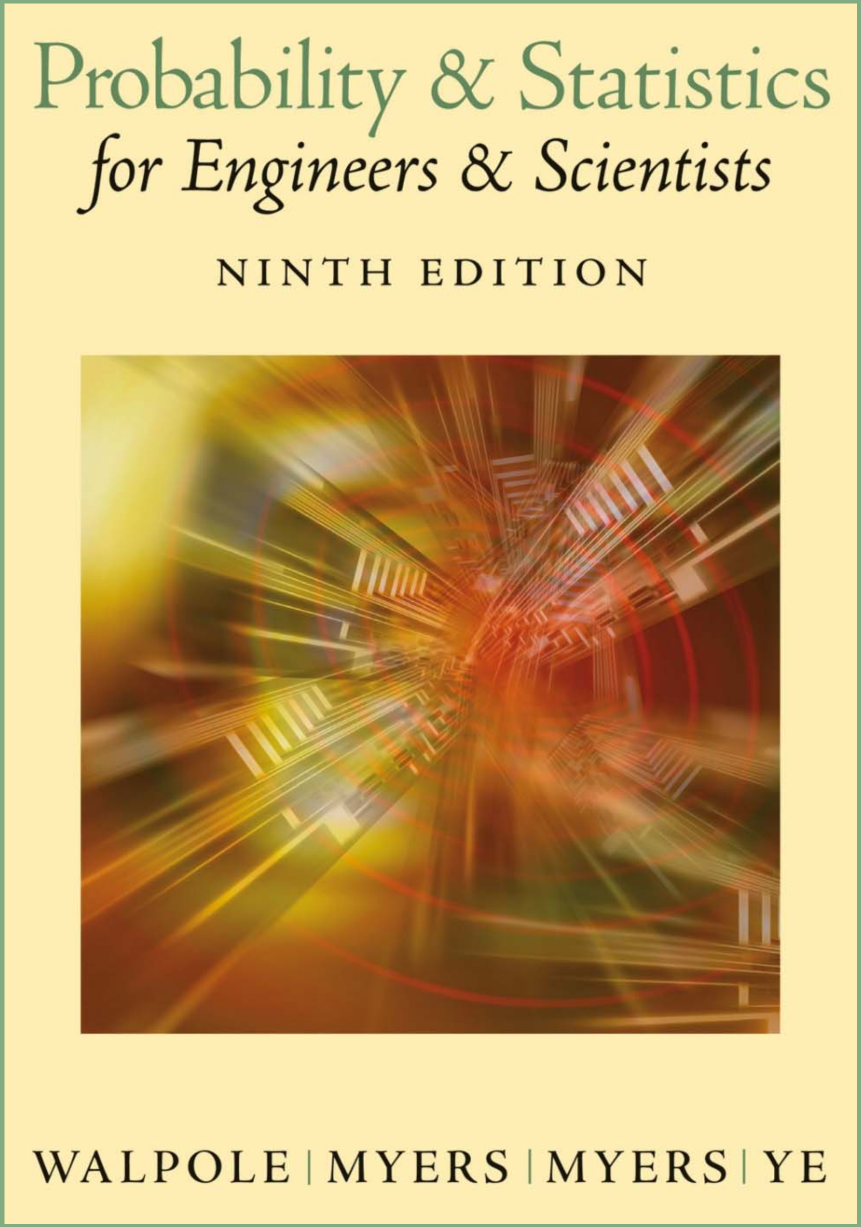
# Chapter 2

## Probability



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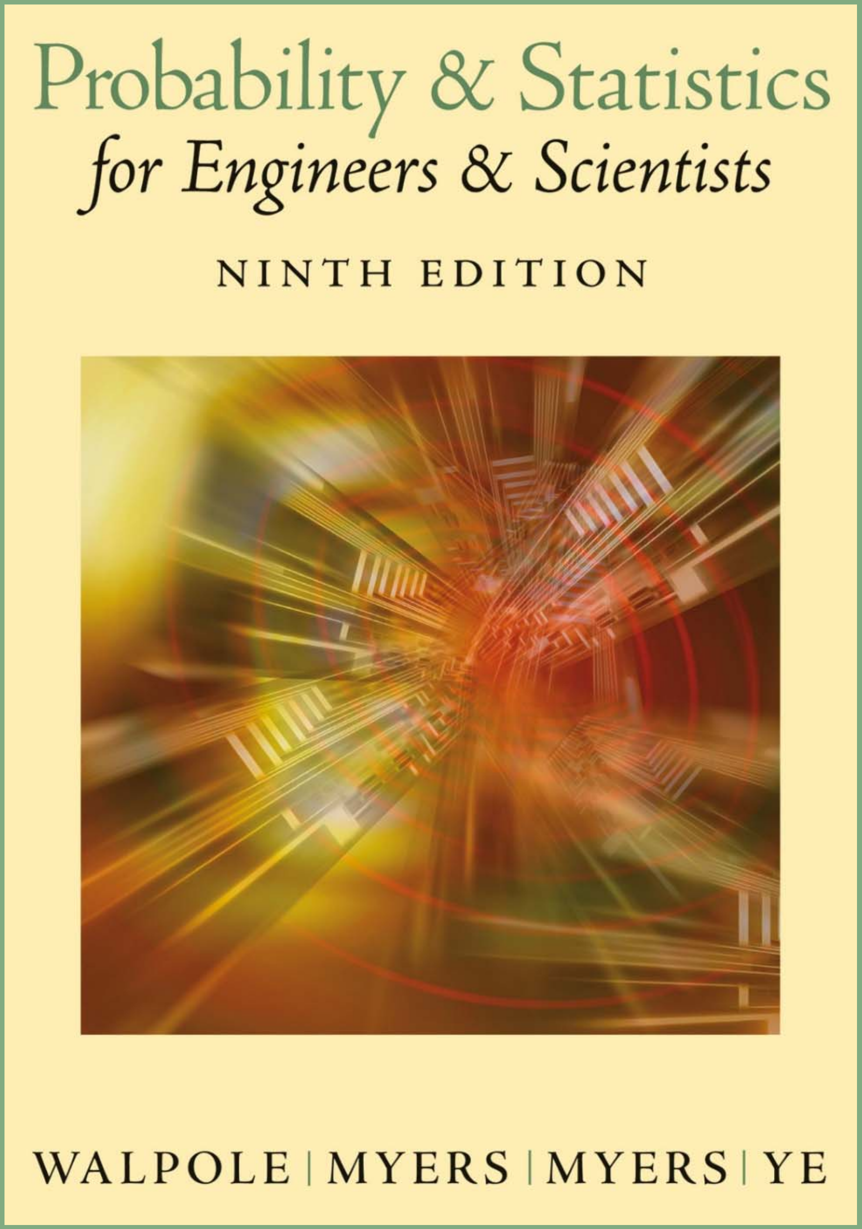
# Section 2.1

## Sample Space



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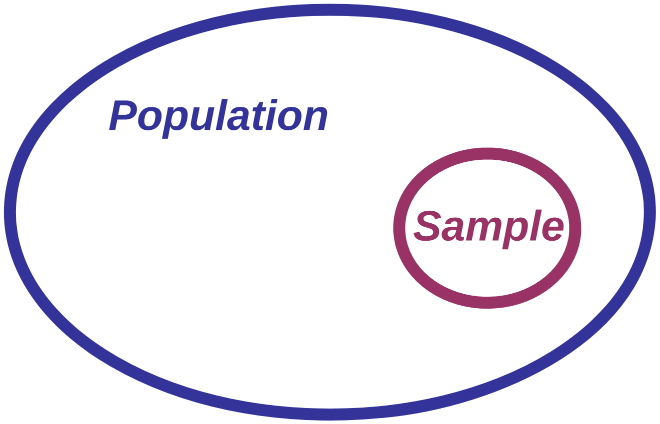
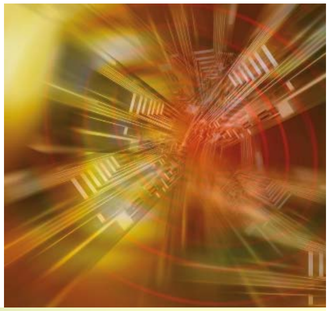
# Population & Sample

Population: A population is the set of all possible observations of interest to the problem, and from which we want to draw conclusion.

Sample: the part of the population from which we collect information.



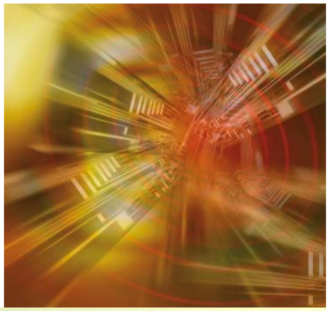


# Experiment

An Statistical Experiment: is some procedure (or process) that we do and it results in a set of data.

A random experiment: is an experiment we do not know its exact outcome in advance but we know the set of all possible outcomes.

# Definition 2.1: Sample Space

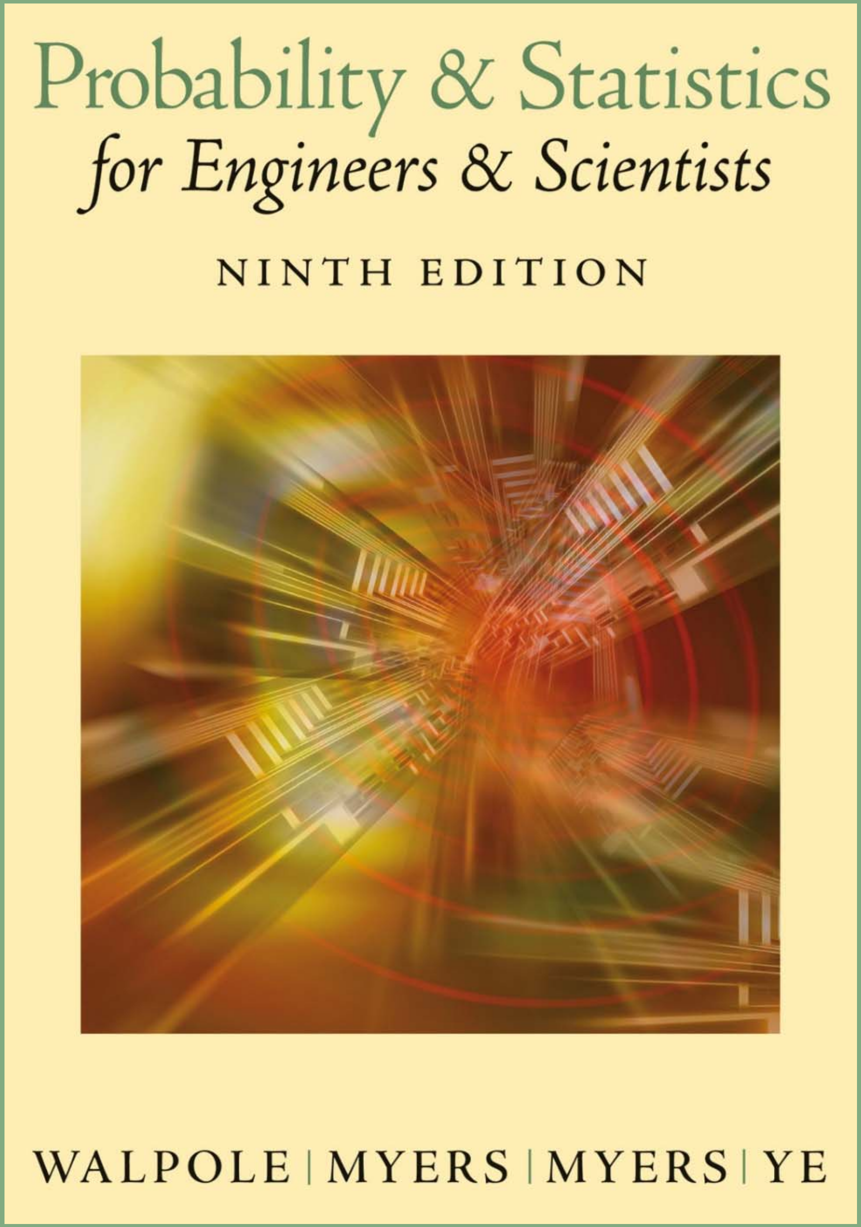


The set of **all possible outcomes** of a statistical experiment is called the sample space and is denoted by S.

Each outcome (element or member) of the sample space S is called a **sample point**.

# Section 2.2

## Events



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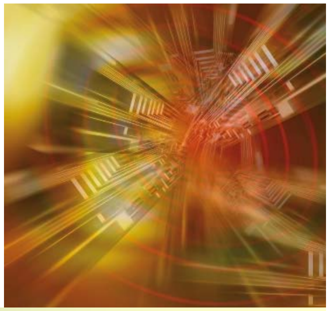
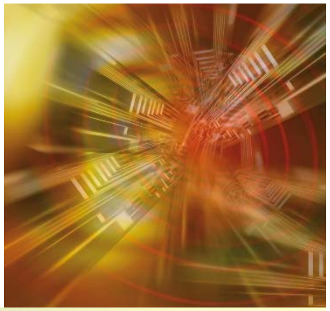
# Definition 2.2



An **event** is a subset of a sample space.

- We say that an event A occurs if the outcome (the result) of the experiment is an element of A.
- $\phi \subseteq S$ is an event (ϕ is called the impossible event)
- $S \subseteq S$ is an event (S is called the sure event)

# Example



**Selecting a ball from a box containing 6 balls numbered 1,2,3,4,5 and 6. (or tossing a die)**

**This experiment has 6 possible outcomes**

**The sample space is $S=\{1,2,3,4,5,6\}$.**

**Consider the following events:**

- **$E_1$= getting an even number $=\{2,4,6\}\subseteq S$**
- **$E_2$ = getting a number less than $4=\{1,2,3\}\subseteq S$**
- **$E_3$ = getting 1 or $3=\{1,3\}\subseteq S$**
- **$E_4$ = getting an odd number$=\{1,3,5\}\subseteq S$**
- **$E_5$ = getting a negative number$=\{\ \}=\phi \subseteq S$**
- **$E_6$ = getting a number less than 10 $= \{1,2,3,4,5,6\}=S\subseteq S$**

**Notation:**

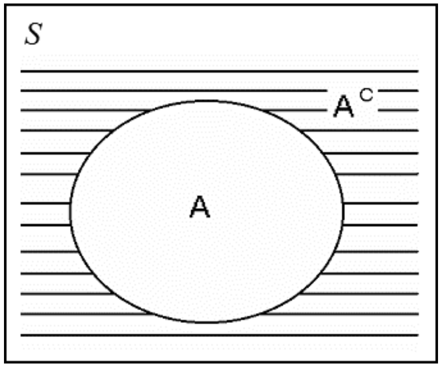
- **n(S)= no. of outcomes (elements) in S.**
- **n(E)= no. of outcomes (elements) in the event E.**

# Definition 2.3

The **complement** of an event $A$ with respect to $S$ is the subset of all elements of $S$ that are not in $A$. We denote the complement of $A$ by the symbol $A'$.

**Complement of The Event A:**

- **$A^c$ or $A'$**
- **$A^c = \{x \in S: \ x \notin A\}$**
- **$A^c$ consists of all points of S that are not in A.**
- **$A^c$ occurs if A does not.**



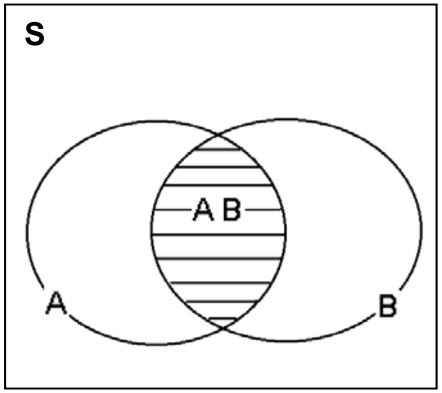
**Venn diagram**

# Definition 2.4

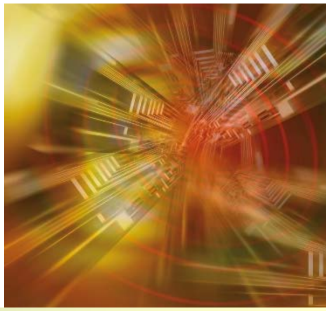
The **intersection** of two events $A$ and $B$, denoted by the symbol $A \cap B$, is the event containing all elements that are common to $A$ and $B$.

**Intersection:**

- **$A \cap B = AB = \{x \in S\text{: } x \in A \text{ and } x \in B\}$**
- **$A \cap B$ Consists of all points in both A and B.**
- **$A \cap B$ Occurs if both A and B occur together.**

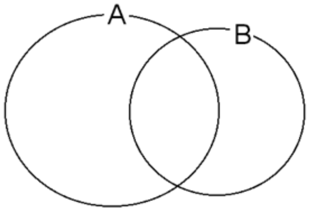


# Definition 2.5

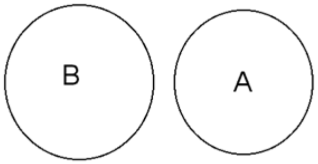


Two events $A$ and $B$ are **mutually exclusive**, or **disjoint**, if $A \cap B = \phi$, that is, if $A$ and $B$ have no elements in common.

**Two events A and B are mutually exclusive (or disjoint) if and only if $A \cap B = \phi$; that is, A and B have no common elements (they do not occur together).**



**$A \cap B \neq \phi$**
***A* and *B* are not mutually exclusive**



**$A \cap B = \phi$**
***A* and *B* are mutually exclusive (disjoint)**

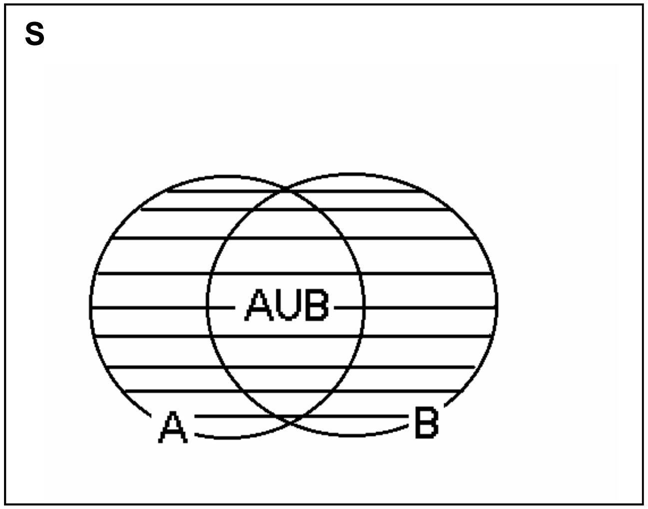
# Definition 2.6

The **union** of the two events $A$ and $B$, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to $A$ or $B$ or both.

$A \cup B = \{x \in S: x \in A \text{ or } x \in B\}$

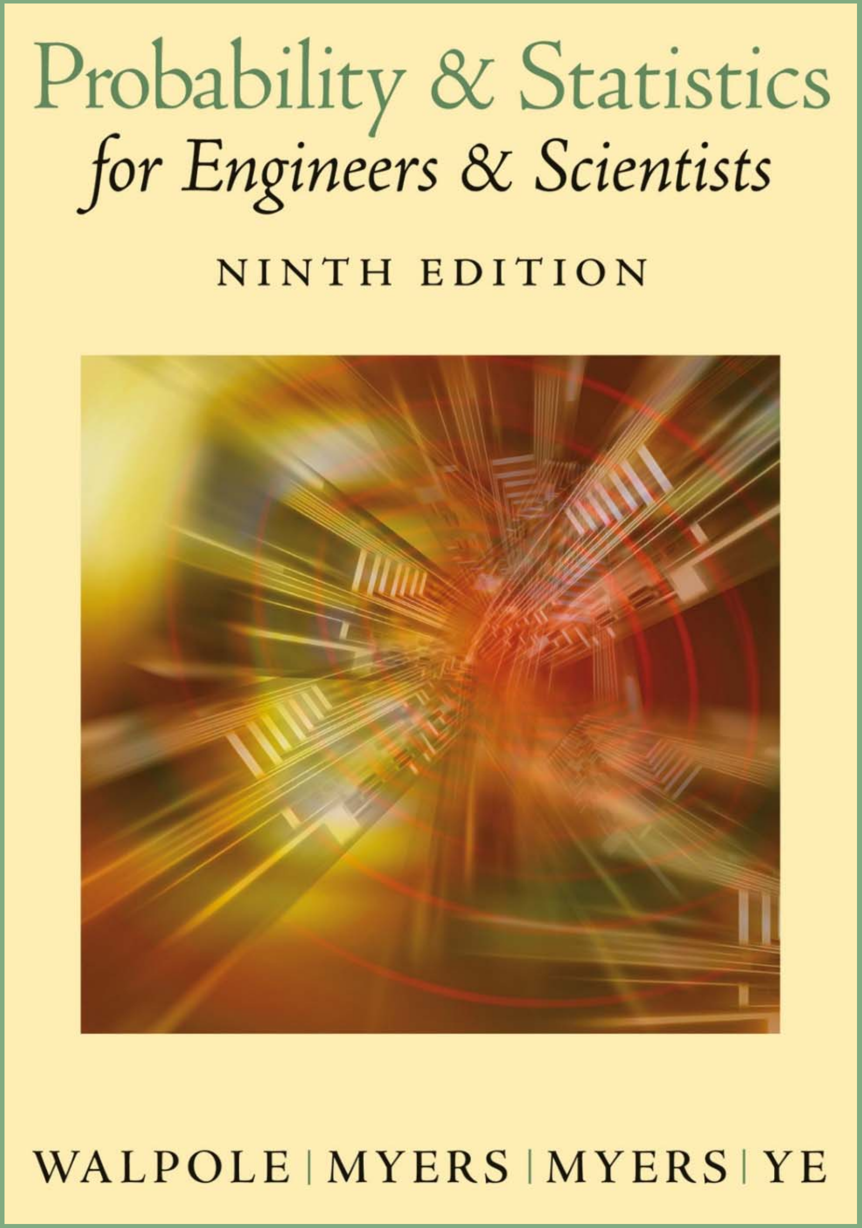
$A \cup B$ Consists of all outcomes in A or in B or in both A and B.

- $A \cup B$ Occurs if A occurs, or B occurs, or both A and B occur. That is $A \cup B$ Occurs if at least one of A and B occurs.



# Section 2.3

## Counting Sample Points



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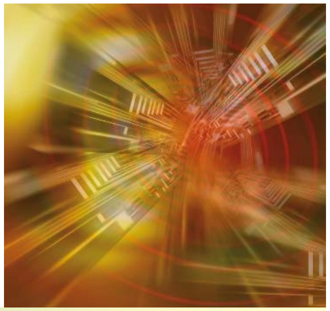
# Rule 2.1

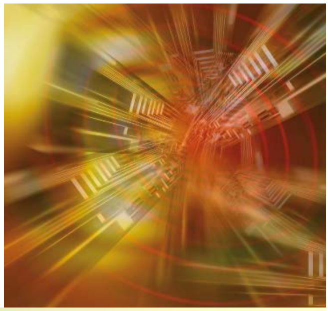


- There are many counting techniques which can be used to count the number points in the sample space (or in some events) without listing each element.
- In many cases, we can compute the probability of an event by using the counting techniques.

If an operation can be performed in $n_1$ ways, and if for each of these ways a second operation can be performed in $n_2$ ways, then the two operations can be performed together in $n_1 n_2$ ways.

If an operation can be performed in $n_1$ ways, and if for each of these a second operation can be performed in $n_2$ ways, and for each of the first two a third operation can be performed in $n_3$ ways, and so forth, then the sequence of $k$ operations can be performed in $n_1 n_2 \cdots n_k$ ways.

# Generalized Multiplication Rule



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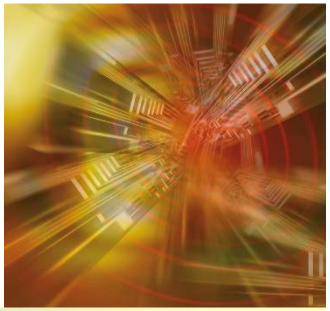
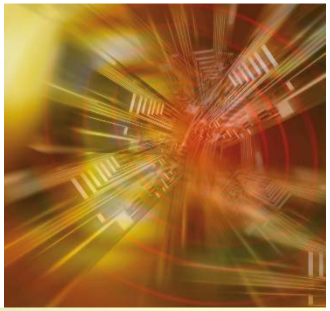
# Example



Lunch choices



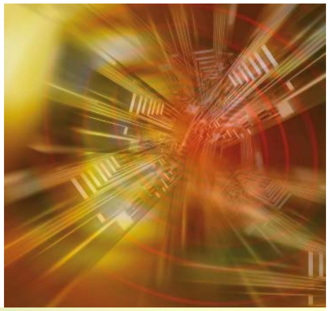
# Combinations

In many problems, we are interested in the number of ways of selecting r objects from n objects **without regard to order**. These selections are called combinations.

The number of combinations of *n* distinct objects taken *r* at a time is denoted by $\binom{n}{r}$ and is given by:

$$\binom{n}{r} = \frac{n!}{r!\ (n-r)!}; \qquad r = 0, 1, 2, \ldots, n$$

$$= n(n-1)(n-2)\cdot\cdot\cdot(n-r+1)$$



# Combinations

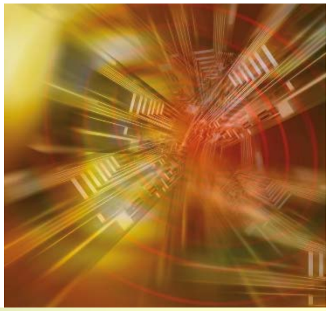
Example:
Ann, Barry, Chris, and Dan should from a committee consisting of two persons, i.e. unordered without replacement.

Number of possible combinations:

$$\binom{4}{2} = \frac{4!}{2!2!} = 6$$

Writing it out : AB AC AD BC BD CD

# Permutation



In many problems, we are interested in the number of ways of selecting r objects from n objects where order matters. These selections are called Permutation.

For any non-negative integer $n$, $n!$, called "$n$ factorial," is defined as

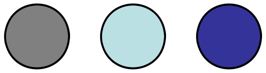
$$n! = n(n-1)\cdots(2)(1),$$

with special case $0! = 1$.

The number of permutations of $n$ objects is $n!$.

# Permutation

Example: Ordering *n* different objects



**There are**

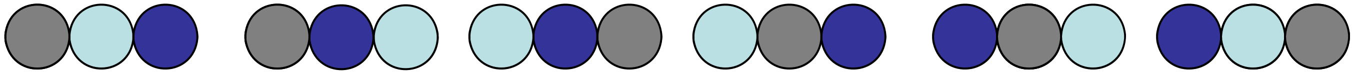
- **n ways of selecting the first object**
- **n -1 ways of selecting second object**

  ⋮
- **1 way of selecting the last object**

**”n factorial”**

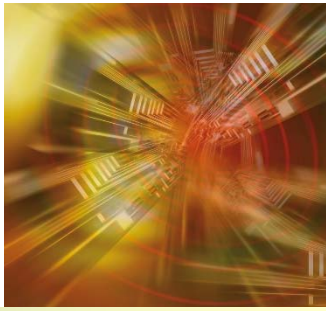
**n · (n -1) · ··· · 1 = n ! ways**

**The multiplication rule**



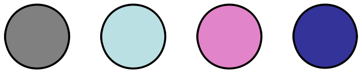
**3!= 6**

# Permutation

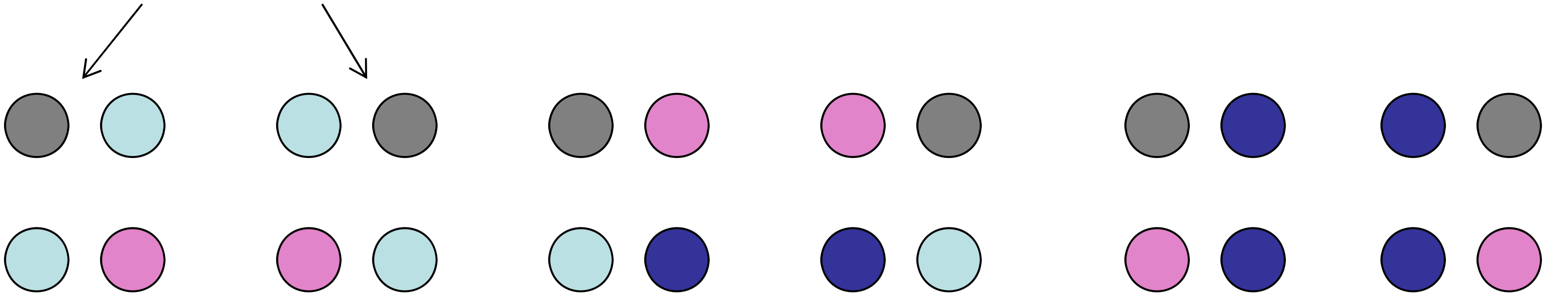


Example:

Select 2 out of 4 different balls ordered and without replacement

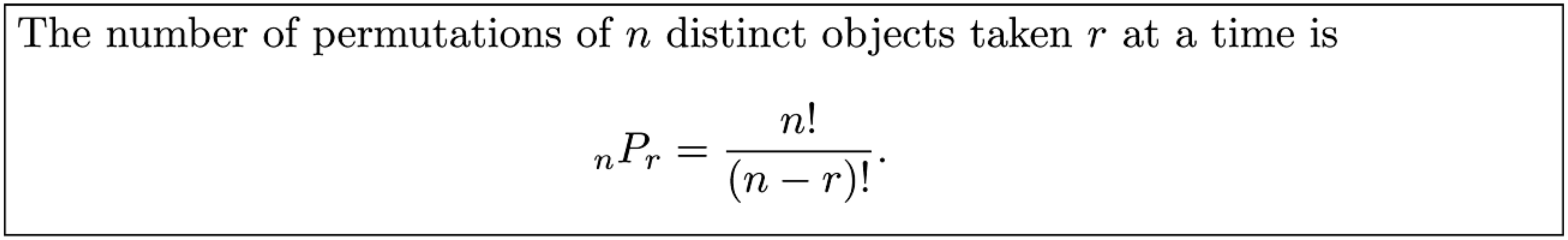


**Number of** possible **combinations:** ${}_4P_2 = \frac{4!}{(4-2)!} = 12$



**Notice: Order matters!**

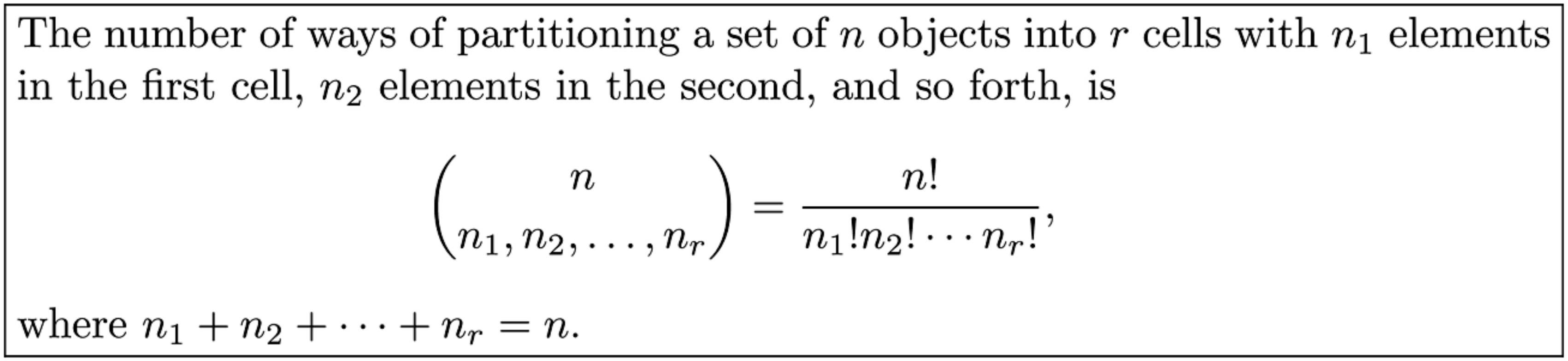
# Permutation



The number of permutations of $n$ distinct objects taken $r$ at a time is

$${}_nP_r = \frac{n!}{(n-r)!}.$$

# Permutation



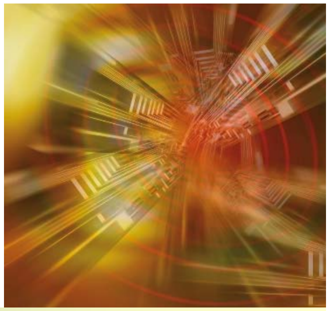
The number of ways of partitioning a set of $n$ objects into $r$ cells with $n_1$ elements in the first cell, $n_2$ elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!},$$

where $n_1 + n_2 + \cdots + n_r = n$.

In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

# Summary of counting

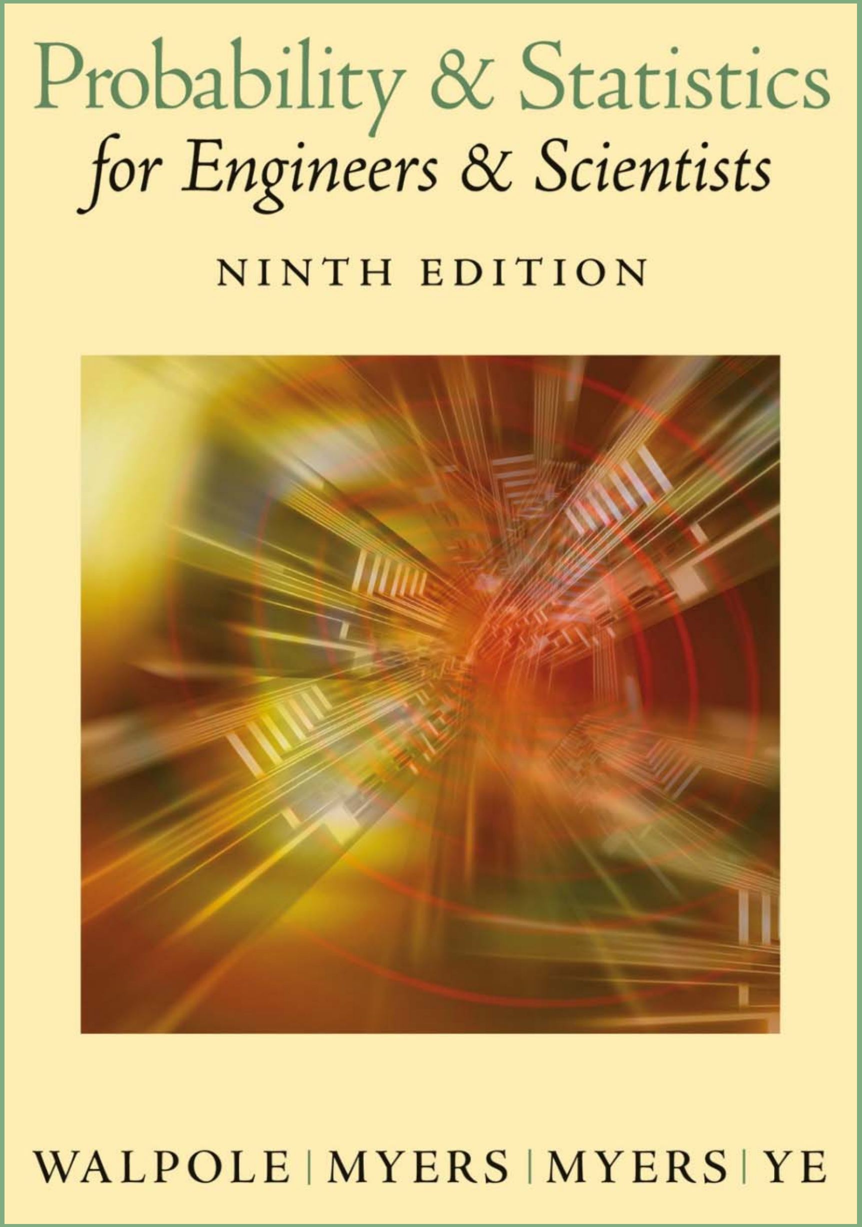


Number of possible ways of selecting r objects from a set of n destinct elements:

| | **Without replacement** | **With replacement** |
|---|---|---|
| **Ordered** | ${}_nP_r = \frac{n!}{(n-r)!}$ | $n^r$ |
| **Unordered** | $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ | - |

# Section 2.4

## Probability of an Event



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# Definition of Probability

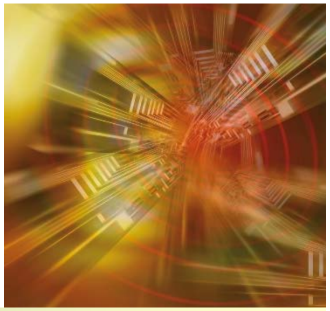
The **probability** of an event $A$ is the sum of the weights of all sample points in $A$. Therefore,

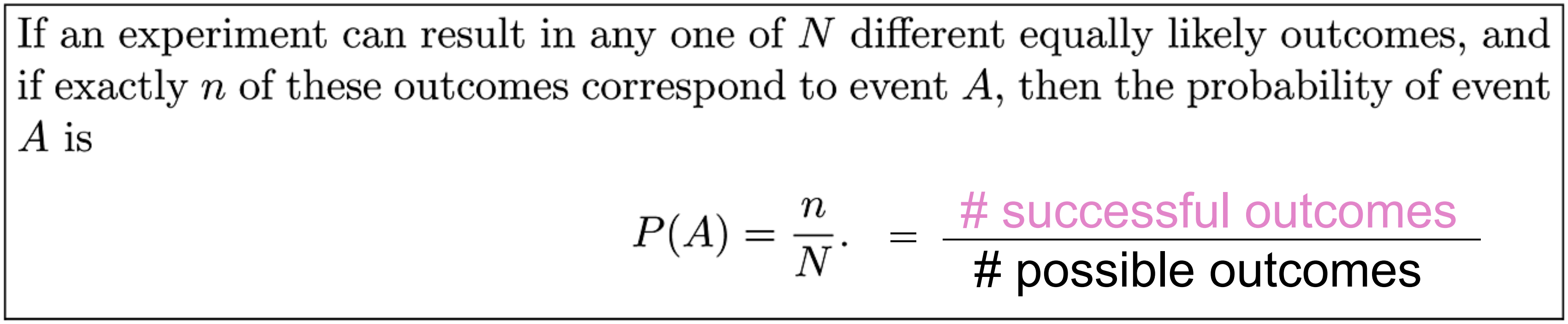
$$0 \leq P(A) \leq 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1.$$

Furthermore, if $A_1$, $A_2$, $A_3$, $\ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots .$$

# Definition of Probability





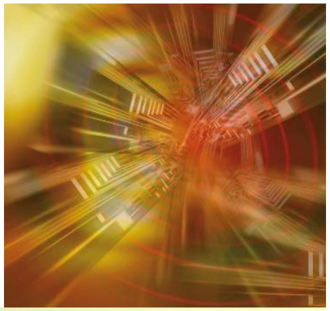
If an experiment can result in any one of $N$ different equally likely outcomes, and if exactly $n$ of these outcomes correspond to event $A$, then the probability of event $A$ is

$$P(A) = \frac{n}{N}. \quad = \frac{\text{\# successful outcomes}}{\text{\# possible outcomes}}$$

Example: Rolling a dice

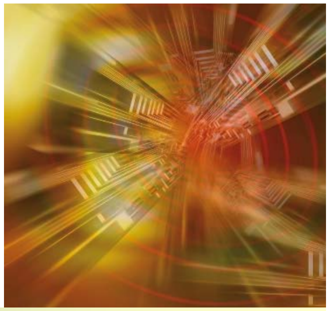
$$P(\text{even number})$$

$$= \frac{3}{6} = \frac{1}{2}$$



# Example of Probability 1

A balanced coin is tossed twice. What is the probability that at least one head occurs?



# Example of Probability 1

A balanced coin is tossed twice. What is the probability that at least one head occurs?

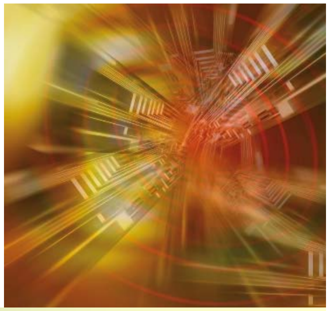
Solution:

S = {HH, HT, TH, TT}

A = {at least one head occurs}= {HH, HT, TH}

Since the coin is balanced, the outcomes are equally likely; i.e., all outcomes have the same weight or probability.

| **Outcome** | **Weight (Probability)** | |
|---|---|---|
| **HH**<br>**HT**<br>**TH**<br>**TT** | **P(HH) = w**<br>**P(HT) = w**<br>**P(TH) = w**<br>**P(TT) = w** | **4w =1 ⇔ w =1/4 = 0.25**<br>**P(HH)=P(HT)=P(TH)=P(TT)=0.25** |
| **sum** | **4w=1** | |

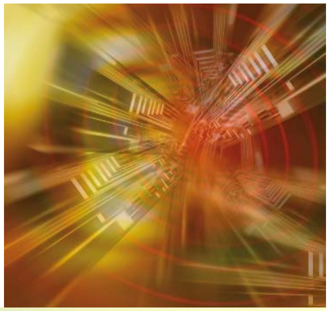


# Example of Probability 1

The probability that at least one head occurs is:
P(A) = P({at least one head occurs})=P({HH, HT, TH})
= P(HH) + P(HT) + P(TH)
= 0.25+0.25+0.25
= 0.75

$$P(A)=\frac{n(A)}{n(S)}=\frac{n(A)}{N}=\frac{no.\ of\ outcomes\ \ in\ \ A}{no.\ of\ outcomes\ \ in\ \ S}$$

# Example of Probability 2

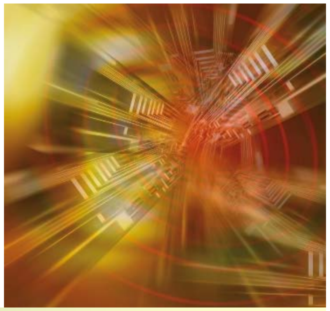


Example: Quality control
A batch of 20 units contains 8 defective units.

Select 6 units (unordered and without replacement).

Event A:  no defective units in our random sample.

# Example of Probability 2



Example: Quality control
A batch of 20 units contains 8 defective units.

Select 6 units (unordered and without replacement).

Event A: no defective units in our random sample.

Number of possible samples: $N = \binom{20}{6}$ **(# possible)**

Number of samples without defective units: $n = \binom{12}{6}$

**(# successful)**

$$P(A) = \frac{\binom{12}{6}}{\binom{20}{6}} = \frac{12!6!14!}{6!6!20!} = \frac{77}{3230} = 0.024$$

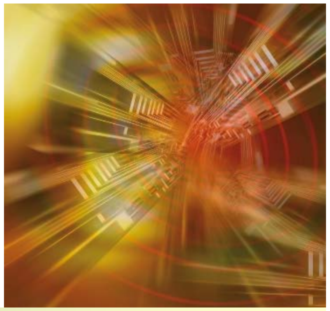
# Example of Probability 3

**Example: continued**

**Event B: exactly 2 defective units in our sample**

**Number of samples with exactly 2 defective units:**

# Example of Probability 3



**Example: continued**

**Event B: exactly 2 defective units in our sample**

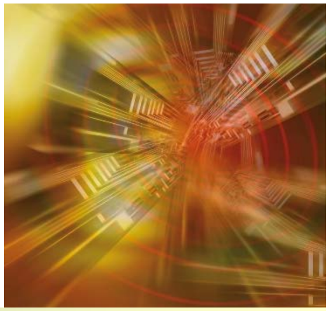
**Number of samples with exactly 2 defective units:**

$$n = \binom{12}{4} \cdot \binom{8}{2}$$

**(# successful)**

$$P(B) = \frac{\binom{12}{4} \cdot \binom{8}{2}}{\binom{20}{6}} = \frac{12!8!6!14!}{4!8!2!6!20!} = 0.3576$$

# Example of Probability 4



Example: In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

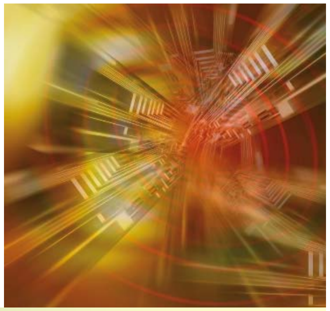
Solution:

Experiment: selecting 5 cards from 52 cards.

n(S) = no. of outcomes of the experiment of selecting 5 cards from 52 cards.

$$= \binom{52}{5} = \frac{52!}{5! \times 47!} = 2598960$$

# Example of Probability 4



The outcomes of the experiment are equally likely because the selection is made at random.

Define the event A = {holding 2 aces and 3 jacks}

n(A) = **no. of ways of selecting 2 aces and 3 jacks**

= (no. of ways of selecting 2 aces) × (no. of ways of selecting 3 jacks)

= (no. of ways of selecting 2 aces from 4 aces) × (no. of ways of selecting 3 jacks from 4 jacks)

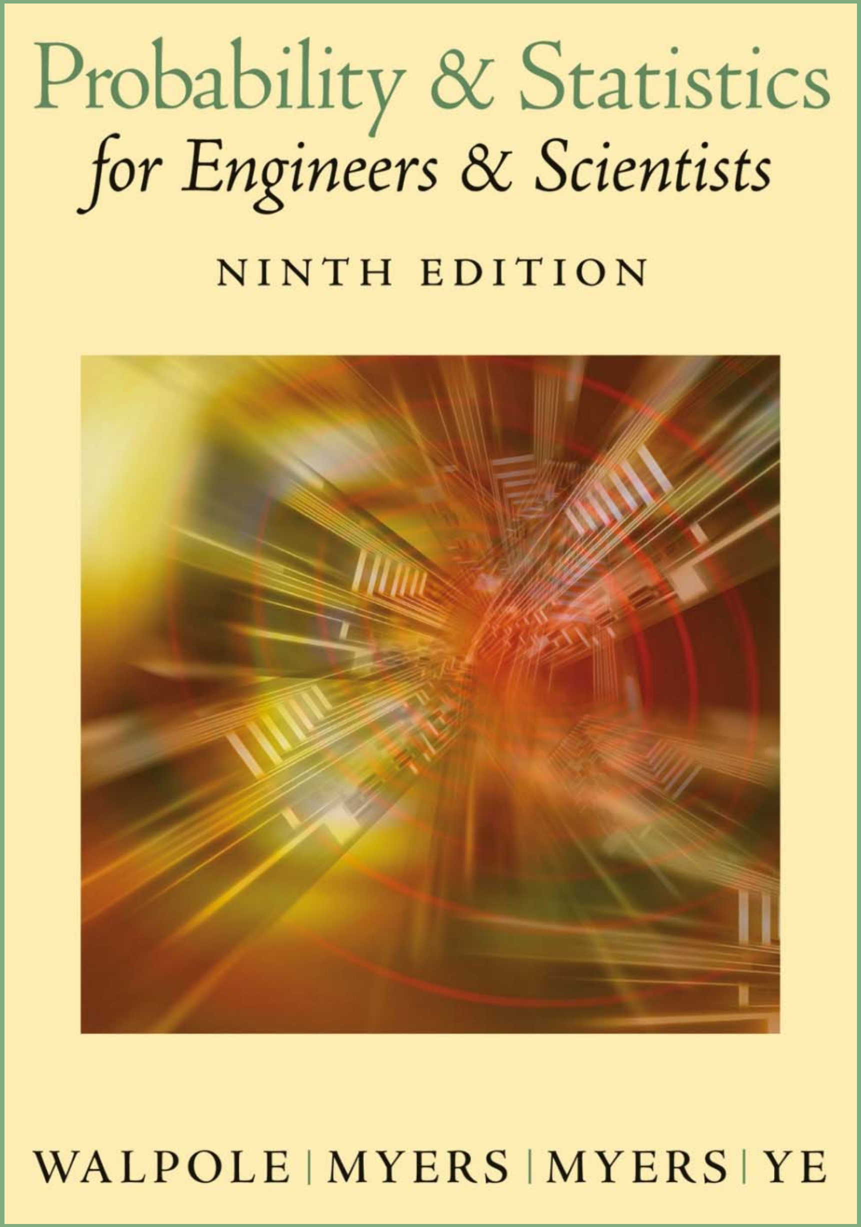
$$=\binom{4}{2}\times\binom{4}{3} = \frac{4!}{2!\times 2!}\times\frac{4!}{3!\times 1!} = 6\times 4 = 24$$

Conclusion: P(A )= P({holding 2 aces and 3 jacks })

$$=\frac{n(A)}{n(S)}=\frac{24}{2598960}=0.000009$$

# Section 2.5

## Additive Rules



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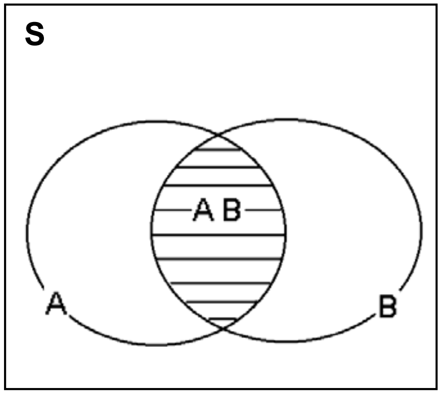
# Theorem 2.7

If $A$ and $B$ are two events, then

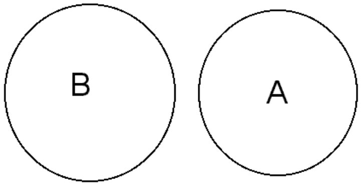
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



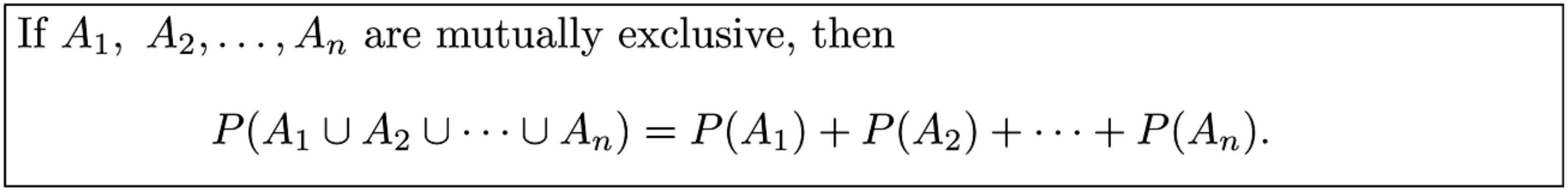
# Corollary 2.1

If $A$ and $B$ are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$



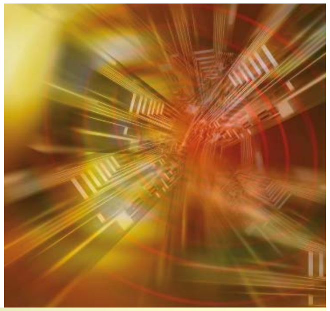
# Corollary 2.2



If $A_1,\ A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

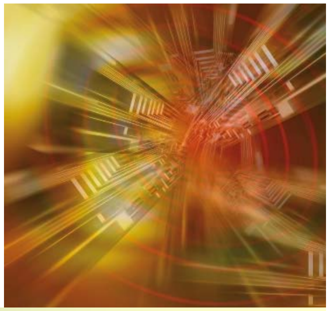
# Corollary 2.3



If $A_1,\ A_2,\ldots,A_n$ is a partition of sample space $S$, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1.$$

# Theorem 2.8



For three events $A$, $B$, and $C$,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

**Not required**

# Theorem 2.9

If $A$ and $A'$ are complementary events, then

$$P(A) + P(A') = 1.$$

# Summary

*Note: Two event Problems:*

Total area= P(S)=1 in Venn diagrams, consider the probability of an event A as the area of the region corresponding to the event A.

* Total area= $P(S)=1$

** Examples:*

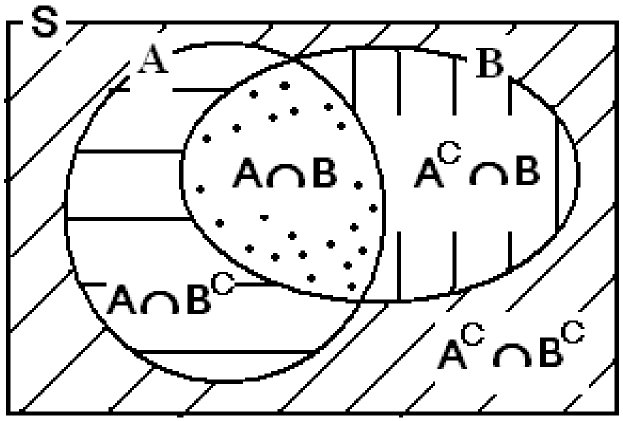
$P(A)= P(A\cap B)+ P(A\cap B^C)$

$P(A\cup B)= P(A) + P(A^C\cap B)$

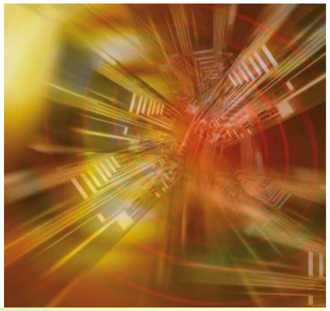
$P(A\cup B)= P(A) + P(B) - P(A\cap B)$

$P(A\cap B^C)= P(A) - P(A\cap B)$

$P(A^C\cap B^C)= 1 - P(A\cup B)$



**Total area= $P(S)=1$**



# Example

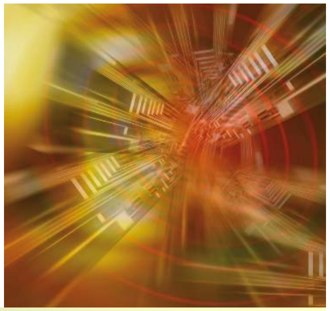
The probability that Paula passes Mathematics is 2/3, and the probability that she passes English is 4/9. If the probability that she passes both courses is 1/4, what is the probability that she will:

(a) pass at least one course?

(b) pass Mathematics and fail English?

(c) fail both courses?

# Example



The probability that Paula passes Mathematics is 2/3, and the probability that she passes English is 4/9. If the probability that she passes both courses is 1/4, what is the probability that she will:

(a) pass at least one course?
(b) pass Mathematics and fail English?
(c) fail both courses?

*Solution:*

Define the events: $M$={Paula passes Mathematics}
$E$={Paula passes English}

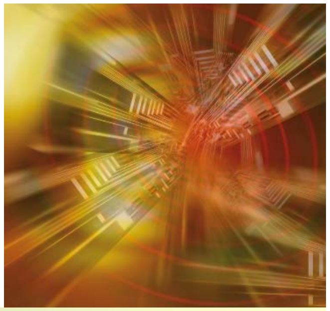
We know that $P(M)=2/3$, $P(E)=4/9$, and $P(M\cap E)=1/4$.

(a) Probability of passing at least one course is:

$$P(M\cup E)= P(M) + P(E) - P(M\cap E)$$

$$=\frac{2}{3}+\frac{4}{9}-\frac{1}{4}=\frac{31}{36}$$

# Example



(b) Probability of passing Mathematics and failing English is:

$P(M \cap E^C) = P(M) - P(M \cap E)$

$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

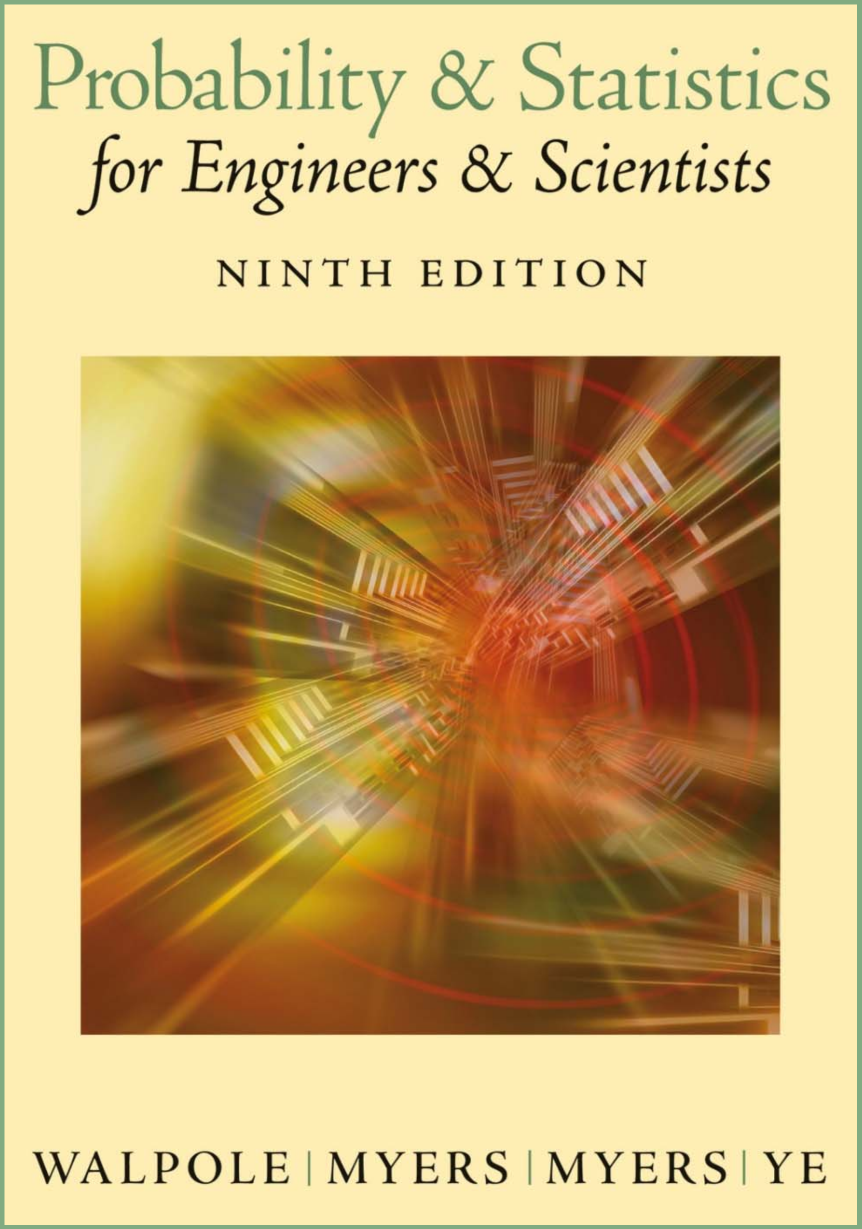
(c) Probability of failing both courses is:

$P(M^C \cap E^C) = 1 - P(M \cup E)$

$$= 1 - \frac{31}{36} = \frac{5}{36}$$

# Section 2.6

## Conditional Probability, Independence, and the Product Rule



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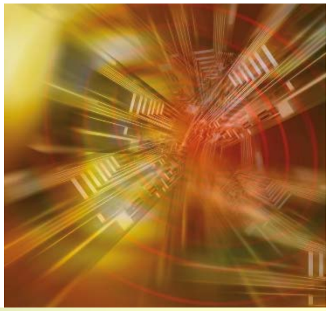
# Definition 2.10



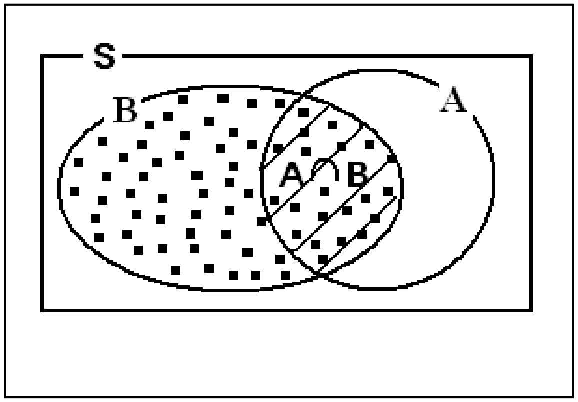
The conditional probability of $B$, given $A$, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

# Definition 2.10

1. $P(A|B)=\dfrac{P(A\cap B)}{P(B)}=$

$$=\frac{n(A\cap B)/n(S)}{n(B)/n(S)}=\frac{n(A\cap B)}{n(B)};\ \textit{for equally likely outcomes case}$$



**P(*S*)=Total area=1**

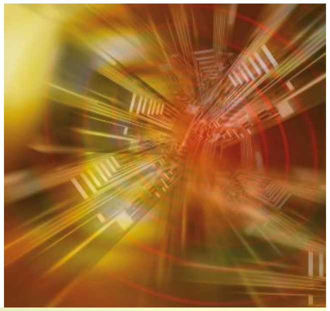
2. $P(B|A)=\dfrac{P(A\cap B)}{P(A)}$

3. $P(A\cap B)=P(A)\,P(B|A)$

$$=P(B)\,P(A|B)$$

**(Multiplicative Rule=Theorem 2.13)**

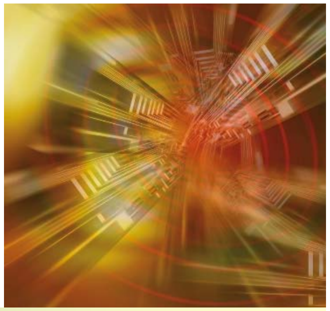
# Theorem 2.10



If in an experiment the events $A$ and $B$ can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

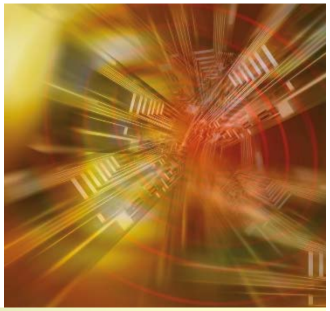
# Example



Example page 59:
The distribution of employed/unemployed amongst men and women in a small town.

| | Employed | Unemployed | Total |
|---|---|---|---|
| Man | 460 | 40 | 500 |
| Woman | 140 | 260 | 400 |
| Total | 600 | 300 | 900 |

# Example



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| | Employed | Unemployed | Total |
|---|---|---|---|
| Man | 460 | 40 | 500 |
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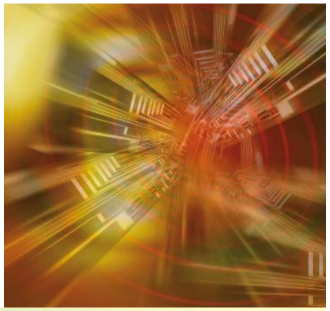
| Total | 600 | 300 | 900 |

$$P(\text{man} \mid \text{employed}) = \frac{P(\text{man \& employd})}{P(\text{employd})} = \frac{460/900}{600/900} = \frac{460}{600} = \frac{23}{30} = 76.7\%$$

$$P(\text{man}|\text{unemployed}) = \frac{P(\text{man \& unemployed})}{P(\text{unemployed})} = \frac{40/900}{300/900} = \frac{40}{300} = \frac{2}{15} = 13.3\%$$

# Example



339 physicians are classified as given in the table below. A physician is to be selected at random.

(1) Find the probability that:

(a) the selected physician is aged 40 – 49

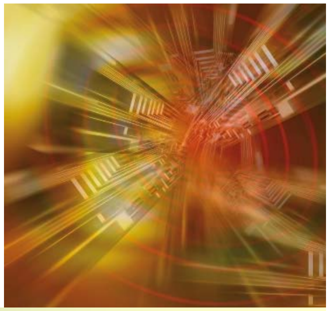
(b) the selected physician smokes occasionally

(c) the selected physician is aged 40 – 49 and smokes occasionally

(2) Find the probability that the selected physician is aged 40 – 49 given that the physician smokes occasionally.

| | | Smoking Habit | | | |
|---|---|---|---|---|---|
| | | Daily ($B_1$) | Occasionally ($B_2$) | Not at all ($B_3$) | Total |
| Age | 20 - 29 ($A_1$) | 31 | 9 | 7 | 47 |
| | 30 - 39 ($A_2$) | 110 | 30 | 49 | 189 |
| | 40 - 49 ($A_3$) | 29 | 21 | 29 | 79 |
| | 50+ ($A_4$) | 6 | 0 | 18 | 24 |
| | Total | 176 | 60 | 103 | 339 |

# Example



$n(S) = 339$   equally likely outcomes.

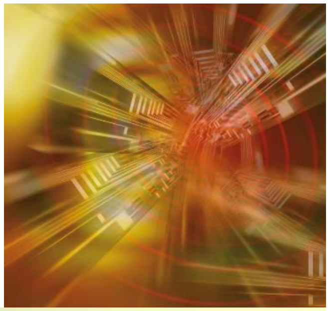
Define the following events:

$A_3$ = the selected physician is aged 40 – 49

$B_2$ = the selected physician smokes occasionally

$A_3 \cap B_2$ = the selected physician is aged 40 – 49 and smokes occasionally

# Example



*Solution:*

(1) (a) $A_3$ = the selected physician is aged 40 – 49

$$P(A_3) = \frac{n(A_3)}{n(S)} = \frac{79}{339} = 0.2330$$

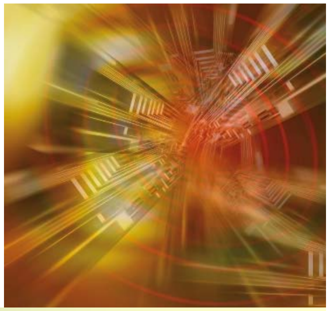
(b) $B_2$ = the selected physician smokes occasionally

$$P(B_2) = \frac{n(B_2)}{n(S)} = \frac{60}{339} = 0.1770$$

(c) $A_3 \cap B_2$ = the selected physician is aged 40 – 49 and smokes occasionally.

$$P(A_3 \cap B_2) = \frac{n(A_3 \cap B_2)}{n(S)} = \frac{21}{339} = 0.06195$$

# Example



*Solution:*

(2) $A_3|B_2$ = the selected physician is aged 40 – 49 given that the physician smokes occasionally

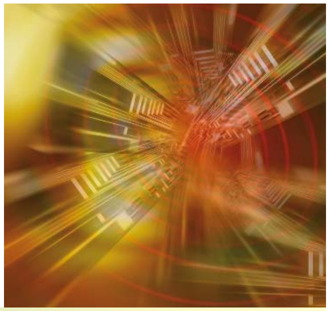
$$\text{(i)}P(A_3 \mid B_2) = \frac{n(A_3 \cap B_2)}{n(B_2)} = \frac{21}{60} = 0.35$$

$$\text{(ii)}P(A_3 \mid B_2) = \frac{P(A_3 \cap B_2)}{P(B_2)} = \frac{0.06195}{0.1770} = 0.35$$

(iii) We can use the restricted table directly:

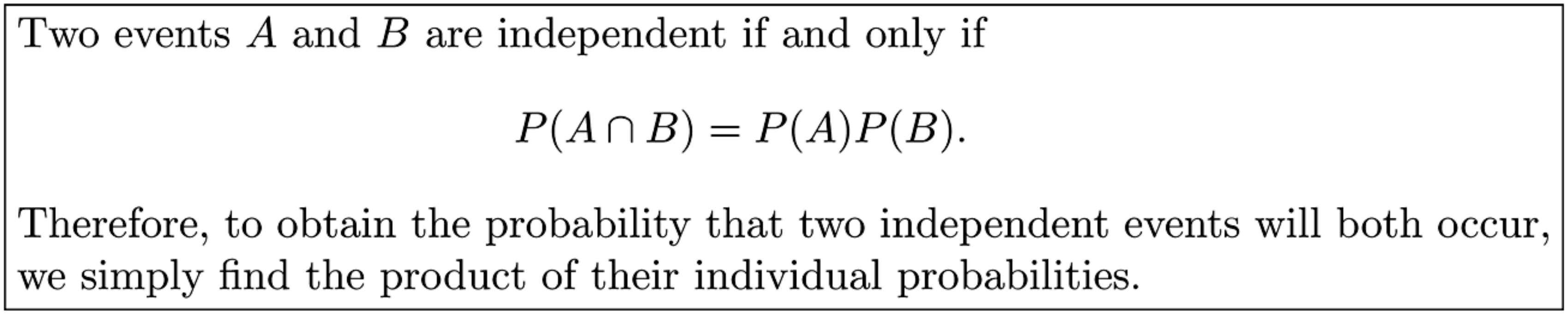
$$P(A_3 \mid B_2) = \frac{21}{60} = 0.35$$

Notice that P($A_3|B_2$)=0.35 > P($A_3$)=0.233.
The conditional probability does not equal unconditional probability; i.e., P($A_3|B_2$) $\neq$ P($A_3$) ! What does this mean?



# Theorem 2.11

## Independent Events:

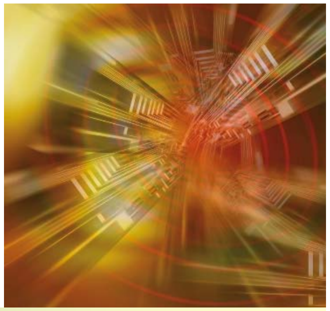


Two events $A$ and $B$ are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

# Example



Suppose we have a fuse box containing 20 fuses of which 5 are defective (D) and 15 are non-defective (N). If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Define the following events:
$A$ = {the first fuse is defective}
$B$ = {the second fuse is defective}
$A \cap B$={the first fuse is defective and the second fuse is
defective} = {both fuses are defective}

# Example

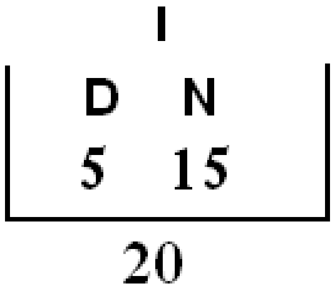
*Solution:*

We need to calculate $P(A \cap B)$.

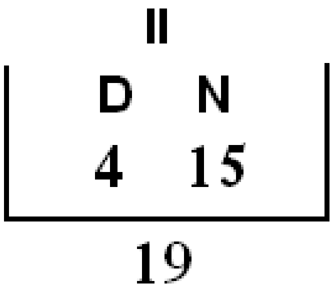
$P(A) = \frac{5}{20}$

$P(B|A) = \frac{4}{19}$

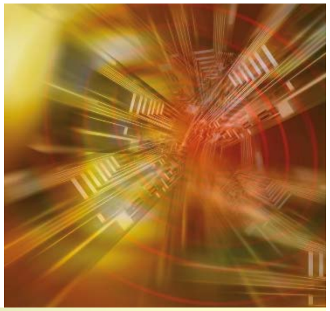
$$P(A \cap B) = P(A)\,P(B|A) = \frac{5}{20} \times \frac{4}{19} = 0.052632$$



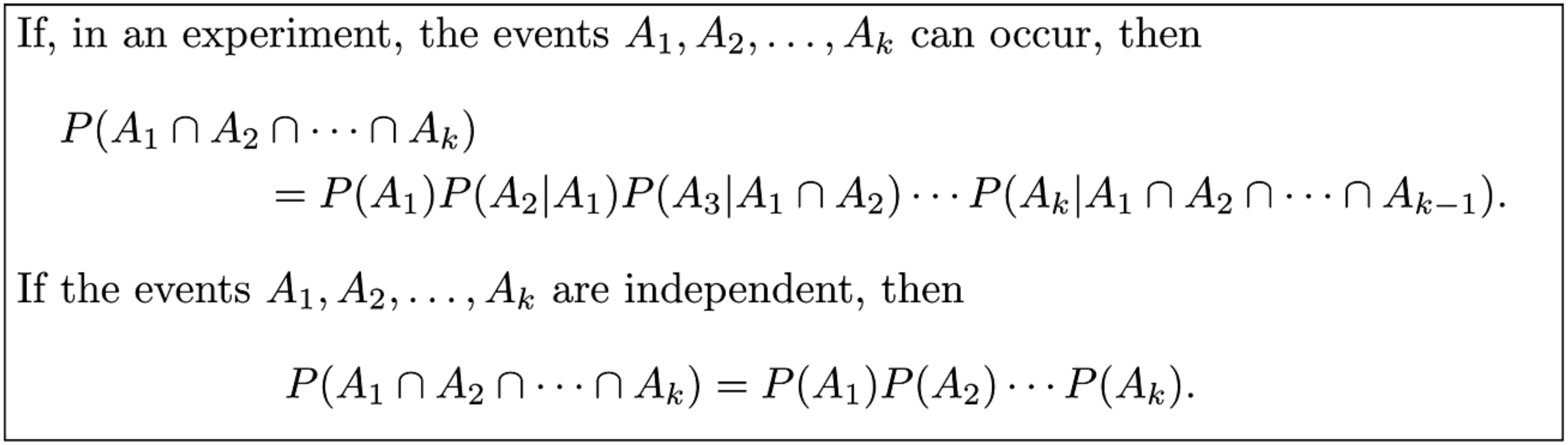
First Selection



Second Selection: given that the first is defective (D)



# Theorem 2.12



If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2) \cdots P(A_k).$$

($k$=3)

- If $A_1$, $A_2$, $A_3$ are 3 events, then:

$$\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(A_1)\ \mathrm{P}(A_2 | A_1)\ \mathrm{P}(A_3 | A_1 \cap A_2)$$

- If $A_1$, $A_2$, $A_3$ are 3 independent events, then:

$$\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(A_1)\ \mathrm{P}(A_2)\ \mathrm{P}(A_3)$$

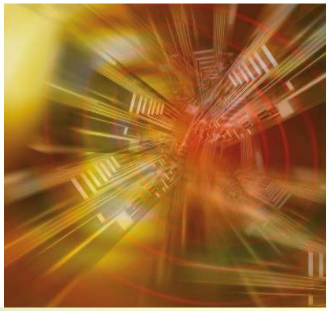
# Example

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Fined P($A_1 \cap A_2 \cap A_3$), where the events $A_1$, $A_2$, and $A_3$ are defined as follows:

$A_1$ = {the 1-st card is a red ace}

$A_2$ = {the 2-nd card is a 10 or a jack}

$A_3$ = {the 3-rd card is a number greater than 3 but less than 7}



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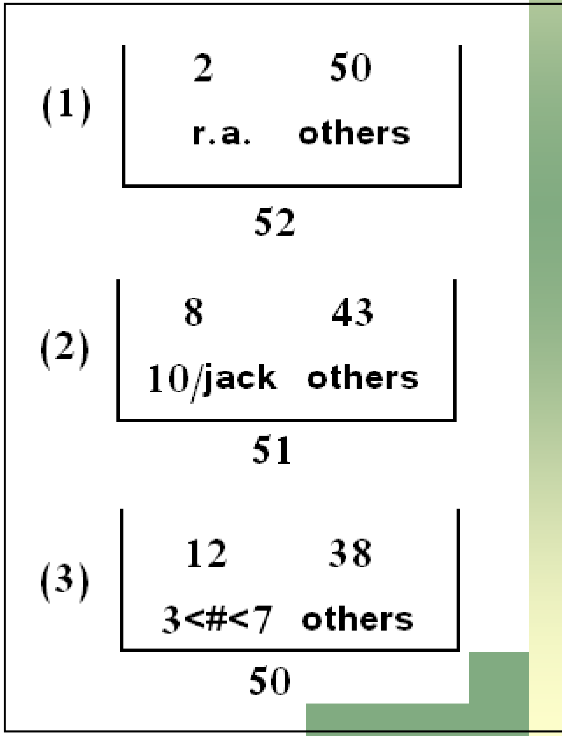
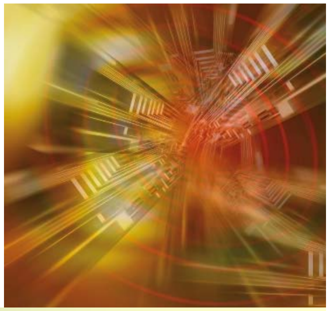


*Solution:*

P($A_1$) = 2/52
P($A_2$ |$A_1$) = 8/51
P($A_3$| $A_1 \cap A_2$) = 12/50
P($A_1 \cap A_2 \cap A_3$)
= P($A_1$) P($A_2$| $A_1$) P($A_3$| $A_1 \cap A_2$)

$$= \frac{2}{52} \times \frac{8}{51} \times \frac{12}{50} = \frac{192}{132600} = 0.0014479$$

# Definition 2.12

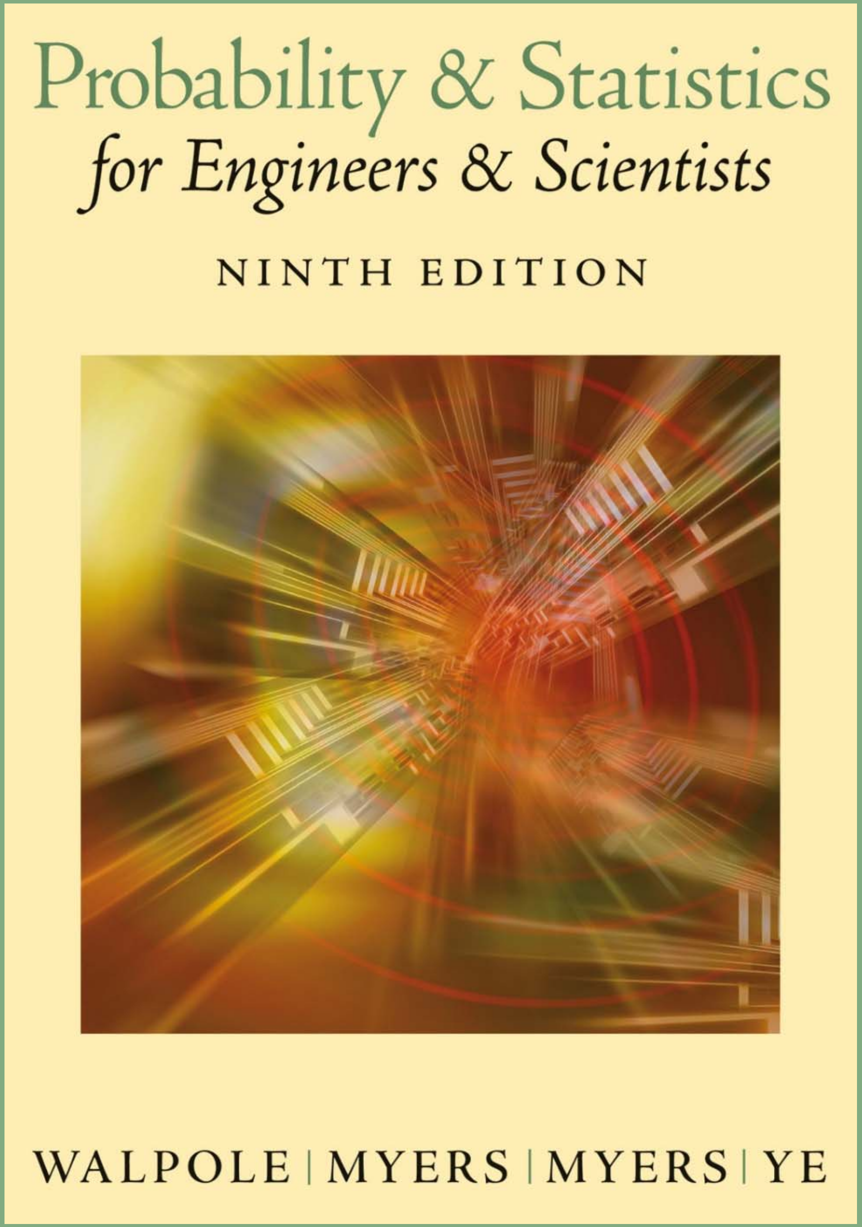


A collection of events $\mathcal{A} = \{A_1, \ldots, A_n\}$ are mutually independent if for any subset of $\mathcal{A}$, $A_{i_1}, \ldots, A_{i_k}$, for $k \leq n$, we have

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}).$$

# Section 2.7

## Bayes' Rule



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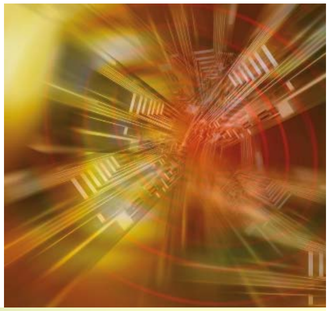


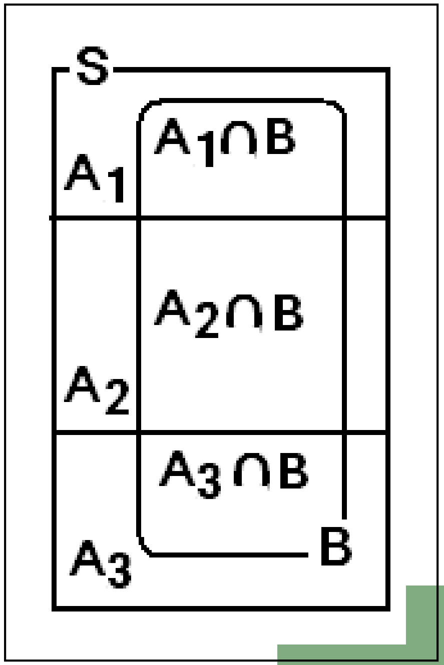
# Bayes' Rule

*Definition:*

The events $A_1$, $A_2$,…, and $A_n$ constitute a partition of the sample space *S* if:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = S$$

$$A_i \cap A_j = \phi, \quad \forall\, i \neq j$$



# Bayes' Rule

*Definition:*

The events $A_1$, $A_2$,…, and $A_n$ constitute a partition of the sample space $S$ if:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = S$$

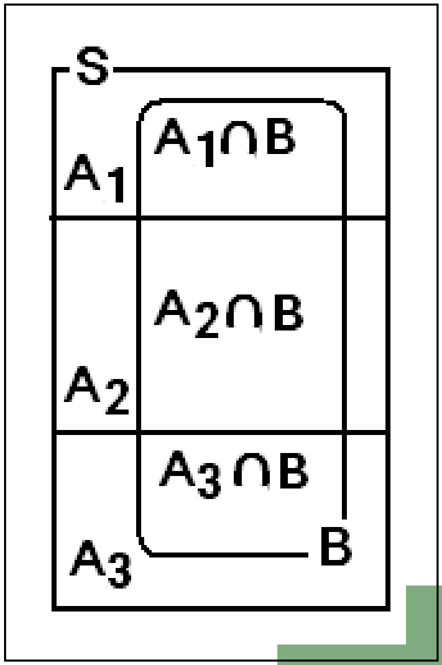
$$A_i \cap A_j = \phi, \quad \forall\ i \neq j$$

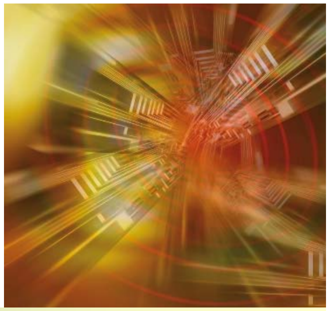
*Theorem 2.16:* (Total Probability)

If the events $A_1$, $A_2$,…, and $A_n$ constitute a partition of the sample space $S$ such that $P(A_k) \neq 0$ for $k=1, 2, \ldots, n$, then for any event $B$:

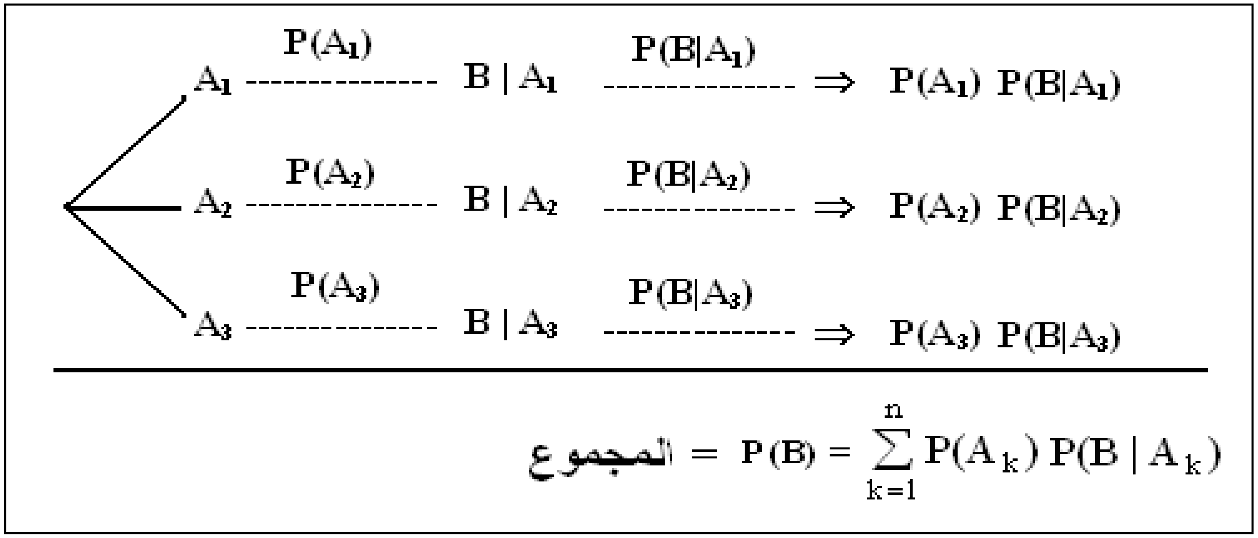
$$P(B) = \sum_{k=1}^{n} P(A_k)\,P(B \mid A_k)$$

$$= \sum_{k=1}^{n} P(A_k \cap B)$$





# Bayes' Rule



$A_1$ $P(A_1)$ $B \mid A_1$ $P(B|A_1)$ $\Rightarrow$ $P(A_1)\,P(B|A_1)$

$A_2$ $P(A_2)$ $B \mid A_2$ $P(B|A_2)$ $\Rightarrow$ $P(A_2)\,P(B|A_2)$

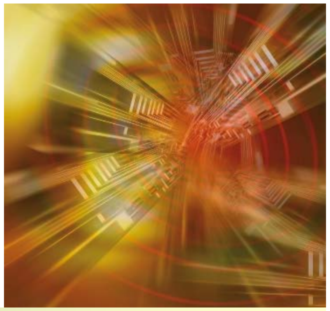
$A_3$ $P(A_3)$ $B \mid A_3$ $P(B|A_3)$ $\Rightarrow$ $P(A_3)\,P(B|A_3)$

المجموع $= P(B) = \sum_{k=1}^{n} P(A_k)\,P(B \mid A_k)$

*Example 2.38:*

Three machines $A_1$, $A_2$, and $A_3$ make 20%, 30%, and 50%, respectively, of the products. It is known that 1%, 4%, and 7% of the products made by each machine, respectively, are defective. If a finished product is randomly selected, what is the probability that it is defective?

# Example



*Example 2.38:*

Three machines $A_1$, $A_2$, and $A_3$ make 20%, 30%, and 50%, respectively, of the products. It is known that 1%, 4%, and 7% of the products made by each machine, respectively, are defective. If a finished product is randomly selected, what is the probability that it is defective?

Define the following events:

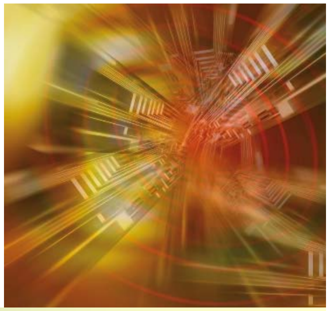
$B$ = {the selected product is defective}

$A_1$ = {the selected product is made by machine $A_1$}

$A_2$ = {the selected product is made by machine $A_2$}

$A_3$ = {the selected product is made by machine $A_3$}

# Example



*Solution:*

$$P(A1) = \frac{20}{100} = 0.2; \quad P(B \mid A1) = \frac{1}{100} = 0.01$$

$$P(A2) = \frac{30}{100} = 0.3; \quad P(B \mid A2) = \frac{4}{100} = 0.04$$

$$P(A3) = \frac{50}{100} = 0.5; \quad P(B \mid A3) = \frac{7}{100} = 0.07$$

$$P(B) = \sum_{k=1}^{3} P(A_k) P(B \mid A_k)$$

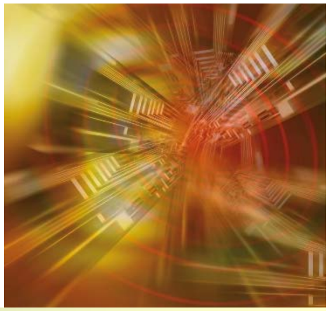
$= P(A_1)\ P(B|A_1) + P(A_2)\ P(B|A_2) + P(A_3)\ P(B|A_3)$

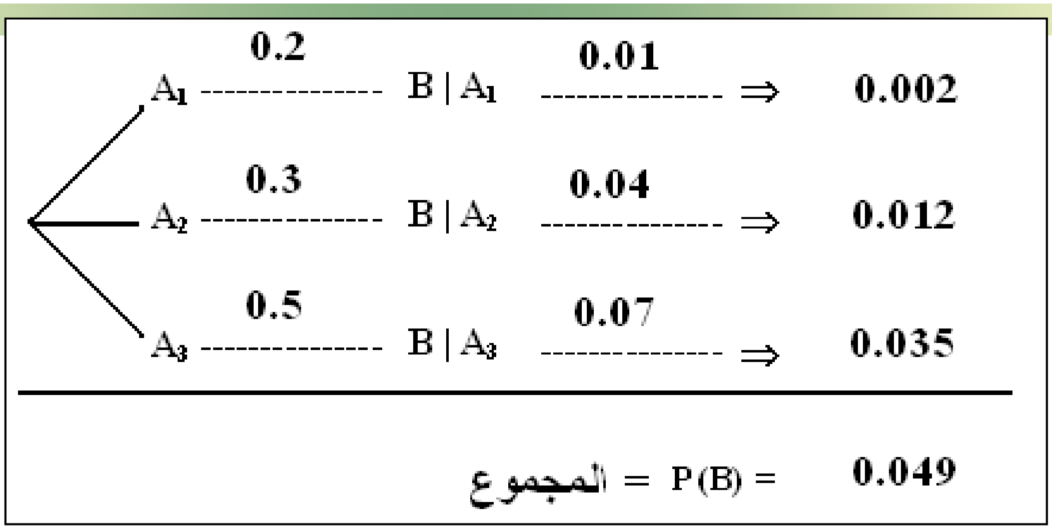
$= 0.2\times0.01 + 0.3\times0.04 + 0.5\times0.07$

$= 0.002 + 0.012 + 0.035$

$= 0.049$

# Example





***Question:***

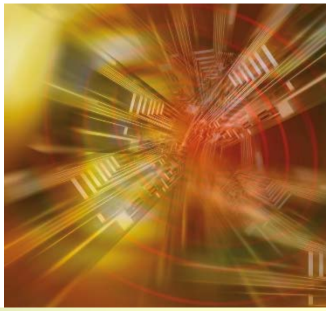
**If it is known that the selected product is defective, what is the probability that it is made by machine $A_1$?**

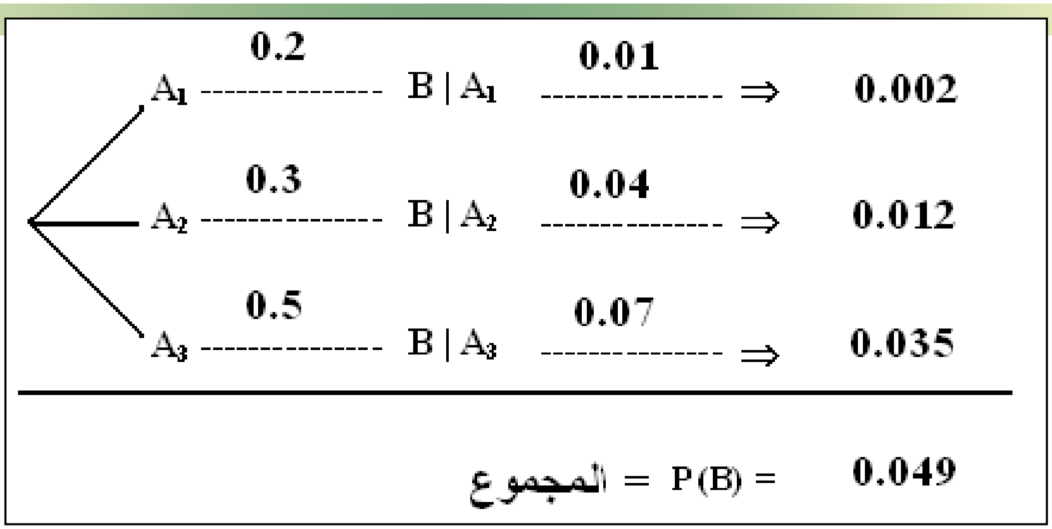
**Answer:**

$$P(A1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.2 \times 0.01}{0.049} = \frac{0.002}{0.049} = 0.0408$$

**This rule is called Bayes' rule.**

# Example





***Question:***

**If it is known that the selected product is defective, what is the probability that it is made by machine $A_1$?**

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$$P(A1|B)=\frac{P(A_1\cap B)}{P(B)}=\frac{0.2\times 0.01}{0.049}=\frac{0.002}{0.049}=0.0408$$

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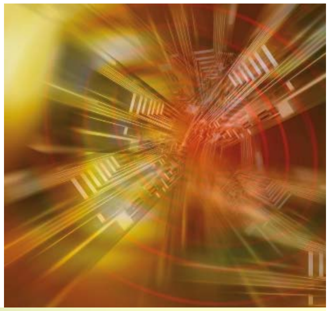
# Bayes' Rule



**If the events $A_1, A_2, \ldots,$ and $A_n$ constitute a partition of the sample space $S$ such that $P(A_k) \neq 0$ for $k=1, 2, \ldots, n$, then for any event $B$ such that $P(B) \neq 0$:**

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)} = \frac{P(A_i)P(B|A_i)}{P(B)}$$

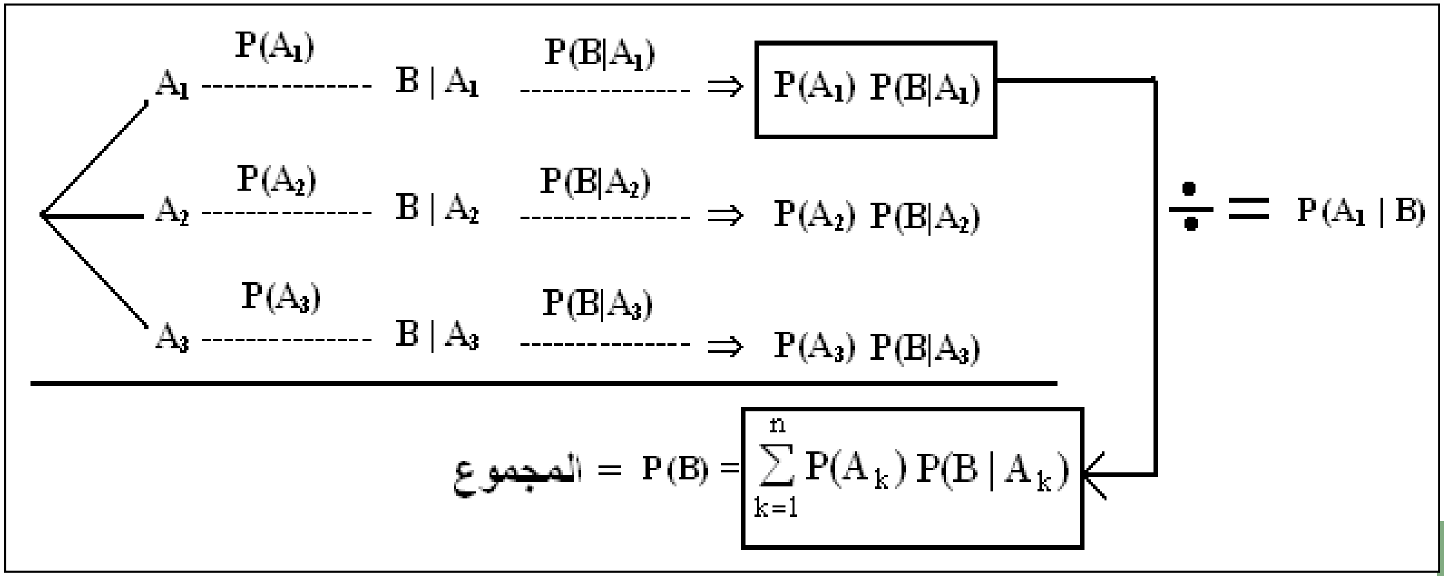
**for $i = 1, 2, \ldots, n$.**

# Bayes' Rule

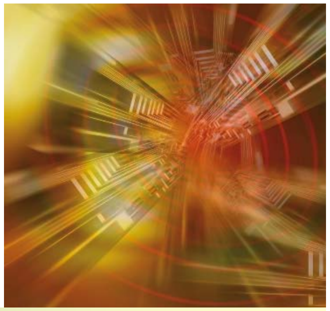
**If the events $A_1, A_2, \ldots,$ and $A_n$ constitute a partition of the sample space $S$ such that $P(A_k) \neq 0$ for $k=1, 2, \ldots, n$, then for any event $B$ such that $P(B) \neq 0$:**

$$P(A_i | B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i) P(B | A_i)}{\sum_{k=1}^{n} P(A_k) P(B | A_k)} = \frac{P(A_i) P(B | A_i)}{P(B)}$$

**for $i = 1, 2, \ldots, n$.**



# Example



## Lung disease & Smoking

According to ”The American Lung Association” 7% of the population suffers from a lung disease, and 90% of these are smokers. Amongst people without any lung disease 25.3% are smokers.

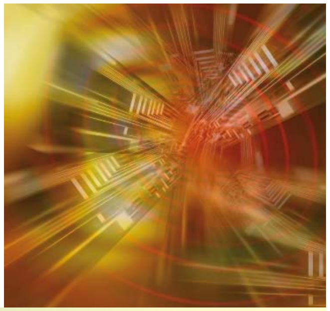
**Events:**
**A: person has lung disease**
**B: person is a smoker**

**Probabilities:**
**P(A) = 0.07**
**P(B|A) = 0.90**
**P(B|A´ ) = 0.253**

**What is the probability that at smoker suffers from a lung disease?**

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')} = \frac{0.9 \cdot 0.07}{0.9 \cdot 0.07 + 0.253 \cdot 0.93} = 0.211$$

# Example



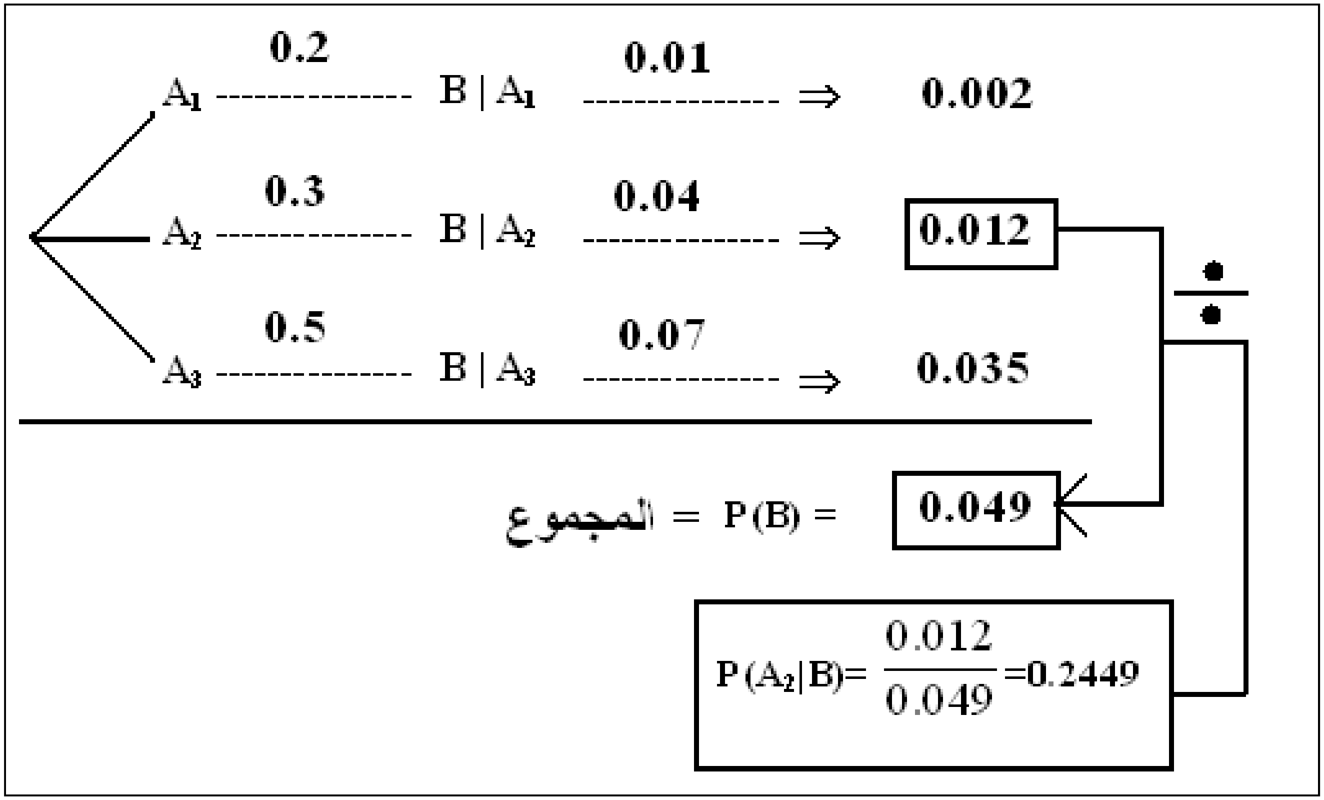
***Example 2.39:***

**In Example 2.38, if it is known that the selected product is defective, what is the probability that it is made by:**

**(a) machine $A_2$?**

**(b) machine $A_3$?**

# Example



$A_1$ 0.2 $B \mid A_1$ 0.01 ⇒ 0.002

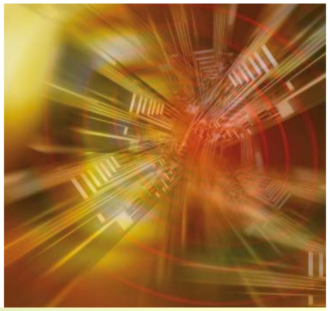
$A_2$ 0.3 $B \mid A_2$ 0.04 ⇒ $\boxed{0.012}$

$A_3$ 0.5 $B \mid A_3$ 0.07 ⇒ 0.035

÷

المجموع = $P(B) = \boxed{0.049}$

$$P(A_2|B)= \frac{0.012}{0.049} = 0.2449$$



# Example

*Solution:*

$$(a)P(A_2|B) = \frac{P(A_2)P(B|A_2)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)} = \frac{P(A_2)P(B|A_2)}{P(B)}$$

$$= \frac{0.3 \times 0.04}{0.049} = \frac{0.012}{0.049} = 0.2449$$

$$(b)P(A_3|B) = \frac{P(A_3)P(B|A_3)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)} = \frac{P(A_3)P(B|A_3)}{P(B)}$$

$$= \frac{0.5 \times 0.07}{0.049} = \frac{0.035}{0.049} = 0.7142$$