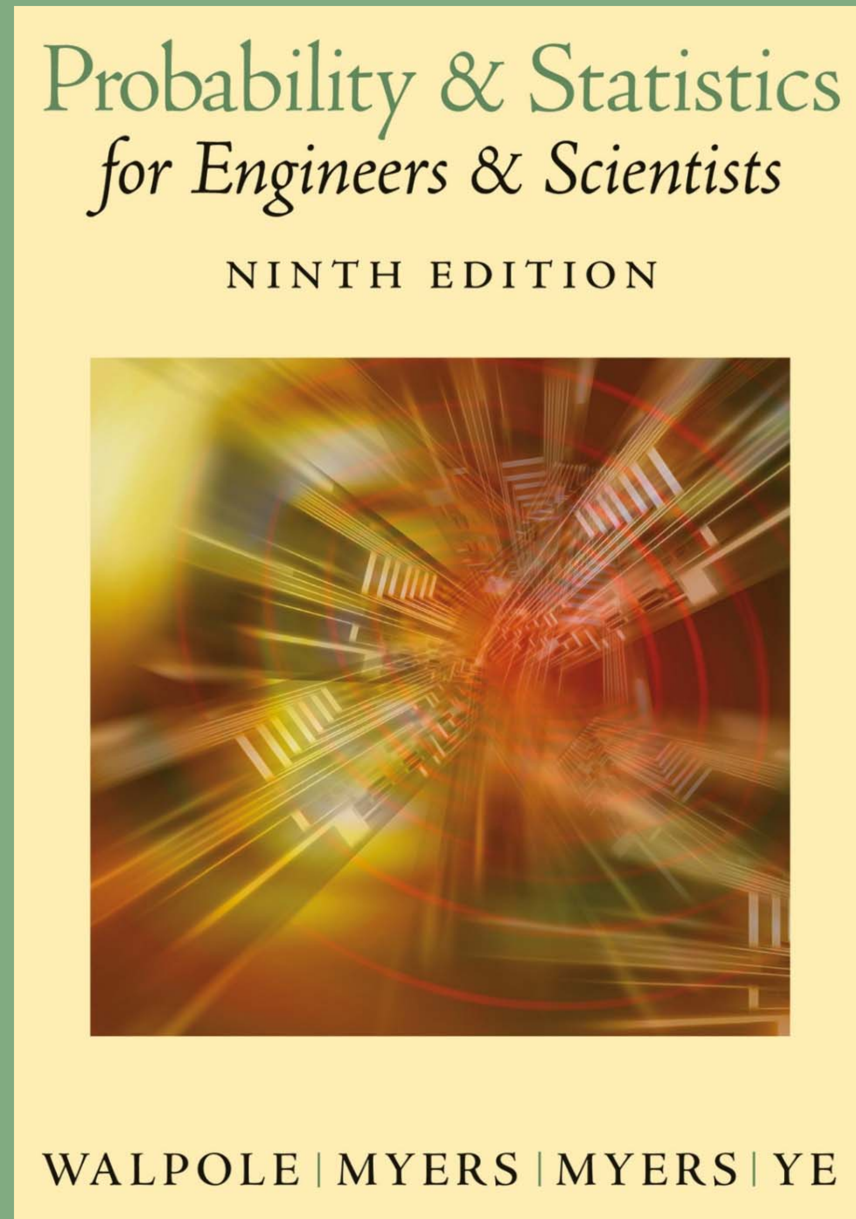
Chapter 3

Random Variables and Probability Distributions



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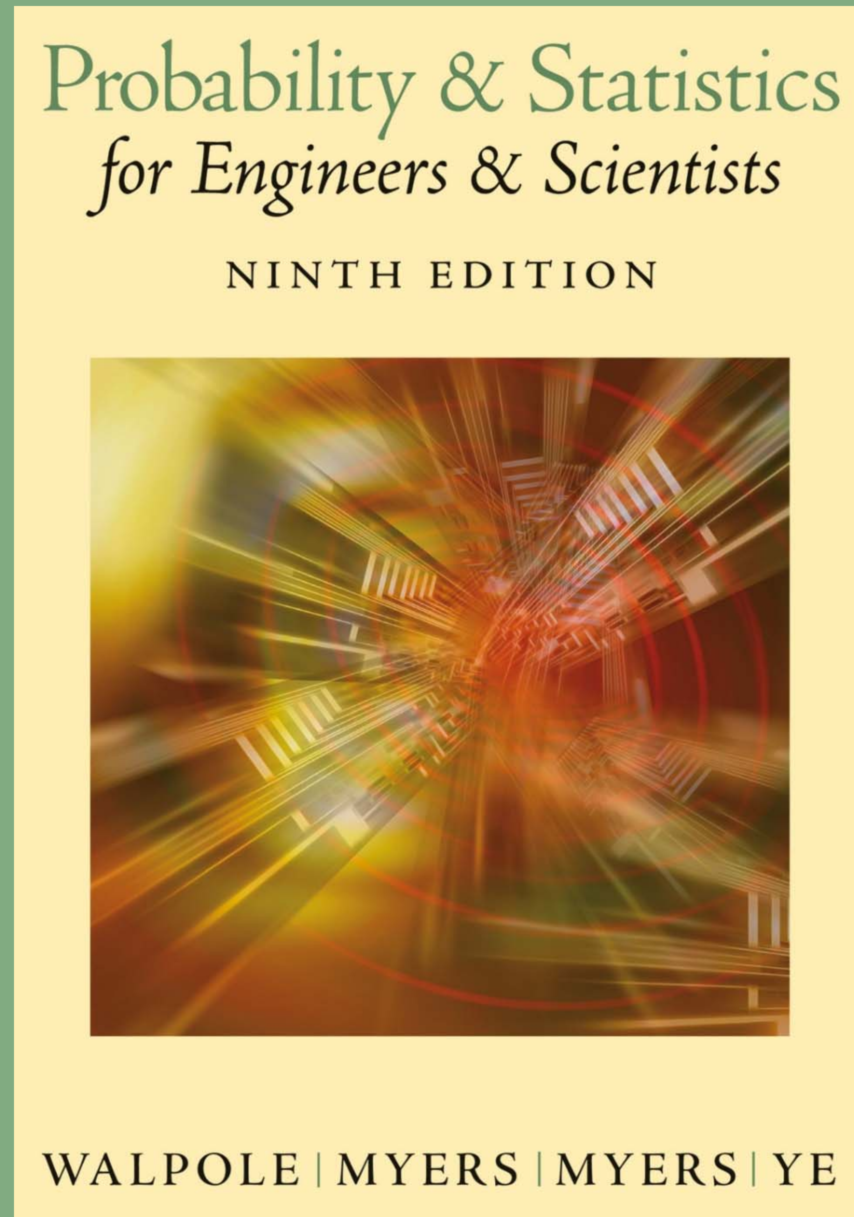
Section 3.1

Concept of a Random Variable



Concept of a Random Variable:

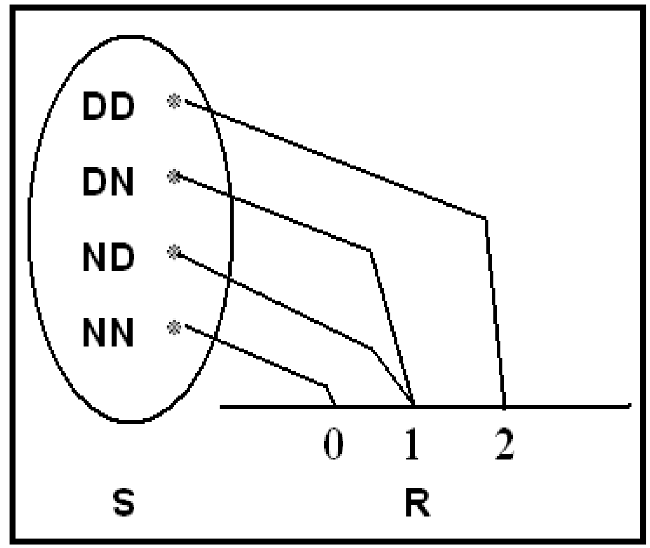
- In a statistical experiment, it is often very important to allocate numerical values to the outcomes.

Example:

- Experiment: testing two components. (D=defective, N=non-defective)
- Sample space: S={DD,DN,ND,NN}
- Let X = number of defective components when two components are tested.
- Assigned numerical values to the outcomes are:

Example

Sample point (Outcome)	Assigned Numerical Value (x)
DD	2
DN	1
ND	1
NN	0



❑Notice that, the set of all possible values of the random variable X is {0, 1, 2}.

Definition 3.1

Definition 3.1:

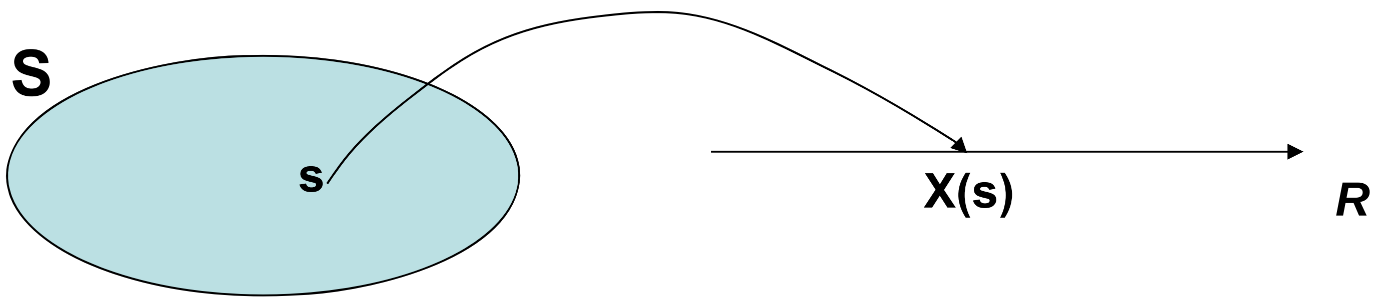
A random variable X is a function that associates each element in the sample space with a real number (i.e., $X : S \to \mathbb{R}$.)

Notation: " X " denotes the random variable .

" x " denotes a value of the random variable X.

Definition 3.1

In an experiment a number is often attached to each outcome.



Definition:

A random variable X is a function defined on S, which takes values on the real axis

$$X:\ \underbrace{S}_{\text{Sample space}} \rightarrow \underbrace{R}_{\text{Real numbers}}$$

Types of RV

· A random variable X is called a discrete random variable if its set of possible values is countable, i.e.,

.$x \in \{x_1, x_2, \ldots, x_n\}$ or $x \in \{x_1, x_2, \ldots\}$

· A random variable X is called a continuous random variable if it can take values on a continuous scale, i.e.,

.$x \in \{x: a < x < b; a, b \in R\}$

· In most practical problems:

o A discrete random variable represents count data, such as the number of defectives in a sample of *k* items.

o A continuous random variable represents measured data, such as height.

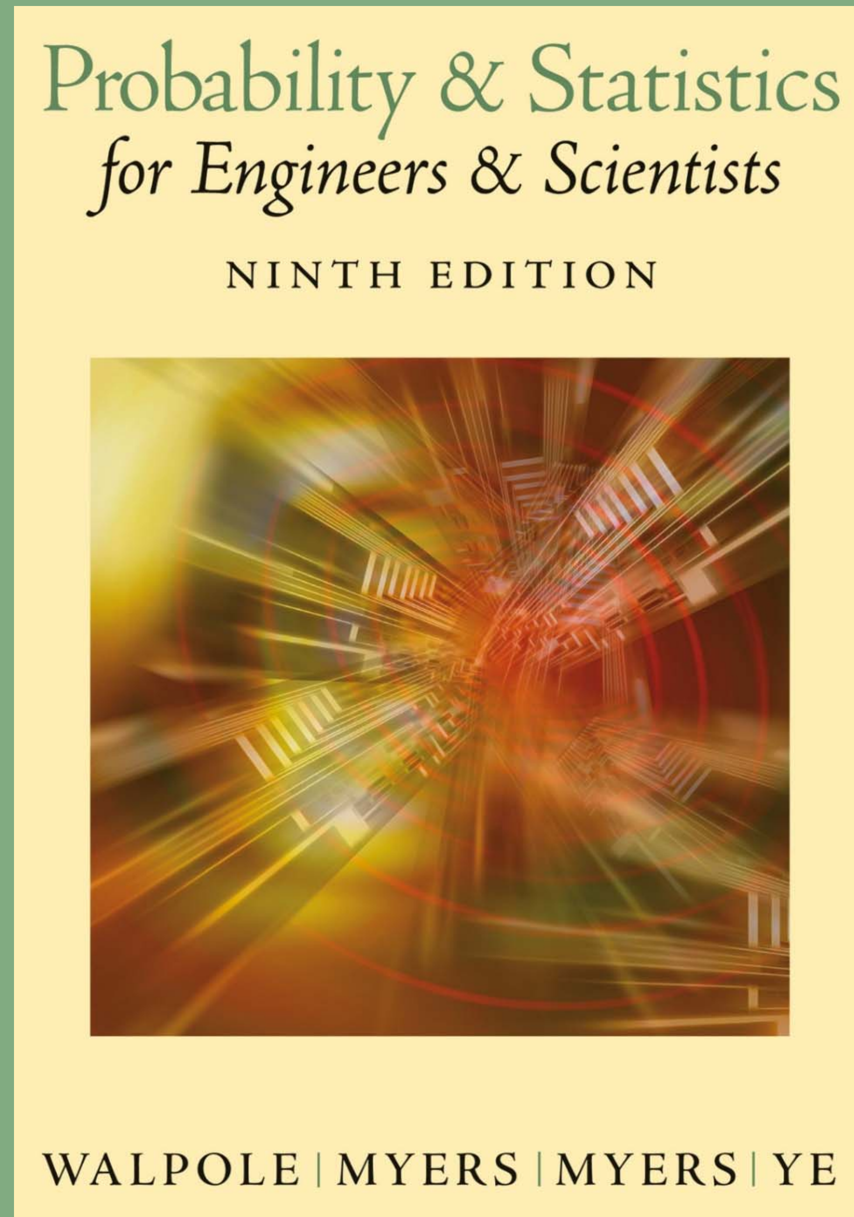
Example

Example:

Random variable	Type	
Number of eyes when rolling a dice	discrete	
The sum of eyes when rolling two dice	discrete	counting
Number of children in a family	discrete	
Age of first-time mother	discrete	
Time of running 5 km	continuous	
Amount of sugar in a coke	continuous	measure
Height of males	continuous	

Section 3.2

Discrete Probability Distribution



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3.2 Discrete Probability Distributions

- A discrete random variable X assumes each of its values with a certain probability.

Example:

- Experiment: tossing a non-balance coin 2 times independently.
- H= head , T=tail
- Sample space: S={HH, HT, TH, TT}
- Suppose $P(H)=\frac{1}{2}P(T) \Leftrightarrow P(H)=1/3$ and $P(T)=2/3$
- Let X= number of heads

Example

Sample point (Outcome)	Probability	Value of X (x)
HH	P(HH)=P(H) P(H)=$1/3\times1/3$ = 1/9	2
HT	P(HT)=P(H) P(T)=$1/3\times2/3$ = 2/9	1
TH	P(TH)=P(T) P(H)=$2/3\times1/3$ = 2/9	1
TT	P(TT)=P(T) P(T)=$2/3\times2/3$ = 4/9	0

- The possible values of X are: 0, 1, and 2.
- X is a discrete random variable.

Example

- Define the following events:

Event (X=x)	Probability = P(X=x)
(X=0)={TT}	P(X=0) = P(TT)=4/9
(X=1)={HT,TH}	P(X=1) =P(HT)+P(TH)=2/9+2/9=4/9
(X=2)={HH}	P(X=2) = P(HH)= 1/9

- The possible values of X with their probabilities are:

X	0	1	2	Total
P(X=x)=f(x)	4/9	4/9	1/9	1.00

The function f(x)=P(X=x) is called the probability function (probability distribution) of the discrete random variable X.

Definition 3.4

The set of ordered pairs $(x, f(x))$ is a **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum_x f(x) = 1$,
3. $P(X = x) = f(x)$.

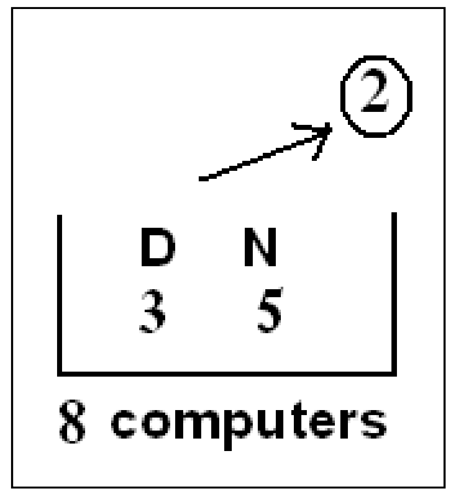
Example:
For the previous example, we have:

X	0	1	2	Total
f(x)= P(X=x)	4/9	4/9	1/9	$\sum_{x=0}^{2} f(x) = 1$

Example

$P(X<1) = P(X=0)=4/9$
$P(X\leq 1) = P(X=0) + P(X=1) = 4/9+4/9 = 8/9$
$P(X\geq 0.5) = P(X=1) + P(X=2) = 4/9+1/9 = 5/9$
$P(X>8) = P(\phi) = 0$
$P(X<10) = P(X=0) + P(X=1) + P(X=2) = P(S) = 1$

Example 3.3:
A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective.
If a school makes a random purchase of 2 of these computers, find the probability distribution of the number of defectives.



Example

Solution:

We need to find the probability distribution of the random variable: X = the number of defective computers purchased.

Experiment: selecting 2 computers at random out of 8

n(S) = $\binom{8}{2}$ equally likely outcomes

The possible values of X are: x=0, 1, 2.
Consider the events:

$$(X=0)=\{0D \text{ and } 2N\} \Rightarrow n(X=0)=\binom{3}{0}\times\binom{5}{2}$$

$$(X=1)=\{1D \text{ and } 1N\} \Rightarrow n(X=1)=\binom{3}{1}\times\binom{5}{1}$$

$$(X=2)=\{2D \text{ and } 0N\} \Rightarrow n(X=2)=\binom{3}{2}\times\binom{5}{0}$$

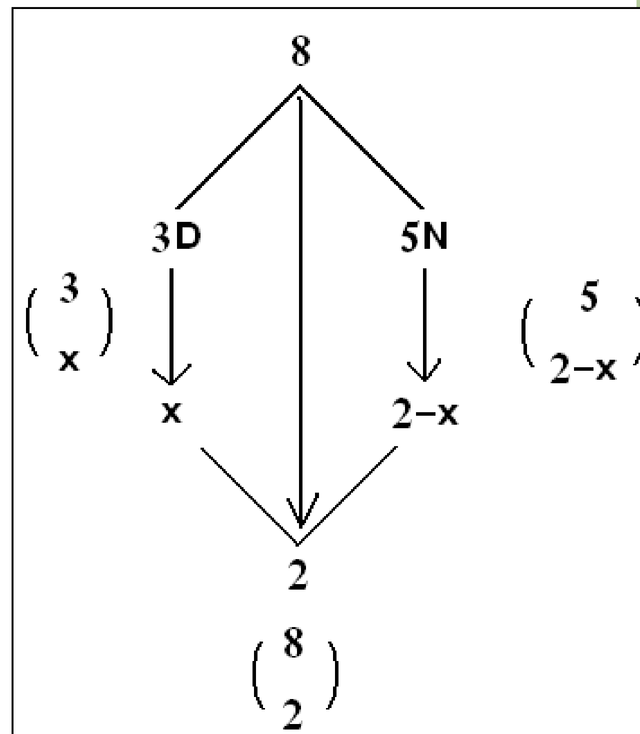
$$f(0)=P(X=0)=\frac{n(X=0)}{n(S)}=\frac{\binom{3}{0}\times\binom{5}{2}}{\binom{8}{2}}=\frac{10}{28}$$

$$f(1) = P(X=1) = \frac{n(X=1)}{n(S)} = \frac{\binom{3}{1} \times \binom{5}{1}}{\binom{8}{2}} = \frac{15}{28}$$

$$f(2) = P(X=2) = \frac{n(X=2)}{n(S)} = \frac{\binom{3}{2} \times \binom{5}{0}}{\binom{8}{2}} = \frac{3}{28}$$

In general, for x=0,1, 2, we have:

$$f(x)=P(X=x)=\frac{n(X=x)}{n(S)}=\frac{\binom{3}{x}\times\binom{5}{2-x}}{\binom{8}{2}}$$



The probability distribution of X is:

x	0	1	2	Total
f(x)= P(X=x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	**1.00**

$$f(x) = P(X = x) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{5}{2-x}}{\binom{8}{2}}; & x = 0,\ 1,\ 2 \\ 0;\ otherwise & \end{cases}$$

Definition 3.5

The **cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

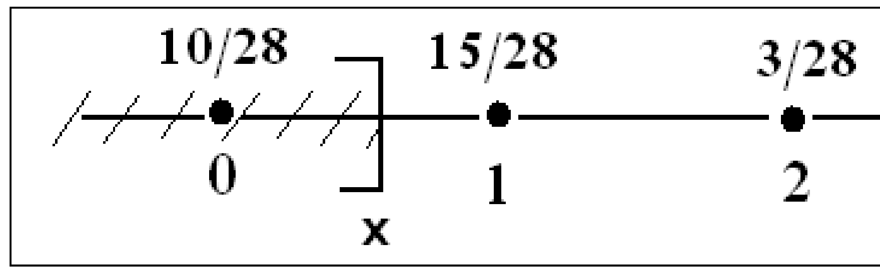
$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad \text{for } -\infty < x < \infty.$$

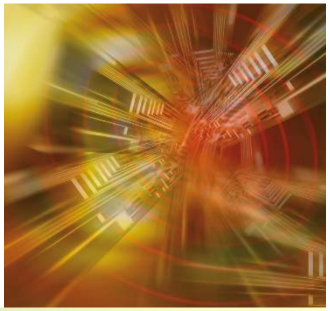
Example

Example:

Find the CDF of the random variable X with the probability function:

X	0	1	2
f(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$





Example

Example:

Find the CDF of the random variable X with the probability function:

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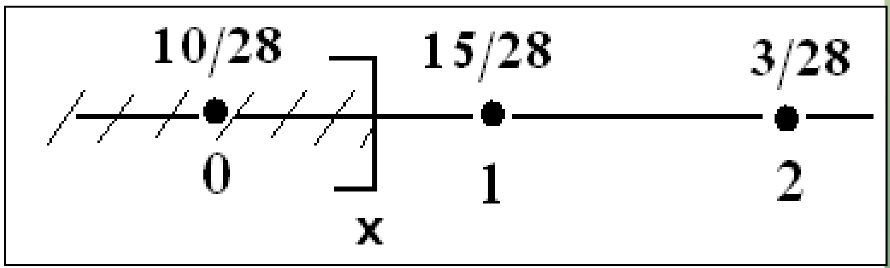
f(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$



Solution:

F(x)=P(X≤x) for $-\infty<x<\infty$

For x<0: **F(x)=0**

For 0≤x<1: **F(x)=P(X=0)=** $\frac{10}{28}$

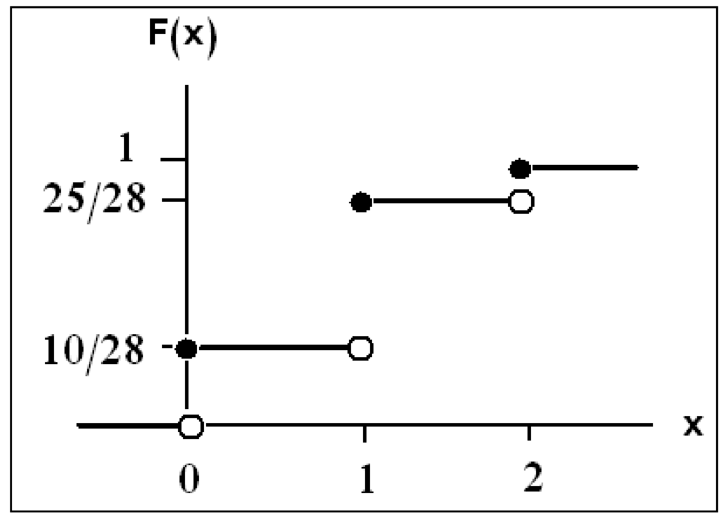
For 1≤x<2: **F(x)=P(X=0)+P(X=1)=** $\frac{10}{28}+\frac{15}{28}=\frac{25}{28}$

For x≥2: **F(x)=P(X=0)+P(X=1)+P(X=2)=** $\frac{10}{28}+\frac{15}{28}+\frac{3}{28}=1$

Example

The CDF of the random variable X is:

$$F(x) = P(X \leq x) = \begin{cases} 0 \;; & x < 0 \\ \dfrac{10}{28} \;; & 0 \leq x < 1 \\ \dfrac{25}{28} \;; & 1 \leq x < 2 \\ 1 \;; & x \geq 2 \end{cases}$$



Conclusion of CDF

Suppose that the probability function of X is:

x	x_1	x_2	x_3	…	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	…	$f(x_n)$
$F(x)$	$F(x_1)$	$F(x_2)$	$F(x_3)$	…	$F(x_n)$

Where $x_1 < x_2 < \dots < x_n$. Then:

$F(x_i) = f(x_1) + f(x_2) + \dots + f(x_i)$; $i=1, 2, \dots, n$

$F(x_i) = F(x_{i-1}) + f(x_i)$; $i=2, \dots, n$

$f(x_i) = F(x_i) - F(x_{i-1})$

$P(X>x_i) = 1 - F(x_i)$

Example:

In the previous example,

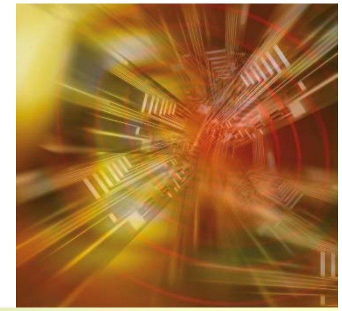
$P(0.5 < X \leq 1.5) = F(1.5) - F(0.5) = \frac{25}{28} - \frac{10}{28} = \frac{15}{28}$

Example

Example: **Flip three coins   X : # heads   X : S $\rightarrow$ {0,1,2,3}**

Outcome	Value of X	Probability function	Cumulative dist. Func.
TTT	X=0	$f(0) = P(X=0) = 1/8$	$F(0) = P(X \leq 0) = 1/8$
HTT, TTH, THT	X=1	$f(1) = P(X=1) = 3/8$	$F(1) = P(X \leq 1) = 4/8$
HHT, HTH, THH	X=2	$f(2) = P(X=2) = 3/8$	$F(2) = P(X \leq 2) = 7/8$
HHH	X=3	$f(3) = P(X=3) = 1/8$	$F(3) = P(X \leq 3) = 1$

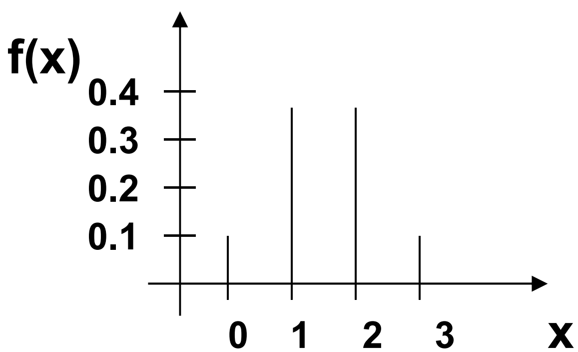
Example



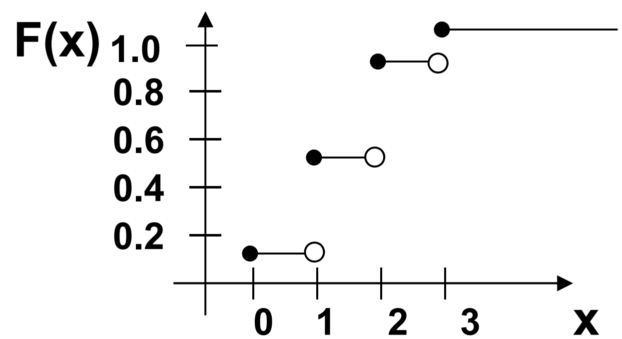
Example: Flip three coins X : # heads $X : S \rightarrow \{0,1,2,3\}$

Outcome	Value of X	Probability function	Cumulative dist. Func.
TTT	$X=0$	$f(0) = P(X=0) = 1/8$	$F(0) = P(X \leq 0) = 1/8$
HTT, TTH, THT	$X=1$	$f(1) = P(X=1) = 3/8$	$F(1) = P(X \leq 1) = 4/8$
HHT, HTH, THH	$X=2$	$f(2) = P(X=2) = 3/8$	$F(2) = P(X \leq 2) = 7/8$
HHH	$X=3$	$f(3) = P(X=3) = 1/8$	$F(3) = P(X \leq 3) = 1$

Probability function:

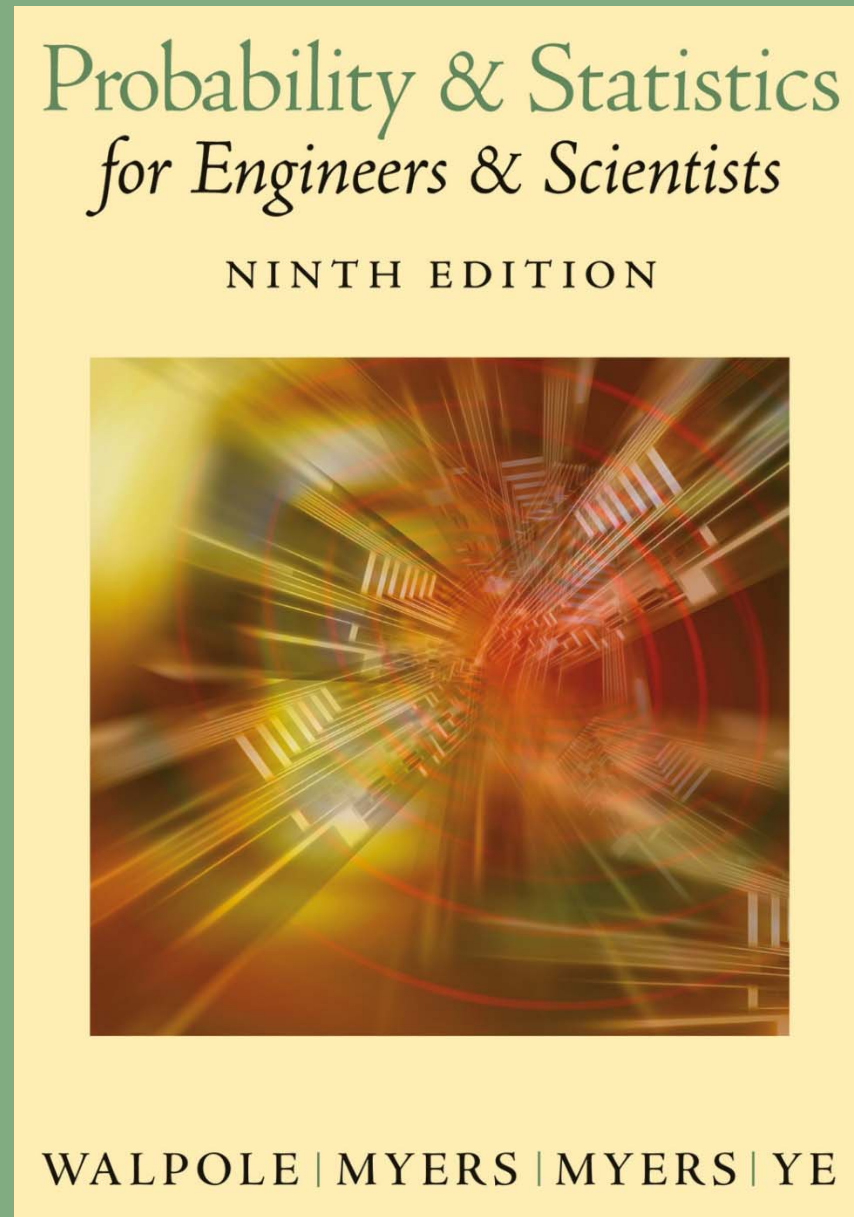


Cumulative distribution function:



Section 3.3

Continuous Probability Distributions



Continuous random variable

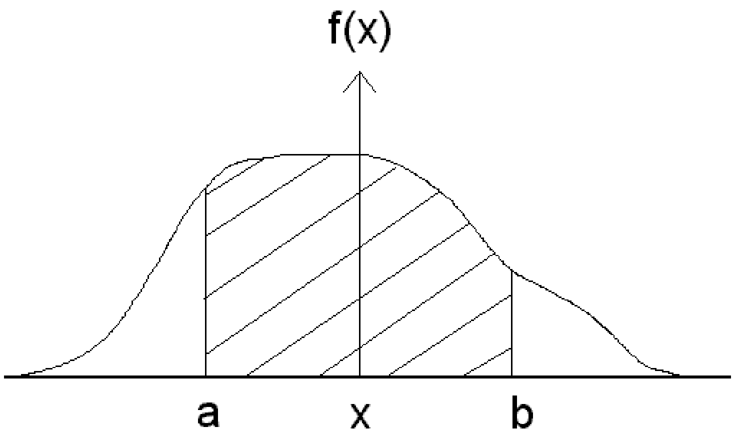
A ***continuous random variable*** is one that can assume an **uncountable** number of values.

→ We cannot list the possible values because there is an infinite number of them.

→ Because there is an infinite number of values, the probability of each individual value is virtually 0.

Thus, we can determine the probability of a *range of values* only.

For any continuous random variable, X, there exists a non-negative function f(x), called the **probability density function (p.d.f)** through which we can find probabilities of events expressed in term of X.



f: R $\rightarrow$ [0, ∞)

P(a < X < b) = $\int_a^b f(x)\,dx$

= area under the curve of f(x) and over the interval (a,b)

P(X$\in$A) = $\int_A f(x)\,dx$

= area under the curve of f(x) and over the region A

Definition 3.6

The function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if

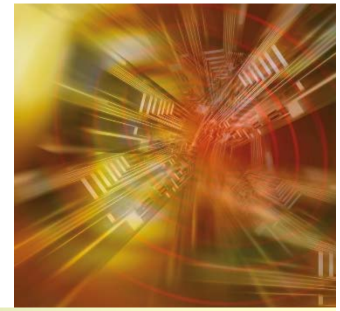
1. $f(x) \geq 0$, for all $x \in R$.
2. $\int_{-\infty}^{\infty} f(x)\ dx = 1$.
3. $P(a < X < b) = \int_a^b f(x)\ dx$.

Note:

For a continuous random variable X, we have:

1. $f(x) \neq P(X=x)$ (in general)
2. $P(X=a) = 0$ for any $a \in R$
3. $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$
4. $P(X \in A) = \int_A f(x)\,dx$

Continuous Random Variables



- In addition to a CDF, every *Continuous R.V.* has a probability *density* function (PDF).

- The PDF is defined by using the CDF and taking its derivative:

$$f(x) = \frac{d}{dx} F(x)$$

- Using this, given any set *A*, we can write that the probability *X* takes a value in *A* is

$$\int_A f(x)dx$$

Example

Example 3.6:

Suppose that the error in the reaction temperature, in °C, for a controlled laboratory experiment is a continuous random variable X having the following probability density function:

$$f(x) = \begin{cases} \frac{1}{3}x^2 \; ; -1 < x < 2 \\ 0 \; ; \; elsewhere \end{cases}$$

1. **Verify that (a) $f(x) \geq 0$ and (b)** $\int_{-\infty}^{\infty} f(x)\,dx = 1$
2. **Find $P(0 < X \leq 1)$**

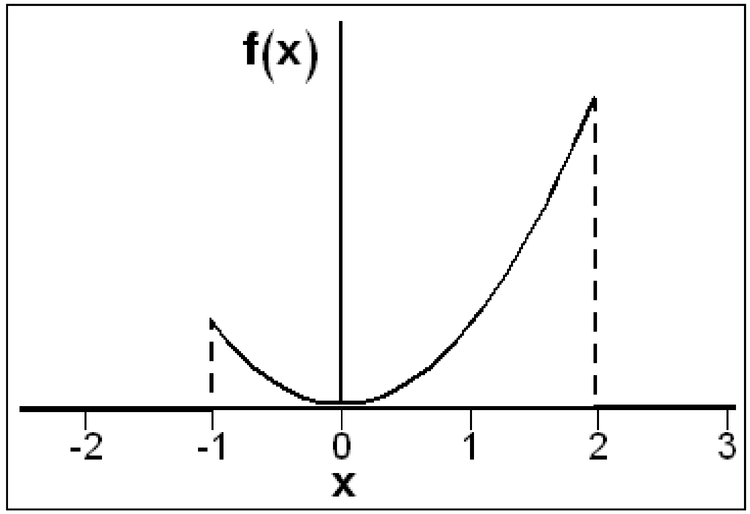
Example

Solution:

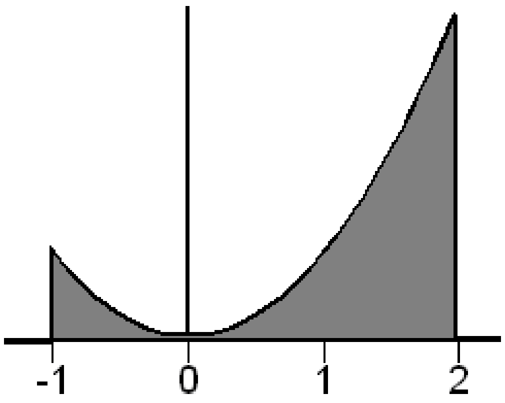
X = the error in the reaction temperature in ^{o}C.

X is continuous r. v.

$$f(x)=\begin{cases}\frac{1}{3}x^2 \;;\; -1<x<2\\ 0\;;\; elsewhere\end{cases}$$



1. (a) $f(x) \geq 0$ because $f(x)$ is a quadratic function.

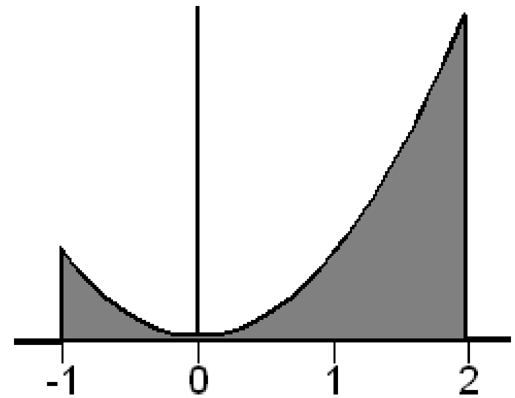


1. (a) $f(x) \geq 0$ because f(x) is a quadratic function.

(b) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{2} \frac{1}{3}x^2\,dx + \int_{2}^{\infty} 0\,dx$$

$$= \int_{-1}^{2} \frac{1}{3}x^2\,dx = \left[\frac{1}{9}x^3 \;\Big|\; \begin{matrix} x=2 \\ x=-1 \end{matrix} \right]$$

$$= \frac{1}{9}(8-(-1)) = 1$$



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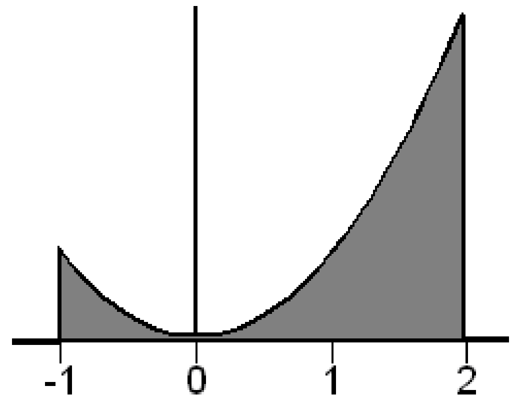
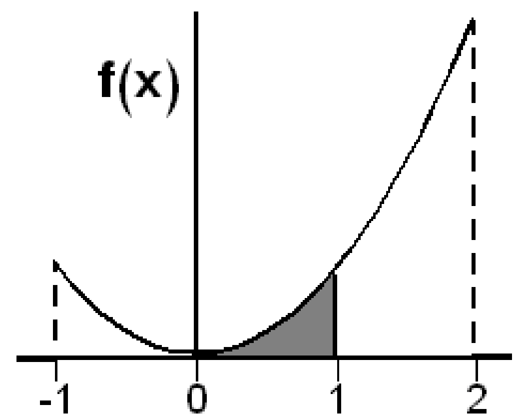


2. P(0<X≤1) = $$\int_0^1 f(x)\,dx = \int_0^1 \frac{1}{3}x^2\,dx$$

$$= \left[\frac{1}{9}x^3 \,\middle|\, \begin{matrix} x=1 \\ x=0 \end{matrix}\right]$$

$$= \frac{1}{9}(1-(0))$$

$$= \frac{1}{9}$$



Definition 3.7

The **cumulative distribution function** $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt, \quad \text{for } -\infty < x < \infty.$$

Result:

$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$

Example

Example 3.6:

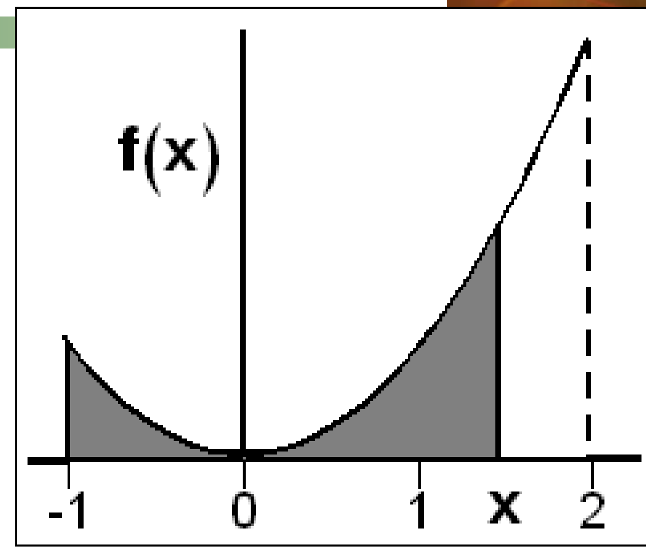
Suppose that the error in the reaction temperature, in ^{o}C, for a controlled laboratory experiment is a continuous random variable X having the following probability density function:

$$f(x)=\begin{cases}\frac{1}{3}x^{2} \ ; -1<x<2 \\ 0 \ ; \ elsewhere\end{cases}$$

1. Find the CDF
2. Using the CDF, find $P(0<X\leq 1)$.

Solution:

$$f(x)=\begin{cases}\frac{1}{3}x^2 \; ; -1<x<2\\ 0\;;\; elsewhere\end{cases}$$



For x< –1:

F(x) = $\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$

For –1≤x<2:

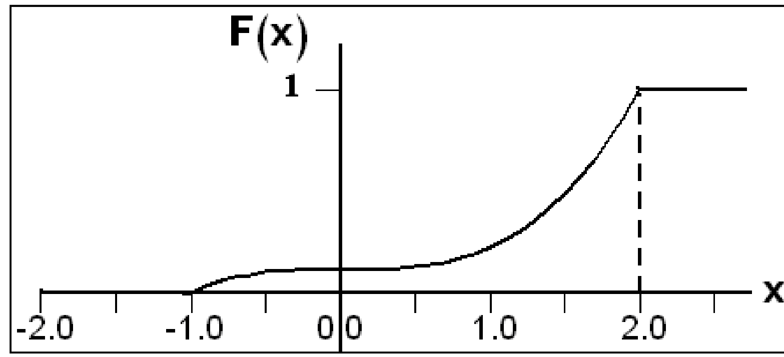
F(x) = $\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{x} \frac{1}{3}t^2\,dt$

$$=\int_{-1}^{x}\frac{1}{3}t^2\,dt$$

$$=\left[\frac{1}{9}t^3 \;\middle|\; \begin{matrix} t=x \\ t=-1\end{matrix}\right]=\frac{1}{9}(x^3-(-1))=\frac{1}{9}(x^3+1)$$

For x≥2:

F(x) = $$\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{2} \frac{1}{3}t^2\,dt + \int_{2}^{x} 0\,dt = \int_{-1}^{2} \frac{1}{3}t^2\,dt = 1$$



Therefore, the CDF is:

$$F(x) = P(X \le x) = \begin{cases} 0 & ;\ x < -1 \\ \frac{1}{9}(x^3 + 1) & ;\ -1 \le x < 2 \\ 1 & ;\ x \ge 2 \end{cases}$$

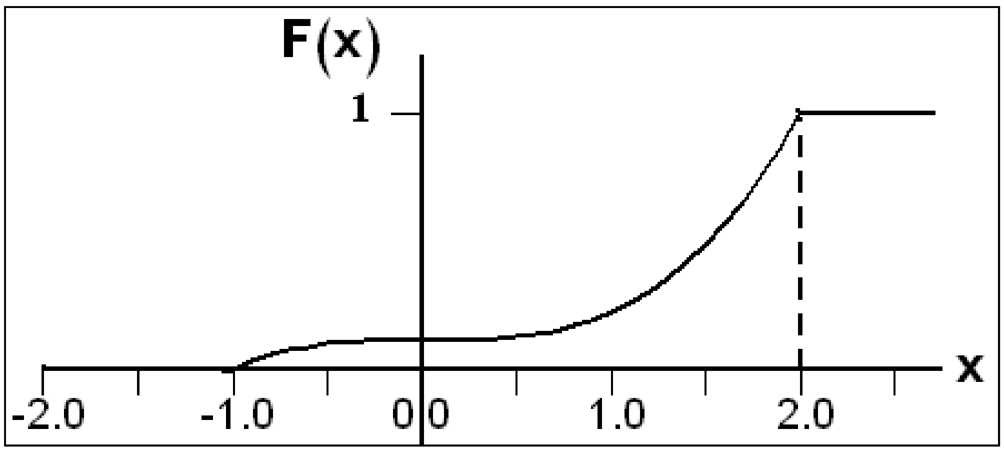
2. Using the CDF,
P(0<X≤1) = F(1) – F(0) = $\frac{2}{9} - \frac{1}{9} = \frac{1}{9}$

For x≥2:

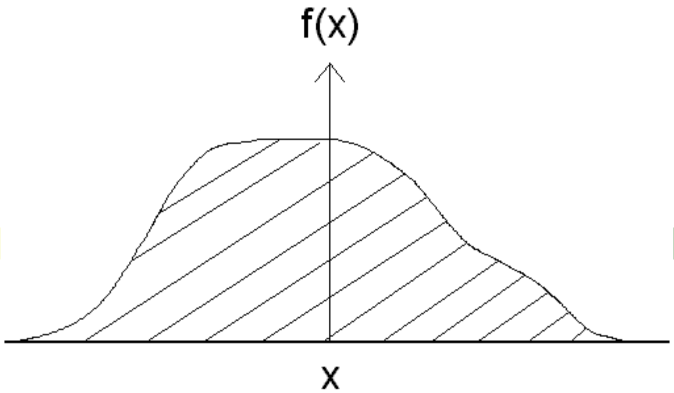
F(x) = $\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{2} \frac{1}{3}t^2\,dt + \int_{2}^{x} 0\,dt = \int_{-1}^{2} \frac{1}{3}t^2\,dt = 1$

Therefore, the CDF is:

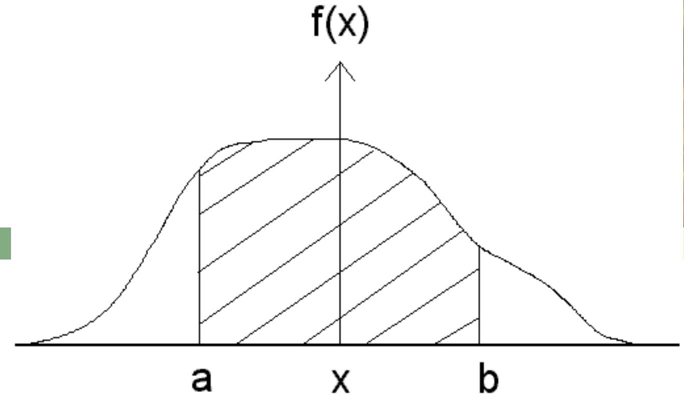
$$F(x) = P(X \le x) = \begin{cases} 0 \ ; x < -1 \\ \frac{1}{9}(x^3+1) \ ; -1 \le x < 2 \\ 1 \ ; x \ge 2 \end{cases}$$



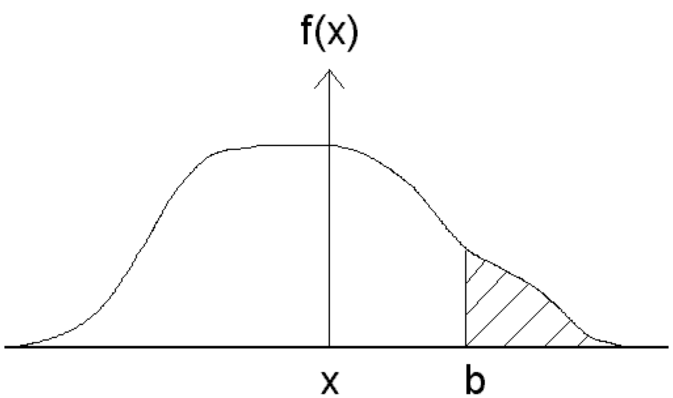
2. Using the CDF,
P(0<X≤1) = F(1) – F(0) = $\frac{2}{9} - \frac{1}{9} = \frac{1}{9}$



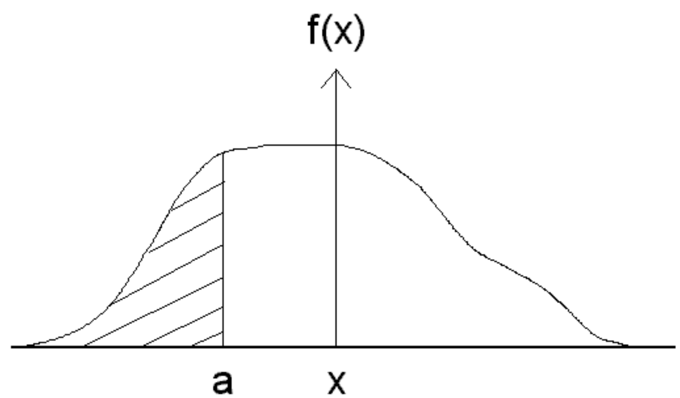
$$area = P(-\infty \le X \le +\infty)$$
$$= \int_{-\infty}^{+\infty} f(x)dx$$



$$area = P\left(a \le X \le b\right)$$
$$= \int_{a}^{b} f(x)dx$$



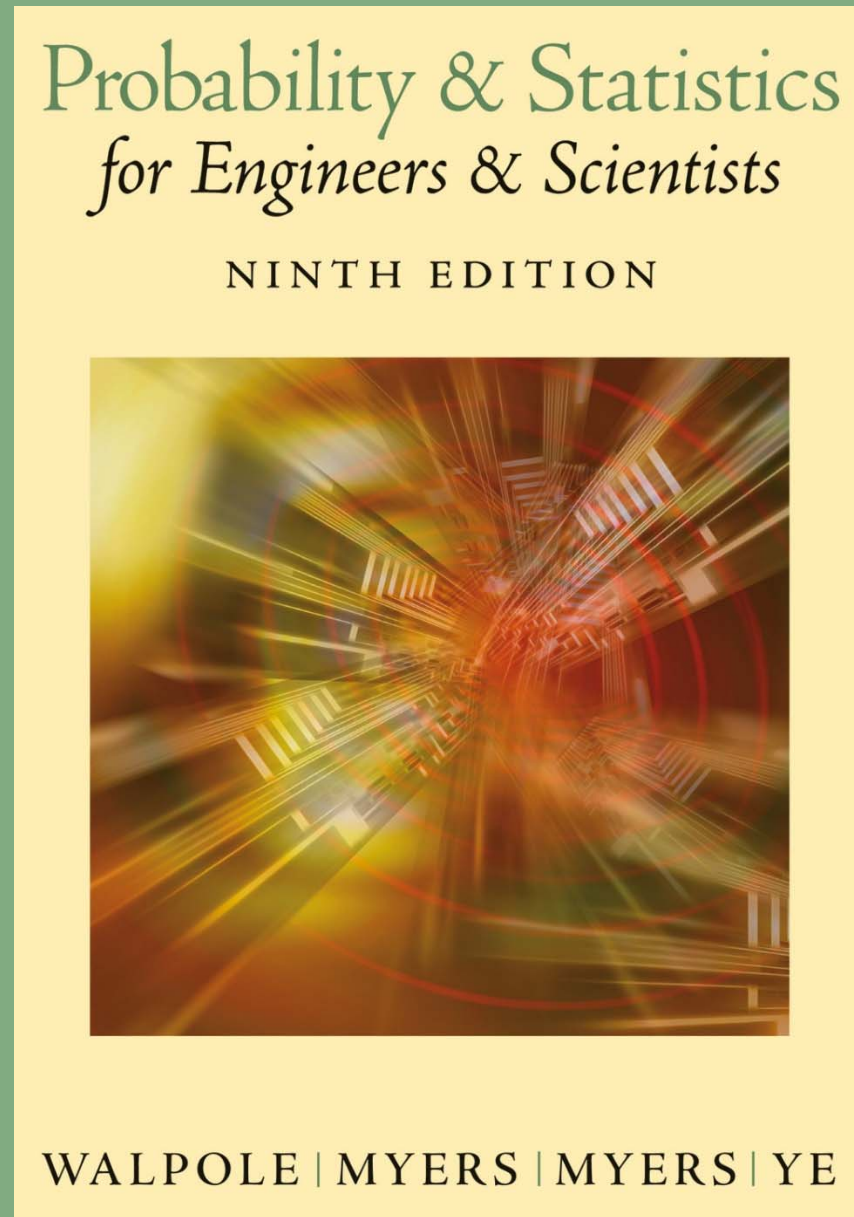
$$area = P\left(X \ge b\right)$$
$$= \int_{b}^{\infty} f(x)\, dx$$



$$area = P\left(X \le a\right)$$
$$= \int_{-\infty}^{a} f(x)dx$$

Section 3.4

Joint Probability Distributions



Motivation

When we record more than one characteristic from each population unit, the outcome variable is multivariate, e.g.

Example:

A bridge hand (13 cards) is selected from a deck of 52 cards.

X = the number of spades in the hand.

Y = the number of hearts in the hand.

In this example we will define:

$$p(x,y) = P[X = x,\ Y = y]$$

The possible values of *X* are 0, 1, 2, …, 13

The possible values of *Y* are also 0, 1, 2, …, 13 and $X + Y \leq 13$.

The number of ways of choosing the *y* **hearts** for the hand

The number of ways of choosing the *x* **spades** for the hand

The number of ways of completing the hand with **diamonds** and **clubs.**

$$p(x,y) = P[X = x, Y = y] = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{13-x-y}}{\binom{52}{13}}$$

The total number of ways of choosing the 13 cards for the hand

Joint probability function (discrete)

Definition:
Let X and Y be two discrete random variables.
The joint probability function f(x,y) for X and Y
Is defined by

1. $f(x,y) \geq 0$ for all x og y
2. $\sum_x \sum_y f(x,y) = 1$
3. $P(X = x, Y = y) = f(x,y)$ (the probability that both X = x and Y=y)

For a set A in the xy plane: $$P((X,Y) \in A) = \sum\sum_{(x,y) \in A} f(x,y)$$

Example

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

(a) the joint probability function $f(x, y)$,

(b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

Example

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

(a) the joint probability function $f(x, y)$,

(b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

Solution (a) :

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}},$$

for $x = 0, 1, 2$; $y = 0, 1, 2$; and $0 \leq x + y \leq 2$.

	$f(x,y)$	$x = 0$	$x = 1$	$x = 2$	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Example

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

(a) the joint probability function $f(x, y)$,

(b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

Solution (b) :

$$\begin{aligned} P[(X,Y) \in A] &= P(X + Y \leq 1) = f(0,0) + f(0,1) + f(1,0) \\ &= \frac{3}{28} + \frac{3}{14} + \frac{9}{28} = \frac{9}{14}. \end{aligned}$$

Marginal probability function

Definition: $X = S1,\ Y = S2$

Let X and Y denote two discrete random variables with joint probability function

$$p(x,y) = P[X = x,\ Y = y]$$

Then

$p_X(x) = P[X = x]$ is called the marginal probability function of X.

and

$p_Y(y) = P[Y = y]$ is called the marginal probability function of Y.

Marginal probability function

Note: Let y_1, y_2, y_3, … denote the possible values of Y.

$$p_X(x) = P[X = x, Y = S_2]$$

$$= P\left[\{X = x, Y = y_1\} \cup \{X = x, Y = y_2\} \cup \ldots\right]$$

$$= P\left[X = x, Y = y_1\right] + P\left[X = x, Y = y_2\right] + \ldots$$

$$= p\left(x, y_1\right) + p\left(x, y_2\right) + \ldots$$

$$= \sum_j p\left(x, y_j\right) = \sum_y p\left(x, y\right)$$

Thus the marginal probability function of X, $p_X(x)$ is obtained from the joint probability function of X and Y by summing $p(x,y)$ over the possible values of Y.

Marginal probability function

Also

$$p_Y(y) = P[X = S_1, Y = y]$$

$$= P\left[\{X = x_1, Y = y\} \cup \{X = x_2, Y = y\} \cup \dots\right]$$

$$= P\left[X = x_1, Y = y\right] + P\left[X = x_2, Y = y\right] + \dots$$

$$= p\left(x_1, y\right) + p\left(x_2, y\right) + \dots$$

$$= \sum_i p\left(x_i, y\right) = \sum_x p\left(x, y\right)$$

Definition: Marginal Probability Mass Functions

If X and Y are discrete random variables with joint probability mass function $f_{XY}(x, y)$, then the **marginal probability mass functions** of X and Y are

$$f_X(x) = P(X = x) = \sum_y f_{XY}(x, y) \quad \text{and} \quad f_Y(y) = P(Y = y) = \sum_x f_{XY}(x, y) \tag{5-2}$$

where the first sum is over all points in the range of (X, Y) for which $X = x$ and the second sum is over all points in the range of (X, Y) for which $Y = y$

Joint distribution
Marginal probability function

	$f(x,y)$	x = 0	1	2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

- $P(Y=1) = 3/14+3/14+0 = 3/7$
- $P(X=2) = 3/28+0+0 = 3/28$

Joint distribution (constinuous)
Joint density function

Definition:
Let X og Y be two continuous random variables.
The joint density function f(x,y) for X and Y is defined by

1. $f(x,y) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$
3. $P(a < X < b,\ c < Y < d) = \int_{c}^{d} \int_{a}^{b} f(x, y)\, dx\, dy$

For a region A in the xy-plane: $P[(X,Y) \in A] = \iint_{A} f(x,y)\, dxdy$

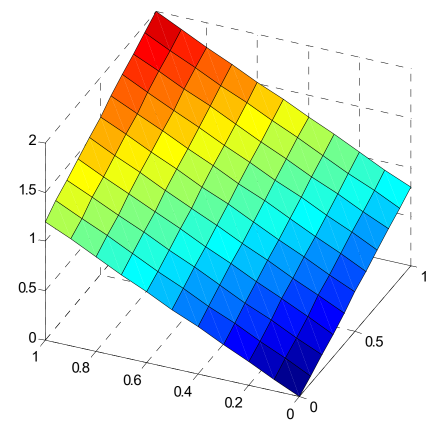
Example

A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify condition 2 of Definition 3.9. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

(b) Find $P[(X,Y) \in A]$, where $A = \{(x,y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$.



Solution

(a) The integration of $f(x, y)$ over the whole region is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\;dx\;dy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\;dx\;dy$$

$$= \int_0^1 \left(\frac{2x^2}{5} + \frac{6xy}{5}\right)\Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{2}{5} + \frac{6y}{5}\right) dy = \left(\frac{2y}{5} + \frac{3y^2}{5}\right)\Big|_0^1 = \frac{2}{5} + \frac{3}{5} = 1.$$

(b) To calculate the probability, we use

$$P[(X,Y) \in A] = P\left(0 < X < \frac{1}{2}, \frac{1}{4} < Y < \frac{1}{2}\right)$$

$$= \int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)\;dx\;dy$$

$$= \int_{1/4}^{1/2} \left(\frac{2x^2}{5} + \frac{6xy}{5}\right)\Big|_{x=0}^{x=1/2} dy = \int_{1/4}^{1/2} \left(\frac{1}{10} + \frac{3y}{5}\right) dy$$

$$= \left(\frac{y}{10} + \frac{3y^2}{10}\right)\Big|_{1/4}^{1/2}$$

$$= \frac{1}{10}\left[\left(\frac{1}{2} + \frac{3}{4}\right) - \left(\frac{1}{4} + \frac{3}{16}\right)\right] = \frac{13}{160}.$$

Joint distribution
Marginal density function

Definition**:**
Let X and Y be two continuous random variables with joint density function f(x,y).
The marginal density function for X is given by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

The marginal density function for Y is given by

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Example

Joint density f(x,y) for X and Y:

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Marginal density function for X:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{0}^{1} \frac{2}{5}(2x+3y)\,dy$$

$$= \left[\frac{2}{5}2xy + \frac{1}{5}3y^2\right]_0^1 = \frac{4}{5}x + \frac{3}{5}$$

Joint distribution (Independence)

Definition:
Two random variables X and Y (continuous or discrete) with joint density/probability functions f(x,y) and marginal density/probability functions g(x) and h(y), respectively, are said to be independent if and only if

$$f(x,y) = g(x)\,h(y) \qquad \text{for all } x,y$$

Example

Joint density f(x,y) for X and Y:

$$f(x,y)=\begin{cases}\frac{2}{5}(2x+3y) & 0\le x\le 1, 0\le y\le 1\\ 0 & \text{otherwise}\end{cases}$$

Marginal density function for X:

$$g(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$=\frac{4}{5}x+\frac{3}{5}$$

Marginal density function for Y:

$$h(y)=\int_{-\infty}^{\infty} f(x,y)\,dx$$

$$=\frac{6}{5}y+\frac{2}{5}$$

Are g(x) and h(y) independent?

Conditional Probability

Definition: Let X and Y denote two random variables with joint probability density function $f(x,y)$ and marginal densities $f_X(x)$, $f_Y(y)$ then the conditional density of Y given $X = x$

$$f_{Y|X}\left(y|x\right) = \frac{f\left(x,y\right)}{f_X\left(x\right)}$$

conditional density of X given $Y = y$

$$f_{X|Y}\left(x|y\right) = \frac{f\left(x,y\right)}{f_Y\left(y\right)}$$

Example

Joint density f(x,y) for X and Y:

$$f(x,y)=\begin{cases}\frac{2}{5}(2x+3y) & 0\le x\le 1, 0\le y\le 1\\ 0 & \text{otherwise}\end{cases}$$

Conditional probability function for Y given X:

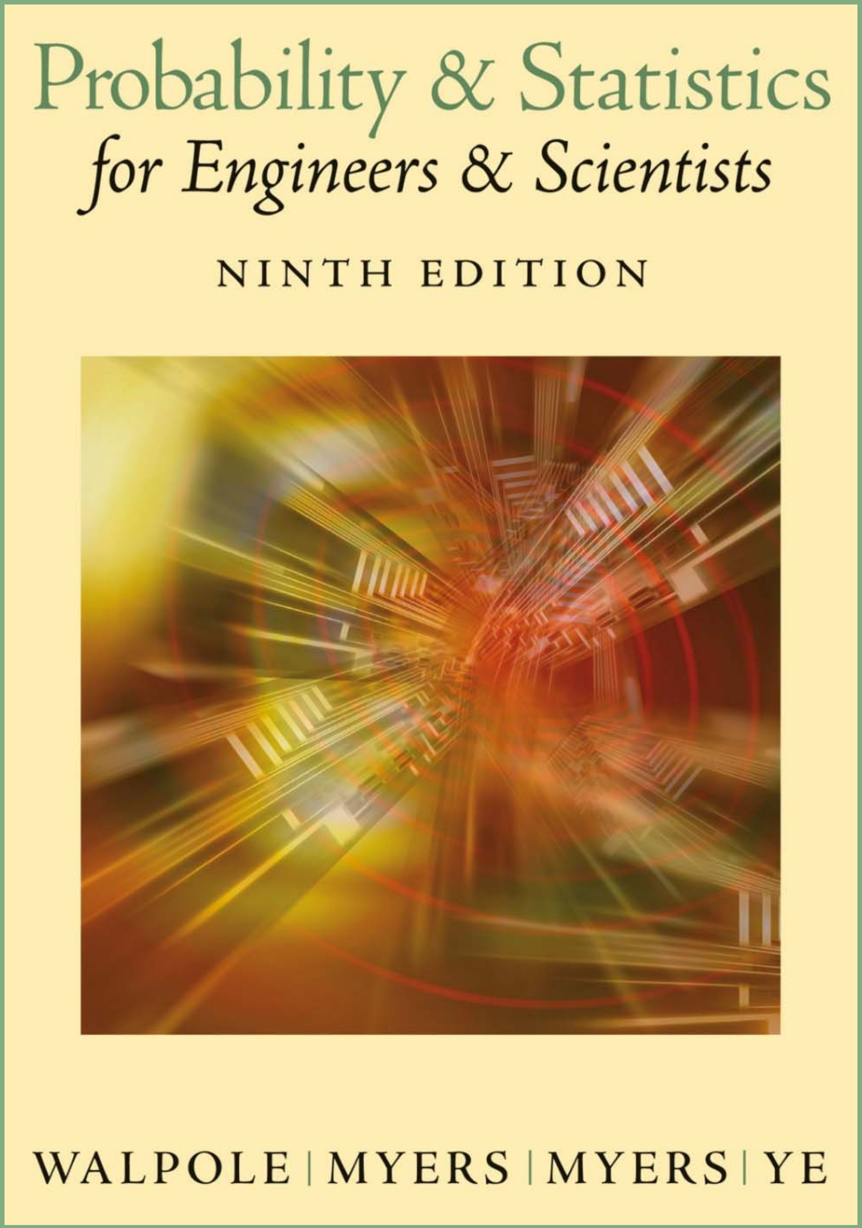
$$f(y\,|\,x)=\frac{\frac{2}{5}(2x+3y)}{\frac{4}{5}x+\frac{3}{5}}$$

Conditional probability function for X given Y:

$$f(x\,|\,y)=\frac{\frac{2}{5}(2x+3y)}{\frac{6}{5}y+\frac{2}{5}}$$

Chapter 3

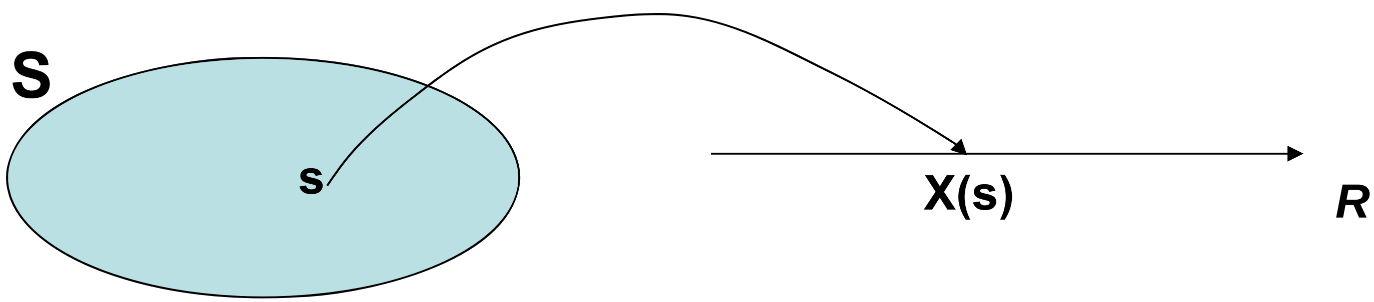
Review



PEARSON
Addison Wesley

Definition 3.1

In an experiment a number is often attached to each outcome.



Definition:

A random variable X is a function defined on S, which takes values on the real axis

$$X: S \to R$$

Sample space (S) **Real numbers** (R)

Definition 3.4

The set of ordered pairs $(x, f(x))$ is a **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum_{x} f(x) = 1$,
3. $P(X = x) = f(x)$.

Definition 3.5

The **cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad \text{for } -\infty < x < \infty.$$

Definition 3.7

The **cumulative distribution function** $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt, \quad \text{for } -\infty < x < \infty.$$

Result:

$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$

Definition 3.8

The function $f(x, y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y),
2. $\sum\limits_{x} \sum\limits_{y} f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

For any region A in the xy plane, $P[(X, Y) \in A] = \sum \sum\limits_{A} f(x, y)$.

Definition 3.9

The function $f(x, y)$ is a **joint density function** of the continuous random variables X and Y if

1. $f(x, y) \geq 0$, for all (x, y),
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ dx\ dy = 1$,
3. $P[(X, Y) \in A] = \int \int_A f(x, y)\ dx\ dy$, for any region A in the xy plane.

Definition 3.10

The **marginal distributions** of X alone and of Y alone are

$$g(x) = \sum_y f(x, y) \quad \text{and} \quad h(y) = \sum_x f(x, y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

for the continuous case.

Definition 3.11

Let X and Y be two random variables, discrete or continuous. The **conditional distribution** of the random variable Y given that $X = x$ is

$$f(y|x) = \frac{f(x,y)}{g(x)}, \text{ provided } g(x) > 0.$$

Similarly, the conditional distribution of X given that $Y = y$ is

$$f(x|y) = \frac{f(x,y)}{h(y)}, \text{ provided } h(y) > 0.$$

Definition 3.13

Let $X_1, X_2, \ldots, X_n$ be n random variables, discrete or continuous, with joint probability distribution $f(x_1, x_2, \ldots, x_n)$ and marginal distribution $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$, respectively. The random variables $X_1, X_2, \ldots, X_n$ are said to be mutually **statistically independent** if and only if

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

for all $(x_1, x_2, \ldots, x_n)$ within their range.