Linear Algebra

Matrices

Matrix norm

Norm of an $m \times n$ matrix A, denoted $\|A\|$:

$$\|A\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}$$

RMS value of matrix entries tell us the typical size of entries, independent of matrix dimensions

$$RMS = \|A\| / \sqrt{m \times n}$$

- $\|A + B\| \leq \|A\| + \|B\|$
- Distance of two matrices: $\|A - B\|$
- $\|A\|^2 = \|a_1\|^2 + \|a_2\|^2 + \ldots + \|a_n\|^2$

Matrix - vector multiplication

$$y = Ax$$

$$y_i = \sum_{j=1}^{n} A_{ij} \cdot x_j$$

or $y = a_1 \cdot x_1 + a_2 \cdot x_2 + \ldots + a_n \cdot x_n$

$\Rightarrow$ y is linear combination of columns of A

The columns of A are linearly independent if

$$Ax = 0 \text{ implies } x = 0$$

Properties of matrix-vector multiplication:

- $A(u + v) = Au + Av$
- $(A + B)u = Au + Bu$
- $(\alpha A)u = \alpha(Au)$

Complexity of matrix-vector multiplication Ax:

$$2mn$$

Geometric transformation

Scaling Scaling is mapping $y = ax$, where a is scalar.
This can be expressed as $y = Ax$ with $A = aI$

Dilation Stretches or shrinks vector x along different axes

$$y = Dx \text{ where } D = \text{diag}(d_1, d_2)$$

Rotation Rotate x by θ radians counterclockwise:

$$y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x$$

Vũ Viết Thư – 1963 (My dad wrote his name)

Vũ Duy Tùng (My mom wrote my name)

Image cropping

Suppose x is an image with $M * N$ - pixel (M, N are even). Let y is an $(M/2) \times (N/2)$ image that giving the upper left corner of image x. Then:

$$y = Ax$$

, where the i^{th} row of A is $e_{k_i}^T$ with k_i is the index of pixel in x corresponds to i^{th} pixel in y

Permutation matrix

An $n \times n$ permutation matrix is one in which each column is a unit vector, each row is a transpose of unit vector

Ex:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow Ax = (x_3, x_1, x_2)$$

Convolution

The convolution of an n-vector a and m-vector b denoted as $c = a * b$

$$c_k = \sum_{i+j-1=k} (a_i b_j), \quad k = 1, \ldots, m+n-1$$

- Properties of convolution:
  + Symmetric: $a * b = b * a$
  + Associative: $(a * b) * c = a * (b * c)$
  + For fixed a, $a * b$ is a linear func of b and vice verse. $\Rightarrow a * b = T(a) b = T(b) a$

Audio filtering

If the n vector is an audio signal, and a is a vector
The vector $y = a * x$ is called a filter audio signal
with filter coefficients a

2-D convolution

Suppose A is an $m \times n$ matrix and B is an $p \times q$ matrix
There convolution is an $(m+p-1)(n+q-1)$ matrix

$$C_{rs} = \sum_{i+k-1=r,\ j+\ell-1=s} A_{ij} B_{k\ell}$$

Image blurring $Y = X * B$ (point spread function PSF)

Ex:

$$B = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$$