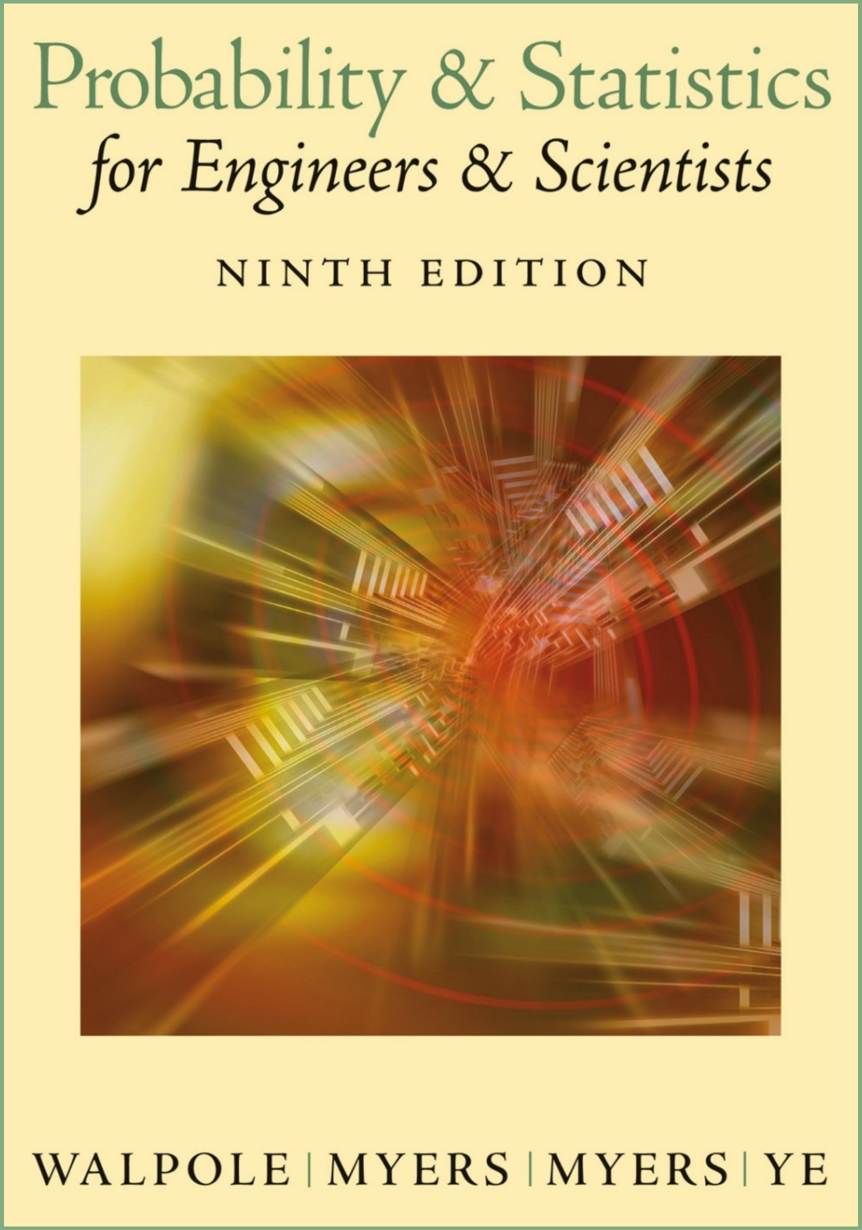
Chapter 6

Some Continuous Probability Distributions



Continuous distributions

Five important continuous distributions:

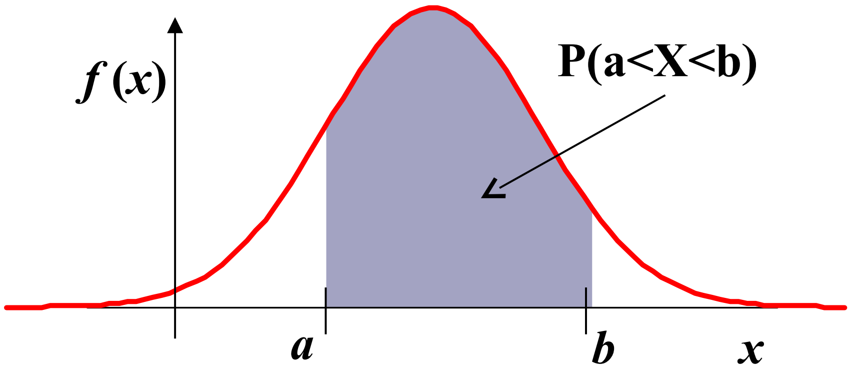
1. **uniform distribution (contiuous)**
2. **Normal distribution**
3. χ^2 **–distribution ["ki-square"]**
4. **t-distribution**
5. **F-distribution**

A reminder

Definition:
Let X: S → R be a continuous random variable.

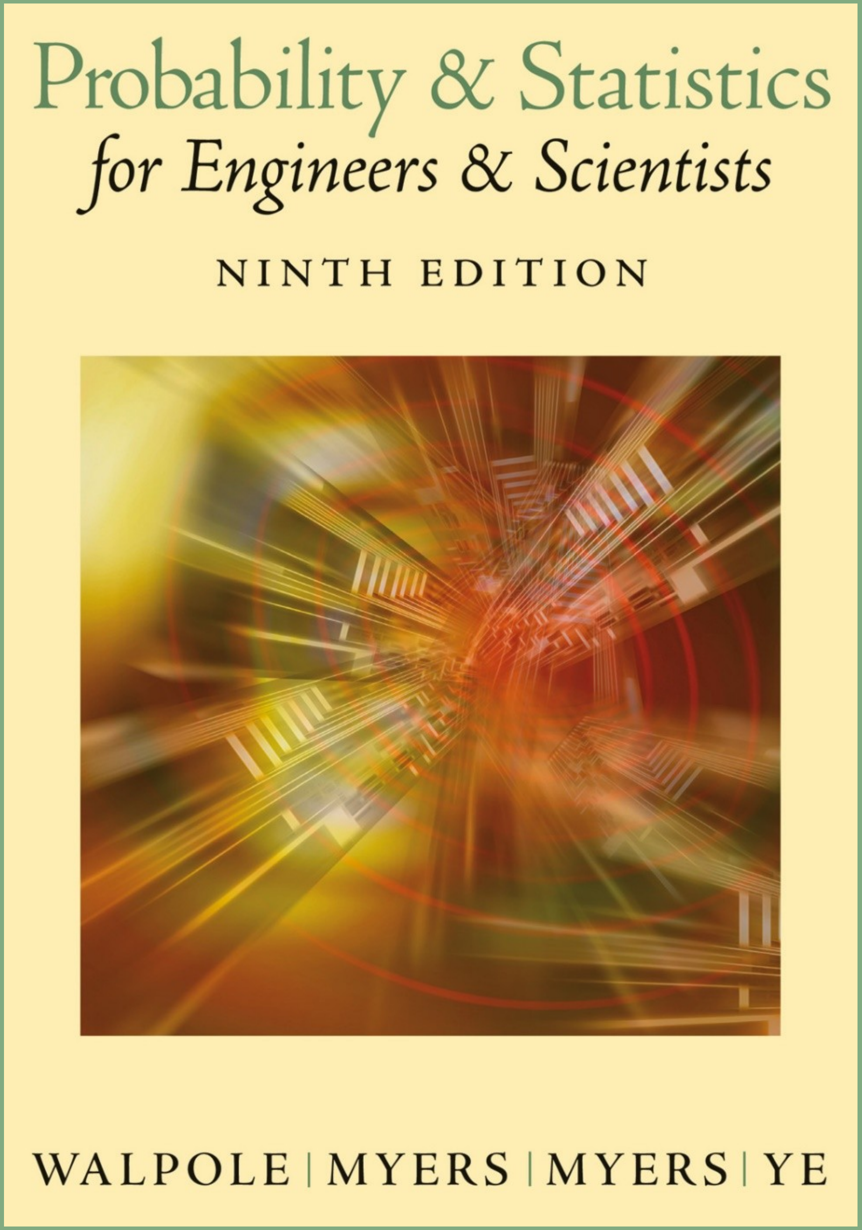
A density function for X, $f(x)$, is defined by:

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a < X < b) = \int_{a}^{b} f(x)\,dx$



Section 6.1

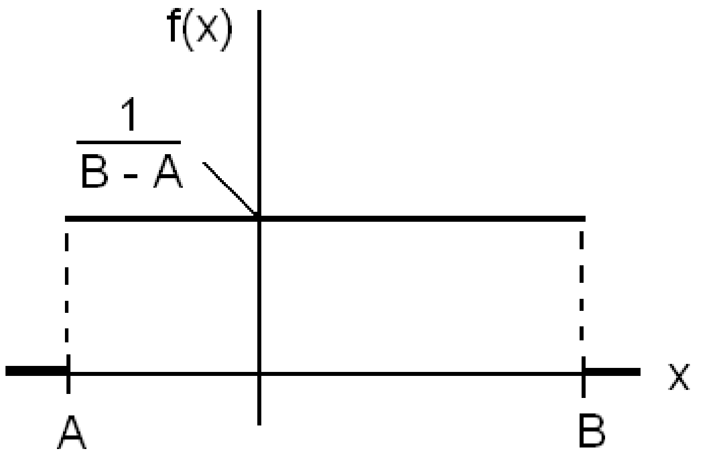
Continuous Uniform Distribution



6.1 Continuous Uniform distribution:

(Rectangular Distribution)
The probability density function of the continuous uniform random variable X on the interval [*A*, *B*] is given by:

$$f(x) = f(x; A, B) = \begin{cases} \dfrac{1}{B - A} & ; \ A \le x \le B \\ 0 & ; \ elsewhere \end{cases}$$



We write X~Uniform(*A*,*B*).

Theorem 6.1:

The mean and the variance of the continuous uniform distribution on the interval [*A*, *B*] are:

$$\mu = \frac{A + B}{2}$$

$$\sigma^2 = \frac{(B - A)^2}{12}$$

Example 6.1:

Suppose that, for a certain company, the conference time, X, has a uniform distribution on the interval [0,4] (hours).

(a) What is the probability density function of X?

(b) What is the probability that any conference lasts at least 3 hours?

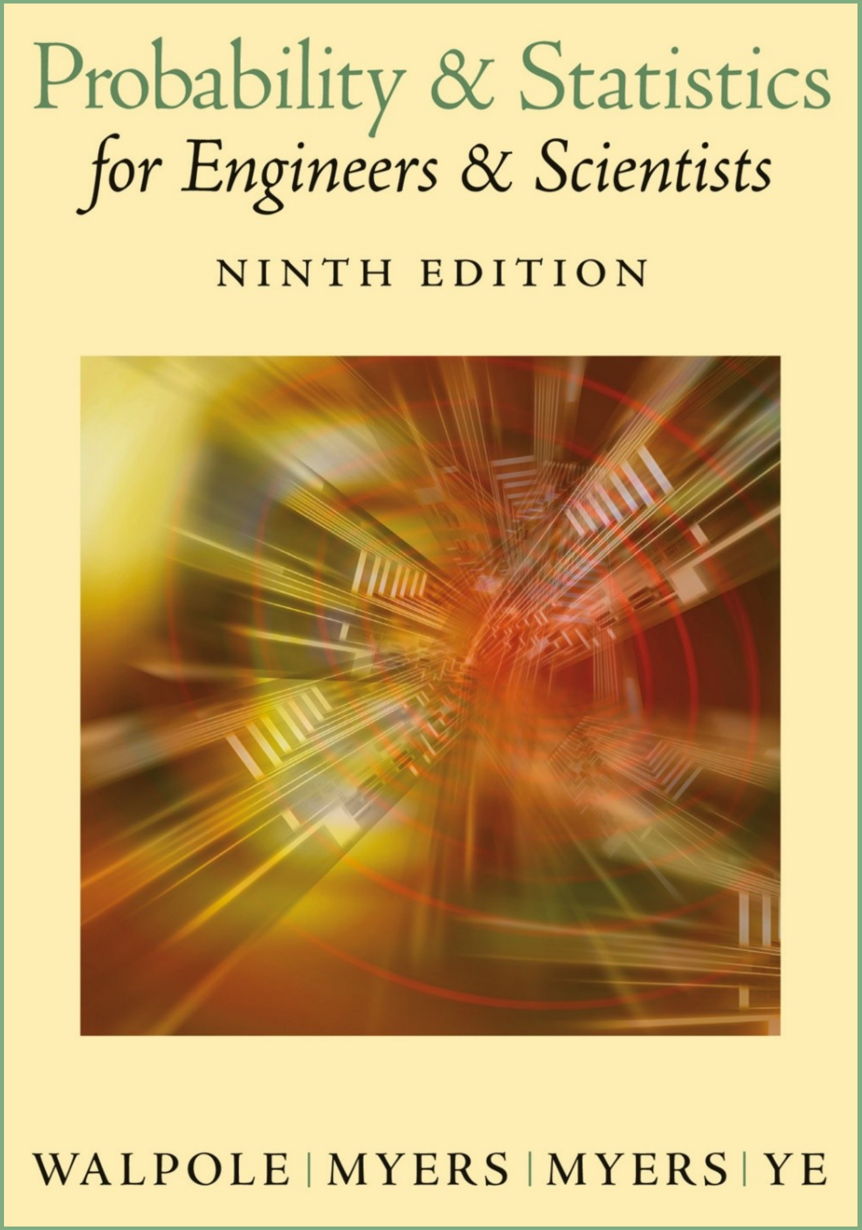
Solution:

$$(a)\ f(x) = f(x;0,4) = \begin{cases} \frac{1}{4} & ;\ 0 \le x \le 4 \\ 0 & ;\ \text{elsewhere} \end{cases}$$

$$(b)\ p(X \ge 3) = \int_3^4 f(x)\,dx = \int_3^4 \frac{1}{4}\,dx = \frac{1}{4}$$

Section 6.2

Normal Distribution



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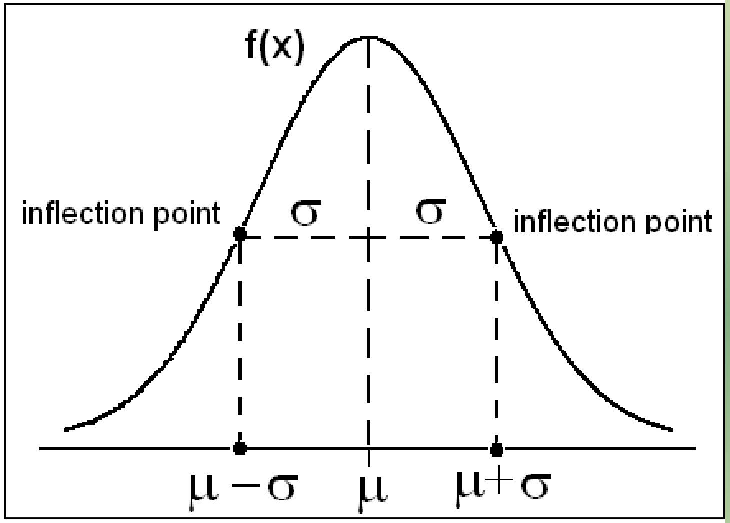
6.2 Normal Distribution:

- The normal distribution is one of the most important distributions.

- Many measurable characteristics are normally or approximately normally distributed, such as, height and weight.

- The graph of the probability density function (pdf) of a normal distribution, called the normal curve, is a bell-shaped curve.



- The pdf of the normal distribution depends on two parameters: mean = E(X)= μ and variance =Var(X) = σ^2.
- If the random variable X has a normal distribution with mean μ and variance σ^2, we write:

$$X \sim \text{Normal}(\mu,\sigma) \quad \text{or} \quad X \sim N(\mu,\sigma)$$

- The pdf of X ~ Normal(μ,σ) is given by:

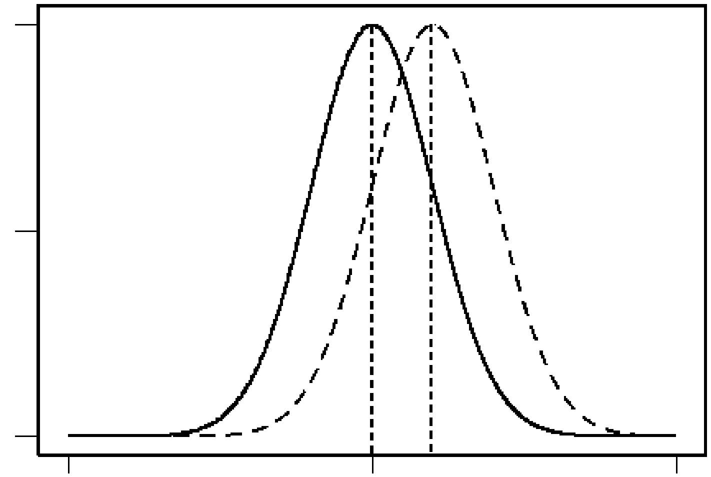
$$f(x) = n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \quad \begin{cases} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{cases}$$

- The location of the normal distribution depends on μ and its shape depends on σ.

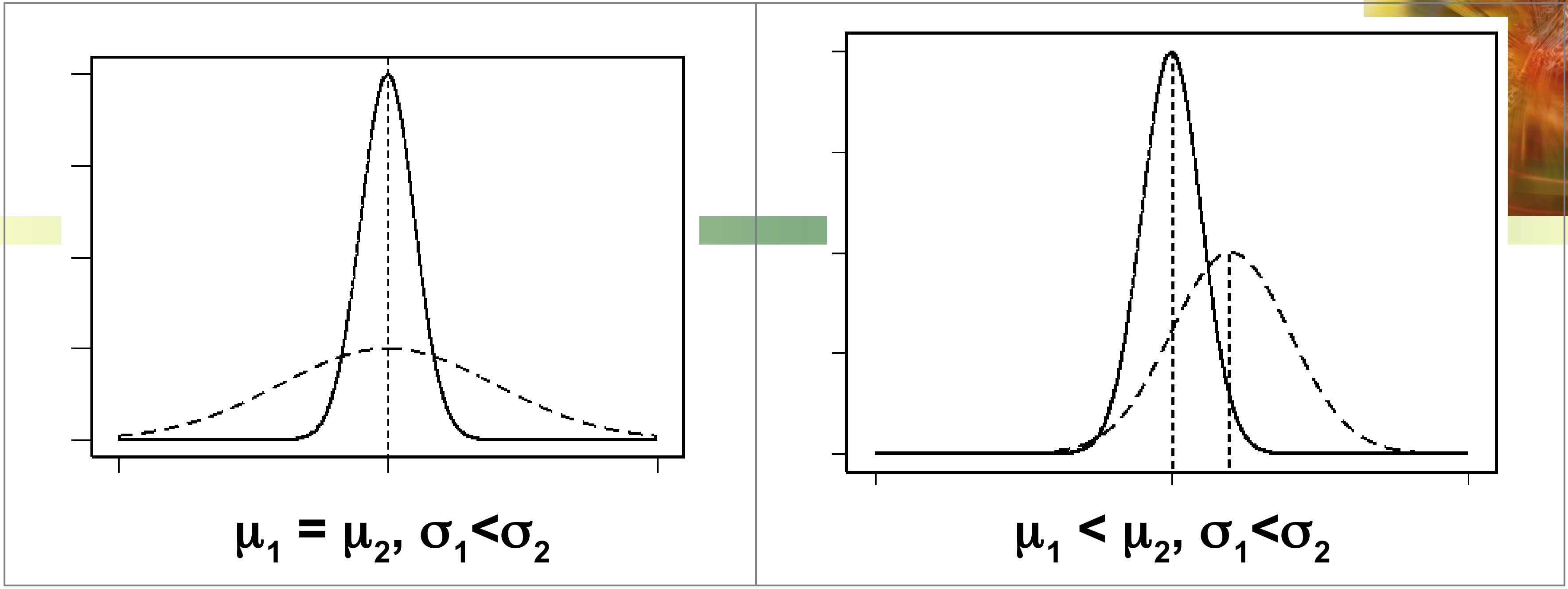
Suppose we have two normal distributions:

________ $N(\mu_1, \sigma_1)$

----------- $N(\mu_2, \sigma_2)$



$\mu_1 < \mu_2,\ \sigma_1 = \sigma_2$

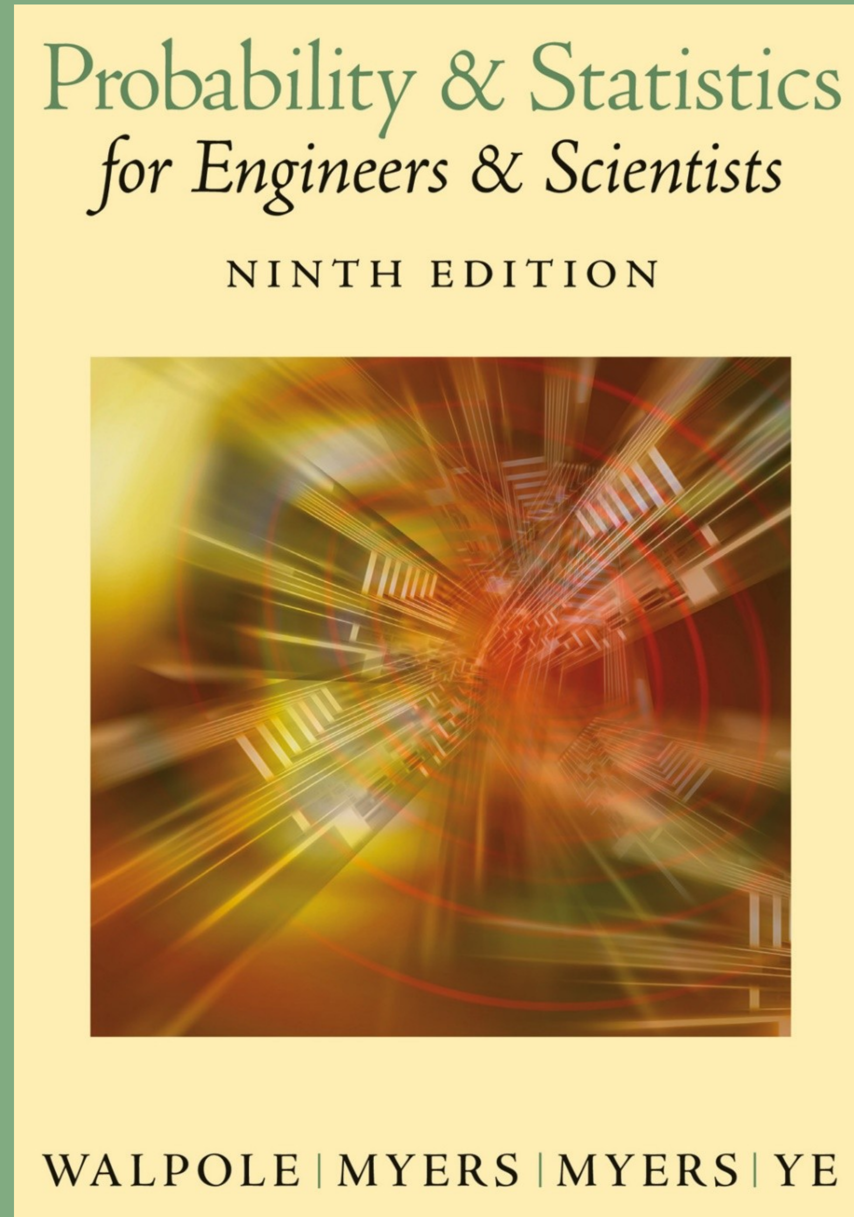


- **Some properties of the normal curve $f(x)$ of $N(\mu,\sigma)$:**

1. **$f(x)$ is symmetric about the mean μ.**
2. **$f(x)$ has two points of inflection at $x = \mu \pm \sigma$.**
3. **The total area under the curve of $f(x) = 1$.**
4. **The highest point of the curve of $f(x)$ at the mean μ.**

Section 6.3

Areas under the Normal Curve



Standard Normal Distribution

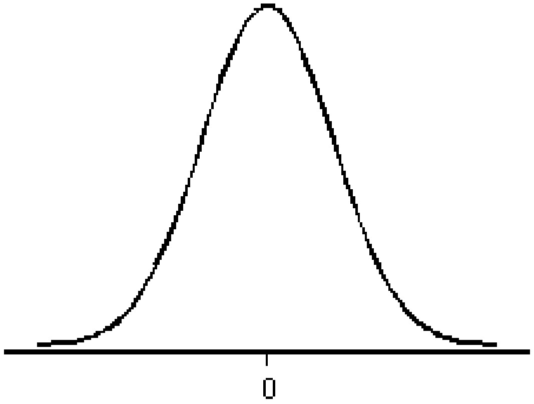
Definition 6.1:

The Standard Normal Distribution:

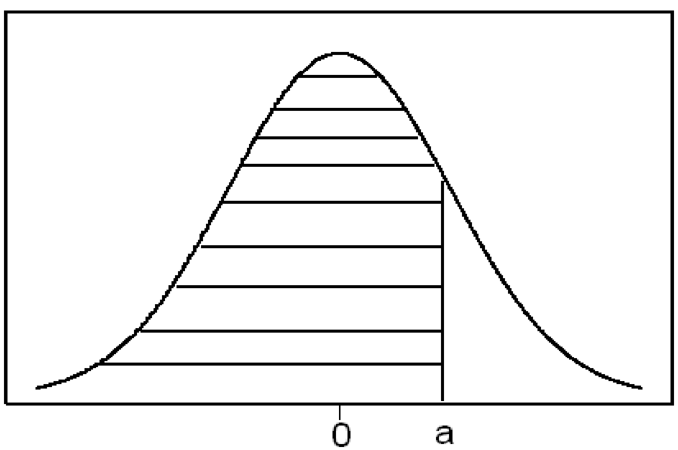
• The normal distribution with mean $\mu=0$ and variance $\sigma^2=1$ is called the standard normal distribution and is denoted by Normal(0,1) or N(0,1). If the random variable Z has the standard normal distribution, we write Z~Normal(0,1) or Z~N(0,1).

• The pdf of Z~N(0,1) is given by:

$$f(z)=n(z;0,1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$



- The standard normal distribution, Z~N(0,1), is very important because probabilities of any normal distribution can be calculated from the probabilities of the standard normal distribution.
- Probabilities of the standard normal distribution Z~N(0,1) of the form $P(Z \leq a)$ are tabulated (Table A.3, p681).



$$P(Z \leq a) = \int_{-\infty}^{a} f(z)\,dz$$

$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz$$

= from the table

We can transfer any normal distribution X~N(μ,σ) to the standard normal distribution, Z~N(0,1) by using the following result.

Result: If X~N(μ,σ), then $Z = \frac{X - \mu}{\sigma} \sim N(0,1)$

Example

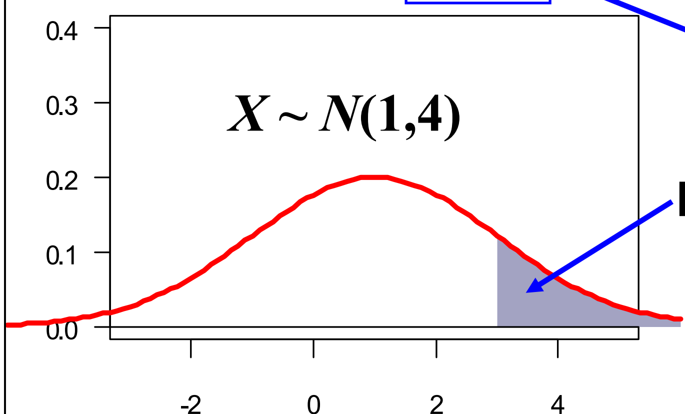
Theorem: Standardise

If $X \sim N(\mu,\sigma^2)$ **then** $\dfrac{X-\mu}{\sigma} \sim N(0,1)$

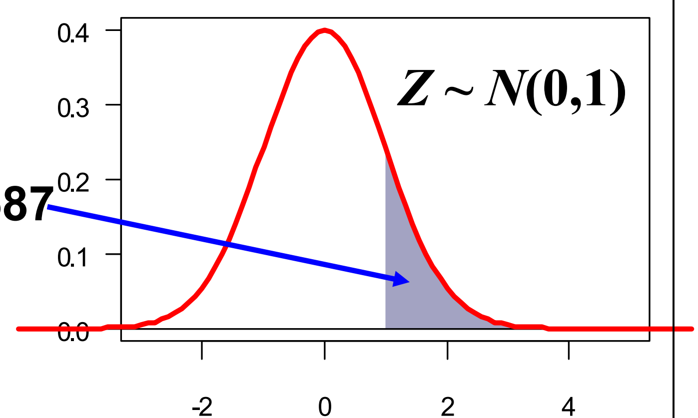
Example cont.: $X \sim N(1,4)$**. What is** $P(X>3)$ **?**

$$P(X>3) = P\left(\boxed{\frac{X-1}{2}} > \frac{3-1}{2}\right) = P(Z>1) = 1 - P(Z \le 1) = 0.1587$$

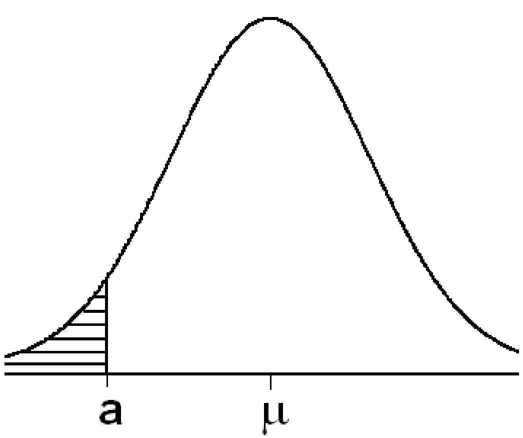
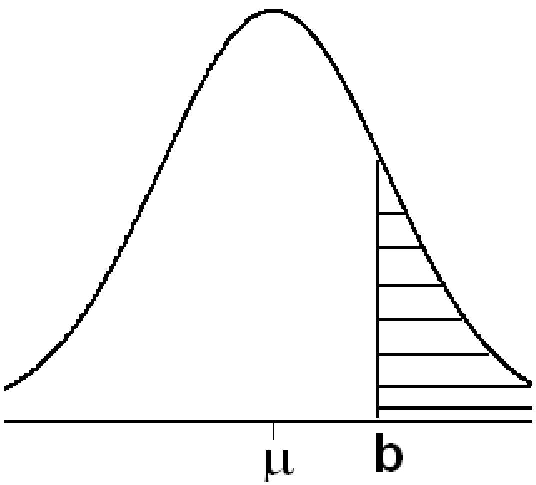
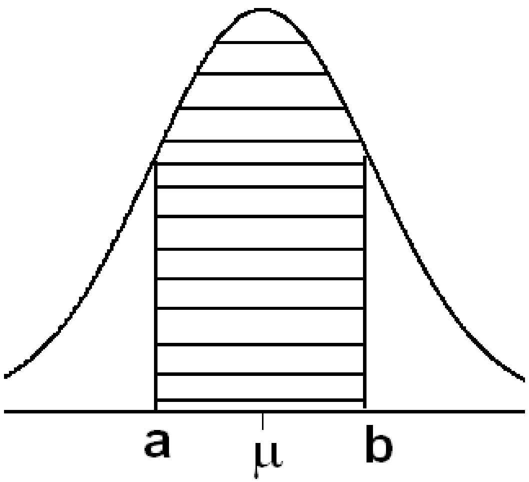
$Z \sim N(0,1)$



Equal areas = 0.1587

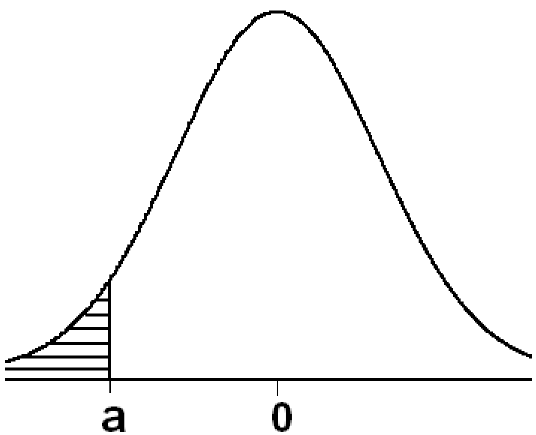


Areas Under the Normal Curve of X~N(μ,σ)
The probabilities of the normal distribution N(μ,σ) depends on μ and σ.

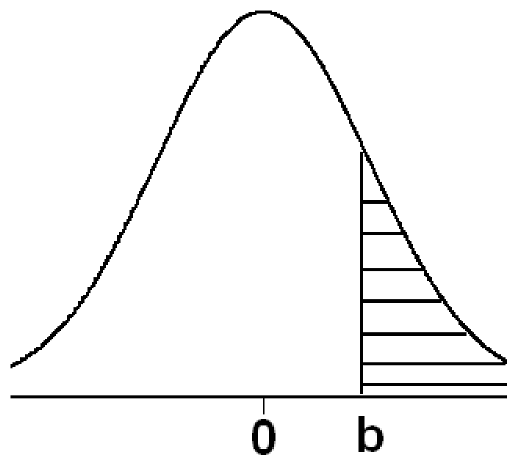
		
$P(X < a) = \int_{-\infty}^{a} f(x)\,dx$	$P(X > b) = \int_{b}^{\infty} f(x)\,dx$	$P(a < X < b) = \int_{a}^{b} f(x)\,dx$

Probabilities of Z~N(0,1):

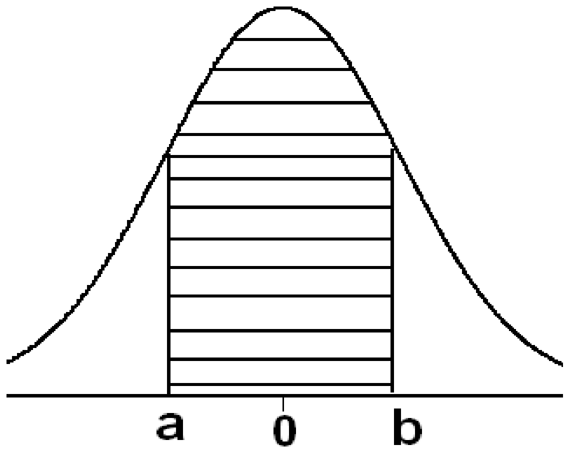
Suppose Z ~ N(0,1).



$P(Z \le a)$ = From Table (A.3)



$P(Z \ge b) = 1 - P(Z \le b)$

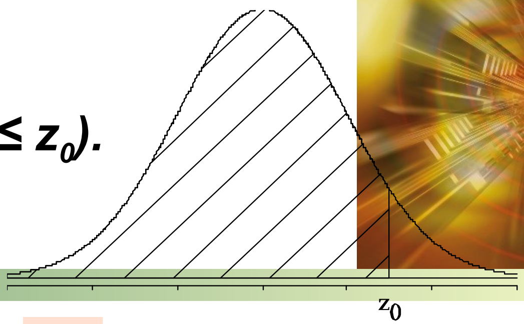


$P(a \le Z \le b) =$

$P(Z \le b) - P(Z \le a)$

Note: $P(Z=a)=0$ for every a .

The values listed in Table I of the Appendix are *P(Z ≤ z_0)*.



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	[illegible]	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	[illegible]	[illegible]	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	[illegible]	[illegible]	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	[illegible]	[illegible]	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	[illegible]	[illegible]	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	[illegible]	[illegible]	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	[illegible]	[illegible]	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	[illegible]	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

.9750 is the area under *N*(0,1) left of z_0 = 1.96

Thus, *P(Z ≤ 1.96)* = 0.9750

Example:

Suppose $Z \sim N(0,1)$.

(1) $P(Z \leq 1.50) = 0.9332$

(2) $P(Z \geq 0.98) = 1 - P(Z \leq 0.98)$
$= 1 - 0.8365$
$= 0.1635$

(3)
$P(-1.33 \leq Z \leq 2.42)$
$= P(Z \leq 2.42) - P(Z \leq -1.33)$
$= 0.9922 - 0.0918$
$= 0.9004$

(4) $P(Z \leq 0) = P(Z \geq 0) = 0.5$

Z	0.00	0.01	…
:	⋮		
1.5 ⇒	0.9332		
:			

Z	0.00	…	0.08
:	:	:	⋮
:	…	…	⋮
0.9⇒	⇒	⇒	0.8365

Z	…	0.02	0.03
:	:	⋮	⋮
-1.3	⇒		0.0918
:		⋮	
2.4	⇒	0.9922	

Example:

Suppose Z~N(0,1). Find the value of *k* such that
P(Z≤*k*)= 0.0207.

Solution:

.*k* = -2.04

Z	…	0.04	
:	:	□ □	
-2.0	⇐⇐	0.0207	
:			

Probabilities of X~N(μ,σ):

- ***Result:*** ***X*** **~N(μ,σ)** $\Leftrightarrow \quad Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$
- $X \le a \Leftrightarrow \dfrac{X-\mu}{\sigma} \le \dfrac{a-\mu}{\sigma} \Leftrightarrow Z \le \dfrac{a-\mu}{\sigma}$

$$1)\ P(X \le a) = P\left(Z \le \frac{a-\mu}{\sigma}\right)$$

$$2)\ P(X \ge a) = 1 - P(X \le a) = 1 - P\left(Z \le \frac{a-\mu}{\sigma}\right)$$

$$3)\ P(a \le X \le b) = P(X \le b) - P(X \le a) = P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$$

4) $P(X=a)=0$ for every a.

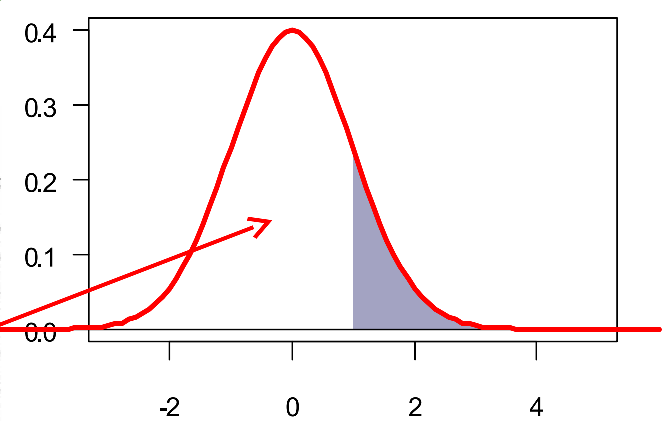
5) $P(X\le\mu) = P(X\ge\mu)=0.5$

Example 6.4: **Reading assignment**
Example 6.5: **Reading assignment**
Example 6.6: **Reading assignment**

Normal distribution
Standard normal distribution

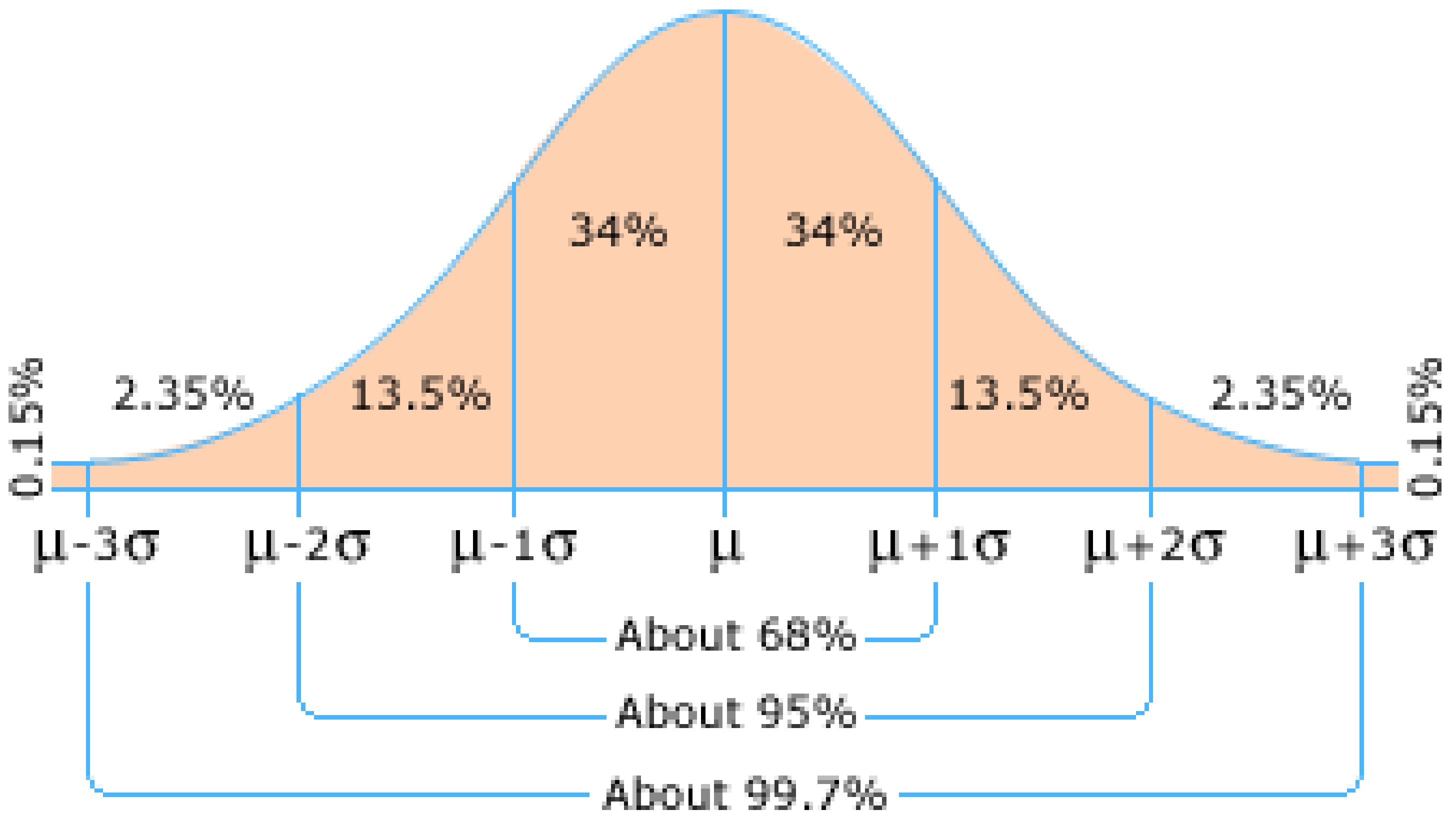
Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06			
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0		
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0		
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0		
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0		
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0		
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	[illegible]	[illegible]	[illegible]
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	$\boxed{0.8413}$	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319



$$P(Z \leq 1) = 0.8413 \quad \Rightarrow \quad P(X > 3) = 1 - P(Z \leq 1) = 1 - 0.8413 = 0.1587$$

The Empirical Rule for any $N(\mu,\sigma)$



Example:

Suppose that the hemoglobin level for healthy adults males has a normal distribution with mean $\mu=16$ and variance $\sigma^2=0.81$ (standard deviation $\sigma=0.9$).

(a) Find the probability that a randomly chosen healthy adult male has hemoglobin level less than 14.

(b) What is the percentage of healthy adult males who have hemoglobin level less than 14?

Solution:

Let ***X*** **= the hemoglobin level for a healthy adult male**

X **~ N(μ,σ)= N(16, 0.9).**

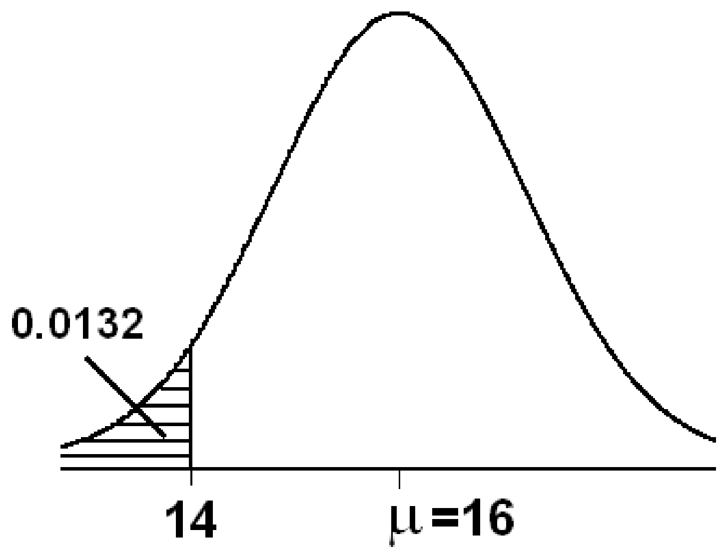
(a) $$P(X \le 14) = P\left(Z \le \frac{14-\mu}{\sigma}\right) = P\left(Z \le \frac{14-16}{0.9}\right)$$

$$= P(Z \le -2.22) = 0.0132$$

(b) The percentage of healthy adult males who have hemoglobin level less than 14 is

$$P(X \leq 14) \times 100\% = 0.01320\times$$

Therefore, 1.32% of healthy adult males have h



Example:

Suppose that the birth weight of American babies has a normal distribution with mean μ=3.4 and standard deviation σ=0.35.

(a) Find the probability that a randomly chosen American baby has a birth weight between 3.0 and 4.0 kg.

(b) What is the percentage of American babies who have a birth weight between 3.0 and 4.0 kg?

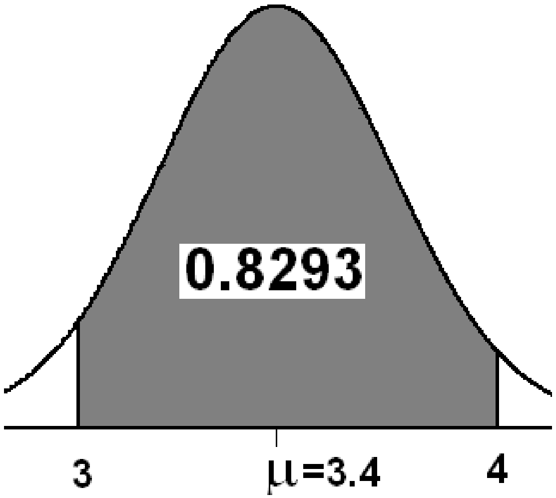
Solution:

X = birth weight of a American baby

$\mu = 3.4 \quad \sigma = 0.35 \quad (\sigma^2 = 0.35^2 = 0.1225)$

$X \sim N(3.4, 0.35)$

(a) $P(3.0<X<4.0)=P(X<4.0)-P(X<3.0)$



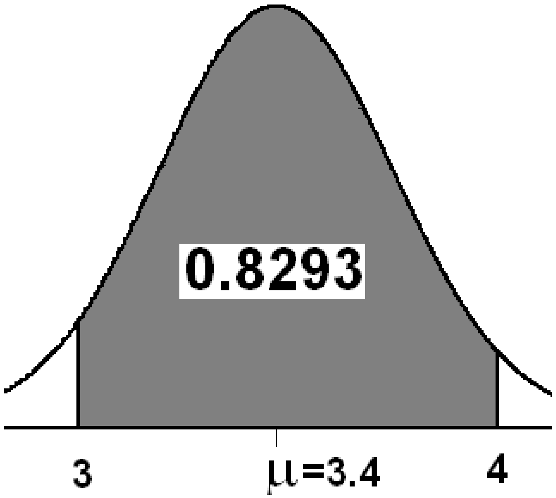
$$= P\left(Z \le \frac{4.0-\mu}{\sigma}\right) - P\left(Z \le \frac{3.0-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{4.0-3.4}{0.35}\right) - P\left(Z \le \frac{3.0-3.4}{0.35}\right)$$

$= P(Z \le 1.71) - P(Z \le -1.14)$

$= 0.9564 - 0.1271$

$= 0.8293$



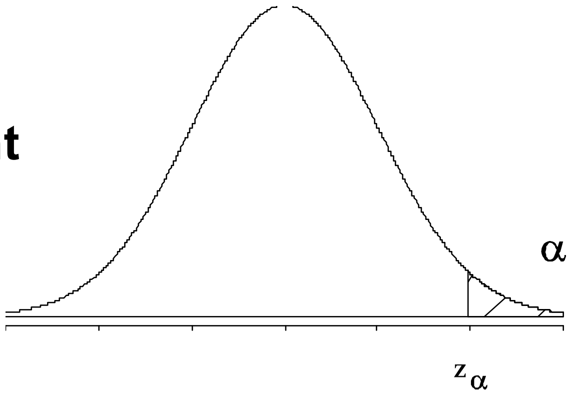
(b) **The percentage of american babies who have a birth weight between 3.0 and 4.0 kg is**

$$P(3.0<X<4.0) \times 100\% = 0.8293 \times 100\% = 82.93\%$$

Standard Score (z-score)

We often need to use the *Z*-value associated with specific *<u>“tail” areas</u>* of the standard normal distribution.

Let Z_{α} **represent the *Z*-value associated with a right hand “tail area” of .**



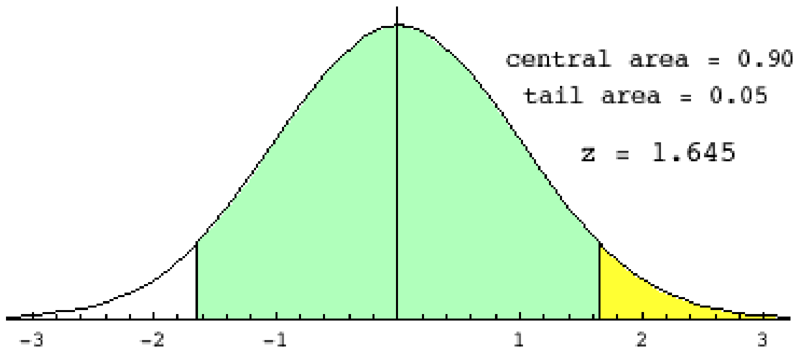
Z_{α} **is that value for *Z* such that** $P(Z > z_{\alpha}) = \alpha$

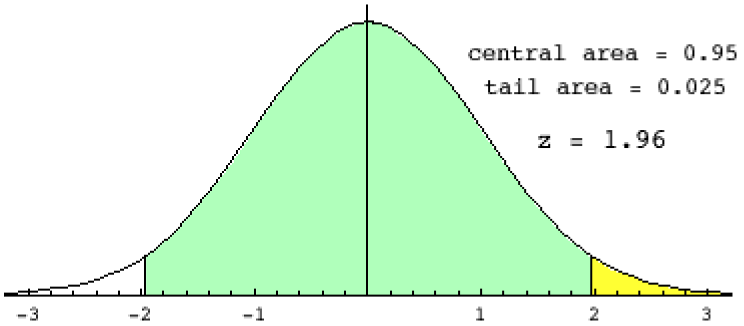
$$1.0 - P(Z \leq z_{\alpha}) = \alpha$$

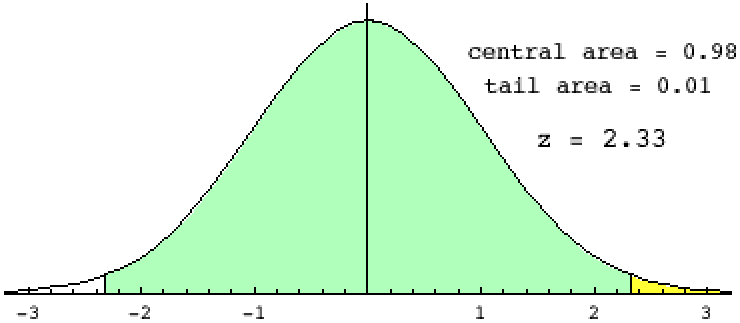
$$P(Z \leq z_{\alpha}) = 1.0 - \alpha$$

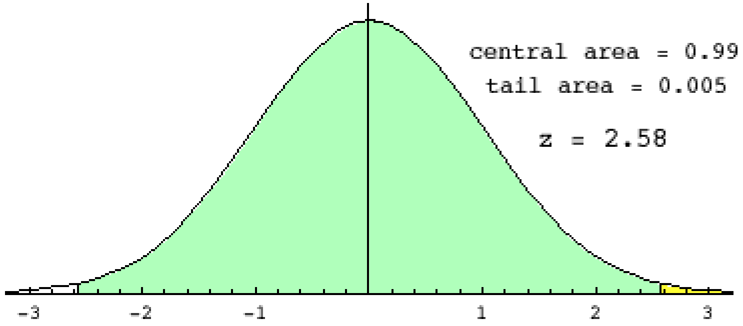
$$TableValue = 1.0 - \alpha$$

Standard Score (z-score)

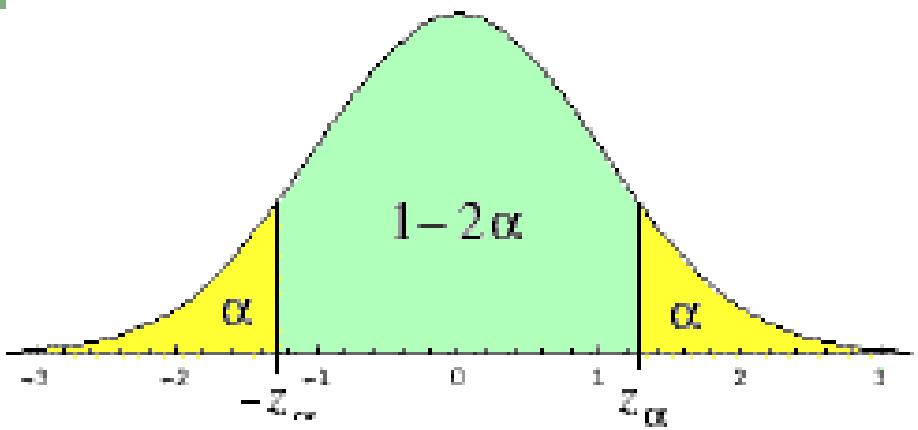








Standard Score (z-score)



α = tail area	central area = 1 – 2α	z_α
0.10	0.80	$z_{.10}$ = **1.28**
0.05	0.90	$z_{.05}$ = **1.645**
0.025	0.95	$z_{.025}$ = **1.96**
0.01	0.98	$z_{.01}$ = **2.33**
0.005	0.99	$z_{.005}$ = **2.58**

Notation:

$P(Z \geq Z_{\alpha}) = \alpha$

Result:

$Z_{\alpha} = -Z_{1-\alpha}$

Example:

$Z \sim N(0,1)$

$P(Z \geq Z_{0.025}) = 0.025$

$P(Z \geq Z_{0.95}) = 0.95$

$P(Z \geq Z_{0.90}) = 0.90$

Example:

$Z \sim N(0,1)$

$Z_{0.025} = 1.96$

$Z_{0.95} = -1.645$

$Z_{0.90} = -1.285$

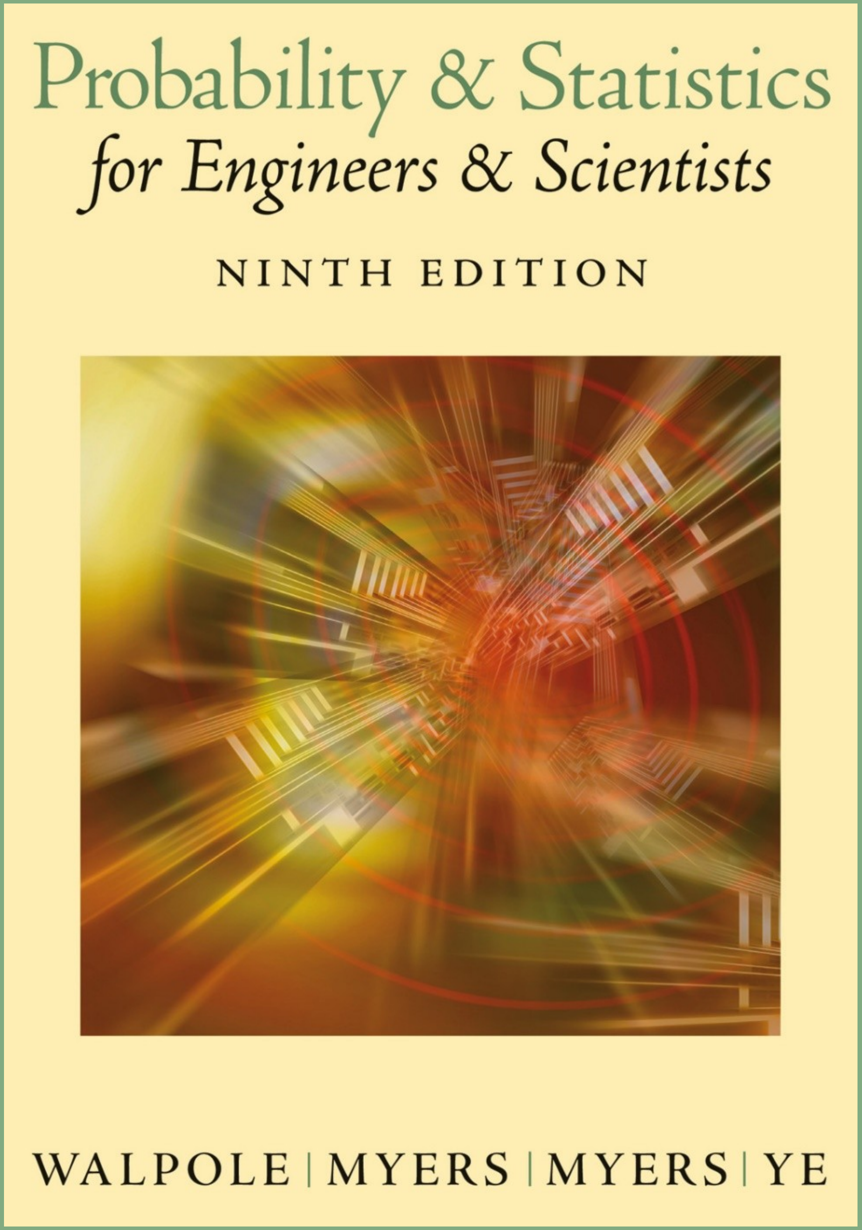
Z	…	0.06	
:	:	⋮ ⋮	
1.9	⇐⇐	0.975	

$P(Z \geq Z_{0.025}) = 0.025$

$Z_{0.025} = 1.96$

Section 6.4

Applications of the Normal Distribution



Example 6.9:

In an industrial process, the diameter of a ball bearing is an important component part. The buyer sets specifications on the diameter to be 3.00±0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that, in the process, the diameter of a ball bearing has a normal distribution with mean 3.00 cm and standard deviation 0.005 cm. On the average, how many manufactured ball bearings will be scrapped?

Example 6.7: **Reading assignment**

Example 6.8: **Reading assignment**

Solution:

μ=3.00

σ=0.005

X=diameter

X~N(3.00, 0.005)

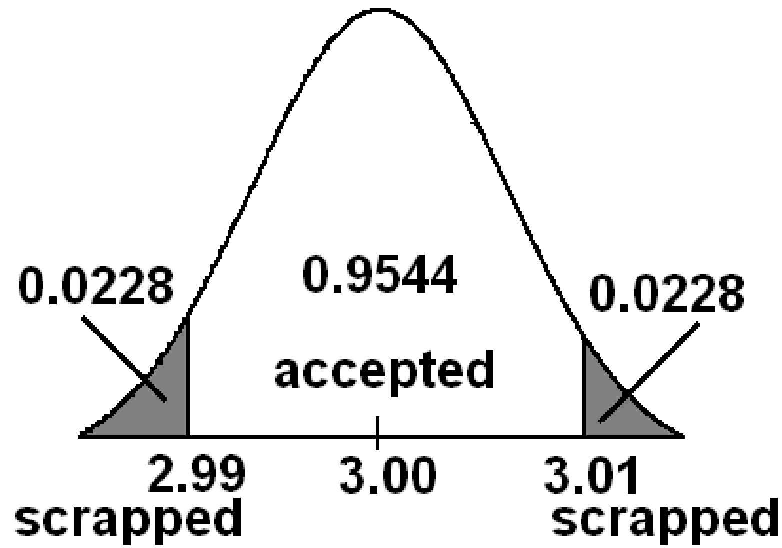
The specification limits are:

3.00±0.01

x_1=Lower limit=3.00-0.01=2.99

x_2=Upper limit=3.00+0.01=3.01

$P(x_1<X< x_2)$=P(2.99<X<3.01)=P(X<3



$$=P\left(Z \le \frac{3.01-\mu}{\sigma}\right) - P\left(Z \le \frac{2.99-\mu}{\sigma}\right)$$

$$=P\left(Z \le \frac{3.01-3.00}{0.005}\right) - P\left(Z \le \frac{2.99-3.00}{0.005}\right)$$

$= P(Z \leq 2.00) - P(Z \leq -2.00)$

$= 0.9772 - 0.0228$

$= 0.9544$

Therefore, on the average, 95.44% of manufactured ball bearings will be accepted and 4.56% will be scrapped?

Example 6.10:

Gauges are use to reject all components where a certain dimension is not within the specifications 1.50±*d*. It is known that this measurement is normally distributed with mean 1.50 and standard deviation 0.20. Determine the value *d* such that the specifications cover 95% of the measurements.

Solution:

μ=1.5
σ=0.20
X= measurement
X~N(1.5, 0.20)

The specification limits are:
1.5±d
x_1=Lower limit=1.5-d
x_2=Upper limit=1.5+d
$P(X > 1.5+d) = 0.025 \Leftrightarrow P(X < 1.5+d) = 0.975$
$P(X < 1.5-d) = 0.025$
$P(X < 1.5-d) = 0.025$

$$\Leftrightarrow P\left(\frac{X-\mu}{\sigma} \le \frac{(1.5-d)-\mu}{\sigma}\right) = 0.025$$

$$\Leftrightarrow P\left(Z \le \frac{(1.5-d)-\mu}{\sigma}\right) = 0.025$$

$$\Leftrightarrow P\left(Z \le \frac{(1.5-d)-1.5}{0.20}\right) = 0.025$$

$$\Leftrightarrow P\left(Z \le \frac{-d}{0.20}\right) = 0.025$$

Z	…	0.06	
:	:		
-1.9	⇐⇐	0.025	

$$P\left(Z \le \frac{-d}{0.20}\right) = 0.025$$

$$\frac{-d}{0.20} = -1.96$$

$$\text{Note}: \frac{-d}{0.20} = z_{0.025}$$

$$\Leftrightarrow \frac{-d}{0.20} = -1.96$$

$$\Leftrightarrow -d = (0.20)(-1.96)$$

$$\Leftrightarrow d = 0.392$$

The specification limits are:

x_1=Lower limit=1.5-d = 1.5 - 0.392 = 1.108

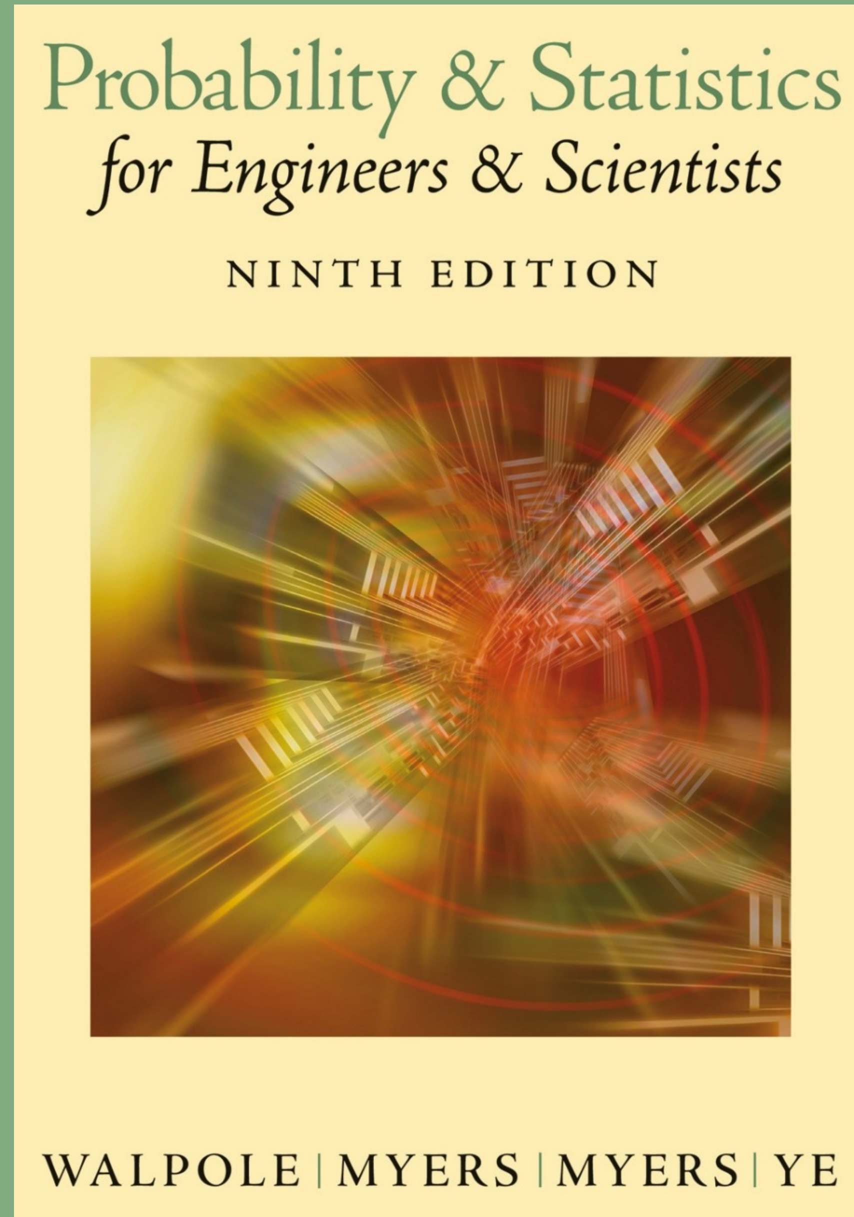
x_2=Upper limit=1.5+d=1.5+0.392= 1.892

Therefore, 95% of the measurements fall within the specifications (1.108, 1.892).

Example 6.11: **Reading assignment**
Example 6.12: **Reading assignment**
Example 6.13: **Reading assignment**
Example 6.14: **Reading assignment**

Section 6.5

Normal Approximation to the Binomial

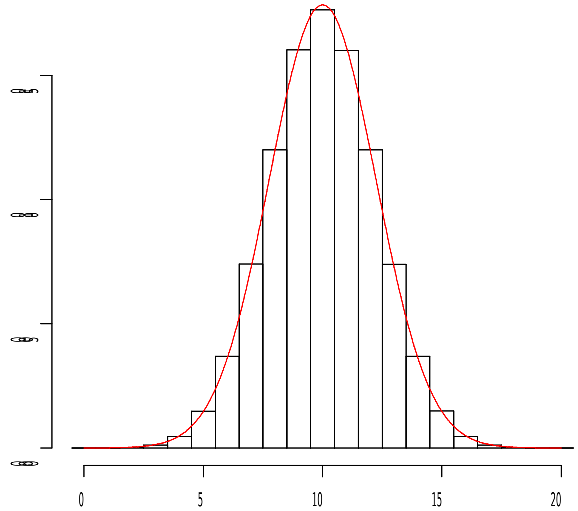


If X is binomial distributed with parameters n and p, then

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

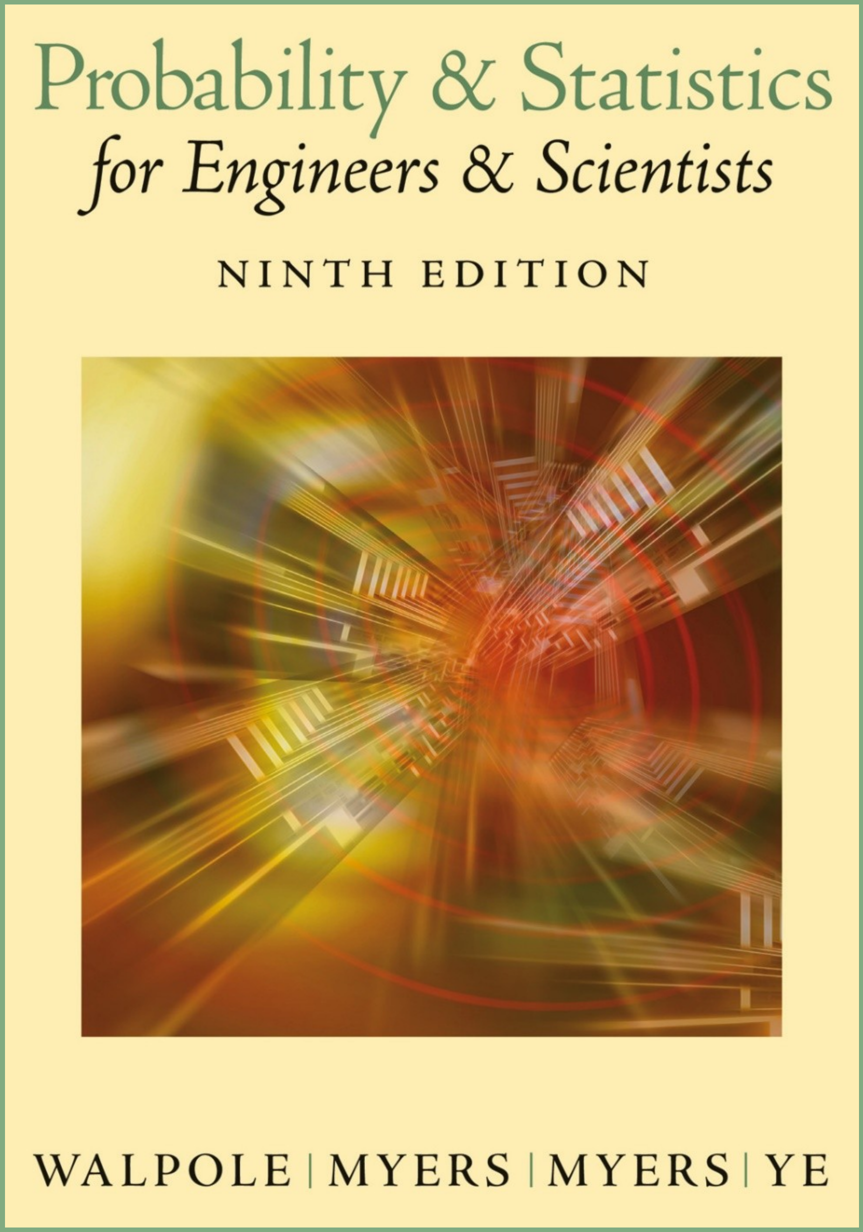
is approximately normal distributed.

Rule of thumb:
If $np>5$ and $n(1-p)>5$,
then the approximation
is good



Section 6.6

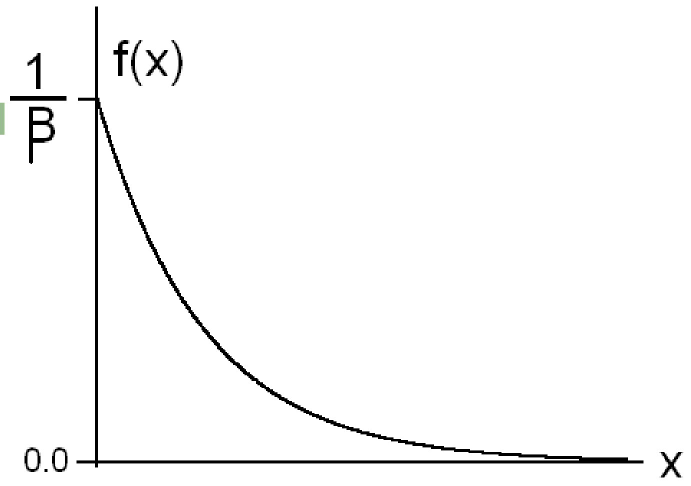
Exponential Distributions



6.6 Exponential Distribution:

Definition:

The continuous random variable X has an exponential distribution with parameter β, if its probability density function is given by:



$$f(x) = \begin{cases} \dfrac{1}{\beta}\, e^{-x/\beta} & ; x > 0 \\ 0 & ; elsewhere \end{cases}$$

and we write X~Exp(β)

Theorem:

If the random variable X has an exponential distribution with parameter β, i.e., X~Exp(β), then the mean and the variance of X are:

$$E(X) = \mu = \beta$$

$$Var(X) = \sigma^2 = \beta^2$$

Example 6.17:

Suppose that a system contains a certain type of component whose time in years to failure is given by *T*. The random variable *T* is modeled nicely by the exponential distribution with mean time to failure β=5. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Solution:

β=5

***T*~Exp(5)**

The pdf of *T* is:

$$f(t) = \begin{cases} \frac{1}{5} e^{-t/5} & ; t > 0 \\ 0 & ; elsewhere \end{cases}$$

The probability that a given component is still functioning after 8 years is given by:

$$\mathrm{P(T > 8)} = \int_{8}^{\infty} \mathrm{f(t)\ dt} = \int_{8}^{\infty} \frac{1}{5}\ \mathrm{e}^{-\mathrm{t}/5}\ \mathrm{dt} = \mathrm{e}^{-8/5} = 0.2$$

Now define the random variable:

X= number of components functioning after 8 years out of 5 components

X~ Binomial(5, 0.2) **(*n*=5, *p*= P(*T*>8)= 0.2)**

$$f(x) = P(X = x) = b(x;5,0.2) = \begin{cases} \binom{5}{x} 0.2^{x}\ 0.8^{5-x} & ;\ x = 0, 1, \ldots, 5 \\ 0\ ; & otherwise \end{cases}$$

The probability that at least 2 are still functioning at the end of 8 years is:

$$P(X \geq 2) = 1 - P(X<2) = 1 - [P(X=0)+P(X=1)]$$

$$= 1 - [\binom{5}{0} 0.2^{0}\ 0.8^{5-0} + \binom{5}{1} 0.2^{1}\ 0.8^{5-1}]$$

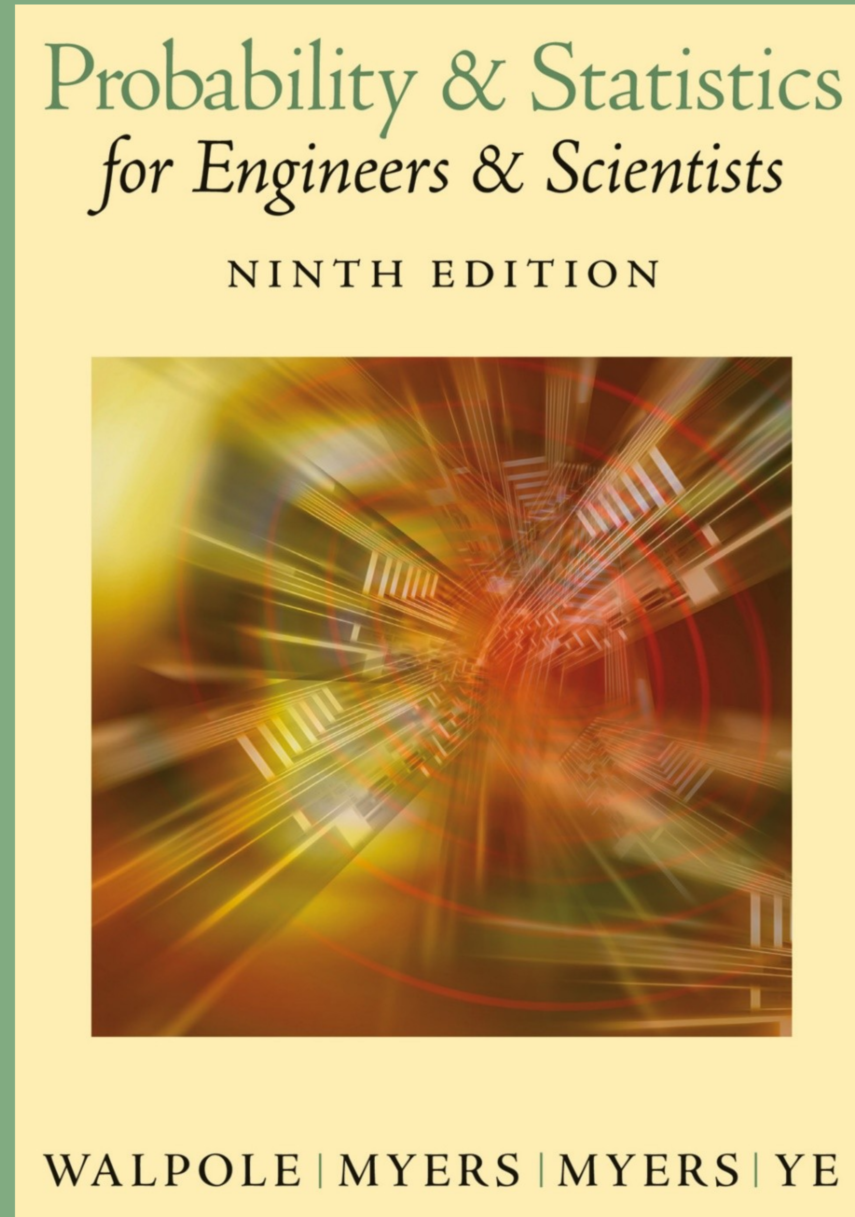
$$= 1 - [0.8^{5} + 5 \times 0.2 \times 0.8^{4}]$$

$$= 1 - 0.7373$$

$$= 0.2627$$

Section 6.6

Gamma Distributions



Gamma Function

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

$$\Gamma(\alpha+1) = \int_0^\infty y^{\alpha} e^{-y} dy \quad \text{Integrating by Parts:}$$

$$u = y^{\alpha} \Rightarrow du = \alpha y^{\alpha-1} dy$$

$$dv = e^{-y} dy \Rightarrow v = -e^{-y}$$

$$\Rightarrow \Gamma(\alpha+1) = \int_0^\infty y^{\alpha} e^{-y} dy = uv - \int v\,du = -y^{\alpha} e^{-y}\Big|_0^\infty + \int_0^\infty \alpha y^{\alpha-1} e^{-y} dy =$$

$$= -0 - (-0) + \alpha \int_0^\infty y^{\alpha-1} e^{-y} dy = \alpha\Gamma(\alpha) \quad \text{(Recursive Property)}$$

Note that if α is an integer, $\Gamma(\alpha) = (\alpha - 1)!$

Consider the integral: $\int_0^\infty y^{\alpha-1} e^{-y/\beta} dy$ Letting $x = y/\beta$:

$$\Rightarrow y = x\beta \Rightarrow dy = \beta dx$$

$$\Rightarrow \int_0^\infty y^{\alpha-1} e^{-y/\beta} dy = \int_0^\infty (x\beta)^{\alpha-1} e^{-x} \beta dx = \beta^{\alpha} \int_0^\infty x^{\alpha-1} e^{-x} dx = \beta^{\alpha}\Gamma(\alpha)$$

Gamma Distribution

- Family of Right-Skewed Distributions
- Random Variable can take on positive values only
- Used to model many biological and economic characteristics
- Can take on many different shapes to match empirical data

$$f(y) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-y/\beta} & y > 0, \alpha, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

Gamma Distribution - Expectations

$$E(Y)=\int_0^\infty y\left[\frac{1}{\Gamma(\alpha)\beta^\alpha}y^{\alpha-1}e^{-y/\beta}\right]dy=\frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty y^{\alpha}e^{-y/\beta}dy=$$

$$=\frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty y^{(\alpha+1)-1}e^{-y/\beta}dy=\frac{1}{\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha+1)\beta^{\alpha+1}=\frac{\Gamma(\alpha+1)\beta}{\Gamma(\alpha)}=$$

$$=\frac{\alpha\Gamma(\alpha)\beta}{\Gamma(\alpha)}=\alpha\beta$$

$$E\left(Y^2\right)=\int_0^\infty y^2\left[\frac{1}{\Gamma(\alpha)\beta^\alpha}y^{\alpha-1}e^{-y/\beta}\right]dy=\frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty y^{\alpha+1}e^{-y/\beta}dy=$$

$$=\frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty y^{(\alpha+2)-1}e^{-y/\beta}dy=\frac{1}{\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha+2)\beta^{\alpha+2}=\frac{\Gamma(\alpha+2)\beta^2}{\Gamma(\alpha)}=$$

$$=\frac{(\alpha+1)\Gamma(\alpha+1)\beta^2}{\Gamma(\alpha)}=\frac{(\alpha+1)\alpha\Gamma(\alpha)\beta^2}{\Gamma(\alpha)}=(\alpha+1)\alpha\beta$$

$$\Rightarrow V(Y)=E\left(Y^2\right)-\left[E(Y)\right]^2=(\alpha+1)\alpha\beta-(\alpha\beta)^2=\alpha^2\beta^2+\alpha\beta^2-\alpha^2\beta^2=\alpha\beta^2$$

$$\Rightarrow \sigma=\beta\sqrt{\alpha}$$

Family of Gamma Distributions

- The gamma distribution defines a family of which other distributions are special cases.
- Important applications in waiting time and reliability analysis.
- Special cases of the Gamma Distribution
  - Exponential Distribution when $\alpha = 1$
  - Chi-squared Distribution when

$$\alpha = \frac{\nu}{2} \; and \; \beta = 2,$$

Where ν is a positive integer

Standard Gamma Distribution

The standard gamma distribution has $\beta = 1$

The probability density function of the standard Gamma distribution is:

$$f(x;\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha - 1} e^{-x} \quad \textbf{for} \quad x \geq 0$$

And is 0 otherwise

Gamma Distribution - Example

Suppose the reaction time X of a randomly selected individual to a certain stimulus has a standard gamma distribution with $\alpha = 2$ sec. Find the probability that reaction time will be

(a) between 3 and 5 seconds

(b) greater than 4 seconds

Solution

Since

$$P(3 \le X \le 5) = F(5) - F(3) = F^*(5;2) - F^*(3;2)$$

$$F^*(3;2) = \int_0^3 \frac{1}{\Gamma(2)} y e^{-y} dy = 0.801$$

$$F^*(5;2) = \int_0^5 \frac{1}{\Gamma(2)} y e^{-y} dy = 0.960$$

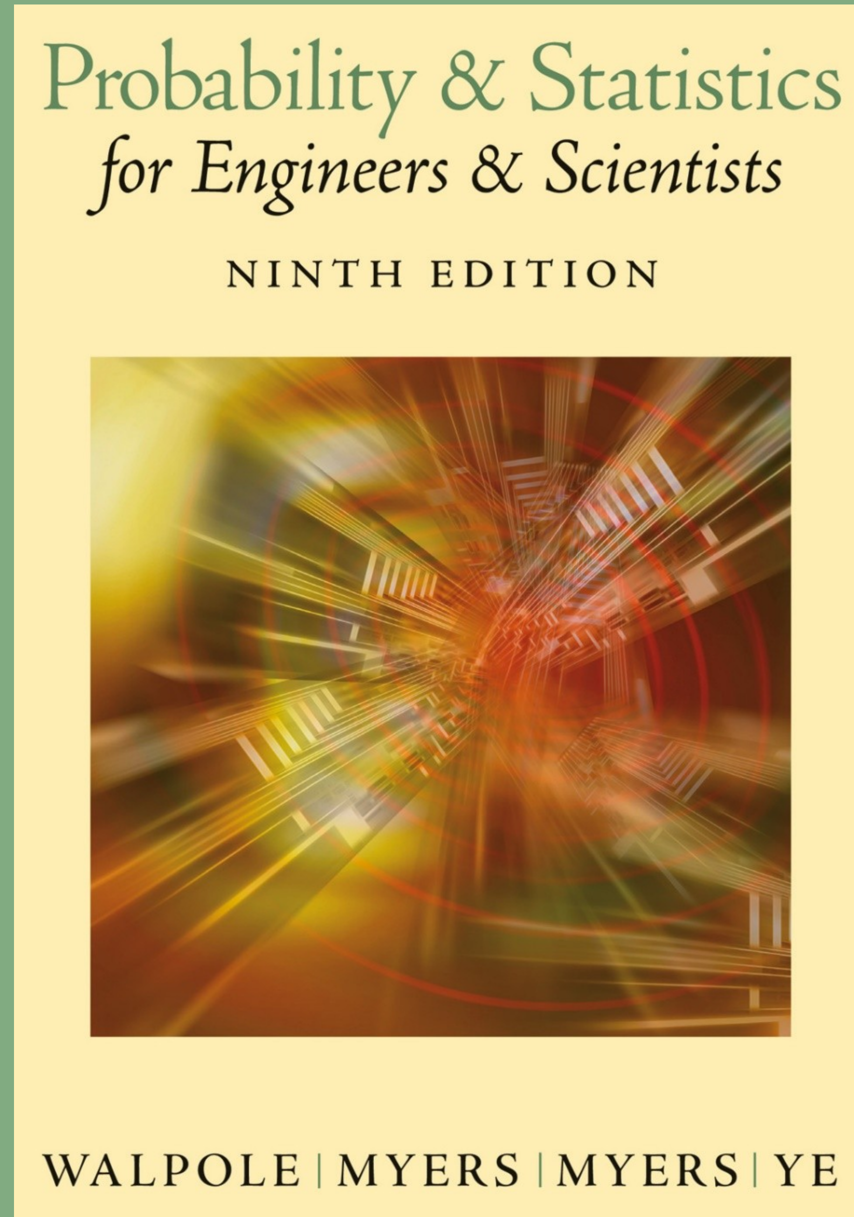
$$P(3 \le x \le 5) = 0.960 - 0.801 = 0.159$$

The probability that the reaction time is more than 4 sec is

$$P(X > 4) = 1 - P(X \le 4) = 1 - F^*(4; 2) = 1 - 0.908$$
$$= 0.092$$

Section 6.7

Chi-Squared Distributions



The χ^2 distribution
Definition

Definition: (alternative to Walpole, Myers, Myers & Ye)
If $Z_1, Z_2, \ldots, Z_n$ are independent random variables, where

$$Z_i \sim N(0,1), \qquad \text{for } i = 1,2,\ldots,n,$$

then the distribution of

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_n^2 = \sum_{i=1}^{n} Z_i^2$$

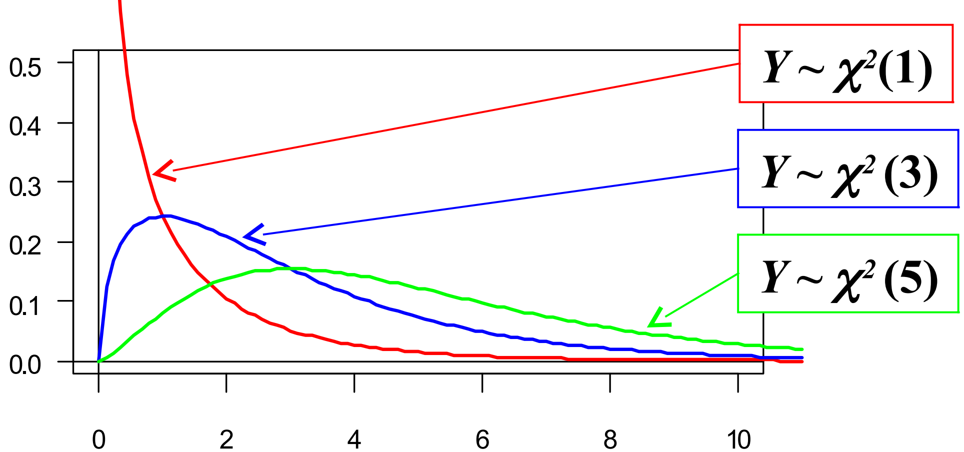
is the χ^2-distribution with n degrees of freedom.

Notation: $Y \sim \chi^2(n)$

The χ^2 distribution
Definition

Definition: A continuous random variable X follows a χ^2-distribution with n degrees of freedom if it has density function

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} \quad \text{for } x > 0$$



Antag $Y \sim \chi^2(n)$

$E(Y) = n$

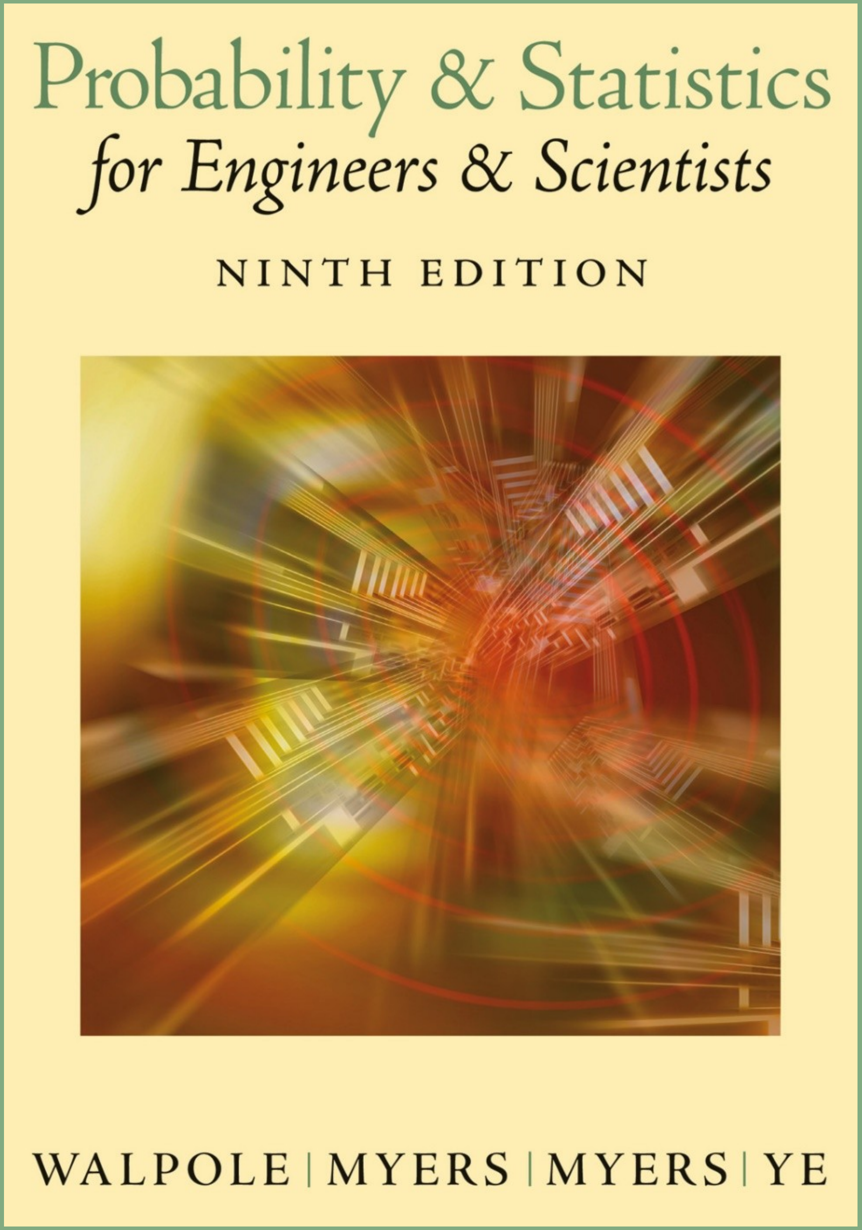
$Var(Y) = 2n$

$E(Y/n) = 1$

$Var(Y/n) = 2/n$

Section 6.7

T Distributions



PEARSON
Addison Wesley

t-distribution
Definition

Definition:

Let $Z \sim N(0,1)$ and $V \sim \chi^2(n)$ be two independent random variables. Then the distribution of

$$T = \frac{Z}{\sqrt{V/n}}$$

is called the *t*-distribution with n degrees of freedom.

Notation: $\mathrm{T} \sim t(n)$

The T probability density function

What does t look like mathematically? (You may at least recognize some resemblance to the normal distribution function…)

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

Where:

v is the degrees of freedom

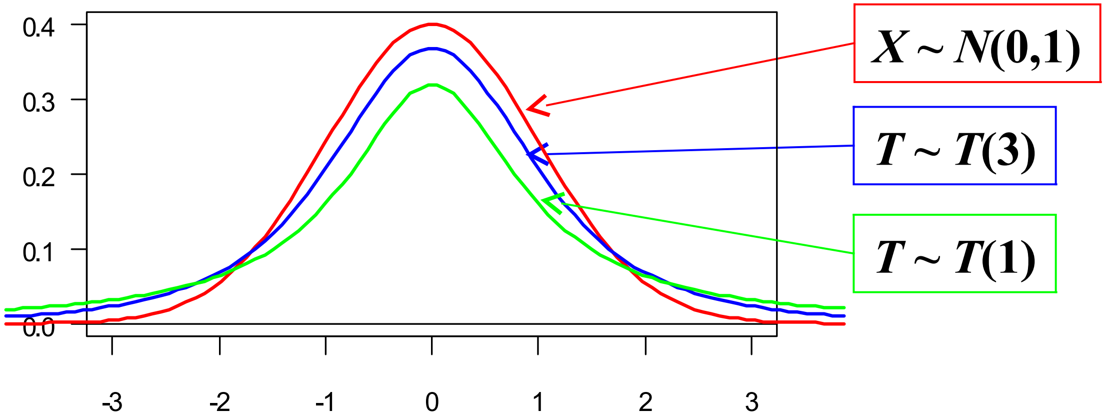
(gamma) is Γ the Gamma function

is the constant Pi (3.14...)

Degrees of Freedom

- If the sample size is n, then t is said to have n – 1 degrees of freedom.
- We use df to denote degrees of freedom.

t-distribution
Compared to standard normal



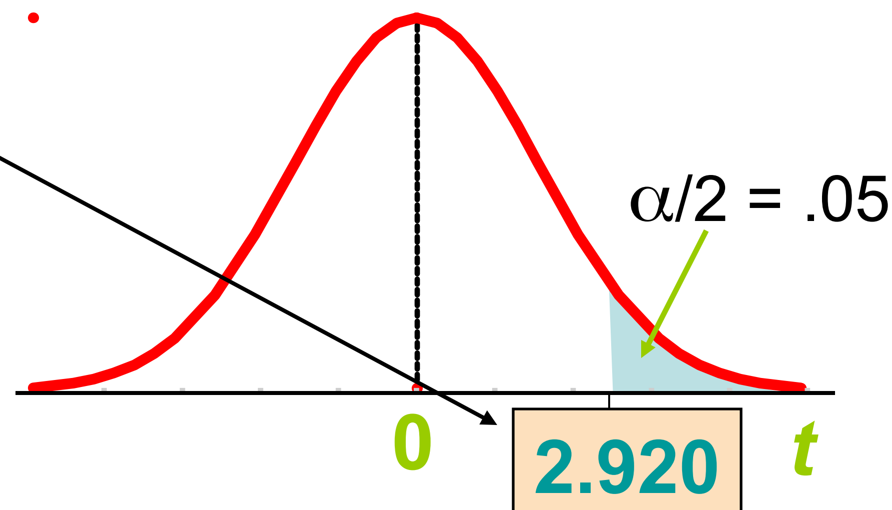
- **The *t*-distribution is symmetric around 0**
- **The *t*-distribution is more flat than the standard normal**
- **The more degrees of freedom the more the *t*-distribution looks like a standard normal**

Student's t Table

	Upper Tail Area		
df	.25	.10	**.05**
1	1.000	3.078	6.314
2	0.817	1.886	**2.920**
3	0.765	1.638	2.353

The body of the table contains t values, not probabilities

Let: n = 3
df = n - 1 = 2
α = .10
α/2 =.05



from "Statistics for Managers" Using Microsoft® Excel 4th Edition, Prentice-Hall 2004

t distribution values

With comparison to the Z value

Confidence Level	t (10 d.f.)	t (20 d.f.)	t (30 d.f.)	Z
.80	1.372	1.325	1.310	1.28
.90	1.812	1.725	1.697	1.64
.95	2.228	2.086	2.042	1.96
.99	3.169	2.845	2.750	2.58

Note: $t \longrightarrow Z$ as n increases

F-distribution
Definition

Definition:

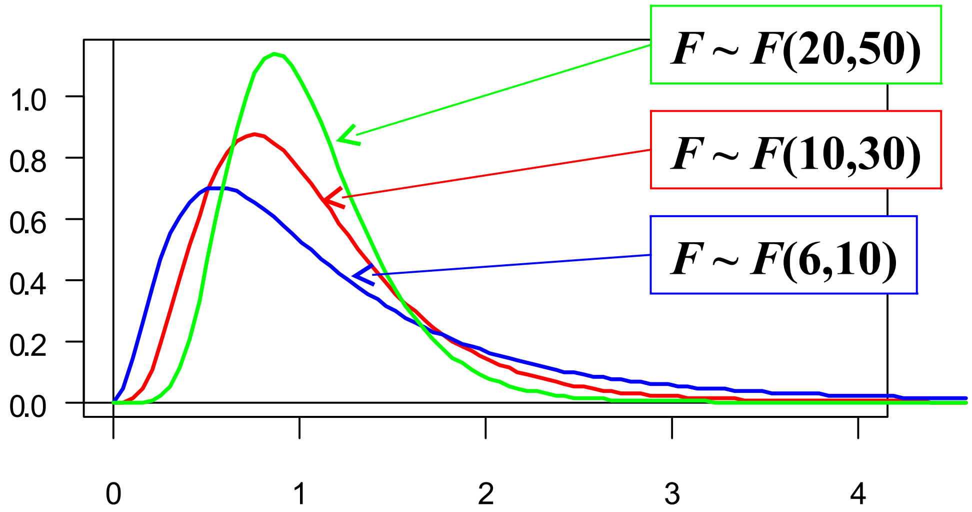
Let $U \sim \chi^2(n_1)$ and $V \sim \chi^2(n_2)$ be two independent random variables. Then the distribution of

$$F = \frac{U/n_1}{V/n_2}$$

is called the F-distribution with n_1 and n_2 degrees of freedom.

Notation: $F \sim F(n_1, n_2)$ Critical values: Table A.6

F-distribution
Example



Section 6.11

Review