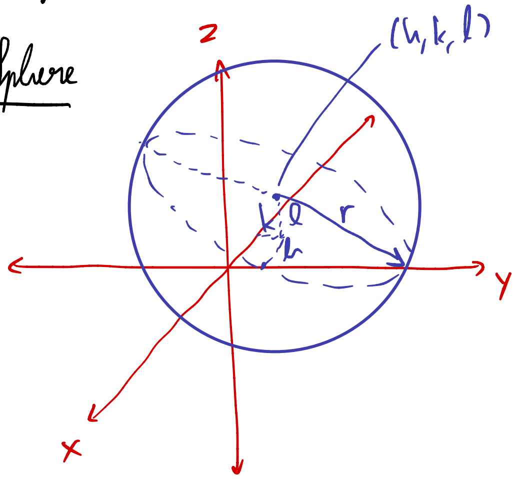
Surfaces of Revolutions

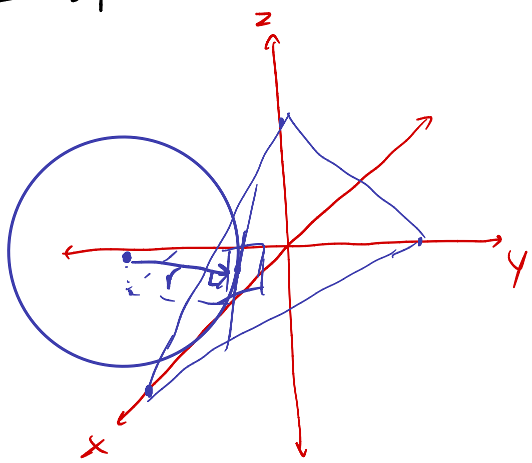
1. Sphere



สมการทรงกลม (Sphere)

$$\boxed{(x-h)^2+(y-k)^2+(z-l)^2=r^2}$$

eg. Sphere @ (2, −3, 1) สัมผัส $x+2y+2z=10$



$x+2y+2z-10=0$, $(2,-3,1)$

$$d=\frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}$$

$$d=\frac{|1\cdot 2-2\cdot 3+2\cdot 1-10|}{\sqrt{1^2+2^2+2^2}}=\frac{12}{3}$$

$\therefore d=4=r$

$\rightarrow (x-2)^2+(y+3)^2+(z-1)^2=16$

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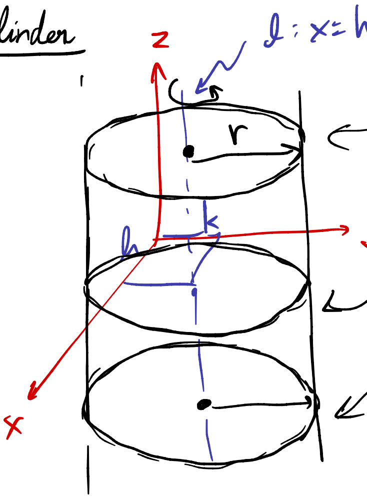
2. Cylinder



planar $z=0$: $(x-h)^2+(y-k)^2=r^2$

z $\forall$ any : $(x-h)^2+(y-k)^2=r^2$

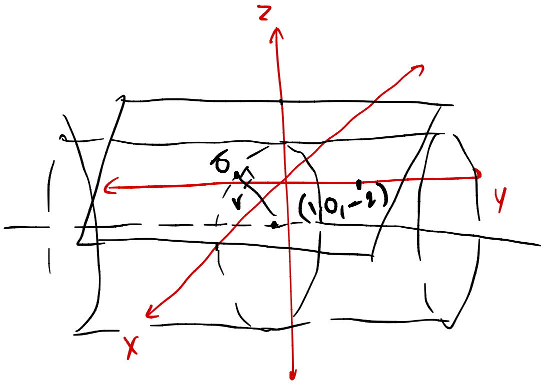
$\therefore$ สมการทรงกระบอก (Cylinder)

$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

Eq. is invariant to z

(Any combination of (x,y), (y,z), (x,z)).

eg. CYL สัมผัส $3x+4z=10$, $x=1, z=-2$ เป็นแกน



$(1,0,-2)$, $3x+4z-10=0$

$$d=\frac{|3(1)-4(2)-10|}{\sqrt{3^2+4^2}}$$

$$d=\frac{15}{5}=3=r$$

$$\rightarrow (x-1)^2+(z+2)^2=9$$

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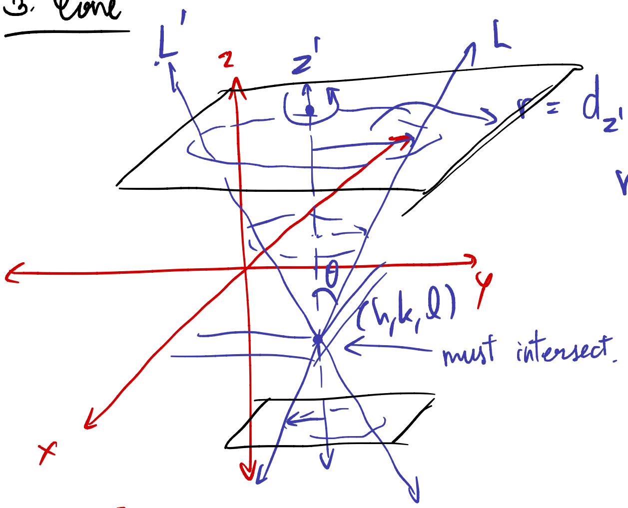
3. Cone



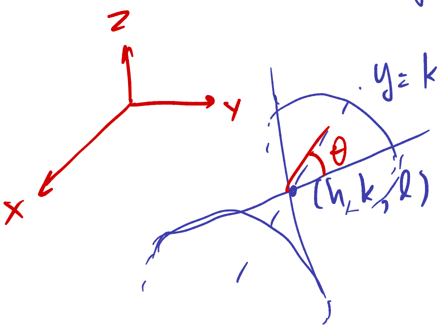
$r = d_{z'l} = |z'\tan\theta|$

$r = |(z-l)\tan\theta|$

ที่ $(x,y)=(h,k)$

สมการกรวย (cone)

$$\rightarrow \boxed{(x-h)^2+(y-k)^2=(z-l)^2\underbrace{\tan^2\theta}_{=k}}$$



Any combinations of $(x,y,z),(y,z,x),(x,z,y)$

$$\rightarrow (y-k)^2+(z-l)^2=(x-h)^2\tan^2\theta$$

*แทนแล้วจริง

eg. กรวยที่ $(3,-1,2)$, ขนาน X , มี $x=3+t, y=-1-t, z=2+2t$ อยู่บนกราฟ

set : $(y+1)^2+(z-2)^2=k(x-3)^2$

$(-1-t+1)^2+(2+2t-2)^2=k(3+t-3)^2$

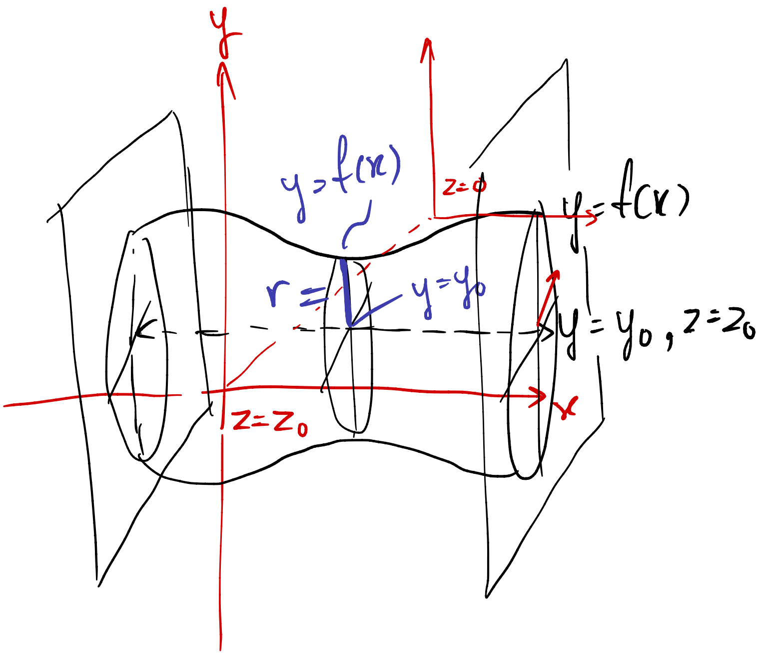
$5t^2=kt^2 \rightarrow k=5$

$\tan^2\theta=5$

$\tan\theta=\sqrt{5}$

$\therefore$ สมการคือ $(y+1)^2+(z-2)^2=5(x-3)^2$

หรือถ้าขึ้นต่อ ๆ เพื่อ profile



ในแนวขนาน X (plane ⊥ x)

$$(\Delta y)^2 + (\Delta z)^2 = r^2 \quad (r = |f(x) - y_0|)$$

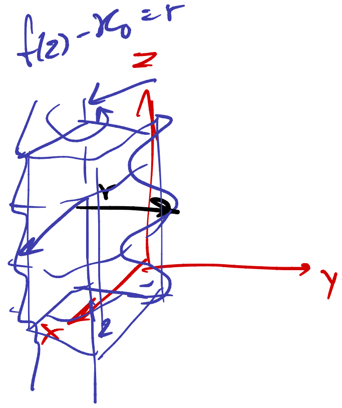
$$\boxed{(y-y_0)^2 + (z-z_0)^2 = (f(x)-y_0)^2}$$

eg. หมุน $4y = x^2 + 4,\ z = 1$

รอบ $y = -1,\ z = 1$ เปลี่ยนแกนนอนหมุน

$\rightarrow f(x) = \frac{x^2}{4} + 1$ บน plane $z = 1$

$$\therefore (y+1)^2 + (z-1)^2 = \left(\frac{x^2}{4} + 2\right)^2$$



eg. หมุน $x = \sin z,\ y = 0$ รอบ $x = 2,\ y = 0$

$\rightarrow f(z) = \sin z$ บน plane $y = 0$

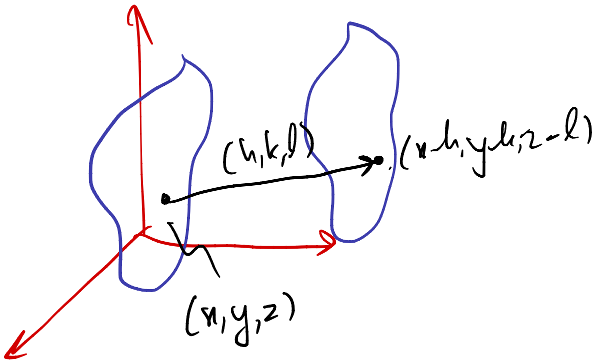
$$\therefore (x-2)^2 + y^2 = (\sin z - 2)^2$$

$$\text{or } (x-2)^2 + y^2 = (2 - \sin z)^2$$

Transformational Geometry

① Translation (offset)

$$(x,y,z)-(h,k,l) = (x-h,\ y-k,\ z-l)$$



② Dilation & Contraction (scale)

$$\begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/a \\ 1/b \\ 1/c \end{bmatrix}$$

$$(x,y,z) \longrightarrow \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$$

$a>1$: dilation

$0<a<1$: contraction

$a<0$: reflection + scaling

eg. $x^2+y^2+z^2=1$; 2 เท่า X, -3 เท่า Y, $\frac{1}{2}$ เท่า Z

$$\longrightarrow \frac{x^2}{4}+\frac{y^2}{9}+4z^2=1$$

eg. $(2x+3)^2+\frac{1}{4}y^2+(z-1)^2=1$ กับ $x^2+y^2+z^2=1$

สัมพันธ์กันอย่างไร:

Ⓐ

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x+3 \\ \frac{1}{2}y \\ z-1 \end{bmatrix}$$

scale $x = \frac{1}{2}$, $\Delta x = -3$

scale $y = 2$, $\Delta z = 1$

Ⓑ

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2x+3 \\ \frac{1}{2}y \\ z-1 \end{bmatrix}$$

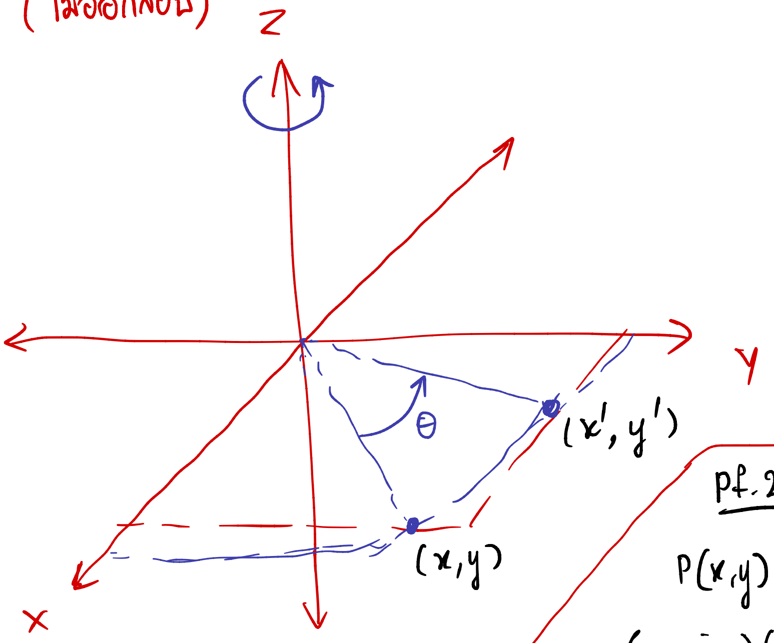
$\Delta x = -3/2$, scale $x = 1/2$

$\Delta z = 1$, scale $y = 2$

③ Rotation

- 2D ✓
- 3D (Hard) → 2D + 2D + 2D

(ไม่ออกสอบ)



✱ $\hat{i} \times \hat{j} = \hat{k}$ (XY) รอบ Z

$\hat{j} \times \hat{k} = \hat{i}$ (YZ) รอบ X

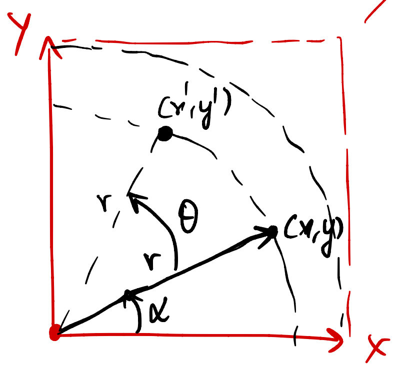
$\hat{k} \times \hat{i} = \hat{j}$ (ZX) รอบ Y

มุมนับจาก... หมุนหา...

Rotate about XY Plane

$$\boxed{\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}}$$

Pf.)



Pf. 2)

$P(x,y) = x + iy \rightarrow P'(x', y') = x' + iy'$

$(x + iy)(\cos\theta + i\sin\theta) =$

$\underbrace{(x\cos\theta - y\sin\theta)}_{= x'} + i\underbrace{(x\sin\theta + y\cos\theta)}_{= y'}$

- $x = r\cos\alpha$,
- $x' = r\cos(\alpha + \theta)$,

  $x' = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

  $x' = x\cos\theta - y\sin\theta$ ■

- $y = r\sin\alpha$
- $y' = r\sin(\alpha + \theta)$

  $y' = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$

  $y' = y\cos\theta + x\sin\theta$

  $y' = x\sin\theta + y\cos\theta$ ■

✱ Rotation matrix

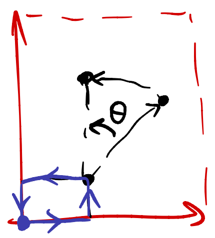
$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Alternatively,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

⑦ จุด (x', y') ได้จากการหมุน (x, y) รอบ (x_0, y_0) ด้วยมุม θ

แนวคิด



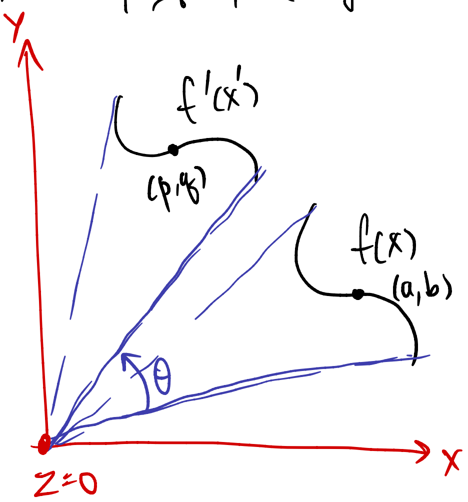
① เลื่อนทุกอย่าง กลับไป $(0,0)$ ด้วย $\vec{V_0} = (-x_0, -y_0)$

② หมุนตามปกติ

③ เลื่อนกลับมาที่เดิมด้วย $-\vec{V_0} = (x_0, y_0)$

$$\therefore \boxed{\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{③} + \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{②} \underbrace{\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}}_{①}}$$

④ สำหรับทุกๆ จุด (x, y) บน $f(x,y) = z$ ทำได้ในทำนองเดียวกัน



แทน x	ด้วย	$x\cos\theta + y\sin\theta$
แทน y	ด้วย	$-x\sin\theta + y\cos\theta$

Pf. Goal: (a,b) ในเทอม (p,q)

$a = \ldots p \ldots q$

$b = \ldots p \ldots q$

$f(a,b) = 0$

ต้องการหมุน $(p,q) \rightarrow (a,b)$ ***(รอบ O ด้วย $-\theta$)

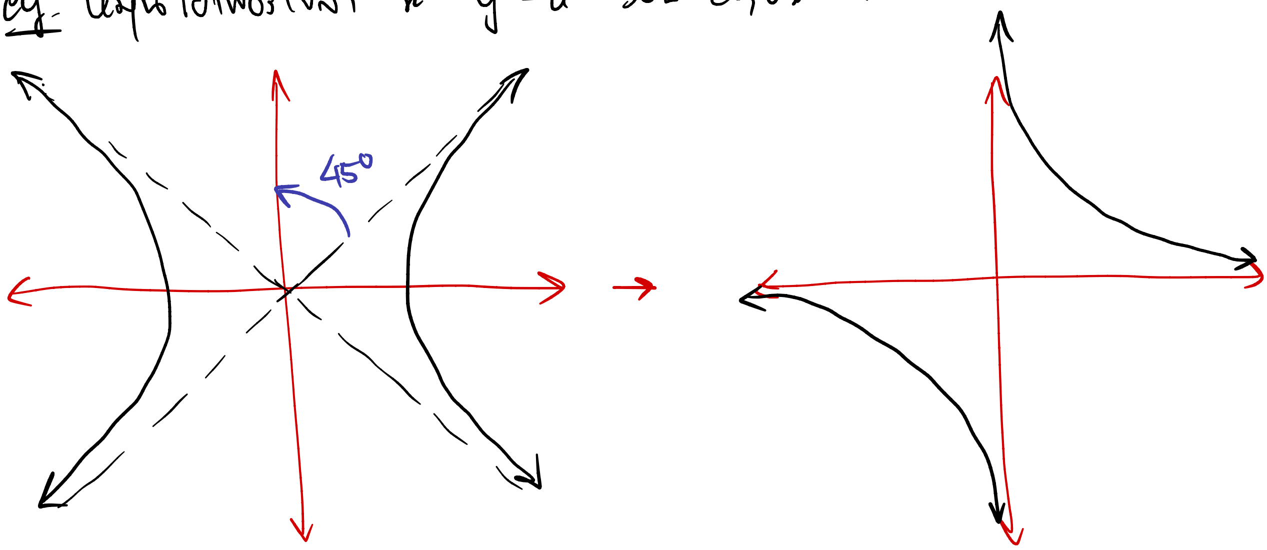
$\rightarrow$ จะได้ $\sin(-\theta) = -\sin\theta$ (กลับเครื่องหมายหน้า $\sin\theta$)

$\therefore a = p\cos\theta + q\sin\theta$

$b = -p\sin\theta + q\cos\theta$

จะได้ $f(a,b) = f(p\cos\theta + q\sin\theta, -p\sin\theta + q\cos\theta)$

eg. หมุนไฮเพอร์โบลา $x^2 - y^2 = a^2$ รอบ $(0,0)$ 45°



จ $x^2 - y^2 = a^2$,

$$x^2 - y^2 - a^2 = 0 \overset{set}{=} f(x,y)$$

$\theta = 45^\circ \longrightarrow \sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

ได้ $f(x', y') = f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$

$$0 = (x^2\cancel{\cos^2\theta} + y^2\cancel{\sin^2\theta} + \cancel{2}xy\,\cancel{\sin\theta}\cancel{\cos\theta})$$
$$- (x^2\cancel{\sin^2\theta} + y^2\cancel{\cos^2\theta} - \cancel{2}xy\,\cancel{\sin\theta}\,\cancel{\cos\theta})$$
$$- a^2$$

$$a^2 = 2xy$$

$$\therefore xy = \frac{1}{2}a^2 \ \#$$

Quadric Surfaces

$x^2 + y^2 + z^2 = 1 \longrightarrow$ ellipsoid

$x^2 + y^2 = z^2 \longrightarrow$ cone   ← ตัดพอดี

$x^2 + y^2 = z^2 + 1 \longrightarrow$ hyperboloid one sheet   ← เพิ่มคอคอด

ไม่ตัดกับแกนหมุน

(เกิดจาก )

$x^2 + y^2 = z^2 - 1 \longrightarrow$ hyperboloid two sheet   ← เอาคอคอดออก

$x^2 + y^2 = z \longrightarrow$ elliptic paraboloid   ← cross section

$x^2 - y^2 = z \longrightarrow$ hyperbolic paraboloid   ← cross section

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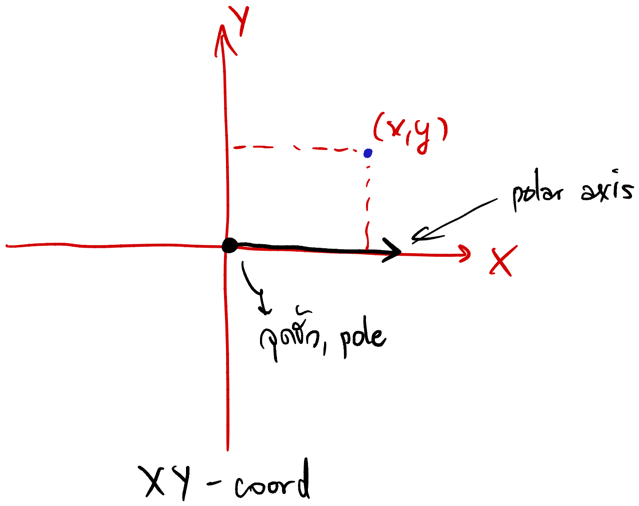
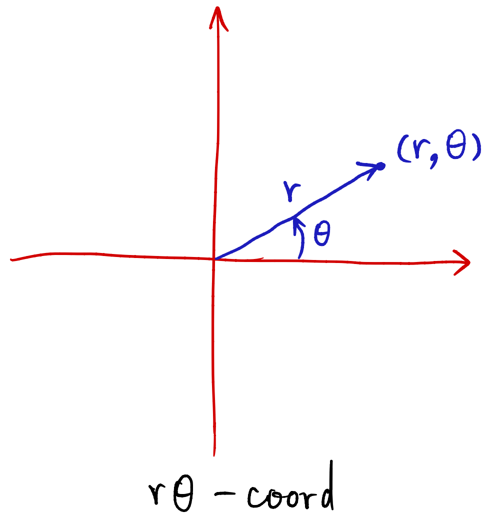
Polar Coordinates

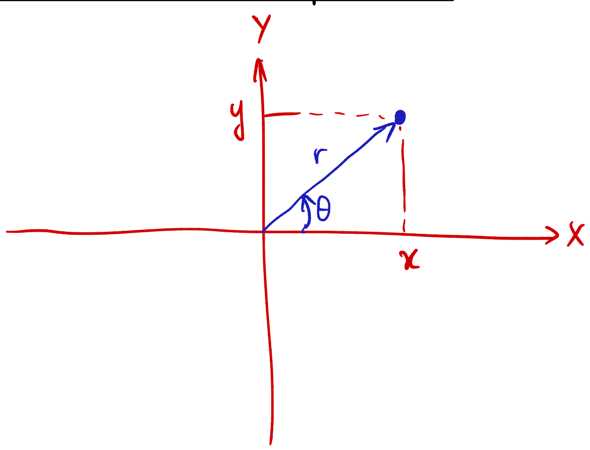




* If $r=0$: จะเป็น pole ($\theta =$ ใดๆ)
* $A(r,\theta)$ เป็นจุดเดียวกับ $A'(r, \theta \pm 2n\pi)$ → ครบวง
* $B(-r,\theta)$ เป็นจุดเดียวกับ $B'(r, \theta \pm (2m+1)\pi)$ → สะท้อน (−)

• รูปทั่วไป (General Form) ของ (r,θ) คือ $\boxed{((-1)^n r, \theta + n\pi) ; n \in \mathbb{Z}}$

XY – Polar Transformation



$$C \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$P \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

* r → +, −

* θ : atan( ) + $2n\pi$

! ระวัง, $\tan\theta$ มีคาบเป็น π ไม่ใช่ 2π
อาจเจอตัวผลลัพธ์ผิด เช่น
XY(3,3) → $(3\sqrt{2}, 45^\circ)$
~~$(3\sqrt{2}, 225^\circ)$~~ ผิด!
แต่ tanθ เท่ากัน
r^2 เท่ากัน

เขียน P แบบนี้ได้

$$P \begin{cases} r^2 = x^2 + y^2 \\ \theta = \arctan\left(\frac{y}{x}\right) \pm 2n\pi \end{cases}$$

eg. XY$(\sqrt{3}, -3)$ → (r,θ)

$\tan\theta = -\sqrt{3} \rightarrow \theta = -60^\circ$

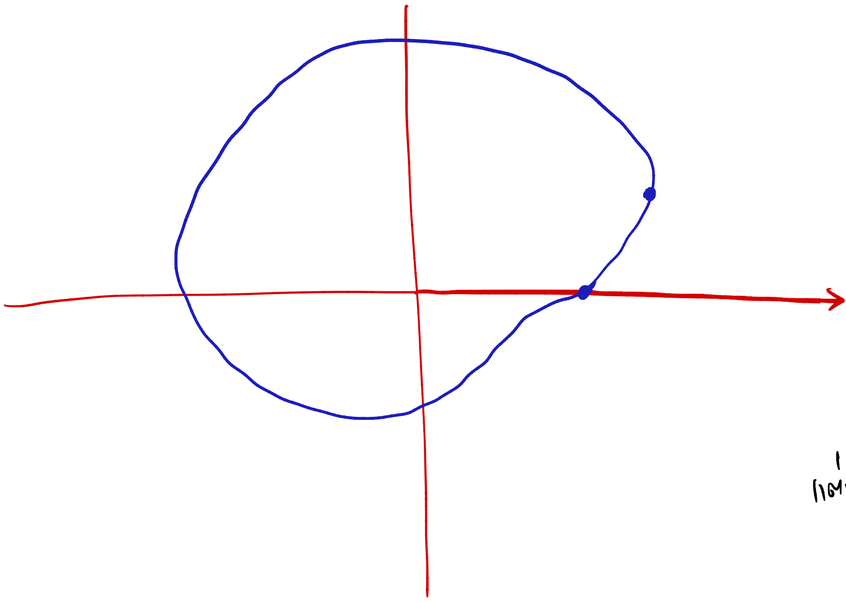
$r = 2\sqrt{3}$

$\therefore P(2\sqrt{3}, -60^\circ)$

$= P(2\sqrt{3}(-1)^n, -60^\circ + n\pi)$

Graph

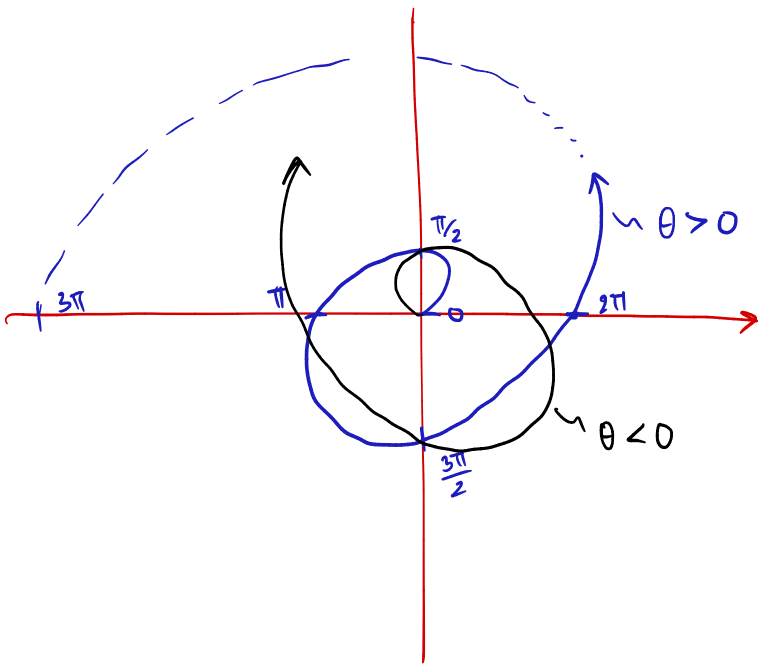
$r = 3 + \sin\theta + \sin 2\theta$



θ	r
0°	3
30°	$3 + 0.5 + 0.8$
60°	$3 + 0.8 + \dots$
⋮	⋮
360°	3
⋮	⋮
∞	

แต่หยุดที่ $0^\circ - 360^\circ$ $\because$ $\sin\theta$ มี $T = 2\pi$
$\sin 2\theta$ มี $T = \pi$
(ครบ loop)

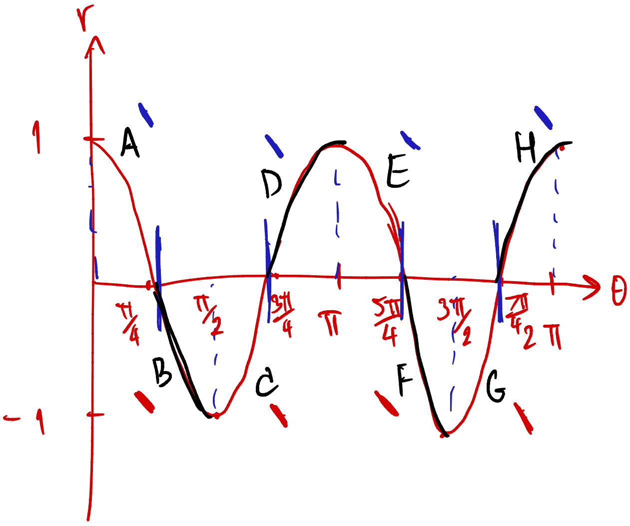
$r = \theta$

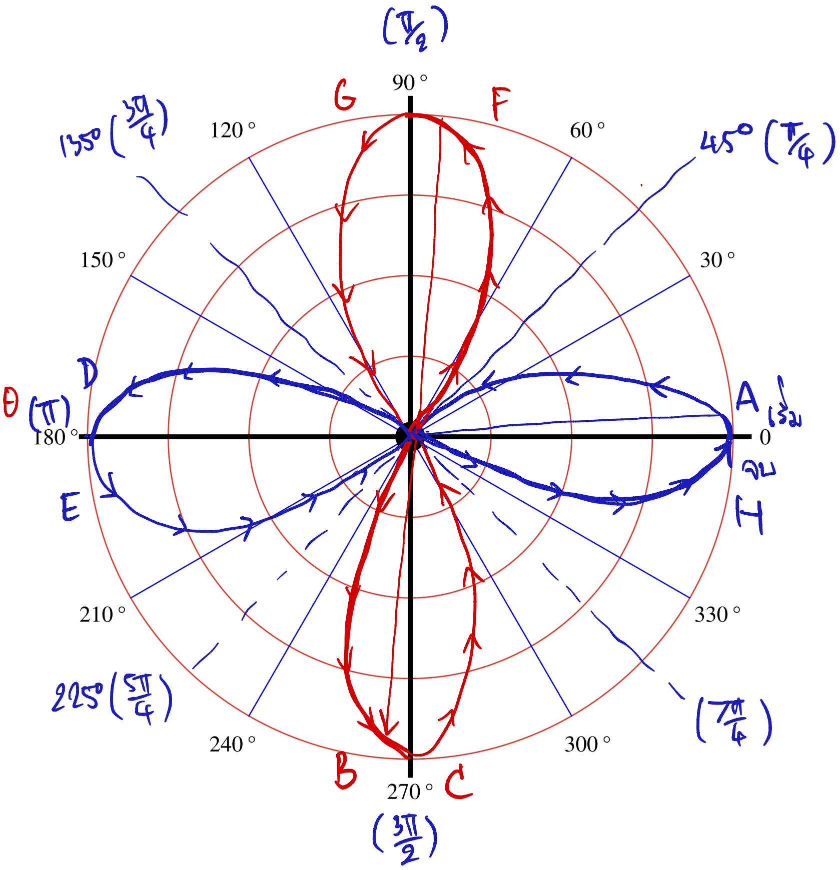


θ	r
0	0
$\pi/3$	$\pi/3$
π	π
2π	2π
3π	3π
⋮	⋮

θ	r
0	0
$-\pi/3$	$-\pi/3$
$-\pi$	$-\pi$
-2π	-2π
-3π	-3π

Plot $\boxed{r = \cos 2\theta}$





แปลง XY ⟷ Polar

eg. $ax + by = c \longrightarrow ar\cos\theta + br\sin\theta = c$ ✓

$$r = \frac{c}{a\cos\theta + b\sin\theta}$$ ✓

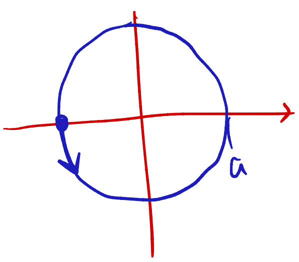
$$a\cos\theta + b\sin\theta = \frac{c}{r}$$ ✓

eg. $x^2 + y^2 = a^2 \longrightarrow r^2 = x^2 + y^2 \rightarrow r^2 = a^2$

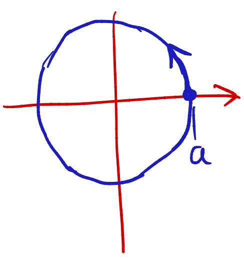
$r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$\therefore r^2 = a^2$

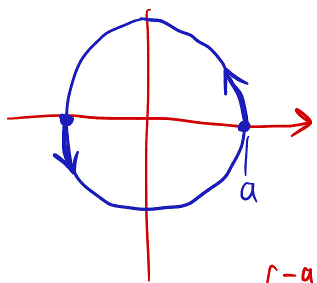
Note



$r = -a$



$r = a$



$r^2 = a^2 : r = \begin{cases} -a \\ a \end{cases}$

eg. $y^2 = 4px$

$$(r\sin\theta)^2 = 4pr\cos\theta$$

$$r^{\cancel{2}}\sin^2\theta = 4p\cancel{r}\cos\theta$$

$$r(\theta) = \frac{4p\cos\theta}{\sin^2\theta} = \frac{4p}{\sin\theta\tan\theta} = 4p\csc\theta\cot\theta$$

eg. $r = 2\cos\theta \rightarrow r^2 = \underline{2r\cos\theta}$

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$

eg. $r = 2\cos\theta + 4\sin\theta \rightarrow r^2 = 2r\cos\theta + 4r\sin\theta$

$$x^2 + y^2 = 2x + 4y$$

$$(x-1)^2 + (y-2)^2 = 5$$

eg. $r^2 = 2a^2\sin 2\theta \rightarrow r^2 = 4a^2\sin\theta\cos\theta$

$$r^4 = 4a^2(r\sin\theta)(r\cos\theta)$$

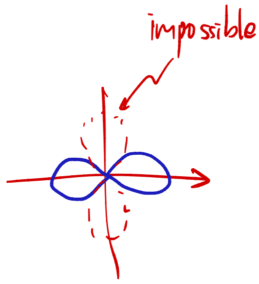
$$x^4 + 2x^2y^2 + y^4 = 4a^2xy$$

$$\frac{x^3}{y} + 2xy + \frac{y^3}{x} = 4a^2$$

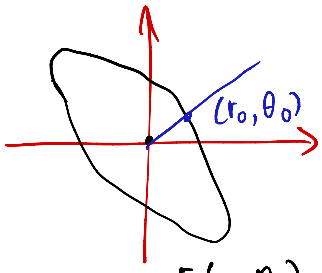
eg. $r^2 = 2a^2\cos 2\theta \rightarrow r^2 = 4a^2(\cos^2\theta - \sin^2\theta)$

$$r^4 = 4a^2(r^2\cos^2\theta - r^2\sin^2\theta)$$

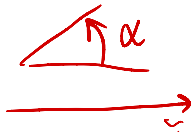
$$x^4 + 2x^2y^2 + y^4 = 4a^2(x^2 - y^2)$$



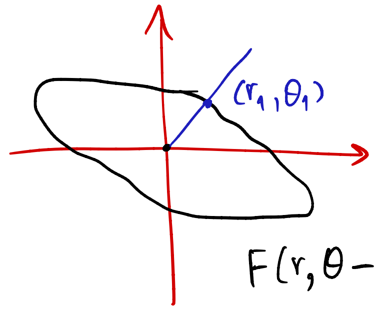
Rotation of Polar Coordinates



$F(r, \theta) = 0$



หมุนรอบจุดขั้ว
ด้วยมุม $+\alpha$
(ทวนเข็ม)



$F(r, \theta - \alpha) = 0$

↳ $r_0 = r_1$

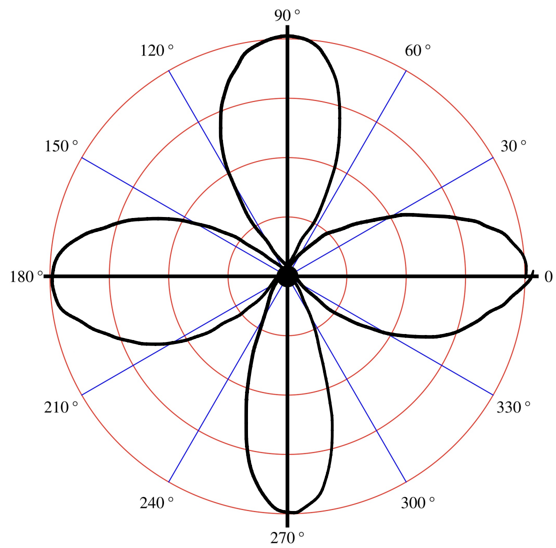
$\theta_0 + \alpha = \theta_1$ → ตัวเส้นในสมการใหม่ ต้องใส่ $\theta - \alpha$ จะได้ค่าเท่ากับค่าเดิม

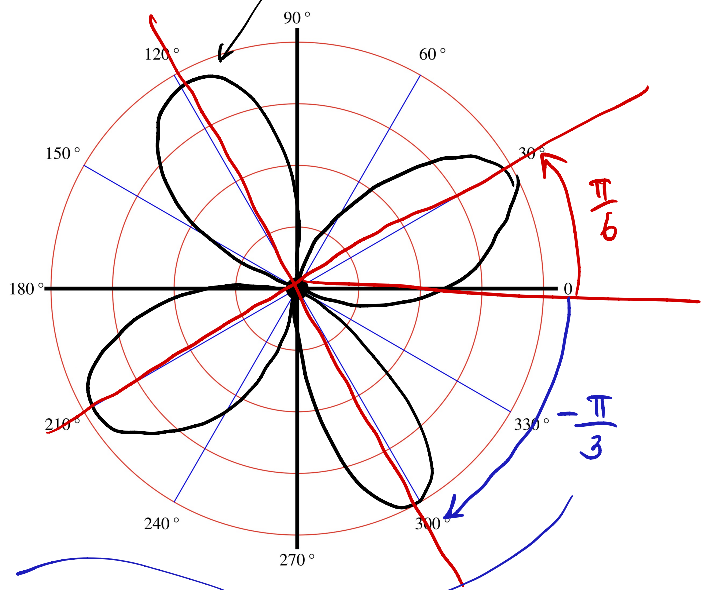
$(\theta_0 = \theta_1 - \alpha)$

eg. หมุน $r = \cos 2\theta$ ด้วย $\frac{\pi}{6}$

$r = \cos\left(2\left(\theta - \frac{\pi}{6}\right)\right)$

$r = \cos\left(2\theta - \frac{\pi}{3}\right)$





ได้ $r = \cos\left(2\theta + \frac{2\pi}{3}\right)$: สมการรูปต่างกัน หน้าตากราฟเหมือนกัน

↓

$r = -\cos\left(2\theta - \frac{\pi}{3}\right)$ → "สมการเทียบเท่า"

eg. $r = \cos\theta + \sqrt{3}\sin\theta$ ← รูปรวมสั้นได้

$\oplus$ $a\cos\theta + b\sin\theta$

$= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right)$

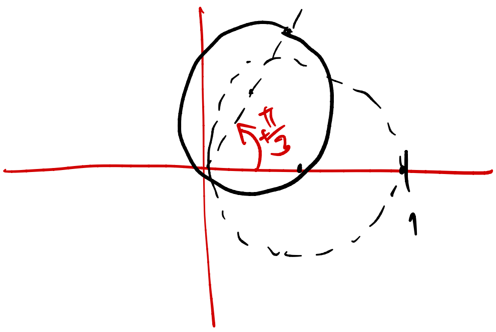
$= \sqrt{a^2+b^2}\left(\cos(\theta-\alpha)\right)$

$r = 2\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$

cosα หรือ sinα   sinα หรือ cosα

$r = 2\left(\_\_\_\_\cos\theta + \_\_\_\_\sin\theta\right)$

$r = 2\cos\left(\theta - \frac{\pi}{3}\right)$ ซึ่งได้จากการหมุน $r = 2\cos\theta$ รอบจุดขั้วด้วยมุม $+\frac{\pi}{3}$



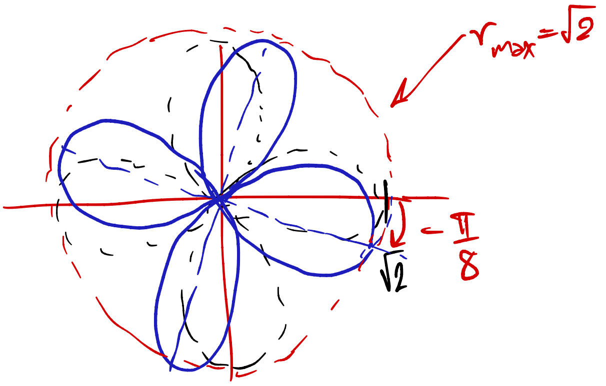
eg. $r = \cos 2\theta - \sin 2\theta$

$r = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 2\theta - \frac{1}{\sqrt{2}}\sin 2\theta\right)$

** คิด $f(r,\theta) \rightarrow f(r, \theta - \Delta\theta)$

$r = \sqrt{2}\left(\cos 2\theta\cos\frac{\pi}{4} - \sin 2\theta\sin\frac{\pi}{4}\right)$

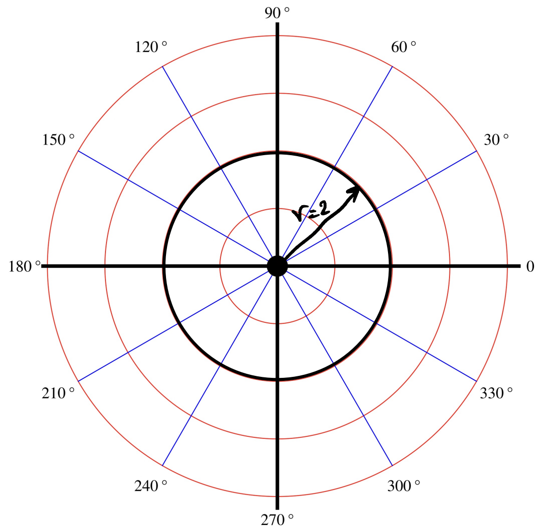
$r = \sqrt{2}\cos\left(2\theta + \frac{\pi}{4}\right) = \sqrt{2}\cos\left(2\left(\theta + \frac{\pi}{8}\right)\right)$ ซึ่งได้จากการหมุน $r = \sqrt{2}\cos 2\theta$ รอบจุดขั้ว ด้วยมุม $-\frac{\pi}{8}$



Equivalent Equations

Remind: รูปทั่วไปของ Polar Form : $((-1)^m r,\ \theta \pm m\pi)$ $\begin{cases} m=2k : (r, \theta + 2k\pi) \\ m = 2k-1 : (-r, \theta + (2k-1)\pi) \end{cases}$

eg. จุดต่อไปนี้อยู่บนกราฟหรือไม่?



$r=2$:

$(2, 0°)$ ✓

$(2, 60°)$ ✓

$(-2, 0°) \xrightarrow{(-1)^n, \pi} (2, \pi)$ ✓

$(-2, 60°) \xrightarrow{(-1)^n, \pi} (2, 60° + \pi)$ ✓

$r = 2\cos 2\theta$:

$(1, 30°)$ $\quad 1 \overset{?}{=} 2\cos 60°$ ✓

$(1, 60°)$ $\quad 1 \overset{?}{=} 2\cos 120° = -2\cos 60°$ ✗

$-1 \overset{?}{=} 2\cos(120° + 2\pi) = 2\cos 120° = -2\cos 60°$ ✓

} ไม่ valid ในสมการ แต่อยู่บนกราฟของสมการ

$\neq$ แต่...

$(0, 40°)$ $\quad 0 \overset{?}{=} 2\cos 80°$ ✓

เป็นจุดขั้ว มีพิกัดหลากหลาย ซึ่งเป็นจุดเดียวกันกับ $(0, 45°)$ ซึ่งอยู่บนกราฟแน่ๆ
(แทนแล้ว valid)

$(1, 90°)$ $\quad 1 \neq 2\cos 180°$ ✗

$-1 \neq 2\cos(180° + 2\pi)$ ✗

$\vdots$

$-1 \neq 2\cos(180° + 2m\pi) = -2$ ✗

} ✗

Pf. $\left((-1)^n, \frac{\pi}{2} + n\pi\right)$: $\quad (-1)^n = \cos 2\left(\frac{\pi}{2} + n\pi\right)$

$(-1)^n = \cos(\pi + 2n\pi)$

$(-1)^n = -2$ $\quad$ เป็นไปไม่ได้!

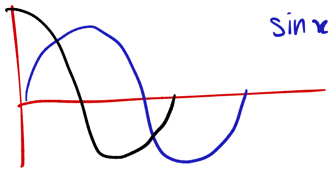
$\rightarrow$ $F(r, \theta) = 0 \sim G(r, \theta) = 0 \longleftrightarrow F, G$ เป็นรูปเดียวกัน

<u>One of which</u>: $F(r,\theta)=0 \longrightarrow F(r,\theta+2k\pi)$

$\longrightarrow F(-r,\theta+(2k-1)\pi)$

$\Big\}\ F((-1)^n r,\theta+n\pi)$ เทียบเท่าแน่ ┐

$\cos x = \sin(x+\frac{\pi}{2})$



eg. $r=\sin\frac{\theta}{2} \rightarrow F(r,\theta)=r-\sin\frac{\theta}{2}=0$

1. $n=2k$: $r=\sin\left(\frac{\theta+2k\pi}{2}\right)$

$r=\sin\left(\frac{\theta}{2}+k\pi\right)$

$r=-\sin\frac{\theta}{2}$ *

2. $n=2k+1$: $r=\sin\left(\frac{\theta+2k\pi+\pi}{2}\right)$

$r=\sin\left(\frac{\theta}{2}+\frac{\pi}{2}+k\pi\right)$ → k คู่ / k คี่

$r=-\sin\left(\frac{\theta}{2}+\frac{\pi}{2}\right)$

$r=-\cos\frac{\theta}{2}$ *

→ $n=2k$: $r=-\cos\left(\frac{\theta}{2}+k\pi\right)$

$r=\cos\frac{\theta}{2}$ *

$\therefore\ r=\underbrace{\sin\frac{\theta}{2},\ -\sin\frac{\theta}{2}}_{*},\ \underbrace{\cos\frac{\theta}{2},\ -\cos\frac{\theta}{2}}_{*}$

$|r| \qquad \left|\sin\frac{\theta}{2}\right| \qquad \left|\cos\frac{\theta}{2}\right|$

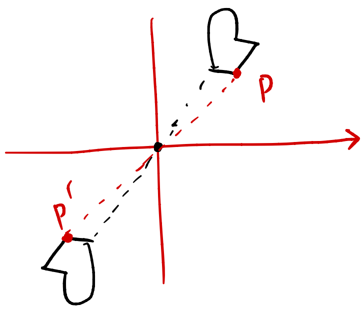
$r^2 \qquad \sin^2\frac{\theta}{2} \qquad \cos^2\frac{\theta}{2}$

* <u>ข้อควรรู้</u>

ไม่ใช่คำตอบทั้งหมดที่เป็นไปได้
อาจมีคำตอบในรูปอื่นอีก
ที่เห็นมาจากรูปทั่วไป อย่างเดียว.

<u>Symmetry</u> (สมมาตร)

1.) เทียบกับจุดขั้ว $\longleftrightarrow$ หมุน $F(r,\theta)$ ด้วย π



เช่น $p(r,\theta) \rightarrow p'(r,\theta+(2k+1)\pi)$

$\qquad\qquad\ \searrow p'(-r,\theta+2k\pi)$ } P, P' สมมาตรเทียบจุดขั้ว

$$\boxed{F(r,\theta)=0 \rightarrow F'(r,\theta+(2k+1)\pi)=0 \quad;\quad F'(-r,\theta+2k\pi)=0}$$

2. เทียบแกนเชิงขั้ว ($\theta_0 = 0°$)

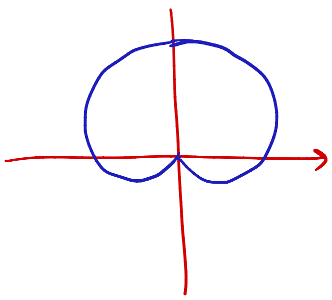
$$F(r,\theta) = 0 \rightarrow F'((-1)^n r, -\theta + n\pi) = 0$$

3. เทียบเส้นตรง $\theta_0 = 90°$

$$F(r,\theta) = 0 \rightarrow F'((-1)^{n+1}, -\theta + n\pi) = 0$$

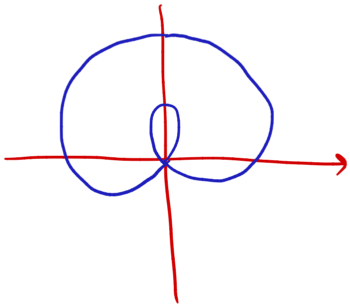
รูปกราฟ

1. Limaçon

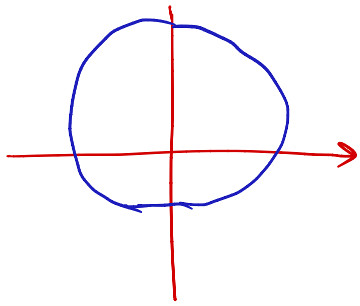


$r = 1 + \sin\theta$

"cardiod"

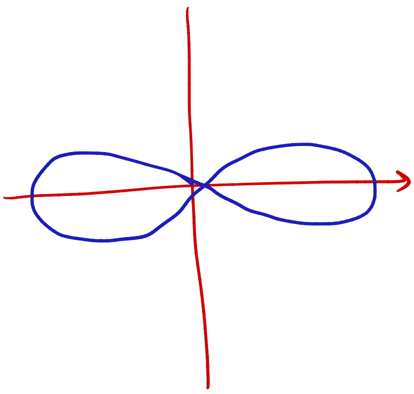


$r = 1 + 2\sin\theta$

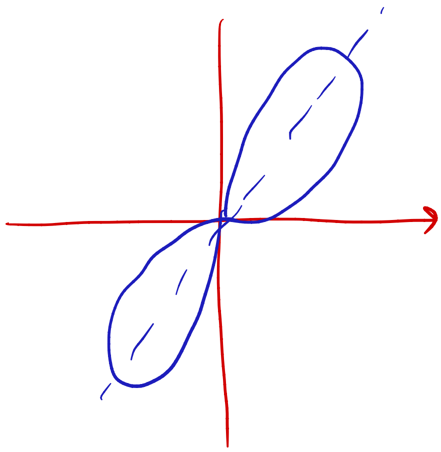


$r = 2 + \sin\theta$

2. Lemniscate



$r^2 = 2a^2 \cos 2\theta$



$r^2 = 2a^2 \sin 2\theta$

3. Rose Curves

$$\left.\begin{array}{l} r = a\cos(n\theta) \\ r = a\sin(n\theta) \end{array}\right]$$

→ $n \in \mathbb{Z}$ ได้ Rose Curve

→ $n \in \mathbb{Z}$ ได้ Self-Intersecting Curve

4. Conic Section

$$r = \frac{a}{1 - e\sin\theta}$$

5. Archemedean Spiral

$$r = a + b\theta$$

การหาจุดตัดกราฟเชิงขั้ว

$r = f(\theta)$

1. Solve $\boxed{f(\theta) = (-1)^n g(\theta + n\pi)}$: $(f(\theta), \theta)$
2. Check ว่า f, g ผ่านจุดขั้วไหม? : $(0, \theta)$: Pole

eg. หาจุดตัดทั้งหมดของ $r = 2\cos 2\theta$ และ $r = 1$

$$f(\theta) = (-1)^n g(\theta + n\pi) \quad ; \quad n \in \mathbb{Z}$$

① $n = 2k$

$r \rightarrow \underline{1} = 2\cos 2\theta$

$\cos 2\theta = \frac{1}{2}$

$2\theta = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$ ($2\pi - \frac{\pi}{3}$)

$\theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$

และ $r = 1$

② $n = 2k-1$

$r \rightarrow \underline{-1} = 2\cos 2\theta$

$\cos 2\theta = -\frac{1}{2}$

$2\theta = \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}$

$\theta = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$

และ $r = -1$

③ จุดขั้ว : $r = 1$ ไม่ผ่านจุดขั้ว ไม่ต้องตรวจสอบ

$\therefore$ จุดขั้ว ไม่ใช่จุดตัด

$\therefore$ จุดตัดมี 8 จุด : $(1, -\frac{\pi}{6}), (1, \frac{\pi}{6}), (1, -\frac{5\pi}{6}), (1, \frac{5\pi}{6})$,

$(-1, -\frac{\pi}{3}), (-1, \frac{\pi}{3}), (-1, -\frac{2\pi}{3}), (-1, \frac{2\pi}{3})$.

eg. $r = 3\sin\theta$ และ $r = 1+\sin\theta$

"$f(\theta)$" "$g(\theta)$"

$$f(\theta) = (-1)^n g(\theta + n\pi)$$
$$3\sin\theta = (-1)^n(1+\sin(\theta+n\pi)) \quad ; \quad n \in \mathbb{Z}$$

① $n = 2k$

$$3\sin\theta = 1+\sin\theta$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad ; \quad r = \frac{3}{2}$$

② $n = 2k-1$

$$3\sin\theta = -1+\sin\theta$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6} \quad ; \quad r = -\frac{3}{2}$$

③ จุดขั้ว ?

f: $0 = 3\sin\theta$, $\therefore \theta = 0, 2\pi$

g: $0 = 1+\sin\theta$, $\therefore \theta = \frac{3\pi}{2}$

ผ่านจุดขั้วด้วย $(0,0)$

$(-1)^n$, $+n\pi$

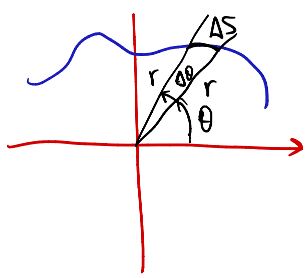
จุดเดียวกัน

$\therefore$ ตัดที่จุด $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), \left(-\frac{3}{2}, \frac{7\pi}{6}\right), \left(-\frac{3}{2}, \frac{11\pi}{6}\right), (0,0)$ (3 จุดต่างกัน)

จุดเดียวกัน

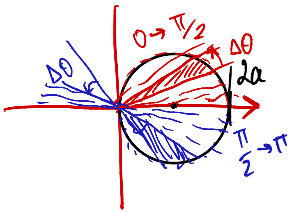
$(-1)^n$, $+n\pi$

การหาพื้นที่กราฟเชิงขั้ว



$$r = f(\theta)$$
$$\Delta S = r\Delta\theta$$
$$\mathcal{A} = \frac{1}{2}\Delta S \cdot r = \frac{1}{2} r \cdot r \cdot \cancel{\sin\Delta\theta}^{\approx \Delta\theta} = \cancel{\pi} r^2\left(\frac{\Delta\theta}{2\cancel{\pi}}\right)$$
$$\mathcal{A} = \frac{1}{2} r^2 \Delta\theta$$
$$\therefore \mathcal{A} = \frac{1}{2}\int_a^b r^2\,d\theta$$
$$A = \frac{1}{2}\int_a^b f(\theta)^2\,d\theta$$

eg. หาพื้นที่ภายในวงกลม $r = 2a\cos\theta$



$$A = \int_0^{\pi} \frac{1}{2} r^2 \, d\theta$$

$$A = \int_0^{\pi} 2a^2 \cos^2\theta \, d\theta \quad ; \quad \cos^2\theta = \frac{1+\cos 2\theta}{2} \ * \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \ *$$

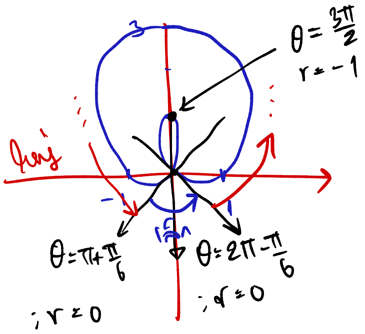
$$A = \frac{a^2}{2} \int_0^{\pi} (1+\cos 2\theta)\, d(2\theta)$$

$$A = \frac{a^2}{2}\left[2\theta + \cancel{\sin 2\theta}^{\,0}\right]_0^{\pi}$$

$$A = a^2\theta \Big|_0^{\pi} = \pi a^2 \ *$$

∴ พื้นที่ของกราฟเป็น πa^2

eg. Limaçon $r = 1 + 2\sin\theta$



1.) พท. วงใหญ่ $\rightarrow \left[-\frac{\pi}{6}, \frac{7\pi}{6}\right]$

2.) พท. วงเล็ก $\rightarrow \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$

2.) ; $$A = \int_{7\pi/6}^{11\pi/6} \frac{1}{2} r^2 \, d\theta$$

$$A = \int_{7\pi/6}^{11\pi/6} \frac{1}{2}(1+2\sin\theta)^2 \, d\theta = \int \frac{1}{2} + 2\sin\theta + 2\sin^2\theta \, d\theta$$

$$= \left[\frac{1}{2}\theta - 2\cos\theta + \theta - \frac{\sin 2\theta}{2}\right]$$

$$= \left[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin 2\theta}{2}\right]$$

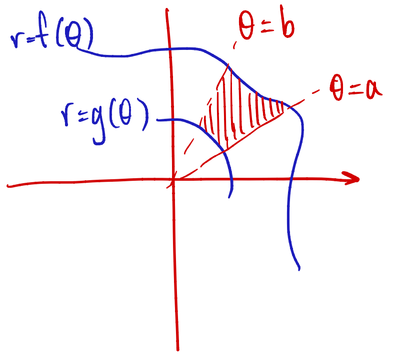
$$A = (\ \ldots\ )\Big|_{7\pi/6}^{11\pi/6}$$

$$A = \pi - \frac{3\sqrt{3}}{2} \quad \text{(วงเล็ก)}$$

1.) ; $$A = \int_{-\pi/6}^{7\pi/6} \frac{1}{2}(1+2\sin\theta)^2 \, d\theta = (\ \ldots\ )\Big|_{-\pi/6}^{7\pi/6} = 2\pi + \frac{3\sqrt{3}}{2}$$

สังเกต: $$\int_0^{2\pi} \frac{1}{2}(1+2\sin\theta)^2 \, d\theta = A_1 + A_2 \quad \text{(วงใหญ่ + วงเล็ก)}$$

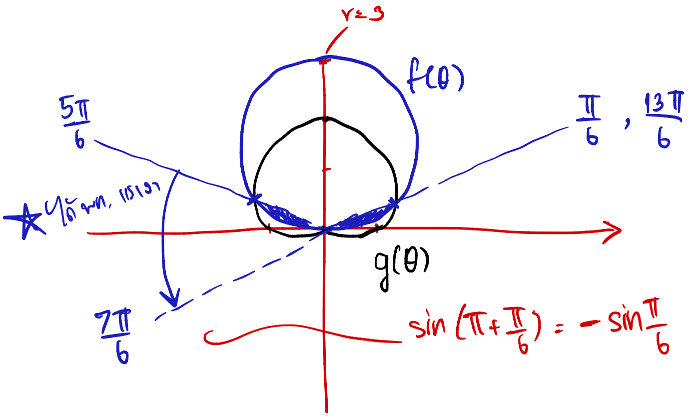
Area Difference



$$A = \frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 \, d\theta$$

eg. $r = 3\sin\theta$ กับ $r = 1+\sin\theta$
(วงกลม) (cardiod)

1. Area ภายในวงกลม นอก Cardiod
2. Area ใน Cardiod นอก วงกลม
3. Intersecting area



จุดตัด: $3\sin\theta = 1+\sin\theta$
$\sin\theta = \frac{1}{2}$ และ จุดขั้ว
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$f(\theta) = 3\sin\theta$ $\qquad g(\theta) = 1+\sin\theta$
$f^2(\theta) = 9\sin^2\theta$ $\qquad g^2(\theta) = 1+\sin^2\theta + 2\sin\theta$

1.) $A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} 9\sin^2\theta - 1 - \sin^2\theta - 2\sin\theta \, d\theta$

$A = \frac{1}{2}\int 8\sin^2\theta - 2\sin\theta - 1 \, d\theta$

$A = \frac{1}{2}\int \overset{3}{\cancel{4}} - 4\cos 2\theta - 2\sin\theta - \cancel{1} \, d\theta$

$A = \frac{1}{2}\left(3\theta - 2\sin 2\theta + 2\cos\theta\right)\Big|_{\pi/6}^{5\pi/6}$

$A = \frac{1}{2}\left(3\cdot\frac{4\pi}{6} + \cancel{2}\sin\frac{2\pi}{3} + \cancel{2}\sin\frac{\pi}{3} + \cancel{2}\cos\frac{5\pi}{6} - \cancel{2}\cos\frac{\pi}{6}\right)$
$\quad = \frac{\sqrt{3}}{2} \quad = \frac{\sqrt{3}}{2} \quad = -\frac{\sqrt{3}}{2} \quad = \frac{\sqrt{3}}{2}$

$A = \pi$

2.) $A = \frac{1}{2}\int_{5\pi/6}^{7\pi/6} 9\sin^2\theta \, d\theta - \frac{1}{2}\int_{5\pi/6}^{13\pi/6} (1+\sin\theta)^2 \, d\theta$

3.) $A = \underbrace{\int_0^{\pi/6} 9\sin^2\theta \, d\theta}_{\text{เฉพาะวงกลม}} + \underbrace{\int_{\pi/6}^{\pi/2} (1+\sin\theta)^2 \, d\theta}_{\text{cardiod}}$

★ หรือ ใช้สมมาตร แบ่งครึ่งช่อง แล้ว ( ) × 2

CALCULUS OF VARIATION : MULTIVARIABLE CALCULUS

Chapter 3 (A) : Multivariable Derivative

1.) Partial Derivative

eg. $z = f(x,y)$: $z = x^2y^3 + \arctan(x^2 + \sin x^3)$

จงหา f_x , f_y , f_{xy}

$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial y \partial x} \to \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

$$\frac{\partial z}{\partial x} = 2xy^3 + \frac{2x + 3x^2\cos x^3}{1 + x^2 + \sin x^3}$$

$$\frac{\partial z}{\partial y} = 3x^2y^2$$

$$\frac{\partial z^2}{\partial y \partial x} = 6xy^2 \quad \Big| \quad \frac{\partial z^2}{\partial x \partial y} = 6xy^2$$

บังเอิญ ?

- ให้ $w = f(x,y,z)$; $(a,b,c) \in D_f^3$

$$f_x(a,b,c) = \left.\frac{\partial f}{\partial x}\right|_{(a,b,c)} = \lim_{h \to 0} \frac{f(a+h,b,c) - f(a,b,c)}{h}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b,c)} = \lim_{h \to 0} \frac{f(a,b+h,c) - f(a,b,c)}{h}$$

$$\left.\frac{\partial f}{\partial z}\right|_{(a,b,c)} = \lim_{h \to 0} \frac{f(a,b,c+h) - f(a,b,c)}{h}$$

---

eg. $f(x,y,z) = \begin{cases} x+y+z & xyz = 0 \\ |x|+y+z & xyz \neq 0 \end{cases}$

(1) $f_x(0,1,1)$ มีค่าหรือไม่ ?

$\hookrightarrow \lim_{h \to 0} \frac{f(h,1,1) - f(0,1,1)}{h} = \lim_{h \to 0} \frac{(|h|+1+1) - (0+1+1)}{h}$

$= \lim_{h \to 0} \frac{|h|}{h} \to$ หาค่าไม่ได้

$\therefore$ limit DNE.

$\therefore f_x(0,1,1)$ ไม่มีค่า.

---

eg.

$$f:\mathbb{R}\to\mathbb{R}\ ;\ f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{elsewhere}\\ 0 & (0,0)\end{cases}$$

หา $f_x(0,y)$ และ $f_y(x,0)$

หา $f_{xy}(0,0)$ และ $f_{yx}(0,0)$

Clairaut's Theorem

$f: D\subset\mathbb{R}^2\to\mathbb{R}\ ;\ (a,b)\in D$

$$f_{xy}\ \wedge\ f_{yx}\ \text{are continuous}\ \to\ f_{xy}(a,b)=f_{yx}(a,b)$$

Leibniz Derivative Notation

$$f_{xyz}=\frac{\partial^3 f}{\partial z\,\partial y\,\partial x}=\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)$$

---

eg. หา f_{xyz} ; $f(x,y,z)=xy^2z^3+\underset{\substack{\downarrow\\0}}{\ln(1+y^2+z^2)}+\underset{\substack{\downarrow\\0}}{\ln(1+x^2+z^2)}+\underset{\substack{\downarrow\\0}}{\ln(1+x^2+y^2)}$

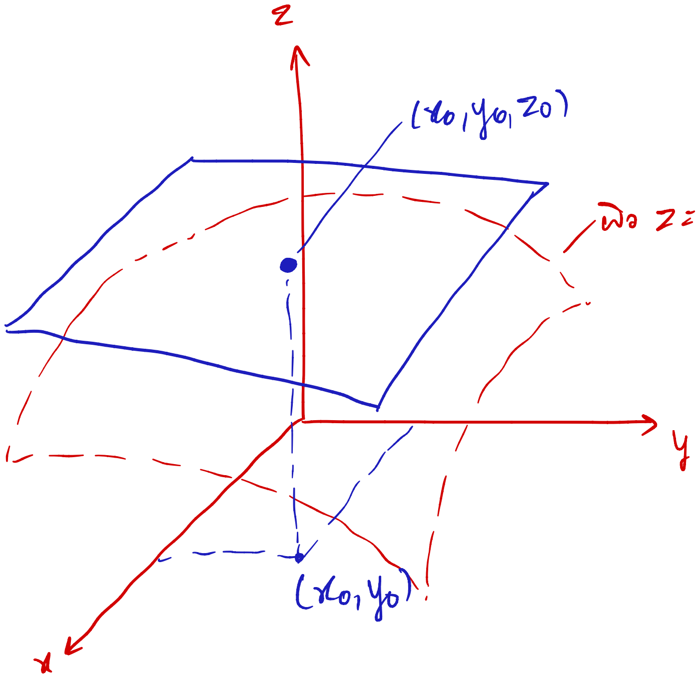
$$f_x=y^2z^3+G_1(x,z)+G_2(x,y)$$

$$f_{xy}=2yz^3+0+G_2'(x,y)$$

$$f_{xyz}=6yz^2+0+0$$

$$\therefore f_{xyz}=6yz^2$$

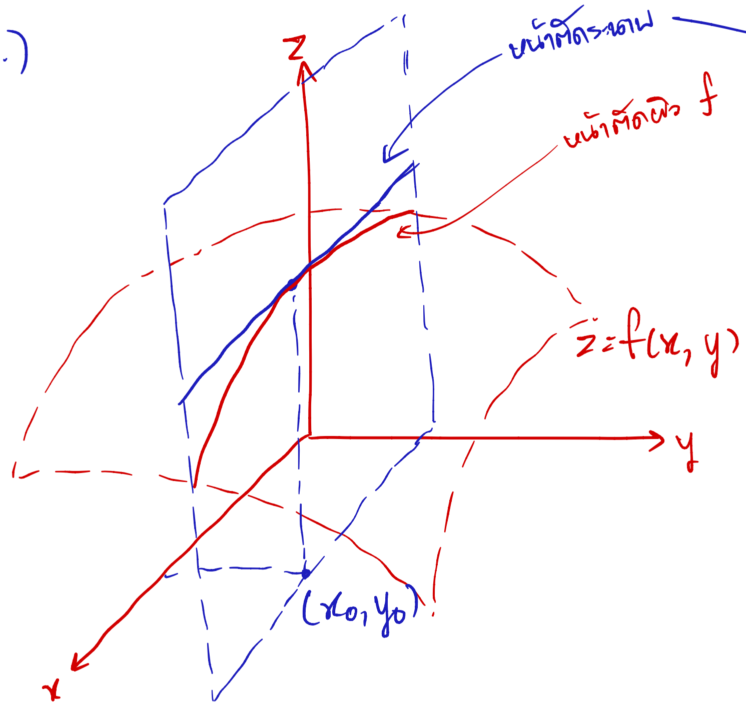
ระนาบสัมผัส



ระนาบมีสมการในรูป

$$\boxed{\begin{aligned} z-z_0 &= a(x-x_0)+b(y-y_0); \\ a &= f_x(x_0,y_0) \\ b &= f_y(x_0,y_0) \end{aligned}}$$

(Dv.)



สมการ $z-z_0 = a(x-x_0)$ บน $y=y_0$

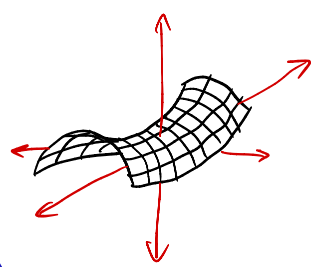
ความชันที่ (x_0, y_0, z_0)

$\therefore a = f_x(x_0,y_0) = \left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$

- และในทำนองเดียวกัน

$z-z_0 = b(y-y_0)$ บน $x=x_0$

$\therefore b = f_y(x_0,y_0) = \left.\dfrac{\partial z}{\partial y}\right|_{(x_0,y_0)}$



$$\vec{N} = \left\langle \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}, \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}, 1 \right\rangle$$

และ $P_0(x_0, y_0, z_0)$

eg. สมการระนาบสัมผัส $z = 2x^2 - 5y^2$ ที่ $(2,1,3)$

$f_x = 4x \rightarrow f_x(2,1) = 8$

$f_y = -10y \rightarrow f_y(2,1) = -10$

$\rightarrow$ สมการคือ $z-3 = 8(x-2) - 10(y-1)$ *

$\therefore z = 8x - 10y - 3$

Key Point : ที่ $z = z_0$ ของผิว S และระนาบสัมผัส T, ณ รอบๆ (x_0, y_0, z_0)

S → f, T → L

บน S และ T จะมีค่า z ที่ใกล้กันมากๆ

"Proximinality" ในข้อสอบ (Proof ใน code 2)

→ นำไปคิด Linear Approximation คือ $f(x,y) \approx L(x,y)$

→ นำไปสู่ $f(x_1, x_2, x_3, \dots, x_n) \approx L(x_1, x_2, x_3, \dots, x_n)$ * ใช้ได้ตั้งแต่ 1 มิติขึ้นไป

→ สูตรทั่วไป (General Form) $f(x_1,\dots,x_n) \approx f(a_1,\dots,a_n) + \sum_{i=1}^{n} f_{x_i}(a_1,\dots,a_n)(x_i - a_i)$

$f(x) : y_0$
$f(x,y) : z_0$
$f(x,y,z) : w_0$ ...

e.g. $f(x,y) = \ln(xy)$

1.) สมการระนาบสัมผัสที่ $(1,1,0)$

2.) ประมาณ $f(1.1, 0.95)$

→ 1.) $f_x = \frac{1}{xy} \cdot \frac{\partial}{\partial x}(xy)$

$\therefore f_x = \frac{1}{x}$, $f_y = \frac{1}{y}$

$f_x(1,1) = 1$, $f_y(1,1) = 1$

ระนาบเป็น

$z = (x-1) + (y-1)$

$\therefore z = x + y - 2$

$L(x,y) = x + y - 2$

→ 2.) $f(x,y) \approx L(x,y)$

$f(1.1, 0.95) \approx 1.1 + 0.95 - 2$

$= 0.05$

$f(x,y,z) = \ln(xyz)$ ที่ $(1,1,1,0)$ (x, y, z, w) : $f(1.1, 0.95, 1) = ?$

$f_x = \frac{1}{\cancel{xyz}} \cdot \cancel{yz} = \frac{1}{x}$, $f_y = \frac{1}{y}$, $f_z = \frac{1}{z}$

$= 1$, $= 1$, $= 1$

$L(x,y,z) = w_0 + (x - \underset{1}{\cancel{x_0}}) + (y - \underset{1}{\cancel{y_0}}) + (z - \underset{1}{\cancel{z_0}})$

$= 0 + (1.1 + 0.95 + 1) - 3$

$= 0.05$ ✓

Chain Rule

e.g. $\quad w = x^3y + xyz^2 \ , \quad x = \cos t \ , \quad y = \sin t \ , \quad z = t+1$

จงหา $\left.\dfrac{dw}{dt}\right|_{t=0}$

$$\begin{tikzcd}
 & w & \\
x \arrow[ur] & y \arrow[u] & z \arrow[ul] \\
t \arrow[u] & t \arrow[u] & t \arrow[u]
\end{tikzcd}$$

"Differentials / Chain Rule"

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}}$$

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

$\dfrac{\partial w}{\partial x} = 3x^2y + yz^2$ $\qquad$ $\dfrac{dx}{dt} = -\sin t$ $\qquad$ $t=0:\ x = 1 \quad -\sin t = 0$

$\dfrac{\partial w}{\partial y} = x^3 + xz^2$ $\qquad$ $\dfrac{dy}{dt} = \cos t$ $\qquad$ $y = 0 \quad \cos t = 1$

$\dfrac{\partial w}{\partial z} = xy$ $\qquad$ $\dfrac{dz}{dt} = 1$ $\qquad$ $z = 1$

$$\therefore \frac{dw}{dt} = (-\sin t)(3x^2y + yz^2) + (\cos t)(x^3 + xz^2) + xy$$

$$= 2 \ \#$$

e.g. $\quad PV = nRT \quad : \quad P \to -a\,\%/h \quad V \to +b\,\%/h \quad T \to -c\,\%/h \quad n \to ?$

ของ X

$$* \quad a\% = \frac{a}{100} \times X$$

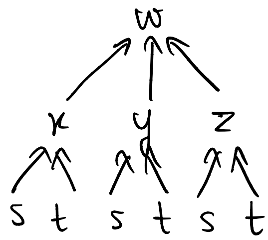
$$n = \frac{1}{R}\cdot\frac{PV}{T} \quad : \quad n(P,V,T) = \frac{1}{R}\left(\frac{PV}{T}\right)$$

$$\frac{dn}{dt} = \underbrace{\frac{\partial n}{\partial P}}_{\frac{V}{RT}}\cdot\overbrace{\frac{dP}{dt}}^{-a\%P} + \underbrace{\frac{\partial n}{\partial V}}_{\frac{P}{RT}}\cdot\overbrace{\frac{dV}{dt}}^{+b\%V} + \underbrace{\frac{\partial n}{\partial T}}_{-\frac{PV}{RT^2}}\cdot\overbrace{\frac{dT}{dt}}^{-c\%T}$$

$$= -a\%\cancel{P}\,\frac{n}{\cancel{P}} + b\%\cancel{V}\,\frac{n}{\cancel{V}} + c\%\cancel{T}\,\frac{n}{\cancel{T}}$$

$$\frac{dn}{dt} = (-a+b+c)\%n \longrightarrow n \text{ ปป. } (-a+b+c)\ \% \text{ ต่อชั่วโมง.}$$

Partial Chain Rule



$$\boxed{\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}}$$

as well as $\frac{\partial w}{\partial t}$.

Implicit Differentiation

Ⓘ $F(x,y) = 0$; $y = f(x) \rightarrow F(x, f(x)) = 0$

$$\frac{d}{dx}F(x,y) = 0$$

$$\underbrace{\frac{\partial F}{\partial x}}_{=F_x}\cdot\underbrace{\cancel{\frac{dx}{dx}}}_{1} + \underbrace{\frac{\partial F}{\partial y}}_{=F_y}\cdot\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Review Cramer's Rule

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\therefore x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Ⓘⓘ $F(x,y,z) = 0$, $G(x,y,z) = 0$; $y = f(x)$, $z = g(x)$

$\rightarrow F(x, f(x), g(x)) = 0$, $G(x, f(x), g(x)) = 0$

$$\begin{array}{cc} y & z \\ \uparrow & \uparrow \\ x & x \end{array}$$

$$\begin{cases} \frac{d}{dx}F(x,y,z) = 0 \rightarrow F_x\underbrace{\cancel{\frac{dx}{dx}}}_{1} + F_y\frac{dy}{dx} + F_z\frac{dz}{dx} = 0 \\ \frac{d}{dx}G(x,y,z) = 0 \rightarrow G_x\underbrace{\cancel{\frac{dx}{dx}}}_{1} + G_y\frac{dy}{dx} + G_z\frac{dz}{dx} = 0 \end{cases}$$

$$\begin{cases} F_y\frac{dy}{dx} + F_z\frac{dz}{dx} = -F_x \\ G_y\frac{dy}{dx} + G_z\frac{dz}{dx} = -G_x \end{cases}$$ จะได้ $$\begin{bmatrix} F_y & F_z \\ G_y & G_z \end{bmatrix}\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = -\begin{bmatrix} F_x \\ G_x \end{bmatrix}$$

$$\therefore \frac{dy}{dx} = -\frac{\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}}, \quad \frac{dz}{dx} = -\frac{\begin{vmatrix} F_y & F_x \\ G_y & G_x \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}}$$

"Jacobian"

$$\boxed{\frac{\partial(F,G)}{\partial(x,y)} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}$$

จะได้ $\dfrac{dy}{dx} = -\dfrac{\left|\dfrac{\partial(F,G)}{\partial(x,z)}\right|}{\left|\dfrac{\partial(F,G)}{\partial(y,z)}\right|}$, $\dfrac{dz}{dx} = -\dfrac{\left|\dfrac{\partial(F,G)}{\partial(y,x)}\right|}{\left|\dfrac{\partial(F,G)}{\partial(y,z)}\right|}$

General Form

มี
$$\begin{cases} F_1(x_1, x_2, \dots, x_n) = 0 \\ F_2(x_1, x_2, \dots, x_n) = 0 \\ \quad\vdots \\ F_m(x_1, x_2, \dots, x_n) = 0 \end{cases} \quad ; \; n > m$$

จะได้ $\dfrac{\partial x_i}{\partial x_j} = -\dfrac{\left|\dfrac{\partial(F_1, F_2, \dots, F_m)}{\partial(x_1, x_2, \text{แทรก}, x_m)}\right|}{\left|\dfrac{\partial(F_1, F_2, \dots, F_m)}{\partial(x_1, x_2, \dots, x_m)}\right|}$ ← Top-most child

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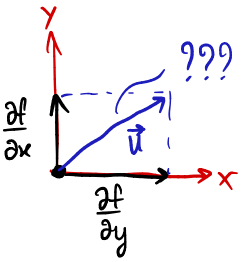
Directional Derivative (เป็นค่าสเกลาร์)

จาก $f_x(x_0, y_0) = \lim\limits_{h\to 0} \dfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$,

$f_y(x_0, y_0) = \lim\limits_{h\to 0} \dfrac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$

ก็คือ Derivative ใน 2 ทิศทาง

เมื่อพิจารณารวมกัน



จะได้ Derivative แบบมีทิศ $= \lim\limits_{h\to 0} \dfrac{f((x_0, y_0) + h\cdot\vec{u}) - f(x_0, y_0)}{h}$

same.

<u>def</u> ให้ $\vec{u} = (a,b)$ และ $\|\vec{u}\| = 1$, $D_{\vec{u}} f(x_0, y_0) = \lim\limits_{h\to 0} \dfrac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$

* ต้องเป็น unit vector.

- $f_x(x_0, y_0) = D_{\hat{i}} f(x_0, y_0)$, $f_y(x_0, y_0) = D_{\hat{j}} f(x_0, y_0)$
- สำหรับ f n ตัวแปร : $f(\vec{x}_0)$; $\vec{x}_0 = (x_1, x_2, \dots, x_n)$:

$$D_{\vec{u}} f(\vec{x}_0) = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

- $D_{\vec{u}} f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \dots, f_n(\vec{x})) \cdot \vec{u}$

$$D_{\vec{u}} f(x,y) = (f_x, f_y) \cdot \vec{u}$$

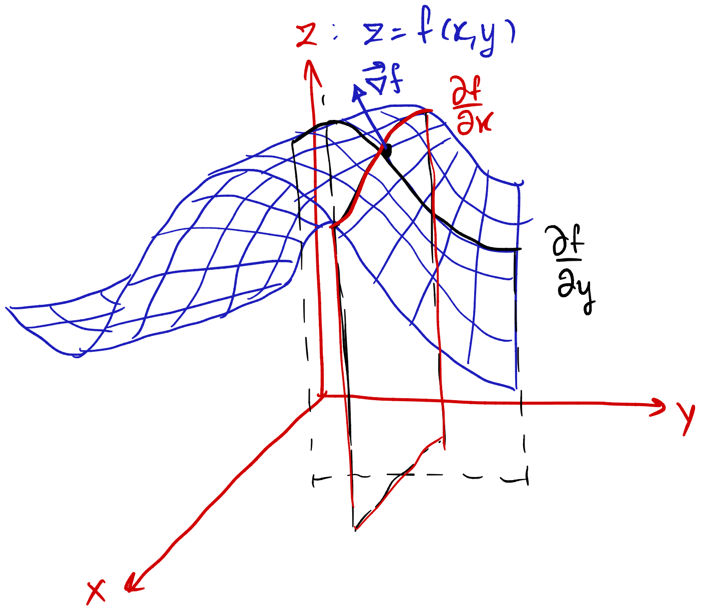
Gradient Vector (เป็นเวกเตอร์)

$$z = f(x,y) \rightarrow \vec{\nabla} f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle$$

$$w = f(x,y,z) \rightarrow \vec{\nabla} f(x,y,z) = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z) \rangle$$

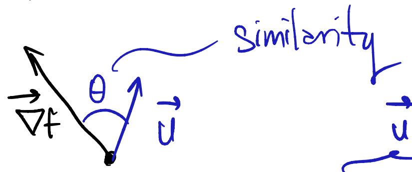
$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\boxed{D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}}$$



✳ Gradient คือความชัน (อัตราการเปลี่ยนแปลง)

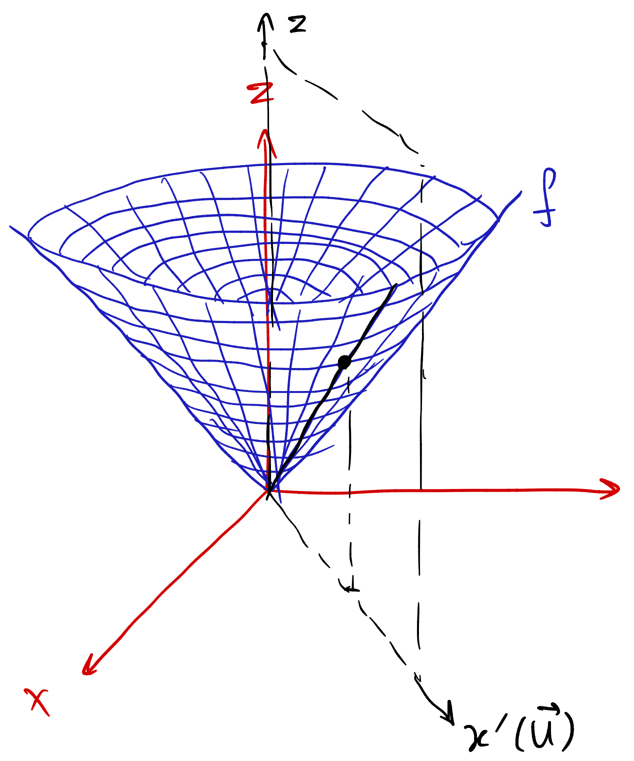
ถ้ามา dot กับ ทิศทางที่อยากหา $\vec{u}$
จะได้ความสัมพันธ์ว่า $\vec{u}$ ตรงกับทิศการเปลี่ยนแปลงมากแค่ไหน



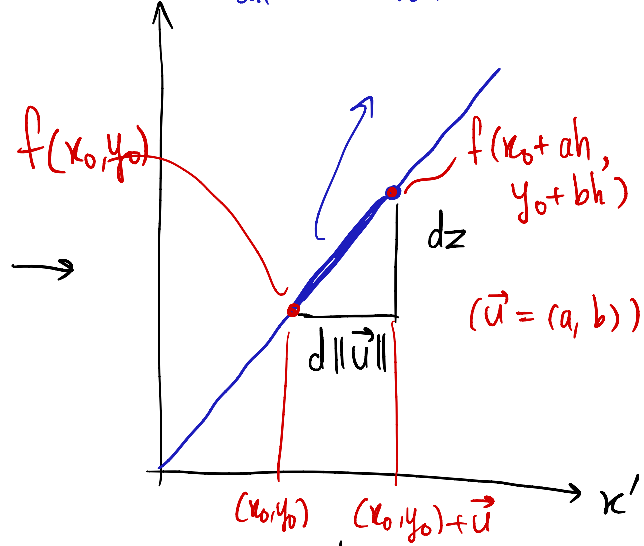
$\vec{u}$ ที่ไปทางเดียวกับ $\vec{\nabla} f$

$$D_{\vec{u}} f_{max} \text{ เมื่อ } = \vec{\nabla} f \cdot \left(\frac{\vec{\nabla} f}{\|\vec{\nabla} f\|} \right)$$

(pf. next page)



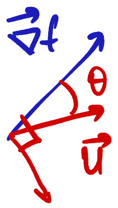
$\frac{dz}{d\|\vec{u}\|}$ คือ Directional Derivative.



"อัตราการเปลี่ยนแปลงของ f
ตามทิศทางของ $\vec{u}$"

(pf.) $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \|\vec{\nabla} f\| \|\vec{u}\| \cos\theta$

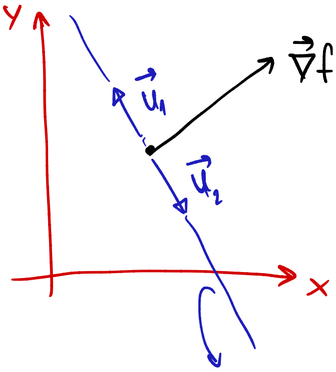
$$D_{\vec{u}} f = \|\vec{\nabla} f\| \cos\theta$$



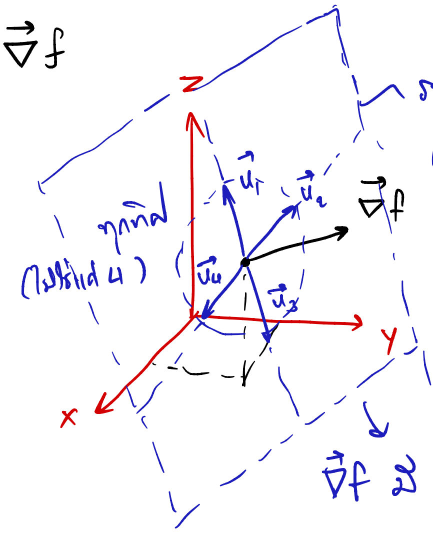
□ $D_{\vec{u}} f(\vec{x}_0)$ มีค่าสูงสุดเมื่อ $\underbrace{\vec{u} \parallel \vec{\nabla} f \text{ และ } sgn(\vec{u}) = sgn(\vec{\nabla} f)}_{\text{“}\vec{u}\text{ ทิศเดียวกับ } \vec{\nabla} f\text{”}}$

$\vec{\nabla} f \cdot \vec{u} = 0$

□ $D_{\vec{u}} f(\vec{x}_0) = 0$ เมื่อ $\vec{u} \perp \vec{\nabla} f$



$\vec{\nabla} f$ มี $\vec{u}_\perp$ เป็น L



ระนาบตั้งฉากกับ $\vec{\nabla} f$ ($\vec{\nabla} f$ เป็น $\vec{N}$ ของ M)

$\vec{\nabla} f$ มี $\vec{u}_\perp$ เป็น M

eg. ให้ $f(x,y) = x^2 y$

หา $\vec{u}$ ที่ทำให้ 1.) $D_{\vec{u}} f(3,2)$ สูงสุด (1)

2.) $D_{\vec{u}} f(3,2)$ ต่ำสุด (−1)

3.) $D_{\vec{u}} f(3,2) = 0$

$\vec{\nabla} f = \langle 2xy, x^2 \rangle$

$\vec{\nabla} f(3,2) = \langle 12, 9 \rangle$; 1.) $\therefore \vec{u} = \frac{1}{15}\langle 12, 9\rangle = \langle \frac{4}{5}, \frac{3}{5} \rangle$

2.) $\vec{u} = -\frac{1}{15}\langle 12, 9\rangle = \langle -\frac{4}{5}, -\frac{3}{5} \rangle$

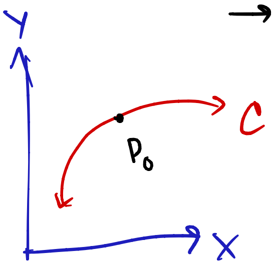
3. $\vec{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle , \langle \frac{3}{5}, -\frac{4}{5} \rangle$ $(m_1 \cdot m_2 = -1)$

Tangent Line and Surface

จากวิธีการหาเส้น/ผิวสัมผัสเดิม $\longrightarrow$ ใช้ Gradient $\vec{\nabla} f$ หา vector$_\perp$ $f(x,y)=k$ และ

vector$_\perp$ $f(x,y,z)=k$; k เป็น constant

Thm. 1. $f(x,y)$, $P_0(x_0,y_0)$ อยู่บน $C: f(x,y)=k$



$\rightarrow \vec{\nabla} f(x_0,y_0) \perp C$ ที่ P_0 จะได้ $\underbrace{\vec{\nabla} f(x_0,y_0)}_{\text{Vector}\perp} \cdot \underbrace{(x-x_0, y-y_0)}_{\text{Vector } \alpha} = 0$

($\vec{A}$ ที่ต้องการหา)

$$\vec{\nabla} f(x_0,y_0) \cdot (x-x_0, y-y_0) = 0$$

$$(f_x, f_y) \cdot (x-x_0, y-y_0) = 0$$

$\square$ $L: f_x(x-x_0) + f_y(y-y_0) = 0 \Big|_{(x_0,y_0)}$

2. ในทำนองเดียวกัน $f(x,y,z)$, $P_0(x_0,y_0,z_0)$ บน $S: f(x,y,z)=k$

จะได้ $\vec{\nabla} f(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0$

$\square$ $M: f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0 \Big|_{(x_0,y_0,z_0)}$

Gradient Vector Field

*Field
- Scalar : $f(x,y)$
- Vector : $\vec{F}(x,y)$

□ $\vec{\nabla} f(\vec{x}_0)$ ตั้งฉากกับ ผิว / โค้ง $f(\vec{x}) = k$ ที่ $\vec{x} = \vec{x}_0$

eg. $f(x,y) = x^2 - y^2$