S y \rho\, dV \qquad I_y = \iiint_S (x^2+z^2)\rho\, dV$$

$$\bar{z} = \frac{1}{m}\iiint_S z \rho\, dV \qquad I_z = \iiint_S (x^2+y^2)\rho\, dV$$

e.g. ทรงตัน E ความหนาแน่น $\rho(x,y,z)$ มีมวล $m=2$ โดย

$$(\bar{x},\bar{y},\bar{z}) = (1,2,-1) \;,\; \langle I_x, I_y, I_z \rangle = \langle 5, 10, 7 \rangle$$

จงหา 1) $\displaystyle\iiint_E (x+1)\rho(x,y,z)\,dV = \iiint_E x\overset{dm}{\underline{\rho\,dV}} + \iiint_E \overset{dm}{\underline{\rho\,dV}}$

$$= m\cdot\bar{x} + m$$

$$= 4 \quad \square$$

2.) $\displaystyle\iiint_E (x+y-z)\rho(x,y,z)\,dV = m\bar{x} + m\bar{y} - m\bar{z}$

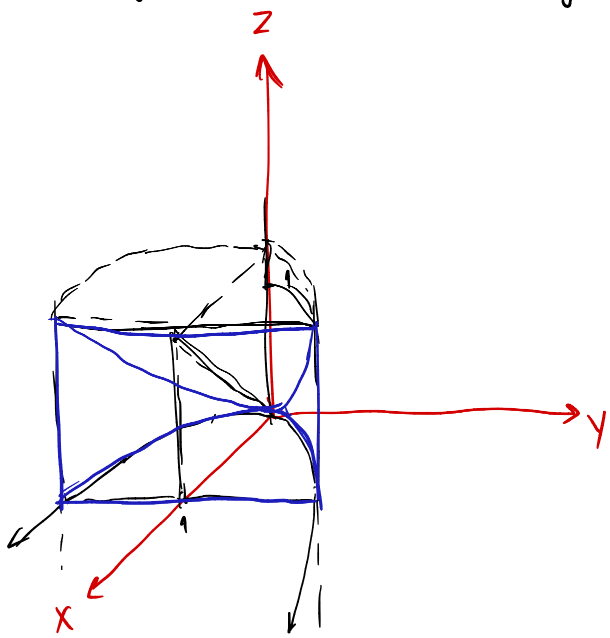
3.) $\displaystyle\iiint_E z^2\rho(x,y,z)\,dV$ Using: $(x^2+z^2) + (z^2+y^2) - (x^2+y^2) = 2z^2$

$$= \frac{I_y + I_x - I_z}{2}$$

4.) $\displaystyle\iiint_E (x^2-y^2)\rho(x,y,z)\,dV = I_y - I_x$

5.) $\displaystyle\iiint_E (y+1)^2\rho\,dV = \frac{I_x + I_z - I_y}{2} + 2m\bar{y} + m$

e.g. หา $(\bar{x}, \bar{y}, \bar{z})$ ของทรงตัน E ที่ล้อมด้วย $x=y^2$, $x=z$, $z=0$, $x=1$



$$V = \iiint_E dV = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} dz\,dx\,dy = \frac{4}{5}$$

$$\bar{x} = \frac{\cancel{\rho}}{M}\iiint_E x\,dV = \frac{1}{V}\iiint_E x\,dV \qquad M = \cancel{\rho} V$$

$$\bar{x} = \frac{1}{V}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} x\,dz\,dx\,dy$$

$$= \frac{5}{4}\int_{-1}^{1}\int_{y^2}^{1} x^2\,dx\,dy$$

$$= \frac{5}{4}\int_{-1}^{1} \frac{1}{3} - \frac{y^6}{3}\,dy$$

$$= \frac{5}{12}\left(y - \frac{y^7}{7}\right)\Big|_{-1}^{1}$$

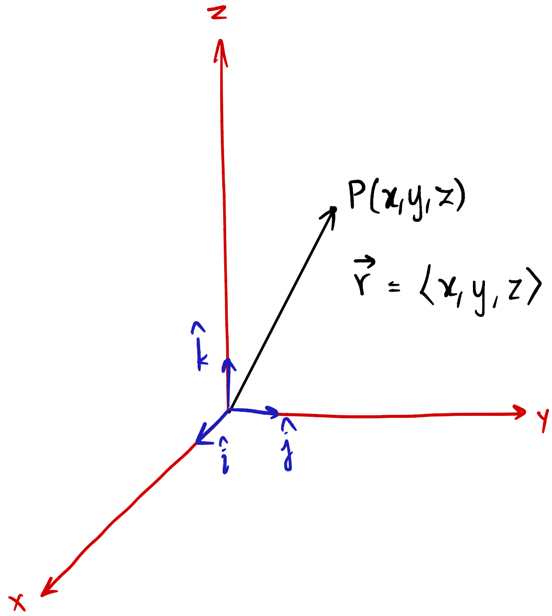
$$= \frac{5}{12}\left(1 - \frac{1}{7} + 1 - \frac{1}{7}\right) = \left(2 - \frac{2}{7}\right)$$

$$\bar{x} = \frac{5}{6}\left(\frac{6}{7}\right) = \frac{5}{7} \quad \square$$

$\bar{y} = 0$ (สมมาตร) ?

$\bar{z} = ?$

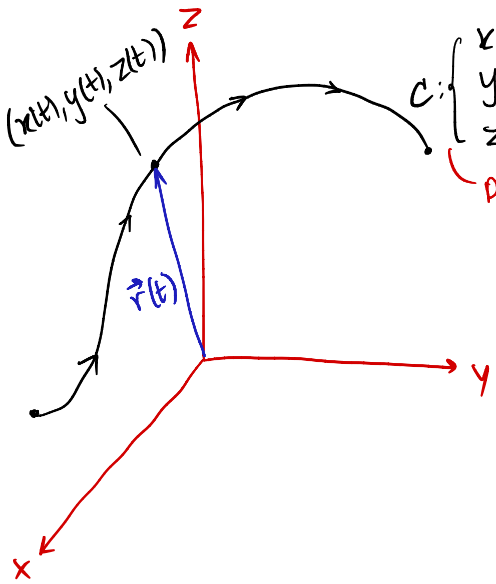
VECTOR CALCULUS



$\vec{r} = \langle x, y, z \rangle = x\hat{i} + y\hat{j} + z\hat{k}$

$f(x,y,z) = f(\vec{r})$

$\vec{F}(x,y,z) = \vec{F}(\vec{r})$



parameters

$$C: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

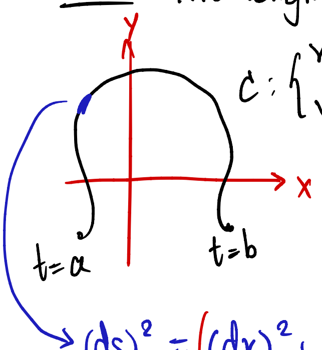
Paramatized form.

- Vector-valued Function

$\vec{r} = \vec{r}(t)$

$$\frac{d\vec{r}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$$

<u>Review</u> : Arc length



$$C: \begin{cases} x = x(t) \\ y = y(t) \end{cases}, \quad a \le t \le b$$

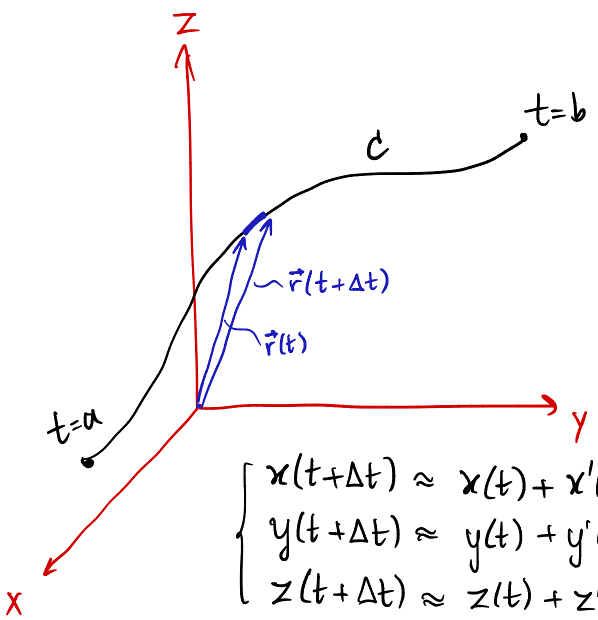
$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$(ds)^2 = \left(\frac{(dx)^2}{(dt)^2} + \frac{(dy)^2}{(dt)^2} \right) (dt)^2$$

$$\therefore\ ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Arc length in 3D

$$C: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in [a,b]$$



- $L = \displaystyle\int_a^b \underbrace{\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt}$
- $ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\; dt$

$$\begin{cases} x(t+\Delta t) \approx x(t) + x'(t)\Delta t \\ y(t+\Delta t) \approx y(t) + y'(t)\Delta t \\ z(t+\Delta t) \approx z(t) + z'(t)\Delta t \end{cases}$$

$$\begin{cases} \Delta x = x'(t)\,\Delta t \; ; \; dx = x'(t)\,dt \\ \Delta y = y'(t)\,\Delta t \; ; \; dy = y'(t)\,dt \\ \Delta z = z'(t)\,\Delta t \; ; \; dz = z'(t)\,dt \end{cases}$$

Differentiating a vector

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

- $\dfrac{d\vec{r}}{dt} = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$
- $\|\vec{r}(t)\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$
- $d\vec{r} := \vec{r}\,'(t)\,dt$ bbn: $d\vec{r} = \langle dx, dy, dz \rangle$
- $ds = \|d\vec{r}\| = \|\vec{r}\,'(t)\|\,dt$

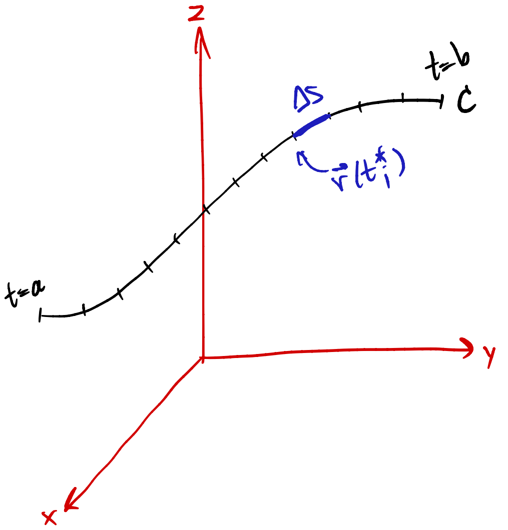
Integrals of VVF

Thm. กำหนด $f(x,y,z)$

- $\displaystyle\int_C f(x,y,z)\,ds = \int_{t=a}^{b} f(x(t),y(t),z(t))\cdot\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$

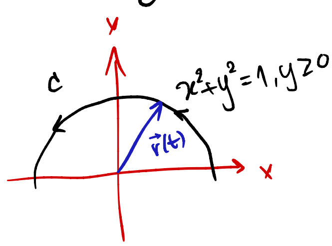
- $\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,\|d\vec{r}\|$

Pf. Riemann Sum



จะได้ $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{r}(t_i^*))\,\Delta s$

e.g. $\displaystyle\int_C (2+x^2y)\,ds$



$\vec{r}(t) = \langle x(t), y(t)\rangle = \langle \cos t, \sin t\rangle \; ; \; 0\le t\le \pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t\rangle$

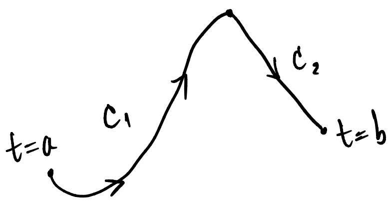
$d\vec{r} = \vec{r}\,'(t)\,dt = \langle -\sin t, \cos t\rangle\,dt$

$ds = \|d\vec{r}\| = \underbrace{\|\langle -\sin t, \cos t\rangle\|}_{\sqrt{\sin^2 t+\cos^2 t}}\,dt = dt$

$$\therefore \int_C (2+x^2y)\,ds = \int_0^{\pi} (2+\cos^2 t\sin t)\,dt$$

$$= 2\pi + -\int_0^{\pi}\cos^2 t\,d\cos t = 2\pi - \frac{\cos^3 t}{3}\Big|_0^{\pi}$$

Multiple connecting curves



$$\int_C f(\vec{r})\,ds = \int_{C_1} f(\vec{r})\,ds + \int_{C_2} f(\vec{r})\,ds$$

e.g. $\int_C xyz\,ds$: $x = 2\sin t$, $y = t$, $z = 2\cos t$, $0 \le t \le \pi$

$\vec{r} = \langle 2\sin t, t, 2\cos t\rangle$

$ds = \|d\vec{r}\| = \|\vec{r}\,'(t)\|\,dt$

$ds = \|\langle 2\cos t, 1, -2\sin t\rangle\|\,dt$

$ds = \sqrt{5}\,dt$

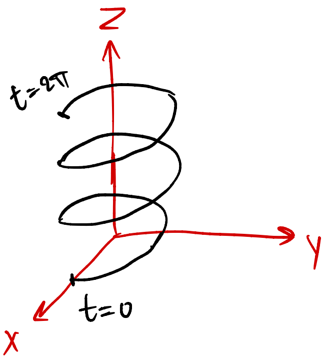
$$\int_C xyz\,ds = \frac{\sqrt{5}}{2}\int_0^{\cancel{\pi}\ 2\pi} 2\cdot\overbrace{2\sin t\cos t}^{\sin 2t}\cdot t\,\cancel{dt}\ d(2t) = \frac{\sqrt{5}}{2}\int_0^{2\pi} u\sin u\,du$$

$$= \frac{\sqrt{5}}{2}\left(u\cos u + \sin u\right)\Big|_0^{2\pi} = \frac{\sqrt{5}}{2}\cdot 2\pi = \sqrt{5}\pi \quad \square$$

+: ความยาวของ C : $L_C = \int_C ds$

มวลของ C : $m_C = \int_C \rho(x,y,z)\,ds$ λ

* e.g. $C: x=\cos 3t,\ y=\sin 3t,\ z=4t,\quad 0\le t\le 2\pi,\quad \rho(x,y,z)=x^2+y^2+z-1$ เส้นลวด



$$\vec{r}=\langle \cos 3t, \sin 3t, 4t\rangle$$

$$ds=\|d\vec{r}\| = \|\vec{r}\,'(t)\|\,dt$$

$$ds=\|\langle -3\sin t, 3\cos t, 4\rangle\|\,dt$$

$$ds=\sqrt{3^2+4^2} = 5\,dt$$

มวล $\quad M=\int_C \rho\, ds = \int_0^{2\pi}\left(\cancel{\cos^2 3t+\sin^2 3t}^{\,1}+4t-\cancel{1}\right)5\,dt$

$$=20\int_0^{2\pi} t\,dt = 40\pi^2$$

ความยาว $\quad L=\int_C ds = \int_0^{2\pi} 5\,dt$

$$=10\pi$$

Integrals of Vector-valued Functions

ให้ $\vec{F}(x,y,z)$ เป็นสนามเวกเตอร์ (vector field)

กำหนด $C: x=x(t),\ y=y(t),\ z=z(t),\ a\le t\le b$; $\vec{r}=\langle x,y,z\rangle$

$\vec{r}=\vec{r}(t)=\langle x(t), y(t), z(t)\rangle$

dot

□ $$\int_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

โดย field $\vec{F}=\langle P,Q,R\rangle$ หรือ $\vec{F}(\vec{r})=\langle P(\vec{r}), Q(\vec{r}), R(\vec{r})\rangle$

□ $$\int_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_C \underbrace{\begin{bmatrix}P\\Q\\R\end{bmatrix}}_{\vec{F}}\cdot\underbrace{\begin{bmatrix}dx\\dy\\dz\end{bmatrix}}_{d\vec{r}} = \int_C P\,dx+Q\,dy+R\,dz$$

เป็น Notation (สัญลักษณ์) แยกไม่ได้!

e.g. คำนวณ line integral $\int_C z\,dx + x\,dy + y\,dz$ $\quad \int_C \langle z, x, y\rangle \cdot d\vec{r}$

$C:\ x = t^2,\ y = t^3,\ z = t^2\ ;\quad 0 \le t \le 1$

$dx = 2t\,dt\ ,\ dy = 3t^2 dt,\ dz = 2t\,dt$

$$I = \int_0^1 2t^3 dt + \int_0^1 3t^4 dt + \int_0^1 2t^4 dt = \frac{3}{2} \quad \square$$

e.g. $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$, $C:\ x = t,\ y = t^2,\ z = t^3,\quad 0 \le t \le 1$

$\vec{r} = \langle t, t^2, t^3\rangle$

$d\vec{r} = \vec{r}\,'(t)\,dt = \langle 1, 2t, 3t^2\rangle\,dt$

$dx = dt$, $dy = 2t\,dt$, $dz = 3t^2 dt$ $\quad d\vec{r} = \langle dx, dy, dz\rangle = \langle 1, 2t, 3t^2\rangle\,dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \langle xy, yz, zx\rangle \cdot \langle 1, 2t, 3t^2\rangle\,dt$$

$$= \int_C xy + 2yzt + 3zxt^2\,dt$$

$$= \int_0^1 t^3 + 2t^6 + 3t^6\,dt = \int_0^1 5t^6 + t^3\,dt$$

$$= \left(\frac{5}{7}t^7 + \frac{1}{4}t^4\right)\Big|_0^1 = \frac{27}{28} \quad \square$$

Line Integrals : Fundamental Theorem (Gradient Theorem)

จาก $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$ (Fundamental Theorem of Calculus)

ได้ $\displaystyle\int_C \vec{\nabla} f \cdot d\vec{r}$ จากจุด $\vec{r}(a)$ ไป $\vec{r}(b)$ $= f(\vec{r}(b)) - f(\vec{r}(a))$

หากให้ $\vec{F} = \vec{\nabla} f$ (อนุ) จะได้ $\displaystyle\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

Proof. ให้ $C : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ จาก $\vec{r}(a)$ ไป $\vec{r}(b)$

$$\begin{aligned}
\vec{\nabla} f \cdot d\vec{r} &= \vec{\nabla} f \cdot \vec{r}\,'(t)\,dt \\
&= \langle f_x, f_y, f_z \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle\,dt \\
&= \langle f_x x'(t), f_y y'(t), f_z z'(t) \rangle\,dt \quad \text{differentials} \\
&= \left[\frac{d}{dt} f(x(t), y(t), z(t)) \right] dt
\end{aligned}$$

Let $g(t) = f(x(t), y(t), z(t)) = f(\vec{r}(t))$

$$\therefore \quad \vec{\nabla} f \cdot d\vec{r} = g'(t)\,dt$$

$$\int_C \vec{\nabla} f \cdot d\vec{r} = \int_{t=a}^{t=b} g'(t)\,dt = g(b) - g(a)$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

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e.g. หา $\int_C y\,dx + x\,dy$; $C: x = e^t\cos 2t,\ y = e^{-t}\sin 2t,\ 0 \le t \le \frac{\pi}{8}$

$dx = (-2e^t\sin 2t + e^t\cos 2t)\,dt$, $dy = (2e^{-t}\cos 2t - e^{-t}\sin 2t)\,dt$

$y\,dx = (-2\sin^2 2t + \sin 2t\cos 2t)\,dt$, $x\,dy = (2\cos^2 2t - \sin 2t\cos 2t)\,dt$

$$= \int_{t=0}^{\pi/8} (\cos^2 2t - \sin^2 2t)\,2\,dt = \frac{2}{4}\int_{4\cdot 0}^{4\cdot \pi/8} \cos 4t\, d(4t) = \left(\frac{1}{2}\sin 4t\right)\Big|_0^{\pi/2} = 0 \quad \square$$

sol. $I = \int_C \langle y, x\rangle \cdot \langle dx, dy\rangle = \int_C \vec{F}\cdot d\vec{r}$; $\vec{F} = \langle y, x\rangle$

$*\ f = xy$: $\vec{\nabla} f = \langle y, x\rangle = \vec{F}$ ตอบ. จะได้ $I =$

$= \sin 2t\cos 2t$

$= \frac{1}{2}\sin 4t \longrightarrow I = f(\frac{\pi}{8}) - f(0) = 0 \quad \square$

e.g. หา $\int_C yz\,dx + xz\,dy + (xy+2z)\,dz$ เมื่อ C เป็น linear จาก $(1,0,-2) \to (4,6,3)$

$$I = \int_C \langle yz, xz, xy+2z\rangle \cdot \langle dx, dy, dz\rangle$$

$\vec{F} = \vec{\nabla} f = \langle yz, xz, xy+2z\rangle$; $f = xyz + z^2$ ✓ (ตอบ.)

$\therefore I = f(4,6,3) - f(1,0,-2) = 72 + 9 - 4 = 77 \quad \square$

(Please check.)

□ Conservative Vector Field : $\boxed{\vec{F} = \vec{\nabla} f}$ โดย f เป็นศักย์

หาก $\vec{F} = \langle P, Q, R \rangle$ จะได้ $\langle P, Q, R \rangle = \langle f_x, f_y, f_z \rangle$

และถ้า P, Q, R diff'able และต่อเนื่อง จะได้ $\boxed{P_y = Q_x\ ,\ P_z = R_x\ ,\ Q_z = R_y \leftrightarrow \vec{F} \text{ อนุ.}}$

"Path Independence"

$$\int_{P_1} \vec{F} \cdot d\vec{r} = \int_{P_2} \vec{F} \cdot d\vec{r}$$

<u>การหาศักย์ f</u>

$P = f_x \rightarrow f = \int P\,dx + g(y,z)$

$Q = f_y \rightarrow f_y = G = \frac{\partial}{\partial y}\int P\,dx + \cancel{g_y}\ \tilde{P}(y,z)$

$R = f_z \rightarrow \ldots$

<u>e.g.</u> เป็นอนุ? หาศักย์

$\vec{F}_1 = (2x - 3y)\hat{i} + (-3x + 4y - 8)\hat{j}$

$\vec{F}_2 = \langle e^x \cos y , e^x \sin y \rangle$

$\vec{F}_3 = \langle y^2 z + 2xz^2 , 2xyz , xy^2 + 2x^2 z \rangle$

<u>Sol.</u> $\vec{F}_1$: $\left.\begin{aligned} P &= 2x - 3y \\ Q &= -3x + 4y - 8 \end{aligned}\right\} P_y = Q_x$

$P_y = -3$

$Q_x = -3$ ✓

$P = f_x \rightarrow f = \int (2x - 3y)\,dx + g(y)$

$\therefore f = x^2 - 3xy + g(y)$ f_y

$Q = f_y \rightarrow -\cancel{3x} + 4y - 8 = 0 - \cancel{3x} + g'(y)$

$\therefore g'(y) = 4y - 8$

$g(y) = \int 4y - 8\,dy + C = 2y^2 - 8y + C$

$\therefore f = x^2 - 3xy + 2y^2 - 8y + C$ □

* e.g. $\vec{F}(x,y,z) = \langle 2xy,\ x^2+2yz,\ A(x,y) \rangle$ เป็นสนามอนุรักษ์

หา • $A(x,y)$; $A(0,0) = 2$

• $f(x,y,z)$ ศักย์ของ $\vec{F}$

Sol $P_z = R_x$: $0 = A_x$ —(1)

$Q_z = R_y$: $2y = A_y$ —(2)

$A_y = 2y$

$A = \int 2y\,dy + f(x)$

$A = \underline{y^2} + f(x) = C_1 + \underline{g(y)}$

$\therefore g(y) = y^2$

$f(x) = C$

$\therefore A = C + y^2$

(1) : $A = \int 0\,dx + g(y) = C_1 + g(y)$

(2) : $2y = g'(y) \rightarrow g(y) = y^2 + C_2$

$\therefore A = C_1 + C_2 + y^2 = C + y^2$

$A(0,0) = 2$: $C = 2 \rightarrow A = 2 + y^2$

Sol $P = f_x$: $f = \int 2xy\,dx + g(y,z)$

$f = x^2y + g(y,z)$

f_y

$Q = f_y$: $x^2 + 2yz = \underline{x^2 + g_y}$

$\therefore g_y = 2yz \rightarrow g = \int 2yz\,dy + h(z) = y^2z + h(z)$

$\therefore f = x^2y + y^2z + h(z)$

f_z

$R = f_z$: $2 + y^2 = y^2 + h'(z)$

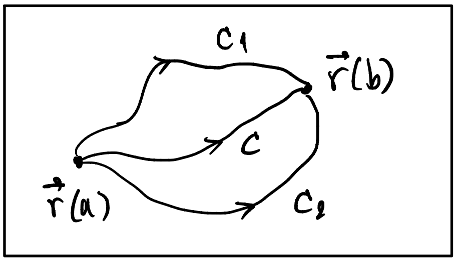
$h'(z) = 2 \rightarrow h(z) = 2z + C$

$\therefore f = x^2y + y^2z + 2z + C$

□ Path Independence

ให้ $\vec{F} = \vec{\nabla} f$ (อนุรักษ์) ; จาก Gradient Thm., $\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

โดย $C, C_1, C_2 : \vec{r}(a)$ ไป $\vec{r}(b)$



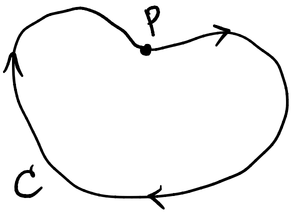
$\vec{F}$ เป็น Path Independence ก็ต่อเมื่อ

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

$*$ $\boxed{\vec{F} \text{ conservative} \longleftrightarrow \vec{F} \text{ path independent}}$ $*$

□ Closed curve

C เป็น Closed curve เมื่อ $\vec{r}(a) = \vec{r}(b)$; $C : \vec{r}(a)$ ไป $\vec{r}(b)$

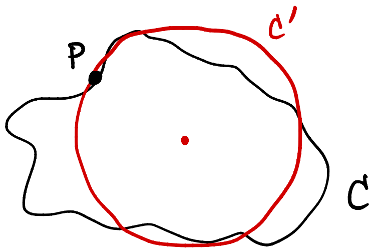


□ Circulation ของ $\vec{F}$ รอบ C $= \oint_C \vec{F}\cdot d\vec{r}$

Gauss's law for Φ_B

หาก $\vec{F}$ conserve/path independent แล้ว $\oint_C \vec{F}\cdot d\vec{r} = 0$

ดังนั้น เราสามารถเปลี่ยน C ให้ง่ายขึ้นได้



$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{\nabla} f\cdot d\vec{r} = f(P) - f(P) = 0 \quad \blacksquare$$

$\underline{\text{e.g.}}$ Check for path independence ของ

$$\int_C (1-ye^{-x})\,dx + e^{-x}\,dy \quad \text{ให้ } C: (0,1) \to (1,2)$$

$$\vec{F} = \langle 1-ye^{-x}, e^{-x} \rangle = \langle P, Q \rangle$$

$\underline{\text{Check}}$ $P_y = Q_x \iff$ อนุ $\iff$ p.i.

$$\frac{\partial}{\partial y}(1-ye^{-x}) = \frac{\partial}{\partial x} e^{-x}$$

$$-e^{-x} = -e^{-x} \quad \checkmark$$

$\therefore$ Integral ไม่ขึ้นกับวิถี $\iff \vec{F} = \vec{\nabla} f$

$\underline{\text{Sol 1.}}$ หา f (ยาก)

$\underline{\text{Sol 2.}}$ เปลี่ยน path : $x = t,\ y = 1+t$

$C: (0,1) \to (1,2)$

$\left.\begin{array}{l} x: 0 \to 1 \\ y: 1 \to 2 \end{array}\right\}$ let $x = t$, $y = x+1$

$\therefore y = t+1$

$$\therefore \int_C (1-ye^{-x})\,dx + e^{-x}\,dy = \int_0^1 \cancel{(}1-(1+t)e^{-t}\cancel{)}\,dt + e^{-t}\,dt$$

$$= \int_0^1 1 - te^{-t}\,dt = 1 + e^{-1} + e^{-t}\Big|_0^1 \quad \square$$

$\underline{\text{e.g.}}$ $\displaystyle\int_C yze^{-xy}\,dx + xze^{-xy}\,dy - e^{-xy}\,dz \ ; \ 1 \le t \le 2$

$$C: \vec{r}(t) = \langle t^2-1, t^2+1, t^2-2t \rangle$$

$\underline{\text{Sol.}}$ $\vec{F} = \langle \overset{P}{yze^{-xy}}, \overset{Q}{xze^{-xy}}, \overset{R}{-e^{-xy}} \rangle$

$P_y \overset{?}{=} Q_x$: $ze^{-xy} - xyze^{-xy} = ze^{-xy} - xyze^{-xy}$ ✓

$P_z \overset{?}{=} R_x$: $ye^{-xy} = ye^{-xy}$ ✓

$Q_z \overset{?}{=} R_y$: $xe^{-xy} = xe^{-xy}$ ✓

$\therefore \vec{F}$ มี integral อิสระจากวิถี

$$\vec{r}(1) = \langle 0, 2, -1 \rangle, \quad \vec{r}(2) = \langle 3, 5, 0 \rangle$$

C': $\vec{r} = \langle 0,2,-1 \rangle + t\langle 3,3,1 \rangle$ → เส้นตรง $\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$

$\to x = 3t,\ y = 2+3t,\ z = -1+t$

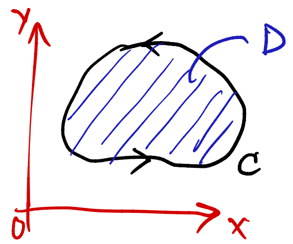
$dx = 3\,dt,\ dy = 3\,dt,\ dz = dt$

$$\int_{C'} \cdots = \int_0^1 \underline{t \xrightarrow{\text{แทน}} (x,y,z)}\,dt = \cdots \quad \square$$

□ Green's Theorem for Simply-connected domain (2D)

$\vec{F} = \langle P, Q \rangle$, $P = P(x,y)$, $Q = Q(x,y)$

ให้ D เป็น simply-connected domain :



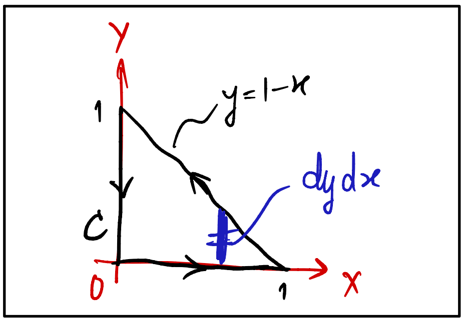
→ C เป็น simple closed curve. * ให้ทวนเข็มเป็น +C → ตามเข็มเป็น −C

กล่าวว่า

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA}$$

$\oint_C \vec{F}\cdot d\vec{r}$

(โจทย์ให้ C มา)



e.g. $\oint_C x^4\,dx + xy\,dy$

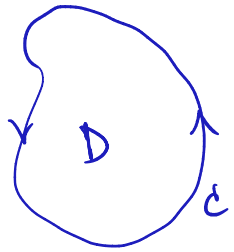
sol. D เป็นเชื่อมเดี่ยว, C อย่างง่าย, ปิด, ทิศ +

$$\oint_C x^4\,dx + xy\,dy = \iint_D \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^4)\,dA$$

$$= \iint_D y\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx$$

$$= 1/6 \quad \square$$

e.g. จงแสดงว่า: $\underbrace{\oint_C x\,dy}_{①} = \underbrace{-\oint_C y\,dx}_{②} = \underbrace{\frac{1}{2}\oint_C x\,dy - y\,dx}_{③}$



$\vec{F} = \langle P, Q \rangle$

①: $P = 0$, $Q = x$

$$\oint_C x\,dy = \iint_D (x)_x - (0)_y \, dA = \iint_D dA$$

②: $P = -y$, $Q = 0$

$$-\oint_C y\,dx = \iint_D (0)_x - (-y)_y \, dA = \iint_D dA$$

③: $P = -\frac{y}{2}$, $Q = \frac{x}{2}$

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \iint_D \left(\frac{x}{2}\right)_x - \left(-\frac{y}{2}\right)_y dA$$

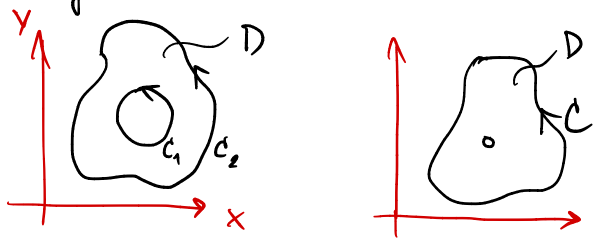
$$= \iint_D dA$$

$\square$

□ Green's Theorem for Multiply-connected Domain (D)

• For simply-connected domain : $\oint_C \vec{F}\cdot d\vec{r} = \oint P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$

– Multiply-connected Domain



e.g. $\oint \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ โดย C เป็นวงกลมหนึ่งหน่วย

We get: $P = \frac{-y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2}\,dy$

$x \neq 0, y \neq 0$ พร้อมกัน

$\therefore D = \{(x,y) : x^2+y^2 \leq 1 \, , \, (x,y) \neq (0,0)\}$

In this case, $\oint_C P\,dx + Q\,dy \neq \iint_D Q_x - P_y \, dA$

→ We let $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$ → $dx = -\sin t\,dt$, $dy = \cos t\,dt$

$\therefore \oint_C P\,dx + Q\,dy = \int_0^{2\pi} (-\sin t)^2\,dt + (\cos t)^2\,dt = 2\pi$ → ถูกต้อง

→ $Q_x - P_y = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0$ → ผิด

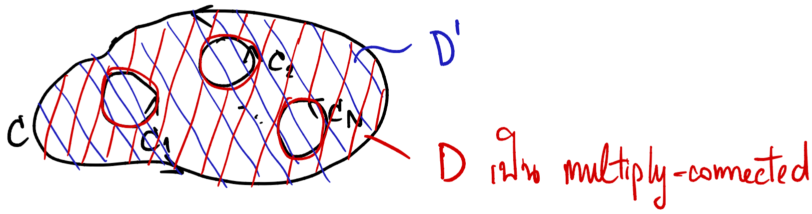
$\therefore \iint_D Q_x - P_y \, dA = 0$

$\therefore \oint P\,dx + Q\,dy \neq \iint_D Q_x - P_y \, dA$ ■

Def.

ให้ C เป็น simple curve, D' มีขอบล้อมด้วย C ที่ P,Q diff'able & continuous

โดยสมมติ D' มี ช่องโหว่ N ช่อง , แต่ละช่องล้อมด้วย $C_1, C_2, C_3, \dots, C_N$ (ทวนเข็ม)



ฉะนั้น D อยู่ใน C แต่อยู่นอก C_i ; $i = 1, 2, 3, \dots, N$

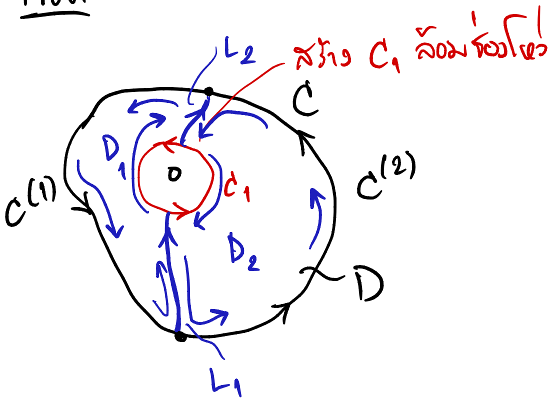
$$\boxed{\oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA + \sum_{i=1}^{N} \oint_{C_i} P\,dx + Q\,dy}$$

เขียนในรูปเชิงความหมายทาง geometric :

$$\boxed{\oint_C P\,dx + Q\,dy - \sum_{i=1}^{N} \oint_{C_i} P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA}$$

$\because$ บริเวณจากโค้งทั้งหมด C ลบ ช่องโหว่ทั้งหมด C_i ออก ได้พื้นที่ .

Proof



- ให้ C แบ่งเป็น $C^{(1)}$ และ $C^{(2)}$
- แบ่ง $D \to D_1, D_2$ $(D = D_1 \cup D_2)$

จะได้ D_1 วิ่งตาม Path $C^{(1)}, -C_1$ และ

D_2 วิ่งตาม Path $C^{(2)}, -C_1$ ด้วย

โดย D_1, D_2 เป็น simply-connected domains

$$D_1 \to C^{(1)} \cup L_1 \cup (-C_1^{(1)}) \cup L_2$$

$$D_2 \to C^{(2)} \cup (-L_2) \cup (-C_1^{(2)}) \cup (-L_1)$$

Green's Thm. on D_1 : $\displaystyle\oint_{C^{(1)} \cup L_1 \cup (-C_1^{(1)}) \cup L_2} P\,dx + Q\,dy = \iint_{D_1} Q_x - P_y \, dA$

$$\to \int_{C^{(1)}} P\,dx + Q\,dy + \cancel{\int_{L_1} P\,dx + Q\,dy} \overset{-}{+} \int_{-C_1^{(1)}} P\,dx + Q\,dy + \cancel{\int_{L_2} P\,dx + Q\,dy} = \iint_{D_1} Q_x - P_y \, dA \quad \text{—(1)}$$

D_2 :

$$\to \int_{C^{(2)}} P\,dx + Q\,dy \overset{-}{+} \cancel{\int_{-L_2} P\,dx + Q\,dy} \overset{-}{+} \int_{-C_1^{(2)}} P\,dx + Q\,dy \overset{-}{+} \cancel{\int_{-L_1} P\,dx + Q\,dy} = \iint_{D_2} Q_x - P_y \, dA \quad \text{—(2)}$$

(1) + (2) : $\displaystyle\oint_C P\,dx + Q\,dy - \oint_{C_1} P\,dx + Q\,dy = \iint_{D_1 \cup D_2} Q_x - P_y \, dA = \iint_D Q_x - P_y \, dA$

$$\therefore \oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA + \oint_{C_1} P\,dx + Q\,dy \quad \blacksquare$$

Special case : $\vec{F}$ เป็นสนามอนุ. $(\vec{F} = \langle P, Q\rangle)$

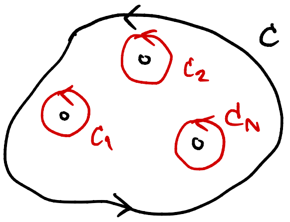
$\therefore Q_x = P_y$:

$$\boxed{\oint_C P\,dx + Q\,dy = \sum_{i=1}^{N} \oint_{C_i} P\,dx + Q\,dy}$$

eg. ใน C มีจุด $(x_1, y_1), \dots, (x_N, y_N)$ ที่เป็นช่องโหว่ (ไม่อยู่ในโดเมน)

(กล่าวคือ ถ้าสมมติ P,Q จะ diff'able และ continuous ทุกจุดใน C ยกเว้น (x_i, y_i))

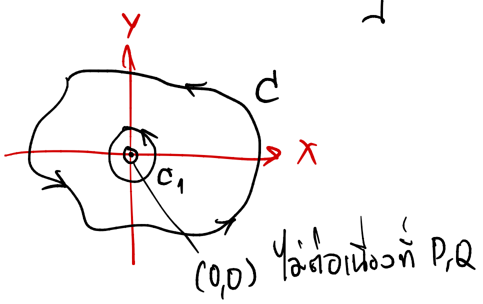
กำหนด C_i เป็นวงกลมจุดศูนย์กลาง (x_i, y_i) มีรัศมีน้อยๆ ให้ C_i อยู่ภายใน C



$$\oint_C P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \dots + \oint_{C_N} P\,dx + Q\,dy$$

* **e.g.** ให้ $F(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ จงแสดงว่า $\oint_C \vec{F}\cdot d\vec{r} = 2\pi$

เมื่อ C เป็นเส้นโค้งใดๆ รอบ O



ตรวจสอบ $Q_x = \frac{y^2 - x^2}{(x^2+y^2)^2}$, $P_y = \frac{y^2 - x^2}{(x^2+y^2)}$ $\Big\}$ $Q_x = P_y$ ✓

(ตอบ) ให้ C_1 วงกลมรัศมี r รอบ $(0,0) \to C_1 : x = r\cos t,\ y = r\sin t,\ 0 \le t \le 2\pi$, r น้อยๆ

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \oint_{C_1} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-r\sin t}{r^2}, \frac{r\cos t}{r^2} \right\rangle \cdot \langle -r\sin t, r\cos t \rangle\, dt$$

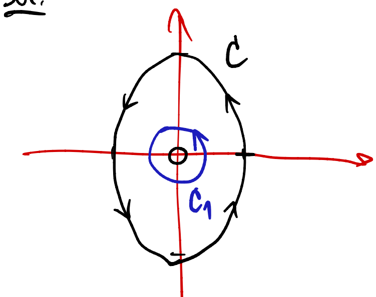
$$= \int_0^{2\pi} \frac{\cancel{r^2}\sin^2 t}{\cancel{r^2}} + \frac{\cancel{r^2}\cos^2 t}{\cancel{r^2}}\, dt$$

$$= \int_0^{2\pi} dt = 2\pi \quad \square$$

$*$ e.g. จงหา $\displaystyle\oint_C \frac{x^2y}{(x^2+y^2)^2}\,dx - \frac{x^3}{(x^2+y^2)^2}\,dy$

เมื่อ C เป็นโค้ง $x=2\cos t,\ y=3\sin t,\ 0\le t\le 2\pi$, C วิ่งจาก $0\to 2\pi$.

sol.



เป็นหลายเชิง.

$$P=\frac{x^2y}{(x^2+y^2)^2}\ ;\ P_y=\frac{x^4-3x^2y^2}{(x^2+y^2)^3}$$

$$Q=\frac{-x^3}{(x^2+y^2)^2}\ ;\ Q_x=\frac{x^4-3x^2y^2}{(x^2+y^2)^3}$$

$$\Big\}\ Q_x=P_y$$

เลือก C_1 เป็นวงกลมรัศมี 1

$$C_1:\ x=\cos t,\ y=\sin t,\ 0\le t\le 2\pi$$

$$dx=-\sin t\,dt,\ dy=\cos t\,dt$$

ใช้ Green's Thm. แบบหลายเชิง ที่ $Q_x=P_y$:

$$\oint_C P\,dx+Q\,dy=\oint_{C_1}P\,dx+Q\,dy$$

$$=\int_0^{2\pi}\frac{\cos^2t\cdot\sin t}{1^2}(-\sin t)-\frac{\cos^3t}{1^2}(\cos t)\,dt$$

$$=\int_0^{2\pi}(-\cos^2t\sin^2t)-\cos^4t\,dt$$

$$=\int_0^{2\pi}-\cos^2t\,(\underbrace{\sin^2t+\cos^2t}_{1})\,dt$$

$$=-\int_0^{2\pi}\frac{1+\cos 2t}{2}\,d\cancel{t}\ \frac{2t}{2}$$

$$=-\pi$$

$\square$

Remind

$$\cos^2x=\frac{1+\cos 2x}{2}$$

$$\sin^2x=\frac{1-\cos 2x}{2}$$