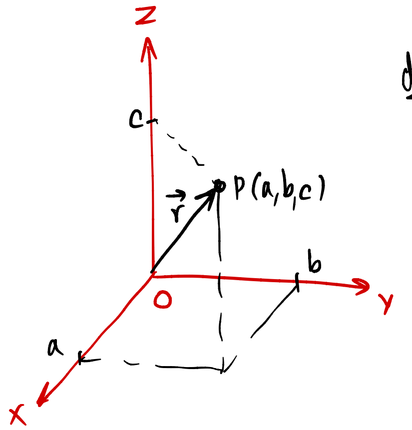
Vector , 3D Space



$\underline{\text{def}}$ $\vec{r} = (a,b,c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

- $\vec{B} - \vec{A} = (b_1 - a_1 , b_2 - a_2 , b_3 - a_3)$ $(\vec{OB}, \vec{OA})$
- $\vec{A} + \vec{B} = (a_1 + b_1 , a_2 + b_2 , a_3 + b_3)$
- $\vec{0} = (0, 0, 0)$

$*$ $\vec{A} \parallel \vec{B} \leftrightarrow A = cB$

Euclidean Norm

: $\underline{\text{def}}$ $\vec{A} = (a,b,c) \longrightarrow \|\vec{A}\| = A = \sqrt{a^2 + b^2 + c^2}$

$\underline{\text{def}}$ $\hat{e}_A = \hat{A} = \dfrac{\vec{A}}{\|\vec{A}\|}$: $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, $\hat{k} = (0,0,1)$

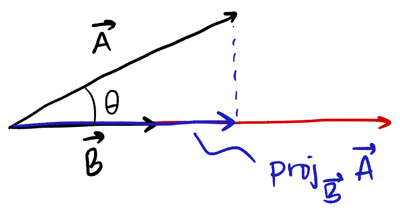
Scalar Product

: $\underline{\text{def}}$ $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$

- $\vec{A} \perp \vec{B} \leftrightarrow \vec{A} \cdot \vec{B} = 0$
- $\vec{A} \cdot \vec{A} = \|A\|^2$
- $\vec{A} \cdot \vec{B} = \|A\| \, \|B\| \cos\theta$; $\theta = \angle(\vec{A}, \vec{B})$

$\oplus$ $\forall \vec{A} \, (\vec{0} \perp \vec{A})$

Projection of Vectors



$\underline{\text{def}}$ $\text{proj}_{\vec{B}} \vec{A} = (\vec{A} \cdot \hat{B}) \hat{B}$

- $\text{proj}_{\vec{B}} \vec{A} = \dfrac{(\vec{A} \cdot \vec{B})}{\|B\|^2} \vec{B}$
- $\text{proj}_{a\vec{B}} \vec{A} = \text{proj}_{b\vec{B}} \vec{A}$; $a \neq b \neq 0$
- $\text{proj}_{k\vec{A}} \vec{A} = \vec{A}$

$\vec{A} \cdot \vec{B} = AB\cos\theta$

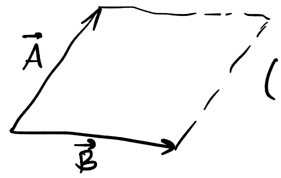
$\vec{A} \cdot \hat{B} = A\cos\theta$

$\vec{A} \cdot \hat{B} = \| \text{proj}_{\vec{B}} \vec{A} \|$

Cross Product (Vector Product)

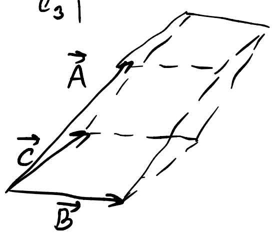
def $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$

- $\vec{A}\times\vec{B} = 0 \longleftrightarrow \vec{A}\parallel\vec{B}$
- $\vec{A}\times\vec{B} \perp \vec{A}$ and $\vec{B}$
- $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$
- $\|\vec{A}\times\vec{B}\| = \|A\|\|B\|\sin\theta$; $\theta = \angle(\vec{A},\vec{B})$
  - → พื้นที่ (flux)



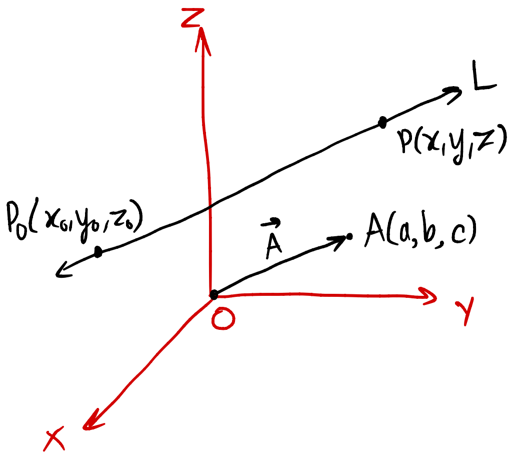
(Parallelogram)

- $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$
- $\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
  - → $|\vec{A}\cdot(\vec{B}\times\vec{C})|$ = ปริมาตรทรง



(Parallelepiped)

Line (เส้นตรง)



"Direction Vector"

- $\overrightarrow{P_0P} \parallel \vec{A}$
- $\overrightarrow{P_0P} = t\vec{A}$ → **def** $\vec{P} = \vec{P_0} + t\vec{A}$; $t\in\mathbb{R}$

$\overrightarrow{P_0P} = \vec{P}-\vec{P_0}$

$$(x,y,z) = (x_0,y_0,z_0) + t(a,b,c)$$

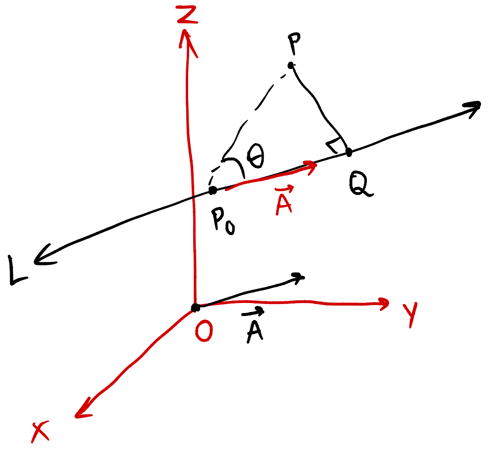
- $\vec{A}$ is direction vector → $k\vec{A}$ is direction vector.
- $P_0(x_0,y_0,y_0)$, $P_1(x_1,y_1,z_1)$ ผ่าน L →
  $\overrightarrow{P_0P_1} = (x_1-x_0, y_1-y_0, z_1-z_0)$ เป็น direction vector.

- Parametric Equation : $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$
- Symmetric Equation : $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

หาก L อยู่บน Plane YZ → $x = x_0$

หาก L ขนานแกน Z → $x = x_0$, $y = y_0$

- Distance between Line and Point



- □ $\|\overrightarrow{PQ}\| = \|\overrightarrow{P_0P}\| \sin\theta = \|\overrightarrow{P_0P} \times \hat{A}\|$

- □ $\overrightarrow{P_0Q} = \text{proj}_{\vec{A}} \overrightarrow{P_0P} = (\overrightarrow{P_0P} \cdot \hat{A})\hat{A}$

  $\downarrow$

  $\vec{Q} - \vec{P_0} = (\overrightarrow{P_0P} \cdot \hat{A})\hat{A}$

- □ $\vec{Q} = \vec{P_0} + (\overrightarrow{P_0P} \cdot \hat{A})\hat{A}$

- Angle between 2 direction vectors of Lines. (Angle between 2 Lines)

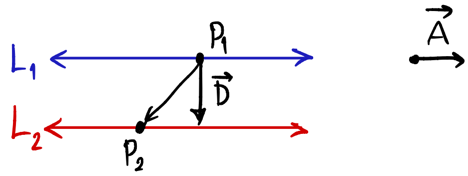
  $L_1 : \vec{A_1}$, $L_2 : \vec{A_2}$; $\angle(L_1, L_2) = \angle(\vec{A_1}, \vec{A_2})$

- □ From $\vec{A_1} \cdot \vec{A_2} = \|\vec{A_1}\| \|\vec{A_2}\| \cos\theta \rightsquigarrow \angle(\vec{A_1}, \vec{A_2}) = \arccos \dfrac{\vec{A_1} \cdot \vec{A_2}}{\|\vec{A_1}\| \|\vec{A_2}\|}$; $\theta \in (0, \pi)$

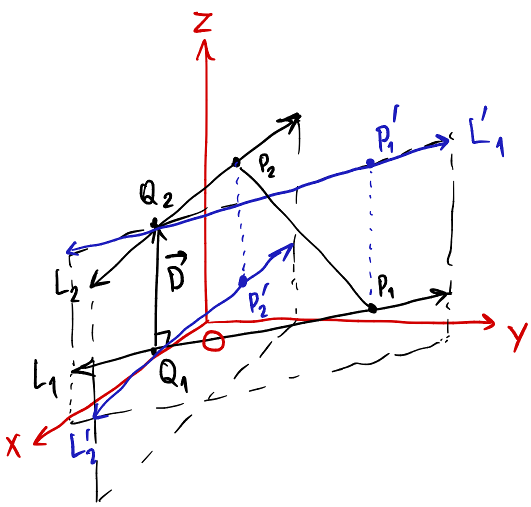
- 2 Lines :
  1. $L_1 \parallel L_2 \longrightarrow \vec{A_1} = k\vec{A_2}$ & หาระยะห่างได้
  2. $L_1 \nparallel L_2$
     - ตัดกัน $\longrightarrow$ แก้ระบบสมการ $\begin{cases} L_1 \\ L_2 \end{cases}$ หา (x, y, z) ตัด
     - ไม่ตัดกัน (ต่างระนาบ) $\rightarrow$ หาระยะห่างได้

- Distance between 2 Lines

  □ $L_1 \parallel L_2 \longrightarrow \vec{D} = \overrightarrow{P_1P_2} \times \hat{A}$

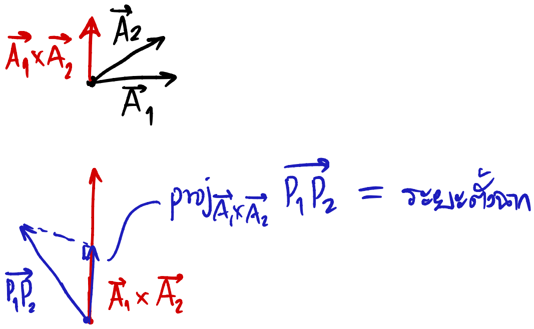


  □ $L_1 \nparallel L_2 \longrightarrow \vec{D} = (\overrightarrow{P_1P_2} \cdot \hat{e}_{\vec{A_1} \times \vec{A_2}})\hat{e}_{\vec{A_1} \times \vec{A_2}}$

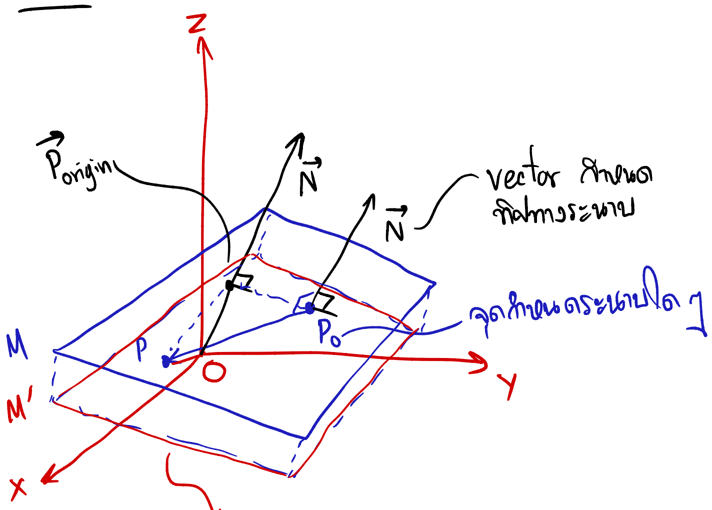


□ หาจุด Q_1, Q_2 ได้จาก L_1, L_2' หรือ L_1', L_2

Note



Plane



Parallel plane with same $\vec{N}$

but different P, P_0

$(\{P \mid P \in M_1\} \cap \{P \mid P \in M_2\} = \emptyset ; M_1 \parallel M_2)$

หาก Plane เดียวกัน แทนจุดใดๆ จะได้เป็นจริงเสมอ

จุด P ใดๆ อยู่บนระนาบ $\leftrightarrow$

def $\quad \overrightarrow{P_0P} \cdot \vec{N} = 0$

□ $(\vec{P} - \vec{P_0}) \cdot \vec{N} = 0$

$\therefore \vec{P} \cdot \vec{N} = \vec{P_0} \cdot \vec{N}$

- Vector Equation

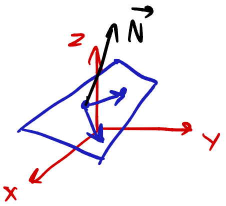
$(x, y, z) \cdot (a, b, c) = (x_0, y_0, z_0) \cdot (a, b, c)$

- Cartesian Equation

$ax + by + cz - d = 0 ; \; d = ax_0 + by_0 + cz_0$

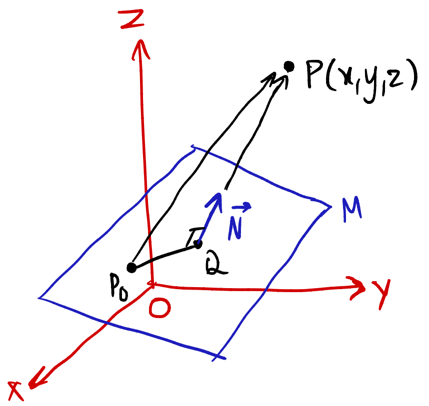
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

□ $\vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$



Line on a plane : $\vec{N} = \vec{A_1} \times \vec{A_2}$

- Distance between a point and a plane



□ $\overrightarrow{QP} = (\overrightarrow{P_0P} \cdot \hat{N}) \hat{N}$

□ $D = \dfrac{|ax + by + cz - d|}{\sqrt{a^2 + b^2 + c^2}}$ "สมการระนาบ แทนจุด"

- Plane and Line

□ $\vec{A} \cdot \vec{N} \neq 0$: ตัดระนาบ 1 จุด

L $\angle(\vec{A}, \vec{N}) = \arccos \dfrac{\vec{A} \cdot \vec{N}}{\|\vec{A}\| \|\vec{N}\|}$

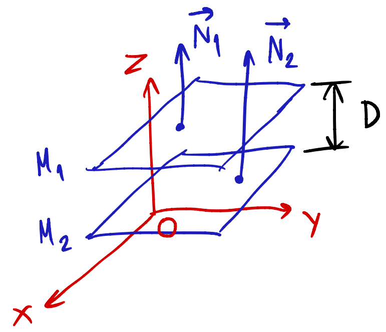
$\therefore \angle(L, M) = \angle(\vec{A}, M) = \frac{\pi}{2} - \angle(\vec{A}, \vec{N})$

$= \arcsin \dfrac{\vec{A} \cdot \vec{N}}{\|\vec{A}\| \|\vec{N}\|}$

□ $\vec{A} \cdot \vec{N} = 0$: ขนานกับระนาบ

- $D \neq 0$: ลอยจากระนาบ
- $D = 0$: อยู่บนระนาบ

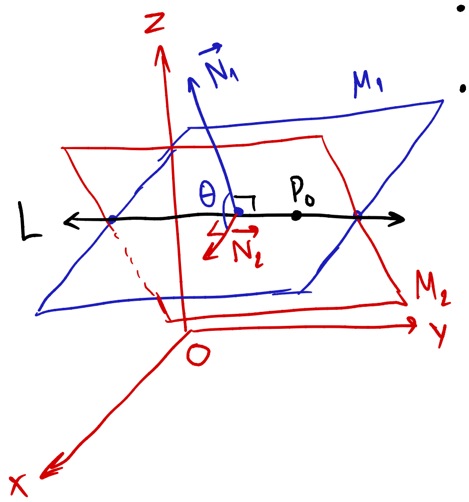
- Plane and Plane



- $\square$ $\vec{N_1} \parallel \vec{N_2} \rightarrow M_1 \parallel M_2$
  - $\square$ $D = \dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$;

$M_1 : ax+by+cz-d_1=0$, $M_2 : ax+by+cz-d_2=0$.

- $\square$ $\vec{N_1} \nparallel \vec{N_2} \rightarrow$ M_1 ตัดกับ M_2 ได้เส้นตรงที่มี Independent variable 1 ตัว



- แก้สมการ M_1 , M_2 เหลือสมการ 2 ตัวแปร (x,y) หรือ (y,z) หรือ (x,z)
- $\vec{A} = \vec{N_1} \times \vec{N_2}$ หาจุด $P_0(x_0,y_0,z_0)$ ที่เป็นจริงใน M_1, M_2

$\square$ $\theta = \arccos \dfrac{\vec{N_1} \cdot \vec{N_2}}{\|\vec{N_1}\|\|\vec{N_2}\|}$ $\quad (\angle(M_1,M_2) = \angle(\vec{N_1},\vec{N_2}) = \theta)$

Vector-valued Functions

def $D \subset \mathbb{R}$: 1. 2D Functions $\leftrightarrow \vec{F} : D \to \mathbb{R}^2 \leftrightarrow \vec{F}(t) = (f(t), g(t))$

2. 3D Functions $\leftrightarrow \vec{F} : D \to \mathbb{R}^3 \leftrightarrow \vec{F}(t) = (f(t), g(t), h(t))$

def $\lim\limits_{t\to a} \vec{F}(t) = \left(\lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t)\right)$

และ $\vec{F}$ ต่อเนื่องที่ $t=a \leftrightarrow \lim\limits_{t\to a}\vec{F}(t)$ exists and equals $\vec{F}(a)$

def $\vec{F}'(t) = (f'(t), g'(t), h'(t))$

def $\int_a^b \vec{F}(t)\,dt = \left(\int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\right)$

$\square$ Derivative ของ VVF ใช้สมบัติเดียวกับ Derivative ปกติได้เลย เช่น :

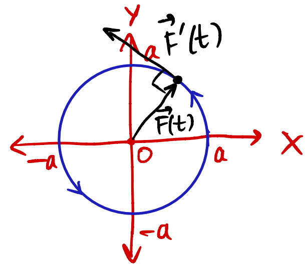
- $\square$ $\frac{d}{dt}(\vec{F}\cdot\vec{G}) = \vec{F}\cdot\vec{G}' + \vec{F}'\cdot\vec{G}$ — Dot
- $\square$ $\frac{d}{dt}(\vec{F}\times\vec{G}) = \vec{F}\times\vec{G}' + \vec{F}'\times\vec{G}$ — Cross

□ $\frac{d}{dt}\|\vec{F}(t)\|^2 = \frac{d}{dt}\left(\vec{F}(t)\cdot\vec{F}(t)\right) = 2\vec{F}(t)\cdot\vec{F}'(t)$

□ $\left(\frac{d}{dt}\|\vec{F}(t)\|^2 = 0 \leftrightarrow \|\vec{F}(t)\| = c\right) \rightarrow \left(\vec{F}(t) \perp \vec{F}'(t) \leftrightarrow \vec{F}(t)\cdot\vec{F}'(t) = 0\right)$

เช่น $\vec{F}(t) = (a\cos t, a\sin t)$

$\|\vec{F}(t)\| = a$, $\vec{F}'(t) = (-a\sin t, a\cos t)$

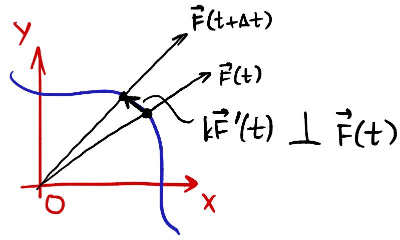


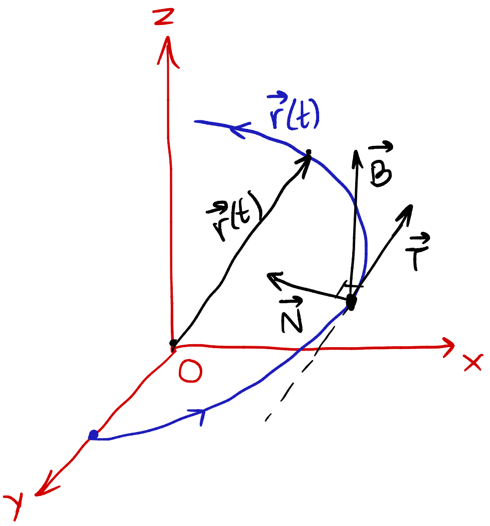
ในฟิสิกส์ : การเคลื่อนที่แบบวงกลม $\vec{r} = \vec{F}(t)$, $\vec{r}' = \underset{\text{Tangent}}{\vec{v}} = \vec{F}'(t)$

และ $\vec{a} = \vec{F}''(t)$

□ ใช้ Definition of Limit หา $\vec{F}'(t)$:

$$\vec{F}'(t) = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$$





• TNB Frame , Frenet–Serret Formulas

□ $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$

□ $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$

□ $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

} Unit vector

• VVF components : $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\vec{v}(t)}{v(t)}$

$\therefore \vec{v}(t) = v(t)\vec{T}(t)$

□ $\vec{v}'(t) = \vec{a}(t) = \underbrace{v'(t)\vec{T}(t)}_{\vec{a}_T} + \underbrace{v(t)\cdot\vec{N}(t)\cdot\|\vec{T}'(t)\|}_{\vec{a}_N}$

• VVF arc length □ $L = \int_a^b \|\vec{r}'(t)\|\,dt$

□ $s(t) = \int_a^t \|\vec{r}'(u)\|\,du \longrightarrow \frac{ds}{dt} = \|\vec{r}'(t)\|$

• Curvature : □ $\vec{T}(s) \rightarrow \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{\vec{T}'(t)}{\|\vec{r}'(t)\|} = \kappa(t)$ ⇝ Curvature

□ Radius of curvature : $\rho(t) = \frac{1}{\kappa(t)}$