Modeling Multi-Body Spacecraft Using Momentum State Dynamics (MSD)

Eric Stoneking

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Objectives

- To formulate momentum state dynamics for Freespace, adding these capabilities
  - Tree topology
  - Flexible modes
  - 2-DOF and 3-DOF gimbal joints
  - Slider joints
  - Lockable joints
- By request, we seek to preserve the decoupling between the differential equations of motion

$F = \dot{p}$ The rest is just accounting.

Rigid Bodies

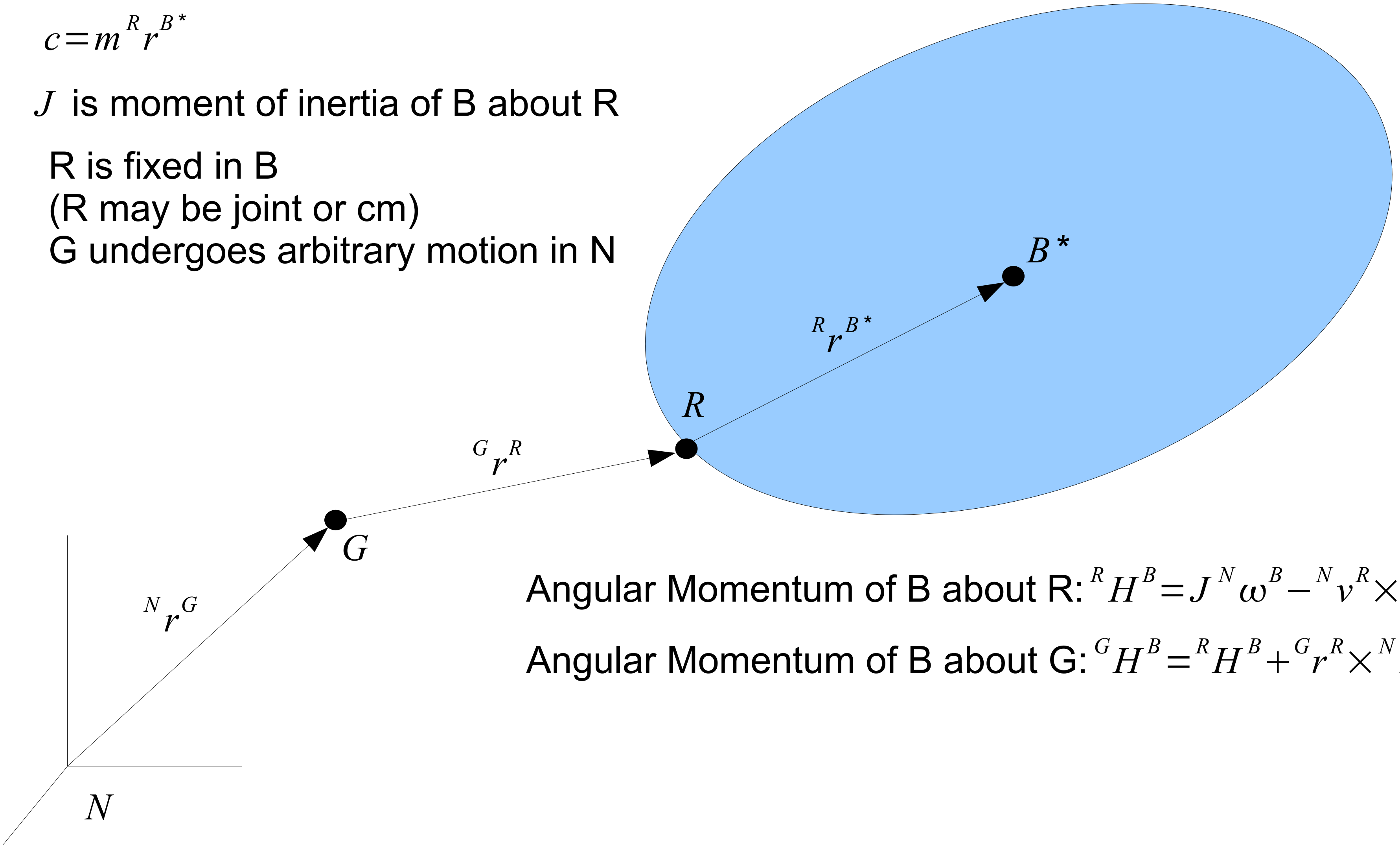
Linear and Angular Momentum

Linear Momentum: ${}^{N}P^{B^*} = m\,{}^{N}v^{B^*} = m\,{}^{N}v^{R} + {}^{N}\omega^{B} \times c$

$c = m\,{}^{R}r^{B^*}$

J is moment of inertia of B about R

R is fixed in B
(R may be joint or cm)
G undergoes arbitrary motion in N



Angular Momentum of B about R: ${}^{R}H^{B} = J\,{}^{N}\omega^{B} - {}^{N}v^{R} \times c$

Angular Momentum of B about G: ${}^{G}H^{B} = {}^{R}H^{B} + {}^{G}r^{R} \times {}^{N}P^{B^*}$

MSD Equations of Motion

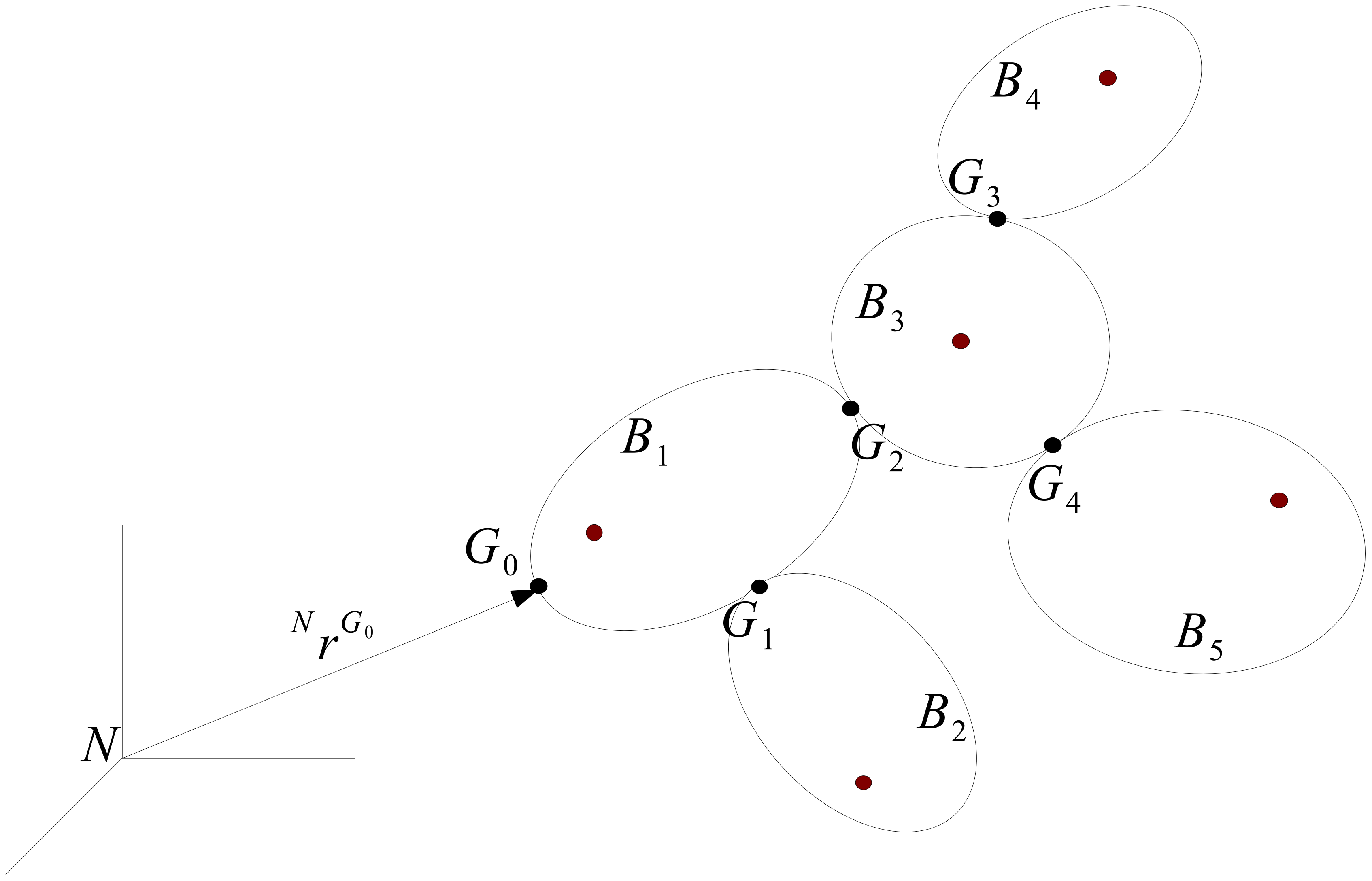
$$ {}^{N}\dot{P}^{B*} \equiv \frac{{}^{N}d\,{}^{N}P^{B*}}{dt} = F $$

$$ {}^{G}\dot{H}^{B} \equiv \frac{{}^{B}d\,{}^{G}H^{B}}{dt} = T_{G} - {}^{N}v^{G} \times {}^{N}P^{B*} - {}^{N}\omega^{B} \times {}^{G}H^{B} $$

Since $\dot{H}$ and $\dot{P}$ appear alone on the lefthand side, the EOM may be integrated without solving a system of equations for them.

However, since H and P depend explicitly on ω and v, a system of equations must still be solved at each evaluation of the EOM.

Example Problem



Composite Momenta

Let P_k be the linear momentum of the assembly of bodies outboard of the joint G_k

$$\{P_k\} = \begin{Bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} {}^N P^{B_1^*} + {}^N P^{B_2^*} + {}^N P^{B_3^*} + {}^N P^{B_4^*} + {}^N P^{B_5^*} \\ {}^N P^{B_2^*} \\ {}^N P^{B_3^*} + {}^N P^{B_4^*} + {}^N P^{B_5^*} \\ {}^N P^{B_4^*} \\ {}^N P^{B_5^*} \end{Bmatrix}$$

Let H_k be the angular momentum of all bodies outboard of joint G_k, about G_k.

$$\{H_k\} = \begin{Bmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \\ H_4 \end{Bmatrix} = \begin{Bmatrix} {}^{G_0} H^{B_1} + {}^{G_0} H^{B_2} + {}^{G_0} H^{B_3} + {}^{G_0} H^{B_4} + {}^{G_0} H^{B_5} \\ {}^{G_1} H^{B_2} \\ {}^{G_2} H^{B_3} + {}^{G_2} H^{B_4} + {}^{G_2} H^{B_5} \\ {}^{G_3} H^{B_4} \\ {}^{G_4} H^{B_5} \end{Bmatrix}$$

$[\{H_k\}, P_0]$ are the momentum states chosen by Hughes in his text, and used in Freespace.

Handling Non-spherical Joints

When all joints are spherical, the momentum states $[\{H_k\}, P_0]$ form a minimum-dimension set of dynamical variables.

When some joints are 1-DOF or 2-DOF gimbals, it is advisable to reduce the order of the dynamical state accordingly.

Queen, London, and Gonzalez handled 1-DOF gimbals with scalar momentum states, which are the projection of H_k along the hinge axis.

We extend this technique to 2-DOF and 3-DOF gimbals using the idea of joint partials, as introduced in our development of multibody dynamics using Kane's method.

Joint Partials

Consider two bodies connected by a gimballed joint. The relative angular velocity of the outer body, B, with respect to the inner body, A, may be written as a function of the gimbal angles, θ, and the gimbal rates, $\sigma \equiv \dot{\theta}$. For example, a Body-3, 2-1-3 Euler rotation through $(\theta_{1,}\theta_{2,}\theta_3)$ gives:

$$ {}^{A}\omega^{B} = \underbrace{\begin{bmatrix} c_2 s_3 & c_3 & 0 \\ c_2 c_3 & -s_3 & 0 \\ -s_2 & 0 & 1 \end{bmatrix}}_{\Gamma} \underbrace{\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix}}_{\sigma} $$

where $s_2 = \sin(\theta_2), c_2 = \cos(\theta_2),$ etc, and the components of ω are expressed in the frame of the outer body, B.

The matrix Γ is called the matrix of joint partials. Each Euler sequence, including one- and two-DOF gimbals, has its own Γ. The dimensions of Γ are 3 x NDOF.

For a spherical joint, $\sigma = \omega$, and $\Gamma = U$, the identity matrix.

Joint Partials and Momentum States

Noting that the columns of Γ are the components of the gimbal axes expressed in the joint's outer body frame, we define the scalar momentum states

$$h_{kj} \equiv H_k \cdot \Gamma_{kj},$$

or the 1-, 2-, or 3-DOF

$$h_k = \Gamma_k^T H_k$$

The minimum-dimension angular momentum state vector then becomes:

$$\{h_k\} = \begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \end{Bmatrix} = \begin{Bmatrix} {}^{G_0}H^{B_1} + {}^{G_0}H^{B_2} + {}^{G_0}H^{B_3} + {}^{G_0}H^{B_4} + {}^{G_0}H^{B_5} \\ \Gamma_1^T \left({}^{G_1}H^{B_2} \right) \\ \Gamma_2^T \left({}^{G_2}H^{B_3} + {}^{G_2}H^{B_4} + {}^{G_2}H^{B_5} \right) \\ \Gamma_3^T \left({}^{G_3}H^{B_4} \right) \\ \Gamma_4^T \left({}^{G_4}H^{B_5} \right) \end{Bmatrix}$$

Construction Preliminaries

Using joint partials, we can systematically construct expressions for $[\{H_k\}, P_0]$ as functions of the minimum-dimension state vector, $u \equiv [\sigma_k, v_0]$.

To begin, let's define some terms:

$$\omega_i \equiv {}^N\omega^{B_i}$$

$$v_k \equiv {}^N v^{G_k}$$

${}^iC^j$ is the direction cosine matrix between B_i and B_j,
so the components of a vector may be transformed thus: ${}^i v = {}^iC^j\, {}^j v$

ρ_{ik} is the vector from R_i to G_k. It is expressed in N.

The cross-product matrix is denoted by: $\tilde{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$, $\quad x \times y = \tilde{x}\, y$

Assembly Notation

As we assemble the system, we adopt a not-uncommon notation to distinguish system-level properties from body-level properties. In cases where a common symbol is used for both, the system-level vectors are enclosed in braces, $\{\cdot\}$, and system-level matrices are enclosed in brackets, $[\cdot]$ to reduce confusion.

For example, we use J to denote the inertia tensor of a generic single body. J_i denotes the inertia tensor of the i th body, and $[J]$ denotes a system matrix formed by assembling the single-body inertia tensors:

$$[J] \equiv \text{diag}[J_1 \quad J_2 \quad J_3 \quad \ldots]$$

Linear and Angular Velocities

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{Bmatrix} = \underbrace{\begin{bmatrix} U & 0 & 0 & 0 & 0 & 0 \\ {}^{2}C^{1} & \Gamma_1 & 0 & 0 & 0 & 0 \\ {}^{3}C^{1} & 0 & \Gamma_2 & 0 & 0 & 0 \\ {}^{4}C^{1} & 0 & {}^{4}C^{3}\Gamma_2 & \Gamma_3 & 0 & 0 \\ {}^{5}C^{1} & 0 & {}^{5}C^{3}\Gamma_2 & 0 & \Gamma_4 & 0 \end{bmatrix}}_{\Omega} \underbrace{\begin{Bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ v_0 \end{Bmatrix}}_{u}$$

$$\begin{Bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & U \\ \tilde{\rho}_{20}\,{}^{N}C^{1} & \tilde{\rho}_{21}\,{}^{N}C^{2}\Gamma_1 & 0 & 0 & 0 & U \\ \tilde{\rho}_{30}\,{}^{N}C^{1} & 0 & \tilde{\rho}_{32}\,{}^{N}C^{3}\Gamma_2 & 0 & 0 & U \\ \tilde{\rho}_{40}\,{}^{N}C^{1} & 0 & \tilde{\rho}_{42}\,{}^{N}C^{3}\Gamma_2 & \tilde{\rho}_{43}\,{}^{N}C^{4}\Gamma_3 & 0 & U \\ \tilde{\rho}_{50}\,{}^{N}C^{1} & 0 & \tilde{\rho}_{52}\,{}^{N}C^{3}\Gamma_2 & 0 & \tilde{\rho}_{54}\,{}^{N}C^{5}\Gamma_4 & U \end{bmatrix}}_{V} \underbrace{\begin{Bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ v_0 \end{Bmatrix}}_{u}$$

Body Momenta

Using:

$$ {}^{N}P^{B^*} = m\,{}^{N}v^{R} - c \times {}^{N}\omega^{B}, $$

$$ {}^{R}H^{B} = J\,{}^{N}\omega^{B} + c \times {}^{N}v^{R}, $$

Let

$$ [m] \equiv \text{diag}\left[m_1 U \quad m_2 U \quad m_3 U \quad m_4 U \quad m_5 U \right] $$

$$ [J] \equiv \text{diag}\left[J_1 \quad J_2 \quad J_3 \quad J_4 \quad J_5 \right] $$

$$ [\tilde{c}] \equiv \text{diag}\left[{}^{N}C^{1}\tilde{c}_1 \quad {}^{N}C^{2}\tilde{c}_2 \quad {}^{N}C^{3}\tilde{c}_3 \quad {}^{N}C^{4}\tilde{c}_4 \quad {}^{N}C^{5}\tilde{c}_5 \right] $$

yielding

$$ \left\{ {}^{N}P^{B^*} \right\} = \left[[m]V - [\tilde{c}]\Omega \right] u $$

$$ \left\{ {}^{R}H^{B} \right\} = \left[[J]\Omega - [\tilde{c}]^{T} V \right] u $$

Composite Momenta

Using

$$^{G}H^{B} = {}^{R}H^{B} + {}^{G}r^{R} \times {}^{N}P^{B*}$$

Let

$$S \equiv \begin{bmatrix} U & {}^{1}C^{2} & {}^{1}C^{3} & {}^{1}C^{4} & {}^{1}C^{5} \\ & U & & & \\ & & U & {}^{3}C^{4} & {}^{3}C^{5} \\ & & & U & \\ & & & & U \end{bmatrix}, \qquad R \equiv \begin{bmatrix} 0 & {}^{1}C^{N}\tilde{\rho}_{20} & {}^{1}C^{N}\tilde{\rho}_{30} & {}^{1}C^{N}\tilde{\rho}_{40} & {}^{1}C^{N}\tilde{\rho}_{50} \\ & 0 & & & \\ & & 0 & {}^{3}C^{N}\tilde{\rho}_{42} & {}^{3}C^{N}\tilde{\rho}_{52} \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

yielding

$$\{H_k\} = S\{{}^{R}H^{B}\} - R\{{}^{N}P^{B*}\}$$

or

$$\{H_k\} = \left[(S[J] + R[\tilde{c}])\Omega - (S[\tilde{c}]^T + R[m])V\right]u$$

$$\{h_k\} = [\Gamma_k]^T\{H_k\} = [\Gamma_k]^T\left[(S[J] + R[\tilde{c}])\Omega - (S[\tilde{c}]^T + R[m])V\right]u$$

Similarly,

$$\Sigma \equiv \begin{bmatrix} U & U & U & U & U \\ & U & & & \\ & & U & U & U \\ & & & U & \\ & & & & U \end{bmatrix} \text{ gives } \{P_k\} = \Sigma\left[[m]V - [\tilde{c}]\Omega\right]u$$

Partial Momenta

Define

$$A = (S[J] + R[\tilde{c}])\Omega - (S[\tilde{c}]^T + R[m])V$$

$$\Lambda = \begin{bmatrix} U & U & U & U & U \end{bmatrix} \begin{bmatrix} [m]V - [\tilde{c}]\Omega \end{bmatrix}$$

so

$$\{H_k\} = A\,u, \qquad \{h_k\} = [\Gamma_k]^T A\,u$$

$$P_0 = \Lambda\,u$$

In perfect analogy to Ω and V, A and Λ are, respectively, the partial angular momenta and partial linear momenta relating $\{H_k\}$ and P_0 to the dynamic states, u.

MSD Equations of Motion

Integrate:

$$\begin{Bmatrix} \dot{P}_0 \\ \{\dot{h}_k\} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ [\Gamma_k]^T \left\{ T_k - v_k \times P_k - \sum_i \omega_i \times {}^{G_k}H^{B_i} \right\} \end{Bmatrix}$$

(where F_0 is the total resultant force on the spacecraft, applied at the center of mass, and T_k is the resultant torque about G_k of all forces and moments acting on bodies outboard of G_k)

Solve for u:

$$\begin{Bmatrix} P_0 \\ \{h_k\} \end{Bmatrix} = \begin{bmatrix} \Lambda \\ [\Gamma_k]^T A \end{bmatrix} u$$

Construct $\{\omega_i\}, \{v_k\}$:

$$\{\omega_i\} = \Omega u$$
$$\{v_k\} = V u$$

Repeat

Locking a Single DOF

To lock the jth DOF of joint G_k, simply set the corresponding element of u to zero, and eliminate the corresponding $\dot{h}_{kj}$ from the EOM to be integrated.

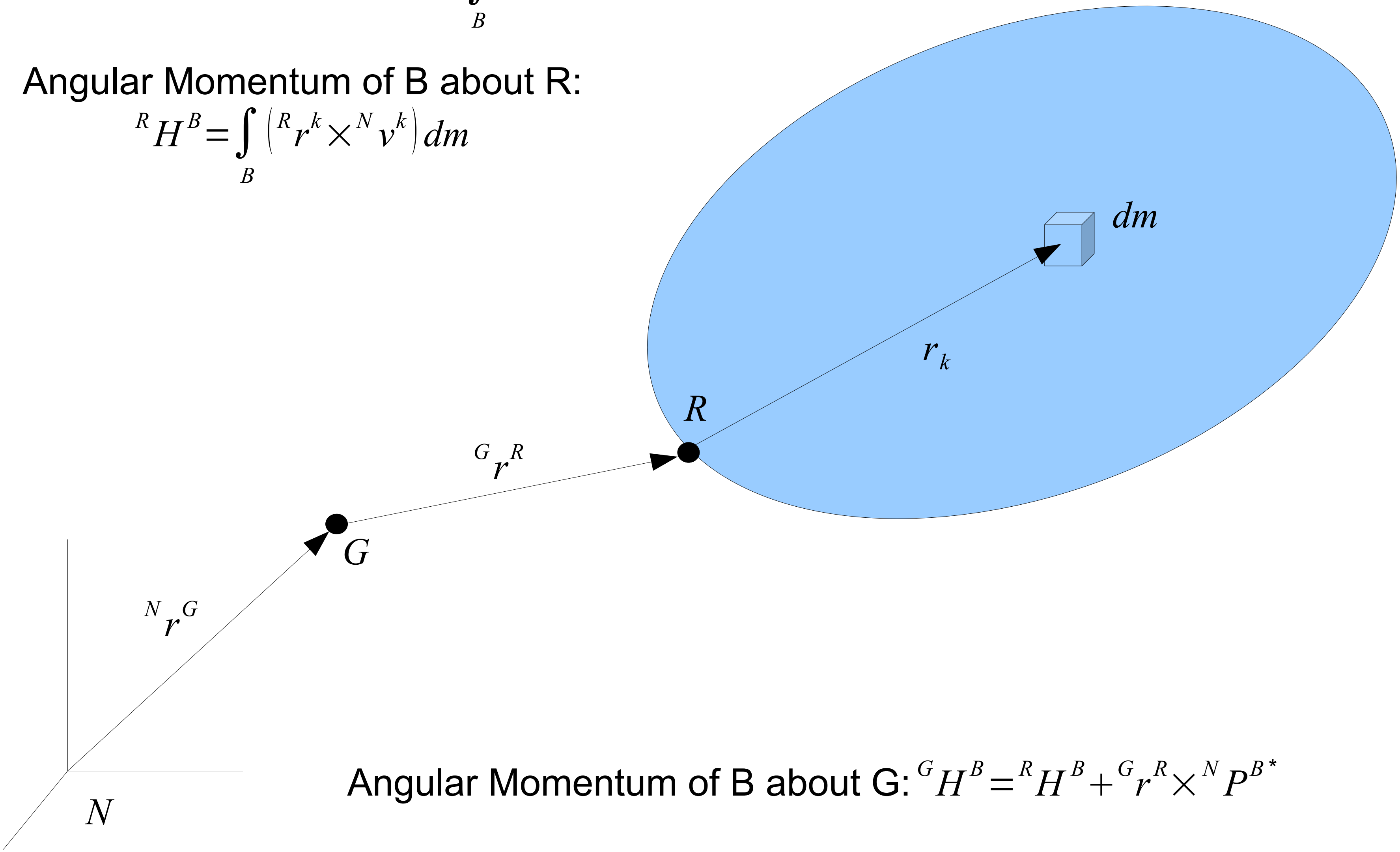
Flexible Bodies

Linear and Angular Momentum

Linear Momentum: ${}^{N}P^{B^*} = \int_B {}^{N}v^{k}\, dm$

Angular Momentum of B about R:

$$ {}^{R}H^{B} = \int_B \left({}^{R}r^{k} \times {}^{N}v^{k} \right) dm $$



Angular Momentum of B about G: ${}^{G}H^{B} = {}^{R}H^{B} + {}^{G}r^{R} \times {}^{N}P^{B^*}$

Linear and Angular Momentum

Linear Momentum:

$$
\begin{aligned}
{}^{N}P^{B*} &= \int_B {}^{N}v^{k}\, dm \\
&= m\, {}^{N}v^{R} + {}^{N}\omega^{B} \times (c + P_{\Psi}\eta) + P_{\Psi}\xi
\end{aligned}
$$

Angular Momentum of B about R:

$$
\begin{aligned}
{}^{R}H^{B} &= \int_B \left({}^{R}r^{k} \times {}^{N}v^{k}\right) dm \\
&= \int_B (r_k + \Psi_k \eta) \times \left[{}^{N}v^{R} + {}^{N}\omega^{B} \times (r_k + \Psi_k \eta) + \Psi_k \xi\right] dm \\
&= J\, {}^{N}\omega^{B} - {}^{N}v^{R} \times (c + P_{\Psi}\eta) + (H_{\Psi} + Q_{\Psi}\eta)\xi - {}^{N}\omega^{B} \cdot (R_{\Psi} + R_{\Psi}^{T})\eta
\end{aligned}
$$

Linear and Angular Velocities

$$\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\\ \omega_5\end{Bmatrix}=\underbrace{\begin{bmatrix}U & 0 & 0 & 0 & 0 & 0\\ {}^2C^1 & \Gamma_1 & 0 & 0 & 0 & 0\\ {}^3C^1 & 0 & \Gamma_2 & 0 & 0 & 0\\ {}^4C^1 & 0 & {}^4C^3\Gamma_2 & \Gamma_3 & 0 & 0\\ {}^5C^1 & 0 & {}^5C^3\Gamma_2 & 0 & \Gamma_4 & 0\end{bmatrix}}_{\Omega_u}\underbrace{\begin{Bmatrix}\sigma_0\\ \sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ v_0\end{Bmatrix}}_{u}+\underbrace{\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ {}^2C^1\Theta_{11} & -\Theta_{21} & 0 & 0 & 0\\ {}^3C^1\Theta_{12} & 0 & -\Theta_{32} & 0 & 0\\ {}^4C^1\Theta_{12} & 0 & {}^4C^3(\Theta_{33}-\Theta_{32}) & -\Theta_{43} & 0\\ {}^5C^1\Theta_{12} & 0 & {}^5C^3(\Theta_{34}-\Theta_{32}) & 0 & -\Theta_{54}\end{bmatrix}}_{\Omega_\xi}\underbrace{\begin{Bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\\ \xi_5\end{Bmatrix}}_{\xi}$$

$$\begin{Bmatrix}v_0\\ v_1\\ v_2\\ v_3\\ v_4\end{Bmatrix}=\underbrace{\begin{bmatrix}0 & 0 & 0 & 0 & 0 & U\\ \tilde{\rho}_{20}{}^NC^1 & \tilde{\rho}_{21}{}^NC^2\Gamma_1 & 0 & 0 & 0 & U\\ \tilde{\rho}_{30}{}^NC^1 & 0 & \tilde{\rho}_{32}{}^NC^3\Gamma_2 & 0 & 0 & U\\ \tilde{\rho}_{40}{}^NC^1 & 0 & \tilde{\rho}_{42}{}^NC^3\Gamma_2 & \tilde{\rho}_{43}{}^NC^4\Gamma_3 & 0 & U\\ \tilde{\rho}_{50}{}^NC^1 & 0 & \tilde{\rho}_{52}{}^NC^3\Gamma_2 & 0 & \tilde{\rho}_{54}{}^NC^5\Gamma_4 & U\end{bmatrix}}_{V_u}\underbrace{\begin{Bmatrix}\sigma_0\\ \sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ v_0\end{Bmatrix}}_{u}+$$

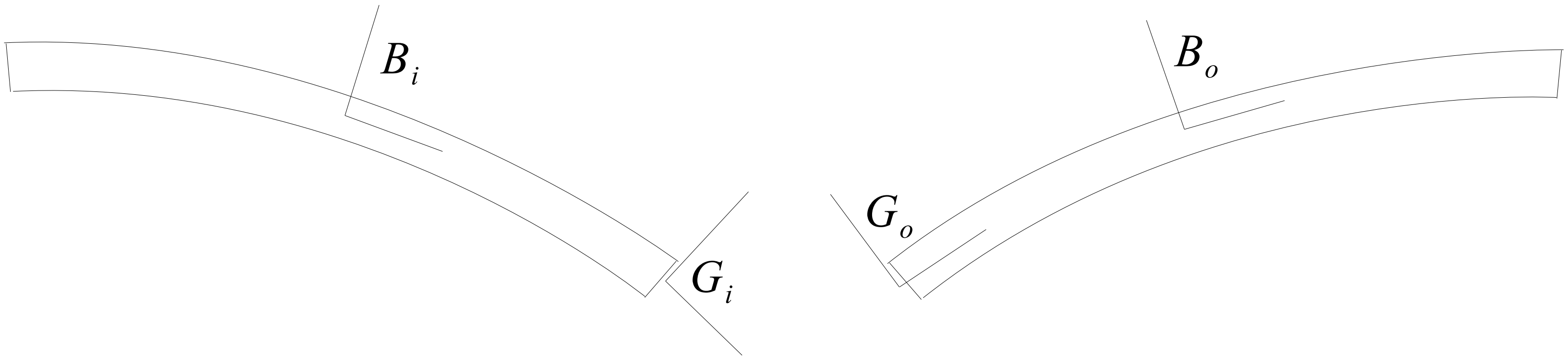
$$\underbrace{\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ {}^NC^1\Psi_{11}+\tilde{\rho}_{21}{}^NC^1\Theta_{11} & -{}^NC^2\Psi_{21}-\tilde{\rho}_{21}{}^NC^2\Theta_{21} & 0 & 0 & 0\\ {}^NC^1\Psi_{12}+\tilde{\rho}_{32}{}^NC^1\Theta_{12} & 0 & -{}^NC^3\Psi_{32}-\tilde{\rho}_{32}{}^NC^3\Theta_{32} & 0 & 0\\ {}^NC^1\Psi_{12}+\tilde{\rho}_{42}{}^NC^1\Theta_{12} & 0 & {}^NC^3(\Psi_{33}-\Psi_{32})+\tilde{\rho}_{43}{}^NC^3\Theta_{33}-\tilde{\rho}_{42}{}^NC^3\Theta_{32} & -{}^NC^4\Psi_{43}-\tilde{\rho}_{43}{}^NC^4\Theta_{43} & 0\\ {}^NC^1\Psi_{12}+\tilde{\rho}_{52}{}^NC^1\Theta_{12} & 0 & {}^NC^3(\Psi_{34}-\Psi_{32})+\tilde{\rho}_{54}{}^NC^3\Theta_{34}-\tilde{\rho}_{52}{}^NC^3\Theta_{32} & 0 & -{}^NC^5\Psi_{54}-\tilde{\rho}_{54}{}^NC^5\Theta_{54}\end{bmatrix}}_{V_\xi}\underbrace{\begin{Bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\\ \xi_5\end{Bmatrix}}_{\xi}$$

Effect of Flexibility on Rotation at Joints

At joint G_k, the relative attitude of the outer body, B_o, with respect to the inner body, B_i, must account for the rotation due to the flexibility of both bodies at the joint, as well as the "rigid-body" rotation due to the gimballed or spherical joint.

Introducing the intermediate frames G_o and G_i affixed, respectively, to the interfaces between the "rigid" joint and the outer and inner bodies,

$$ {}^{o}C^{i} = \underbrace{{}^{Bo}C^{Go}}_{C(\Theta_{ko}\eta_o)} \underbrace{{}^{Go}C^{Gi}}_{C(\theta)} \underbrace{{}^{Gi}C^{Bi}}_{C(\Theta_{ki}\eta_i)} $$



Effect of Flexibility on $\tilde{\rho}_{ik}$

Recall that $\tilde{\rho}_{ik}$ denotes the vector from the reference point R_i (fixed in the i th body, B_i) to the k th joint, G_k.

We keep the same notation, but keep in mind that this vector now includes deflections due to flexibility ($\Psi\ \eta$) as appropriate over the traversal from R_i to G_k.

Body Momenta

Using:

$$ {}^{N}P^{B^*} = m\,{}^{N}v^{R} - (c + P_\Psi \eta) \times {}^{N}\omega^{B} + P_\Psi \xi\,, $$

$$ {}^{R}H^{B} = J\,{}^{N}\omega^{B} - {}^{N}v^{R} \times (c + P_\Psi \eta) + (H_\Psi + Q_\Psi \eta)\xi - {}^{N}\omega^{B} \cdot (R_\Psi + R_\Psi^T)\eta\,, $$

Let

$$ \underline{J}_i \equiv (J - \eta^T (R_\Psi + R_\Psi^T))_i $$

$$ [m] \equiv \text{diag}\begin{bmatrix} m_1 U & m_2 U & m_3 U & m_4 U & m_5 U \end{bmatrix} $$

$$ [\underline{J}] \equiv \text{diag}\begin{bmatrix} \underline{J}_1 & \underline{J}_2 & \underline{J}_3 & \underline{J}_4 & \underline{J}_5 \end{bmatrix} $$

$$ [\underline{\tilde{c}}] \equiv \text{diag}\begin{bmatrix} {}^{N}C^{1}(c + P_\Psi \eta)_1^{\times} & {}^{N}C^{2}(c + P_\Psi \eta)_2^{\times} & {}^{N}C^{3}(c + P_\Psi \eta)_3^{\times} & {}^{N}C^{4}(c + P_\Psi \eta)_4^{\times} & {}^{N}C^{5}(c + P_\Psi \eta)_5^{\times} \end{bmatrix} $$

$$ [P_\Psi] \equiv \text{diag}\begin{bmatrix} {}^{N}C^{1} P_{\Psi 1} & {}^{N}C^{2} P_{\Psi 2} & {}^{N}C^{3} P_{\Psi 3} & {}^{N}C^{4} P_{\Psi 4} & {}^{N}C^{5} P_{\Psi 5} \end{bmatrix} $$

$$ [\underline{H}_\Psi] \equiv \text{diag}\begin{bmatrix} (H_\Psi + Q_\Psi \eta)_1 & (H_\Psi + Q_\Psi \eta)_2 & (H_\Psi + Q_\Psi \eta)_3 & (H_\Psi + Q_\Psi \eta)_4 & (H_\Psi + Q_\Psi \eta)_5 \end{bmatrix} $$

yielding

$$ \{{}^{N}P^{B^*}\} = \left[[m]V_u - [\underline{\tilde{c}}]\Omega_u\right]u + \left[[m]V_\xi - [\underline{\tilde{c}}]\Omega_\xi + [P_\Psi]\right]\xi $$

$$ \{{}^{R}H^{B}\} = \left[[\underline{J}]\Omega_u - [\underline{\tilde{c}}]^T V_u\right]u + \left[[\underline{J}]\Omega_\xi - [\underline{\tilde{c}}]^T V_\xi + [\underline{H}_\Psi]\right]\xi $$

Composite Momenta

Using

$$^{G}H^{B} = {}^{R}H^{B} + {}^{G}r^{R} \times {}^{N}P^{B*}$$

Let

$$S \equiv \begin{bmatrix} U & {}^{1}C^{2} & {}^{1}C^{3} & {}^{1}C^{4} & {}^{1}C^{5} \\ & U & & & \\ & & U & {}^{3}C^{4} & {}^{3}C^{5} \\ & & & U & \\ & & & & U \end{bmatrix}, \qquad R \equiv \begin{bmatrix} 0 & {}^{1}C^{N}\tilde{\rho}_{20} & {}^{1}C^{N}\tilde{\rho}_{30} & {}^{1}C^{N}\tilde{\rho}_{40} & {}^{1}C^{N}\tilde{\rho}_{50} \\ & 0 & & & \\ & & 0 & {}^{3}C^{N}\tilde{\rho}_{42} & {}^{3}C^{N}\tilde{\rho}_{52} \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

yielding

$$\{H_k\} = S\{{}^{R}H^{B}\} - R\{{}^{N}P^{B*}\}$$

or

$$\{H_k\} = \left[(S[\underline{J}] + R[\tilde{\underline{c}}])\,\Omega_u - (S[\tilde{\underline{c}}]^T + R[m])\,V_u\right]u + \left[(S[\underline{J}] + R[\tilde{\underline{c}}])\,\Omega_\xi - (S[\tilde{\underline{c}}]^T + R[m])\,V_\xi + S[\underline{H}_\Psi] - R[P_\Psi]\right]\xi$$

$$\{h_k\} = [\Gamma_k]^T\{H_k\}$$

Similarly,

$$\Sigma \equiv \begin{bmatrix} U & U & U & U & U \\ & U & & & \\ & & U & U & U \\ & & & U & \\ & & & & U \end{bmatrix} \text{ gives } \{P_k\} = \Sigma\left\{\left[[m]V_u - [\tilde{\underline{c}}]\Omega_u\right]u + \left[[m]V_\xi - [\tilde{\underline{c}}]\Omega_\xi + [P_\Psi]\right]\xi\right\}$$

Partial Momenta

Define

$$A_\Psi = \left[(S[\underline{J}] + R[\underline{\tilde{c}}])\,\Omega_u - (S[\underline{\tilde{c}}]^T + R[m])\,V_u \quad (S[\underline{J}] + R[\underline{\tilde{c}}])\,\Omega_\xi - (S[\underline{\tilde{c}}]^T + R[m])\,V_\xi + S[\underline{H}_\Psi] - R[P_\Psi] \right]$$

$$\Lambda_\Psi = \left[U \quad U \quad U \quad U \quad U \right] \left[[m]\,V_u - [\underline{\tilde{c}}]\,\Omega_u \quad [m]\,V_\xi - [\underline{\tilde{c}}]\,\Omega_\xi + [P_\Psi] \right]$$

so

$$\{H_k\} = A_\Psi \begin{Bmatrix} u \\ \xi \end{Bmatrix}, \qquad \{h_k\} = [\Gamma_k]^T A_\Psi \begin{Bmatrix} u \\ \xi \end{Bmatrix}$$

$$P_0 = \Lambda_\Psi \begin{Bmatrix} u \\ \xi \end{Bmatrix}$$

Flexible Body Momenta

The momentum states discussed so far are extensions of the rigid-body momentum states. We still need some equations of motion for the flexible modes of each body. Ken London suggests we use the momentum states:

$$\Pi \equiv \int_B \Psi_k^T \, {}^N v^k \, dm$$

with the equations of motion:

$$\dot{\Pi} = \int_B \Psi_k^T \, df$$

Expressing ${}^N v^k$ as:

$${}^N v^k = {}^N v^R + {}^N \omega^B \times ({}^R r^k + \Psi_k \eta) + \Psi_k \xi$$

and referring to modal integrals defined in Kane_NBody_Flex slides, we immediately obtain

$$\Pi = P_\Psi^T \, {}^N v^R + (H_\Psi + Q_\Psi \eta)^T \, {}^N \omega^B + M_\Psi \xi$$

or

$$\{\Pi\} = \begin{bmatrix} [\underline{H}_\Psi]^T \Omega_u + [P_\Psi]^T V_u & [\underline{H}_\Psi]^T \Omega_\xi + [P_\Psi]^T V_\xi + [M_\Psi] \end{bmatrix} \begin{Bmatrix} u \\ \xi \end{Bmatrix}$$

For compactness, we introduce Φ_Ψ,

$$\{\Pi\} = \Phi_\Psi \begin{Bmatrix} u \\ \xi \end{Bmatrix}$$

MSD Equations of Motion

Integrate:

$$\begin{Bmatrix} \dot{P}_0 \\ \{\dot{h}_k\} \\ \{\dot{\Pi}_i\} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ [\Gamma_k]^T \left\{ T_k - v_k \times P_k - \sum_i \omega_i \times {}^{G_k}H^{B_i} \right\} \\ \left\{ \int_{B_i} \Psi^T df \right\} \end{Bmatrix}$$

(where F_0 is the total resultant force on the spacecraft, applied at the center of mass, and T_k is the resultant torque about G_k of all forces and moments acting on bodies outboard of G_k).

As discussed in the Kane_NBody_Flex slides, $\int_{B_i} \Psi^T df$ comprises spring and damping forces, $-C_\Psi \xi - K_\Psi \eta$, as well as any external forces or moments that have zero resultants on the body, but non-zero projections into the modal space.

Solve for u, ξ:

$$\begin{Bmatrix} P_0 \\ \{h_k\} \\ \{\Pi\} \end{Bmatrix} = \begin{bmatrix} \Lambda_\Psi \\ [\Gamma_k]^T A_\Psi \\ \Phi_\Psi \end{bmatrix} \begin{Bmatrix} u \\ \xi \end{Bmatrix}$$

Construct $\{\omega_i\}, \{v_k\}$:

$$\{\omega_i\} = \Omega_u u + \Omega_\xi \xi$$
$$\{v_k\} = V_u u + V_\xi \xi$$

Repeat