Partial Velocities for Rotary and Translational Joints

Eric Stoneking
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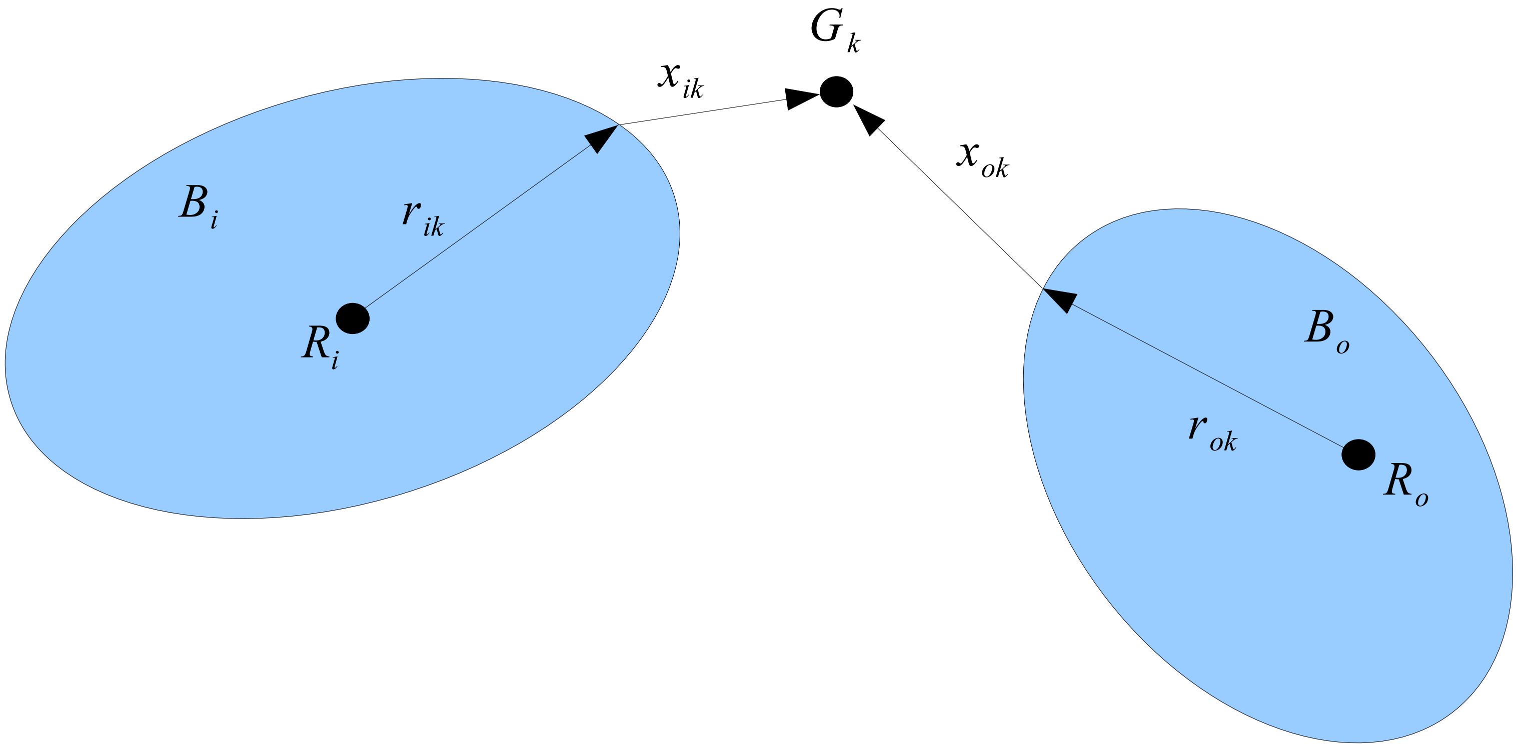
"$F = ma$. The rest is just accounting."

What is a Joint?

- A joint is a set of constraints on the relative motion of two bodies
- We will consider rotational (spherical or gimballed) joints and translational (1-, 2-, or 3-DOF slider) joints
- We will develop partial angular velocities and partial (linear) velocities
- These may be used in multi-body dynamic formulations
  - Either Kane's or Hughes's method

Relative Motion of Two Bodies

Consider two free bodies, B_i and B_o, and a point G_k.
R_i is a reference point fixed in B_i. r_{ik} is fixed in B_i.
R_o, r_{ok} are similarly fixed in B_o.
G_k, x_{ik}, and x_{ok} are, in general, free to move.



Relative Motion Variables

Let d_{ik} be a set of translational (kinematic) state variables, and let Δ_{ik} be a matrix of translational joint partials, so that:

$$x_{ik} = \Delta_{ik} d_{ik}$$

Let s_{ik} be the time rate of change of d_{ik}:

$$s_{ik} = \dot{d}_{ik}$$

We will use s_{ik} as translational dynamic state variables.
Similarly,

$$x_{ok} = \Delta_{ok} d_{ok}$$
$$s_{ok} = \dot{d}_{ok}$$

Let σ_k be a set of rotational dynamic state variables, with the associated matrix of rotational joint partials, Γ_k

Relative Velocity and Acceleration

The velocity of G_k may then be used to obtain a constraint equation relating the velocities of R_i, R_o, and our new state variables:

$$ {}^N v^{G_k} = v_i + \omega_i \times (r_{ik} + \Delta_{ik} d_{ik}) + \Delta_{ik} s_{ik} = v_o + \omega_o \times (r_{ok} + \Delta_{ok} d_{ok}) + \Delta_{ok} s_{ok} $$

Also, $\omega_o = \omega_i + \Gamma_k \sigma_k$

Rearranging, we obtain

$$ v_o = v_i + \omega_i \times (r_{ik} + \Delta_{ik} d_{ik}) + \Delta_{ik} s_{ik} - \omega_o \times (r_{ok} + \Delta_{ok} d_{ok}) - \Delta_{ok} s_{ok} $$

Differentiating, we relate the acceleration of R_o to that of R_i:

$$ \begin{aligned} a_o = {} & a_i + \alpha_i \times (r_{ik} + \Delta_{ik} d_{ik}) + \omega_i \times [\omega_i \times (r_{ik} + \Delta_{ik} d_{ik})] + 2\omega_i \times \Delta_{ik} s_{ik} + \Delta_{ik} \dot{s}_{ik} \\ & - \alpha_o \times (r_{ok} + \Delta_{ok} d_{ok}) - \omega_o \times [\omega_o \times (r_{ok} + \Delta_{ok} d_{ok})] - 2\omega_o \times \Delta_{ok} s_{ok} - \Delta_{ok} \dot{s}_{ok} \end{aligned} $$

To implement Kane's equation, we'll also need the remainder acceleration:

$$ \begin{aligned} a_{ro} = {} & a_{ri} + \alpha_{ri} \times (r_{ik} + \Delta_{ik} d_{ik}) + \omega_i \times [\omega_i \times (r_{ik} + \Delta_{ik} d_{ik})] + 2\omega_i \times \Delta_{ik} s_{ik} \\ & - \alpha_{ro} \times (r_{ok} + \Delta_{ok} d_{ok}) - \omega_o \times [\omega_o \times (r_{ok} + \Delta_{ok} d_{ok})] - 2\omega_o \times \Delta_{ok} s_{ok} \end{aligned} $$

Joint Partials

The joint partial matrices Γ and Δ allow us to conveniently handle large classes of joints. The relative positions and velocities are always three-dimensional vector quantities, but our joint state variables don't have to be. The joint partials map these joint states from state space into 3-space.

As an example, consider a joint which consists of a 2-DOF gimbal (a Body rotation through a 231 Euler sequence), and a slider aligned with the Z axis of B_o. We write:

$$\Gamma_k = \begin{bmatrix} \sin\theta_2 & 0 \\ \cos\theta_2 & 0 \\ 0 & 1 \end{bmatrix}$$

to go with

$$\sigma_k = \{\sigma_1 \quad \sigma_2\}^T$$

and

$$\Delta_{ok} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

to go with the scalar s_{ok}.

Since there are no translational DOF on the B_i side of G_k, Δ_{ik} and s_{ik} vanish entirely.

Partial Velocities

Append translation joint states to the state vector as derived in previous work:

$$\{u \quad s \quad \xi\} = \{\omega_1 \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad v_1 \quad s_{11} \quad s_{21} \quad s_{12} \quad s_{32} \quad s_{33} \quad s_{43} \quad s_{34} \quad s_{54} \quad \xi_1 \quad \xi_2 \quad \xi_3 \quad \xi_4 \quad \xi_5\}$$

Previously-obtained partial angular velocities and partial velocities, Ω_u, V_u, Ω_ξ, and V_ξ, are unchanged from previous development.

For our 5-body example, partial velocities related to translational joints may be found to be:

$$\Omega_s = 0$$

$$V_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ {}^N C^1 \Delta_{11} & -{}^N C^2 \Delta_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & {}^N C^1 \Delta_{12} & -{}^N C^3 \Delta_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & {}^N C^1 \Delta_{12} & -{}^N C^3 \Delta_{32} & {}^N C^3 \Delta_{33} & -{}^N C^4 \Delta_{43} & 0 & 0 \\ 0 & 0 & {}^N C^1 \Delta_{12} & -{}^N C^3 \Delta_{32} & 0 & 0 & {}^N C^3 \Delta_{34} & -{}^N C^5 \Delta_{54} \end{bmatrix}$$

Then the system-level partial velocities are:

$$\Omega = [\Omega_u \quad \Omega_s \quad \Omega_\xi]$$

$$V = [V_u \quad V_s \quad V_\xi]$$

Path Vectors

Recall that we define β_i to be the vector from the origin of B_i to the origin of B_1, and ρ_{ik} to be the vector from the origin of B_i to the joint G_k.

When we considered only rotational joints, these path vectors were just combinations of r_{ik} and r_{ok}. Now, they also depend on any translational motion of the joints that they span.

Conclusion

- Translational joints are handled much the same as rotary joints
  - Joint partials are key concept
- Expressions for partial velocities and remainder accelerations have been presented
  - Expressions for path vectors are left for implementation
- With these building blocks, translational joints may be incorporated into either Kane's or Hughes's formulations