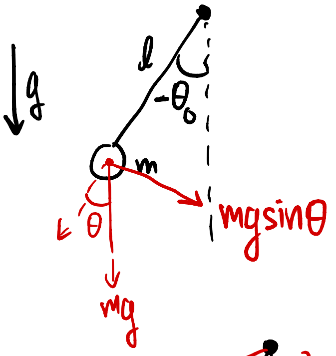
SINGLE PENDULUM

$x = \theta R \qquad \dot{x} = v = \dot{\theta} R$

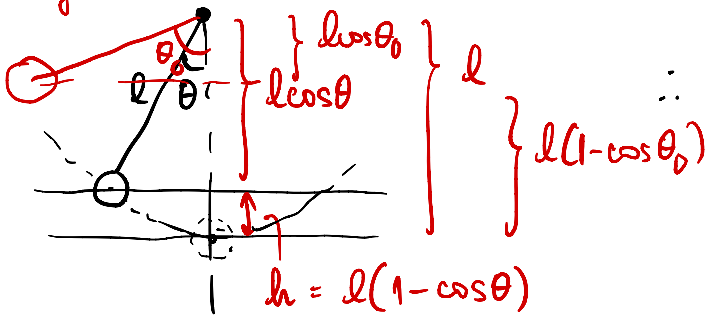
$\ddot{x} = \ddot{\theta} R$



$$m\ddot{x}_T = mg\sin(\theta) = m\ddot{\theta} l$$

Newton:
$$-g\sin\theta = \ddot{\theta} l$$

$$\therefore \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \quad ; \quad \theta(0) = \theta_0$$
$$\dot{\theta}(0) = 0$$



E-cons.:

$$\therefore \frac{1}{2}\cancel{m}\underbrace{v^2}_{(\dot{\theta} l)^2} + \cancel{m}gl(1-\cos\theta) = \cancel{m}gl(1-\cos\theta_0)$$

$$\frac{1}{2}\dot{\theta}^2 l^{\cancel{2}} = g\cancel{l}(1-\cos\theta_0 - 1 + \cos\theta)$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$$

$$dt = \sqrt{\frac{l}{2g}}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,d\theta$$

$$\int_0^{T/4} dt = \int_0^{\theta_0}\sqrt{\frac{l}{2g}}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,d\theta$$

$$\therefore T = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,d\theta \,,$$

$$\sin^2\xi = \frac{1}{2}(1 - \cos 2\xi)$$

$$\therefore \cos 2\xi = 1 - 2\sin^2\xi \quad ; \quad \xi = \frac{\theta}{2}$$

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

$$= \frac{\cancel{4}\,2}{\cancel{\sqrt{2}\,2}\sqrt{2}}\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{1}{\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}\,d\theta$$

$$= 2\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{1}{\sin\left(\frac{\theta_0}{2}\right)\sqrt{1 - \frac{\sin^2(\theta/2)}{\sin^2(\theta_0/2)}}}\,d\theta$$

$\sin(\theta/2) = \sin\phi\sin(\theta_0/2) \rightarrow \cos(\theta/2) = \sqrt{1-\sin^2(\theta/2)}$

$$; \quad \sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} \quad \Big| \quad \arcsin(1) = \pi/2, \quad \phi = 0$$

$$= 2\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{2\cancel{\sin(\theta_0/2)}\cancel{\cos\phi}}{\cancel{\sin(\theta_0/2)}\cancel{\sqrt{1-\sin^2\phi}}\sqrt{1-\sin^2\phi\sin^2(\theta_0/2)}}\,d\phi$$

$$\cos\phi\, d\phi = \frac{\cos\left(\frac{\theta}{2}\right)}{2\sin(\theta_0/2)}\,d\theta$$

$$d\theta = \frac{2\sin(\theta_0/2)\cos\phi}{\cos(\theta/2)}\,d\phi$$

$$d\theta = \frac{2\sin(\theta_0/2)\cos\phi}{\sqrt{1 - \sin^2\phi\sin^2(\theta_0/2)}}\,d\phi$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{1}{\sqrt{1-\sin^2\phi\sin^2(\theta_0/2)}}\,d\phi$$

$$m = \sin(\theta_0/2) \rightarrow K(m) = \int_0^{\pi/2}\frac{1}{\sqrt{1 - m^2\sin^2\theta}}\,d\theta$$

$$\rightarrow T = 4\sqrt{\frac{l}{g}}\,K\left(\sin\left(\frac{\theta_0}{2}\right)\right)$$

K Power Series

$$\int_0^{\pi/2} \sin^{2i}\phi \, d\phi$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}$$

$$= 2\int_0^{\pi/2} \sin^{2x-1}(t)\cos^{2x-1}(t)\,dt$$

	D	I
$+$	$\sin^{2i-1}\phi$	$\sin\phi$
$-$	$\sin^{2i-2}\phi\cos\phi(2i-1)$	$-\cos\phi$

$(1-\sin^2\phi)$

$$= -\sin^{2i-1}\phi\cos\phi\Big|_0^{\pi/2} + (2i-1)\int_0^{\pi/2}\sin^{2i-2}\phi\,\cancel{\cos^2}\phi\,d\phi$$

$$1\int_0^{\pi/2}\sin^{2i}\phi\,d\phi = (2i-1)\int_0^{\pi/2}\sin^{2i-2}\phi\,d\phi - (2i-1)\int_0^{\pi/2}\sin^{2i}\phi\,d\phi \quad -\sin^{2i-1}\phi\cos\phi\Big|_0^{\pi/2}$$

$$\int_0^{\pi/2}\sin^{2i}\phi\,d\phi = \frac{1}{2i}\left[(2i-1)\int_0^{\pi/2}\sin^{2i-2}\phi\,d\phi - \underbrace{\sin^{2i-1}\phi\cos\phi\Big|_0^{\pi/2}}_{=0}\right]$$

$$\int_0^{\pi/2}\sin^{2i}\phi\,d\phi = \frac{2i-1}{2i}\int_0^{\pi/2}\sin^{2i-2}\phi\,d\phi$$

$$f(x) = (1-x)^{-1/2}$$

$$f'(x) = \frac{1}{2}(1-x)^{-3/2}$$

$$f^{(2)}(x) = \frac{1}{2}\cdot\frac{3}{2}(1-x)^{-5/2}$$

$$f^{(3)}(x) = \frac{1\cdot3\cdot5}{2\cdot2\cdot2}(1-x)^{-7/2}$$

$$\vdots$$

$$f^{(k)}(x) = \frac{(2k-1)!!}{2^n}(1-x)^{-\frac{2k+1}{2}}$$

$$f^{(k)}(0) = \frac{(2k-1)!!}{2^n}$$

$$\rightarrow f(x) = \sum_{k\geq0}\frac{(2k-1)!!}{k!\cdot2^n}x^n$$

$$K(m) = \int_0^{\pi/2}\frac{1}{\sqrt{1-\underbrace{m^2\sin^2\theta}_{x}}}\,d\theta$$

$$= \int_0^{\pi/2}\sum_{n\geq0}\frac{(2n-1)!!}{n!2^n}m^{2n}\sin^{2n}\theta\,d\theta$$

$$= \sum_{n\geq0}\frac{(2n-1)!!}{n!\,2^n}m^{2n}\int_0^{\pi/2}\sin^{2n}\theta\,d\theta$$

$$0 = 1-1 = 2\left(\tfrac{1}{2}\right)-1 \quad y$$

$$2n = 2n+1-1 = 2\left(n+\tfrac{1}{2}\right)-1 \quad x$$

$$= \frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(n+1)}$$

$$= \frac{\Gamma\left(\frac{2n+1}{2}\right)\cdot\sqrt{\pi}}{2\cdot n!}$$

$$= \sum_{n\geq0}\frac{(2n-1)!!}{(n!)^2\cdot2^n}m^{2n}\,\Gamma\left(\frac{2n+1}{2}\right)\cdot\frac{\sqrt{\pi}}{2}$$

$$* \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(1+\tfrac{1}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \Gamma\left(1+\tfrac{3}{2}\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\cdot\tfrac{3}{2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{7}{2}\right) = \Gamma\left(1+\tfrac{5}{2}\right) = \frac{1\cdot3\cdot5}{2^3}\sqrt{\pi}$$

$$\vdots$$

$$\Gamma\left(\overset{n+\frac{1}{2}}{\frac{2n+1}{2}}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

$$K(m) = \sum_{n\geq 0} \frac{(2n-1)!!}{(n!)^2 \cdot 2^n} m^{2n} \cdot \frac{(2n-1)!!}{2^n} \sqrt{\pi} \cdot \frac{\sqrt{\pi}}{2}$$

$$K(m) = \frac{\pi}{2} \sum_{n\geq 0} \left[\frac{(2n-1)!!}{2^n \cdot n!}\right]^2 m^{2n}$$

$2^n \cdot n! = (2n)!!$

$n=0; \quad 1 = 1$

$n=1; \quad 2\times 1! = 2!!$

$n=2; \quad 2^2 \times [2!] = 4!! = 4\times 2$

$n=3; \quad 2^3 \cdot [3!] = 6\times 4\times 2 = 6!!$ ✓

$$\therefore K(m) = \frac{\pi}{2} \sum_{n\geq 0} \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 m^{2n}$$

$$m = \sin(\theta_0/2)$$

$$T = 4\sqrt{\frac{l}{g}} K(\sin(\theta_0/2))$$

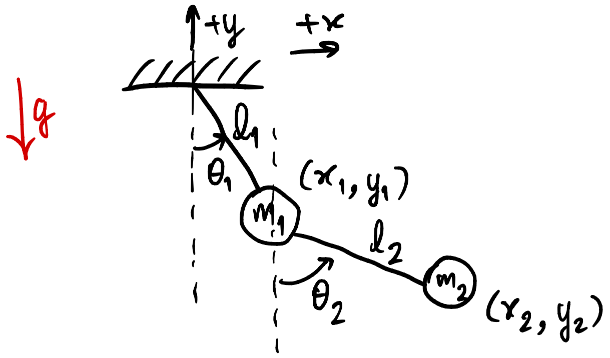
$$T = \cancel{4}2 \sqrt{\frac{l}{g}} \cdot \frac{\pi}{\cancel{2}} \left[1 + \frac{1^2}{2^2}\sin^2\left(\frac{\theta_0}{2}\right) + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\sin^2\left(\frac{\theta_0}{2}\right) + \dots\right]$$

$$T = 2\pi\sqrt{\frac{l}{g}}\left[\dots\right]$$

Conclusion :

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}} \sum_{n\geq 0} \left(\frac{(2n-1)!!}{(2n)!!} \sin^n \frac{\theta_0}{2}\right)^2}$$

DOUBLE PENDULUM (*Lagrangian Mechanics*)



$$\begin{cases} x_1 = l_1 \sin\theta_1 & (1) \\ y_1 = -l_1\cos\theta_1 & (2) \\ x_2 = l_1\sin\theta_1 + l_2 \sin\theta_2 & (3) \\ y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2 & (4) \end{cases}$$

$$\rightarrow \quad \dot{x}_1 = \dot{\theta}_1 l_1 \cos\theta_1 \qquad (5)$$

$$\dot{y}_1 = \dot{\theta}_1 l_1 \sin\theta_1 \qquad (6)$$

$$\dot{x}_2 = \dot{\theta}_1 l_1 \cos\theta_1 + \dot{\theta}_2 l_2 \cos\theta_2 \qquad (7)$$

$$\dot{y}_2 = \dot{\theta}_1 l_1 \sin\theta_1 + \dot{\theta}_2 l_2 \sin\theta_2 \qquad (8)$$

Tot. V: $\quad V = m_1 g y_1 + m_2 g y_2$

$$V = -(m_1+m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \quad \text{—} (9)^{*}$$

Tot. K: $\quad K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

$$K = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$K = \frac{1}{2} m_1 \dot{\theta}_1^2 l_1^2 \cancelto{1}{(\sin^2\theta_1 + \cos^2\theta_1)} + \frac{1}{2} m_2 \Big[(\dot{\theta}_1 l_1 \cos\theta_1 + \dot{\theta}_2 l_2 \cos\theta_2)^2 + (\dot{\theta}_1 l_1 \sin\theta_1 + \dot{\theta}_2 l_2 \sin\theta_2)^2 \Big]$$

$$K = \frac{1}{2} m_1 \dot{\theta}_1^2 l_1^2 + \frac{1}{2} m_2 \dot{\theta}_1^2 l_1^2 + \frac{1}{2} m_2 \dot{\theta}_2^2 l_2^2 + m_2 \dot{\theta}_1 \dot{\theta}_2 l_1 l_2 (\cos\theta_1\cos\theta_2 + \sin\theta_1 \sin\theta_2)$$

$$K = \frac{1}{2}(m_1+m_2)\dot{\theta}_1^2 l_1^2 + \frac{1}{2} m_2 \dot{\theta}_2^2 l_2^2 + m_2 \dot{\theta}_1 \dot{\theta}_2 l_1 l_2 \cos(\theta_2 - \theta_1) \quad \text{—} (10)^{*}$$

Lagrangian non-conservative force: $\quad Q_i = \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_i} - \frac{\partial V}{\partial q_i}$

$$\rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad ; \quad L = \overset{K}{T} - V \Big|_{(10)-(9)}^{**}$$

$$\theta_1 ; \quad \frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_2-\theta_1) - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_2 - \theta_1)$$

$$\frac{\partial L}{\partial \theta_1} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1) - (m_1+m_2) g l_1 \sin\theta_1$$

$$\rightarrow (m_1+m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_2-\theta_1) - \underline{m_2 l_1 l_2 \dot{\theta}_2} (\dot{\theta}_2 - \cancel{\dot{\theta}_1}) \underline{\sin(\theta_2-\theta_1)} - \cancel{m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1)}$$
$$+ (m_1+m_2) g l_1 \sin\theta_1 = 0$$

$$(m_1+m_2) \cancel{l_1} l_1 \ddot{\theta}_1 + m_2 \cancel{l_1} l_2 \ddot{\theta}_2 \underbrace{\cos(\theta_2-\theta_1)}_{=1} - \underbrace{m_2 \cancel{l_1} l_2 \dot{\theta}_2^2 \sin(\theta_2-\theta_1)}_{=0} + (m_1+m_2) g \cancel{l_1} \underbrace{\sin\theta_1}_{=\theta_1} = 0$$

Small Angle $\rightarrow$

$$\boxed{(m_1+m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2-\theta_1) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2-\theta_1) + (m_1+m_2) g \sin\theta_1 = 0} \quad (A)$$

θ_2;
$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_2-\theta_1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2-\theta_1) - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_2-\theta_1)$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1) - m_2 g l_2 \sin\theta_2$$

$$\rightarrow \cancel{m_2} l_2^{\cancel{2}} \ddot{\theta}_2 + \cancel{m_2} l_1 \cancel{l_2} \ddot{\theta}_1 \cos(\theta_2-\theta_1) - \cancel{m_2} l_1 \cancel{l_2} \dot{\theta}_1 (\cancel{\dot{\theta}_2} - \dot{\theta}_1) \sin(\theta_2-\theta_1) + \cancel{m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1)}$$
$$+ \cancel{m_2} g \cancel{l_2} \sin\theta_2 = 0$$

$$\boxed{l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_2-\theta_1) + l_1 \dot{\theta}_1^2 \sin(\theta_2-\theta_1) + g \sin\theta_2 = 0} \quad (B)$$

Lin Eq:
$$\begin{bmatrix} (m_1+m_2) l_1 & m_2 l_2 \cos(\theta_2-\theta_1) \\ l_1 \cos(\theta_2-\theta_1) & l_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2-\theta_1) - (m_1+m_2) g \sin\theta_1 \\ -l_1 \dot{\theta}^2 \sin(\theta_2-\theta_1) - g \sin\theta_2 \end{bmatrix}$$

or $\rightarrow$ multiply m_2 (Row) both sides for uniformity

Can be written as: $\underbrace{\cancel{[m]^{-1}[m]}}_{I} \ddot{\vec{\theta}} = [m]^{-1} \cdot \vec{F}$

$$\ddot{\vec{\theta}} = [m]^{-1} \cdot \vec{F}$$

Let $\vec{y} = \begin{bmatrix} \dot{\vec{\theta}} \\ \vec{\theta} \end{bmatrix}$, $\dot{\vec{y}} = \begin{bmatrix} \ddot{\vec{\theta}} \\ \dot{\vec{\theta}} \end{bmatrix} = \begin{bmatrix} [m^{-1}] \cdot \vec{F} \\ \dot{\vec{\theta}} \end{bmatrix} = \vec{G}$

Therem: $\dot{y} \approx \frac{y_t - y_{t_0}}{\Delta t}$

$$\rightarrow y_t = y_{t_0} + \dot{y} \Delta t$$

Integrating machine

$$\therefore \underline{\vec{y}_{t+1} = \vec{y}_t + \Delta t \cdot \vec{G}}$$

The Principle of Stationary Action

QUANTITY: $\quad S = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$

For $t_1 \leq t \leq t_2$, $x(t_1) = x_1$, $x(t_2) = x_2$

If $x_0(t)$ yields a stationary S, $x_a(t) = x_0(t) + \alpha\beta(t)$,
$[\alpha \wedge \beta(t) :\rightarrow \beta(t_1) = \beta(t_2) = 0]$

$$\frac{\partial}{\partial a} S[x_a(t)] = \frac{\partial}{\partial a}\int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial a}\, dt \quad \partial\ ???$$

$$\frac{\partial S}{\partial a} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a}\frac{\partial x_a}{\partial a} + \frac{\partial L}{\partial \dot{x}_a}\frac{\partial \dot{x}_a}{\partial a} \right) dt \ ;$$

$$\beta = \frac{\partial x_a}{\partial a} , \quad \dot{\beta} = \frac{\partial \dot{x}_a}{\partial a}$$

$$\frac{\partial S}{\partial a} = \int_{t_1}^{t_2} \frac{\partial L}{\partial x_a}\beta\, dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}_a}\dot{\beta}\, dt$$

$$\int \frac{\partial L}{\partial \dot{x}_a}\dot{\beta}\, dt = \frac{\partial L}{\partial \dot{x}_a}\beta - \int \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_a}\right)\beta\, dt$$

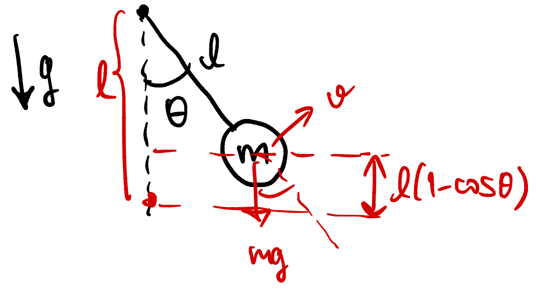
$$\frac{\partial S[x_a(t)]}{\partial a} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a} \right)\beta\, dt + \frac{\partial L}{\partial \dot{x}_a}\beta \Big|_{t_1}^{t_2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_a}\beta \Big|_{t_1}^{t_2} - \frac{\partial S}{\partial a} \right) = \cancel{\frac{d}{dt}}\cancel{\int_{t_1}^{t_2}} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a} - \frac{\partial L}{\partial x_a} \right)\beta\, \cancel{dt}$$

$-Q$

$$\frac{1}{\beta}\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_a}\beta\right) - \frac{\partial S}{\partial a} \right\} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a} - \frac{\partial L}{\partial x_a}$$

$$\rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_0} - \frac{\partial L}{\partial x_0} = 0$$



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{\theta}^2 R^2$$

$$V = mgl(1-\cos\theta)$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(\cos\theta - 1)$$

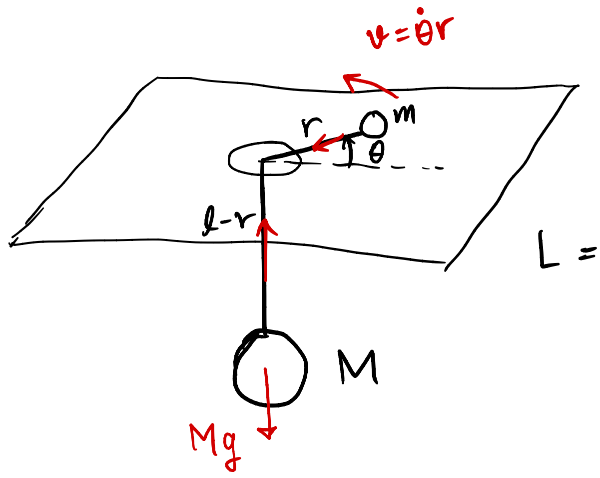
L-E: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \quad , \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m l^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\cancel{m} l^{\cancel{2}} \ddot{\theta} + \cancel{m} g \cancel{l} \sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$



$$\mathcal{L} = \frac{1}{2} M \dot{r}^2 + \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \dot{\theta}^2 r^2 - Mg(r - l)$$

$$L = \sum \vec{r} \times m\vec{v} = 0 : \frac{d}{dt}(m r^2 \dot{\theta}) = 0 \ ; L = m r^2 \dot{\theta}$$

$$\dot{\theta} = \frac{L}{m r^2}$$

$$(M+m)\ddot{r} = m\dot{\theta}^2 r - Mg$$

$$(M+m)\ddot{r} = \frac{L^2}{m r^3} - Mg = 0$$

Circ. Motion $\ddot{r} = \dot{r} = 0$,

$$r_0^3 = \frac{L^2}{mMg} \longleftrightarrow m\dot{\theta}^2 r_0 = Mg$$

r) $$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = M\dot{r} + m\dot{r} \ , \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = (M+m)\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} = m\dot{\theta}^2 r - Mg$$

θ) $$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} \ , \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \longrightarrow r^{\cancel{2}} \ddot{\theta} = -2 \cancel{r} \dot{r} \dot{\theta}$$

$$-2\dot{r}\dot{\theta} = r\ddot{\theta}$$