Vectors, Mapping, Linearity

Vectors

Vector x : $\vec{x} = \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \left\langle \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \right\rangle}_{\text{column vector}}$; $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ $(\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R})$

$\vec{x}^T = \langle x_1, x_2, \ldots, x_n \rangle \rightsquigarrow$ row vector

e.g. 2D vector represents 2D coordinate system.

Real Vectors
- $\to$ Magnitude (Euclidean norm) : $\|\vec{x}\| = \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\vec{x}^T \vec{x}}$
- $\to$ Direction : θ

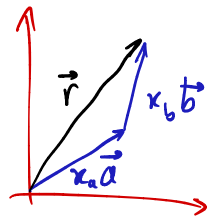
Operations on Vectors (Binary Ops.)

1. Addition : $\langle x_1, \ldots, x_n \rangle + \langle y_1, \ldots, y_n \rangle = \langle x_1 + x_2, \ldots, x_n + y_n \rangle$
2. Scalar Multiplication : $k \langle x_1, \ldots, x_n \rangle = \langle kx_1, \ldots, kx_n \rangle$

Linear Combination : "weighed sum"

x_i = coefficient of $\vec{a}_i$

$$\vec{a} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots + x_i \vec{a}_i + \ldots + x_n \vec{a}_n$$



$\vec{r} = x_a \vec{a} + x_b \vec{b}$

($\vec{r}$ สามารถเขียนในรูป linear comb. ของ $\vec{a}, \vec{b}$ ได้)

e.g. $x\langle 2,1 \rangle + y \langle -1,2 \rangle = \langle 8,-1 \rangle \quad \leftrightarrow \quad \begin{cases} 2x - y = 8 \\ x + 2y = -1 \end{cases}$

มี solution เดียว $\quad 3\langle 2,1 \rangle - 2\langle -1,2 \rangle = \langle 8,1 \rangle$.

def Standard Basis (Vectors)

$$\left.\begin{aligned} \vec{e}_1 &= \langle 1,0,0,\ldots,0 \rangle \\ \vec{e}_2 &= \langle 0,1,0,\ldots,0 \rangle \\ \vec{e}_3 &= \langle 0,0,1,\ldots,0 \rangle \\ &\vdots \\ \vec{e}_n &= \langle 0,0,0,\ldots,1 \rangle \end{aligned}\right\}$$

Any $\vec{x} = \langle x_1, x_2, x_3, \ldots, x_n \rangle$ can be expressed with standard basis :

$$\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \ldots + x_3 \vec{e}_3$$

In 3D Euclidean space :

$$\begin{aligned} \vec{e}_x &= \langle 1,0,0 \rangle = \hat{x} = \hat{i} \\ \vec{e}_y &= \langle 0,1,0 \rangle = \hat{y} = \hat{j} \\ \vec{e}_z &= \langle 0,0,1 \rangle = \hat{z} = \hat{k} \end{aligned}$$

Matrices

$$M_{m\times n} = \underbrace{\begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix}}_{n \text{ cols.}} \Big\} \; m \text{ rows}$$

Matrix-Vector Multiplication

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \longleftrightarrow M\vec{x} = \vec{u}$$

In this example,

$M : \mathbb{R}^4 \to \mathbb{R}^3$

└ Some mapping / transformation.

$$= \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} x_2 + \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} x_3 + \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix} x_4$$

Eigenvectors & Eigenvalues

$M_m(\mathbb{R})$ | $M_{m,n}(\mathbb{R}) \simeq \mathbb{R}^{mn}$

$\vec{x} \in \mathbb{R}^m$, $A \in M(m, \mathbb{R})$, $\lambda \in \mathbb{R}$

There is eigenvector $\vec{x}$ and eigenvalue λ (scalar) s.t. $A\vec{x} = \lambda\vec{x}$

e.g. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

$A\vec{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$: $\lambda_x = 3$, $\vec{x}$ is eigenvector.

$A\vec{y} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$: $\lambda_y = -1$, $\vec{y}$ is eigenvector.

Transformation (Map)

$F : \mathbb{R}^n \to \mathbb{R}^m$: $A\vec{x} = \vec{y}$; $A \in M_{m,n}(\mathbb{R})$

e.g. $\mathbb{R}^n \to \mathbb{R}^m$ $(n \geq m)$

Interesting properties: Domain, Range, Pre-Image, Image, 1-1, onto

e.g. $\mathbb{R}^n \to \mathbb{R}^n$

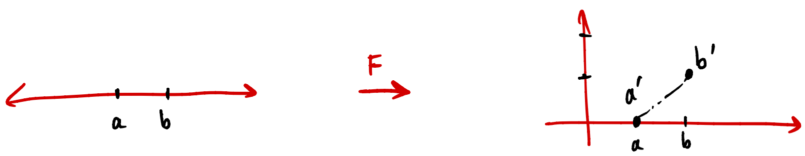
e.g. $J : \mathbb{R}^2 \to \mathbb{R}^2 : J\langle x, y\rangle^T = \langle -y, x\rangle^T$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(Dilation)

Expansion, Contraction

e.g. $F : \mathbb{R} \to \mathbb{R}^2 : F(x) = \langle x, x-1\rangle$ เป็น 1-1 ✓, onto ?



F ทำให้ space expands.

Linearity

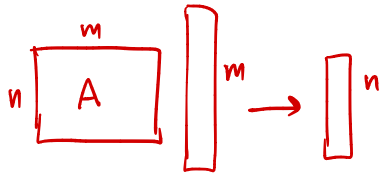
<u>def</u> $F: \mathbb{R}^n \to \mathbb{R}^m$ is linear if $\wedge \begin{cases} 1. & \vec{x},\vec{y} \in \mathbb{R}^m \to F(\vec{x}+\vec{y}) = F(\vec{x}) + F(\vec{y}) \\ 2. & \vec{x} \in \mathbb{R}^m,\ c \in \mathbb{R} \to F(c\vec{x}) = cF(\vec{x}) \end{cases}$

i.e. $\mathbb{R}$-module homomorphic in addition (Additive map)

F is a linear map. / $\mathbb{R}$-module homomorphism.

<u>e.g.</u> Matrix-Vector multiplication is linear.

<u>def</u> Matrix of Linear Transformation



$T: \mathbb{R}^m \to \mathbb{R}^n : T(\vec{x}) = A\vec{x}\ ;\ A \in M_{n,m}(\mathbb{R})$

<u>e.g.</u> $\vec{x} = x_1\vec{e}_1 + \ldots + x_n\vec{e}_n$

$T(\vec{x}) = x_1 T(\vec{e}_1) + \ldots + x_n T(\vec{e}_n)$ ✓

$\to A\vec{x} = A(x_1\vec{e}_1 + \ldots + x_n\vec{e}_n)\ ;$ Let $A = [\vec{a}_{i1}, \ldots, \vec{a}_{in}]$

$A\vec{x} = x_1 \underbrace{\vec{a}_{i1}^{\,T}\vec{e}_1} + \ldots + x_n \underbrace{\vec{a}_{in}^{\,T}\vec{e}_n}$

$A\vec{x} = x_1 T(\vec{e}_1) + \ldots + x_n T(\vec{e}_n)$ ✓

* $A\vec{x} = c_1 A x_1 + c_2 A x_2 + \ldots c_m A x_m$

* $A\vec{x} = \begin{vmatrix} r_1 A \cdot \vec{x} \\ r_2 A \cdot \vec{x} \\ \vdots \\ r_n A \cdot \vec{x} \end{vmatrix}$

Linear System

$T: \mathbb{R}^m \to \mathbb{R}^n : T(\vec{x}) = A\vec{x}$; $\vec{x} \in \mathbb{R}^m$, $A \in M_{n,m}(\mathbb{R})$.

Always

; $\begin{cases} T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) \\ T(k\vec{x}) = kT(\vec{x}) \text{ ; } k \in \mathbb{R} \end{cases}$

Pre-image problem : If $\vec{b}$ is image of T. How can we find all $T^{-1}(\vec{b})$

(Finding $\vec{x}$ from $A\vec{x} = \vec{b}$

e.g. $\underset{A}{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}} \longleftrightarrow \begin{cases} x - y = -1 \\ y - z = -1 \\ -x + z = 2 \end{cases}$

Gauss-Jordan Strategy : Reduced Row Echelon Form (RRE)

Augmented Matrix : $\begin{array}{cccc} x & y & z & \vec{b} \\ \end{array}$ $\left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 2 \end{array}\right)$

e.g. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \end{array}\right)$: $\langle x_1, x_2, x_3 \rangle = \langle -2, 3, t \rangle$

$x_1 = -2$
$x_2 = 3$
$x_3 = t$ (free)

* RE	1.	0's row on the bottom of all rows.
	2.	Leading entry goes lower right.
	3.	Below leading entries are 0's.
* RRE	4.	Leading entries are 1's. (leading 1) → pivot
(3.')	5.	Leading entry's column has other entries be 0's.

e.g. $\left(\begin{array}{cc|c} 1 & a & b \\ 0 & 0 & 0 \end{array}\right)$

pivot　free

$\to x_1 + a x_2 = b$

$\to x_1 = b - a x_2$

$\to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} + \begin{pmatrix} -a \\ 1 \end{pmatrix} s$; $\forall s \in \mathbb{R}$

: $x_2 = s$
$x_1 = b - as$ ✓

Elimination : Generating RRE

1. Push all zeroes to the bottom.
2. Push row with largest leftmost element to the top.
3. Top row $\longrightarrow$ make pivot.
4. Repeats to make it RE
5. Swap to make it RRE.

Elementary Row Operations

1. Swap.
2. Scalar multiplication.
3. Add/Sub. scalar multiple.

Case Homogeneous

e.g.

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right) \longleftrightarrow \begin{cases} x_1 - 2x_3 = 0 \\ x_2 + 3x_3 = 0 \end{cases}$$

$x_3 = \text{free}$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} t \; ; \forall t \in \mathbb{R}$$

e.g.

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \quad a = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ (homogenous)}$$

$$\begin{pmatrix} 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longleftrightarrow \begin{cases} x_2 + 3x_3 - 2x_5 = 0 \\ x_4 + 4x_5 = 0 \end{cases}$$

x_3 free, x_5 free, x_1 free

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t_1 \underbrace{\begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{= h_1} + t_2 \underbrace{\begin{pmatrix} 0 \\ 2 \\ 0 \\ -4 \\ 1 \end{pmatrix}}_{= h_2} + t_3 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{= h_3}$$

$; \forall t_1, t_2, t_3 \in \mathbb{R}$.

$$\left.\begin{aligned} H &= (h_1 \;\; h_2 \;\; h_3) \\ \vec{t} &= (t_1, t_2, t_3) \\ \vec{x} &= (x_1, x_2, x_3, x_4, x_5) \end{aligned}\right\} \quad H\vec{t} = \vec{x}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ -3 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

e.g.
$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 0 \\ 3 \\ 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}$$

<u>Nonhomogeneous</u> * Sol. = particular + homogeneous

e.g.
$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \longrightarrow \begin{cases} x_1 + 2x_2 = 7 \\ x_4 = 3 \end{cases}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 3 \end{pmatrix} + t_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

* $T: R^m \to R^n$ is 1-1 $\leftrightarrow$ RRE has pivot in every column.
* $T: R^m \to R^n$ is onto $\leftrightarrow$ RRE has pivot in every row.

<u>Geometric Vectors</u> : $\vec{v}$: Addition / Scalar multiplication } Linear combination

Element-wise / Entry-wise / Hadamard Product: $\vec{v} \odot \vec{v} = \vec{v} * \vec{v}$

Vector length from Euclidean norm $\|\vec{v}\| = \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

Normalized (Unit Vector) : $\hat{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Laws of Cosine $\quad C^2 = A^2 + B^2 - 2AB\cos \overset{\theta}{\angle AB}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

$$\|\vec{a} \cdot \vec{b}\| \leq |\vec{a}||\vec{b}|$$

<u>Line</u> : Line spanned by $\vec{v} \leftrightarrow a\vec{v}$; $a \in \mathbb{R}^*$

$L: \vec{p} + t\vec{v}$ $\left(L: \vec{P} = \vec{P}_0 + t\vec{A} \right)$

- Consider $ax + by = 0$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$

  Let $\vec{x} = \langle x, y \rangle$, $\vec{a} = \langle a, b \rangle$, $\vec{x} \cdot \vec{a} = 0$,

  meaning $\vec{x}$, $\vec{a}$ are <u>orthogonal</u> (perpendicular).

- Consider $ax + by = c \rightarrow \vec{x}_0 \cdot \vec{a}$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = c$

  Let $\vec{x} = \langle x, y \rangle$, $\vec{a} = \langle a, b \rangle$, $\vec{x} \cdot \vec{a} = c$,

  $\hookrightarrow \vec{x} = \vec{x}_p + t\vec{n}$

  $(a \ \ b \mid c) \sim (1 \ \ b/a \mid c/a)$

  $\therefore \underbrace{(x, y)}_{\vec{x}} = \underbrace{(c/a, 0)}_{\vec{x}_p} + t \underbrace{(-b/a, 1)}_{\vec{n}}$

<u>e.g</u> Line through $(2,3)$, perpendicular to $(1,-2)$

$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0 \rightarrow x - 2y = 0$

$\hookrightarrow x - 2y = b$; $x = 2, y = 3 \rightarrow b = -4$

$\Big\} \ x - 2y = -4$

<u>Plane</u> $\vec{x} \cdot \vec{a} = d \rightarrow \vec{x}_0 \cdot \vec{a}$; $\vec{x} = (x, y, z)$, $\vec{a} = (a, b, c)$

$\hookrightarrow \vec{x} = \vec{x}_p + t_1 \vec{n}_1 + t_2 \vec{n}_2$

$(a \ \ b \ \ c \mid d) \sim (1 \ \ b/a \ \ c/a \mid d/a)$

$\therefore (x, y, z) = (d/a, 0, 0) + t_1(-b/a, 1, 0) + t_2(-c/a, 0, 1)$

$\rightarrow$ Solving plane from 3 points : $(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)$.

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d \\ a_2 & b_2 & c_2 & d \\ a_3 & b_3 & c_3 & d \end{array} \right)$$

H ผ่าน origin $:= \vec{a}^{\perp}$

Hyperplane (H) in $\mathbb{R}^m$ is a flat in $\mathbb{R}^m$ of dimension $m-1$.

e.g. H of $\mathbb{R}^2$ is line, H of $\mathbb{R}^3$ is plane, ...

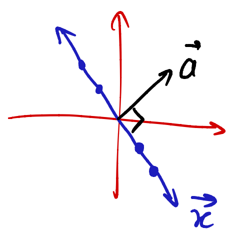
flat dim 1   flat dim 2

$\vec{a}^{\perp}$ is homogeneous solution set.

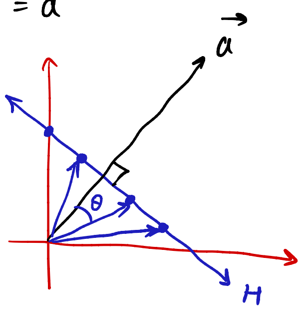
$H: \vec{x}_p + \vec{a}^{\perp}$

Intersection(s) of n H of $\mathbb{R}^m$ yields solution of flat $\dim_{max}$ m.
(usually $m-n$ when all n H are independent)

$\oplus$ $\vec{a}\cdot\vec{x} = 0$



$\vec{a}\cdot\vec{x} = d$



$\vec{a}\cdot\vec{x} = ax\cos\theta = d$

$\therefore \cos\theta = \dfrac{d}{ax}$

<u>Linearly Independent</u>

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ are linearly independent $\leftrightarrow$ $[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n \mid \vec{0}_m]$ has only trivial (particular) solution.

$[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]\vec{x}$; $\vec{x} \in \mathbb{R}^n$

dependent $\leftrightarrow$ $\vec{x}$ has homogeneous parts and $\vec{x} \neq \vec{0}_n$

Span : A set of linear combinations of vectors.

Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subseteq \mathbb{R}^m$; $\vec{v}_i \in \mathbb{R}^m$

$H = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p : c_i \in \mathbb{R}\}$

Perp : "Orthogonal Complement" Intersection of hyperplanes $a_i^\perp$. & $S^\perp$ is a subspace of $\mathbb{R}^m$

$\{a_1^\perp, a_2^\perp, \ldots, a_k^\perp\} = \ldots$ & $\{\}^\perp = \mathbb{R}^m$

Properties : $\vec{0}_m \subseteq \text{Span}\{\ldots\} \rightarrow \text{Span}\,\emptyset = \{\vec{0}_m\}$

Closure: Let $\vec{u}, \vec{v} \in H$, $\vec{u}+\vec{v} \in H$ and $c\vec{u} \in H$.

$\therefore$ H is a <u>subspace</u> of $\mathbb{R}^m$. ($\mathbb{R}^m$ is largest subspace and

i.e., $\vec{0}_m$ (hyperplane is not necessarily subspace.) but is "affine subspace" $\{\vec{0}_m\}$ is smallest subspace in $\mathbb{R}^m$)

* If Origin $\notin$ H, then H is not a subspace of $\mathbb{R}^m$ *

$H = \text{Span}\{\vec{v}_i \ldots\}$; $\vec{v}_i \in \mathbb{R}^m \rightarrow$ H is subspace of $\mathbb{R}^m$ spanned by $\vec{v}_i$.

Column Space (Row Space

Let $A = [\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n]$; $\vec{v}_i \in \mathbb{R}^m \rightarrow A \in M_{m,n}(\mathbb{R})$,

$\text{Col}\,A = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ – Span of column of A.

Row A $= \{A^T\vec{x} : \vec{x} \in \mathbb{R}^m\}$

$\text{Col}\,A = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\} \because = \{x_1\vec{v}_1 + \ldots + x_n\vec{v}_n\}$

* $\vec{b} \in \text{Col}\,A \leftrightarrow (A \mid \vec{b})$ has solutions.

* Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m : \vec{x} \mapsto A\vec{x}$; $A \in M_{m,n}(\mathbb{R})$,

$\underset{\text{(image)}}{\text{range}}\,T = T[\mathbb{R}^n] = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\} = \text{Col}\,A$

[right] **Null Space** (Left nullspace

$\text{Nul}\,A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}_m\}$ – Perp. to Row space ($(\text{Row}\,A)^\perp = \text{Nul}\,A$)

* $\text{Nul}\,A \leq \mathbb{R}^n$; $A \in M_{m,n}(\mathbb{R})$

$\text{Nul}\,A^T = \{\vec{x} \in \mathbb{R}^m : A^T\vec{x} = \vec{0}_n\}$

Perp. to column space ($(\text{Col}\,A)^\perp = \text{Nul}\,A^T$)

Let $\leq$ mean "is a subspace of".

$\vec{y}A = \vec{0} \rightarrow A^T\vec{y}^T = \vec{0}^T : A^T\vec{x} = \vec{0}$

Basis : $B \subseteq H$, $H \leq \mathbb{R}^m$ and B is linearly independent and $\operatorname{Span} B = H$.

Standard Basis of $\mathbb{R}^m$: $B = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m\}$; $\vec{e}_j$ is column of I_m.

* Pivot columns of A form a basis of $\operatorname{Col} A$. $\operatorname{Nul} A = \left\{\begin{pmatrix}5\\1\\1\end{pmatrix}t\right\} = \operatorname{Span}\left\{\begin{pmatrix}5\\1\\1\end{pmatrix}\right\}$

e.g. $A = \begin{pmatrix} 1 & -3 & -2 \\ 0 & 1 & -1 \\ -2 & 3 & 7 \end{pmatrix} \sim \begin{array}{c} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \\ P & P & \end{array} \end{array}$ because B is linearly independant.

$\operatorname{Col} A = \ldots$

$\therefore$ $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \right\}$ is a basis of $\operatorname{Col} A$.

Let $H \leq \mathbb{R}^m$ and B be basis of H. If C is another basis of H, then $|B| = |C|$. **Dimension** is number of vectors in basis, i.e., $|B|$.

→ $\operatorname{Span} \overset{B}{\emptyset} = \{\vec{0}_m\}$, $\dim \{\vec{0}_m\} = 0$

→ $\dim \mathbb{R}^m = m$; $B = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m\}$

→ $\operatorname{rank} A = \dim \operatorname{Col} A$ (# of pivot columns in A) fixed variables

from basis (lin. indep.) → spans subspace $H \leq \mathbb{R}^m$

→ $\operatorname{nullity} A = \dim \operatorname{Nul} A$ (# of homogeneous (free) variables.)

Rank Theorem : $A \in M_{m,n}(\mathbb{R}) \to \operatorname{rank} A + \operatorname{nullity} A = n$

Invertible Matrix Theorem

These statements are equivalent.

1. A is invertible
2. $A \sim I_n$
3. A has n pivots
4. $A\vec{x} = \vec{0}$ has only trivial sol.
5. Columns of A are lin. indep. **
6. $T: \vec{x} \mapsto A\vec{x}$ is 1–1.
7. $A\vec{x} = \vec{b}$ has solution for any $\vec{b}$
8. $\det A \neq 0$
9. $\mathrm{Col}\,A = \mathbb{R}^n$; $\mathrm{rank}\,A = n \rightarrow \mathcal{B}$ is of $\mathbb{R}^n$
10. $\mathrm{Nul}\,A = \{\vec{0}_n\}$; $\mathrm{nullity}\,A = 0$

Perp $\Leftrightarrow$ Span

- Explicit representation : $S = \mathrm{Span}\,V$
- Implicit representation : $S = P^\perp$

$\Big\}\ S \leq \mathbb{R}^n$

★ Perp $\rightarrow$ Span : Given P, find V s.t. $P^\perp = \mathrm{Span}\,V$

$P = \{\text{rows of } A\}$, $\mathrm{Nul}\,A$: perp to row space

$\therefore \mathrm{nul}\,A = P^\perp = \{\vec{x} : A\vec{x} = \vec{0}\}$

★ Span $\rightarrow$ Perp : Given V find P s.t. $P^\perp = \mathrm{Span}\,V \rightarrow$ Solve for $\mathrm{Col}\,A$

confusing!

* $(\mathrm{Col}\,A)^\perp = \mathrm{Nul}\,A^T$

Span(...)? $\sim$ $\mathrm{Col}\,A = (\mathrm{Nul}\,A^T)^\perp$

Span of columns

* $(\mathrm{Row}\,A)^\perp = \mathrm{Nul}\,A$

$\therefore A = (v_1\ v_2 \cdots v_n)$

$A^T = \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix}$

Change of Basis (Plus!)

→ Let $H \le \mathbb{R}^m$ and $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$: Span $B = H$, (Basis B)

$\forall \vec{x} \in H$: $\exists! c_i \in \mathbb{R}$: $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p$: $\vec{c}$ = sol. of $(\vec{v}_1\ \vec{v}_2 \dots \vec{v}_p \mid \vec{x})$ $= [\vec{x}]_B$

$$[\vec{x}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix}$$: Coordinate vector relative to basis B.

also $\mathbb{R}$-module homomorphic in addition. $[\vec{x}]_B + [\vec{y}]_B = [\vec{x}+\vec{y}]_B$

→ Let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of $\mathbb{R}^n$, $C = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ of $\mathbb{R}^m$, $T : \mathbb{R}^n \to \mathbb{R}^m : \vec{x} \mapsto A\vec{x}$; $A \in M_{m,n}(\mathbb{R})$

j^{th} column of A is $[T(\vec{v}_j)]_C$: $[T]_B^C = \left([T(\vec{v}_1)]_C\ [T(\vec{v}_2)]_C \dots [T(\vec{v}_n)]_C\right) = A$

Matrix for T relative to bases B and C.

→ From $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$, $T(\vec{x}) = c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n)$.

$\therefore [T(\vec{x})]_C = c_1[T(\vec{v}_1)]_C + \dots + c_n[T(\vec{v}_n)]_C$ / $\{v_i\} = B$

$\therefore [T(\vec{x})]_C = [T]_B^C [\vec{x}]_B$

→ Mapping

$$\begin{array}{ccc} \vec{x} & \xrightarrow{\quad T \quad} & T(\vec{x}) \\ \downarrow \text{change of basis to } B & & \downarrow \text{change of basis to } C \\ [\vec{x}]_B & \xrightarrow{\quad [T]_B^C \times \quad} & [T]_B^C[\vec{x}]_B = [T(\vec{x})]_C \end{array}$$

→ Special Case $T : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto Ax$; $A \in M_n(\mathbb{R})$

$[T]_B$

→ Composition $S : \mathbb{R}^n \to \mathbb{R}^m$, $T : \mathbb{R}^m \to \mathbb{R}^p$

1. $T \circ S : \mathbb{R}^n \to \mathbb{R}^p$
2. If B is of $\mathbb{R}^n$, C is of $\mathbb{R}^m$, D is of $\mathbb{R}^p$, then

$$[T \circ S]_B^D = [T]_C^D [T]_B^C$$