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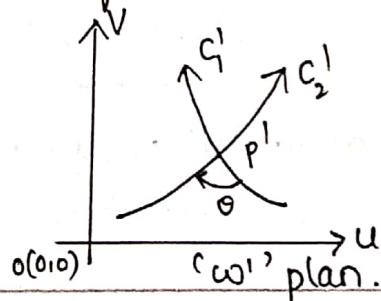
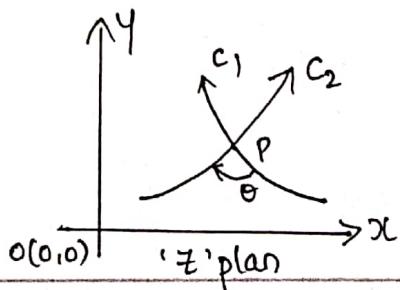
MODULE - 02

CONFORMAL MAPPING (OR) TRANSFORMATION.

Definition:

If the angle between any 2 curves in the Magnitude and its direction is called conformal transformation(or)

Suppose 2 curves c_1, c_2 in the ' z ' plan intersect at the point ' p ', and the corresponding curves c'_1, c'_2 in the ' w ' plan intersect at ' p' , and if the angle of intersection of the curves ' p ' is the same as the angle of intersection of the curves at ' p ' in Magnitude and direction, then this transformation is called the Conformal transformation.



Condition for $w=f(z)$ to represent a Conformal transformation:

1. Necessary Condition: If $w=f(z)$ represents a Conformal transformation of a domain (D) in the ' z ' plan into a domain (D') of the ' w ' plan, then $f(z)$ is analytic function of z in D .
2. Sufficient Condition: let $w=f(z)$ be an analytic function of z in a region (D) of the ' z ' plan and let $f'(z) \neq 0$ inside ' D ' then the mapping $w=f(z)$ is Conformal at the points of D .

1. Discussion of transformation $w=e^z$.

Given, $w=f(z)=e^z$ is analytic

$\therefore f(z)$ is differentiable.

$$\Rightarrow f'(z) = e^z, z \neq 0$$

$$\text{we have, } w=f(z)=u+iv=e^z \rightarrow (1)$$

$$\text{let } z=x+iy$$

$$\therefore (1) \Rightarrow u+iv = e^x + iy$$

$$\Rightarrow u+iv = e^x \cdot e^{iy}$$

$$\Rightarrow u+iv = e^x (\cos y + i \sin y)$$

$$\Rightarrow u+iv = e^x \cos y + e^x \sin y$$

$$\therefore u = e^x \cos y, v = e^x \sin y \rightarrow (2) \rightarrow (3)$$

By eliminating ' y ', we have

$$(2)^2 + (3)^2 = u^2 + v^2 = (e^x \cos y)^2 + (e^x \sin y)^2$$

$$\therefore u^2 + v^2 = e^{2x} (\cos^2 y + \sin^2 y)$$

$$\Rightarrow u^2 + v^2 = e^{2x} \rightarrow (4)$$

By eliminating 'x', we have

$$\frac{(3)}{(2)} \Rightarrow \frac{v}{u} = \frac{e^x \sin y}{e^x \cos y} = \frac{\sin y}{\cos y} = \tan y.$$

$$\Rightarrow v = \underline{\tan y \cdot u} \rightarrow (5)$$

Case 1: let $x = c_1 = \text{constant}$, we have

$$\text{eqn (4)} \Rightarrow u^2 + v^2 = e^{2c_1} = r_1 \text{ (Say)}$$

$$\Rightarrow u^2 + v^2 = r_1^2 \rightarrow (6)$$

' r_1 ' represents a circle having the centre as origin with radius ' r_1 '.

Case 2: let $y = c_2 = \text{constant}$, we have

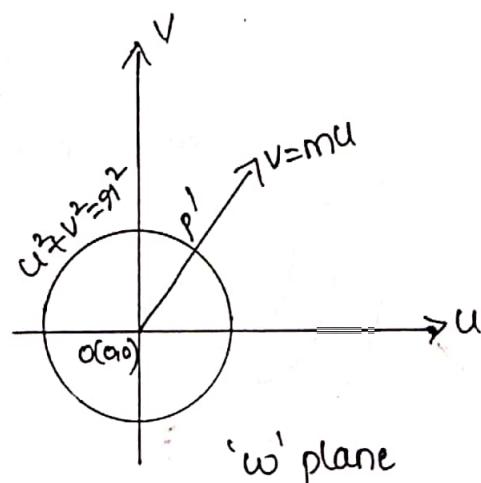
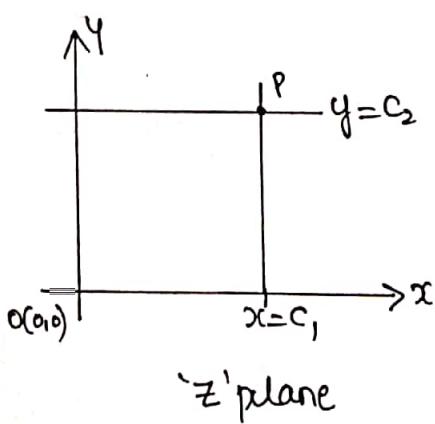
$$\text{eqn (5)} \Rightarrow v = \underline{\tan c_2 \cdot u}$$

$$\Rightarrow v = \tan c_2 \cdot u$$

$$\therefore v = m \cdot u \text{ where } m = \tan c_2 = \text{slope}$$

$\rightarrow (7)$

represents a straight line passing through the origin having the slope 'm'.



2. Discussion of transformation $w = z^2$.

Given, $w = f(z) = z^2$ is analytic
 $\therefore f(z)$ is differentiable.

$$\Rightarrow f'(z) = 2z, z \neq 0.$$

$$\text{we have, } w = f(z) = u + iv = z^2 \rightarrow (1)$$

$$\text{let } z = x + iy$$

$$(1) \Rightarrow u + iv = z^2$$

$$\Rightarrow u + iv = (x + iy)^2 = x^2 + i^2y^2 + 2ixy$$

$$\Rightarrow u + iv = (x^2 - y^2) + i(2xy) \rightarrow *$$

$$\therefore u = x^2 - y^2, v = 2xy \quad \begin{matrix} \rightarrow (2) \\ \rightarrow (3) \end{matrix}$$

By eliminating (u, v, x, y) continuously we have,

Case 1: let $u = c_1$ = Constant

$$\therefore (2) \Rightarrow x^2 - y^2 = c_1 \rightarrow (4)$$

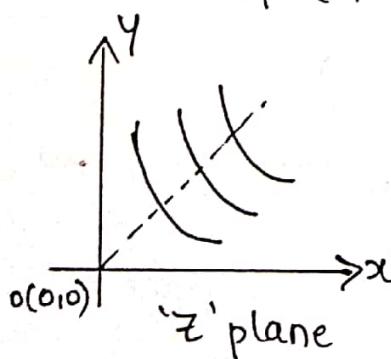
eqn (4) represents a hyperbola.

Case 2: let $v = c_2$

$$\therefore (3) \Rightarrow 2xy = c_2$$

$$\Rightarrow xy = \frac{c_2}{2} \rightarrow (5)$$

eqn (5) represents a rectangular hyperbola.



Case 3: let $x = c_3$

$$\therefore (2) \Rightarrow u = c_3^2 - y^2$$

$$y^2 = c_3^2 - u \rightarrow (c)$$

$$\therefore (3) \Rightarrow v = 2xy$$

$$2c_3y = v$$

$$y = \frac{v}{2c_3} \Rightarrow y^2 = \frac{v^2}{4c_3^2} \rightarrow (d)$$

\therefore From eqn (c) & (d)

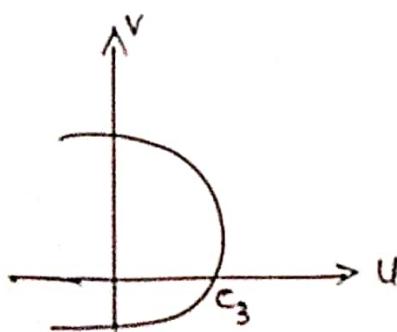
$$\frac{v^2}{4c_3^2} = c_3^2 - u$$

$$\Rightarrow v^2 = 4c_3^2(c_3^2 - u)$$

$$\Rightarrow v^2 = -4c_3^2(u - c_3^2)$$

$$(v-0)^2 = -4c_3^2(u - c_3^2) \rightarrow (e)$$

\therefore eqn (e) represents a parabola, symmetric about 'u' axis in the -ve direction having the vertex at $(c_3^2, 0)$



Case 4: let $y = c_4$

$$\therefore (2) \Rightarrow u = x^2 - c_4^2$$

$$x^2 = u + c_4^2 \rightarrow (f)$$

$$\therefore (3) \Rightarrow v = 2xy$$

$$v = 2x c_4$$

$$x = \frac{v}{2c_4} \Rightarrow x^2 = \frac{v^2}{4c_4^2} \rightarrow (g)$$

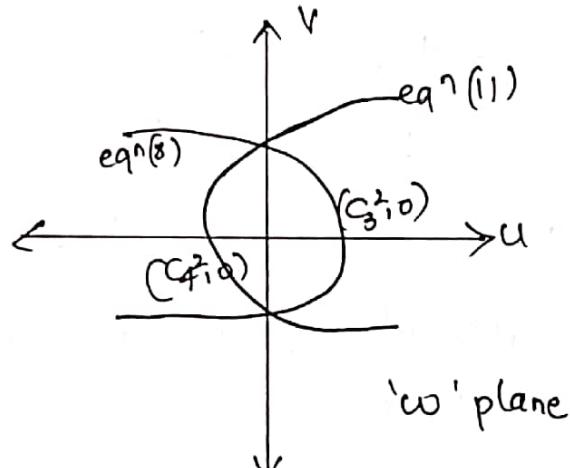
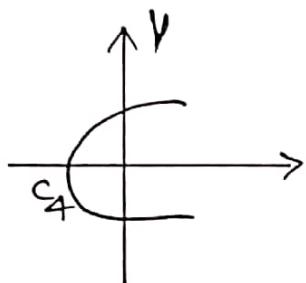
\therefore From eqn (9) & (10)

$$\frac{v^2}{4c_4^2} = u + c_4^2$$

$$v^2 = 4c_4^2(u + c_4^2)$$

$$(v - 0)^2 = 4c_4^2(u - (-c_4^2)) \rightarrow (11)$$

\therefore eqn (11) represents a parabola symmetric about 'u' axis in the +ve direction having the vertex at $(-c_4^2, 0)$.



Discussion of the transformation $w = z + \frac{1}{z}$, $z \neq 0$.

Given, $w = f(z) = z + \frac{1}{z}$ is analytic,

$\therefore f(z)$ is differentiable.

$$\Rightarrow f'(z) = z + \frac{1}{z}, z \neq 0$$

$$= 1 - \frac{1}{z^2}, z \neq 1$$

$$\text{let } z = re^{i\theta}$$

$$\therefore \text{WKT, } w = f(z) = u + iv = z + \frac{1}{z}$$

$$\Rightarrow u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$\Rightarrow u + iv = re^{i\theta} + \frac{1}{r} e^{-i\theta}$$

$$\Rightarrow u+iv = r(\cos\theta + i \sin\theta) + \frac{1}{r}(\cos\theta - i \sin\theta)$$

$$= (r\cos\theta + i r\sin\theta) + \left(\frac{1}{r}\cos\theta - i \frac{1}{r}\sin\theta\right)$$

$$\Rightarrow u+iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$\therefore u = \left(r + \frac{1}{r}\right)\cos\theta \rightarrow (1), \quad v = \left(r - \frac{1}{r}\right)\sin\theta \rightarrow (2)$$

$$\therefore (1) \Rightarrow \frac{u}{\left(r + \frac{1}{r}\right)} = \cos\theta \rightarrow (3) \quad \therefore (2) \Rightarrow \frac{v}{\left(r - \frac{1}{r}\right)} = \sin\theta \rightarrow (4)$$

$$\therefore (3)^2 + (4)^2 \frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = \cos^2\theta + \sin^2\theta$$

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1 \rightarrow (5)$$

From eqn (1) $\Rightarrow \frac{u}{\cos\theta} = r + \frac{1}{r} \rightarrow (6) \quad \left| \frac{v}{\sin\theta} = r - \frac{1}{r} \rightarrow (7) \right.$

$$\therefore (6)^2 - (7)^2 \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2$$

$$\Rightarrow \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4r\left(\frac{1}{r}\right) \quad \left| (a+b)^2 - (a-b)^2 = 4ab \right.$$

$$\Rightarrow \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4$$

$$\Rightarrow \frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1$$

$$\Rightarrow \frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \rightarrow (8)$$

Case 1: When $r=c_1$, we can have eqn (5) $\Rightarrow \frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$
represents Ellipse at Centre origin & vertex @ $(\pm 2, 0)$

Case 9: If $0 = c_2$ is a constant, we have

$$\text{eqn (8)} \Rightarrow \frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$$

represents a hyperbola at the points $(\pm 2, 0)$

Also the ellipse in the ' w ' plane, we have $|z| = H$

$$= \sqrt{x^2 + y^2} = H$$

$$\Rightarrow x^2 + y^2 = H^2$$

represents a circle in the ' z ' plane and for the hyperbola in the ' w ' plane, we have $\alpha = \text{amplitude of } z$

$$\alpha = \tan^{-1}\left(\frac{y}{x}\right)$$

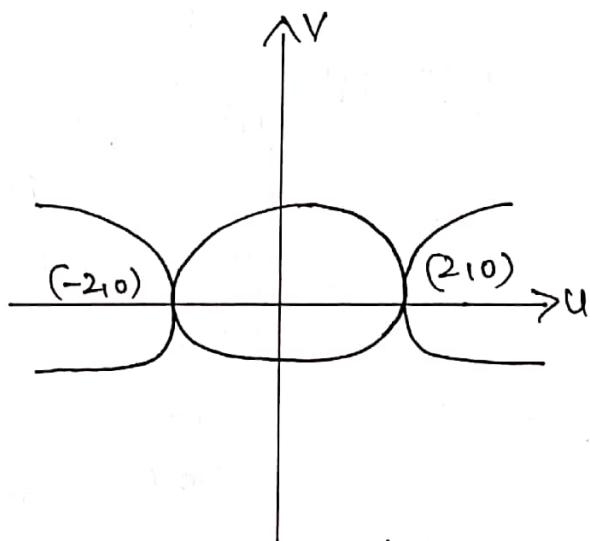
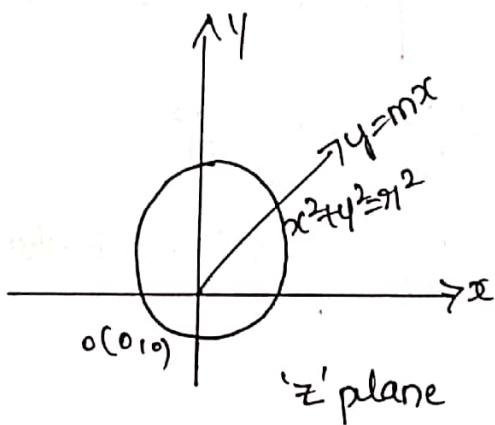
$$\tan \alpha = m = \text{Slope}$$

$$y/x = \tan \alpha$$

$$y = x \tan \alpha$$

$$\boxed{y = mx}$$

represents a straight line.



Bilinear Transformation:

A transformation of the form $w = \frac{az+b}{cz+d}$, where a, b, c, d are real or complex constants, such that

$ad - bc \neq 0$ is called a Bilinear transformation or

linear fractional transformation or Möbius transformation.

The ratio $w = \frac{az+b}{cz+d}$ can also be expressed as

$cwz + dw - az - b = 0$ is linear both in ' z ' & ' w ', hence it is called a bilinear transformation and if $\frac{dw}{dz} \neq 0$, hence it is the conformal.

Properties of Bilinear transformation:

- (1) The transformation $w = \frac{az+b}{cz+d}$ sets of a one-two-one corresponds between the points of closed ' z -plane and the closed w -plane.
- (2) If $ad - bc = 0$, then ' w ' is either a constant (or) meaningless.
- (3) Invariant points (or) fixed points:

If a point ' z ' maps into itself i.e., $w = z$ under the bilinear transformation then the point is called invariant point (or) fixed points of bilinear transformation.

Eg: Suppose invariant of $w = z^2$ are the solutions of equations

$$\Rightarrow z^2 - z = 0$$

$$\Rightarrow z(z-1) = 0$$

$$\therefore z=0, z=1.$$

(4). Cross ratio: If z_1, z_2, z_3, z_4 are four distinct points then

the ratio is called the cross ratio of these points and

$$\text{it is denoted by } (z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

(5) A Bilinear transformation preserves the cross ratio of the 4 points @ let w_1, w_2, w_3, w_4 be the images of 4 distinct points of z_1, z_2, z_3, z_4 in the z -plane under a Bilinear transformation, then

$$\Rightarrow (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

$$\Rightarrow \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$=$$

Problems:

1. Find the Bilinear transformation, which map the points $z=1, i, -1$ into $w=1, 0, -i$.

Given, $z_1=1, z_2=i, z_3=-1$ and $w_1=1, w_2=0, w_3=-i$

WKT, from the one-to-one-bilinear transformation

$$\Rightarrow (w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-i)(0+i)}{(i-0)(-i-w)} = \frac{(z-1)(i+1)}{(z-i)(1-z)}$$

$$\Rightarrow \frac{i(w-i)}{i(w+i)} = \frac{(z-1)(1+i)}{(z+1)(1-i)}$$

$$\Rightarrow \frac{w-i}{w+i} = \frac{(z+1)(1+i)}{(z+1)(1-i)}$$

$$\Rightarrow (w-i)(z+i)(1-i) = (w+i)(z-i)(1+i)$$

$$\Rightarrow (w-i)(z-i)z+i-i = (w+i)(z+i)z-i-i$$

$$\Rightarrow w^2 - iwz + w - iz^2 - iz - z^2 + i - 1 = w^2 + iwz - w - iz^2 + iz - z^2 - i + 1$$

$$\Rightarrow -iwz + iwz + w + i = iz + iz + i + 1$$

$$\Rightarrow -2iwz + 2w = 2iz + 2$$

$$\Rightarrow 2w(1-iz) = 2(1+iz)$$

$$\Rightarrow \boxed{w = \frac{(1+iz)}{(1-iz)}} \\ =$$

2. Find the Bilinear transformation, which map the points $z=1, i, -1$ into $w=2, i, -2$. Also find the invariant points of transformation.

Given, $z=1, i, -1$, $z_1=1$, $z_2=i$, $z_3=-1$ and

$$w_1=2, w_2=i, w_3=-2$$

WKT, Bilinear transformation,

$$w = \frac{az+b}{cz+d} \rightarrow (1)$$

$$\therefore \text{when } z=1, w=2$$

$$\therefore (1) \Rightarrow 2 = \frac{a(1)+b}{c(1)+d} = \frac{a+b}{d+c}$$

$$\Rightarrow a+b = 2d+2c$$

$$\Rightarrow a+b-2c-2d=0 \rightarrow (2)$$

$$\therefore \text{when } z=i, w=i$$

$$\therefore (1) \Rightarrow i = \frac{ai+b}{ci+d} \Rightarrow ai+b = i(ci+d)$$

$$\Rightarrow ai+b = -c+id$$

$$\Rightarrow ai+b+c-id=0 \rightarrow (3)$$

\therefore when $z = -1$, $w = -2$

$$\therefore (1) \Rightarrow -2 = \frac{a(-1)+b}{c(-1)+d}$$

$$\Rightarrow -2 = \frac{-a+b}{-c+d}$$

$$\Rightarrow -a+b = -2(-c+d)$$

$$\Rightarrow -a+b = +2c-2d$$

$$\Rightarrow -a+b-2c+2d=0 \rightarrow (4)$$

Now eqn (2) - (3)

$$\Rightarrow (a+\beta-2c-2d) - (a\dot{i}+\beta+c-i\dot{d}) = 0$$

$$\Rightarrow a(1-i) - 3c + (i-2)d = 0$$

Now eqn (3) - (4)

$$\Rightarrow (a\dot{i}+\beta+c-i\dot{d}) - (-a+\beta-2c+2d) = 0$$

$$\Rightarrow a(1+i) + 3c - d(i+2) = 0 \rightarrow (5)$$

$$\begin{array}{cccc} \frac{a}{1-i} & \frac{c}{-3} & \frac{d}{i-2} & \frac{a}{1-i} \\ \cancel{1+i} & \cancel{-3} & \cancel{i-2} & \cancel{1-i} \\ \cancel{+3} & & \cancel{- (i+2)} & \cancel{1+i} \end{array}$$

$$\Rightarrow \frac{a}{3(i+2)-3(i-2)} = \frac{c}{(i-2)(1+i)+(1-i)(i+2)} = \frac{d}{3(1-i)+3(1+i)}$$

$$\Rightarrow \frac{a}{3(i+2)-3(i-2)} = \frac{c}{i^2-2i-i^2-2i+i^2+2i+i^2-2i} = \frac{d}{3(1-i)+3(1+i)}$$

$$\Rightarrow \frac{a}{12} = \frac{c}{-2i} = \frac{d}{6} = k$$

$$\Rightarrow a = 12k, \quad c = -2ik, \quad d = 6k$$

$$\therefore \text{eqn}(2) \Rightarrow 12k + b + 4ik - 1k = 0 \\ \Rightarrow b = -4ik$$

$$\therefore (1) \Rightarrow w = \frac{12kz - 4ik}{-2iz + 6k}$$

$$\Rightarrow w = \frac{12z - 4i}{-2iz + 6}$$

$$\Rightarrow w = \frac{6z - 2i}{-iz + 3} \rightarrow (6)$$

when $w = z$

$$(6) \Rightarrow z = \frac{6z - 2i}{-iz + 3}$$

$$\Rightarrow z(-iz + 3) = 6z + 2i$$

$$\Rightarrow z(-iz + 3) - 6z + 2i = 0$$

$$\Rightarrow -iz^2 + 3z - 6z + 2i = 0$$

$$\Rightarrow -i^2 z^2 - 3z + 2i^2 = 0 \quad (\text{multiply by } i)$$

$$\Rightarrow z^2 - 3zi - 2 = 0$$

$$\Rightarrow z = \frac{3i \pm \sqrt{9i^2 - 4(1)(-2)}}{2(1)}$$

$$\Rightarrow z = \frac{3i \pm \sqrt{-9 + 8}}{2} = \frac{3i \pm \sqrt{-1}}{2} = \frac{3i \pm \sqrt{i^2}}{2}$$

$$\Rightarrow z = \frac{3i + i}{2} \quad \text{Case 1: } z = \frac{3i + i}{2} = \frac{4i}{2} = 2i$$

$$\quad \text{Case 2: } z = \frac{3i - i}{2} = \frac{2i}{2} = i$$

$$\therefore z = 2i, i$$

3. Find the Bilinear transformation, which maps $z = \infty, i, 0$ into $w = -1, -i, 1$. Also find the fixed points of the transformation.

Soln For any real or complex constants

WKT, Bilinear transformation, $w = \frac{az+b}{cz+d} \rightarrow (1)$

$$\therefore (1) \Rightarrow w = \frac{a+b/z}{c+d/z} \rightarrow (2)$$

and given $z_1 = \infty, z_2 = i, z_3 = 0$ and $w_1 = -1, w_2 = -i, w_3 = 1$.

\therefore when $z = \infty, \therefore w = -1$

$$\therefore (2) \Rightarrow -1 = \frac{a+0}{c+0} = \frac{a}{c}$$

$$a = -c$$

$$\underline{a+c=0} \rightarrow (3)$$

\therefore when $z = i, w = -i$

$$\therefore (2) \Rightarrow -i = \frac{ai+b}{ci+d}, \Rightarrow -i(ci+d) = ai+b$$

$$\Rightarrow ai+b - ci^2 - di = 0 \rightarrow (4)$$

\therefore when $z = 0, w = 1$

$$\therefore (2) \Rightarrow 1 = \frac{b}{d}$$

$$\Rightarrow b-d=0 \rightarrow (5)$$

$\therefore (3) + (4)$

$$a(1+i) + b + di = 0 \rightarrow (6)$$

$$\therefore (5) \Rightarrow 0 \cdot a + b - d = 0 \rightarrow (7)$$

$$\begin{array}{cccc} \frac{a}{1+i} & \frac{b}{1} & \frac{c}{i} & \frac{a}{1+i} \\ 0 & 1 & -1 & 0 \end{array}$$

$$\frac{a}{(-1-i)} = \frac{b}{(1+i)} = \frac{d}{(1+i)}$$

$$\frac{a}{-(1+i)} = \frac{b}{(1+i)} = \frac{d}{(1+i)} = k$$

$$\frac{a}{-1} = \frac{b}{1} = \frac{d}{1} = k$$

$$a = -k, b = k, d = k$$

$$\text{and eqn (3)} \quad a+c=0$$

$$c = -a = -(-k) = \underline{\underline{k}}$$

WKT,

$$w = \frac{az+b}{cz+d} = \frac{-kz+k}{kz+k} = \frac{-z+1}{z+1}$$

$$\boxed{w = \frac{-z+1}{z+1}} \rightarrow (2)$$

$$(2) \Rightarrow z = \frac{-z+1}{z+1}, \text{ To find invariant points, let } w=z$$

$$z(z+1) = (-z+1)$$

$$\Rightarrow (z^2+z)(z-1)=0$$

$$\Rightarrow z^2+2z-1=0$$

$$z = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2(1)} = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm \sqrt{8}}{2}$$

$$\Rightarrow z = \frac{-2 \pm 2\sqrt{2}}{2} = \frac{-2(-1 \pm \sqrt{2})}{2} = -1 \pm \sqrt{2}$$

$$\Rightarrow z = -1 + \sqrt{2}, \quad z = -1 - \sqrt{2}$$

4. Find the Bilinear transformation, which map the points $\underline{\underline{z=1, i, -1}}$ into $w=0, 1, \infty$.

Soh WKT, for any real or complex constants of a, b, c, d ,

the Bilinear transformation is $w = \frac{az+b}{cz+d} \rightarrow (1)$

$$\therefore (1) \Rightarrow \frac{1}{w} = \frac{cz+d}{az+b} \rightarrow (2)$$

and given, $z_1=1, z_2=i, z_3=-1$ and $w_1=0, w_2=1, w_3=\infty$

$$\therefore \text{when } z=1, w=0 \therefore (1) \Rightarrow 0 = \frac{a+b}{c+d}$$

$$\Rightarrow a+b=0 \rightarrow (3)$$

\therefore when $\bar{z} = i$, $w = 1$

$$\Rightarrow (1) \Rightarrow 1 = \frac{ai+b}{ci+d}$$

$$ai+b - ci - d = 0 \rightarrow (4)$$

\therefore when $\bar{z} = -1$, $w = \infty$

$$\Rightarrow (2) \Rightarrow \frac{1}{\infty} = \frac{c(-1)+d}{a(-1)+b} = \frac{-c+d}{-a+b}$$

$$0 = -c+d \rightarrow (5)$$

eqⁿ (4) - (3)

$$(i-1)a - ci - d = 0 \rightarrow (6)$$

$$\therefore (5) \Rightarrow 0 \cdot a - c + d = 0 \rightarrow (7)$$

$$\begin{array}{cccc} \frac{a}{(i-1)} & \frac{c}{-i} & \frac{d}{-1} & \frac{a}{(i-1)} \\ 0 & -1 & 1 & 0 \end{array}$$

$$\frac{a}{i-1} = \frac{c}{-(i-1)} = \frac{d}{-(i-1)} = k$$

$$\Rightarrow a = -k(i+i), \quad c = -k(i-i), \quad d = -k(i-i)$$

$$c = k(i-i) \quad d = k(i-i)$$

$$\Rightarrow \text{eq}^n (3) \quad b = -a$$

$$b = -(-k(i+i))$$

$$b = k(i+i)$$

$$\therefore w = \frac{-k(i+i)\bar{z} + k(i+i)}{k(i-i)\bar{z} + k(i-i)}$$

$$w = \frac{-(i+i)\bar{z} + (i+i)}{(i-i)\bar{z} + k(i-i)}$$

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Complex Integration:

Suppose $w=f(z)$ be a continuous complex valued function over a region 'R', for any complex variable $z=x+iy$ in the curve 'c'. Then the complex integration of $f(z)$ from the point 'p' to 'q' can be defined as $\int f(z) \cdot dz$ and which will be evaluated by dividing the interval into 'n' no. of parts.

Line Integral of a Complex valued function:

Let $f(z) = u(x,y) + iv(x,y)$ be a continuous complex valued function over a region 'R' of any complex variable $z=x+iy$ in the curve 'c', then the line of integral of $f(z)$ can be defined as $\int_C f(z) \cdot dz = \int_C (u+iv)(dx+idy)$

$$= \int_C (u \cdot dx + iu \cdot dy + iv \cdot dx - v \cdot dy)$$

$$= \int_C (udx - vdy) + i \int_C (vdx + udy)$$

Problems:

- Evaluate $\int_C |z|^2 \cdot dz$, where 'c' is of square with the vertices $(0,0), (1,0), (1,1), (0,1)$.

Given, $\int_C |z|^2 \cdot dz$
 $f(z) = |z|^2$, where $z = x+iy$
 $\Rightarrow |z| = \sqrt{x^2+y^2}$

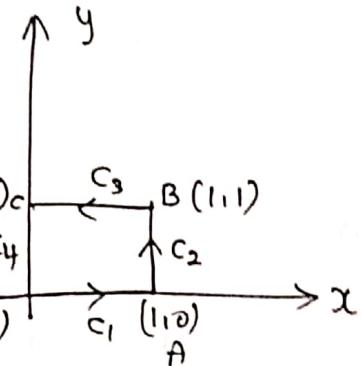
$$\Rightarrow |z|^2 = x^2 + y^2$$

$$\text{and } dz = dx + idy$$

$$\begin{aligned} \therefore \text{wkT} \int_C f(z) dz &= \int_C (x^2 + y^2)(dx + idy) \rightarrow (1) \\ &= \int_{C_1} (x^2 + y^2)(dx + idy) + \int_{C_2} (x^2 + y^2)(dx + idy) + \\ &\quad \int_{C_3} (x^2 + y^2)(dx + idy) + \int_{C_4} (x^2 + y^2)(dx + idy) \rightarrow (2) \end{aligned}$$

(i) Along the curves, $C_1 \Rightarrow y=0$
 $dy=0$

$$\begin{aligned} \therefore \int_C (x^2 + y^2)(dx + idy) &= \int_{x=0}^1 (x^2)(dx + i0) \quad (0,0)_C \xrightarrow[C_4]{\quad} O(0,0) \xrightarrow[C_1]{\quad} (1,0)_A \xrightarrow[C_2]{\quad} B(1,1) \\ &= \int_0^1 (x^2)(dx) \\ &= \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3} \end{aligned}$$



(ii) Along the curves, $C_2 \Rightarrow x=1, dx=0$

$$\begin{aligned} \therefore \int_C (x^2 + y^2)(dx + idy) &= \int_{y=0}^1 (1+y^2)(idy) \\ &= i \int_0^1 (1+y^2) dy \Rightarrow i \left[y + \frac{y^3}{3} \right]_0^1 \\ &= i \left(1 + \frac{1}{3} \right) = \frac{4i}{3} \end{aligned}$$

(iii) Along the curves, $C_3 \Rightarrow y=1, dy=0$

$$\begin{aligned} \therefore \int_C (x^2 + y^2)(dx + idy) &= \int_{x=1}^0 (x^2 + 1) dx \\ &= \left. \frac{x^3}{3} + x \right|_1^0 \Rightarrow 0 - \left(\frac{1}{3} + 1 \right) = -\frac{4}{3} \end{aligned}$$

(iv) Along the curves, $C_4 \Rightarrow x=0, dx=0$

$$\therefore \int_C (x^2 + y^2)(dx + idy) = \int_{y=1}^0 y^2 \cdot idy \Rightarrow i \left. \frac{y^3}{3} \right|_1^0 \Rightarrow 0 - \frac{i}{3} = -\frac{i}{3}$$

$$\begin{aligned}\therefore \int_C f(z) dz &= \frac{1}{3} + 4\frac{i}{3} - \frac{4}{3} - \frac{i}{3} \\ &= -1 + i \\ &= -1 + \underline{i}\end{aligned}$$

Cauchy's Integral Theorem (or) Fundamental Theorem.

Statement: If a function $f(z)$ is analytic out all the points within and on a closed contour 'c' then $\int_C f(z) dz = 0$.

Proof: Given, $w = f(z) = u(x, y) + i v(x, y)$ is analytic function over a region 'R' at any complex variable $z = x + iy$ of a curve 'c', we have $\int_C f(z) dz = \int_C (u + iv)(dx + idy)$.

$$\Rightarrow \int_C f(z) dz = \int_C (u dx - v dy) + i(v dx + u dy) \rightarrow (1)$$

By the Green's Theorem, WKT

$$\int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\therefore (1) \Rightarrow \int_C f(z) dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy \rightarrow (2)$$

Given, $f(z) = u + iv$ is analytic and it can satisfy the C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\therefore (2) \Rightarrow \int_C f(z) dz = \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

$$\Rightarrow \int_C f(z) dz = 0 + i 0$$

$$\Rightarrow \int_C f(z) dz = 0$$

Problem:

1. Verifying Cauchy's Theorem for the function $f(z) = z^2$, where 'C' is the square having the vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$.

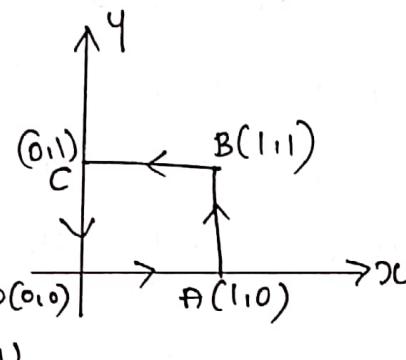
Given, The Complex valued function $f(z) = z^2$, where 'C' is a closed square having the vertices $O(0,0)$, $A(1,0)$, $B(1,1)$ and $C(0,1)$.

$$\text{WKT, } z = x + iy$$

$$\Rightarrow dz = dx + idy$$

$$\therefore f(z) \stackrel{def}{=} z^2 \cdot dz$$

$$\Rightarrow f(z) \cdot dz = (x+iy)^2 (dx+idy)$$



$$\text{and } \int_C f(z) dz = \int_{OA} f(z) dz + \int_{AB} f(z) dz + \int_{BC} f(z) dz + \int_{CO} f(z) dz \rightarrow (2)$$

Along OA $\Rightarrow y=0$

$$dy=0$$

$$\int_{OA} f(z) dz = \int_{OA} (x+i(0))^2 \cdot (dx+i(0))$$

$$= \int_{OA} x^2 \cdot dx = \int_{x=0}^1 x^2 \cdot dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

Along AB $\Rightarrow x=1$

$$dx=0$$

$$\int_{AB} f(z) dz = \int_{AB} (1+iy)^2 \cdot (0+idy)$$

$$= i \int_{AB} (1+2iy-y^2) dy$$

$$= i \int_{y=0}^1 (1+2iy-y^2) dy = i \left| y + iy^2 - \frac{y^3}{3} \right|_0^1$$

$$\int_{AB} f(z) dz = \int_{AB} \left(1+i - \frac{1}{z}\right) dz = \int_{AB} \left(1+i - \frac{1}{z}\right) dz = 1 + \frac{2i}{3}$$

$$\int_{AB} f(z) dz = 1 + \frac{2i}{3}$$

\therefore Along BC $\Rightarrow y=1, dy=0$

$$\begin{aligned} \int_{BC} f(z) dz &= \int_{BC} (x+i)^2 \cdot (dx+idy) \\ &= \int_{BC} (x^2 + 2ix - 1) dx \\ &= \int_{x=1}^0 (x^2 + 2ix - 1) dx \\ &= \left(\frac{x^3}{3} + 2i \frac{x^2}{2} - x \right) \Big|_1^0 \\ &= 0 - \left(\frac{1}{3} + i - 1 \right) \\ &= -\left(i - \frac{2}{3}\right) = \underline{\underline{\frac{2}{3} - i}} \end{aligned}$$

\therefore Along CO $\Rightarrow x=0, dx=0$

$$\begin{aligned} \int_{CO} f(z) dz &= \int_{CO} (0+iy)^2 \cdot (0+idy) \\ &= \int_{CO} i^2 y^2 \cdot idy \\ &= \int_{CO} -y^2 \cdot idy \Rightarrow -i \int_{y=1}^0 y^2 \cdot dy \\ &= -i \left| \frac{y^3}{3} \right|_1^0 \\ &= -i \left(0 - \frac{1}{3} \right) = \underline{\underline{\frac{1}{3}}} \end{aligned}$$

$$\therefore (1) \Rightarrow \int_C f(z) \cdot dz = \frac{1}{3} - 1 + \frac{2i}{3} + \frac{2}{3} - i + \frac{i}{3}$$

$$\int_C f(z) dz = 0$$

\therefore hence Cauchy's Theorem Verified.

Theorem : Cauchy's Integral formula.

Statement : If $f(z)$ is analytic inside and on a simple closed curve ' C ' and if ' a ' is any point within ' C ', then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \cdot dz$

Proof : Given $f(z)$ is an analytic function inside and on a simple closed curve ' C '.

Since ' a ' is a point within ' C ', we shall enclose it by a circle ' C_1 ' and with $z=a$ as a center and ' r_1 ' as a radius, such that C_1 lies entirely within ' C '.

Therefore the function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the region b/w ' C ' and ' C_1 '.

$$\text{we have } \int_C \frac{f(z)}{(z-a)} dz = \int_{C_1} \frac{f(z)}{(z-a)} dz \rightarrow (1)$$

$$\text{WKT, } |z-a| = r_1$$

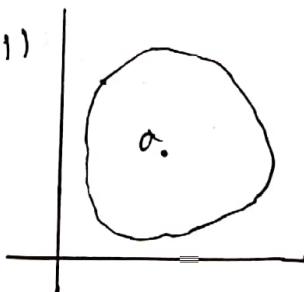
$$\Rightarrow z-a = r_1 e^{i\theta}$$

$$\Rightarrow z = a + r_1 e^{i\theta}$$

$$\Rightarrow \frac{dz}{d\theta} = (0 + r_1 i e^{i\theta})$$

$$\Rightarrow dz = r_1 i e^{i\theta} d\theta$$

$$\therefore (1) \Rightarrow \int_C \frac{f(z)}{z-a} \cdot dz = \int_{0=0}^{2\pi} \frac{f(a + r_1 e^{i\theta})}{r_1 e^{i\theta}} \cdot r_1 i e^{i\theta} d\theta$$



$$\Rightarrow \int_C \frac{f(z)}{z-a} \cdot dz = i \int_0^{2\pi} f(a + re^{i\theta}) d\theta \rightarrow (2)$$

when $a=0$,

$$\therefore (2) \Rightarrow \int_C \frac{f(z)}{z-a} dz = i \int_{\theta=0}^{2\pi} f(a) \cdot d\theta.$$

$$\begin{aligned} \Rightarrow \int_C \frac{f(z)}{z-a} dz &= i \cdot f(a) \cdot \int_0^{2\pi} 1 \cdot d\theta \\ &= i \cdot f(a) \cdot [0]_0^{2\pi} \\ &= i \cdot f(a) (2\pi) \end{aligned}$$

$$\Rightarrow \int_C \frac{f(z)}{(z-a)} dz = i \cdot f(a) \cdot 2\pi$$

$$\therefore f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz$$

Generalized Cauchy's Integral Formula:

If $f(z)$ is analytic inside and on a simple closed curve ' C ' and if ' a ' is any point within ' C ', then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \cdot dz$$

$$\text{also } f^{(n)}(a) \cdot \frac{2\pi i}{n!} = \int_C \frac{f(z)}{(z-a)^{n+1}} \cdot dz$$

Problems:

- Evaluate $\int_C \frac{e^{2z}}{(z+1)(z+2)} dz$, where ' C ' is a circle $|z|=3$.

Sol. let $I = \int_C \frac{e^{2z}}{(z+1)(z+2)} dz \rightarrow (1)$

Since $C: |z|=3$ is a circle with the center '0' (zero) and radius '3'.

$z = -1$ and $z = -2$ are inside of the circle.

$\therefore f(z) = e^{2z}$ is analytic at all points except $z = -1$ and $z = -2$.

$$\therefore \frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$\Rightarrow 1 = A(z+2) + B(z+1) \rightarrow (2)$$

when $z = -2$

, when $z = -1$

$$\therefore (2) \Rightarrow 1 = -B$$

$$\therefore (2) \Rightarrow 1 = A$$

$$B = -1$$

$$A = 1$$

$$\therefore \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$\therefore \frac{e^{2z}}{(z+1)(z+2)} = \frac{e^{2z}}{z+1} - \frac{e^{2z}}{z+2}$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)(z+2)} dz = \int_C \frac{e^{2z}}{z-(-1)} dz - \int_C \frac{e^{2z}}{z-(-2)} dz$$

$$= 2\pi i e^{2(-1)} - 2\pi i e^{2(-2)}$$

$$= 2\pi i e^{-2} - 2\pi i e^{-4}$$

$$= 2\pi i (e^{-2} - e^{-4})$$

$$= 2\pi i \left(\frac{1}{e^2} - \frac{1}{e^4} \right) = 2\pi i \left(\frac{e^4 - e^2}{e^6} \right)$$

$$= 2\pi i \left(\frac{e^2 - 1}{e^4} \right)$$

Q. Evaluate $\int_C \frac{e^{2z}}{(z+1)(z+2)} dz$, where 'c' is a circle $|z|=3$.

Sol. let $I = \int_C \frac{e^{2z}}{(z+1)(z+2)} . dz \rightarrow (1)$

Since C : $|z|=3$ is a circle with the center '0' (zero) &

Radius 3.

$z = -1$ and $z = 2$ are inside the circle.

$\therefore f(z) = e^{2z}$ is analytic at all points except $z = -1$ and $z = 2$.

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2} \rightarrow (2)$$

$$1 = A(z-2) + B(z+1)$$

when $z = -1$, when $z = +2$

$$1 = A(-3)$$

$$\boxed{A = -1/3}$$

$$1 = B(3)$$

$$\boxed{B = 1/3}$$

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{1}{3(z-2)} - \frac{1}{3(z+1)}$$

$$\therefore \frac{e^{2z}}{(z+1)(z-2)} = \frac{e^{2z}}{3(z-2)} - \frac{e^{2z}}{3(z+1)}$$

$$\begin{aligned}\therefore \int_C \frac{e^{2z}}{(z+1)(z-2)} dz &= \int_C \frac{e^{2z}}{3(z-2)} dz - \int_C \frac{e^{2z}}{3(z+1)} dz \\ &= \int_C \frac{e^{2z}}{3(z-(2))} dz - \int_C \frac{e^{2z}}{3(z-(-1))} dz \\ &= \frac{2\pi i}{3} \cdot e^{2(2)} - \frac{2\pi i}{3} e^{2(-1)} \\ &= \frac{2\pi i}{3} e^4 - \frac{2\pi i}{3} e^{-2} \\ &= \frac{2\pi i}{3} (e^4 - e^{-2}) = \frac{2\pi i}{3} \left(e^4 - \frac{1}{e^2}\right)\end{aligned}$$

3. Evaluate $\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz$, where 'c' is a circle $|z|=3$.

Sol. let $I = \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz \rightarrow (1)$

Since $C : |z|=3$ is a circle with center '0' and radius 3.

$\therefore z=1$ and $z=2$ are inside of the circle.

$\therefore f(z) = \sin(\pi z^2) + \cos(\pi z^2)$ is analytic at all points except $z=1$ and $z=2$.

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow (2)$$

$$1 = A(z-2) + B(z-1)$$

when $z=1$, when $z=2$

$$1 = A(-1)$$

$$\boxed{A = -1}$$

$$1 = B(1)$$

$$\boxed{B = 1}$$

From eqn (2)

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\therefore \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-2} . dz - \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-1} . dz$$

where Substitute

$$z=2 \text{ and } z=1$$

$$= 2\pi i (\sin 4\pi + \cos 4\pi) -$$

$$2\pi i (\sin \pi + \cos \pi)$$

$$= 2\pi i (0+1) - 2\pi i (0-1)$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

4. Evaluate $\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2(z-2)} dz$, where $C : |z|=3$

S.I let $I = \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2(z-2)} dz \rightarrow (1)$

Since $C : |z|=3$ is a circle with center '0' and radius '3'

$z=2$ and $z=1$ are inside of the circle

$\therefore f(z) = \sin(\pi z^2) + \cos(\pi z^2)$ is a analytic function at all the points except $z=1$ and $z=2$.

$$\therefore \frac{1}{(z-1)^2(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)}$$

$$1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2 \rightarrow (2)$$

when $z=1$, when $z=2$

$$1 = B(-1)$$

$B = -1$

$$1 = C$$

$C = 1$

$$\therefore A+C=0$$

$$A = -C = -1$$

$A = -1$

$$\therefore \frac{1}{(z-1)^2(z-2)} = \frac{-1}{(z-1)} - \frac{1}{(z-1)^2} + \frac{1}{(z-2)}$$

$$\begin{aligned} \therefore \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2) dz}{(z-1)^2(z-2)} &= \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-2)} dz - \\ &\quad \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-1} dz - \\ &\quad \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2} dz \end{aligned}$$

$$= 2\pi i (\sin 4\pi + \cos 4\pi) - 2\pi i (\sin \pi + \cos \pi) - \dots$$

$$- \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^{1+1}} dz$$

$$= 2\pi i (0+) - 2\pi i (0-) - \frac{2\pi i}{1!} f'(1) \rightarrow (3)$$

$$\text{where } f(z) = \sin \pi z^2 + \cos \pi z^2$$

$$f'(z) = (\cos \pi z^2 - \sin \pi z^2) 2\pi z$$

$$\begin{aligned}\therefore f'(1) &= (\cos \pi - \sin \pi)^{2\pi} \\ &= (-1 - 0)^{2\pi} \\ &= \underline{\underline{-2\pi}}\end{aligned}$$

\therefore from eqn (3)

$$\begin{aligned}\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz &= 2\pi i + 2\pi i - \frac{2\pi i}{1!}(-2\pi) \\ &= 2\pi i + \frac{4\pi^2 i}{1} \\ &= \underline{\underline{4\pi i (1+\pi)}}\end{aligned}$$

5. Evaluate $\int_C \frac{z^2 - z + 1}{z-1} dz$ where 'c' is a circle (i) $|z|=1$ (ii) $|z|=\frac{1}{2}$

So let $I = \int_C \frac{z^2 - z + 1}{z-1} dz \rightarrow (i)$

(i) Since $C: |z|=1$ is a circle with the centre '0' and radius 1.
③ origin
 $z=1$ is on the circle $|z|=1$.

Hence $f(z) = z^2 - z + 1$ is analytic except $z=1$.

$$\begin{aligned}\therefore \int_C \frac{z^2 - z + 1}{z-1} dz &= 2\pi i (1^2 - 1 + 1) \\ &= 2\pi i \\ &= \underline{\underline{2\pi i}}\end{aligned}$$

(ii) Since $C: |z|=\frac{1}{2}$ can represent the circle with centre as origin and radius $\frac{1}{2}$.

$z=1$ is outside of the circle $|z|=\frac{1}{2}$

$$\therefore \int_C \frac{z^2 - z + 1}{z-1} dz = 0 \quad (\because \text{outside } \text{No values})$$