HOMEWORK 4

1. In class, you were introduced to a divide-and-conquer algorithm for solving the change-making problem. The following recurrence relation indicates the number of times the algorithm's basic operation is executed:

$$T(n) = \begin{cases} \sum_{i=1}^{\lfloor n/2 \rfloor} \big(T(i) + T(n-i)\big) + \Theta(n) & n > 2 \\ 1 & n = 2 \\ 0 & n = 1 \end{cases}$$

Prove that:

$$T(n) \in \Omega(2^n)$$

2. In class, we designed a divide-and-conquer algorithm for solving the Closest-pair problem. Prove that the search area of the size $2\delta \times \delta$ contains no more than 8 points.

3. Josephus problem
   a. Prove the following formula by induction:
   $$J(2^k + i) = 2i + 1, \forall i \in [2^k - 1]$$
   b. Prove that $J(n)$ can be obtained by a 1-bit cyclic shift left of n itself. For example, $J(6) = J(110_2) = 101_2 = 5$ and $J(9) = J(1001_2) = 11_2 = 3$.

4. (It's not a mandatory question, +2 marks) Where does the following formula come from:

$$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$$