Heat Transfer with Finite Elements

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December 2022

For isotropic homogeneous materials

|  |  |  |
| --- | --- | --- |
|  |  | (1) |

where is specific heat, is density, is temperature, is time, is thermal conductivity, is the internal heat generation rate per unit volume, and are space components. Repeated indices are summed. In some cases, it may be convenient to write the equations in terms of thermal diffusivity . Assuming

|  |  |  |
| --- | --- | --- |
|  |  | (2) |

The notation is used instead of for compactness; summation over is still implied. Using Galerkin method, are a set of functions that are used for both the basis of the field variable and for weighting the residual. Integrate the weighted residual

|  |  |  |
| --- | --- | --- |
|  |  | (3) |

Distribute

|  |  |  |
| --- | --- | --- |
|  |  | (4) |

Substitute discretized

|  |  |  |
| --- | --- | --- |
|  |  | (5) |

where are the temperature values at nodes. are functions of space only, so

|  |  |  |
| --- | --- | --- |
|  |  | (6) |

and

|  |  |  |
| --- | --- | --- |
|  |  | (7) |

Substitute

|  |  |  |
| --- | --- | --- |
|  |  | (8) |

Rearrange

|  |  |  |
| --- | --- | --- |
|  |  | (9) |

Move nodal temperature values outside of the integrals

|  |  |  |
| --- | --- | --- |
|  |  | (10) |

Integrate the conduction term by parts

|  |  |  |
| --- | --- | --- |
|  |  | (11) |

where is the surface normal component in direction . Substituting into Equation (10)

|  |  |  |
| --- | --- | --- |
|  |  | (12) |

Recognizing heat flux on the boundary can be specified as a natural boundary condition

|  |  |  |
| --- | --- | --- |
|  |  | (13) |

The right-hand side is written accordingly

|  |  |  |
| --- | --- | --- |
|  |  | (14) |

where is the portion of the boundary where heat flux is specified. Temperature may be specified at the remaining boundary and enforced through matrix algebra.

Write Equation (14) as a matrix equation

|  |  |  |
| --- | --- | --- |
|  |  | (15) |

where

|  |  |  |
| --- | --- | --- |
|  |  | (16) |

Degrees of freedom are constrained where nodal temperatures are specified, so the matrices are partitioned into free () and constrained () degrees of freedom

|  |  |  |
| --- | --- | --- |
|  |  | (17) |

The top row of Equation (17) is used to solve for the unknown temperatures

|  |  |  |
| --- | --- | --- |
|  |  | (18) |

## Steady-State Response

For steady state: and can be obtained using pure matrix algebra

|  |  |  |
| --- | --- | --- |
|  |  | (19) |

## Transient Response

Solve Equation (18) for for the form needed for typical time integration with initial conditions.

|  |  |  |
| --- | --- | --- |
|  |  | (20) |

## Thermal Modes

The thermal modes are determined considering the dynamics of the free degrees of freedom

|  |  |  |
| --- | --- | --- |
|  |  | (21) |

Assuming

|  |  |  |
| --- | --- | --- |
|  |  | (22) |

where is a matrix of eigenvectors and is a diagonal matrix of eigenvalues, the generalized eigenvalue equation is

|  |  |  |
| --- | --- | --- |
|  |  | (23) |

The physical degrees of freedom can be transformed to modal degrees of freedom using the eigenvectors

|  |  |  |
| --- | --- | --- |
|  |  | (24) |

## Modal Equations

Starting with Equation (18)

write in modal coordinates

|  |  |  |
| --- | --- | --- |
|  |  | (25) |

Premultiply by

|  |  |  |
| --- | --- | --- |
|  |  | (26) |

Conductance and capacitance matrices corresponding to free degrees of freedom are now diagonalized; this is a property of the eigenvectors.

|  |  |  |
| --- | --- | --- |
|  |  | (27) |

Assuming and rearranging

|  |  |  |
| --- | --- | --- |
|  |  | (28) |

Arranging terms for time integration

|  |  |  |
| --- | --- | --- |
|  |  | (29) |

Equation (29) is analogous to Equation (20), but inverting is trivial and analysis degrees of freedom can be reduced through modal truncation.

## Time-Analytic Solution to Modal Equations

A time-analytic solution is possible by treating the dynamic solution as a perturbation from steady state

|  |  |  |
| --- | --- | --- |
|  |  | (30) |

and assuming constant and . The steady state model solution is

|  |  |  |
| --- | --- | --- |
|  |  | (31) |

And the dynamic model solution is

|  |  |  |
| --- | --- | --- |
|  |  | (32) |

The analytic solution to Equation (32) for initial value is

|  |  |  |
| --- | --- | --- |
|  |  | (33) |

Given initial condition , initial modal coordinates are

|  |  |  |
| --- | --- | --- |
|  |  | (34) |

and initial dynamic modal coordinates are

|  |  |  |
| --- | --- | --- |
|  |  | (35) |

Combining Equation (30) and Equation (33)

|  |  |  |
| --- | --- | --- |
|  |  | (35) |

Equation (35) is an analytic expression for the modal coordinates at any time. The modal basis requires spatial discretization and an eigenvalue solution. The result has spatial discretization error, optionally modal truncation error, but no time discretization error. The result can be transformed to physical temperatures using Equation (24).