Kane's Method Applied to the Dynamics of Spacecraft Composed of Multiple Flexible Bodies

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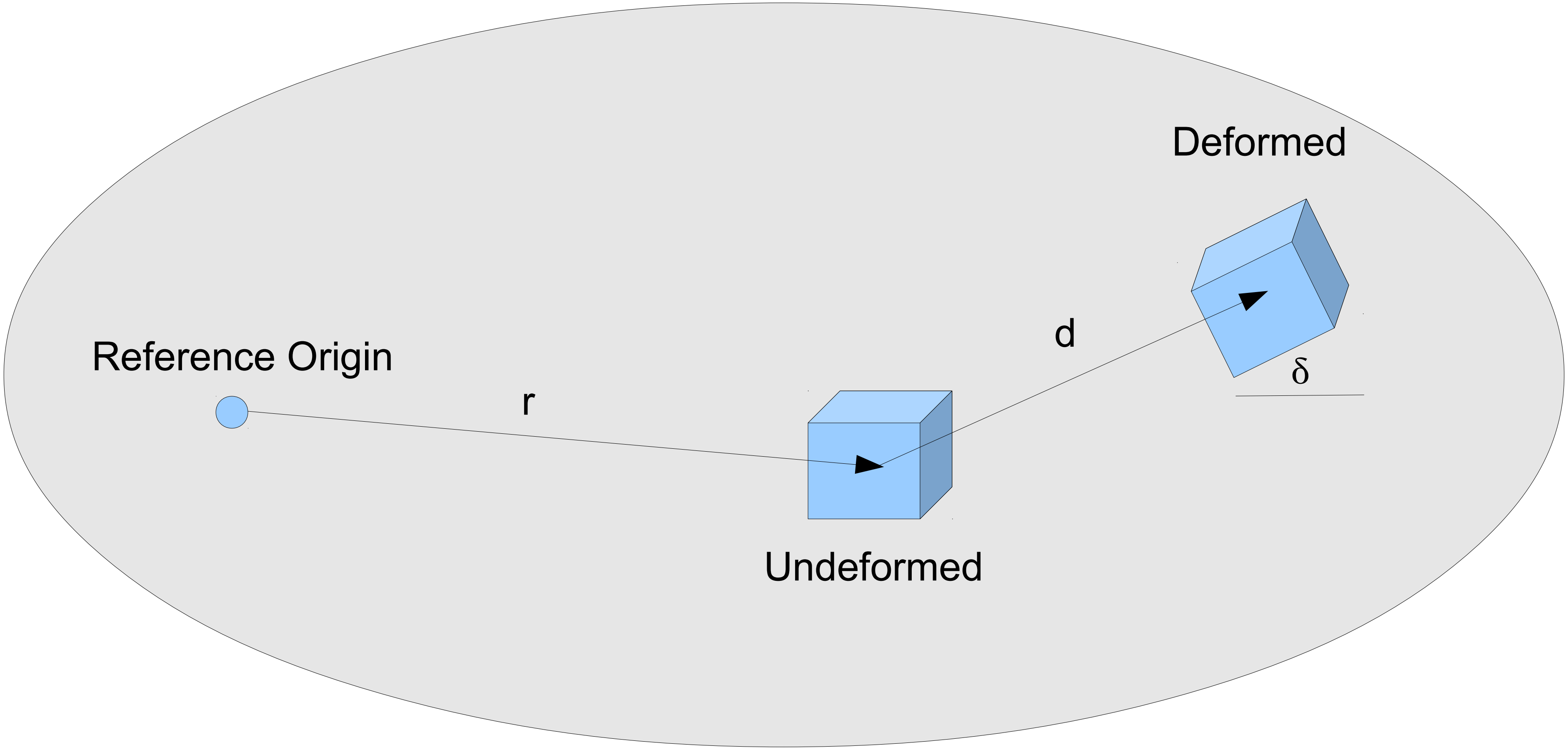
“$F = ma$. The rest is just accounting.”

Modelling A Flexible Body

Deflection of a Mass Element



Velocity and Acceleration of a Mass Element

Define translational and angular deformation velocities:

$$
\begin{aligned}
e_k &= \dot{d}_k \\
\epsilon_k &= \dot{\delta}_k
\end{aligned}
$$

Then we can express the total (including rigid and flexible motion) velocity and acceleration of the mass element as:

$$
\begin{aligned}
&\breve{v}_k = v + \omega \times (r_k + d_k) + e_k \\
&\breve{a}_k = a + \omega \times \left[\omega \times (r_k + d_k) \right] + \alpha \times (r_k + d_k) + \dot{e}_k + 2\,\omega \times e_k
\end{aligned}
$$

where the undecorated ω, v, α, and a describe the motion of the body's reference point

This Could Get Out of Hand...

- To formulate dynamic EOM, we need the acceleration of every mass element in the system
  - For a particle, there are only 3 DOF. Easy.
    - $F = ma$
  - For a rigid body, there are only 6 DOF. Not bad.
    - $F = ma, \quad T = I\alpha - \omega \times H$
- For a flexible body, how do we avoid infinite DOF?
  - Assumed modes, truncated to N most interesting

Assumed Modes (1)

Consider flexible deformation to be a linear combination of a finite number of independent modes. For mass element k,

Translational deflection — Translational Mode Shapes

$$\begin{Bmatrix} d_k \\ \delta_k \end{Bmatrix} = \begin{bmatrix} \Psi_k \\ \Theta_k \end{bmatrix} \{\eta\} \qquad \text{and} \qquad \xi \equiv \dot{\eta}$$

Angular deflection — Rotational Mode Shapes — Modal Coordinates

Q: Where do mode shapes come from?

A1: Solid mechanics of beam, rod, column, etc, OR

A2: NASTRAN or other Finite Element Modelling software, OR

A3: Specify by axis, frequency, damping, modal mass

Assumed Modes (2)

The units of η, Ψ, Θ are somewhat arbitrary, as long as d, δ have units of length, angle. One common choice is “mass-normalized”:

$$[\eta] = m\sqrt{kg}, \quad [\Psi] = \frac{1}{\sqrt{kg}}, \quad [\Theta] = \frac{rad}{m\sqrt{kg}}$$

Another obvious choice is simply dimensionless modal coordinates:

$$[\eta] = 1, \quad [\Psi] = m, \quad [\Theta] = rad$$

Assumed Modes (3)

- Assuming N modes, Ψ_k, Θ_k are each of dimension 3xN for each mass element.
- In practice, we are only interested in the motions of a few points, not every mass element
  - Joint motions are needed for multibody kinematics
  - Motion of sensor locations may be needed for proper measurement modeling
  - Points of application of forces (e.g. thrusters) may be needed to properly capture modal excitation
- But we do need to account for the effects of flexibility integrated over the entire body

Free-Free vs. Cantilever Mode Shapes

- Free-Free mode shapes assume body is unconstrained
  - Flexible motion does not move mass center
  - Use mass center as reference point for acceleration formulation
    - Some modal integrals integrate to zero!
  - Multibody kinematics must account for deflections of joints
- Cantilever mode shapes fix body at some point which is not the mass center
  - Use that point as reference point for acceleration formulation
  - If that point is a joint, some terms drop out of the multibody kinematics
    - Non-contributing constraint forces at the joints are eliminated from the equations of motion (but Kane's method eliminates non-contributing forces anyway)
  - Modal integrals are non-zero (as are first mass moments of rigid body)

Equations of Motion for One Flexible Body

Formulating Equations of Motion (1)

For a rigid body, Kane's equations may be written

$$\Omega^T \{T - I\,\alpha - \omega \times H\} + V^T \{F - m\,a\} = 0$$

For a flexible body, we combine Newton's law with the moment about our reference point, and integrate over the body:

$$\int \{\Omega^T [d\tau - (r+d) \times \breve{a}\,dm] + V^T [df - \breve{a}\,dm]\} = 0$$

Using the generalized speeds $\begin{Bmatrix} u \\ \xi \end{Bmatrix}$, we may write:

$$\breve{\omega} = \omega = [\Omega_u \quad \Omega_\xi] \begin{Bmatrix} u \\ \xi \end{Bmatrix}, \quad \breve{v} = v + \Psi\,\xi = [V_u \quad V_\xi + \Psi] \begin{Bmatrix} u \\ \xi \end{Bmatrix}$$

where Ω_u, Ω_ξ, V_u, and V_ξ account for motion of the reference point. These partial velocities are constant over a body.

For a single body, Ω_u, Ω_ξ, V_u, and V_ξ are trivial (U, 0, U, 0, respectively), so it's tempting to drop them from the formalism. They will play a key role, however, when we assemble multi-body systems.

Formulating Equations of Motion (2)

The acceleration terms may be integrated over the body:

$$
\begin{aligned}
\int \breve{a}\, dm &= \int \left[a + \omega \times \omega \times (r + \Psi\, \eta) + \alpha \times (r + \Psi\, \eta) + \Psi\, \dot{\xi} + 2\,\omega \times \Psi\, \xi \right] dm \\
&= ma + \omega \times \omega \times (c + P_\psi \eta) + \alpha \times (c + P_\psi \eta) + P_\psi \dot{\xi} + 2\,\omega \times P_\psi \xi
\end{aligned}
$$

¿

$$
\begin{aligned}
\int (r+d) \times \breve{a}\, dm &= \int (r + \Psi\, \eta) \times \left[a + \omega \times \omega \times (r + \Psi\, \eta) + \alpha \times (r + \Psi\, \eta) + \Psi\, \dot{\xi} + 2\,\omega \times \Psi\, \xi \right] dm \\
&= c \times a + P_\psi \eta \times a + J\,\alpha + \omega \times J\,\omega + \left(H_\psi + Q_\Psi \eta \right) \dot{\xi}
\end{aligned}
$$

(Most terms that are second-order in η, ξ have been dropped.)

Some useful “modal integrals” appear:

$$
c = \int r dm, \quad P_\psi = \int \Psi\, dm, \quad J = -\int r^{\times} r^{\times}\, dm, \quad H_\psi = \int r^{\times} \Psi\, dm
$$

(If our reference point is the center of mass, then $J \rightarrow I$, and $c = P_\psi = H_\psi = 0$)

Q_Ψ is a 3xNxN tensor that will be defined on slide 17.

Formulating Equations of Motion (3)

So the equations of motion are (so far):

$$\begin{bmatrix} \Omega_u^T \\ \Omega_\xi^T \end{bmatrix} \left\{ T - \left[J\alpha + \omega \times J\omega + (c + P_\psi \eta) \times a + (H_\psi + Q_\Psi \eta)\dot{\xi} \right] \right\} +$$

$$\begin{bmatrix} V_u^T \\ V_\xi^T \end{bmatrix} \left\{ F - \left[ma + \alpha \times (c + P_\psi \eta) + \omega \times (\omega \times (c + P_\psi \eta)) + P_\psi \dot{\xi} + 2\omega \times P_\psi \xi \right] \right\} +$$

$$\left\{ \begin{matrix} 0 \\ \int \Psi^T \left[df - \breve{a}\, dm \right] \end{matrix} \right\} = 0$$

More Modal Integrals (1)

We've already introduced these modal integrals:

$$c=\int r dm, \quad P_\Psi=\int \Psi\, dm, \quad J=-\int r^\times r^\times dm, \quad H_\Psi=\int r^\times \Psi\, dm$$

In the derivation to follow, we will also need:

$$M_\Psi=\int \Psi^T \Psi\, dm, \quad \underbrace{L_{ijp}}_{3\times 3\times N}=\int \Psi_{ip} r_j\, dm, \quad \underbrace{N_{ijpq}}_{3\times 3\times N\times N}=\int \Psi_{ip}\Psi_{jq}\, dm$$

L_{ijp} and N_{ijpq} capture all possible products of Ψ with r or with itself. We could, in fact, construct H_Ψ and M_Ψ from elements of L_{ijp} and N_{ijpq}, respectively.

Some other tensors derived from L_{ijp} and N_{ijpq} will also be needed. We define them in the following slides, and label them Q_Ψ, R_Ψ, and S_Ψ.

More Modal Integrals (2)

It will be convenient to denote the rows of Ψ as $\Psi = \begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{Bmatrix}$ and partially expand L_{ijp} and N_{ijpq}:

$$L_{ijp} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \int \begin{bmatrix} \Psi_1 r_1 & \Psi_1 r_2 & \Psi_1 r_3 \\ \Psi_2 r_1 & \Psi_2 r_2 & \Psi_2 r_3 \\ \Psi_3 r_1 & \Psi_3 r_2 & \Psi_3 r_3 \end{bmatrix} dm$$

$$N_{ijpq} = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix} = \int \begin{bmatrix} \Psi_1^T \Psi_1 & \Psi_1^T \Psi_2 & \Psi_1^T \Psi_3 \\ \Psi_2^T \Psi_1 & \Psi_2^T \Psi_2 & \Psi_2^T \Psi_3 \\ \Psi_3^T \Psi_1 & \Psi_3^T \Psi_2 & \Psi_3^T \Psi_3 \end{bmatrix} dm$$

L_{ij} is an N-element vector, and N_{ij} is an N-by-N matrix.

More Modal Integrals (3)

We can then introduce:

$$Q_\Psi \equiv \begin{bmatrix} N_{32} - N_{23} \\ N_{13} - N_{31} \\ N_{21} - N_{12} \end{bmatrix} \qquad (3 \times N \times N)$$

$$R_\Psi \equiv \begin{bmatrix} -L_{22} - L_{33} & L_{21} & L_{31} \\ L_{12} & -L_{33} - L_{11} & L_{32} \\ L_{13} & L_{23} & -L_{11} - L_{22} \end{bmatrix} \qquad (3 \times N \times 3)$$

$$S_\Psi \equiv \begin{bmatrix} -N_{22} - N_{33} & N_{21} & N_{31} \\ N_{12} & -N_{33} - N_{11} & N_{32} \\ N_{13} & N_{23} & -N_{11} - N_{22} \end{bmatrix} \qquad (3 \times N \times N \times 3)$$

Expansion of $\int \Psi^T df$

df refers to any forces, internal or external, acting on the flexible body. For example, suppose the body material obeys Hooke's Law:

$$df_K = -Sd = -S\Psi\eta\, dV$$

where S is the (3 x 3) material elasticity matrix relating stress and strain. (See Appendix)
Substituting, we find the familiar modal stiffness force term:

$$f_K = \int \Psi^T df_K = \int_V (-\Psi^T S \Psi \eta)\, dV$$

$$f_K = -K_\Psi \eta$$

where K_Ψ is the familiar modal stiffness matrix.
Similar arguments may be used for a Coulomb damping term, $f_C = -C_\Psi \xi$, and any other internal forces required.

The effect of external forces and torques on flexible modes is denoted by:

$$\breve{F} = \int \Psi^T df_{ext}$$

$$\breve{T} = \int \Theta^T d\tau_{ext}$$

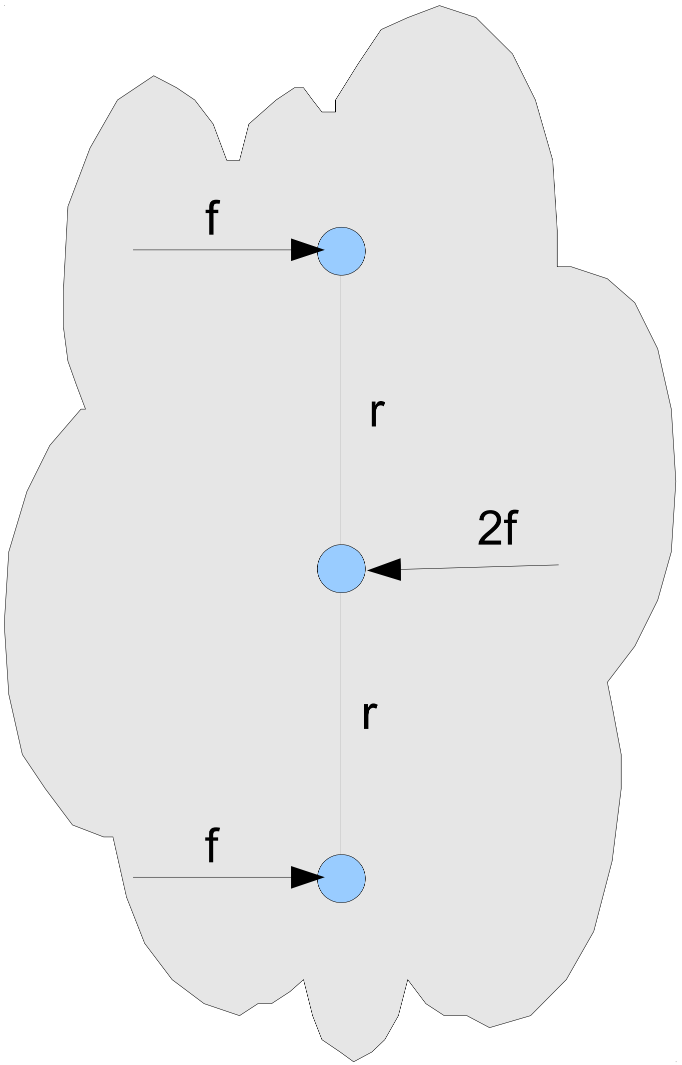
See following slides for an illustrative example.

Flexible Mode Excitation by External Forces and Torques

- In the development so far, F and T denote the resultant force and torque exerted on the body by external forces and torques, applied to the reference point.
- To capture the excitation of flexible modes by external forces and torques, their points of application must be projected into the modal state space.

Modal Excitation Example 1

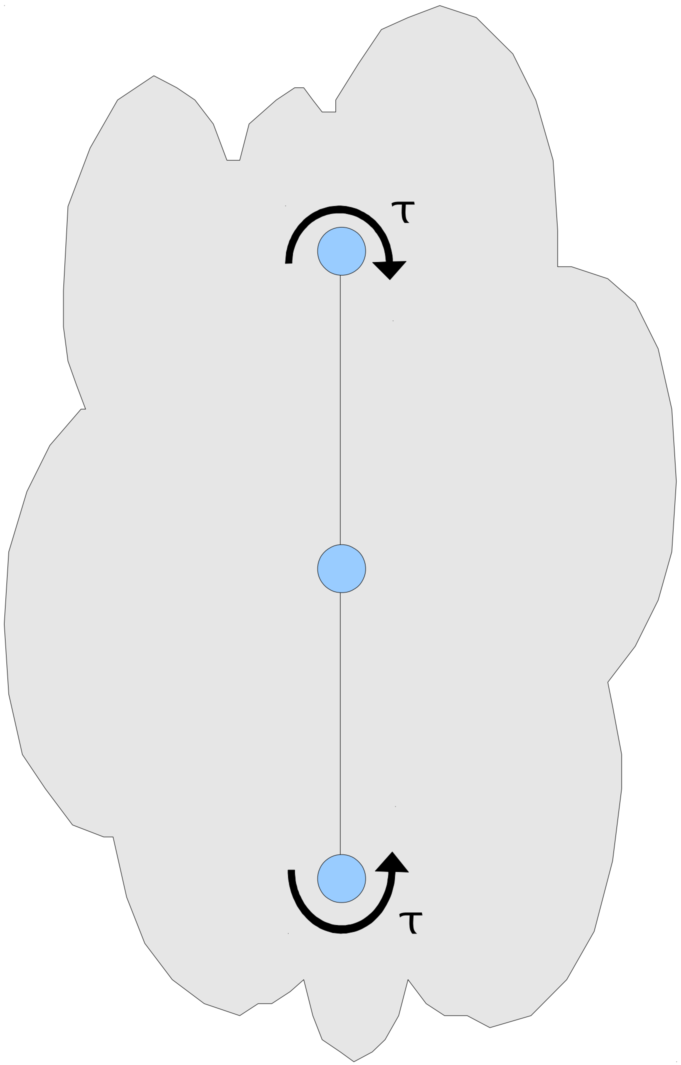
- Consider a flexible body with free-free mode shapes and reference point at the mass center
- The force system shown has zero resultant force and torque, but excites a bending mode
- Moral: Modeling of applied forces and torques must account for distribution over the body to capture influence on flexible modes.



$$F = \int df = 0$$
$$T = \int r \times df = 0$$
$$\breve{F} = \int \Psi^T \, df \neq 0$$

Modal Excitation Example 2

- Consider a flexible body with free-free mode shapes and reference point at the mass center
- The force system shown has zero resultant force and torque, but excites a bending mode
- Moral: Modeling of applied forces and torques must account for distribution over the body to capture influence on flexible modes.



$$F = \int df = 0$$
$$T = \int r \times df = 0$$
$$\breve{T} = \int \Theta^T d\tau \neq 0$$

Expansion of $-\int \Psi^T \breve{a}\, dm$

$$-\int \Psi^T \breve{a}\, dm = -\int \Psi^T \left\{ a + \omega \times \left[\omega \times (r + \Psi\, \eta) \right] + \alpha \times (r + \Psi\, \eta) + \Psi\, \dot{\xi} + 2\, \omega \times \Psi\, \xi \right\} dm$$

Expand each term, taking constants out of the integral. Terms 1, 4, and 6 are short work. The remaining terms will be expanded in following slides.

Term 1: $-\int \Psi^T a\, dm = -\left(\int \Psi^T dm\right) a = -P_\Psi^T a$

Term 2: $-\int \Psi^T \left[\omega \times (\omega \times r) \right] dm$

Term 3: $-\int \Psi^T \left[\omega \times (\omega \times \Psi\, \eta) \right] dm$

Term 4: $-\int \Psi^T (\alpha \times r)\, dm = -\int (r \times \Psi) \cdot \alpha\, dm = -H_\Psi^T \alpha$

Term 5: $-\int \Psi^T (\alpha \times \Psi\, \eta)\, dm$

Term 6: $-\int \Psi^T \Psi\, \dot{\xi}\, dm = -\left(\int \Psi^T \Psi\, dm\right) \dot{\xi} = -M_\Psi \dot{\xi}$

Term 7: $-2 \int \Psi^T (\omega \times \Psi\, \xi)\, dm$

Expansion of Term 2

Term 2: $-\int \Psi^T [\omega \times (\omega \times r)]\, dm$

Expand to $-\int \begin{bmatrix} \Psi_1^T & \Psi_2^T & \Psi_3^T \end{bmatrix} \begin{Bmatrix} \omega_2(\omega_1 r_2 - \omega_2 r_1) - \omega_3(\omega_3 r_1 - \omega_1 r_3) \\ \omega_3(\omega_2 r_3 - \omega_3 r_2) - \omega_1(\omega_1 r_2 - \omega_2 r_1) \\ \omega_1(\omega_3 r_1 - \omega_1 r_3) - \omega_2(\omega_2 r_3 - \omega_3 r_2) \end{Bmatrix} dm$

then
$$\begin{aligned} -\int & \Psi_1^T [r_1(-\omega_2^2 - \omega_3^2) + r_2 \omega_1 \omega_2 + r_3 \omega_1 \omega_3] \\ & + \Psi_2^T [r_1 \omega_2 \omega_1 + r_2(-\omega_3^2 - \omega_1^2) + r_3 \omega_2 \omega_3] \\ & + \Psi_3^T [r_1 \omega_3 \omega_1 + r_2 \omega_3 \omega_2 + r_3(-\omega_1^2 - \omega_2^2)]\, dm \end{aligned}$$

and finally:

$$\begin{aligned} -\int \Psi^T [\omega \times (\omega \times r)]\, dm = {} & L_{11}(\omega_2^2 + \omega_3^2) + L_{22}(\omega_3^2 + \omega_1^2) + L_{33}(\omega_1^2 + \omega_2^2) \\ & - (L_{12} + L_{21})\omega_1 \omega_2 - (L_{13} + L_{31})\omega_1 \omega_3 - (L_{23} + L_{32})\omega_2 \omega_3 \end{aligned}$$

For brevity in the ensuing development, we use R_Ψ, so

$$-\int \Psi^T [\omega \times (\omega \times r)]\, dm = -\omega \cdot R_\Psi \cdot \omega$$

Expansion of Term 3

Term 3: $-\int \Psi^T \left[\omega \times (\omega \times \Psi\,\eta)\right] dm$

Follow the example of term 2, replacing r with $\Psi\,\eta$:

$$-\int \left[\Psi_1^T \quad \Psi_2^T \quad \Psi_3^T\right] \left\{\begin{matrix} \omega_2(\omega_1\Psi_2\eta-\omega_2\Psi_1\eta)-\omega_3(\omega_3\Psi_1\eta-\omega_1\Psi_3\eta) \\ \omega_3(\omega_2\Psi_3\eta-\omega_3\Psi_2\eta)-\omega_1(\omega_1\Psi_2\eta-\omega_2\Psi_1\eta) \\ \omega_1(\omega_3\Psi_1\eta-\omega_1\Psi_3\eta)-\omega_2(\omega_2\Psi_3\eta-\omega_3\Psi_2\eta) \end{matrix}\right\} dm$$

then

$$\begin{aligned} -\int \Psi_1^T&\left[\Psi_1(-\omega_2^2-\omega_3^2)+\Psi_2\omega_1\omega_2+\Psi_3\omega_1\omega_3\right]\eta \\ +\Psi_2^T&\left[\Psi_1\omega_2\omega_1+\Psi_2(-\omega_3^2-\omega_1^2)+\Psi_3\omega_2\omega_3\right]\eta \\ +\Psi_3^T&\left[\Psi_1\omega_3\omega_1+\Psi_2\omega_3\omega_2+\Psi_3(-\omega_1^2-\omega_2^2)\right]\eta\, dm \end{aligned}$$

and finally:

$$\begin{aligned} -\int \Psi^T\left[\omega\times(\omega\times\Psi\,\eta)\right]dm = [&N_{11}(\omega_2^2+\omega_3^2)+N_{22}(\omega_3^2+\omega_1^2)+N_{33}(\omega_1^2+\omega_2^2) \\ &-(N_{12}+N_{21})\omega_1\omega_2-(N_{13}+N_{31})\omega_1\omega_3-(N_{23}+N_{32})\omega_2\omega_3]\,\eta \end{aligned}$$

For brevity in the ensuing development, we use S_Ψ so

$$-\int \Psi^T\left[\omega\times(\omega\times\Psi\,\eta)\right]dm = -\omega\cdot S_\Psi\cdot\omega\,\eta$$

Expansion of Term 5

Term 5: $-\int \Psi^T(\alpha\times\Psi\,\eta)\,dm$

Expand as: $-\int \begin{bmatrix}\Psi_1^T & \Psi_2^T & \Psi_3^T\end{bmatrix}\begin{Bmatrix}\alpha_2\Psi_3\eta-\alpha_3\Psi_2\eta\\ \alpha_3\Psi_1\eta-\alpha_1\Psi_3\eta\\ \alpha_1\Psi_2\eta-\alpha_2\Psi_1\eta\end{Bmatrix}dm$

which reduces to:

$$-\int \Psi^T(\alpha\times\Psi\,\eta)\,dm=-\eta^T\begin{bmatrix}(N_{32}-N_{23}) & (N_{13}-N_{31}) & (N_{21}-N_{12})\end{bmatrix}\alpha$$

For brevity in the ensuing development, we use Q_Ψ, so

$$-\int \Psi^T(\alpha\times\Psi\,\eta)\,dm=-\eta^T Q_\Psi^T\,\alpha$$

Expansion of Term 7

Term 7: $-2\int \Psi^T(\omega\times\Psi\,\xi)\,dm$

As with term 5, expand as: $-2\int \begin{bmatrix}\Psi_1^T & \Psi_2^T & \Psi_3^T\end{bmatrix}\begin{Bmatrix}\omega_2\Psi_3\xi-\omega_3\Psi_2\xi \\ \omega_3\Psi_1\xi-\omega_1\Psi_3\xi \\ \omega_1\Psi_2\xi-\omega_2\Psi_1\xi\end{Bmatrix}dm$

which reduces to:

$$-2\int \Psi^T(\omega\times\Psi\,\xi)\,dm=-2\,\xi^T\left[(N_{32}-N_{23})\quad(N_{13}-N_{31})\quad(N_{21}-N_{12})\right]\omega$$

For brevity in the ensuing development, we use Q_Ψ, so

$$-2\int \Psi^T(\omega\times\Psi\,\xi)\,dm=-2\,\xi^T Q_\Psi^T\,\omega$$

Equations of Motion for One Flexible Body

Reassembling, and including forces due to material stiffness and damping as examples of internal forces, we obtain

$$\breve{F}+\breve{T}-M_{\Psi}\dot{\xi}-C_{\Psi}\xi-K_{\Psi}\eta-P_{\Psi}^{T}a-(H_{\Psi}^{T}+\eta^{T}Q_{\Psi}^{T})\alpha-\omega\cdot R_{\Psi}\cdot\omega-\omega\cdot S_{\Psi}\cdot\omega\eta-2\xi^{T}Q_{\Psi}^{T}\omega=0$$

Inserting into the overall equations of motion from slide 15:

$$\begin{bmatrix}\Omega_{u}^{T}\\ \Omega_{\xi}^{T}\end{bmatrix}\left\{T-\left[J\alpha+\omega\times J\omega+(c+P_{\psi}\eta)\times a+(H_{\psi}+Q_{\Psi}\eta)\dot{\xi}\right]\right\}+$$

$$\begin{bmatrix}V_{u}^{T}\\ V_{\xi}^{T}\end{bmatrix}\left\{F-\left[ma+\alpha\times(c+P_{\psi}\eta)+\omega\times(\omega\times(c+P_{\psi}\eta))+P_{\psi}\dot{\xi}+2\omega\times P_{\psi}\xi\right]\right\}+$$

$$\begin{Bmatrix}0\\ \breve{F}+\breve{T}-M_{\Psi}\dot{\xi}-C_{\Psi}\xi-K_{\Psi}\eta-P_{\Psi}^{T}a-(H_{\Psi}^{T}+\eta^{T}Q_{\Psi}^{T})\alpha-\omega\cdot R_{\Psi}\cdot\omega-\omega\cdot S_{\Psi}\cdot\omega\eta-2\xi^{T}Q_{\Psi}^{T}\omega\end{Bmatrix}=0$$

Simplified EOM for One Free-Free Body

For a single body, the equations of motion may be written:

$$\begin{bmatrix} J & (c+P_\Psi\eta)^\times & (H_\Psi+Q_\Psi\eta) \\ -(c+P_\Psi\eta)^\times & mU & P_\Psi \\ (H_\Psi+Q_\Psi\eta)^T & P_\Psi^T & M_\Psi \end{bmatrix} \begin{Bmatrix} \dot\omega \\ \dot v \\ \dot\xi \end{Bmatrix} = \begin{Bmatrix} T-\omega\times J\,\omega \\ F-\omega\times[\omega\times(c+P_\Psi\eta)]-2\,\omega\times P_\Psi\xi \\ \breve F+\breve T-C_\Psi\xi-K_\Psi\eta-\omega\cdot R_\Psi\cdot\omega-\omega\cdot S_\Psi\cdot\omega\eta-2\,\xi^T Q_\Psi^T\omega \end{Bmatrix}$$

Using the mass center as the reference point, and dropping nonlinear rigid-flexible coupling terms:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & mU & 0 \\ 0 & 0 & M_\Psi \end{bmatrix} \begin{Bmatrix} \dot\omega \\ \dot v \\ \dot\xi \end{Bmatrix} = \begin{Bmatrix} T-\omega\times I\,\omega \\ F \\ \breve F+\breve T-C_\Psi\xi-K_\Psi\eta \end{Bmatrix}$$

which is reassuringly familiar. Note that the flexible modes are only coupled to rigid-body motion through excitation by $\breve F$ and $\breve T$!

Observations

- If modes are mass-normalized, then

$$M_{\Psi} = U\,, \qquad K_{\Psi} = \begin{bmatrix} \ddots & & \\ & \omega_{\Psi}^{2} & \\ & & \ddots \end{bmatrix}$$

- K_{Ψ} is sometimes assumed to be diagonal
  - But the full stress-strain relation leads to off-diagonal stiffness elements
  - These off-diagonal elements become important for “fast” motions
  - Neglecting them yields poor (even nonphysical) results

Equations of Motion for Multiple Flexible Bodies in a Tree Topology

The Rest Is Just Accounting

Substitute $\alpha = \Omega_u \dot{u} + \Omega_\xi \dot{\xi} + \alpha_r, \;\; a = V_u \dot{u} + V_\xi \dot{\xi} + a_r$ into equations of motion,

$$\begin{bmatrix} \Omega_u^T \\ \Omega_\xi^T \end{bmatrix} \left\{ T - \left[J\alpha + \omega \times J\omega + (c + P_\psi \eta) \times a + (H_\psi + Q_\Psi \eta)\dot{\xi} \right] \right\} +$$

$$\begin{bmatrix} V_u^T \\ V_\xi^T \end{bmatrix} \left\{ F - \left[ma + \alpha \times (c + P_\psi \eta) + \omega \times (\omega \times (c + P_\psi \eta)) + P_\psi \dot{\xi} + 2\omega \times P_\psi \xi \right] \right\} +$$

$$\left\{ \begin{matrix} 0 \\ \breve{F} + \breve{T} - M_\Psi \dot{\xi} - C_\Psi \xi - K_\Psi \eta - P_\Psi^T a - (H_\Psi^T + \eta^T Q_\Psi^T)\alpha - \omega \cdot R_\Psi \cdot \omega - \omega \cdot S_\Psi \cdot \omega \eta - 2\xi^T Q_\Psi^T \omega \end{matrix} \right\} = 0$$

and group terms on $\dot{u}, \dot{\xi}$:

Equations of Motion

Let

$$Z_1 = J\,\Omega_u + (c + P_\Psi \eta) \times V_u$$
$$Z_2 = m V_u - (c + P_\Psi \eta) \times \Omega_u$$
$$Z_3 = J\,\Omega_\xi + (c + P_\Psi \eta) \times V_\xi + (H_\Psi + Q_\Psi \eta)$$
$$Z_4 = m V_\xi - (c + P_\Psi \eta) \times \Omega_\xi + P_\Psi$$
$$Z_5 = T - J\,\alpha_r - \omega \times J\,\omega - (c + P_\Psi \eta) \times a_r$$
$$Z_6 = F - m\,a_r - \alpha_r \times (c + P_\Psi \eta) - \omega \times [\omega \times (c + P_\Psi \eta)] - 2\,\omega \times P_\Psi \eta$$

Then

$$\begin{bmatrix} \Omega_u^T Z_1 + V_u^T Z_2 & \Omega_u^T Z_3 + V_u^T Z_4 \\ \Omega_\xi^T Z_1 + V_\xi^T Z_2 + P_\Psi^T V_u + (H_\Psi^T + \eta^T Q_\Psi^T)\,\Omega_u & \Omega_\xi^T Z_3 + V_\xi^T Z_4 + M_\Psi + P_\Psi^T V_\xi + (H_\Psi^T + \eta^T Q_\Psi^T)\,\Omega_\xi \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{\xi} \end{Bmatrix} =$$

$$\begin{Bmatrix} \Omega_u^T Z_5 + V_u^T Z_6 \\ \Omega_\xi^T Z_5 + V_\xi^T Z_6 + \breve{F} + \breve{T} - C_\Psi \xi - K_\Psi \eta - P_\Psi^T a_r - (H_\Psi^T + \eta^T Q_\Psi^T)\,\alpha_r - \omega \cdot R_\Psi \cdot \omega - \omega \cdot S_\Psi \cdot \omega\,\eta - 2\,\xi^T Q_\Psi^T \omega \end{Bmatrix}$$

Equations of Motion (with Bases)

Let

$$Z_1 = J\,\Omega_u + (c + P_\Psi \eta) \times {}^B C^N V_u$$

$$Z_2 = m V_u - {}^N C^B (c + P_\Psi \eta) \times \Omega_u$$

$$Z_3 = J\,\Omega_\xi + (c + P_\Psi \eta) \times {}^B C^N V_\xi + (H_\Psi + Q_\Psi \eta)$$

$$Z_4 = m V_\xi - {}^N C^B \left[(c + P_\Psi \eta) \times \Omega_\xi - P_\Psi \right]$$

$$Z_5 = T - J\,\alpha_r - \omega \times J\,\omega - (c + P_\Psi \eta) \times {}^B C^N a_r$$

$$Z_6 = F - m\,a_r - {}^N C^B \left\{ \alpha_r \times (c + P_\Psi \eta) + \omega \times \left[\omega \times (c + P_\Psi \eta) \right] + 2\,\omega \times P_\Psi \eta \right\}$$

Then

$$\begin{bmatrix} \Omega_u^T Z_1 + V_u^T Z_2 & \Omega_u^T Z_3 + V_u^T Z_4 \\ \Omega_\xi^T Z_1 + V_\xi^T Z_2 + P_\Psi^T\, {}^B C^N V_u + (H_\Psi^T + \eta^T Q_\Psi^T)\Omega_u & \Omega_\xi^T Z_3 + V_\xi^T Z_4 + M_\Psi + P_\Psi^T\, {}^B C^N V_\xi + (H_\Psi^T + \eta^T Q_\Psi^T)\Omega_\xi \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{\xi} \end{Bmatrix} =$$

$$\begin{Bmatrix} \Omega_u^T Z_5 + V_u^T Z_6 \\ \Omega_\xi^T Z_5 + V_\xi^T Z_6 + \breve{F} + \breve{T} - C_\Psi \xi - K_\Psi \eta - P_\Psi^T\, {}^B C^N a_r - (H_\Psi^T + \eta^T Q_\Psi^T)\alpha_r - \omega \cdot R_\Psi \cdot \omega - \omega \cdot S_\Psi \cdot \omega\,\eta - 2\,\xi^T Q_\Psi^T \omega \end{Bmatrix}$$

Remainder Accelerations

We have expressed linear and angular accelerations as:

$$a = V_u \dot{u} + V_\xi \dot{\xi} + a_r$$

$$\alpha = \Omega_u \dot{u} + \Omega_\xi \dot{\xi} + \alpha_r$$

where a_r and α_r simply denote whatever terms in a and α do not depend on $\dot{u}$ and $\dot{\xi}$. For lack of a better term, we call these the remainder accelerations.

Consider an inner body, B_i, and an outer body, B_o, connected by joint G_k.
We may write the recursive expressions:

$$\alpha_{ro} = \alpha_{ri} + \dot{\Gamma}_k \sigma_k + \omega_o \times \Gamma_k \sigma_k + \omega_i \times \Theta_{ik} \xi_i - \omega_o \times \Theta_{ok} \xi_o$$

$$a_{ro} = a_{ri} + \omega_i \times (\omega_i \times \breve{r}_{ik}) - \omega_o \times (\omega_o \times \breve{r}_{ok}) + \alpha_{ri} \times \breve{r}_{ik} - \alpha_{ro} \times \breve{r}_{ok} + 2(\omega_i \times \Psi_{ik} \xi_i - \omega_o \times \Psi_{ok} \xi_o)$$

Partial Velocities

Partial velocities, $V=[V_u \;\; V_\xi]$ and partial angular velocities $\Omega=[\Omega_u \;\; \Omega_\xi]$ reflect the kinematics of the multibody spacecraft as a whole. It is easiest to define them by way of example.

Consider the five-body spacecraft illustrated on the next slide. We will write linear and angular velocities for all bodies, and look for helpful patterns to emerge.

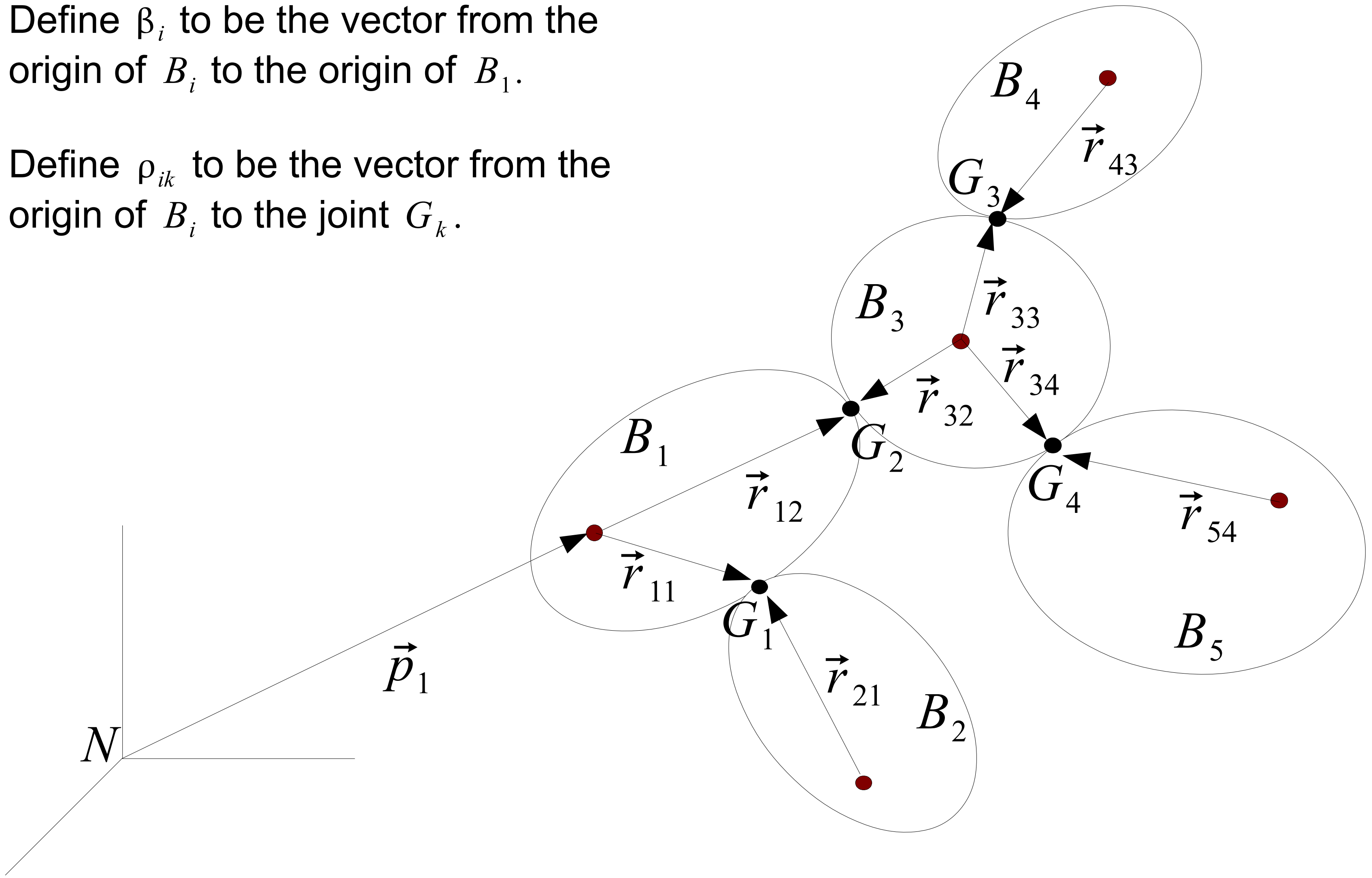
As with the remainder accelerations, we can form recursive expressions for the angular and linear velocity of an outer body (B_o) in terms of its inner body (B_i) and the joint (G_k) they share:

$$\omega_o=\omega_i+\Gamma_k\sigma_k+\Theta_{ik}\xi_i-\Theta_{ok}\xi_o$$
$$v_o=v_i+\omega_i\times\breve{r}_{ik}-\omega_o\times\breve{r}_{ok}+\Psi_{ik}\xi_i-\Psi_{ok}\xi_o$$

Example Problem

Define β_i to be the vector from the origin of B_i to the origin of B_1.

Define ρ_{ik} to be the vector from the origin of B_i to the joint G_k.



Example Partial Velocities

$$\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\\ \omega_5\end{Bmatrix}=\underbrace{\begin{bmatrix}U & 0 & 0 & 0 & 0 & 0\\ {}^2C^1 & \Gamma_1 & 0 & 0 & 0 & 0\\ {}^3C^1 & 0 & \Gamma_2 & 0 & 0 & 0\\ {}^4C^1 & 0 & {}^4C^3\Gamma_2 & \Gamma_3 & 0 & 0\\ {}^5C^1 & 0 & {}^5C^3\Gamma_2 & 0 & \Gamma_4 & 0\end{bmatrix}}_{\Omega_u}\underbrace{\begin{Bmatrix}\omega_1\\ \sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ v_1\end{Bmatrix}}_{u}+\underbrace{\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ {}^2C^1\Theta_{11} & -\Theta_{21} & 0 & 0 & 0\\ {}^3C^1\Theta_{12} & 0 & -\Theta_{32} & 0 & 0\\ {}^4C^1\Theta_{12} & 0 & {}^4C^3(\Theta_{33}-\Theta_{32}) & -\Theta_{43} & 0\\ {}^5C^1\Theta_{12} & 0 & {}^5C^3(\Theta_{34}-\Theta_{32}) & 0 & -\Theta_{54}\end{bmatrix}}_{\Omega_\xi}\underbrace{\begin{Bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\\ \xi_5\end{Bmatrix}}_{\xi}$$

$$\begin{Bmatrix}v_1\\ v_2\\ v_3\\ v_4\\ v_5\end{Bmatrix}=\underbrace{\begin{bmatrix}0 & 0 & 0 & 0 & 0 & U\\ \tilde\beta_2\,{}^NC^1 & \tilde\rho_{21}\,{}^NC^2\Gamma_1 & 0 & 0 & 0 & U\\ \tilde\beta_3\,{}^NC^1 & 0 & \tilde\rho_{32}\,{}^NC^3\Gamma_2 & 0 & 0 & U\\ \tilde\beta_4\,{}^NC^1 & 0 & \tilde\rho_{42}\,{}^NC^3\Gamma_2 & \tilde\rho_{43}\,{}^NC^4\Gamma_3 & 0 & U\\ \tilde\beta_5\,{}^NC^1 & 0 & \tilde\rho_{52}\,{}^NC^3\Gamma_2 & 0 & \tilde\rho_{54}\,{}^NC^5\Gamma_4 & U\end{bmatrix}}_{V_u}\underbrace{\begin{Bmatrix}\omega_1\\ \sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ v_1\end{Bmatrix}}_{u}+$$

$$\underbrace{\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ {}^NC^1\Psi_{11}+\tilde\rho_{21}\,{}^NC^1\Theta_{11} & -{}^NC^2\Psi_{21}-\tilde\rho_{21}\,{}^NC^2\Theta_{21} & 0 & 0 & 0\\ {}^NC^1\Psi_{12}+\tilde\rho_{32}\,{}^NC^1\Theta_{12} & 0 & -{}^NC^3\Psi_{32}-\tilde\rho_{32}\,{}^NC^3\Theta_{32} & 0 & 0\\ {}^NC^1\Psi_{12}+\tilde\rho_{42}\,{}^NC^1\Theta_{12} & 0 & {}^NC^3(\Psi_{33}-\Psi_{32})+\tilde\rho_{43}\,{}^NC^3\Theta_{33}-\tilde\rho_{42}\,{}^NC^3\Theta_{32} & -{}^NC^4\Psi_{43}-\tilde\rho_{43}\,{}^NC^4\Theta_{43} & 0\\ {}^NC^1\Psi_{12}+\tilde\rho_{52}\,{}^NC^1\Theta_{12} & 0 & {}^NC^3(\Psi_{34}-\Psi_{32})+\tilde\rho_{54}\,{}^NC^3\Theta_{34}-\tilde\rho_{52}\,{}^NC^3\Theta_{32} & 0 & -{}^NC^5\Psi_{54}-\tilde\rho_{54}\,{}^NC^5\Theta_{54}\end{bmatrix}}_{V_\xi}\underbrace{\begin{Bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\\ \xi_5\end{Bmatrix}}_{\xi}$$

Observations

Similar patterns arise in Ω_ξ, V_ξ as arose in rigid-body Ω_u, V_u.

These patterns are a product of the recursive nature of the tree.

Non-zero entries trace the path from a body back to the root.

If body origins are at joints, then

$$\rho_{21} = \rho_{32} = \rho_{43} = \rho_{54} = 0$$
$$\Psi_{21}, \Psi_{32}, \Psi_{43}, \Psi_{54} = 0$$
$$\Theta_{21}, \Theta_{32}, \Theta_{43}, \Theta_{54} = 0$$

Conclusion

- Equations of motion have been derived for a system of flexible bodies connected by (spherical or gimballed) joints in a tree topology
- This formulation supports both free-free modes and cantilever modes

Appendix: Stress-Strain Relation

Assuming a bulk material with a modulus of elasticity, E, and Poisson's ratio, ν, stress (σ) and strain (ϵ) are related by

$$\sigma = S\epsilon$$

where

$$S = \frac{E}{1-\nu-2\nu^2}\begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}$$

Steel, as an example, has $E = 200$ GPa
For many materials, $\nu = 0.2 - 0.3$