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Stochastic approach to snow avalanches modeling

Master's Thesis in Mathematics

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Abstract

The primary purpose of this thesis is to develop and investigate a stochastic snow avalanche model. General description of the process behind the snow avalanches, leads to the theory of piecewise deterministic Markov processes (PDMP), which has been studied thoroughly over the past three decades. A brief description of the mathematical theory necessary to develop and analyze a model constructed within the framework of PDMP is given. The thesis then presents a PDMP formulation of the general snow avalanche model, in which a distinction between the snow depth at the slope and the amount of fresh snow during a period of a snowfall is made. The studies of expectations of the general snow avalanche model leads to a partial differential equation, which eventually turns out to be too complicated to provide any analytical results. For that reason, a simplified model in which the snowfall periods are replaced by positive jumps is developed. The simplified model is then analyzed in stationary cases, which are proved to exist. Finally, a brief discussion and comparison of the snow avalanche models already available in the literature with the ones developed in previous chapters is presented.

Popular summary

The aim of this thesis is to present and study a mathematical description of the snow avalanches behavior, which are considered to be a major natural hazard in mountainous areas. In order to do that, the thesis introduces the mathematical notions, which are necessary to understand and fully present the definition of a piecewise deterministic Markov process (PDMP), assuming that the reader is familiar with some basic facts from the probability and calculus theory. Using the PDMP theory, the thesis presents how to model and analyze the behavior of a processes with deterministic trajectories randomly interrupted by instantaneous jumps. In particular, a general snow avalanche model is derived, which turned out to be too complicated to obtain any analytical results. For that reason a simplified model is proposed and studied in stationary cases, which are proved to exist. Some applications of the Laplace transform and relations to queueing theory are also presented. Finally, models already available in the literature are formulated in the PDMP framework and compared with the ones presented in the thesis.

Contents

1 Introduction

Snow avalanche is a rapidly moving snow mass, which is descending along a steep slope. It is considered to be a major natural hazard in mountainous areas throughout the world, endangering human life and infrastructure.
Each year, the number of snow avalanche fatalities speaks for itself, when it comes to the importance of studying and predicting this natural phenomenon. As it has been reported in [8], just for the European Alps, the number of snow avalanche fatalities during the winter of 2014/2015 was 132. According to the media, in the winter 2015/2016 this number has decreased to 67. In January 2017, the destructive impact of this natural hazard has been reminded again, when a snow avalanche descended on a hotel in Italy, killing 29 people.

An avalanche formation is a result of complex interaction between terrain, snowpack and weather conditions. The authors of [13] summarize and describe the most contributory factors: terrain, new snow, wind and temperature. Crucial to our considerations is the statement that terrain is an essential and the only factor that is constant over time. Moreover, the systematic identification of potential snow avalanche starting zones has been restricted to slope angles between 30° and 50° assuming that there is no forest. Those two arguments convinced us to omit the impact of terrain on snow avalanche occurrence in our stochastic model, which is described in the next section.

Instead, we focus mainly on the snowpack. According to [13], the accumulation of fresh snow is the strongest forecasting parameter and is closely related to avalanche danger. Nevertheless, it has been proved that the new snow depth alone is not sufficient to explain avalanche activity. Therefore modeling the snowpack by making a distinction between the snow depth at the slope and the amount of fresh snow during a period of a snowfall might give more insight on the behavior of an avalanche formation process. The impact of wind and temperature on the occurrence of a snow avalanche is mainly connected to the snow depth and stability. However, the effect of both factors on the snowpack stability is rather complex and thus hardly tractable.

Our goal is to develop and investigate a stochastic snow avalanche model in a more general form than it has already been presented in the few references available.

In [7] the authors present a stochastic approach to so called growth-collapse processes, which describe the cycles of periods of steady growth followed by a sudden avalanche. Further studies of such processes have been presented in [3], where the authors make use of the theory of piecewise deterministic Markov processes. However, certain assumptions of growth-collapse processes do not coincide entirely with the behavior of snow avalanches which are our interest of study.

In [10] a model of a stochastic dynamics of snow avalanche is presented. The aim of that paper is to describe the dynamics of the snow depth at a slope. In order to do that, the authors of [10] consider deterministic trajectories randomly interrupted by instantaneous jumps, which are occurring according to two different compound Poisson processes.

In our paper we build upon the ideas from previously mentioned papers and develop a more general snow avalanche model.

The outline of this paper is as follows. In the following section of Chapter 1 we introduce our general snow avalanche model and its main assumptions. In Chapter 2 we briefly recall all the mathematical preliminaries, which are necessary to introduce and analyze piecewise deterministic Markov processes. A mathematical formulation and description of the general snow avalanche model is presented in Chapter 3, which is then simplified and thoroughly analyzed in Chapter 4. The models from [10] and [7] are compared to the general snow avalanche model and presented in the formulation of piecewise deterministic Markov processes in Chapter 5.

1.1 General snow avalanche model

In this section we introduce the general snow avalanche model. As it was mentioned before, the impact of slope angle on snow avalanches occurrence is constant over time, therefore we will not include it in the model. Instead, one might simply think that we consider a slope with a potential snow avalanche starting zone. Two other main factors having crucial impact on snow avalanches are taken into account, namely by $X(t)$ we denote the actual snow depth on a slope, while the variable $Y(t)$ stands for the depth of falling snow at time t.

Since the snowfalls occur randomly over time we introduce a variable $I(t) \in \{0,1\}$ and a function $r_{I(t)}$, such that $r_0(Y(t)) = 0$ and $r_1(Y(t))$ is some nonnegative deterministic function describing the rate of snowfall for $t \geq 0$. In other words, if $I(t) = 0$, then there is no snowfall, whereas if $I(t) = 1$, then the snow is falling according to some rate function r_1 at time t. Other weather factors, like wind or temperature, which are causing a decrease in the snow depth are having an impact on $X(t)$ according to some function ρ_i, where in case of $I(t) = 0$ the function

$\rho_0 = \rho_0(X(t), Y(t))$ depends solely on the actual snow depth on a slope, and when $I(t) = 1$, the function $\rho_1 = \rho_1(X(t), Y(t))$ depends also on the depth of falling snow at time t. We note, that the fresh snow is added to the snow lying on the slope implicitly through function ρ_1.

Naturally, we would like for $X(\cdot)$ and $Y(\cdot)$ to be nonnegative. Therefore we assume that at the boundary, i.e. when there is no snow at the slope,

$$r_1(0), \rho_1(0,0) > 0 \text{ and } r_0(0), \rho_0(0,0) = 0.$$

In Figure 1 we present a plot of a sample path of the general snow avalanche model, separately for $X(\cdot)$ and $Y(\cdot)$, from which it is also clear how to obtain the plot in terms of $I(\cdot)$. Note that the dotted lines, visible on the plots, represent the jumps caused by snow avalanches.

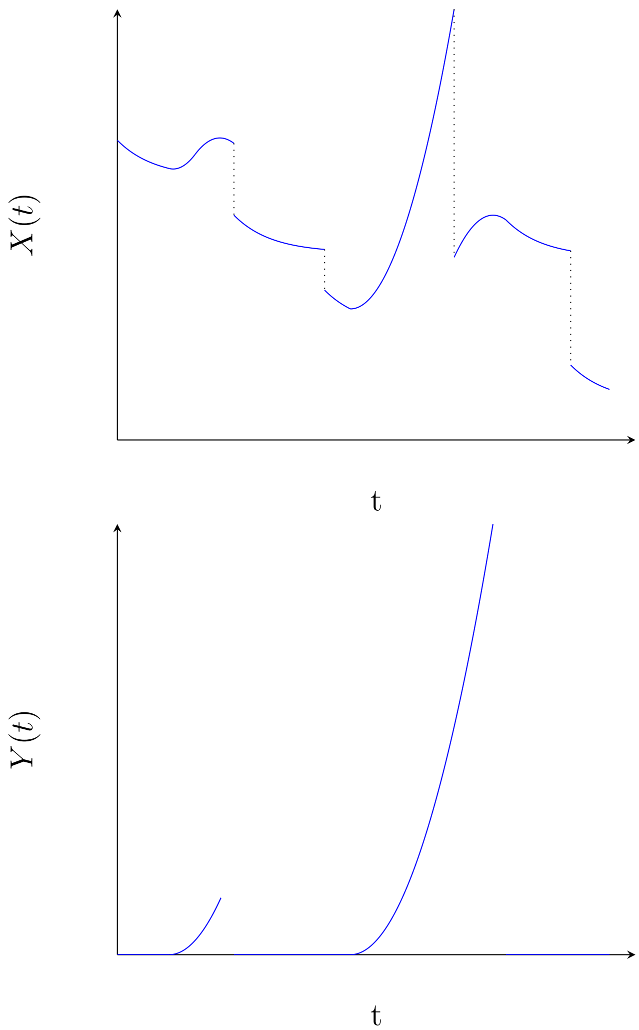

Fig. 1: Example of a sample path of the general snow avalanche model

2 Preliminaries

In this chapter we focus on mathematical theory which will be necessary to formally describe and analyze the general snow avalanche model. The main part of this chapter is the definition of piecewise deterministic Markov processes and their properties, which require some facts from martingale and differential generators theory. Throughout this chapter we assume that all random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a Borel space $(E, \mathcal{E})$. In this chapter we also use the following notation

- L_1 is the set of all integrable random variables $X \colon \Omega \to \mathbb{R}$,
- $B(E)$ is the set of all real valued and bounded measurable functions on E,
- $\mathcal{M}(E)$ is the set of all real valued and measurable functions on E.

2.1 Martingales

We begin this chapter by briefly revising basic facts from martingale theory. For a more advanced summary we refer the reader to the book of Davis [4].
First of all we recall the definition of conditional expectation and its few properties.

Definition 1. *(i) Let $X \in L_1$ be a random variable and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Any $\mathcal{G}$-measurable random variable Z such that*

$$\int_G X d\mathbb{P} = \int_G Z d\mathbb{P} \qquad \text{for all } G \in \mathcal{G}$$

is called the conditional expectation of X given $\mathcal{G}$ and denoted as $Z = \mathbb{E}(X|\mathcal{G})$.

(ii) The conditional expectation of $X \in L_1$ given some random variable Y is defined as

$$\mathbb{E}(X|Y) = \mathbb{E}(X|\sigma(Y)),$$

where $\sigma(Y)$ is the σ-algebra generated by Y.

Theorem 1. *Let $X \in L_1$ and $\mathcal{G}, \mathcal{H}$ be sub-σ-fields of $\mathcal{F}$.*

1. *If $Y \in L_1$ and $\alpha, \beta \in \mathbb{R}$, then $\mathbb{E}(\alpha X + \beta Y|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \beta\mathbb{E}(Y|\mathcal{G})$.*
2. *If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$.*
3. *If X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.*

4. *If Y is a bounded and $\mathcal{G}$-measurable random variable, then* $\mathbb{E}(XY|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})Y$.

5. *If $\mathcal{G} \subset \mathcal{H}$, then* $\mathbb{E}\left[\mathbb{E}(X|\mathcal{H})|\mathcal{G}\right] = \mathbb{E}(X|\mathcal{G})$.

For proofs of the above properties we refer the reader to [11].

Definition 2. *(i) A family of sub-σ-fields $(\mathcal{F}_t)_{t\geq 0}$ of $\mathcal{F}$ is called a filtration if*

$$\mathcal{F}_s \subset \mathcal{F}_t \qquad \text{for } s \leq t.$$

(ii) We call a stochastic process $(X(t))_{t\geq 0}$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$, if for every $t \geq 0$, $X(t)$ is an $\mathcal{F}_t$-measurable random variable.

(iii) A random variable $S : \Omega \to [0, \infty]$ such that

$$\{S \leq t\} \in \mathcal{F}_t, \qquad \text{for all } t \geq 0$$

is called a stopping time.

Now we are able to introduce the notion of a martingale.

Definition 3. *We call a process $(X(t))_{t\geq 0}$ a martingale of filtration $(\mathcal{F}_t)_{t\geq 0}$ if*

(i) $(X(t))_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$,

(ii) $X(t) \in L_1$, for every $t \geq 0$,

(iii) $\mathbb{E}(X(t)|\mathcal{F}_s) = X(s)$ almost surely for $s \leq t$.

As it will turn out later, a very important role in piecewise deterministic Markov processes is played by the local martingale which is defined as follows.

Definition 4. *A process $(X(t))_{t\geq 0}$ is a local martingale if*

(i) there exists a sequence of increasing stopping times $(S_n)_{n\in\mathbb{N}}$ such that $S_n \to \infty$ almost surely,

(ii) $X_n(t) := X(\min\{t, S_n\})$ is a martingale which is uniformly integrable, i.e.

$$\lim_{c\to\infty} \sup_{t\geq 0} \mathbb{E}(|X_n(t)|\mathbb{1}_{|X_n(t)|>c}) = 0 \qquad \text{for every } n \in \mathbb{N}.$$

2.2 Stochastic processes

In Chapter 5 we describe snow avalanche models presented in papers [10] and [7]. In our considerations we will use a slightly different approach than in these papers, but in order to compare all of the models, we briefly summarize some basic facts from stochastic processes described in [1] and [14].

Definition 5. *Let $(X(t))_{t\geq 0}$ be a stochastic process.*

(i) We say that the process $(X(t))_{t\geq 0}$ has independent increments if for every $n \in \mathbb{N}$ and all $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ the random variables

$$X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent.

(ii) We say that the process $(X(t))_{t\geq 0}$ has stationary increments if

$$X(t) - X(s) \stackrel{d}{=} X(t-s) - X(0)$$

for every $0 \leq s < t < \infty$.

The Poisson process is one of the most important stochastic processes used in various applications, including snow avalanche models. For that reason we introduce the definition of renewal and Poisson processes which are closely related to each other.

Definition 6. *Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of non-negative random variables and $(N(t))_{t\geq 0}$ be a counting process defined for every $t \geq 0$ as*

$$N(t) = \begin{cases} 0, & \text{if } X_1 > t, \\ \sup\{n \in \mathbb{N} : X_1 + \ldots + X_n \leq t\}, & \text{if } X_1 \leq t. \end{cases}$$

(i) The counting process $(N(t))_{t\geq 0}$ is called a renewal process if

$$0 < \mathbb{E}(X_1) < \infty.$$

(ii) The counting process $(N(t))_{t\geq 0}$ is called a Poisson process with rate λ if for every $n \in \mathbb{N}$, X_n is exponentially distributed with parameter λ, i.e.

$$\mathbb{P}(X_n \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

To end this short section we introduce the definition of a compound Poisson process.

Definition 7. *Let $(N(t))_{t\geq 0}$ be a Poisson process with rate λ. We call a stochastic process $(X(t))_{t\geq 0}$ a compound Poisson process if it can be represented by*

$$X(t) = \sum_{n=1}^{N(t)} D_n, \qquad t \geq 0,$$

where $(D_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables that are independent of the process $(N(t))_{t\geq 0}$.

2.3 The Laplace transform

As it will be visible in Chapter 4, the Laplace transform is a useful mathematical tool, commonly applied, for example, in the field of integro-differential equations. In this section we briefly present its definition and some basic properties, which can be found with proofs in [6].

Definition 8. *The Laplace transform of a function $f : [0, \infty) \to \mathbb{R}$, denoted as $\mathcal{L}(f(x))$, is the function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ defined as*

$$\tilde{f}(s) = \int_0^\infty f(x)e^{-sx}dx.$$

Property 1. *For any $a, b \in \mathbb{R}$*

$$\mathcal{L}(af(x) + bg(x)) = a\mathcal{L}(f(x)) + b\mathcal{L}(g(x)),$$

if $f(x)$ and $g(x)$ are functions whose Laplace transforms exist.

Property 2. *If $\mathcal{L}(f(x)) = \tilde{f}(s)$, then $\mathcal{L}(xf(x)) = -\frac{d}{ds}\tilde{f}(s)$.*

Property 3. *Let $\mathcal{L}(f(x)) = \tilde{f}(s)$. Then for any differentiable function $f(x)$*

$$\mathcal{L}\left(\frac{d}{dx}f(x)\right) = s\tilde{f}(s) - f(0).$$

2.4 Markov operators and stability

We devote this section to introduce the notions of Markov operators and semigroups, which are very useful in studying asymptotic properties of some systems. For more details we refer the reader to [12].

Throughout this section we let the triple $(E, \mathcal{E}, m)$ be a σ-finite measure space and let L^1 be the Banach space of all integrable functions $f \colon E \to \mathbb{R}$ with norm $\|f\| = \int_E |f(x)|m(dx)$.

Definition 9. *Let $D \subset L^1$ be the set of all densities, i.e.*

$$D = \{f \in L^1 : \|f\| = 1, f \geq 0\}.$$

A linear operator $\mathcal{P} : L^1 \to L^1$ is called a Markov operator if $\mathcal{P}(D) \subset D$.

Definition 10. *A family $\{\mathcal{P}(t)\}_{t\geq 0}$ of Markov operators is called a Markov semigroup if*

(i) $\mathcal{P}(0) = Id$,

(ii) $\mathcal{P}(t+s) = \mathcal{P}(t)\mathcal{P}(s)$ *for* $s, t \geq 0$,

(iii) the function $t \mapsto \mathcal{P}(t)f$ is continuous for every $f \in L^1$.

We finish this section with the following definition.

Definition 11. *Let $\{\mathcal{P}(t)\}_{t\geq 0}$ be a Markov semigroup*

(i) A density $f_* \in D$ is called invariant if $\mathcal{P}(t)f_* = f_*$ for all $t > 0$.

(ii) If there exists an invariant density $f_* \in D$ such that

$$\lim_{t\to\infty} \|\mathcal{P}(t)f - f_*\| = 0 \quad \text{for } f \in D,$$

then the Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ is called asymptotically stable.

The asymptotic stability of a Markov semigroup generated by some differential equation means that, starting from a density, the solution of the equation converges to the invariant density.

2.5 Differential generators of Markov processes

In this section we introduce the notions of strong and extended differential generators of Markov processes, which will play a crucial role in the analysis of models presented in the rest of the paper.

Definition 12. *A collection $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X(t))_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$, where*

- $(\Omega, \mathcal{F})$ *is a measurable space,*
- $(\mathcal{F}_t)_{t\geq 0}$ *is a filtration,*
- $(X(t))_{t\geq 0}$ *is a family of random variables adapted to* $(\mathcal{F}_t)_{t\geq 0}$,

- *for every $x \in E$, $\mathbb{P}_x$ is a probability measure on $(\Omega, \mathcal{F})$, such that $(X(t))_{t\geq 0}$ is a Markov process on $(\Omega, \mathcal{F}, \mathbb{P}_x)$, i.e. for any $f \in B(E)$ and $s \leq t$*

$$\mathbb{E}[f(X(t))|\mathcal{F}_s] = \mathbb{E}[f(X(t))|X(s)],$$

with intial distribution δ_x,

is called a Markov family.

Throughout this section we assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X(t))_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ is a Markov family.

The differential generator of a Markov process is strictly associated with the semigroup $(P_t)_{t\geq 0}$ of operators $P_t : B(E) \to B(E)$, defined for every $t \geq 0$ as

$$P_t f(x) = \mathbb{E}_x f(X(t)). \tag{1}$$

The semigroup property of $(P_t)_{t\geq 0}$ means that for any $s, t \geq 0$, $f \in B(E)$ and $x \in E$ we have

$$P_s(P_t f)(x) = P_{s+t} f(x).$$

Definition 13. *Operator $\hat{\mathcal{A}}$ with domain $\mathcal{D}(\hat{\mathcal{A}}) \subset B(E)$ defined as*

$$\mathcal{D}(\hat{\mathcal{A}}) = \left\{ f \in B(E) : \lim_{t\downarrow 0} \frac{1}{t}(P_t f - f) \text{ exists in } B(E) \right\},$$

$$\hat{\mathcal{A}} f = \lim_{t\downarrow 0} \frac{1}{t}(P_t f - f), \qquad f \in \mathcal{D}(\hat{\mathcal{A}}),$$

is called the strong generator of semigroup $(P_t)_{t\geq 0}$.

Note that Definition 13 is stated for any semigroup, but in this paper we will only consider strong generators of semigroups defined with formula (1).
We now present three important propositions regarding the strong generator in probabilistic terms. Proofs of these statements in analytical terms can be found in [4].

Proposition 1. *Let $W(t,x) = \mathbb{E}_x f(X(t))$ for $f \in \mathcal{D}(\hat{\mathcal{A}})$. Then for every $x \in E$, W satisfies*

$$\frac{\partial}{\partial t} W(t,x) = \hat{\mathcal{A}} W(t,x), \qquad W(0,x) = f(x).$$

Proposition 2. *(Dynkin's formula) For any $f \in \mathcal{D}(\hat{\mathcal{A}})$ the following holds*

$$\mathbb{E}_x f(X(t)) = f(x) + \mathbb{E}_x \int_0^t \hat{\mathcal{A}} f(X(s)) ds, \quad x \in E.$$

Proposition 3. *For $f \in \mathcal{D}(\hat{\mathcal{A}})$ let $(C_t^f)_{t\geq 0}$ be a real-valued process defined by*

$$C_f(t) = f(X(t)) - f(X(0)) - \int_0^t \hat{\mathcal{A}} f(X(s))ds.$$

Then the process $(C_f(t))_{t\geq 0}$ is a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$, for any $x \in E$.

It has been noticed, that there may be other functions f, not necessarily in $\mathcal{D}(\hat{\mathcal{A}})$, for which something similar to above propositions is still true. For that reason, we use the following definition of the extended generator.

Definition 14. *Let $\mathcal{D}(\mathcal{A})$ be the set of measurable functions $f : E \to \mathbb{R}$ such that there exist a measurable function $g : E \to \mathbb{R}$ having the following properties:*

- *for every $x \in E$, the function $t \mapsto g(X(t))$ is integrable P_x - almost surely,*
- *the process $C_f(t) = f(X(t)) - f(X(0)) - \int_0^t g(X(s))ds$ is a local martingale.*

Then writing $g = \mathcal{A}f$ we call $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ the extended generator of the process $(X(t))_{t\geq 0}$.

Remark 1. *Note that if $C_f(t)$ is a martingale, then $\mathbb{E}_x C_f(t) = C_f(0) = 0$. Thus the set $D = \{f \in \mathcal{D}(\mathcal{A}) : C_f(t) \text{ is a martingale}\}$ is the largest class of functions for which Dynkin's formula holds, i.e.*

$$\mathbb{E}_x f(X(t)) = f(x) + \mathbb{E}_x \int_0^t \mathcal{A} f(X(s))ds, \qquad f \in D.$$

2.6 Piecewise deterministic Markov processes

Before we proceed to deriving the general snow avalanche model we introduce some facts from the theory of piecewise deterministic Markov processes (PDMP), which were first described by Davis in [5]. Consistently with [5], in this section we will denote the studied process with a small letter instead of a capital one.

Definition 15. *Let $(E, \mathcal{E})$ be a Borel space with*

$$E = \{(\xi, k) : \xi \in M_k, k \in C\}, \tag{2}$$

where C is a finite set and $M_k \subset \mathbb{R}^{d(k)}$ for $k \in C$ and some function $d : C \to \mathbb{N}$. Furthermore let the state of the process be denoted by $x(t) = (\xi(t), k(t))$. Then the following objects determine the PDMP (or equivalently the probability law of $(x(t))_{t\geq 0}$):

(i) Vector fields $\mathscr{X}_k$, $k \in C$, such that for every $\xi \in \mathbb{R}^{d(k)}$ there exists a unique and global solution $\phi_k(t, \xi)$ in $\mathbb{R}^{d(k)}$ of

$$(3) \qquad \begin{cases} \frac{d}{dt}\zeta(t) = \mathscr{X}_k \zeta(t), \\ \zeta(0) = \xi. \end{cases}$$

(ii) A measurable function $\lambda : E \to \mathbb{R}_+$. We assume that for every $(\xi, k) \in E$ there exists $\epsilon > 0$ such that the function $t \mapsto \lambda(\phi_k(t, \xi), k)$ is integrable for $t \in [0, \epsilon)$.

(iii) A transition measure $Q : (E \cup \Gamma^*) \times \mathcal{E} \to [0, 1]$ such that for every fixed $A \in \mathcal{E}$, $Q(x, A)$ is a measurable function of $x \in E \cup \Gamma^*$, $Q(x, A)$ is a probability measure on $(E, \mathcal{E})$ and $Q(x, \{x\}) = 0$ for every $x \in E$. The set Γ^* is defined using the boundary of M_k (denoted by ∂M_k) as follows

$$\Gamma^* = \bigcup_{k \in C} \partial^* M_k,$$

where

$$\partial^* M_k = \{\zeta \in \partial M_k : \phi_k(t, \xi) = \zeta \text{ for some } (t, \xi) \in \mathbb{R}_+ \times M_k\}.$$

Let $x(0) = (\xi_0, k_0) \in E$ be the starting point of $(x(t))_{t \geq 0}$ and denote

$$t^*(\xi, k) = \inf\{t > 0 : \phi_k(t, \xi) \in \partial^* M_k\}.$$

The motion of our process can be constructed as follows.
Define

$$\bar{F}(t) = \begin{cases} \exp\left(-\int_0^t \lambda(\phi_{k_0}(s, \xi_0), k_0) ds\right), & t < t^*(x(0)), \\ 0, & t \geq t^*(x(0)), \end{cases}$$

and choose a random variable T_1 such that $\mathbb{P}(T_1 > t) = \bar{F}(t)$. Then for $t \leq T_1$ the trajectory of $x(t)$ is given by

$$x(t) = \begin{cases} (\phi_{k_0}(t, \xi_0), k_0), & t < T_1, \\ (\Xi, K), & t = T_1, \end{cases}$$

where (Ξ, K) is a random variable having distribution $Q(\phi_{k_0}(T_1, \xi), \cdot)$. The next inter-jump time $T_2 - T_1$ and post-jump location $x(T_2)$ are chosen in the same manner as above starting from point $x(T_1)$. Doing so with jump times $T_1, T_2, \ldots$, one defines a piecewise deterministic trajectory of the process $(x(t))_{t \geq 0}$.
For the sake of clarity we will write $T_0 = 0$.

Assumption 1. *Let* $N_t(\omega) = \sum_{i=0}^{\infty} \mathbb{1}_{\{\omega\in\Omega: t\geq T_i(\omega)\}}(\omega)$ *be the number of jumps in* $[0,t]$*. Then we assume that for every starting point* $x \in E$ *and* $t \geq 0$

$$\mathbb{E}N_t < \infty.$$

In particular, this assumption implies that $T_n(\omega) \xrightarrow[n\to\infty]{} \infty$ almost surely.

Remark 2. *Under stated assumptions on* λ *we have*

$$\mathbb{P}(T_1 > 0) = \mathbb{P}(T_n - T_{n-1} > 0) = 1, \quad n \in \mathbb{N}.$$

Proposition 4. $(x(t))_{t\geq 0}$ *is a Markov process.*

Proof. Let $T_n \leq t < T_{n+1}$ for some $n \in \mathbb{N}$. We have

$$\mathbb{P}(T_{n+1} - T_n > s) = \begin{cases} \exp\left(-\int_0^s \lambda(\phi_{k(T_n)}(u, \xi(T_n)), k(T_n))du\right), & t < t^*(x(T_n)) \\ 0, & t \geq t^*(x(T_n)) \end{cases}$$

Taking $t < s < t^*(x(T_n))$ we calculate that

$$\mathbb{P}(T_{n+1} > s | T_n = t_n, T_{n+1} > t) = \mathbb{P}(T_{n+1} - t_n > s - t_n | T_n = t_n, T_{n+1} - t_n > t - t_n) =$$

$$= \exp\left(-\int_{t-t_n}^{s-t_n} \lambda(\phi_{k(t_n)}(u, \xi(t_n)), k(t_n))du\right) = \exp\left(-\int_0^{s-t} \lambda(\phi_{k(t)}(u, \xi(t)), k(t))du\right),$$

where we have used the fact, that $k(t_n) = k(t)$ and ϕ has a semigroup property, i.e. $\phi_{k(t)}(u + s, \xi) = \phi_{k(t)}(u, \phi_{k(t)}(s, \xi))$.
Therefore the distribution of T_{n+1} depends only on the current state $(\xi(t), k(t))$, thus since the process begins afresh at T_{n+1}, it satisfies the Markov property. $\square$

Before moving on to the final theorem of this section we have to introduce the set Γ of boundary points in Γ^* which the process never hits. Therefore we define

$$\Gamma = \left\{(\xi, k) \in \Gamma^* : \mathbb{P}_x(T_1 = t) > 0 \text{ for some } t > 0, \text{ with } x = (\phi_k^{-1}(t, \xi), k)\right\}.$$

We conclude this chapter with one of the main results of PDMP theory, which allow us to determine the extended generator of a given process and its domain.

Theorem 2. *Let* $(x(t))_{t\geq 0}$ *be a PDMP satisfying all the assumptions made in the above considerations. Then the domain* $\mathcal{D}(\mathcal{A})$ *of the extended generator* $\mathcal{A}$ *of* $(x(t))_{t\geq 0}$ *is the set of those functions* $f \in \mathcal{M}(E)$ *satisfying:*

(i) for every $(\xi, k) \in E$ *the function* $t \mapsto f(\phi_k(t, \xi), k)$ *is absolutely continuous for* $t \in [0, t^*(\xi, k))$*,*

(ii) the boundary condition, i.e.

$$f(x) = \int_E f(y)Q(x, dy) \qquad \text{for } x \in \Gamma,$$

(iii)

$$\mathbb{E}\left[\sum_{T_n \le t} |f(x(T_n)) - f(x(T_{n^-}))|\right] < \infty \qquad \text{for every } t \ge 0.$$

Furthermore for $f \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}f$ is given at point $x \in E$ by

$$\mathcal{A}f(x) = \mathscr{X} f(x) + \lambda(x) \int_E (f(y) - f(x))Q(x, dy),$$

where $\mathscr{X} f(x)$ denotes $\mathscr{X}_k f((\xi, k))$ for $x = (\xi, k) \in E$.

The proof of the above theorem can be found in [5].

3 PDMP formulation of the general snow avalanche model

In this chapter we present the general snow avalanche model as a PDMP consistently with the definition introduced in Section 2.5. We recall that by $X(t)$ we denote the actual snow depth on a slope, while the variable $Y(t)$ stands for the depth of falling snow at time t. We have also introduced a random variable $I(t) \in \{0, 1\}$, where if $I(t) = 0$, then there is no snowfall, whereas if $I(t) = 1$, then the snow is falling.

3.1 Model description

Let $C = \{0, 1\}$ and $M_i \subset \mathbb{R}^2$ (thus $d(i) = 2$) for $i \in C$, where

$$M_0 = [0, \infty) \times \{0\}, \qquad M_1 = (0, \infty) \times (0, \infty) \cup [0, \infty) \times \{0\}.$$

Then we consider our Borel space $(E, \mathcal{E})$ with

$$E = \bigcup_{i=0}^{1} M_i \times \{i\}.$$

We assume that there exist unique and global solutions $\phi_i(t, \xi)$ in $\mathbb{R}^2$ of the following ordinary differential equations for $i = 0, 1$

$$\begin{cases} X'(t) = \rho_i\left(X(t), Y(t)\right) \\ Y'(t) = r_i\left(Y(t)\right). \end{cases} \tag{4}$$

This system of differential equations describes the behavior of our process in between jumps.

In our process there are two possible types of jump points:

I An avalanche occurs, which causes a jump from point $x > 0$ to some $z < x$ with probability density $p(x, z)$. Note that $p(x, z) = 0$, for $z \geq x$, $x \neq 0$ and $p(0, 0) = 1$.

II New snow starts or stops falling, which with probability 1 causes a jump from point (y, i) to $(0, 1 - i)$.

For every $(x, y, i) \in E$ we define $\lambda(x, y, i) = \nu(x, y, i) + q$, where ν is the rate function of avalanche occurrence and q is the snowfall rate. We assume that λ satisfies all the assumptions from Definition 15(ii), i.e. it is a measurable function, locally integrable along the solutions of (4). We would like to note that we want to avoid any jumps from point $(0,0,1)$ - when there is no snow on the slope and fresh snow has started falling. In that case we would like our system to behave deterministically according to the differential equation (4). Therefore we assume that $\lambda(0,0,1)=0$.

The jump moments T_i, $i=0,1,\ldots$, of occurrence of an avalanche or a snowfall are such that

$$\mathbb{P}(T_{n+1}-T_n>t)=\begin{cases} \exp\left(-\int_0^t \lambda(\phi_i(s,\xi_{T_n}),i)ds\right), & t<t^*(X(T_n),Y(T_n),I(T_n)) \\ 0, & t\geq t^*(X(T_n),Y(T_n),I(T_n)) \end{cases},$$

for all $n\in\mathbb{N}$.

Recall that $\lambda(x,y,i)=\nu(x,y,i)+q$. Then $p_i(x,y)=\nu(x,y,i)/\lambda(x,y,i)$, for $(x,y,i)\neq(0,0,1)$, is the probability that the jump point is of type I, while $1-p_i(x,y)$ is the probability of jump point being of type II. In addition we assume that $p_1(0,0)=1$.

Before defining the transition function, we hereby present all the possible jumps in our model, given that a jump occurs, in the following diagrams.
For $x>0$, $z<x$ and $y\geq 0$ we have

$$(x,0,0)\begin{array}{l}\nearrow (z,0,0) \text{ with probability } p(x,z)p_0(x,0)\\ \\ \searrow (x,0,1) \text{ with probability } 1-p_0(x,0)\end{array}$$

$$(x,y,1)\begin{array}{l}\nearrow (z,y,1) \text{ with probability } p(x,z)p_1(x,y)\\ \\ \searrow (x,0,0) \text{ with probability } 1-p_1(x,y)\end{array}$$

Some formal problems arise when it comes to transitions from points $(0,0,0)$ and $(0,0,1)$, i.e. when there is no snow on the slope. In reality when the system reaches state $(0,0,0)$ it stays there for a while until the snow starts falling again. As it was mentioned earlier, we would like our system to behave deterministically when it is in $(0,0,1)$. To describe the "waiting time", which the system spends in $(0,0,0)$, we would formally have to extend our state space and transition function adding one more variable. Such extension would make our model less clear and significantly hinder the calculations, thus we will not use it. Nevertheless we emphasize it's necessity in "waiting time" considerations. Instead we propose the following transitions

$$
\begin{array}{ccl}
 & \nearrow & (0,0,0) \text{ with probability } p_0(0,0) \\
(0,0,0) & & \\
 & \searrow & (0,0,1) \text{ with probability } 1-p_0(0,0)
\end{array}
$$

and

$$(0,0,1) \to (0,0,1) \text{ with probability } 1.$$

Now we are able to build the transition function

$$Q\left((x,y,i), B_1 \times B_2 \times \{j\}\right) = \begin{cases} p_i(x,y)P(x,B_1)\mathbb{1}_{B_2}(y), & \text{if } j = i, \\ (1-p_i(x,y))\mathbb{1}_{B_1}(x)\mathbb{1}_{B_2}(y), & \text{if } j = 1-i, \end{cases} \tag{5}$$

where $B_1 \subset M_1$, $B_2 \subset M_2$ and

$$P(x,B) = \int_B p(x,z)dz.$$

We introduce the extended generator $\mathcal{A}$ of our process, which as it was mentioned in Theorem 2, is given by the following formula for $f \in \mathcal{D}(\mathcal{A})$

$$\mathcal{A}f(x,y,i) = \mathscr{X}_i f(x,y,i) + \lambda(x,y,i)[Qf(x,y,i) - f(x,y,i)]. \tag{6}$$

In our case

$$\mathscr{X}_i f(x,y,i) = \rho_i(x,y)\frac{\partial}{\partial x}f(x,y,i) + r_i(y)\frac{\partial}{\partial y}f(x,y,i),$$

and

$$Qf(x,y,i) = \int_E f(\bar{x},\bar{y},\bar{i})Q((x,y,i),d\bar{x},d\bar{y},d\bar{i}),$$

which after substituting formula (5) gives

$$Qf(x,y,i) = p_i(x,y)\int_0^x p(x,z)f(z,y,i)dz + (1-p_i(x,y))f(x,0,1-i).$$

Thus the extended generator of our process is given by the following formula

$$\begin{aligned} \mathcal{A}f(x,y,i) = {} & \rho_i(x,y)\frac{\partial}{\partial x}f(x,y,i) + r_i(y)\frac{\partial}{\partial y}f(x,y,i) \\ & + \lambda(x,y,i)\left[p_i(x,y)\int_0^x p(x,z)f(z,y,i)dz \right. \\ & \left. +(1-p_i(x,y))f(x,0,1-i) - f(x,y,i)\right]. \end{aligned} \tag{7}$$

3.2 Expectations

We have already introduced our general snow avalanche model and described it as a piecewise deterministic Markov process. What we would like to do next is to calculate some expectations of our process.

Lemma 1. *Let* $W(t,x,y,i) = \mathbb{E}_{(x,y,i)} f(X(t), Y(t), I(t))$ *for* $f \in \mathcal{D}(\hat{\mathcal{A}})$. *Then for every* $(x,y,i) \in E$ *and* $t \in (0,\infty)$ *the transient behavior of* $W(t,x,y,i)$ *is described by the following equation*

$$
\begin{aligned}
(8) \quad \frac{\partial}{\partial t} W(t,x,y,i) &= \rho_i(x,y) \frac{\partial}{\partial x} W(t,x,y,i) + r_i(y) \frac{\partial}{\partial y} W(t,x,y,i) \\
&\quad + \lambda(x,y,i) \left[p_i(x,y) \int_0^x p(x,z) W(t,z,y,i) dz \right. \\
&\qquad\qquad \left. + (1 - p_i(x,y)) W(t,x,0,1-i) - W(t,x,y,i) \right]
\end{aligned}
$$

and $W(0,x,y,i) = f(x,y,i)$.

Proof. To prove this lemma, we note that the extended generator and strong generator of a process coincide on the domain of the strong generator, i.e.

$$\mathcal{A} f = \hat{\mathcal{A}} f \qquad \text{for } f \in \mathcal{D}(\hat{\mathcal{A}}).$$

Therefore substituting formula (7) in Proposition 1 yields the claim of the lemma. □

Equation (8) is really hard to solve analytically not only in the general case, but also taking some specific functions and distributions, in order to make the calculations easier, does not make it clearer how one could obtain the solution of this equation. However, it is possible to solve (8) numerically, obtaining some results on the expectations of the general snow avalanche model. Some attempts were made to derive the equation for the behavior of the density of our process, but also in this case the model turned out to be too complicated. For that reason we will leave this as an open problem and try to simplify our model in the following chapter.

4 Simplification of the general snow avalanche model

As it was mentioned before, it is really hard to obtain some analytical results from the general snow avalanche model. In this chapter we simplify this model by replacing the snowfall periods with positive jumps. After such procedure our state space is no longer three, but one dimensional.

To be more precise, we do not consider the snowfall periods, thus variables $Y(\cdot)$ and $I(\cdot)$ do not appear in this simplification. Instead we assume that all the new snow is added at once after a snowfall. One could imagine that we simply cut out the periods of the snowfall and glue the other periods together, assuming that the snow depth level differences are positive jumps. We note that in the general snow avalanche model we have allowed for an avalanche to occur during the snowfall period. For that reason, to consider the model presented in this chapter as a simplification of the general model, we have to assume in the latter one that it is impossible for an avalanche to occur in the snowfall period. Additionally we assume that snowfall related jumps sizes are independent of the level of the system, which is another simplification of the assumptions made in the general model.

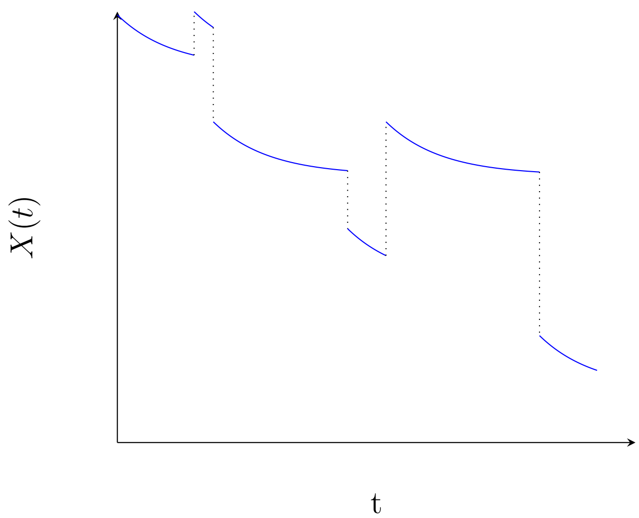

Fig. 2: Example of a sample path of the simplified model

In Figure 2 we present a plot of a sample path of the simplified model. Note that this plot has been obtained from Figure 1 using the method of cutting out the periods of snowfalls, as described above. This particular example shows the importance of the assumption on lack of avalanches during the snowfall periods, since one might easily observe a loss of information on the biggest avalanche from Figure 1.

In between the jumps our model is then described by the following ordinary differential equation for $t \geq 0$

$$X'(t) = -\rho(X(t)), \tag{9}$$

where this time $\rho(\cdot)$ is some nonnegative function of the snow depth, responsible for the deterministic drift. We assume that $\rho(0) = 0$ and that there exists a unique and global solution in $\mathbb{R}$ of (9), which we will denote by $\phi(t, \xi)$. In the simplified model there are two possible types of jump points, :

I An avalanche occurs, which causes a jump from point x to some $z < x$ with probability density $p(x, x-z)$. Note that $p(x, x-z) = 0$ for $z \geq x$.

II There was a snowfall, which causes a jump from point x to some $y > x$ with probability density $b(y-x)$. Note that $b(y-x) = 0$ for $y \leq x$.

In resemblance with the general model, we assume that $\nu(x)$ is the rate function of avalanche occurrence at point $x \in [0, \infty)$ and q is the snowfall rate.

4.1 Derivation of the density equation

For now, our goal is to derive the equation describing the behavior of the density of our process. In this case it can be done in two different ways, which we will describe. The first one is based on infinitesimal arguments, which are more appealing to the intuition of how the system behaves. The second approach is based on the theoretical fundaments of PDMP and can be seen as a more formal way of obtaining the equation mentioned earlier.

Lemma 2. *Let $w(t, x)$ be the density of the simplified model. Then the equation describing the transient behavior of $w(t, x)$ is given for $t, x \in [0, \infty)$ as*

$$\begin{aligned} \frac{\partial}{\partial t} w(t,x) = {} & \frac{\partial}{\partial x}(\rho(x) w(t,x)) - (q + \nu(x)) w(t,x) \\ & + \int_0^x q b(x-y) w(t,y) dy + \int_x^\infty \nu(z) p(z, z-x) w(t,z) dz. \end{aligned} \tag{10}$$

We will prove this lemma in two different ways, namely using an infinitesimal and PDMP approach, which are presented in the following two sections.

4.1.1 Infinitesimal approach

In this section we focus on the behavior of our process during a time interval $(t, t+h)$ for $h > 0$ small enough.
Define for all $t \geq 0$ and $x \in [0, \infty)$ the distribution function

$$W(t,x) = \mathbb{P}(X(t) \leq x) = \int_0^x w(t,y)dy.$$

Furthermore we let $P(\cdot)$ and $B(\cdot)$ be the cumulative distribution functions of avalanche and snowfall jumps sizes, respectively, i.e.

$$P(y,x) = \int_0^x p(y,r)dr, \quad B(x) = \int_0^x b(z)dz.$$

Before we start writing down the equation describing the behavior of our system we briefly describe what could actually happen in $(t, t+h)$.
If we assume that at time $t+h$ the level of our system is below x, i.e. $X(t+h) \leq x$, then one of the following events has occurred in time interval $(t, t+h)$

- there was no jump and the level of our system was decreasing deterministically, such that $X(t+h) \leq x$,
- $X(t) \leq x$ and an avalanche occurred,
- $X(t) = z$ for some $z > x$ and an avalanche of size $A = X(t) - X(t+h)$ occurred,
- $X(t) = y$ for some $y \leq x$ and there was a snowfall of size $S = X(t+h) - X(t)$,
- two or more events occurred with probability $o(h)$.

Note that $A \geq z - x$ since $X(t+h) \leq x$ and $X(t) = z$, thus

$$\mathbb{P}(A \geq z - x | X(t) = z) = 1 - \mathbb{P}(A < z - x | X(t) = z) = 1 - P(z, z - x).$$

Similarly $S \leq x - y$, thus

$$\mathbb{P}(S \leq x - y) = B(x - y).$$

Thus we can describe our system with the following equation

$$\begin{aligned} W(t+h,x) = &\int_0^x (1 - qh - \nu(y)h)dW(t, y + \rho(y)h) + \int_0^x qhB(x-y)dW(t,y) \\ &+ \int_0^x \nu(z)hdW(t,z) + \int_x^\infty \nu(z)h(1 - P(z, z-x))dW(t,z) + o(h). \end{aligned}$$

Recalling that $\int_0^x dW(t,z) = \int_0^x w(t,z)dz = W(t,x)$ we can rewrite this equation as

$$\begin{aligned} W(t+h,x) - W(t,x+\rho(x)h) = h\Big[&- qW(t,x+\rho(x)h) \\ &- \int_0^x \nu(z)dW(t,z+\rho(z)h) + \int_0^x qB(x-y)dW(t,y) \\ &+ \int_0^x \nu(z)dW(t,z) + \int_x^\infty \nu(z)(1-P(z,z-x))dW(t,z) + \frac{o(h)}{h}\Big]. \end{aligned}$$

Dividing both sides of the equation by h and letting $h \downarrow 0$ we obtain

$$\begin{aligned} \lim_{h\downarrow 0} \frac{W(t+h,x) - W(t,x+\rho(x)h)}{h} &= -qW(t,x) \\ &- \int_0^x \nu(z)dW(t,z) + \int_0^x qB(x-y)dW(t,y) \\ &+ \int_0^x \nu(z)dW(t,z) + \int_x^\infty \nu(z)(1-P(z,z-x))dW(t,z). \end{aligned}$$

Consider the left-hand side of the previous equation

$$\begin{aligned} &\lim_{h\downarrow 0} \frac{W(t+h,x) - W(t,x+\rho(x)h)}{h} = \\ &= \lim_{h\downarrow 0} \frac{W(t+h,x) - W(t,x) + W(t,x) - W(t,x+\rho(x)h)}{h} = \\ &= \lim_{h\downarrow 0} \frac{W(t+h,x) - W(t,x)}{h} - \lim_{h\downarrow 0} \frac{W(t,x+\rho(x)h) - W(t,x)}{h} = \\ &= \frac{\partial}{\partial t}W(t,x) - \rho(x)\frac{\partial}{\partial x}W(t,x). \end{aligned}$$

Thus, after an easy rearrangement of terms we have

$$\begin{aligned} \frac{\partial}{\partial t}W(t,x) = \rho(x)\frac{\partial}{\partial x}W(t,x) &- \int_0^x q(1-B(x-y))dW(t,y) \\ &+ \int_x^\infty \nu(z)(1-P(z,z-x))dW(t,z). \end{aligned} \tag{11}$$

To finally obtain the equation on densities, we simply have to differentiate both sides of this equation with respect to x, since $\frac{\partial}{\partial x}W(t,x) = w(t,x)$. This gives

$$\begin{aligned} \frac{\partial}{\partial t}w(t,x) = \frac{\partial}{\partial x}(\rho(x)w(t,x)) - qw(t,x) &+ \int_0^x qb(x-y)dW(t,y) \\ &- \nu(x)w(t,x) + \int_x^\infty \nu(z)p(z,z-x)dW(t,z), \end{aligned}$$

where we note that it was allowed to differentiate under the integral sign.

This completes the proof of Lemma 2.

4.1.2 PDMP approach

In this section we will show that, using the piecewise deterministic Markov processes theory, one can obtain equation (10) in a more formal way.

To begin with we note that in this model the set C from Definition 15 consists of only one element, thus we will simplify the notation by writing $E = [0, \infty)$, i.e. in the whole description of this piecewise deterministic Markov process we neglect the indices which by definition are connected with set C.

For every $x \in [0, \infty)$ we define $\lambda(x) = \nu(x) + q$. Then $\nu(x)/\lambda(x)$ is the probability that the jump point is of type I (avalanche), while $q/\lambda(x)$ is the probability of jump point being of type II (snowfall).

The jump moments T_i, $i = 0, 1, \ldots$, of occurrence of an avalanche or a snowfall are such that

$$\mathbb{P}(T_{n+1} - T_n > t) = \begin{cases} \exp\left(-\int_0^t \lambda(\phi(s, \xi_{T_n}))ds\right), & t < t^*(X(T_n)) \\ 0, & t \geq t^*(X(T_n)) \end{cases},$$

for all $n \in \mathbb{N}$.

For every $x \in [0, \infty)$, $z \leq x$ and $y \geq x$ all the possible jumps of our simplified process can be presented in the following diagram

$$x \begin{array}{l} \nearrow z \text{ with probability } \frac{\nu(x)}{\lambda(x)} p(x, x-z) \\ \\ \searrow y \text{ with probability } \frac{q}{\lambda(x)} b(y-x) \end{array}$$

Therefore the transition function of this process is given for every $x \in E$ and $B \subset E$ as

$$Q(x, B) = \frac{\nu(x)}{\lambda(x)} \int_B p(x, x-z)dz + \frac{q}{\lambda(x)} \int_B b(y-x)dy.$$

Using Theorem 2, we know that the extended generator $\mathcal{A}$ of our process is given for $f \in \mathcal{D}(\mathcal{A})$ by the following formula

$$\mathcal{A}f(x) = -\rho(x)\frac{\partial}{\partial x}f(x) + \lambda(x)[Qf(x) - f(x)].$$

Using the formula for the transition function we can write the previous equation more explicitly as

$$(12)\quad \mathcal{A}f(x) = -\rho(x)\frac{\partial}{\partial x}f(x) + \nu(x)\int_0^x p(x, x-z)f(z)dz \\ + q\int_x^\infty b(z-x)f(z)dz - (\nu(x)+q)f(x).$$

We note that using the forward Kolmogorov equation and assuming that $w(t,x)$ is the density of $X(t)$ at point $x \in [0,\infty)$, for $t \geq 0$, we have

$$\frac{\partial}{\partial t}w(t,x) = \mathcal{G}w(t,x), \tag{13}$$

where $\mathcal{G}$ is a formal adjoint operator of $\mathcal{A}$, i.e. for any functions f, g bounded, smooth enough and integrable with respect to the Lebesgue measure

$$\int_E \mathcal{A}f(x)g(x)dx = \int_E f(x)\mathcal{G}g(x)dx.$$

Thus, to derive the equation describing the behavior of the density of our process, we have to determine the formal adjoint operator $\mathcal{G}$.

Before we begin the calculations, we assume that functions f and g are both smooth, bounded, integrable and at least one of them is equal to zero in infinity. Then we have

$$\int_0^\infty \mathcal{A}f(x)g(x)dx = \int_0^\infty -\rho(x)\frac{\partial}{\partial x}(f(x))g(x)dx \\ + \int_0^\infty \left(\nu(x)\int_0^x p(x,x-z)f(z)dz\right)g(x)dx + \int_0^\infty \left(q\int_x^\infty b(z-x)f(z)dz\right)g(x)dx \\ - \int_0^\infty (\nu(x)+q)f(x)g(x)dx.$$

Integrating by parts the first term of the right-hand side of the above equation we obtain

$$\int_0^\infty \mathcal{A}f(x)g(x)dx = -\rho(x)f(x)g(x)\Big|_0^\infty + \int_0^\infty f(x)\frac{\partial}{\partial x}(\rho(x)g(x))dx \\ + \int_0^\infty\int_0^x \nu(x)p(x,x-z)f(z)g(x)dzdx + \int_0^\infty\int_x^\infty qb(z-x)f(z)g(x)dzdx \\ - \int_0^\infty (\nu(x)+q)f(x)g(x)dx.$$

Recalling that $\rho(0) = 0$, at least one of the two functions f, g is equal to zero in infinity and using Fubini's theorem, we have

$$
\begin{aligned}
\int_0^\infty \mathcal{A}f(x)g(x)dx &= \int_0^\infty f(x)\frac{\partial}{\partial x}(\rho(x)g(x))dx \\
&+ \int_0^\infty f(x)\int_x^\infty \nu(z)p(z,z-x)g(z)dzdx + \int_0^\infty f(x)\int_0^x qb(x-y)g(y)dydx \\
&- \int_0^\infty f(x)(\nu(x)+q)g(x)dx.
\end{aligned}
$$

From the equation above one could easily notice, that the formal adjoint operator $\mathcal{G}$ of the operator $\mathcal{A}$ is given by the following formula for $x \in [0,\infty)$

$$
\begin{aligned}
\mathcal{G}g(x) = {}& \frac{\partial}{\partial x}(\rho(x)g(x)) - (q+\nu(x))g(x) \\
&+ \int_0^x qb(x-y)g(y)dy + \int_x^\infty \nu(z)p(z,z-x)g(z)dz.
\end{aligned}
$$

Substituting this formula in (13) yields (10), which completes the proof.

4.2 Existence and uniqueness of the stationary distribution

In this section we present the proof of existence and uniqueness of the stationary distribution for the simplified model of snow avalanche presented in this chapter.

In Section 4.1 we have derived the equation (10) for the transient behavior of the density of our model. This equation induces a Markov semigroup $\{P(t)\}_{t\geq 0}$ (see e.g. [15]).

Before proceeding to the main statement of this section, we recall that we call a function $f : E \to \mathbb{R}_+$ lower semicontinuous, if

$$\liminf_{x\to y} f(x) \geq f(y).$$

The following theorem and its proof is based on the results of Chapter 7 in [9].

Theorem 3. *Let $(X(t))_{t\geq 0}$ be the PDMP process of the simplified model described in the previous section. Furthermore, assume that:*

- $$\sup_{x\in[0,\infty)} \int_0^\infty b(y)(V(x+y)-V(x))dy < \infty,$$
  where $V(x) = e^{\int_1^x \frac{c_1}{\rho(z)}dz}$ for some $c_1 > 0$ and $V \in \mathcal{D}(\mathcal{A})$,
- *$\rho(\cdot)$ is a C^1 function,*
- *$b(\cdot)$ is lower semicontinuous and $b(x) > 0$ for $x > 0$,*

- $\frac{\partial}{\partial x}\phi(t,x) \neq 0$ *for all* $x > 0$ *and* $t < 0$,
- *function* $t \mapsto \phi(t,x)$ *is decreasing and* $\phi(t,x) \xrightarrow[t\to\infty]{} 0$ *for all* $x \geq 0$.

Then the Markov semigroup $\{P(t)\}_{t\geq 0}$ *induced by equation* (10) *is asymptotically stable, i.e. there exists a unique stationary density satisfying equation* (10).

Proof. The idea of the proof is as follows. First we prove that there exists a nonnegative Borel function h defined on $(0,\infty)\times[0,\infty)\times[0,\infty)$ such that it satisfies the following conditions

1. for every Borel set B, $t > 0$ and $x \geq 0$
$$\mathbb{P}_x(X(t) \in B) \geq \int_B h(t,w,x)dw,$$
2. the function $(w,x) \mapsto h(t,w,x)$ is lower semicontinuous for every $t > 0$,
3. for every $x \in [0,\infty)$ there exists $t > 0$ such that
$$\int_0^\infty h(t,w,x)dw > 0,$$
4. for any Borel set B with a positive Lebesgue measure
$$\int_0^\infty \int_B h(t,w,x)dwdt > 0 \quad \text{a.e.}$$

As it has been pointed out in [9], the existence of function h satisfying all the assumptions above allow us to use theorems stated in [12], which imply that either there exists a unique stationary density satisfying equation (10) or the process X is sweeping from compact subsets of $[0,\infty)$, i.e. for every density u and compact set $F \subset [0,\infty)$
$$\lim_{t\to\infty}\int_0^\infty \mathbb{P}_x(X(t) \in F)u(x)dx = 0.$$
Therefore our goal in this proof is to find a function h satisfying conditions 1-4 and then show that X is not sweeping.

First, let us note that since
$$\mathbb{P}_x(X(t) \in B) = \sum_{n=0}^\infty \mathbb{P}_x(X(t) \in B, T_n \leq t < T_{n+1}),$$
it is sufficient to find a function h such that
$$\mathbb{P}_x(X(t) \in B,\ T_1 \leq t < T_2) \geq \int_B h(t,w,x)dw$$

to satisfy condition 1. Thus we would like to find a lower bound of

$$\mathbb{P}_x(X(t) \in B,\ T_1 \le t < T_2) = \mathbb{P}_x(\phi(t - T_1, X(T_1)) \in B,\ 0 \le t - T_1 < T_2 - T_1).$$

Recall that

$$\mathbb{P}(T_2 - T_1 > t | X(T_1) = z) = e^{-\int_0^t \lambda(\phi(r,z))dr}.$$

Then we have

$$\mathbb{P}_x(X(T_1) \in B) = \int_0^\infty Q(\phi(t_1, x), B)\lambda(\phi(t_1, x))e^{-\int_0^{t_1} \lambda(\phi(r,x))dr}dt_1$$

and

$$\begin{aligned}&\mathbb{P}_x(\phi(t - T_1, X(T_1)) \in B,\ 0 \le t - T_1 < T_2 - T_1)\\ &= \int_0^t \int_0^\infty \mathbb{1}_B(\phi(t - t_1, z))e^{-\int_0^{t-t_1} \lambda(\phi(r,z))dr}\lambda(\phi(t_1, x))e^{-\int_0^{t_1} \lambda(\phi(r,x))dr}Q(\phi(t_1, x), dz)dt_1.\end{aligned}$$

Since for any $w \ge 0$

$$\lambda(w)Q(w, B) \ge q \int_B b(y - w)dy$$

and $b(y - w) = 0$ for $y \le w$, we have

$$\lambda(\phi(t_1, x))Q(\phi(t_1, x), dz) \ge qb(z - \phi(t_1, x))\mathbb{1}_{(0,\infty)}(z - \phi(t_1, x)).$$

Therefore

$$\begin{aligned}&\mathbb{P}_x(\phi(t - T_1, X(T_1)) \in B,\ 0 \le t - T_1 < T_2 - T_1)\\ &\qquad \ge \int_0^t \int_0^\infty \mathbb{1}_B(\phi(t - t_1, z))e^{-\int_0^{t-t_1} \lambda(\phi(r,z))dr}e^{-\int_0^{t_1} \lambda(\phi(r,x))dr}\\ &\qquad\qquad \cdot qb(z - \phi(t_1, x))\mathbb{1}_{(0,\infty)}(z - \phi(t_1, x))dzdt_1.\end{aligned}$$

Integrating by substitution with $w = \phi(t - t_1, z)$, which implies that $z = \phi(t_1 - t, w)$ and $dz = \frac{\partial}{\partial w}\phi(t_1 - t, w)dw$, we conclude that defining function h by the following formula

$$\begin{aligned}h(t, w, x) = q \int_0^t &e^{-\int_0^{t-t_1} \lambda(\phi(r,\phi(t_1-t,w)))dr}e^{-\int_0^{t_1} \lambda(\phi(r,x))dr}b(\phi(t_1 - t, w) - \phi(t_1, x))\\ &\cdot \mathbb{1}_{(0,\infty)}(\phi(t_1 - t, w) - \phi(t_1, x))\frac{\partial}{\partial w}(\phi(t_1 - t, w))dt_1\end{aligned}$$

gives condition 1, i.e.

$$\mathbb{P}_x(X(t) \in B) \ge \int_B h(t, w, x)dw.$$

Note that ϕ is a continuous function, since ρ is a C^1 function. By assumption, b is lower semicontinuous and thus the function $(w,z) \mapsto h(t,w,z)$ is lower semicontinuous for every $t > 0$, proving that function h satisfies condition 2.

Condition 3 is also satisfied by function h, due to the assumption $\frac{\partial}{\partial w}\phi(t,w) \neq 0$ for all $t < 0$.

To show the last condition, we note that it is sufficient to prove that for almost all $x, w \in [0,\infty)$

$$\int_0^\infty h(t,w,x)dt > 0,$$

on account of Fubini's theorem.

Function $t \mapsto \phi(t,x)$ is assumed to be decreasing and $\phi(t,x) \xrightarrow[t\to\infty]{} 0$, which implies that $t \mapsto \phi(t_1 - t, w)$ is increasing and $\phi(t_1 - t, x) \xrightarrow[t\to\infty]{} \infty$. Thus, for t large enough we have $\phi(t_1 - t, w) > \phi(t,x)$. Therefore, due to the fact that $b(x) > 0$ for $x > 0$, it is easy to note that

$$\begin{aligned}\int_0^\infty h(t,w,x)dt = \int_0^\infty \int_0^\infty & \mathbb{1}_{[0,t]}(t_1) q e^{-\int_0^{t-t_1} \lambda(\phi(r,\phi(t_1-t,w)))dr} \\ & \cdot e^{-\int_0^{t_1} \lambda(\phi(r,x))dr} b(\phi(t_1 - t, w) - \phi(t_1, x)) \\ & \cdot \mathbb{1}_{(0,\infty)}(\phi(t_1 - t, w) - \phi(t_1,x)) \frac{\partial}{\partial w}(\phi(t_1 - t, w)) dt_1 dt > 0,\end{aligned}$$

which proves that function h satisfy condition 4.

We have proved that there exists a function h satisfying conditions 1-4. Therefore, as it has already been pointed out, either there exists a unique stationary density satisfying equation (10) or the process X is sweeping from compact subsets of $[0,\infty)$.

Thus to prove our statement we have to show that X is not sweeping. In order to do that assume the contrary, i.e. assume that X is sweeping from compact subsets of $[0,\infty)$. Then for any density u and compact set $F \subset [0,\infty)$ we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \int_0^\infty \mathbb{P}_x(X(t) \in F) u(x) dx dt = 0.$$

From Chebyshev's inequality it follows that for every $a, t > 0$ and $x \in [0,\infty)$

$$\mathbb{P}_x(X(t) \in F_a) \geq 1 - \frac{1}{a}\mathbb{E}_x(V(X(t))),$$

where $F_a = \{x \in [0,\infty) : V(x) \leq a\}$ and V is a nonnegative measurable function. Thus, to obtain a contradiction it is sufficient to prove that for some density u and a continuous function V such that F_a is a compact subset of $[0,\infty)$

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \int_0^\infty \mathbb{E}_x(V(X(s))) u(x) dx ds < \infty.$$

Define $V(x) = e^{\int_1^x \frac{c_1}{\rho(z)}dz}$ for $x \in [0, \infty)$ and some $c_1 > 0$. Recall that the extended generator $\mathcal{A}$ of our simplified model is given by the formula (12). Thus

$$\mathcal{A}V(x) = -c_1 V(x) + \nu(x) \int_0^x p(x, x-z)V(z)dz + q \int_x^\infty b(z-x)V(z)dz - (\nu(x)+q)V(x),$$

where we have used the fact that $\frac{\partial}{\partial x}V(x) = \frac{c_1}{\rho(x)}V(x)$. Note that since function V is increasing, we have

$$\nu(x) \int_0^x p(x, x-z)V(z)dz \leq \nu(x)V(x) \int_0^x p(x, x-z)dz \leq \nu(x)V(x).$$

Therefore

$$\mathcal{A}V(x) \leq -c_1 V(x) + q \int_x^\infty b(z-x)V(z)dz - qV(x).$$

Taking $c_2 = \sup_{x\in[0,\infty)} q \int_0^\infty b(y)(V(x+y) - V(x))dy < \infty$ we obtain

$$\mathcal{A}V(x) \leq -c_1 V(x) + c_2.$$

Dynkin's formula implies that

$$\mathbb{E}_x(V(X(t))) \leq V(x) - c_1 \int_0^t \mathbb{E}_x(V(X(s))ds + c_2 t,$$

thus for any $z \in [0, \infty)$ and $t > 0$ we have

$$\frac{1}{t} \int_0^t \mathbb{E}_x(V(X(s)))ds \leq \frac{c_2}{c_1} + \frac{1}{c_1 t}V(x).$$

Taking density u such that $\int_0^\infty V(z)u(z)dz < \infty$ leads to a contradiction, thus completing the proof. □

4.3 Examples

In previous section we have presented sufficient conditions for existence and uniqueness of the stationary distribution. In this part we present two examples, in which those conditions are satisfied, thus we are able to consider the evolution of the stationary density.

4.3.1 Uniformly distributed avalanche sizes

First let us begin with an example of a simplified avalanche model, where we assume that the deterministic drift is proportional to the snow depth, i.e. $\rho(x) = \rho_1 x$ for some $\rho_1 > 0$ and all $x \in [0, \infty)$. We also assume that $p(x, z) = \frac{1}{x}\mathbb{1}_{(0,x)}(z)$ for $z \geq 0$ and $b(z) = \alpha e^{-\alpha z}$ for $z > 0$ and some $\alpha > 0$, which means that the avalanche and snowfall sizes are uniformly and exponentially distributed, respectively. Moreover for all $x \in [0, \infty)$ we take $\nu(x) = \nu_1 x$, $\nu_1 > 0$.

In the previous section we have given conditions for the existence and uniqueness of the stationary density. Note that in the case of this example, all the conditions are satisfied. Indeed taking $\rho(x) = \rho_1 x$ implies that $V(x) = x$, $\phi(t, x) = xe^{-\rho_1 t}$ for all $x, t \geq 0$. Obviously these functions and $b(\cdot)$ satisfy all the required conditions, proving the existence and uniqueness of the stationary density, which we will denote by $g(\cdot)$. Since the stationary density does not depend on time, using (10) we obtain the following

$$\frac{d}{dx}(\rho(x)g(x)) - \lambda(x)g(x) + \mathcal{P}(\lambda g)(x) = 0, \tag{14}$$

where $\mathcal{P}$ is the transition operator corresponding to Q, i.e. for $f \in \mathcal{D}(\mathcal{A})$

$$\int_0^\infty f(x)\mathcal{P}g(x)dx = \int_0^\infty Qf(x)g(x)dx. \tag{15}$$

For more details on the transition operator and equations of form (10) we refer the reader to [15].

Not only in general, but also in this specific example it is really hard to obtain any analytical results of equation (14) in terms of the stationary density g. However sometimes it is possible to obtain some closed-form results in terms of the Laplace transform, which we have already introduced in Section 2.3. Before we apply this method to equation (14) we define

$$\tilde{g}(s) = \mathcal{L}(g(x)) = \int_0^\infty e^{-sx}g(x)dx.$$

For compactness, we will calculate the Laplace transform of each term of equation (14) separately.
Using the Laplace transform properties (described in Section 2.3) we calculate

$$\mathcal{L}\left(\frac{d}{dx}(\rho(x)g(x))\right) = s\mathcal{L}(\rho(x)g(x)) - \underbrace{\rho(0)g(0)}_{=0} = \rho_1 s\mathcal{L}(xg(x)) = -\rho_1 s\frac{d}{ds}\tilde{g}(s).$$

Similarly

$$\mathcal{L}(\lambda(x)g(x)) = \nu_1\mathcal{L}(xg(x)) + q\mathcal{L}(g(x)) = -\nu_1\frac{d}{ds}\tilde{g}(s) + q\tilde{g}(s).$$

Before we proceed to calculating the Laplace transform of the last term we recall that

$$Qf(x) = \frac{\nu(x)}{\lambda(x)}\int_0^x p(x, x-y)f(y)dy + \frac{q}{\lambda(x)}\int_x^\infty b(y-x)f(y)dy.$$

Since
$$\int_0^x p(x,x-y)e^{-sy}dy = \frac{1}{x}\int_0^x e^{-ys}dy = \frac{1}{xs}\left(1-e^{-sx}\right),$$
and
$$\int_x^\infty b(y-x)e^{-sy}dy = e^{-sx}\int_x^\infty b(y-x)e^{-s(y-x)}dy = e^{-sx}\mathcal{L}(b(x)),$$
we conclude that
$$Qe^{-sx} = \frac{\nu_1}{s\lambda(x)}\left(1-e^{-sx}\right) + \frac{q}{\lambda(x)}e^{-sx}\frac{\alpha}{\alpha+s},$$
where we have used the fact that $\mathcal{L}(\alpha e^{-\alpha x}) = \frac{\alpha}{\alpha+s}$.

Therefore due to equation (15) we have
$$\begin{aligned}\mathcal{L}(\mathcal{P}(\lambda g)(x)) &= \int_0^\infty Qe^{-sx}\lambda(x)g(x)dx \\ &= \int_0^\infty \left(\frac{\nu_1}{s}\left(1-e^{-sx}\right) + qe^{-sx}\frac{\alpha}{\alpha+s}\right)g(x)dx = \frac{\nu_1}{s}\left(1-\tilde{g}(s)\right) + q\frac{\alpha}{\alpha+s}\tilde{g}(s).\end{aligned}$$

Finally we can conclude that taking the Laplace transform of equation (14) results in
$$\left(\nu_1-\rho_1 s\right)\frac{d}{ds}\tilde{g}(s) = \left(\frac{qs}{\alpha+s}+\frac{\nu_1}{s}\right)\tilde{g}(s) - \frac{\nu_1}{s},$$
which can be presented in the following form of a linear ordinary differential equation
$$\frac{d}{ds}\tilde{g}(s) = \frac{qs^2+\nu_1\alpha+\nu_1 s}{(\alpha s+s^2)(\nu_1-\rho_1 s)}\tilde{g}(s) - \frac{\nu_1}{s(\nu_1-\rho_1 s)}.$$

The general solution to the equation above is then given for some $C\in\mathbb{R}$ as
$$\tilde{g}(s) = Ce^{-\int h_1(s)ds} + e^{-\int h_1(s)ds}\left(\int e^{\int h_1(s)ds}h_2(s)ds\right),$$
where
$$h_1(s) = -\frac{qs^2+\nu_1\alpha+\nu_1 s}{(\alpha s+s^2)(\nu_1-\rho_1 s)}, \qquad h_2(s) = -\frac{\nu_1}{s(\nu_1-\rho_1 s)}.$$

Unfortunately, we have been unable to derive a more explicit formula for function $\tilde{g}(\cdot)$. Note that similar calculations hold true for any lower semicontinuous and positive $b(\cdot)$.

Remark 3. *Using the Laplace transform properties, it is also possible to apply the approach presented in this section to some different functions $\rho(\cdot)$, $\nu(\cdot)$, keeping in mind that $\rho(0)=\nu(0)=0$. However this might result in differential equations of higher orders or with shifted frequencies, which are in principle harder to analyze. For example taking $\rho(x)=\rho_1 x^n$ for some $\rho_1>0$ and $n\in\mathbb{N}$ would result in $\mathcal{L}(\frac{d}{dx}(\rho(x)g(x))) = s\rho_1(-1)^n\frac{d^n}{ds^n}\tilde{g}(s)$, while with $\rho(x)=e^{ax}-1$, for some $a\geq 0$, one would obtain $\mathcal{L}(\frac{d}{dx}(\rho(x)g(x))) = \tilde{g}(s-a)-\tilde{g}(s)$.*

4.3.2 Beta distributed avalanche sizes

We consider this example as a modification of the previous one, namely we only change the distribution of an avalanche size. To be more precise, this time we assume that for every $z \in [0, \infty)$ and $x < z$

$$p(z, z-x) = \frac{\beta}{z}\left(\frac{x}{z}\right)^{\beta-1},$$

where $\beta > 0$. This is equivalent to the assumption that the avalanche magnitude $C \in (0, 1)$, which corresponds to an avalanche of size CX, is beta distributed with parameters 1 and $\beta > 0$, i.e. its density is given as

$$f_C(r) = \beta(1-r)^{\beta-1}.$$

The same arguments as in previous example imply that there exists a unique stationary density $g(\cdot)$, which is the solution of

$$\text{(16)} \quad \frac{d}{dx}(\rho(x)g(x)) - \lambda(x)g(x) + \int_0^x qb(x-y)g(y)dy + \int_x^\infty \nu(z)p(z, z-x)g(z)dz = 0.$$

Proposition 5. *If $q = \beta(\rho_1 + \frac{\nu_1}{\alpha})$, then $g(\cdot)$ is the probability density function of the gamma distribution with shape α and rate β, i.e. function defined for $x \in [0, \infty)$ as*

$$g(x) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha x} x^{\beta-1},$$

where $\Gamma(\cdot)$ is the gamma function, is the unique stationary density satysfying (16).

Proof. To prove this proposition we simply note that if for every $x \in [0, \infty)$

$$g(x) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha x} x^{\beta-1},$$

then

$$\int_x^\infty \nu(z)p(z, z-x)g(z)dz = \frac{\nu_1 \beta}{\alpha} g(x),$$

$$\int_0^x qb(x-y)g(y)dy = \frac{q\alpha x}{\beta} g(x)$$

and

$$\frac{d}{dx} g(x) = -\alpha g(x) + \frac{\beta - 1}{x} g(x).$$

Therefore, using basic algebra, equation (16) is equivalent to

$$g(x)\left[\rho_1 - \alpha\rho_1 x + (\beta-1)\rho_1 - q - \nu_1 x + q\frac{\alpha x}{\beta} + \frac{\nu_1\beta}{\alpha}\right] = 0,$$

which is satisfied for all $x \in [0, \infty)$ in case when $q = \beta(\rho_1 + \frac{\nu_1}{\alpha})$, since then all terms within the brackets cancel out. □

4.4 Special case: $M/G/1$ queue with negative customers

In this section we assume that the rate function of an avalanche occurrence is equal to some constant $\nu > 0$, i.e. $\nu(x) = \nu$ for $x \in [0, \infty)$. Furthermore we assume that the deterministic drift is causing a decrease in the snow depth at unit rate, i.e. $\rho(x) = 1$ for $x \in [0, \infty)$.

Under these assumptions, the simplified model presented in this chapter can be also interpreted as an $M/G/1$ queue with negative customers as described in [2], where upon arrival to a queue a negative customer removes a random amount of work from the queue workload (which in our setting is represented by the variable $X(\cdot)$). We would like to note that the authors of [2] assume that the probability of a negative jump depends only on the jump size, which might result in the jump size being bigger than the system's level. Thus it is also necessary to assume that such jumps simply reset the system's level to zero.

Stating this in the framework of our simplified model, we consider an $M/G/1$ queue (i.e. a queueing model with Poisson arrivals, general service times and one server), where positive customers arrive to the queue according to a Poisson process with rate q, while the negative customers arrive to the queue according to a Poisson process with rate ν. The service requirement of a positive customer has an absolutely continuous distribution $B(\cdot)$, with $B(0^+) = 0$ and finite mean β. The reduction of the amount of work in the system by a negative customer has an absolutely continuous distribution $P(\cdot)$, with $P(0^+) = 0$ and finite mean γ. With agreement to the paper [2], we assume the necessary and sufficient ergodicity condition, namely

$$q\beta < 1 + \nu\gamma.$$

Using a level crossing argument, the authors of [2] derive the following equation for the stationary density function of the workload (which exists for $x > 0$ and is denoted by $w(x)$)

$$w(x) = \int_0^x q(1 - B(x - y))dW(y) - \int_x^\infty \nu(1 - P(z - x))dW(z), \quad x > 0 \tag{17}$$

where $W(x) = \int_0^x w(y)dy$ is the stationary cumulative distribution function of the workload.

Note that taking $P(z, z - x) = P(z - x)$, $\nu(x) = \nu$ and $\rho(x) = 1$ in equation (11) and considering the stationary distribution, i.e. $\frac{\partial}{\partial t}W(t, x) = 0$ and $W(t, x) \equiv W(x)$ for all $x \in [0, \infty)$ yields (17).

One of the results of the paper by Boucherie and Boxma [2] is an explicit formula for the Laplace-Stieltjes transform of the workload in case when both $B(\cdot)$ and $P(\cdot)$

are exponentially distributed. In that case,

$$\mathcal{L}(w(x)) = (1+\beta s)\frac{\eta}{\eta+s}, \quad \operatorname{Re} s \geq 0 \tag{18}$$

where

$$\eta = \frac{\beta - \gamma + q\gamma\beta + \nu\gamma\beta - \sqrt{(\gamma - \beta - g\gamma\beta - \nu\gamma\beta)^2 + 4\gamma\beta(1+\nu\gamma - q\beta)}}{2\gamma\beta}. \tag{19}$$

To obtain this result, the authors of [2] had to use facts and arguments from the field of complex analysis, which seems to be a more technical approach than the one presented by us in the previous section. However our method would not lead straightforwardly to the result with exponentially distributed negative jumps.

Nevertheless allowing for the second interpretation of our simplified model as an $M/G/1$ queue with negative customers, we note that in the previous section we have derived an equation for the Laplace transform of the stationary distribution in the case when the server work rate and negative customers arrival rate are state dependent, which to the best of our knowledge have not been studied in the literature before.

5 Comparison with existing models

In this chapter, we compare our general snow avalanche model with already existing ones. Using the piecewise deterministic Markov processes theory in the same way as in Chapters 3 and 4, we analyze the models presented in [10] and [7].

5.1 Modification of the simplified model

First, we consider the model presented by Perona et al. [10]. Following the assumptions of this model, one can already notice, that there are significant differences compared to the assumptions made in our general model.

The main difference between the two considered models is that authors of [10] describe the whole system using only one variable (snow depth), which apart from the deterministic drift, can decrease or increase its value only when one of the two jumps occur (avalanche or a snowfall). To point out the differences precisely - the authors assume, that every avalanche resets the state variable to zero (which is the only way for the process to hit this point) and all the fresh snow is added at once, in a somewhat discrete manner, when the snowfall stops. In our model, considered in Chapter 3, both of these assumptions are stated in a more general way.
We do not assume that the avalanche resets the snow depth x to zero, but decreases the snow depth on a slope to some point, which is chosen from a probability distribution $p(x, \cdot)$. Obviously, taking $\delta_0(\cdot)$ instead of $p(x, \cdot)$ would give us the same assumption in both models considered in this section.
The second assumption made by Perona et al. is also generalized in our model - we simply define two variables, describing the snowfall occurrence and the depth of new snow, which is added in a continuous manner to our system.

We note that the model described by Perona et al. is much closer in its assumptions to the simplified model from Chapter 4, where we also assume that all the fresh snow is added at once (we replace the snowfall periods with jumps). Nevertheless there are still some differences which will be visible in the following description of the process presented in [10], using the PMDP theory.

Since the only way for the process to hit zero is when an avalanche occurs, we fix set $C = \{0, 1\}$, where in this case the indices connected with C states whether the jump point is caused by an avalanche or a snowfall.

Then we consider a Borel space $(E, \mathcal{E})$, where

$$E = \{(0,0)\} \cup (0,\infty) \times \{1\}.$$

Unlike the previous case, in this model the active boundary set Γ is not empty, i.e.

$$\Gamma = \{(0,1)\}.$$

There is no distinction between the snowfall and no-snowfall "periods" and all the new snow is added at once when a jump occurs, thus the differential equation describing our process is here given for $t \geq 0$ as

$$X'(t) = -\rho(X(t)),$$

where the only variable X stands for the snow depth and ρ is a positive deterministic drift given by the formula

$$\rho(x) = \rho_0 + \rho_1 x, \qquad \rho_0, \rho_1 \geq 0.$$

We denote by $\phi(t,\xi)$ the unique and global solution in $\mathbb{R}$ of the above differential equation. We note that according to Definition 15 we should use a separate notation for vector fields $\mathscr{X}_0$, $\mathscr{X}_1$ and its solutions but since one of them is a zero vector field, we will skip the indices for brevity.
In this process there are two possible types of jump points:

I An avalanche occurs, which causes a jump from point x to 0 with probability 1.

II There was a snowfall, which causes a jump from point x to some $z > x$ with probability density $b(z-x)$, where $b(y) = \gamma e^{-\gamma y}$ for $y > 0$.

We define

$$\lambda(x) = \begin{cases} \nu(x) + q & \text{for } x > 0, \\ q & \text{for } x = 0, \end{cases}$$

where q is the snowfall intensity and $\nu(x) = \nu_0 + \nu_1 x$, $\nu_0, \nu_1 \geq 0$ is the rate function of avalanche occurrence.

The jump moments T_i, $i = 0, 1, \ldots$, of occurrence of an avalanche or a snowfall are such that

$$\mathbb{P}(T_{n+1} - T_n > t) = \begin{cases} \exp\left(-\int_0^t \lambda(\phi(s, X(T_n)))ds\right), & t < t^*(X(T_n)), \\ 0, & t \geq t^*(X(T_n)), \end{cases}$$

for all $n \in \mathbb{N}$.

Similarly as in the case of the general model, we note that $(1-\frac{q}{\lambda(x)})$ is the probability that the jump point is of type I, while $\frac{q}{\lambda(x)}$ is the probability of jump point being of type II. Therefore we can describe all the possible jumps in this model, given that a jump occurs, in the following diagrams, for $x>0$ and $z>x$

$$
\begin{array}{ll}
 & (z,1) \text{ with probability } \frac{q}{\lambda(x)}b(z-x) \\
 & \nearrow \\
(x,1) & \\
 & \searrow \\
 & (0,0) \text{ with probability } \left(1-\frac{q}{\lambda(x)}\right)
\end{array}
$$

$$(0,0),(0,1) \rightarrow (z,1) \text{ with probability } b(z).$$

Thus the transition function for this process is of following form

$$Q((x,i),B\times\{j\}) = \frac{q}{\lambda(x)}P(x,B)\delta_1(j) + \left(1-\frac{q}{\lambda(x)}\right)\delta_0(B)\delta_{1-i}(j),$$

where

$$P(x,B) = \int_B b(z-x)dz.$$

Before introducing the extended generator $\mathcal{A}$, we recall that in this process we have $\Gamma = \{(0,1)\}$. For that reason we note that function $f \in \mathcal{D}(\mathcal{A})$ satisfies the boundary condition, which in this case is given by the following equation

$$f(0,1) = Qf(0,1),$$

where

$$Qf(0,1) = \int_0^\infty f(z,i)Q((0,1),dz,di).$$

Using the formula for the transition function we calculate

$$f(0,1) = \int_0^\infty f(z,i)Q((0,1),dz,di) = \int_0^\infty \frac{q}{\lambda(0)}f(z,1)P(0,dz) = \int_0^\infty f(z,1)b(z)dz.$$

The extended generator $\mathcal{A}$ is given for $f \in \mathcal{D}(\mathcal{A})$ and $x>0$ as

$$\mathcal{A}f(x,1) = -\rho(x)\frac{\partial}{\partial x}f(x,1) + \lambda(x)[Qf(x,1) - f(x,1)],$$

where

$$Qf(x,1) = \int_x^\infty f(z,1)\frac{q}{\lambda(x)}b(z-x)dz + \left(1-\frac{q}{\lambda(x)}\right)f(0,0).$$

Considering point $(0,0)$ separately we also have

$$\mathcal{A}f(0,0) = q[Qf(0,0) - f(0,0)],$$

where we note that $Qf(0,0) = Qf(0,1)$.

We will show that using this model description we are able to obtain similar equations as the authors of [10]. More precisely, we will derive the equations for probability density function of X, assuming its existence.
In order to do that, as described in Section 4.1.2, we have to determine the formal adjoint operator $\mathcal{G}$.

Before we begin the calculations, we assume that functions f and g are both smooth, bounded, integrable and at least one of them is equal to zero in infinity. Then we have

$$\int_{(0,\infty)} \mathcal{A}f(x,1)g(x,1)dx + \mathcal{A}f(0,0)g(0,0) = \int_{(0,\infty)} -\rho(x)f(x,1)\frac{\partial}{\partial x}g(x,1)dx \\ + \int_{(0,\infty)} \lambda(x)[Qf(x,1) - f(x,1)]g(x,1)dx + q[Qf(0,0) - f(0,0)]g(0,0).$$

Integrating by parts the first term of the right-hand side of the above equation, we obtain

$$\int_{(0,\infty)} \mathcal{A}f(x,1)g(x,1)dx + \mathcal{A}f(0,0)g(0,0) = -\rho(x)f(x,1)g(x,1)\Big|_0^\infty \\ + \int_{(0,\infty)} f(x,1)\frac{\partial}{\partial x}\left(\rho(x)g(x,1)\right)dx \\ + \int_{(0,\infty)} \lambda(x)\left(\int_x^\infty f(z,1)\frac{q}{\lambda(x)}b(z-x)dz + \left(1 - \frac{q}{\lambda(x)}\right)f(0,0)\right)g(x,1)dx \\ - \int_{(0,\infty)} \lambda(x)f(x,1)g(x,1)dx + q\left(\int_0^\infty f(z,1)b(z)dz\right)g(0,0) - qf(0,0)g(0,0).$$

Recalling that at least one of the two functions f, g is equal to zero in infinity and using Fubini's theorem, we have

$$\int_{(0,\infty)} \mathcal{A}f(x,1)g(x,1)dx + \mathcal{A}f(0,0)g(0,0) = f(0,1)\rho(0)g(0,1) \\ + \int_{(0,\infty)} f(x,1)\left[\frac{\partial}{\partial x}\left(\rho(x)g(x,1)\right) - \lambda(x)g(x,1)\right]dx \\ + \int_{(0,\infty)} f(x,1)\left(\int_0^x qb(x-z)g(z,1)dz + qb(x)g(0,0)\right)dx \\ + f(0,0)\left(\int_{(0,\infty)} \nu(x)g(x,1)dx - qg(0,0)\right),$$

where

$$g(0,1) = \lim_{x\downarrow 0} g(x,1).$$

From the equation above and the boundary condition, one could easily note, that the formal adjoint operator $\mathcal{G}$ of the operator $\mathcal{A}$ is given by the following formulas, which we present separately for $x > 0$ and $x = 0$

$$\mathcal{G}g(x,1) = \frac{\partial}{\partial x}\left(\rho(x)g(x,1)\right) - \lambda(x)g(x,1) + q\int_0^x b(x-z)g(z,1)dz + qb(x)g(0,0) + \rho(0)b(x)g(0,1),$$

$$\mathcal{G}g(0,0) = -qg(0,0) + \int_0^\infty (\nu_0 + \nu_1 x)g(x,1)dx,$$

where the function $x \mapsto g(x,1)$ is absolutely continuous, integrable on $(0,\infty)$ and

$$g(0,1) = \lim_{x\downarrow 0} g(x,1).$$

Applying this to the forward Kolmogorov equation gives us almost the same two equations, as were presented by Perona et al. The only difference is that the limit factor (the one containing $g(x,1)$), which because of the boundary condition is of slightly different form and has to be in the first equation. We note that if one would not consider this additional, binary variable and assume that the boundary condition holds at zero, then the results would be exactly the same.

5.2 Growth-collapse model

We end this chapter by briefly describing the model presented in [7]. The authors of the mentioned paper focused on a growth-collapse process, which describes the cycles of periods of steady growth followed by a sudden avalanche. They assume that this process, denoted by $(X(t))_{t\geq 0}$, is composed of three stochastically independent random 'inputs':

(i) a steady, random inflow (snowfall) $(G(t))_{t\geq 0}$, which is assumed to be a random process with non-negative, stationary and independent increments;

(ii) crash (avalanche) epochs $(\tau_n)_{n\in\mathbb{N}}$, that form a renewal process with inter-crash times forming an i.i.d. sequence of random variables distributed as τ;

(iii) sequence $(C_n)_{n\in\mathbb{N}}$ of avalanches magnitudes, which is an i.i.d. sequence of random variables having values in $(0,1)$.

At the nth avalanche time, i.e. at τ_n, it is assumed that the system level jumps down by the amount $C_n X(\tau_n-)$, thus $X(\tau_n) = (1-C_n)X(\tau_n-)$.

Eliazar and Klafter strongly emphasize in their paper [7] that the renewal structure of the crash epochs renders the growth-colapse model non-Markov. We would like to note that this could actually be fixed, i.e. it is possible to build a Markov process out of a renewal process by creating a new, two component process (for more details see [4], p.40).

Before we describe this system as a piecewise deterministic Markov process, we make some additional assumptions, which will specify the above 'inputs'. We assume that the inflow $(G(t))_{t\geq 0}$ is causing a linear growth of the process at rate 1. Furthermore, we assume that τ is exponentially distributed with parameter λ, i.e. the crash epochs form a Poisson process with intensity λ. Finally, we assume that the avalanche magnitudes $(C_n)_{n\in\mathbb{N}}$ are i.i.d. continuous random variables with density $p(\cdot)$.

To put this model into piecewise deterministic Markov process setting, we note that in this case, the set C from Definition 15 consists of only one element, thus we will simplify the notation by writing $E = [0, \infty)$. Since we have assumed that the process increases linearly at rate 1, we have the following differential equation describing the behavior in between the jumps for $t \geq 0$

$$X'(t) = 1.$$

Assuming the initial condition $X(0) = x_0$ we note that the unique and global solution of this equation is given for $t \in \mathbb{R}$ as

$$\phi(t, x_0) = t + x_0.$$

Thus

$$t^*(X(t)) = \infty$$

for any $t \geq 0$. Unlike the previous cases, in this process there is only one possible type of jump points:

I An avalanche occurs, which causes a jump from point x to $(1-c)x$, for some $c \in (0, 1)$ with probability $p(c)$.

We define $\lambda(x) = \lambda$ for all $x \in E$. Then the jump moments T_i, $i = 0, 1, \ldots$, of occurrence of an avalanche are such that

$$\mathbb{P}(T_{n+1} - T_n > t) = e^{-\lambda t}$$

for all $n \in \mathbb{N}$ and $t \geq 0$.

The transition function of this process is then of the following form

$$Q(x,B) = \int_B p(x,z)dz,$$

where

$$p(x,z) = \frac{1}{x} p\left(1 - \frac{z}{x}\right) \mathbb{1}_{(0,x)}(z).$$

We also note that the extended generator $\mathcal{A}$ is given for $f \in \mathcal{D}(\mathcal{A})$ and $x \geq 0$ as

$$\mathcal{A}f(x) = f'(x) + \lambda \int_0^\infty (f(y) - f(x))Q(x,dy).$$

We do not go into further details of the model presented by Eliazar and Klafter in [7]. Instead, we note that one could let $G(t)$ be a compound Poisson process, as introduced in Definition 7, where $(D_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. positive random variables. Then the model presented in this section would become a special case of the simplified model, presented in Chapter 4, with $\rho(x) = 0$ and $\nu(x) = \nu$ for some $\nu > 0$ and all $x \in [0, \infty)$.

To finish this chapter we note that similar work has been done by Boxma et al. in [3], who presented a growth-collapse model in the PDMP setting, giving a few examples with differently specified probability densities.

6 Conclusion

The models presented in this paper allow us to study the evolution of a snowpack with avalanches occurrence. Due to the random behavior of this natural phenomenon, a stochastic approach to its modeling has been presented, using the theory of piecewise deterministic Markov processes.

A general snow avalanche model has been formulated as a PDMP process, which turned out to be too difficult to analyze. In such cases, a typical method in mathematical modeling, is to simplify the general model, in order to make it more tractable. Successfully, a simplified model has been developed and analyzed within the framework of piecewise deterministic Markov processes. In Chapter 4 we have presented not only how to describe the model mathematically, but also how to prove the existence and uniqueness of a stationary distribution, obtaining its form in two specific examples. First of these examples shows also how to incorporate the Laplace transform in the method of finding the solutions of the integro-differential equations. However, the same example proved that obtaining a closed-form solutions using this approach might be difficult. Another interpretation of our simplified model has also been presented, connecting it to an $M/G/1$ queue with negative customers. This simple observation allows us to state our results on snow avalanches in terms of a queue.

Finally we have compared our general snow avalanche model to the ones presented in already published papers. To obtain a more clear comparison, we have also formulated these models within the framework of PDMP.

In conclusion, using the piecewise deterministic Markov processes theory, one might derive a general mathematical description of a snow avalanche model. However, obtaining analytical results from that type of models is usually a big challenge that requires further studies.

7 Bibliography

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