Module - 2 Differential Calculas

1) **Taylor Series** :- Taylor series expansion of $f(x)$ about the point $x = a$ is given by

$$y(x) = y(a) + (x-a)y_1(a) + \frac{(x-a)^2}{2!}y_2(a) + \frac{(x-a)^3}{3!}y_3(a) + \cdots$$

2) **Maclaurin's Series Expansion** :- If $a = 0$, we get maclaurem series expansion of $y(n)$ given by

$$y(x) = y(0) + x(y_1)(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_2(0) + \cdots$$

① Problems :

Obtain the Taylor Series Expansion of 1) $\log(\cos x)$ $x = \pi/3$ upto the 4^{th} degree term.

Solⁿ :- Taylor series expansion about $x = a$ is given by $y(x)$:

$y(x) = y(\pi/3) + (x-a)y_1(a) + \frac{(x-a)^2}{2!}y_2(a) + \cdots$ upto the 4^{th} degree term about the point $x = \frac{\pi}{3}$ is given by.

$$y(x) = y(\pi/3) + (x-\pi/3)y_1(\pi/3) + \frac{(x-\pi/3)^2}{2!}y_2(\pi/3) + \frac{(x-\pi/3)^3}{3!}y_3(\pi/3) + \frac{(x-\pi/3)^4}{4!}y_4(\pi/3)$$

Let $y(x) = \log(\cos x)$; $y(\pi/3) = \log(\cos\pi/3)$

$$\log(\cos\pi/3) = \log\left(\frac{1}{2}\right) = \overset{0}{\log 1} - \log 2 = \underline{\underline{-\log 2}}$$

$y_1(x) = \tan x$; $y_1(\pi/3) = -\tan\pi/3 = -\sqrt{3}$

$y_2(x) = \sec^2 x$; $y_2(\pi/3) = -\sec^2\left(\frac{\pi}{3}\right) = -(2)^2 = \underline{\underline{-4}}$

$$= -(1 + \tan^2 x)$$

$$y_2(x) = -(1 + y_1^2)$$

differentiating

$$y_3(x) = -(0 + 2y_1 \cdot y_2) \; ; \; y_3(\pi/3) = -2(-\sqrt{3})(-4)$$

$$= -8\sqrt{3}$$

differentiating $y_4(x) = -2[y_1 y_3 + y_2 y_2]$

$$\Rightarrow -2[y_1 y_3 + y_2^2] \; ;$$

$$y_4(\pi/3) = -2[(-\sqrt{3} \times 8\sqrt{3}) + (-4)^2]$$

$$= -2[24 + 16]$$

$$\boxed{= -80}$$

Sub. in equation 1.

$$\log(\cos x) = -\log 2 + (x - \pi/3)(-\sqrt{3}) + \frac{(x-\pi/3)^2}{2} \times (-4) +$$

$$\frac{(x-\pi/3)^3}{6} \times (-8\sqrt{3}) + \frac{(x-\pi/3)^4}{24} \times (-80)$$

$$y(x) = -\log 2 - \sqrt{3}(x - \pi/3) - 2(x-\pi/3)^2 - \frac{4}{3}\sqrt{3}\,(x-\pi/3)^3 - \frac{10}{3}$$

$$- \frac{10}{3}(x - \pi/3)^4$$

2. Expand $\tan x$ about the point $x = \pi/4$ upto 3rd degree.

Sol^n:- Taylor series expansion $x = a$ is given by

$$y(x) = y(a) + (x-a)y_1(a) + \frac{(x-a)^2}{2!} \cdot y_2(a) + \frac{(x-a)^3}{3!} y_3(a)$$

Taylor series expression expansion upto 3rd degree and $x = \pi/4$ given by

$$y(x) = y(\pi/4) + (x - \pi/4)\, y_1(\tfrac{\pi}{4}) + \frac{(x-\pi/4)^2}{2!} \cdot y_2(\tfrac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^3}{3!} \cdot y_3(\pi/4)$$

1) Let $y(x) = \cancel{\log}\ \tan x \rightarrow (2)$

$y\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1 \qquad \boxed{y\left(\frac{\pi}{4}\right) = 1}$

2) Consider $y(x) = \tan x$.

Differentiate w.r.t x.

$y_1(x) = \sec^2 x \longrightarrow (3)$

$y_1\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = (\sqrt{2})^2 = 2 \qquad \boxed{y_1\left(\frac{\pi}{4}\right) = 2}$

3) Consider equation (3), $y_1(x) = \sec^2 x$.

Differentiate w.r.t

$y_2(x) = 2\sec x \sec x \tan x$

$y_2(x) = 2\sec^2 x \tan x$

$y_2(x) = 2 \cdot y_1(x) \cdot y(x) \longrightarrow (4)$

$y_2\left(\frac{\pi}{4}\right) = 2 \cdot y_1\left(\frac{\pi}{4}\right) \cdot y\left(\frac{\pi}{4}\right) = 2 \cdot 2 \cdot 1 = 4$

$\boxed{y_2\left(\frac{\pi}{4}\right) = 4}$

(4) Consider equation 4 $y_2(x) = 2 \cdot y_1(x) \cdot y(x)$

Diff. w.r.t x.

$y_3(x) = 2\left(y_1^2(x) + y_2(x) \cdot y(x)\right)$

$y_3\left(\frac{\pi}{4}\right) = 2\left[y_1^2(\pi/4) + y_2(\pi/4) \cdot y(\pi/4)\right] = 2\left[4 + 4 \cdot 1\right]$

$= 2\left[4 + 4 \cdot 1\right]$

$y_3(\pi/4) = 2 \cdot 8 \qquad \boxed{y_3(\pi/4) = 16}$

Substitute value $y(x) = 1 + (x - \pi/4) \cdot 2 + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} \cdot 4 + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!} \cdot 16$

$y(x) = 1 + 2(x - \pi/4) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ — Answer

3) $y = \sqrt{1+\sin 2x}$ Using Maclaurin's series explain

$$\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \ldots\ldots$$

Solⁿ: Maclaurein's series expansion is given by

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots\ldots \rightarrow (1)$$

Consider $y(x) = \sqrt{1+\sin 2x}$

$$= \sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}$$

$$= \sqrt{(\sin x + \cos x)^2}$$

$y(x) = \sin x + \cos x$; $y(0) = 0+1 = \boxed{1}$

$y_1(x) = \cos x - \sin x$; $y_1(0) = 1-0 = \boxed{1}$

$y_2(x) = -\sin x - \cos x$; $y_2(0) = -0-1 = \boxed{-1}$

$y_3(x) = -\cos x + \sin x$; $y_3(0) = -1+0 = \boxed{-1}$

$y_4(x) = +\sin x + \cos x$; $y_4(0) = 0+1 = \boxed{1}$

Substituting in eq.(1),

$$\sqrt{1+\sin 2x} = 1 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(-1) + \frac{x^4}{24}(1) + \ldots\ldots$$

$$= 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \ldots\ldots$$

4) Expand $\log(\sec x)$ by Maclaurein's series upto the term containing x^6.

Solⁿ Maclaurein's Series is given by

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots\ldots \quad -(1)$$

Given $y(x) = \log(\sec x)$; $y(0) = \log(\sec 0) = \log 1 = \boxed{0}$

$y_1(x) = \frac{1}{\sec x} \times \sec x \tan x = \tan x$; $y_1(x) = \tan x$ $y_1(0) = \tan 0$ $\boxed{0}$

$y_2(x) = \sec^2 x \quad ; \quad y_2(0) = \sec^2 0 = 1^2 = \boxed{1}$

$= (1 + \tan^2 x)$

$= (1 + y_1^2)$

we have $y_2 = 1 + y_1^2$

differentiating

$y_3 = 0 + 2y_1 \cdot y_2 \quad ; \quad y_3(0) = 2\,y_1(0) \cdot y_2(0)$

differentiating $\quad = 2 \times 0 \times 1 = \boxed{0}$

$y_4 = 2\left[y_1 y_3 + y_2 \cdot y_2\right]$

$= 2y_1 y_3 + y_2^2 \quad ; \quad y_4(0) = 2 \times 0 \times 0 + 2 \times 1^2 = \boxed{2}$

differentiating.

$y_5 = 2\left[y_1 \cdot y_4 + y_3 \cdot y_2\right] + 4y_2 \cdot y_3$

$= 2y_1 y_4 + 6y_2 y_3 \quad ; \quad y_5(0) = 0 + 0 = \boxed{0}$

differentiating.

$y_6 = 2\left[y_1 y_5 + y_4 \cdot y_2\right] + 6\left[y_2 y_4 + y_3 \cdot y_3\right]$

$y_6(0) = 2(0 + 2 \times 1) + 6\,(2 \times 1 + 0) = 4 + 12 = \boxed{16}$

Substituting in equation 1,

$$\log(\sec x) = 0 + x(0) + \frac{x^2}{2!} \times (1) + \frac{x^3}{6} \times (0) + \frac{x^4}{24}(2) + \frac{x^5}{120}(0) + \frac{x^6}{720}(16)$$

$$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}$$

⑤ Expand $\log(1+e^x)$ using Maclaurein series till 3^{rd} degree

Solⁿ:- Maclaurein theorom series expansion is given by

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \ldots\ldots$$

Given $y(x) = \log(1+e^x)$; $y(0) = \log(1+e^0) = \log(1+1)$ $= \boxed{\log 2}$

differentiating

$$y_1 = \frac{1}{1+e^x} \times e^x \quad ; \quad y_1(0) = \frac{e^0}{1+e^0} = \boxed{\frac{1}{2}}$$

$$(1+e^x)\, y_1 = e^x$$

differentiating

$$(1+e^x)y_2 + y_1 \cdot e^x = e^x ; (1+e^0)y(0) + \frac{1}{2}\cdot 1 = 1$$

$$2y_2(0) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\boxed{y_2(0) = \frac{1}{4}}$$

differentiating,

$$(1+e^x)y_3 + y_2 \cdot e^x + y_1 e^x + e^x \cdot y_2 = e^x$$

$$(1+1)\, y_3(0) + \frac{1}{4}\cdot 1 + \frac{1}{2}\cdot 1 + 1\cdot\frac{1}{4} = 1$$

$$2y_3(0) + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

$$2y_3(0) = 0$$

$$\boxed{y_3(0) = 0}$$

6) $y = \tan x$ using Maclaurein series till 4^{th} degree.

Solⁿ. $y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3} y_3(0) + \frac{x^4}{4!} y_4(0)$

Consider,

$y(x) = \tan x$; $\tan(0) = \boxed{0}$

i) $y_1 = \sec^2 x = \sec^2(0) = 1^2 = 1 \Rightarrow \boxed{y_1(0) = 1}$

ii) $y_1(x) = \sec^2 x$

Diff. w.r.t. x.

$y_2(x) = 2\sec x \cdot \sec x \tan x$

$y_2(x) = 2\sec^2 x \cdot \tan x$

$y_2(x) = 2 \cdot y_1(x) \cdot y(x)$

iii) $y_2(0) = 2 \cdot 1 \cdot 0 = 0 \rightarrow \boxed{y_2(0) = 0}$

iv) Consider $y_2(x) = 2 \cdot y_1(x) \cdot y(x)$

Diff. w.r.t. x

$y_3(x) = 2(y_1^2(x) + y_2(x) \cdot y(x))$

$y_3(0) = 2(1+0) = 2(1) \cdot 2 \Rightarrow \therefore \boxed{y_3(0) = 2}$

v) Consider $y_3(x) = 2(y_1^2(x) + y_2(x) - y(x))$

Diff. w.r.t. x.

$y_4(x) = 2(2y(x) \cdot y_2(x) + y_3(x) \cdot y(x)) + y_1(x) \cdot y_2(x).]$

$y_4(x) = 6y_1(x) \cdot y_2(x) + y_2(x) \cdot y(x)$

$y_4(0) = 6 \cdot 1 \cdot 0 + 2 \cdot 0 = 0 \quad \boxed{\therefore y_4(0) = 0}$

After substituting, $\boxed{y(x) = x + \frac{x^3}{3}}$

Indeterminate form

If an expression $F(x)$ at $x=a$ assumes forms like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, $0\times\infty$, 0^0, ∞^0, 1^∞ which do not represent any value are called Indeterminate form.

L'Hospital Rule / Theorom

→ To evalute $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are two functions such that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then by L'Hospital Rule. $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

This rule is applicable for $\boxed{\frac{0}{0} \text{ and } \frac{\infty}{\infty}}$ form

Standard limits

1) $\lim_{x\to 0} \frac{\sin x}{x} = 1$
2) $\lim_{x\to 0} \frac{x}{\sin x} = 1$
3) $\lim_{x\to 0} \frac{\tan x}{x} = 1$
4) $\lim_{x\to 0} \frac{x}{\tan x} = 1.$

Note: i) $e^0 = 1$
ii) $e^\infty = \infty$
iii) $e^{-\infty} = 0$
iv) $\log 1 = 0$
v) $\log 0 = -\infty.$
vi) $\log \infty = \infty$

Indeterminate forms of the type $0^0, \infty^0, 1^\infty$

⇒ Working Rule :- Given function will be of the form $\lim_{x\to a} [f(x)]^{g(x)}$, Let $K = \lim_{x \to a} [f(x)]^{g(x)}$

② Take log on both sides so that $\log_e K = \lim_{x \to a} g(x) \cdot \log_e [f(x)]$

③ If the right hand limit is l then $\log_e K = l$ so that $K = e^l$ will be the required limit.

□ Evaluate the following Indeterminate forms.

1) $\lim_{x \to \pi/2} \frac{\log(x - \pi/2)}{\tan x}$

Sol Let $K = \lim_{x \to \pi/2} \frac{\log(x - \pi/2)}{\tan x} = \left(\frac{-\infty}{\infty}\right)$

LHR,

$K = \lim_{x \to \pi/2} \frac{\frac{1}{(x - \pi/2)} \times (1-0)}{\sec^2 x}$

$= \lim_{x \to \pi/2} \frac{\cos^2 x}{x - \pi/2} = \left(\frac{0}{0}\right)$

LHR,

$K = \lim_{x \to \pi/2} \frac{2 \cos x \times -\sin x}{(1-0)}$

$K = \frac{2 \times 0 \times -1}{1} = \frac{0}{1}$ $\boxed{K = 0}$

□ Problems on $1^\infty, 0^0, \infty^0$

1) $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ ~~tan~~

Let $K = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} = (1^\infty)$

$\log_e K = \lim_{x \to a} \log_e \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

$$\log_e k = \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right)\cdot \log_e\left(2-\frac{x}{a}\right)$$

$$= \lim_{x\to a} \frac{\log_e\left(2-\frac{x}{a}\right)}{1/\tan\left(\frac{\pi x}{2a}\right)}$$

$$= \lim_{x\to a} \frac{\log_e\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} = \left(\frac{0}{0}\right)$$

LHR.

$$\log_e k = \lim_{x\to a} \frac{\frac{1}{\left(2-\frac{x}{a}\right)} \times \left(0-\frac{1}{a}\right)}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\times\frac{\pi}{2a}}$$

$$\log_e k = \lim_{x\to a} \frac{-\sin^2\left(\frac{\pi x}{2a}\right)}{-a\left(2-\frac{x}{a}\right)} \times \frac{2a}{\pi}$$

$$= \frac{\sin^2(\pi/2)}{\cancel{a}\times 1} \times \frac{2\cancel{a}}{\pi}$$

$$\log_e k = \frac{1\times 2}{\pi} = \frac{2}{\pi}$$

$$\boxed{k = e^{2/\pi}}$$

2) $\lim\limits_{x\to\pi/2} (\tan x)^{\cos x}$

Solⁿ Let $k = \lim\limits_{x\to\pi/2} (\tan x)^{\cos x} = (\infty^0)$

$\log_e k = \lim\limits_{x\to\pi/2} \log_e (\tan x)^{\cos x}$

$= \lim\limits_{x\to\pi/2} \cos x \log_e (\tan x)$

$= \lim\limits_{x\to\pi/2} \frac{\log_e (\tan x)}{-\sec x} = \left(\frac{\infty}{\infty}\right)$

$\left[\begin{array}{l}\cos \pi/2 = 0 \\ \sec \pi/2 = \frac{1}{0} = \infty\end{array}\right]$

LHR - L'Hospital Rule, Applying.

$\log_e k = \lim\limits_{x\to\pi/2} \frac{1/\tan x \times \sec^2 x}{\sec x \tan x}$

$= \lim\limits_{x\to\pi/2} \frac{\sec^2 x}{\tan x} \times \cos x \cdot \cot x$

$= \lim\limits_{x\to\pi/2} \frac{\sec x}{\tan x} \cdot \cot x$

$= \lim\limits_{x\to\pi/2} \frac{1}{\cancel{\cos x}} \times \frac{\cancel{\cos x}}{\sin} \times \frac{\cos x}{\sin x}$

$= \lim\limits_{x\to\pi/2} \frac{\cos x}{\sin^2 x}$

$\log_e k = \frac{0}{1} = 0$

$\boxed{k = e^0 = 1}$

3) $\lim\limits_{x\to 0} \left(\frac{a^x + b^x + c^x + d^x}{4}\right)^{1/x}$

$k = \lim\limits_{x\to 0} \left(\frac{a^x + b^x + c^x + d^x}{4}\right)^{1/x} = (1^\infty)$

$$\log_e k = \lim_{x\to 0} \log_e \left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{1/x}$$

$$= \lim_{x\to 0} \frac{1}{x} \log\left(\frac{a^x+b^x+c^x+d^x}{4}\right)$$

$$\Rightarrow \lim_{x\to 0} \frac{\log(a^x+b^x+c^x+d^x) - \log 4}{x} = \left(\frac{0}{0}\right)$$

L'Hospital Rule, Applying

$$\log_e k = \lim_{x\to 0} \frac{\frac{1}{(a^x+b^x+c^x+d^x)} \times (a^x\log a + b^x\log b + c^x \log b + d^x \log d)}{1}$$

$$= \frac{1}{4} \times \log a + \log b + \log c + \log d$$

$$= \frac{1}{4} \log(abcd)$$

$$\log_e k = \log_e (abcd)^{1/4}$$

$$\boxed{k = (abcd)^{1/4}}$$

4). $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

Sol $K = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x} = 1^\infty$

$$\log_e k = \lim_{x\to 0} \log_e \left(\frac{\tan x}{x}\right)^{1/x}$$

$$= \lim_{x\to 0} \frac{1}{x} \log_e\left(\frac{\tan x}{x}\right)$$

$$\Rightarrow \lim_{x\to 0} \frac{\log_e\left(\frac{\tan x}{x}\right)}{x} = \left(\frac{0}{0}\right)$$

LHR.

$$\log_e k = \frac{\lim_{x\to 0} \frac{1}{\left(\frac{\tan x}{x}\right)} \times \frac{x\sec^2 x - \tan x \cdot 1}{x^2}}{1}$$

$$= \left(\lim_{x\to 0} \frac{x}{\tan x}\right) \times \left(\lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2}\right)$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2} = \left(\frac{0}{0}\right)$$

LHR,

$$\log_e k = \lim_{x\to 0} \frac{\left(x \cdot 2\sec x \times \sec x \tan x + \sec^2 x \cdot 1\right) - \sec^2 x}{2x}$$

$$= \lim_{x\to 0} \frac{\cancel{2x} \sec^2 x \tan x}{\cancel{2x}}$$

$$\log_e k = \lim_{x\to 0} \sec^2 x \cdot \tan x$$

$$= 1^2 \times 0$$

$$\log_e k = 0$$

$$\boxed{k = e^0 = 1}$$

Partial Differentiation

The Process of finding Partial derivatives is call partial differentiation. If u is a function of two independent variable x, y i.e if $u = f(x,y)$ then the derivative of u with respect to x. when x varies and y remains constant is called the partial derivate of u wrt. x denoted by $\boxed{\frac{\partial u}{\partial x} \text{ (or) } u_x}$.

i.e $\dfrac{\partial u}{\partial x} = u_x = \lim_{\delta x \to 0} \dfrac{f(x+\delta x, y) - f(x,y)}{\delta x}$

Similarly $\dfrac{\partial u}{\partial y} = u_y = \lim_{\delta y \to 0} \dfrac{f(x, y+\delta y) - f(x,y)}{\delta y}$

= The second order partial derivative of u w.r.t x and y are defined as follows.

i) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2} = u_{xx}$

ii) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial x \partial y} = u_{xy}$

iii) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial y \partial x} = u_{yx}$

iv) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial y^2} = u_{yy}$

$\boxed{\text{Note:- } \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}}$

Example :- $\dfrac{d}{dx}(xy) = xy' + y \cdot 1$

$\dfrac{\partial}{\partial x}(xy) = y \cdot 1$

Problems

1). If $u = x^2y + y^2z + z^2x$, then show the $u_x + u_y + u_z = (x+y+z)^2$

Solⁿ. $u = x^2y + y^2z + z^2x$.

differentiating w.r.t. x, y, z

$u_x = y \cdot 2x + 0 + z^2 \cdot 1 = 2xy + z^2$

$u_y = x^2 \cdot 1 + 2y \cdot z + 0 = 2yz + x^2$

$u_z = 0 + y^2 \cdot 1 + x \cdot 2z = y^2 + 2xz$

$\therefore u_x + u_y + u_z = 2xy + z^2 + 2yz + x^2 + 2xz + y^2$

$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

$= (x+y+z)^2$

2) Verify that If, $u = x^y$, verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Solⁿ $u = x^y$

different. p. w.r.t x.

$\frac{\partial u}{\partial x} = y \cdot x^{y-1}$ $\left(\because \frac{d}{dx}(x^n) = nx^{n-1}\right)$

diff. p. w.r.t. y.

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(y \cdot x^{y-1}\right)$

$\frac{\partial^2 u}{\partial y \partial x} = y x^{y-1} \log x + x^{y-1} \cdot 1$ $\left(\because \frac{d}{dy}(a^y) = a^y \log a\right)$

$\boxed{\frac{\partial^2 u}{\partial y \partial x} = x^{y-1} + y \cdot x^{y-1} \cdot \log x}$ —— (1)

Consider $u = x^y$.

Consider $u = x^y$

diff. p. w.r.t y

$$\frac{\partial u}{\partial y} = x^y \log x \qquad \left(\because \frac{d}{dy}(a^y) = a^y \log a\right)$$

diff. p. w.r.t x

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x^y \cdot \log x) = x^y \cdot \frac{1}{x} + \log x \cdot y \cdot x^{y-1}$$

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = x^{y-1} + y \cdot x^{y-1} \log x} \quad \text{—— (2)}$$

from (1) and (2)

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}} \quad \text{—— (3)}$$

* Jacobians

⇒ If u, v are functions of two independent variable x, y then the jacobian of u, v w.r.t. x, y is defined as

$$J = J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

⇒ If u, v, w are function of three independent variable x, y, z then jacobian is defined as

$$J = J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Problems

1) If $x = u(1-v)$, $y = uv$, then evaluate $\frac{\partial(x,y)}{\partial(u,v)}$

Sol^n: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \rightarrow (1)$

Consider. $x = u - uv$. $\qquad y = uv$

$\frac{\partial x}{\partial u} = 1 - v \cdot 1 = 1 - v \qquad \frac{\partial y}{\partial u} = v \cdot 1 = v$

$\frac{\partial x}{\partial v} = 0 - u \cdot 1 = -u \qquad \frac{\partial y}{\partial v} = u \cdot 1 = u$

$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$

$= (1-v)u - (-u \times v)$

$= u - uv + uv$

$= \underline{\underline{u}}$

2) If $x = r\sin\theta \cdot \cos\phi \qquad y = r\sin\theta \cdot \sin\phi \qquad z = r\cos\theta$,

find $\frac{\partial(x,y,z)}{\partial(r,\theta\,\phi)}$

Sol^n $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \quad \text{--- } (1)$$

(1) Consider $x = r\sin\theta \cdot \cos\phi$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \cdot 1.$$

$$\frac{\partial x}{\partial \theta} = r\cos\phi \cdot \cos\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta \cdot \sin\phi$$

(2) Consider $y = r\sin\theta \cdot \sin\phi$

$$\frac{\partial y}{\partial r} = \sin\theta \cdot \sin\phi \cdot 1$$

$$\frac{\partial y}{\partial \theta} = r\sin\phi \cdot \cos\theta$$

$$\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

(3) Considering $z = r\cos\theta$

$$\frac{\partial z}{\partial r} = \cos\theta \cdot 1$$

$$\frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial z}{\partial \phi} = 0$$

Substitutip in equation (1),

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\left[0 + r^2\sin^2\theta\cos\phi\right] - r\cos\theta\cos\phi\left[0 - r\sin\theta\cos\theta\cos\right.$$
$$- r\sin\theta\sin\phi\left[-r\sin^2\theta \cdot \sin\phi - r\cos^2\theta \cdot \sin\phi\right]$$

$= r^2\sin^3\theta \cdot \cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\cdot\sin^2\phi +$
$r^2\sin\theta\cos^2\theta\sin^2\phi$

$= r^2\sin^3\theta\,(\cos^2\phi + \sin^2\phi) + r^2\sin\theta\cos^2\theta\,(\cos^2\phi + \sin^2\phi)$

$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta$

$= r^2\sin\theta\,(\sin^2\theta + \cos^2\theta)$

$= r^2\sin\theta$

Important VTU PEQ

(3) Find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$, where $u = x^2+y^2+z^2$,
$v = xy+yz+zx$, $w = x+y+z$.

Solⁿ:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \longleftarrow (1)$$

(1) Consider $u = x^2+y^2+z^2$

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = 2y$

$\frac{\partial u}{\partial z} = 2z$

(2) Consider $v = xy+yz+zx$

$\frac{\partial v}{\partial x} = y\cdot 1 + 0 + z\cdot 1 = y+z$

$\frac{\partial v}{\partial y} = x\cdot 1 + z\cdot 1 + 0 = x+z$

$\frac{\partial v}{\partial z} = 0 + y\cdot 1 + x\cdot 1 = y+x$

(3) Consider $w = x+y+z$

$\frac{\partial w}{\partial x} = 1+0+0 = 1$

$\frac{\partial w}{\partial y} = 0+1+0 = 1$

$\frac{\partial w}{\partial z} = 0+0+1 = 1$

$\therefore$ Substituting in eq (1)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix} \end{array}$$

$$= 2x\left[(x+z)-(y+x)\right] - 2y\left[(y+z)-(y+x)\right] - 2z\left[(y+z)-(x+z)\right]$$

$$= 2x\left[\cancel{x}+z-y-\cancel{x}\right] - 2y\left[\cancel{y}+z-\cancel{y}+x\right] - 2z\left[y+\cancel{z}-x-\cancel{z}\right]$$

$$= \boxed{\cancel{2xz}} - \cancel{2xy} - 2yz + \cancel{2yx} + 2zy \boxed{\cancel{-2zx}}$$

$$\Rightarrow 0$$

4) If $u = \frac{xy}{z}$ $v = \frac{yz}{x}$ $w = \frac{zx}{y}$, then show that

$$J\left(\frac{u,v,w}{x,y,z}\right) = 4$$

Sol^n.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

① Consider $u = \frac{xy}{z}$

$$\frac{\partial u}{\partial x} = \frac{y}{z}\cdot 1 \qquad \frac{\partial u}{\partial y} = \frac{x}{z}\cdot 1 \qquad \frac{\partial u}{\partial z} = xy \times \frac{-1}{z^2} = \frac{-xy}{z^2}$$

Consider $v = \frac{yz}{x}$

$$\frac{\partial v}{\partial x} = yz \times \frac{-1}{x^2} = \frac{-yz}{x^2} \qquad \frac{\partial v}{\partial y} = \frac{z}{x} \qquad \frac{\partial v}{\partial z} = \frac{y}{x}$$

Consider $w = \frac{zx}{y}$

$$\frac{\partial w}{\partial x} = \frac{z}{y} \cdot 1 \qquad \frac{\partial w}{\partial y} = \frac{-zx}{y^2} \qquad \frac{\partial w}{\partial z} = \frac{x}{y}$$

Substituting in eq(1),

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \\ \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \end{vmatrix}$$

$$= \frac{1}{z^2} \cdot \frac{1}{x^2} \cdot \frac{1}{y^2} \begin{vmatrix} yz & xz & -xy \\ -yz & zx & yx \\ yz & -zx & xy \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}\left[yz(zx^2y + yx^2z) - xz(-y^2xz - y^2xz) - xy(yz^2\cancel{x} - yz^2\cancel{x})\right]$$

One more step

$$= \frac{1}{x^2y^2z^2}\left[yz \times 2zx^2y - xz(-2y^2xz)\right]$$

$$= \frac{1}{x^2y^2z^2}\left[2x^2y^2z^2 + 2x^2y^2z^2\right]$$

$$= \frac{1}{\cancel{x^2y^2z^2}} \times 4\cancel{x^2y^2z^2} = \underline{4}.$$

5) If $x+y+z=u$ $y+z=uv$ $z=uvw$

Evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol^n: $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

Consider $x+y+z=u$

$x+uv=u$

$\boxed{x=u-uv}$

Consider $y+z=uv$

$y+uvw=uv$

$\boxed{y=uv-uvw}$

and $\boxed{z=uvw}$

① Consider $x=u-uv$

$\frac{\partial x}{\partial u} = 1-v\cdot1 = 1-v$

$\frac{\partial x}{\partial v} = 0-u\cdot1 = -u$

$\frac{\partial x}{\partial w} = 0$

② Consider $y=uv-uvw$

$\frac{\partial y}{\partial u} = v\cdot1 - vw\cdot1 = v-vw$

$\frac{\partial y}{\partial v} = u\cdot1 - uw\cdot1 = u-uw$

$\frac{\partial y}{\partial w} = 0 - uv\cdot1 = -uv$

③ Consider $z=uvw$

$\frac{\partial z}{\partial u} = vw$ $\frac{\partial z}{\partial v} = uw$ $\frac{\partial z}{\partial w} = uv$

Sub in (1),

$$\begin{array}{c} \\ \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \end{array} \begin{array}{ccc} + & - & + \\ \end{array}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$= (1-v)\left[(u-uw)(uv) + u^2vw\right] + u\left[(v-vw)(uv) + uv^2w\right] + 0$

$= (1-v)\left[u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right] + u\left[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right]$

$= (1-v)(u^2v) + u(uv^2)$

$= u^2v - u^2v^2 + u^2v^2$

$$\boxed{J = u^2v}$$

6) If $u = x+3y-z^3$ Evaluate $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ at $[1,-1,0]$

$v = 4x^2yz$

$w = 2z^2 - xy$,

Sol^n $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$

- As we know $u = x+3y-z^2$

$\dfrac{\partial u}{\partial x} = 1$, $\dfrac{\partial u}{\partial y} = 3$ $\dfrac{\partial u}{\partial z} = -2z$

- As we know $v = 4x^2yz$

$$v = 4x^2yz$$

$$\frac{\partial v}{\partial x} = 8xyz \qquad \frac{\partial v}{\partial y} = 4x^2z$$

$$\frac{\partial v}{\partial z} = 4x^2y$$

As we know $w = 2z^2 - xy$

$$\frac{\partial w}{\partial x} = -y \qquad \frac{\partial w}{\partial y} = -x \qquad \frac{\partial w}{\partial z} = 4z$$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 3 & -2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

At $(1,-1,0)$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \Rightarrow \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

Solving, $= -4 - 3(4)$

$= -12 - 4$

$= \underline{\underline{-16}}$

Total Differentiation and composite functions

= If $u = f(x,y)$ the total difference for are the exact differentiation of u is defined as $du = \frac{\partial x}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$.

$$\boxed{du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy.}$$

Differentiation of composite function

If $u = f(x,y)$ where x and y are functions of independent variable t, then it is called composite function of single variablity t and can be find $\frac{du}{dt}$ using

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

If $u = f(x,y)$ where both x and y are function of two independent variable (r,s) then u is curved composite function of two variable r,s and we can find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$ using

$$\boxed{\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}} \quad \text{—— } (1)$$

$$\boxed{\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}} \quad \text{—— } (2)$$

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① If $u = x^3y^2 + x^2y^3$ where $x = at^2$ $y = 2at$ find $\frac{du}{dt}$

sol:- $u \to (x, y) \to t$ $\therefore \frac{du}{dt}$ (exist)
$u \to t$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial u}{\partial t} = (y^2 3x^2 + y^3 2x)(2at) + (x^3 \cdot 2y + x^2 \cdot 3y^2)(2a)$$

$$\frac{\partial u}{dt} = (3 \times 4a^2t^2 \times a^2t^4 + 2at^2 \times 8a^3t^3)(2at) + (2a^3t^6 \times 2at + 3a^2t^4 \times 4a^2t^2)(2a)$$

$$\frac{\partial u}{\partial t} = (12a^4t^6 + 16a^4t^5)(2at) + (4a^4t^7 + 12a^4t^6)(2a)$$

$$= 24a^5t^7 + 32a^5t^6 + 8a^5t^7 + 24a^5t^6$$

$$= \underline{56a^5t^6 + 32a^5t^7}$$

② If $u = \tan^{-1}(y/x)$ where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$
find du/dt

solⁿ:- Given $u \to (x, y) \to t$
$u \to t$ & $\frac{du}{dt}$ exist

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Consider, $u = \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times y \times \left(\frac{-1}{x^2}\right)$$

$$\frac{\partial v}{\partial x} = \frac{1}{\frac{x^2+y^2}{x^2}} \times \frac{-y}{x^2}$$

$$\boxed{\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}}$$

Consider $\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} \times 1$

$$\frac{\partial v}{\partial y} = \frac{1}{\frac{x^2+y^2}{x^2}} \times \frac{1}{x}$$

$$\boxed{\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}}$$

Consider $x = e^t - e^{-t}$

$$\frac{dx}{dt} = e^t + e^{-t}$$

$$\frac{dy}{dt} = e^t - e^{-t}$$

Sub in (1)

$$\frac{du}{dt} = \frac{-y}{x^2+y^2} \times (e^t + e^{-t}) + \frac{x}{x^2+y^2} \times (e^t - e^{-t})$$

$$= \frac{-(e^t+e^{-t})^2}{(e^t-e^{-t})^2+(e^t+e^{-t})^2} + \frac{(e^t-e^{-t})^2}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}$$

Denominator :- $(e^t - e^{-t})^2 + (e^t + e^{-t})^2$

$$= e^{2t} + e^{-2t} - 2e^t \cdot e^{-t} + e^{2t} + e^{-2t} + 2e^t \cdot e^{-t}$$

$$= 2e^{2t} + 2e^{-2t}$$

$$= 2(e^{2t} + e^{-2t})$$

$$\Rightarrow \frac{-(e^{2t} + e^{-2t} + 2e^t \cdot e^t) + e^{-2t} - 2e^t \cdot e^{-t}}{2(e^{2t}+e^{-2t})}$$

$$= \frac{-4e^t e^{-t}}{2(e^{2t}+e^{-2t})} = \frac{-2}{e^{2t}+e^{-2t}}$$

4) If $u = f\left(xz, \frac{y}{z}\right)$. P.T $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0$

Sol^n: Let $u = f(p, q)$ where $p = xz$ $q = \frac{y}{z}$.

Given $u \longrightarrow (p, q) \longrightarrow (x, y, z)$

$\therefore u \longrightarrow (x, y, z)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} \quad \text{--- (3)}$$

Consider $p = xz$.

$\frac{\partial p}{\partial x} = z \cdot 1$ $\qquad \frac{\partial p}{\partial y} = 0$ $\qquad \frac{\partial p}{\partial z} = x \cdot 1$

Consider $q = y/z$.

$\frac{\partial q}{\partial x} = 0$ $\qquad \frac{\partial q}{\partial y} = \frac{1}{z} \cdot 1$ $\qquad \frac{\partial q}{\partial z} = y\left(\frac{-1}{z^2}\right)$

Substituting in eq^n (1),

$$\frac{\partial u}{\partial x} = z \cdot \frac{\partial u}{\partial p} + 0 \qquad [x^{ply} \text{ by } x \text{ throughout}]$$

$$\boxed{x\frac{\partial u}{\partial x} = xz\frac{\partial u}{\partial p}} \quad \text{--- (4)}$$

Substituting in eq^n (2),

$$\frac{\partial u}{\partial y} = 0 + \frac{1}{z}\frac{\partial u}{\partial q} \qquad [x^{ply} \text{ by } -y \text{ throughout}]$$

$$\boxed{-y\frac{\partial u}{\partial y} = -\frac{y}{z}\frac{\partial u}{\partial q}} \quad \text{--- (5)}$$

Substituting in eq^n (3),

$$\frac{\partial u}{\partial z} = x\frac{\partial u}{\partial p} - \frac{y}{z^2}\cdot\frac{\partial u}{\partial q} \qquad [x^{ply} \text{ by } -z]$$

$$\boxed{-z\frac{\partial u}{\partial z} = -xz\frac{\partial u}{\partial p} + \frac{y}{z}\frac{\partial u}{\partial q}} \longrightarrow (6)$$

(4) + (5) + (6)

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = \cancel{xz\frac{\partial u}{\partial p}} - \cancel{\frac{y}{z}\frac{\partial u}{\partial q}} \cancel{-xz\frac{\partial u}{\partial p}} + \cancel{\frac{y}{z}\frac{\partial u}{\partial q}}$$

$$= 0 \qquad \text{Hence Proved.}$$

5) If $u = f(y-z, z-x, x-y)$ then P.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol^n:- Let $u = f(p,q,r)$ when $p = y-z$, $q = z-x$, $r = x-y$

$$u \rightarrow (p,q,r) \longrightarrow (x,y,z)$$
$$\therefore u \longrightarrow (x,y,z)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} \quad \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} \quad \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} \quad \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} \quad \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} \longrightarrow (3)$$

= Consideration. Next page.

Consider $p = y - z$

$\frac{\partial p}{\partial x} = 0$ $\quad \frac{\partial p}{\partial y} = 1 - 0 = 1$ $\quad \frac{\partial p}{\partial z} = 0 - 1 = -1$

Consider $q = z - x$

$\frac{\partial q}{\partial x} = 0 - 1 = -1$ $\quad \frac{\partial q}{\partial y} = 0$ $\quad \frac{\partial q}{\partial z} = 1 - 0 = 1$

Consider $r = x - y$

$\frac{\partial r}{\partial x} = 1 - 0 = 1$ $\quad \frac{\partial r}{\partial y} = 0 - 1 = -1$ $\quad \frac{\partial r}{\partial z} = 0$

① $\frac{\partial u}{\partial x} = 0 - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}$ — ④

② $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}$ — ⑤

③ $\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}$ — ⑥

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

6) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then P.T $x u_x + y u_y + z u_z = 0$

Ans: Let $u = f(p, q, r)$ where $p = \frac{x}{y}$ $\quad q = \frac{y}{z}$ $\quad r = \frac{z}{x}$

$$u \longrightarrow (p, q, r) \longrightarrow (x, y, z)$$

$$\therefore \quad u \longrightarrow (x, y, z)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

Consider $p = x/y$

$$\frac{\partial p}{\partial x} = \frac{1}{y} \qquad \frac{\partial p}{\partial y} = x\left(\frac{-1}{y^2}\right) \qquad \frac{\partial p}{\partial z} = 0$$

Consider $q = y/z$

$$\frac{\partial q}{\partial x} = 0 \qquad \frac{\partial q}{\partial y} = \frac{1}{z} \qquad \frac{\partial q}{\partial z} = y\left(\frac{-1}{z^2}\right) = \frac{-y}{z^2}$$

Consider $r = z/x$

$$\frac{\partial r}{\partial x} = z\left(\frac{-1}{x^2}\right) \qquad \frac{\partial r}{\partial y} = 0 \qquad \frac{\partial r}{\partial z} = \frac{1}{x}$$

(1) $$\frac{\partial u}{\partial x} = \frac{1}{y}\cdot\frac{\partial u}{\partial p} + 0 - \frac{z}{x^2}\cdot\frac{\partial u}{\partial r}.$$

x^{ply} by x.

$$\boxed{x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\cdot\frac{\partial u}{\partial r}} \quad \text{—— (4)}$$

(2) $$\frac{\partial u}{\partial y} = \frac{-x}{y^2}\frac{\partial u}{\partial p} + \frac{1}{z}\cdot\frac{\partial u}{\partial q} + 0$$

x^{ply} by y.

$$\boxed{y\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial p} + \frac{y}{z}\frac{\partial u}{\partial q}} \quad \text{—— (5)}$$

(3) $\frac{\partial u}{\partial z} = \frac{-y}{z^2}\frac{\partial u}{\partial q} + \frac{1}{x}\frac{\partial u}{\partial r}$.

x^{ply} by z.

$$\boxed{z \cdot \frac{\partial u}{\partial x} = \frac{-y}{z} \cdot \frac{\partial u}{\partial q} + \frac{z}{x} \cdot \frac{\partial u}{\partial r}} \longrightarrow (6)$$

(4) + (5) + (6)

$$xu_x + yu_y + zu_z = 0$$

☐ Maxima and Minima for an function of 2 variables:

Defn:- A function $f(x,y)$ is said to be have a maximum value at a pt. (a,b) if $\exists$ a nbd of the pt. (a,b) say $(a+h)(b+k)$ [h and k are small] such that $f(a+h, b+k) < f(a,b)$ Also, a f^n $f(x,y)$ is said to have a minimum value at pt. (a,b) if $\exists$ a nbd of the pt. (a,b) say $(a+h, b+k)$ such that $f(a+h, b+k) > f(a,b)$.

Thus $f(a,b)$ is said to be an extreme value of $f(x,y)$ if it is a maximum value or a minimum value.

NOTE:- The neccessary condn for a function $f(x,y)$ to have either a maximum or a minimum at a pt (a,b) are:

$f_x(a,b) = 0$ $\qquad$ $f_y(a,b) = 0$.

The points (a,b) where a and b satisfy $f_x(a,b) = 0$ and $f_y(a,b) = 0$ are called the stationary pts (or) critical points of the f^n. $f(x,y)$.

Working Rule

1) Find the stationary point (x, y) such that $f_x = 0$, $f_y = 0$
2) Find the second order partial derivatives f_{xx}, f_{xy}, f_{yy} denoting $A = f_{xx}$ $B = f_{xy}$ $C = f_{yy}$. Evaluate A, B, C at the stationary points and compute the corresponding value of $AC - B^2$.
3) i) If $AC - B^2 > 0$ and $A < 0$ at the pt (a, b) the f has maximum at (a, b) and $f(a, b)$ is the max value.

ii) If $AC - B^2 > 0$ and $A > 0$ at the pt (a, b) then f has minimum at (a, b) and $f(a, b)$ is the min value.

NOTE :- If $AC - B^2 < 0$ or $AC - B^2 = 0$ or $A = 0$ then the point (x, y) is called the "Saddle point" in these cases.

1) Find the extreme value of the f^n. $x^3 + y^3 - 3x - 12y + 20$.

Sol^n. Let $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

$$f_x = 3x^2 + 0 - 3 - 0 + 0 = 3x^2 - 3$$
$$f_y = 0 + 3y^2 - 0 - 12 + 0 = 3y^2 - 12$$

To find the stationary point (a, b) such that $f_x = 0$, $f_y = 0$

$$3x^2 - 3 = 0 \qquad 3y^2 - 12 = 0$$
$$3x^2 = 3 \qquad 3y^2 = 12$$
$$x^2 = 1 \qquad y^2 = 4$$
$$\boxed{x = \pm 1} \qquad \boxed{y = \pm 2}$$

$\therefore$ The stationary pts are $(1, 2)$ $(1, -2)$ $(-1, 2)$ $(-1, -2)$

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Now $f_{xx} = 6x = A$

$f_{xy} = 0 = B$

$f_{yy} = 6y = C.$

$f(xx, xy, yy)$ \ Point	$(1,2)$	$(+1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$	$6 > 0$	6	-6	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	$12.$	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion.	Minimum point	Saddle point	Saddle point	Maximum Point.

$\therefore$ Min value $= f(1,2)$

$= 1^3 + 2^3 - 3(1) - 24 + 20$

$= \underline{\underline{2}}$

Max value $= f(-1,-2)$

$= (-1)^3 + (-2)^3 + 3 + 24 + 20$

$= \underline{\underline{38}}$

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(2) Find the extreme value of $x^3+3xy^2-15x^2-15y^2+72x$.

Sol: Let $f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$.

$f_x = 3x^2+3y^2\cdot 1-30x-0+72$

$f_y = 0+3x2y-0-30y+0 = 6xy-30y$

To find stationary point (x,y) such that $f_{(x)}=0$ $f_y=0$

i.e. $3x^2+3y^2-30x+72=0$ — (1)

$6xy-30y=0$ — (2)

from (2) $6xy-30y=0$

$6y(x-5)=0$

$6y=0$ $\boxed{y=0}$ $x-5=0$ $\boxed{x=5}$

Subst. $x=5$ in eq (1),

$75+3y^2-150+72=0$

$3y^2-3=0$

$y^2=1$ $\boxed{y=\pm 1}$

Subst. $\boxed{y=0}$ in (1),

$3x^2-30x+72=0$ — Dividing throughout by 3

$x^2-10x+24=0$

$\boxed{x=6,4.}$

$(6,0)$ and $(4,0)$

Substituting the value.

$f_{xx} = A = 6x - 30$

$f_{xy} = B = 6y$

$f_{yy} = C = 6x - 30$

	(5,1)	(5,−1)	(4,0)	(6,0)
$A = 6x - 30$	0	0	−6	6
$B = 6y$	6	−6	0	0
$C = 6x - 30$	0	0	−6	6
Conclusion	Saddle point	Saddle point.	Max point	Min point

$$\begin{aligned}\text{Max Point} &= f(4,0)\\ &= (4)^3 - 15(4)^2 + 72(4)\\ &= 64 - 240 + 288\\ &= 352 - 240\\ \text{Max} &= \underline{112}.\end{aligned}$$

$$\begin{aligned}\text{Min Point} &= f(6,0)\\ &= (6)^3 - 15(6)^2 + 72(6)\\ &= 216 - 540 + 432\\ &= \underline{108}\end{aligned}$$

Self Study

1) Lagrange's Method of multiplier with one susdiary condⁿ

$u(x, y, z)$ $\qquad$ $\phi(x, y, z) = c$

To find stationary value of $u(x, y, z)$ subject to the condition $\phi(x, y, z) = c$, where c is a constant, the following working rule is employed

1) Auxilary f^n $\quad F = u(x, y, z) + \lambda\phi(x, y, z)$

ii) Form $\quad F_x = 0 \quad , F_y = 0 \quad F_z = 0$

iii) $\phi(x, y, z) = 0 \qquad \mu(x, y, z)$

Solve for (x, y, z) and λ from these equaⁿ along with $\phi(x, y, z) = 0$ the values of g $u(x, y, z)$ are the stationary values.

Q- A rectangular box open at the top is to have a volume of 32 cubic feet. find its dimension if ~~x, y, z~~ the T.S.A is minimum.

Solⁿ: Let x, y, z respectively be the length, breadth and height respectively of the rectangular box.

V = Volume

S = Surface Area.

By Data $\quad V = xyz = 32$

$TSA = 2(xy + yz + zx)$

The rectangular box is open from the top the TSA is equal to

$$S = 2(xy + yz + zx) - xy$$

$$S = xy + 2yz + 2zx$$

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To find x, y, z, S is minimum, subject to the condition $xyz = 32$.

Let $f = (xy + 2yz + 2zx) + \lambda(xyz)$

$$F_x = (y + 2z) + \lambda yz$$

$$F_y = (x + 2z) + \lambda xz$$

$$F_z = (2y + 2x) + \lambda \cancel{x}y$$

Form $F_x = 0$ $F_y = 0$ $F_z = 0$

$y + 2z + \lambda yz = 0$; $x + 2z + \lambda xz = 0$; $2y + 2x + \lambda xy = 0$

$$\lambda = \frac{-y - 2z}{yz} \qquad \lambda = \frac{-x - 2z}{xz} \qquad \lambda = \frac{-2y - 2x}{xy}$$

ie $\lambda = \frac{-(y + 2z)}{yz}$ $\lambda = \frac{-(x + 2z)}{xz}$ $\lambda = \frac{-(2y + 2x)}{xy}$

$$\frac{y + 2z}{yz} = \frac{x + 2z}{xz} = \frac{2x + 2y}{xy} \quad \text{——— (1)}$$

Considering 1^{st} 2 terms

$$\frac{y + 2z}{y\cancel{z}} = \frac{x + 2z}{x\cancel{z}}$$

$$x(y + 2z) = y(x + 2x)$$

$$xy + 2xz = xy + 2yz$$

$$\cancel{2}x\cancel{z} = \cancel{2}y\cancel{z}$$

$$\boxed{x = y}$$

Consider 2^{nd} and 3^{rd} terms,

$$\frac{x+2z}{xz} = \frac{2x+2y}{xy}$$

$$y(x+2z) = z(2x+2y)$$

$$yx + 2yz = 2xz + 2yz$$

$$yx = 2xz$$

$$\boxed{y = 2z}$$

$$\therefore \boxed{x = y = 2z}$$

substitute $y = x$ $z = \frac{x}{2}$ in $xyz = 32$ (given conditions)

$$x \cdot x \cdot \frac{x}{2} = 32$$

$$x^3 = 64$$

$\boxed{x = 4}$ (ignoring complex rule)

$\boxed{y = 4}$ $(\because x = y)$

$$z = \frac{x}{2} = \frac{4}{2} = 2$$

$$\boxed{z = 2}$$

$\therefore$ The dimension of x, y, z are 4, 4, 2 respectively.

Homogenous Function

$u = f(x,y)$ of two independent variable is said to be homogenous function of degree n if $f(x,y)$ can be written in the form,

$$x^n \phi(y/x) \quad (\text{or}) \quad y^n \phi(x/y)$$

Ex: $u = x^2y + xy^2$

$= x^3\left(y/x + y^2/x^2\right)$

$= x^3 \phi(y/x)$

$= y^3\left(\frac{x^2}{y^2} + \frac{x}{y}\right)$

$= y^3 \phi(x/y)$

Euler's Theorem

If z is a homogenous function of x and y, of degree 'n', then $x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

1) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$. P.T $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \sin 2u$

Sol^n:- Given $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$\tan u = \frac{x^3+y^3}{x-y} = z$ (say)

$z = \tan u = \frac{x^3\left(1 + y^3/x^3\right)}{x\left(1 - y/x\right)}$

i.e $$z = \tan u = x^2 \frac{\left[(y/x)^0 + (y/x)^3\right]}{\left[(y/x)^0 - (y/x)^1\right]} = x^2 \phi(y/x)$$

$\therefore$ z is a homogeneous function of degree 2 (ie $n=2$)

By Euler's Theorom,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x\cdot\sec^2 u\cdot\frac{\partial u}{\partial x} + y\cdot\sec^2 u\cdot\frac{\partial u}{\partial y} = 2\tan u$$

$$\sec^2 u\left[x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y}\right] = 2\tan u.$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u}$$

$$= \frac{2\sin u}{\cos u} \times \cos^2 u.$$

$$= 2\sin u\cdot\cos u$$

$$= \sin 2u$$