Application of Kane's Method to a General Tree-Topology Spacecraft with Gimballed or Spherical Joints

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Purpose

We seek equations of motion suitable for solution by a computer, using the numerical integrator of your choice. We will obtain a set of coupled equations which are linear in the derivatives of the dynamic state variables:

$$(\Omega^T I \Omega + V^T m V)\dot{u} = \Omega^T (T - \omega \times H - I \alpha_r) + V^T (F - m a_r)$$

This package presents notation, derivations, and an example to help fix the underlying concepts and guide the assembly of these equations for a general tree-topology N-body spacecraft model.

"$F = ma$. The rest is just accounting."

-Unknown

Kane's Equation

A set of scalar equations, one per generalized speed (u_r).

$$F_r + F_r^* = 0 \qquad r = 1, \ldots, N_u$$

where

$$F_r = \sum_k \vec{\omega}_r^k \cdot \vec{T}_k + \sum_k \vec{v}_r^k \cdot \vec{F}_k \qquad \text{(Generalized active force)}$$

$$F_r^* = \sum_k \vec{\omega}_r^k \cdot (-I_k \vec{\alpha}_k - \vec{\omega}^k \times \vec{H}_k) + \sum_k \vec{v}_r^k \cdot (-m_k \vec{a}_k) \qquad \text{(Generalized inertia force)}$$

Rewriting,

$$\sum_k \left[\vec{\omega}_r \cdot (\vec{T} - I \vec{\alpha} - \vec{\omega} \times \vec{H}) \right]_k + \sum_k \left[\vec{v}_r \cdot (\vec{F} - m \vec{a}) \right]_k = 0$$

$\vec{\omega}_r^k$ is called the rth partial angular velocity of the kth body. It is a vector that projects Euler's equation into the state-space dimension of u_r.

$\vec{v}_r^k$ is called the rth partial velocity of the kth body. It is a vector that projects Newton's equation into the state-space dimension of u_r.

How do we find $\vec{\omega}_r$ and $\vec{v}_r$, and how do we assemble a system of equations?

Vectors and Components

- A vector is independent of any reference frame.
- For numerical computation, we need a numerical representation.
- So we often represent a vector as a product of some scalar *components* and a set of *basis vectors*

$$\vec{v} = v_1 \hat{a}_1 + v_2 \hat{a}_2 + v_3 \hat{a}_3$$

$$\vec{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\vec{v} = \boldsymbol{A} v \,,$$
$$v = \boldsymbol{A}^{\boldsymbol{T}} \vec{v}$$

Basis Dyads

- $\vec{v}$ is a *vector*
- v is an *array of components*
- $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are (orthonormal) *basis vectors*
- Hughes calls $\mathbf{A}$ a *vectrix.* I'll use the term *basis dyad*.
- Like a vector, a dyad is independent of any reference frame, but unsuited to computerized manipulation. So we'll eventually seek a matrix representation.

Reference Frames

- A *reference frame* has an origin and three orthonormal basis vectors.
- We will make use of N, a Newtonian (i.e. non-accelerating, or inertial) frame, and several body-fixed frames, B_1, B_2, ... B_{Nb}.
- Note that the direction cosine matrix relating two frames is the product of the frames' basis dyads:
- $${}^{B}C^{A} = \boldsymbol{B}^{T} \boldsymbol{A}$$
- And the components of a vector, expressed in two different reference frames, are related by:

$$\vec{v} = \boldsymbol{A}\,{}^{A}v = \boldsymbol{B}\,{}^{B}v \qquad \rightarrow \qquad {}^{B}v = {}^{B}C^{A}\,{}^{A}v$$

Angular and Linear Velocity

- Angular velocity of B in N:

$$^{N}\vec{\omega}^{B} = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3 = \boldsymbol{B}\,\omega_B$$

- Linear velocity of B's mass center in N:

$$^{N}\vec{v}^{B} = v_1 \hat{n}_1 + v_2 \hat{n}_2 + v_3 \hat{n}_3 = \boldsymbol{N}\,v_B$$

Joint Partials

Consider two bodies connected by a gimballed joint. The relative angular velocity of the outer body, B, with respect to the inner body, A, may be written as a function of the gimbal angles, θ, and the gimbal rates, $\sigma \equiv \dot{\theta}$. For example, a Body-3, 2-1-3 Euler rotation through $(\theta_1, \theta_2, \theta_3)$ gives:

$$ {}^{A}\vec{\omega}^{B} = \boldsymbol{B} \underbrace{\begin{bmatrix} c_2 s_3 & c_3 & 0 \\ c_2 c_3 & -s_3 & 0 \\ -s_2 & 0 & 1 \end{bmatrix}}_{\Gamma} \underbrace{\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix}}_{\sigma} $$

Where $s_2 = \sin(\theta_2), c_2 = \cos(\theta_2),$ etc, and the basis dyad $\boldsymbol{B}$ appears because $\vec{\omega}$ is a vector, and Γ and σ are a matrix and an array.

The matrix Γ is called the *matrix of joint partials*. Each Euler sequence, including one- and two-DOF gimbals, has its own. The dimensions of Γ are 3 x NDOF.

For a spherical joint, $\sigma = \omega$, and $\Gamma = U$, the identity matrix.

From 3-space to State Space

- Consider a system of N_b bodies, connected in a tree topology by (N_g = N_b-1) revolute joints (gimballed or spherical)
- We choose a set of independent generalized speeds:
  - Components of root body's angular velocity
  - Joint angular rates (1-, 2-, or 3-DOF gimbal) or joint angular velocity components (spherical)
  - Components of root body's linear velocity
- We write the generalized speeds and generalized coordinates as

$$u \equiv \begin{Bmatrix} \omega_1 \\ \sigma_1 \\ \vdots \\ \sigma_{Ng} \\ v_1 \end{Bmatrix} \qquad x \equiv \begin{Bmatrix} q_1 \\ \theta_1 \\ \vdots \\ \theta_{Ng} \\ p_1 \end{Bmatrix}$$

From State Space to 3-space

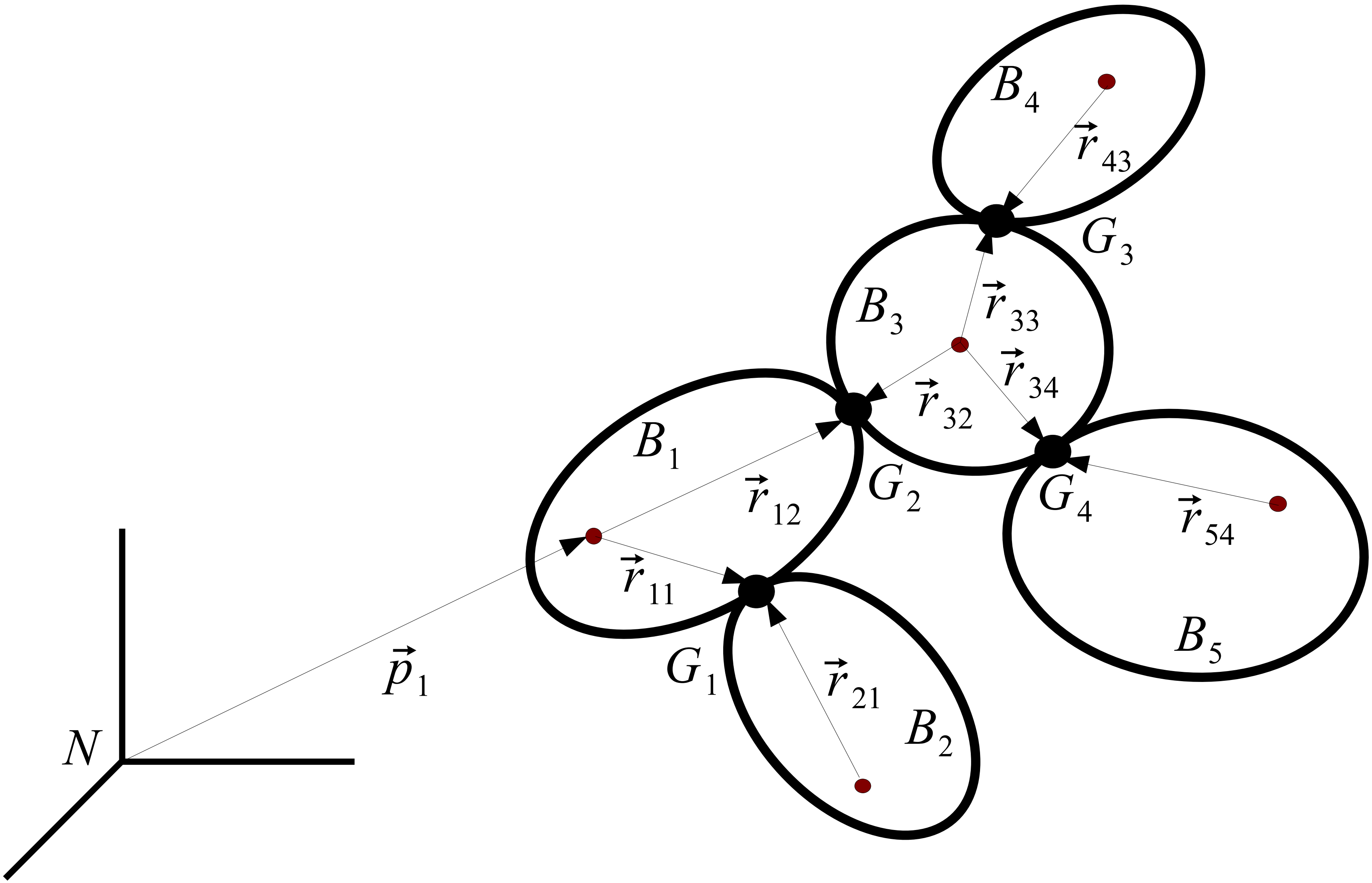
- For Kane's equations, we need $\vec{\alpha}$ and $\vec{a}$ for each rigid body, which we obtain by differentiating $\vec{\omega}$ and $\vec{v}$ of each rigid body.
- Using joint partials, we may construct expressions for these as functions of the generalized speeds.
- We'll look at the structure of these functions by example next.
- For now, simply note that we may write

$$\begin{Bmatrix} \vec{\omega}_1 \\ \vec{\omega}_2 \\ \vdots \\ \vec{\omega}_{Nb} \end{Bmatrix} = \boldsymbol{\Omega}\, u \qquad \begin{Bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_{Nb} \end{Bmatrix} = \boldsymbol{V}\, u$$

Deja Vu All Over Again

- On the preceding slide, we defined $\boldsymbol{\Omega}$ and $\boldsymbol{V}$ to compactly express the angular velocities and velocities of a collection of rigid bodies as linear combinations of a set of independent generalized speeds
- Looking at it another way, the elements of $\boldsymbol{\Omega}$ and $\boldsymbol{V}$ project $\vec{\omega}$ and $\vec{v}$ into the state-space dimension of each element of u.
- $\boldsymbol{\Omega}$ contains the partial angular velocities. $\boldsymbol{V}$ contains the partial velocities.
- In the following slides, we will see how to construct $\boldsymbol{\Omega}$ and $\boldsymbol{V}$ for a general tree-topology N-body spacecraft.

Example Problem



Constructions

Angular velocities are constructed like this:

$$^N\vec{\omega}^2 = {}^N\vec{\omega}^1 + {}^1\vec{\omega}^2 = \boldsymbol{B_1}\omega_1 + \boldsymbol{B_2}\Gamma_1\sigma_1$$

Velocities are constructed like this:

$$^N\vec{v}^2 = {}^N\vec{v}^1 + {}^N\vec{\omega}^1 \times \vec{r}_{11} - {}^N\vec{\omega}^2 \times \vec{r}_{21} = \boldsymbol{N}\, v_1 + (\boldsymbol{B_1}\omega_1) \times (\vec{r}_{11} - \vec{r}_{21}) - (\boldsymbol{B_2}\Gamma_1\sigma_1) \times \vec{r}_{21}$$

Let $\vec{\beta}_i$ be the vector from the mass center of B_i to the mass center of B_1.
For example:

$$\vec{\beta}_4 = \vec{r}_{43} - \vec{r}_{33} + \vec{r}_{32} - \vec{r}_{12}$$

Let $\vec{\rho}_{ij}$ be the vector from the mass center of B_i to joint G_j. For example:

$$\vec{\rho}_{42} = \vec{r}_{43} - \vec{r}_{33} + \vec{r}_{32}$$

Path Tables

Is this body in the path of that body?

	B1	B2	B3	B4	B5
B1	x				
B2	x	x			
B3	x		x		
B4	x		x	x	
B5	x		x		x

Body Path Table

Is this joint in the path of that body?

	(B1)	G1	G2	G3	G4
B1	x				
B2	x	x			
B3	x		x		
B4	x		x	x	
B5	x		x		x

Joint Path Table

Warning: The Path Tables look the same for this example, but they differ in general. (To see this, swap the labels for G3 and G4.)

B1 is not a joint, but its mass center does serve as one, so it is included in the joint path table.

"I do vector calculus just for fun."

-Weird Al Yankovic,
"White and Nerdy"

Useful Notation and Theorems

Cross product matrix: $\tilde{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \qquad x \times y = \tilde{x}\, y$

Double-cross: $\bar{x}\, y = x \times (x \times y)$

General differentiation of a vector: $\frac{{}^{A}d}{dt}\vec{v} = \frac{{}^{B}d}{dt}\vec{v} + {}^{A}\vec{\omega}^{B} \times \vec{v}$

Apply to find angular acceleration, with a joint:

$$ {}^{N}\vec{\alpha}^{B} = \boldsymbol{A}\,\dot{\omega}_{A} + \boldsymbol{B}(\Gamma\,\dot{\sigma} + \dot{\Gamma}\,\sigma) + \boldsymbol{B}(\omega_{B} \times \Gamma\,\sigma) $$

Linear acceleration, with a joint:

$$ {}^{N}\vec{a}^{B} = {}^{N}\vec{a}^{A} + {}^{N}\vec{\omega}^{A} \times ({}^{N}\vec{\omega}^{A} \times \vec{r}_{A}) + {}^{N}\vec{\alpha}^{A} \times \vec{r}_{A} - {}^{N}\vec{\omega}^{B} \times ({}^{N}\vec{\omega}^{B} \times \vec{r}_{B}) - {}^{N}\vec{\alpha}^{B} \times \vec{r}_{B} $$

Angular and Linear Velocities

$$\begin{Bmatrix} \vec{\omega}_1 \\ \vec{\omega}_2 \\ \vec{\omega}_3 \\ \vec{\omega}_4 \\ \vec{\omega}_5 \end{Bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{B}_1 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 \Gamma_1 & 0 & 0 & 0 & 0 \\ \boldsymbol{B}_1 & 0 & \boldsymbol{B}_3 \Gamma_2 & 0 & 0 & 0 \\ \boldsymbol{B}_1 & 0 & \boldsymbol{B}_3 \Gamma_2 & \boldsymbol{B}_4 \Gamma_3 & 0 & 0 \\ \boldsymbol{B}_1 & 0 & \boldsymbol{B}_3 \Gamma_2 & 0 & \boldsymbol{B}_5 \Gamma_4 & 0 \end{bmatrix}}_{\boldsymbol{\Omega}} \underbrace{\begin{Bmatrix} \omega_1 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ v_1 \end{Bmatrix}}_{u}$$

$$\begin{Bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \\ \vec{v}_5 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \boldsymbol{N} \\ \tilde{\beta}_2 \boldsymbol{B}_1 & \tilde{\rho}_{21} \boldsymbol{B}_2 \Gamma_1 & 0 & 0 & 0 & \boldsymbol{N} \\ \tilde{\beta}_3 \boldsymbol{B}_1 & 0 & \tilde{\rho}_{32} \boldsymbol{B}_3 \Gamma_2 & 0 & 0 & \boldsymbol{N} \\ \tilde{\beta}_4 \boldsymbol{B}_1 & 0 & \tilde{\rho}_{42} \boldsymbol{B}_3 \Gamma_2 & \tilde{\rho}_{43} \boldsymbol{B}_4 \Gamma_3 & 0 & \boldsymbol{N} \\ \tilde{\beta}_5 \boldsymbol{B}_1 & 0 & \tilde{\rho}_{52} \boldsymbol{B}_3 \Gamma_2 & 0 & \tilde{\rho}_{54} \boldsymbol{B}_5 \Gamma_4 & \boldsymbol{N} \end{bmatrix}}_{\boldsymbol{V}} \underbrace{\begin{Bmatrix} \omega_1 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ v_1 \end{Bmatrix}}_{u}$$

Note resemblance to joint path table. $\boldsymbol{\Omega}$ and V may be obtained by inspection.

Clearing Basis Dyads

- Let's “convert” vectors and dyads into arrays and matrices.
- Choose which frames to express equations in, and multiply through by the appropriate basis dyad.

Clearing Basis Dyads (2)

$$\begin{bmatrix} \boldsymbol{B}_1^T & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{B}_2^T & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{B}_3^T & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{B}_4^T & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{B}_5^T \end{bmatrix} \begin{Bmatrix} \vec{\omega}_1 \\ \vec{\omega}_2 \\ \vec{\omega}_3 \\ \vec{\omega}_4 \\ \vec{\omega}_5 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{B}_1^T & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{B}_2^T & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{B}_3^T & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{B}_4^T & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{B}_5^T \end{bmatrix} \underbrace{\begin{bmatrix} \boldsymbol{B}_1 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 \boldsymbol{\Gamma}_1 & 0 & 0 & 0 & 0 \\ \boldsymbol{B}_1 & 0 & \boldsymbol{B}_3 \boldsymbol{\Gamma}_2 & 0 & 0 & 0 \\ \boldsymbol{B}_1 & 0 & \boldsymbol{B}_3 \boldsymbol{\Gamma}_2 & \boldsymbol{B}_4 \boldsymbol{\Gamma}_3 & 0 & 0 \\ \boldsymbol{B}_1 & 0 & \boldsymbol{B}_3 \boldsymbol{\Gamma}_2 & 0 & \boldsymbol{B}_5 \boldsymbol{\Gamma}_4 & 0 \end{bmatrix}}_{\boldsymbol{\Omega}} \underbrace{\begin{Bmatrix} \omega_1 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ v_1 \end{Bmatrix}}_{u}$$

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{Bmatrix} = \underbrace{\begin{bmatrix} U & 0 & 0 & 0 & 0 & 0 \\ {}^2C^1 & \Gamma_1 & 0 & 0 & 0 & 0 \\ {}^3C^1 & 0 & \Gamma_2 & 0 & 0 & 0 \\ {}^4C^1 & 0 & {}^4C^3\Gamma_2 & \Gamma_3 & 0 & 0 \\ {}^5C^1 & 0 & {}^5C^3\Gamma_2 & 0 & \Gamma_4 & 0 \end{bmatrix}}_{\Omega} \underbrace{\begin{Bmatrix} \omega_1 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ v_1 \end{Bmatrix}}_{u}$$

Clearing Basis Dyads (3)

$$\begin{bmatrix} \boldsymbol{N}^T & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{N}^T & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{N}^T & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{N}^T & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{N}^T \end{bmatrix} \begin{Bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \\ \vec{v}_5 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{N}^T & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{N}^T & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{N}^T & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{N}^T & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{N}^T \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \boldsymbol{N} \\ \tilde{\beta}_2 \boldsymbol{B}_1 & \tilde{\rho}_{21} \boldsymbol{B}_2 \Gamma_1 & 0 & 0 & 0 & \boldsymbol{N} \\ \tilde{\beta}_3 \boldsymbol{B}_1 & 0 & \tilde{\rho}_{32} \boldsymbol{B}_3 \Gamma_2 & 0 & 0 & \boldsymbol{N} \\ \tilde{\beta}_4 \boldsymbol{B}_1 & 0 & \tilde{\rho}_{42} \boldsymbol{B}_3 \Gamma_2 & \tilde{\rho}_{43} \boldsymbol{B}_4 \Gamma_3 & 0 & \boldsymbol{N} \\ \tilde{\beta}_5 \boldsymbol{B}_1 & 0 & \tilde{\rho}_{52} \boldsymbol{B}_3 \Gamma_2 & 0 & \tilde{\rho}_{54} \boldsymbol{B}_5 \Gamma_4 & \boldsymbol{N} \end{bmatrix}}_{\boldsymbol{V}} \underbrace{\begin{Bmatrix} \omega_1 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ v_1 \end{Bmatrix}}_{u}$$

$$\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & U \\ \tilde{\beta}_2 {}^N C^1 & \tilde{\rho}_{21} {}^N C^2 \Gamma_1 & 0 & 0 & 0 & U \\ \tilde{\beta}_3 {}^N C^1 & 0 & \tilde{\rho}_{32} {}^N C^3 \Gamma_2 & 0 & 0 & U \\ \tilde{\beta}_4 {}^N C^1 & 0 & \tilde{\rho}_{42} {}^N C^3 \Gamma_2 & \tilde{\rho}_{43} {}^N C^4 \Gamma_3 & 0 & U \\ \tilde{\beta}_5 {}^N C^1 & 0 & \tilde{\rho}_{52} {}^N C^3 \Gamma_2 & 0 & \tilde{\rho}_{54} {}^N C^5 \Gamma_4 & U \end{bmatrix}}_{V} \underbrace{\begin{Bmatrix} \omega_1 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ v_1 \end{Bmatrix}}_{u}$$

Accelerations across a Joint

Consider the k th joint, its inner body, $\boldsymbol{B_i}$, and outer body, $\boldsymbol{B_o}$. The angular acceleration of $\boldsymbol{B_o}$ is related to the angular acceleration of $\boldsymbol{B_i}$ and the motion of the joint:

$$^N\vec{\alpha}^o = {}^N\vec{\alpha}^i + \boldsymbol{B_o}(\Gamma_k\dot{\sigma}_k + \dot{\Gamma}_k\sigma_k) + \boldsymbol{B_o}(\omega_o \times \Gamma_k\sigma_k)$$

The acceleration of $\boldsymbol{B_o}$'s mass center is:

$$^N\vec{a}^o = {}^N\vec{a}^i + {}^N\vec{\omega}^i \times ({}^N\vec{\omega}^i \times \vec{r}_{ik}) + {}^N\vec{\alpha}^i \times \vec{r}_{ik} - {}^N\vec{\omega}^o \times ({}^N\vec{\omega}^o \times \vec{r}_{ok}) - {}^N\vec{\alpha}^o \times \vec{r}_{ok}$$

As we construct these expressions, we find that some terms contain $\dot{u}$, and some don't.

We group the latter together, and coin the term 'remainder accelerations', so that we may write

$$\{\alpha\} = \Omega\,\dot{u} + \{\alpha_r\}$$

$$\{a\} = V\,\dot{u} + \{a_r\}$$

The remainder accelerations may be constructed recursively in the same way as α and a.

$$^N\vec{\alpha}^o_r = {}^N\vec{\alpha}^i_r + \boldsymbol{B_o}(\dot{\Gamma}_k\sigma_k + \omega_o \times \Gamma_k\sigma_k)$$

$$^N\vec{a}^o_r = {}^N\vec{a}^i_r + {}^N\vec{\omega}^i \times ({}^N\vec{\omega}^i \times \vec{r}_{ik}) + {}^N\vec{\alpha}^i_r \times \vec{r}_{ik} - {}^N\vec{\omega}^o \times ({}^N\vec{\omega}^o \times \vec{r}_{ok}) - {}^N\vec{\alpha}^o_r \times \vec{r}_{ok}$$

Assembling Kane's Equation

Let: $$m=\begin{bmatrix} m_1 U & & & & \\ & m_2 U & & & \\ & & m_3 U & & \\ & & & m_4 U & \\ & & & & m_5 U \end{bmatrix},\quad I=\begin{bmatrix} I_1 & & & & \\ & I_2 & & & \\ & & I_3 & & \\ & & & I_4 & \\ & & & & I_5 \end{bmatrix},\quad T=\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix},\quad F=\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix},\quad \omega\times H=\begin{Bmatrix} (\omega\times H)_1 \\ (\omega\times H)_2 \\ (\omega\times H)_3 \\ (\omega\times H)_4 \\ (\omega\times H)_5 \end{Bmatrix}$$

Substituting into Kane's equation:

$$\Omega^T\left[T-I(\Omega\,\dot{u}+\alpha_r)-\omega\times H\right]+V^T\left[F-m(V\,\dot{u}+a_r)\right]=0$$

And rearranging:

$$\underbrace{\left(\Omega^T I\,\Omega+V^T m\,V\right)}_{COEF\,(N_u\times N_u)}\dot{u}=\underbrace{\Omega^T\left(T-I\,\alpha_r-\omega\times H\right)+V^T\left(F-m\,a_r\right)}_{RHS\,(N_u\times 1)}$$

Conclusions

- Kane's equation has been rearranged to a form suitable for numerical solution.
  - A system of equations linear in $\dot{u}$
- For a tree-topological spacecraft, partial angular velocity and partial velocity matrices may be deduced by inspection of the path tables.
- The hard part of Kane's method is differentiating vectors
  - Vector theorems help.
  - Notation that handles multiple frames helps, too.