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Homework Assignment 4
Introduction To Design And Analysis Of Algorithms, Fall 2020

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Contents

1 Problem 1

By using divide-and-conquer approach, we can solve the change making problem with the time complexity is represented by the recurrence:

$$T(n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} [T(i) + T(n-i)] + \Theta(n), n > 2$$
$$T(2) = 1, T(1) = 0$$

Replace $\Theta(n)$ by $\lfloor \frac{n}{2} \rfloor$. We have:

$$T(n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} [T(i) + T(n-i)] + \left\lfloor \frac{n}{2} \right\rfloor$$

1.1 Consider the case n is odd number

With odd number, clearly:

$$\begin{aligned} T(n) &= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} [T(i) + T(n-i)] + \left\lfloor \frac{n}{2} \right\rfloor \\ &= T(1) + T(2) + \ldots + T(\left\lfloor \frac{n}{2} \right\rfloor) + T(\left\lfloor \frac{n}{2} \right\rfloor + 1) + \ldots + T(n-1) + \left\lfloor \frac{n}{2} \right\rfloor \\ &= \sum_{i=1}^{n-1} T(i) + \left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

$$\implies 2 \cdot T(n) = 2 \cdot \sum_{i=1}^{n-1} T(i) + n$$

With n, from the equation above:

$$2 \cdot T(n) = 2 \cdot \sum_{i=1}^{n-1} T(i) + n \tag{1}$$

If replace n by $n-1$, $n-1$ is even:

$$2 \cdot T(n-1) = 2 \cdot \sum_{i=1}^{n-2} T(i) + n - 1 + 2 \cdot T(\frac{n-1}{2}) \tag{2}$$

Subtract (2) from (1):

$$2T(n) - 2T(n-1) = 2T(n-1) + 1 - 2 \cdot T(\frac{n-1}{2}) \Leftrightarrow T(n) = 2T(n-1) + 2^{-1} - T(\frac{n-1}{2})$$

Comeback to the familiar form of recurrence equation, easily solve it by using backward substitution:

$$\begin{aligned} T(n) &= 2T(n-1) + 2^{-1} - T(\frac{n-1}{2}) \geq 3T(n-1) + 2^{-1} \\ &= 3 \cdot [3T(n-2) + 2^{-1}] + 2^{-1} = 3^2 \cdot T(n-2) + 3 \cdot 2^{-1} + 2^{-1} \\ &= 3^2 \cdot [3T(n-3) + 2^{-1}] + 3 \cdot 2^{-1} + 2^{-1} = 3^3 \cdot T(n-3) + 3^2 \cdot 2^{-1} + 3 \cdot 2^{-1} + 2^{-1} \\ &= \ldots = 3^i \cdot T(n-i) + \frac{1}{2} \cdot \frac{3^i - 1}{3 - 1} = 3^i \cdot T(n-i) + \frac{3^i - 1}{4} \end{aligned}$$

Base on the initial condition $T(2) = 1, T(1) = 0$.
If $n - i = 1 \implies i = n - 1$:

$$\begin{aligned} T(n) \geq 3^{n-1} \cdot 0 + \frac{3^{n-1} - 1}{4} &= \frac{3^{n-1} - 1}{4} \in \Theta(3^n) \\ &\implies T(n) \in \Omega(2^n) \end{aligned}$$

If $n - i = 2 \implies i = n - 2$:

$$\begin{aligned} T(n) &\geq 3^{n-2} \cdot 1 + \frac{3^{n-2} - 1}{4} \in \Theta(3^n) \\ &\implies T(n) \in \Omega(2^n) \end{aligned}$$

1.2 Consider the case n is even number

With even number, clearly:

$$
\begin{aligned}
T(n) &= \sum_{i=1}^{\frac{n}{2}} [T(i) + T(n-i)] + \frac{n}{2} \\
&= T(1) + T(2) + \ldots + T(\frac{n}{2}) + T(\frac{n}{2}) + \ldots + T(n-2) + T(n-1) + \frac{n}{2} \\
&= \sum_{i=1}^{n-1} T(i) + T(\frac{n}{2}) + \frac{n}{2}
\end{aligned}
$$

If replace n by $n-1$, $n-1$ is odd:

$$2 \cdot T(n-1) = 2 \cdot \sum_{i=1}^{n-2} T(i) + n - 1$$

We can implies some equations like case 1:

$$2T(n) - 2T(n-1) = 2T(n-1) + 1 + 2 \cdot T(\frac{n}{2}) \Leftrightarrow T(n) = 2T(n-1) + 2^{-1} + T(\frac{n}{2})$$

Comeback to the familiar form of recurrence equation, easily solve it by using backward substitution:

$$
\begin{aligned}
T(n) &= 2T(n-1) + 2^{-1} + T(\frac{n}{2}) \geq 2T(n-1) + 2^{-1} \\
&= 2 \cdot [2T(n-2) + 2^{-1}] + 2^{-1} = 2^2 \cdot T(n-2) + 2^0 + 2^{-1} \\
&= 2^2 \cdot [2T(n-3) + 2^{-1}] + 2^0 + 2^{-1} = 2^3 \cdot T(n-3) + 2^1 + 2^0 + 2^{-1} \\
&= \ldots = 2^i \cdot T(n-i) + \frac{2^{i-1} - 1}{2 - 1} + 2^{-1} = 2^i \cdot T(n-i) + 2^{i-1} - \frac{1}{2}
\end{aligned}
$$

Base on the initial condition $T(2) = 1, T(1) = 0$.
If $n - i = 1 \implies i = n - 1$:

$$
\begin{aligned}
T(n) \geq 2^{n-1} \cdot 0 + 2^{n-2} - \frac{1}{2} = 2^{n-2} - \frac{1}{2} \in \Theta(2^n) \\
\implies T(n) \in \Omega(2^n)
\end{aligned}
$$

If $n - i = 2 \implies i = n - 2$:

$$
\begin{aligned}
T(n) \geq 2^{n-2} \cdot 1 + 2^{n-3} - \frac{1}{2} = 3 \cdot 2^{n-3} - \frac{1}{2} \in \Theta(2^n) \\
\implies T(n) \in \Omega(2^n)
\end{aligned}
$$

2 Problem 2

3 Problem 3

Josephus problem

3.1 Problem 3.a

From the observation and analysis of Josephus problem, we have the recurrence:

$$J(1) = 1$$
$$n = 2h,\ J(2h) = 2J(h) - 1$$
$$n = 2h + 1,\ J(2h + 1) = 2J(h) + 1$$

A close-form solution to the two-case recurrence is as follows:

$$J(2^k + i) = 2i + 1,\ i \in [0, \dots, 2^k - 1]$$

With $k = 0 \implies i = 0$:

$$J(1) = 2 \cdot 0 + 1 = 1$$

With $2^k + i = 2m$, then:

$$\begin{aligned} J(2^k + i) &= 2J(2^{k-1} + \frac{i}{2}) - 1 \\ &= 2 \cdot [2 \cdot \frac{i}{2} + 1] - 1 = 2i + 1 \end{aligned}$$

With $2^k + i = 2m + 1$, then:

$$\begin{aligned} J(2^k + i) &= 2J(2^{k-1} + \frac{i-1}{2}) + 1 \\ &= 2 \cdot [\frac{2 \cdot (i-1)}{2} + 1] + 1 \\ &= 2i + 1 \end{aligned}$$

The formula was proved.

3.2 Problem 3.b

We can represent $n = (b_m b_{m-1} \dots b_1 b_0)_2 = b_m \cdot 2^m + b_{m-1} \cdot 2^{m-1} + \dots + b_1 \cdot 2^1 + b_0 \cdot 2^0$.

With $n = 2^k + i \implies i = n - 2^k = 0 \cdot 2^m + b_{m-1} \cdot 2^{m-1} + \dots + b_1 \cdot 2^1 + b_0 \cdot 2^0 = (0 b_{m-1} \dots b_1 b_0)_2$.

So, with $2i$, we have: $2i = (b_{m-1} b_{m-2} \dots b_1 b_0 0)_2$

From the problem 3.a, we proved: $J(2^k + 1) = 2i + 1$:

$$\begin{aligned} \implies 2i + 1 &= (b_{m-1} b_{m-2} \dots b_1 b_0 1)_2 \\ &= (b_{m-1} b_{m-2} \dots b_1 b_0 b_m)_2 \end{aligned}$$

We proved that:

$$J[(b_m b_{m-1} \dots b_1 b_0)_2] = (b_{m-1} b_{m-2} \dots b_1 b_0 b_m)_2$$

In conclusion, J(n) can be obtained by a 1-bit cyclic shift left of n itself.

4 Problem 4

Bonus question: where does the following formula come from:

$$\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{n+1} + 2$$

Clearly, expansion in series of the left side and replace 2 by x:

$$\sum_{i=1}^{n} i \cdot x^i = 1x^1 + 2x^2 + 3x^3 + \dots nx^n$$

To begin with finding the general formula for that sum, we can see that we have $i \cdot x^i$. This observation makes it possible to connect in ideas is a familiar formula, let's call it $g(x)$:

$$g(x) = \frac{x^{n+1} - 1}{x - 1} = 1 + x^1 + x^2 + \dots + x^n$$

Take the derivative of $g(x)$:

$$\frac{d}{dx}(\frac{x^{n+1} - 1}{x - 1}) = 1 + 2x + 3x^2 + \dots nx^{n-1}$$

Multiply x by both sides of the equation:

$$x \cdot \frac{d}{dx}(\frac{x^{n+1} - 1}{x - 1}) = 1x^1 + 2x^2 + 3x^3 + \dots nx^n = \sum_{i=1}^{n} i \cdot x^i$$

Finding the general formula for this sum:

$$\begin{aligned} x \cdot \frac{d}{dx}(\frac{x^{n+1} - 1}{x - 1}) &= x \cdot \frac{(n+1)x^n \cdot (x-1) - (x^{n+1} - 1)}{(x-1)^2} \\ &= x \cdot \frac{nx^{n+1} - nx^n - x^n + 1}{(x-1)^2} \\ &= \frac{x^{n+1} \cdot (nx - n - 1) + x}{(x-1)^2} \end{aligned}$$

Return $x = 2$:

$$\frac{2^{n+1} \cdot (2n - n - 1) + 2}{(2-1)^2} = (n-1) \cdot 2^{n+1} + 2$$

In conclusion:

$$\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{n+1} + 2$$

The formula in problem was proved.