MSc Mathematics

Track: Analysis and Dynamical Systems

Master thesis

Hamiltonian Partial Differential Equations and Symplectic Scale Manifolds

Towards Symplectic Geometry in Infinite Dimensions

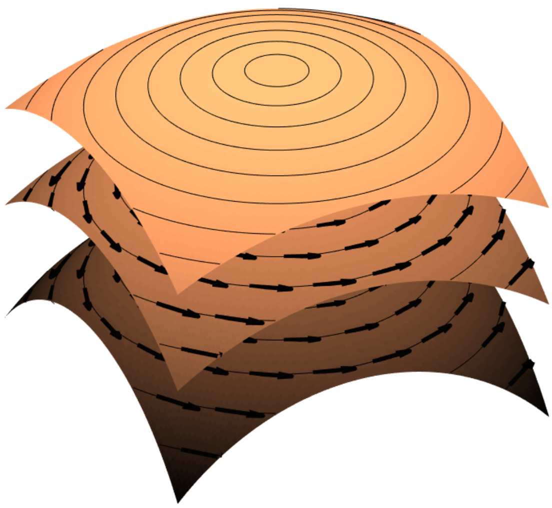
by

João Bernardo Crespo

November 29, 2017

Supervisor: dr. Oliver Fabert

Second examiner: dr. Federica Pasquotto



Department of Mathematics
Faculty of Sciences
VU University Amsterdam

Abstract

This thesis defines symplectic scale manifolds based on Hofer-Wysocki-Zehnder's scale calculus. We introduce Hamiltonian vector fields and flows on these by narrowing down sc-smoothness to what we denote by *strong* sc-smoothness, a concept which effectively formalizes the desired smoothness properties for Hamiltonian functions. The concept is shown to be invariant under sc-smooth symplectomorphisms, whence it is compatible with Hofer's scale manifolds. With this theory, the solutions of a Hamiltonian PDE can be seen as integral curves of the corresponding Hamiltonian vector field. We develop and verify the theory at the hand of the free Schrödinger equation.

Keywords: Hamiltonian PDE, infinite-dimensional symplectic geometry, Hamiltonian vector fields, Hofer-Wysocki-Zehnder scale calculus

Title: Hamiltonian Partial Differential Equations and Symplectic Scale Manifolds
Towards Symplectic Geometry in Infinite Dimensions

Author: João Bernardo Crespo, j.b.farinhapereiracrespo@student.vu.nl, 2555912
Supervisor: dr. Oliver Fabert
Second examiner: dr. Federica Pasquotto

Date: November 29, 2017

Department of Mathematics
VU University Amsterdam
de Boelelaan 1081, 1081 HV Amsterdam
`http://www.math.vu.nl/`

Popular Summary

The space-time trajectory of classical mechanical systems, such as a pendulum, a spring or the celestial system of the earth rotating around the sun, can be described in physics by a system of differential equations called *Hamilton's equations*. Consider the latter example of the earth rotating around the sun. To describe this system, we need six degrees of freedom: the earth can move in the vertical, horizontal and depth directions, and the same holds for the sun. Although this seems to result in a quite complicated system, it turns out that it can be simplified. The momentum of the global system (a three-dimensional vector) is constant since there are no external forces applied, and the total energy (a one-dimensional scalar) is conserved. In total, we thus have four so-called *constants of motion* — the coordinates of the momentum and the energy of the system —, which allow us to reduce the number of degrees of freedom needed to study the system. In essence, this system has only two (six minus four) degrees of freedom and indeed, the motion of the earth around the sun is on a plane.

Mathematics is the art of abstraction and generalization, and it certainly did not leave classical mechanics behind. Hamilton's equations can be put into a more general framework dealt within the field of *symplectic geometry*, where one studies properties of the solutions of generalized forms of these. Also in this general framework, a system with a certain number of degrees of freedom which is subject to constants of motion can be reduced into a lower-dimensional system. In contrast to the earth-sun example though, the reduced system possibly lives on a "curved but locally flat" smooth space, what is formally called a *smooth manifold*. To get an impression of what this might look like, imagine the outer skin of an apple: an ant crawling on it would feel a mostly flat smooth floor under its legs, but a human being would look at it in a completely different way.

In this thesis, we generalize some aspects in the above mentioned field of symplectic geometry to be able to deal with quantum mechanics. Some partial differential equations in physics, for example the Schrödinger equation, can be compared to Hamilton's equations in some sense. The main difference to the case of classical mechanics is that the number of degrees of freedom is infinite, or more formally, we have to work with infinite-dimensional vector spaces. Also in this setting one is interested in reduced systems: two wave functions of Schrödinger's equation which are (complex) multiples of each other are considered to represent the same physical state, and when we take this into account, the vector space turns into an infinite-dimensional manifold. Previous literature had not studied symplectic geometry in a setting which is useful for these partial differential equations when reduced into an infinite-dimensional manifold, and there lies the main contribution of this thesis.

Acknowledgements

I would like to thank my supervisor Oliver Fabert, who gave me the opportunity to work on this interesting project and guided me through the subject. As he can probably confirm, I am someone who can deeply focus on details, and he helped me not to loose the global picture while progressing on the thesis.

I would also like to thank Fabian Ziltener and Federica Pasquotto for their master course on finite-dimensional symplectic geometry, subject which constitutes the foundation of this thesis. It is clearly impossible to understand the infinite-dimensional case without the finite-dimensional background.

Finally, I would like to express my gratitude towards my mother Ana and my boyfriend Jesse, who gave me happiness and fulfillment in the process while supporting me spiritually in more difficult times.

Contents

General Notation

$\mathbb{N}$	natural numbers starting from 1
$\mathbb{N}_0$	natural numbers starting from 0 ($= \mathbb{N} \cup \{0\}$)
δ_{mn}	Kronecker delta
$\mathbb{R}_{\geq 0}$	non-negative real numbers
$\mathbb{R}_{>0}$	positive real numbers
$\overline{\mathbb{R}}$	extended real line (with $-\infty, \infty$)
$\mathbb{F}$	base field for vector spaces ($\mathbb{R}$ or $\mathbb{C}$)
$\mathbb{F}^*$	non-zero elements of base field
$\mathrm{Re}\{\cdot\}, \mathrm{Im}\{\cdot\}$	real, imaginary part
$\overline{(\cdot)}$	complex conjugation
Y^X	set of functions from X to Y
id_X	identity map on X
$\mathbb{1}_A$	indicator function on $A \subseteq X$
diag_X	diagonal map $X \mapsto X \times X$
$f \times g$	Cartesian product of functions f and g
$f \times_{\mathrm{diag}} g$	diagonal product of f and g ($= (f \times g) \circ \mathrm{diag}$)
$\mathrm{Sp}\, A$	linear span of $A \subseteq X$ for a vector space X
$\mathrm{B}(X, Y)$	continuous linear operators from normed space X to Y
$\mathrm{B}(X)$	continuous linear endomorphisms on X ($= \mathrm{B}(X, X)$)
X^*	topological dual of X ($= \mathrm{B}(X, \mathbb{F})$)
L^*	adjoint operator of $L : X \mapsto Y$ ($L^* : Y^* \mapsto X^*$, $f \mapsto f \circ L$)
S^1	the one-dimensional torus (circle)
$L^2(S^1, \mathbb{C})$	square-integrable functions from S^1 to $\mathbb{C}$
$W^{k,2}(S^1, \mathbb{C})$	Levi-Sobolev space of k times weakly differentiable functions from S^1 to $\mathbb{C}$ whose derivatives lie in $L^2(S^1, \mathbb{C})$
$l^2_{\mathbb{F}}$	space of square-summable $\mathbb{Z}$-indexed sequences with values in $\mathbb{F}$
$\hat{u}$	vector in $l^2_{\mathbb{C}}$ with Fourier coefficients of $u \in L^2(S^1, \mathbb{C})$
$\sup, \inf$	supremum, infimum

$\lim, \operatorname{colim}$	limit, colimit
$(\cdot)_\infty, (\cdot)_{-\infty}$	limit, colimit
$\mathrm{S}(X)$	unit sphere of normed space X
$\mathrm{P}(X)$	projectivization of normed space X
$\mathbb{F}\mathrm{P}^d$	real/complex projective space ($= \mathrm{P}(\mathbb{F}^d)$)
$U(x) \subseteq X$	$U \subseteq X$ and $x \in U$
$\dot{u}$	time derivative of u ($= \mathrm{d}u/\mathrm{d}t$)
$\mathrm{T}_p M$	tangent space (or scale) at $p \in M$ for a manifold M
$\mathrm{T}M, \mathrm{T}^*M$	tangent and cotangent bundle of manifold M
d	(exterior) derivative
d_p	(exterior) derivative at $p \in M$ for a manifold M

1 Introduction

More than one hundred eighty years ago, Irish physicist William Rowan Hamilton reworked Newtonian mechanics using a formalism which would have a drastic impact on modern physics and mathematics. His formalism evolved into an unparalleled framework with a broad range of applications, from simple everyday mechanical systems to involved celestial mechanics where multiple bodies interact. Strikingly, many applications supersede the original scope of the formulation: the development of modern quantum mechanics is heavily based on Hamilton's formalism and would not have been possible without his contribution. In parallel to this development in physics, Hamiltonian dynamics has become the *de facto* motivation for modern mathematical areas such as contact and symplectic geometry.

1.1 Finite-dimensional Symplectic Geometry

To gain insight in how classical mechanics and Hamilton's work relates to symplectic geometry, consider a mechanical system of n particles of unit mass in $\mathbb{R}^m$ which are subject to potential forces. This setting gives rise to a dynamical system with $d = nm$ degrees of freedom in the configuration space $\Omega \subseteq \mathbb{R}^d$ open, with coordinates $(q_1, q_2, \ldots, q_d) \in \Omega$ and evolving under the conservation law $E = T + V = \text{constant}$. Here, $T(\dot{q}) = \|\dot{q}\|^2/2$ is the kinetic energy of the system and $V : \Omega \mapsto \mathbb{R}$ is its potential energy, assumed to be a smooth function. As was already known by Newton in the 17^{th} century, the dynamics of this system is governed by the equations

$$\ddot{q}_k = -\frac{\partial V}{\partial q_k}\,,\ k = 1, 2, \ldots, d\,. \tag{1.1}$$

Hamilton's insight was to introduce the conjugate momenta

$$p_k := \frac{\partial (T - V)}{\partial \dot{q}_k} = \dot{q}_k\,, \tag{1.2}$$

thereby expressing the state of the system as a function of both configuration q and momentum p. In more detail, defining the phase space $\mathrm{T}^*\Omega := \Omega \times \mathbb{R}^d \subseteq \mathbb{R}^{2d}$ and a smooth Hamiltonian function given by the total energy of the system — kinetic and potential — $h = E : \mathrm{T}^*\Omega \mapsto \mathbb{R}$, $(q, p) \mapsto T(p) + V(q)$, Equation (1.1) can be rewritten as what are called Hamilton's equations

$$\begin{cases} \dot{q}_k = \dfrac{\partial h}{\partial p_k} \\ \dot{p}_k = -\dfrac{\partial h}{\partial q_k} \end{cases},\ k = 1, 2, \ldots, d\,. \tag{1.3}$$

Although the presented example might seem rather simple, the general derivation of Hamilton's equations holds for a large class of mechanical systems [31].

Hamilton's equations (1.3) exhibit a so-called symplectic structure. A real vector space V of dimension $2d$ is said to be symplectic whenever it is adjoined with a bilinear skew-symmetric form $\omega : V \times V \mapsto \mathbb{R}$ which, similarly to an inner product, identifies V with its dual by means of the isomorphism of vector spaces $\iota_\omega : V \overset{\sim}{\longmapsto} V^*$, $v \mapsto \omega(\cdot, v)$. Symplectic vector spaces are extensively studied in [48, Chapter 2]. These relate seamlessly to Hamilton's equations in the following way. Define the so-called standard symplectic form $\omega : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \mapsto \mathbb{R}$

$$\omega(v, w) = \langle \mathrm{i}v, w \rangle \,, \tag{1.4}$$

where $\mathrm{i} : \mathbb{R}^{2d} \mapsto \mathbb{R}^{2d}$, $v \mapsto \mathrm{i}v$ is the standard complex structure of $\mathbb{R}^{2d} \cong \mathbb{C}^d$, and $\langle \cdot, \cdot \rangle$ is its standard real inner product. It is not difficult to verify that ω is indeed symplectic on $\mathbb{R}^{2d}$. Since ι_ω is an isomorphism, there is a unique smooth vector field $V_h : \mathrm{T}^*\Omega \mapsto \mathbb{R}^{2d}$, called the Hamiltonian vector field generated by h, which satisfies the ω-gradient relation

$$- \,\mathrm{d}h(x) = \omega(\cdot, V_h(x)) \in (\mathbb{R}^{2d})^* \quad \text{for all } x \in \mathrm{T}^*\Omega \,. \tag{1.5}$$

A simple computation shows that the Hamiltonian vector field is given by $V_h = -\mathrm{i}\nabla h$, where ∇h is the gradient of h with respect to $\langle \cdot, \cdot \rangle$, and that Hamilton's equations (1.3) are satisfied for a curve $x = (q, p) \in \mathrm{T}^*\Omega$ if and only if

$$\dot{x} = V_h(x) \,. \tag{1.6}$$

In other words, the phase space trajectory of our dynamical system is an integral curve of the Hamiltonian vector field V_h.

More generally, instead of working with the Euclidean space $\mathbb{R}^{2d}$, we can consider a $2d$-dimensional manifold M [45]: a locally Euclidean space satisfying suitable topological properties, together with a maximal atlas of coordinate charts (homeomorphisms) $\phi_a : U_a \subseteq M \overset{\sim}{\longmapsto} V_a \subseteq \mathbb{R}^{2d}$, $a \in A$, such that the transition maps $\phi_{ab} := \phi_b \, \phi_a^{-1} : \phi_a(U_a \cap U_b) \subseteq \mathbb{R}^{2d} \overset{\sim}{\longmapsto} \phi_b(U_a \cap U_b) \subseteq \mathbb{R}^{2d}$ are diffeomorphisms. A symplectic manifold can then be defined in the same way as a manifold, while requiring the transition maps to be symplectomorphisms: the derivative $\mathrm{d}\phi_{ab}(x) : \mathbb{R}^{2d} \overset{\sim}{\longmapsto} \mathbb{R}^{2d}$ should preserve the standard symplectic form ω for all $x \in \phi_a(U_a \cap U_b)$, in the sense that $\omega(\mathrm{d}\phi_{ab}(x) \cdot v, \mathrm{d}\phi_{ab}(x) \cdot w) = \omega(v, w)$ for all $v, w \in \mathbb{R}^{2d}$. An equivalent definition of a symplectic manifold due to Darboux [48, Theorem 3.15] is that of a manifold M together with a closed two-form which induces, for every $p \in M$, a symplectic form on the tangent space T_pM. The study of symplectic manifolds is the topic of symplectic geometry, and a sound introduction may be found in [48].

Similarly to the Euclidean case, any smooth function $h : M \mapsto \mathbb{R}$ on a symplectic manifold M gives rise to a Hamiltonian vector field — now a smooth section of the tangent bundle — $V_h : M \mapsto \mathrm{T}M$. This vector field is uniquely defined by the relation

$$- \,\mathrm{d}_p h = \omega_p(\cdot, V_h(p)) \in \mathrm{T}_p^*M \quad \text{for all } p \in M \,, \tag{1.7}$$

where $\omega_p : \mathrm{T}_pM \times \mathrm{T}_pM \mapsto \mathbb{R}$ is the induced symplectic form at $p \in M$. An alternative defining equation arises by noting that the symplectic forms ω_p assemble into an isomorphism of vector bundles $\iota_\omega : \mathrm{T}M \overset{\sim}{\longmapsto} \mathrm{T}^*M$, $\mathrm{T}_pM \ni v \mapsto \omega_p(\cdot, v) \in \mathrm{T}^*_pM$. Using this isomorphism, (1.7) can be rewritten as

$$-\,\mathrm{d}h = \iota_\omega \circ V_h : M \mapsto \mathrm{T}^*M\,. \tag{1.8}$$

Once a Hamiltonian vector field V_h is given, its flow $\varphi_h : \mathcal{D} \subseteq \mathbb{R} \times M \mapsto M$ is said to be Hamiltonian and we can regard, at least locally, the integral curves of V_h as solutions of Hamilton's equations. For simplicity of exposition, we only consider complete vector fields in this document, and as such $\mathcal{D} = \mathbb{R} \times M$.

Though at first sight the introduction of symplectic manifolds might seem an unnecessary mathematical artefact, this kind of manifolds arises naturally in the study of Hamiltonian functions with symmetries. Consider, for instance, a finite-dimensional quantum mechanical system on $\mathbb{C}^d$ (*e.g.*, a spin system) with an observable given by a self-adjoint $\mathbb{C}$-linear map $H : \mathbb{C}^d \mapsto \mathbb{C}^d$. A suitable Hamiltonian for this system is the "expectation value function" [6] $h : \mathbb{C}^d \mapsto \mathbb{R}$ given by

$$h(x) = \frac{1}{2}\langle Hx, x\rangle\,. \tag{1.9}$$

If we let the circle $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ act on $\mathbb{C}^d$ by pointwise multiplication, we immediately discover that h is S^1-invariant, in the sense that $h(\lambda x) = h(x)$ for all $\lambda \in S^1$ and $x \in \mathbb{C}^d$. Also, one can check that the fundamental vector field of the action is generated by the S^1-invariant Hamiltonian function (momentum map) $\mu : \mathbb{C}^d \mapsto \mathbb{R}$

$$\mu(x) = \frac{1}{2}(1 - \|x\|^2) \tag{1.10}$$

and that the action is free on $\mu^{-1}(0)$. By the Marsden-Weinstein theorem [2, Theorem 4.3.1], one obtains a natural symplectic manifold structure on

$$\mathbb{C}\mathrm{P}^{d-1} := \mathbb{C}^d \setminus \{0\}\big/\mathbb{C}^* \cong \mu^{-1}(0)\big/S^1 \tag{1.11}$$

such that the induced symplectic form pulls back to the standard from on $\mu^{-1}(0) \subseteq \mathbb{C}^d$ via the canonical projection $\mu^{-1}(0) \mapsto \mathbb{C}\mathrm{P}^{d-1}$. Furthermore, the Hamiltonian function h and flow φ_h descend to functions $\bar{h} : \mathbb{C}\mathrm{P}^{d-1} \mapsto \mathbb{R}$ and $\bar{\varphi}_h : \mathbb{R} \times \mathbb{C}\mathrm{P}^{d-1} \mapsto \mathbb{C}\mathrm{P}^{d-1}$, respectively, and $\bar{\varphi}_h$ is precisely the Hamiltonian flow generated by $\bar{h}$ [2, Theorem 4.3.5]. One can then study the reduced system on the lower-dimensional manifold $\mathbb{C}\mathrm{P}^{d-1}$ and recover the original dynamics therefrom [2, pp. 304–305].

1.2 Hamiltonian Partial Differential Equations

It turns out that certain partial differential equations (PDEs) can be written as flow equations of a Hamiltonian vector field as in (1.6), but using infinite-dimensional symplectic vector spaces. Such PDEs are then said to be Hamiltonian. As an example, take the free Schrödinger equation on the circle

$$\mathrm{i}u_t = -\Delta u \tag{1.12}$$

for an unknown $u : \mathbb{R} \times S^1 \mapsto \mathbb{C}$, $(t, x) \mapsto u(t, x)$, where $\Delta u = u_{xx}$ is the Laplacian operator. To study this equation, introduce the Hilbert space of square-integrable functions on the circle

$$L^2 := L^2(S^1, \mathbb{C}) = \left\{ u : S^1 \mapsto \mathbb{C} : u \text{ measurable and } \int_{S^1} |u(x)|^2 \, \mathrm{d}x < \infty \right\} \tag{1.13}$$

and its standard symplectic form $\omega : L^2 \times L^2 \mapsto \mathbb{R}$ given by (1.4), where now the real inner product and complex structure are the ones of L^2.

As in Section 1.1, we try and define Hamiltonian vector fields and flows for each Hamiltonian map h by requiring

$$-\,\mathrm{d}h(u) = \omega(\cdot, V_h(u)) \tag{1.14}$$

for all $u \in L^2$. With this idea in mind, we observe that the solutions of (1.12) are integral curves of the Hamiltonian vector field generated by the Hamiltonian function

$$h(u) = \frac{1}{2} \int_{S^1} |u_x(a)|^2 \, \mathrm{d}a \,. \tag{1.15}$$

Indeed, integration by parts shows that $\mathrm{d}h(u) \cdot v = \langle u_x, v_x \rangle = -\langle u_{xx}, v \rangle$, whence $V_h(u) = \mathrm{i}u_{xx}$. The corresponding Hamiltonian flow is

$$\varphi_h(t, u) = \mathrm{e}^{\mathrm{i}t\Delta} u \,. \tag{1.16}$$

Further examples and ellaboration on Hamiltonian PDEs can be found in [1, 3, 8, 15, 43, 49].

Although the presented setup seems plausible, it shows a crucial difference with respect to the finite dimensional case. If we inspect the proposed mathematical objects, we see that the Hamiltonian function of (1.15) cannot be defined on the entire space L^2, but only on the dense subset $W^{1,2} = W^{1,2}(S^1, \mathbb{C})$ of weakly differentiable functions with L^2 derivative. Similarly, the vector field V_h is only densely-defined and two derivatives are needed. Rather in contrast to this, the Hamiltonian flow defines a map $\mathbb{R} \times L^2 \mapsto L^2$. We thus recognize that several vector spaces are needed for defining the different objects at stake.

This problem was solved by Kuksin [43], who used Hilbert scales to frame Hamiltonian PDEs. A Hilbert scale is a filtration of Hilbert spaces which are densely and compactly embedded in each other. From L^2, we can build the Levi-Sobolev Hilbert scale $\{W^{k,2}\}_{k \in \mathbb{Z}}$ with $W^{0,2} = L^2$ and extend the real inner product of L^2, hence also ω, to a non-degenerate pairing $W^{k,2} \times W^{-k,2} \mapsto \mathbb{R}$. Since (1.15) defines a smooth map $h : W^{1,2} \mapsto \mathbb{R}$, the usual ω-gradient relation (1.14) produces a vector field $V_h : W^{1,2} \mapsto W^{-1,2}$ which is simply $\mathrm{i}\Delta$ — Kuksin's framework involving Hilbert scales delivers the expected results.

Hilbert scales are particular examples of Banach scales, where in the latter we allow the involved spaces to be Banach. As a matter of fact, both Banach and Hilbert scales are well-known functional analytical objects. Motivated by interpolation theory, Krein and Petunin introduced Banach scales while requiring the scale spaces to satisfy an interpolation inequality [41]. In the same document, the authors defined Hilbert scales departing from a Hilbert space and a positive definite self-adjoint operator. Bonic defined Hilbert scales in a comparable manner and derived properties concerning smoothing projections and commuting operators between limits of two scales [9]. In turn, Dubinsky investigated limits and colimits of Banach scales [18]. A more recent review on Banach scales can be found in [4, Chapter 5]. The theory on Hilbert scales is extensive and has found several applications, such as the study of evolution equations [11, 27, 47], regularization problems [20, 21, 39], spectral theory [17] and, centrally in this thesis, Hamiltonian PDEs [43].

Moving one step further, suppose that we are only interested in nonzero wave functions u of Schrödinger's equation (1.12) up to a nonzero complex scalar. This is the case of interest in physics, where the equivalence classes in the projective Hilbert space

$$\mathrm{P}(L^2) := {}^{L^2 \setminus \{0\}}\!/_{\mathbb{C}^*} \tag{1.17}$$

represent the state of the quantum-mechanical system [6,24]. To describe such a system we desire, by analogy with the finite-dimensional case, to have some local symplectic scale structure on $\mathrm{P}(L^2)$ where we can make sense of basic symplectic geometry as in Section 1.1.

In our path towards this aim, part of the work by Hofer-Wysocki-Zehnder on polyfolds [32, 35, 36, 38] is essential. Departing from a Banach scale they develop the notion of scale calculus, which allows the derivative of a function between scales to be defined only on a dense subspace of higher regularity. Subsequently, they extend scale differentiation inductively to scale smoothness, where an arbitrary number of scale derivatives may be taken. In a similar way as in classical differential geometry [44], [63, Chapter 73], the authors then proceed to introduce smooth scale manifolds (also notated sc$^\infty$-manifolds) locally modeled on Banach scales.

As expected, scale manifolds inherit structures from the underlying local model. Specifically, an sc$^\infty$-manifold M gives rise to a natural filtration $\{M_k\}_{k\in\mathbb{N}_0}$ induced by the local scale structure, and for each point of a filtration subspace M_k, $k \geq 1$, we can associate a partial Banach scale which plays the rôle of the tangent space. Also, a tangent bundle $\pi_{\mathrm{T}M} : \mathrm{T}M \mapsto M^1$ is defined, where $M^1 = \{M_{k+1}\}_{k\in\mathbb{N}_0}$ is the shifted filtration. The shift appearing in the base space of the bundle reflects the higher regularity of the differentiation points.

Scale smoothness and polyfolds were introduced with the purpose of solving problems in symplectic field theory and related areas [22, 37]. Fabert et al. [61] review the theory in a broader perspective, also extending the idea of Banach scales to filtrations of topological spaces. Wehrheim [61] elaborates on the Fredholm theory developed by Hofer-Wysocki-Zehnder in [32–34,36], while noting that a Banach scale can be recovered from its (Frèchet) limit and restricted norms. Gerstenberger [30] works with the limits of Banach scales as well, modifying the scale smoothness and Fredholm theories of Hofer and Wehrheim so as to allow for the application of the Nash-Moser theorem on "tame" Frèchet limits.

On a more general note, the theory developed by Hofer-Wysocki-Zehnder is one in a myriad of concepts which extend classical smoothness and manifolds. A modest selection of other examples includes convenient smoothness [42], Frölicher spaces [57], Chen spaces [7] and diffeological spaces [40]. While the first two examples work on the basis of smooth curves, the last two assign to each (convex or open, respectively) Euclidean subset, a collection of functions on the subset which are specified to be smooth. The advantage of these approaches is that they are closed under natural constructions such as exponentiation [7, 42] (*i.e.*, considering the collection of smooth maps between two objects). We refer the reader to the comparative overview in [57] for further smoothness concepts and a sound discussion.

Between the vast amount of possible choices for smoothness concepts, we claim that scale smoothness is the adequate choice for formalizing symplectic geometry geared towards Hamiltonian PDEs. Indeed, as motivated above, Hamiltonian functions and vector fields are expected to be defined on points of higher regularity compared to the flow. In fact, if we take our guiding example of the free Schrödinger equation and informally differentiate the flow with respect to the time variable, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}t\Delta}u = \mathrm{i}\Delta\mathrm{e}^{\mathrm{i}t\Delta}u = V_h \circ \mathrm{e}^{\mathrm{i}t\Delta}u \tag{1.18}$$

which exists as an element of L^2 whenever $u \in W^{2,2}$. Consequently, the derivative of the flow may only be expected to be densely-defined. Naturally, since linear scales are generalized by scale manifolds, this phenomenon is expected to occur in the latter case as well. Finally, note that scale manifolds suit our motivating example well, as the projective Hilbert space $\mathrm{P}(L^2)$ can be given such an sc-smooth structure (essentially in the same way as $\mathbb{C}\mathrm{P}^d$).

1.3 Main Contribution

The work of Hofer-Wysocki-Zehnder allows us to generalize Banach scales to the manifold context, and it is not difficult to carry the generalization through to sc-smooth vector fields and flows: for example, a vector field is simply an sc-smooth section of the tangent bundle $V : M^1 \mapsto \mathrm{T}M$. Nevertheless, their work lacks a notion of a symplectic structure on an sc-smooth manifold. Naturally, with the lack of symplectic structures comes the lack of Hamiltonian vector fields and flows. Furthermore, it is not clear in what sense a Hamiltonian function should be smooth so as to obtain an sc-smooth vector field via a suitable symplectic gradient relation.

To fill this gap, we propose to define symplectic scale manifolds as scale manifolds locally modeled on a symplectic Banach scale, endowed with a maximal atlas of coordinate charts where all transition maps are symplectomorphisms. The latter condition allows for the definition of a cotangent bundle T^*M and a canonical isomorphism ι_ω between tangent and cotangent bundles. Furthermore, we narrow down the scale smoothness concept of Hofer-Wysocki-Zehnder to what we baptize as *strong* scale smoothness[1]. As it

[1] Not to be confused with the side definition of [38, Remark 1.3].

turns out, a desirable definition for the regularity of a Hamiltonian function requires the test vectors of the scale derivative to be taken from spaces of increasingly lower regularity as the regularity of the differentiation point increases, and the original sc-smoothness concept is too weak to accommodate this requirement. We prove that the concept of strong scale smoothness is invariant under pre-composition with symplectomorphisms, hence it is consistent with symplectic sc-smooth manifolds.

The definition of strong scale smoothness leads to a natural generalization of Hamiltonian vector fields and flows in a symplectic sc-smooth manifold M: for a strongly sc-smooth function $h : M^1 \mapsto \mathbb{R}$, we can interpret its derivative as an sc-smooth section of the cotangent bundle $\mathrm{D}h : M^1 \mapsto \mathrm{T}^*M$ and, as in the finite-dimensional case of (1.8), $\mathrm{D}h$ gives rise to an sc-smooth vector field V_h by means of the bundle isomorphism $\iota_\omega : \mathrm{T}M \overset{\sim}{\longmapsto} \mathrm{T}^*M$ and the symplectic gradient relation

$$-\mathrm{D}h = \iota_\omega \circ V_h : M^1 \mapsto \mathrm{T}^*M\,. \tag{1.19}$$

The contributions of this thesis are developed and presented at the hand of the free Schrödinger equation (1.12) which, in the authors' modest opinion, is simple enough to avoid distractions and, at the same time, serves as a prototypical example exhibiting the core property of Hamiltonian PDEs: the vector field is only densely defined. Correspondingly, the projective Hilbert space $M = \mathrm{P}(L^2)$ is presented as a symplectic sc-smooth manifold locally modeled on the Hilbert scale $X = \{X_k = W^{2k,2}\}_{k\in\mathbb{N}_0}$. The flow $\varphi_h : \mathbb{R} \times X \mapsto X$ of (1.16) is shown to be sc-smooth and Hamiltonian, generated by the strongly sc-smooth Hamiltonian function $h : X^1 \mapsto \mathbb{R}$ of (1.15). In the trend of symplectic reduction, these maps are subsequently seen to descend to maps $\bar{\varphi}_h : \mathbb{R}\times M \mapsto M$ and $\bar{h} : M^1 \mapsto \mathbb{R}$ inheriting the corresponding regularity properties, and $\bar{\varphi}_h$ is concluded to be a Hamiltonian flow generated by $\bar{h}$.

1.4 Organization of Thesis

The remainder of this thesis is organized in three chapters. Chapter 2 reviews linear scale structures, scale calculus by Hofer-Wysocki-Zehnder and scale manifolds. We start introducing the concept of a Banach scale, subsequently showing how to build such a scale departing from a separable infinite-dimensional Hilbert space. In the process, we outline basic notions of linear symplectic geometry on these scales. After the linear part, we review Hofer-Wysocki-Zehnder's scale calculus, presenting and deriving several properties which make this kind of calculus behave in a similar manner as classical Frèchet or finite-dimensional calculus. Finally, we show how to use classical procedures in the construction of manifolds to define scale manifolds and their tangent scales and bundles.

Chapter 3 contains the main contribution of this thesis. We begin by extending the notions of vector fields and flows to scale manifolds, briefly discussing the immensely involved subject of existence and uniqueness within a limited scope. In stark contrast to the finite-dimensional case, sc-smooth flows do not need to exist or be uniquely defined. We continue and define strong scale smoothness on Banach scales, relating it to existing concepts. This new notion allows for the definition of Hamiltonian functions in the scale setting. What is more, we deduce symplectomorphism-invariance of the definition. Subsequently, the concepts introduced for Banach scales are generalized to the manifold setting: we end the chapter defining symplectic scale manifolds, the cotangent bundle, induced isomorphism, strong scale smoothness on symplectic scale manifolds and Hamiltonian vector fields and flows. The explored concepts are solidified with the example of the free Schrödinger equation.

The thesis ends with a short discussion on Chapter 4, supplementing this work with possible future research directions. Therein, we give a lookout of further topics which could be explored on the way to generalize finite-dimensional symplectic geometry towards infinite dimensions.

2 Scale Structures and Manifolds

In this chapter, we introduce Banach scales and a framework which allows us to manipulate these. In essence, a Banach scale is a collection of Banach spaces where each pair of spaces is bonded by a linear continuous map and where the whole structure satisfies certain desirable properties. We explain how to define natural operations on scales, such as translation of the index set, the product of scales, or a scale composed out of the topological duals of each space — the "dual scale". Also, we explain how to form maps between scales and introduce several types of scale maps. We then proceed by narrowing down the scope to Hilbert scales, which are simply Banach scales with the additional structure that each underlying space has a compatible inner product. We lay our focus on separable Hilbert spaces, extensively studied by Kuksin [43], since their structure is simply derivable from a prototypical weighted l^2 space and since they occur frequently in examples.

With the necessary working tools for Banach scales in our pockets, we introduce the notion of sc-smoothness on Banach scales. The presented notion, developed by Hofer-Wysocki-Zehnder [32], provides an alternative concept compared to the classical Frèchet notion. By involving several layers of regularity while defining the derivative of a map of scales, this concept allows for maps which "loose regularity" in their infinitesimal form. Also, in a way similar to Banach manifolds, it gives rise to a manifold structure by endowing a topological space with a (maximal) atlas whose transition maps are sc-diffeomorphisms.

2.1 Banach Scales

We begin by defining projective systems, which are the basic categorical structure that we need to make sense of scales in this thesis. For the complete categorical background underlying this formulation, we refer the reader to [28, 46]. Although Banach scales are frequently dealt with within a simpler framework of a filtration, when introducing dual scales, this simple framework will not suffice. Indeed, as we will see in this section, the dual scale of a filtration is bond by the adjoints of the inclusion maps and the explicit structure of a filtration disappears. Intuitively though, it will still be useful to think of Banach scales as filtrations since the bonding maps will always be injective.

In the following, we turn our attention to the category of locally convex spaces (LCS) over a fixed field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and a non-empty index set $S \subseteq \mathbb{R}$. All maps referred to are to be understood as morphisms in this category, *i.e.*, continuous linear maps. Also, we use the total order relation $<$ on S, which makes S into a directed set.

Definition 2.1. (a) A projective system of LCS on S is a family $\{X_s\}_{s\in S}$ of LCS together with maps $p_{sr} : X_s \mapsto X_r$ for all $s > r \in S$, the so-called bonding maps, such that $p_{sq} = p_{rq} \circ p_{sr}$ for all $s > r > q \in S$.

$$\begin{tikzcd} \dots \arrow[r] & X_s \arrow[r, "p_{sr}"'] \arrow[rr, bend left, "p_{sq}"] & X_r \arrow[r, "p_{rq}"'] & X_q \arrow[r] & \dots \end{tikzcd}$$

(b) A cone to a projective system $\{X_s\}_{s\in S}$ is an LCS Y together with a family of maps $\{f_s : Y \mapsto X_s\}_{s\in S}$, such that $f_r = p_{sr} \circ f_s$ for all $s > r \in S$.

$$\begin{tikzcd} & Y \arrow[dl, "f_s"'] \arrow[dr, "f_r"] & \\ X_s \arrow[rr, "p_{sr}"'] & & X_r \end{tikzcd}$$

(c) A limit of a projective system of LCS $\{X_s\}_{s\in S}$ is defined as a universal cone to the projective system. That is to say, it is a cone $(X_\infty, \{p_{\infty s} : X_\infty \mapsto X_s\}_{s\in S})$ such that for any other cone $(Y, \{f_s : Y \mapsto X_s\})$, there exists a a unique map $\phi : Y \mapsto X_\infty$ such that $f_s = p_{\infty s} \circ \phi$ for all $s \in S$. We also use the notation $\lim_{s\in S} X_s = X_\infty$ and omit the cone maps if clear from context.

$$\begin{tikzcd} & Y \arrow[ddl, "f_s"'] \arrow[d, dotted, "\phi"] \arrow[ddr, "f_r"] & \\ & X_\infty \arrow[dl, "p_{\infty s}"] \arrow[dr, "p_{\infty r}"'] & \\ X_s \arrow[rr, "p_{sr}"'] & & X_r \end{tikzcd}$$

Remark 2.2. (a) Due to the universal property, a projective limit is unique up to a unique isomorphism [28]. In the category of LCS, it always exists and it can be directly realized as the closed subspace of the product $\prod X_s$

$$X_\infty = \left\{ x = (x_s)_{s\in S} \in \prod_{s\in S} X_s : p_{sr}(x_s) = x_r \text{ for all } s > r \right\} \tag{2.1}$$

together with the canonical projections $p_{\infty s} : X_\infty \mapsto X_s$, $x \mapsto x_s$. We also see immediately that X_∞ is Hausdorff whenever all X_s are so.

(b) An important example of a projective system is a descending filtration $\{X_s\}_{s\in S}$ of LCS, $X_s \subseteq X_r$ for $s > r$. In this case, the limit is (isomorphic to) $X_\infty = \bigcap_{s\in S} X_s$ and all maps are simply the inclusions $\iota_{sr} : X_s \hookrightarrow X_r$, $s > r \in S \cup \{\infty\}$.

(c) By reversing the arrows in Definition 2.1(b)–(c), it is possible to define what is called a co-limit of the projective system. We then have co-cones, which are pairs $(Y, \{f_s : X_s \mapsto Y\}_{s\in S})$ such that $f_s = f_r\, p_{sr}$ for all $s > r$, and a co-limit, which is a universal co-cone $(X_{-\infty}, \{p_{s,-\infty} : X_s \mapsto X_{-\infty}\}_{s\in S})$. It satisfies the universal property that for any other co-cone $(Y, \{f_s : X_s \mapsto Y\})$, there exists a a unique map $\phi : X_{-\infty} \mapsto Y$ such that $f_s = \phi\, p_{s,-\infty}$ for all $s \in S$. In the category of LCS it also always exists, although it need not be Hausdorff [25]. We set $\operatorname{colim}_{s\in S} X_s = X_{-\infty}$.

In the following, we will restrict our attention to projective systems where the bonding maps $\iota_{sr} := p_{sr}$ are injective. In this case, the maps can be seen as inclusions and much of the set-theoretic machinery which is valid for the simple case of Remark 2.2(b), where the spaces are nested, carries over to this case. As a matter of fact, when the bonding maps are injective, one can consider the Banach spaces X_s to be subspaces of $X_{-\infty}$ and the projective system to be a descending filtration. The following technical paragraphs introduce notation which bridges the two cases and makes the projective case more intuitive and easy to read.

One should first note that in the injective case, not only the bonding maps ι_{sr}, but also the cone maps $\iota_{\infty s}$ of the limit are injective. For any $s \in S \cup \{\infty\}$ and any $A \subseteq X_s$, we have isomorphic copies $\iota_{sr}(A) \subseteq X_r$ of A for all $r < s$. Hence, it makes sense to define set-theoretical relations on subsets of different spaces $A \subseteq X_s$ and $B \subseteq X_r$, $s > r$, by using the isomorphic copies. We say that $A \subseteq B$ if $\iota_{sr}(A) \subseteq B$ and use the inclusion map $\iota_{sr} : A \overset{\sim}{\longmapsto} \iota_{sr}(A) \subseteq B$. In this setting, if Z is a set and $f : B \mapsto Z$ is a function, then we can restrict f to A by using the above inclusion map, $f|_A := f\, \iota_{sr} : A \mapsto Z$. Dually, if $f : Z \mapsto A$, we can embed the codomain of f in B obtaining $f \equiv \iota_{sr}\, f : Z \mapsto B$.

Similarly, for $C \subseteq X_s$ and $D \subseteq X_r$, $s > r$, we can define $D \subseteq C$ if $D \subseteq \iota_{sr}(C)$ and in that case, we identify D with $\iota_{sr}^{-1}(D) \subseteq C$. With this definition, we can easily describe the restriction of functions $f : Z \mapsto B$, $A \subseteq B$ on the codomain. Namely, by the following simple lemma, f restricts to $f : Z \mapsto A$ if and only if $f(Z) \subseteq A$.

Lemma 2.3. *Let X, Y and Z be sets (vector spaces) and $\iota : X \hookrightarrow Y$ be an injective (linear) map. Then a (linear) map $f : Z \mapsto Y$ lifts to a (linear) map $\tilde{f} : Z \mapsto X$ if and only if $f(Z) \subseteq \iota(X)$. Moreover, if $\tilde{f}$ exists, it is unique.*

$$\begin{tikzcd} & X \arrow[d, hook, "\iota"] \\ Z \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"] & Y \end{tikzcd}$$

Proof. If there is such a commutative diagram, clearly $f(Z) \subseteq \iota(X)$. On the other hand, if this last condition holds, then the fiber of $f(z)$ under ι is non-empty for all $z \in Z$. Due to the injectivity of ι, it consists of a single element which defines $\tilde{f}$ uniquely. For the case of vector spaces, linearity is trivially checked, *e.g.*, through a diagram chase. ◇

Still for $A \subseteq X_s$ and $B \subseteq X_r$, $s > r$, we define $A \cap B \subseteq X_s$ by $A \cap \iota_{sr}^{-1}(B)$, which is isomorphic to $\iota_{sr}(A) \cap B \subseteq X_r$. Finally, we define a system of subsets on X, $A \subseteq X$, to be a collection of subsets $A = \{A_s\}_{s\in S}$ with $A_s \subseteq X_s$ on each $s \in S$, such that $A_s \subseteq A_r$ for all $s > r$. It is called a system of open subsets if each A_s is open in X_s. Such a collection allows us to restrict the projective system to subsets by restricting the bonding maps $\iota_{sr} : A_s \hookrightarrow A_r$. For two Banach scales X and Y on S and two systems of subsets $A \subseteq X$ and $B \subseteq Y$, we define the cartesian product $A \times B := \{A_s \times B_s\}_{s\in S}$. If the domains of X and Y differ, say the domains are S and S', respectively, we restrict them to the common domain $S \cap S'$ before taking the cartesian product. We also define, for $\tau \in \mathbb{R}$, the shifted system $A^\tau := \{A_{s+\tau}\}_{s\in S-\tau}$. Unless otherwise mentioned, we do not use the complete shifted system on $S - \tau$, but restrict it to $S \cap (S - \tau)$ instead.

Before introducing Banach scales, we make a remark regarding bounded operators between Banach spaces. Unless otherwise mentioned, for Banach spaces X and Y, we endow $\mathrm{B}(X, Y) := \{L : X \mapsto Y \text{ linear and continuous}\}$ with the operator norm $\|L\| := \sup_{x\in X,\|x\|_X\le 1} \|L(x)\|_Y$. This norm makes $\mathrm{B}(X, Y)$ into a Banach space and its induced topology is called the strong or norm topology. $X^* := \mathrm{B}(X, \mathbb{F})$ then denotes the topological dual of X with operator norm. Yet, occasionally it will be convenient to work with a weaker topology on $\mathrm{B}(X, Y)$ called the compact-open topology. This is the topology with subbasis given by sets of the form $\{L \in \mathrm{B}(X, Y) : L(K) \subseteq U\}$, where $K \subseteq X$ is compact and $U \subseteq Y$ is open. The compact-open topology on $\mathrm{B}(X, Y)$ has the property that for a metric space A and a map $f : A \times X \mapsto Y$ with $f(a, \cdot) \in \mathrm{B}(X, Y)$ for all $a \in A$, f is continuous if and only if the induced map $\bar{f} : A \mapsto \mathrm{B}(X, Y)$, $a \mapsto f(a, \cdot)$ is continuous [26]. If $\mathrm{B}(X, Y)$ has the strong topology, only the "if" part is valid in general.

After the technical introduction, we are now ready to define Banach scales and the corresponding morphisms. We use real-indexed scales since they occur naturally in applications [43] and can be easily restricted to discrete scales, as shown further on in Proposition 2.5.

Definition 2.4. A Banach scale on S (over $\mathbb{F}$) is a projective system $X := \{X_s\}_{s\in S}$ of LCS where all X_s are Banach spaces, the bonding maps $\iota_{sr} : X_s \mapsto X_r$ are injective, $s > r \in S$, and

(a) the limit X_∞ is dense in X_s for all $s \in S$;

(b) the bonding maps $\iota_{sr} : X_s \mapsto X_r$, $s > r \in S$ are compact operators.

The space X_s is called the s^{th} layer or level of X, $s \in S$. If each X_s is completely normable but no specific norm is available, we call X a Banachable scale. If $S = -S$ and $0 \in S$, then X_0 is called the center of the scale[1]. Also, for a property of a Banach(able) space $\mathcal{P}$, *e.g.*, reflexivity or separability, X is said to have property $\mathcal{P}$ whenever all X_s have property $\mathcal{P}$, $s \in S$.

Compactness of the bonding maps is crucial in applications [32] and allows for a chain rule when we introduce calculus in this framework. We should also note that X_∞ is a Frèchet space due to the countable cofinality of S [29, 54]. A trivial example of a scale is the constant scale $X_s = X$, $s \in S$ for a finite-dimensional vector space X. In fact, this is

[1]This is not to be confused with the center as defined in [9], which is the limit of the scale instead.

the only possible scale if one of the spaces is finite-dimensional, since finite-dimensional subspaces of a normed space are closed. On the other hand, due to the compactness requirement, the same construction is not a scale if X is infinite-dimensional, unless I is a singleton. The same argument shows that all scales $X = \{X_s\}_{s\in S}$ built out of infinite-dimensional Banach spaces X_s are proper, *i.e.*, $X_s \subsetneq X_r$ for all $s > r$.

We proceed with two important propositions which allow us to construct new scales departing from old ones by means of natural operations such as restriction, translation, products and taking duals of the individual Banach spaces.

Proposition 2.5. *Let $X = \{X_s\}_{s\in S}$, $S \subseteq \mathbb{R}$ be a Banach scale and let $S' \subseteq S$ be a non-empty subset. Then $X|_{S'} := \{X_s\}_{s\in S'}$ is a Banach scale with bonding maps $\iota'_{sr} := \iota_{sr}$, $s > r \in S'$. Furthermore, if $s_0 := \sup_{\mathbb{R}} S' = \sup_{\mathbb{R}} S$ and $s_0 \in S \implies s_0 \in S'$ (where we allow $\infty = \infty$), then the limits $X_\infty := \lim_{s\in S} X_s$ and $X'_\infty := \lim_{s\in S'} X_s$ are uniquely isomorphic.*

Proof. It is clear that the projective system $\{X_s\}_{s\in S}$ restricts to an injective projective system on S' with compact bonding maps. To see that the limit X'_∞ is dense in each X_s, $s \in S'$, we distinguish two cases. If $s_0 := \sup_{\mathbb{R}} S' = \sup_{\mathbb{R}} S$ and $s_0 \in S \implies s_0 \in S'$, then $S' \subseteq S$ is cofinal, meaning that for each $r \in S$ there is $s \in S'$ such that $s \geq r$. This implies that the limits X_∞ and X'_∞ are uniquely isomorphic [28], hence X'_∞ is dense in X_s for all $s \in S'$.

On the other hand, if $\sup_{\mathbb{R}} S' < \sup_{\mathbb{R}} S$ or $\sup_{\mathbb{R}} S' \in S \backslash S'$, there exists $s_0 \in S$ such that $s_0 > s$ for all $s \in S'$. Now, $(X_{s_0}, \{\iota_{s_0 s}\}_{s\in S'})$ is a cone from X_{s_0} to the projective system on S', and by the universal property of X'_∞, there exists a unique map $\kappa_{s_0} : X_{s_0} \mapsto X'_\infty$ such that $\iota'_{\infty s}\kappa_{s_0} = \iota_{s_0 s}$ for all $s \in S'$. The injectivity of $\iota_{s_0 s}$ for some $s \in S'$ implies the injectivity of κ_{s_0}. Consequently, since X_{s_0} is dense in X_s for all $s \in S'$ by assumption, we conclude that also X'_∞ is dense in X_s for all $s \in S'$. ◇

Corollary 2.6. *If $\{X_s\}_{s\in S}$, $S \subseteq \mathbb{Z}$ is a discrete Banach scale with limit X_∞ and $S' \subseteq S$ is a non-empty subset, then the limit of the induced sub-scale is X_∞ if S' is unbounded above and $X_{\max S'}$ otherwise.*

Proof. If S' is unbounded above, then so is S and we are in the case $\sup_{\mathbb{R}} S' = \sup_{\mathbb{R}} S = \infty$. Otherwise, $\{\max S'\} \subseteq S'$ induces a sub-scale of the scale on S' which meets the condition of Proposition 2.5. ◇

Proposition 2.7. *Let $X := \{X_s\}_{s\in S}$ and $Y := \{Y_s\}_{s\in S}$ be Banach scales on $S \subseteq \mathbb{R}$ with (co-)limits $X_{\pm\infty}$, $Y_{\pm\infty}$ and maps ι^X_{sr} and ι^Y_{sr}, $s > r \in S \cup \{\pm\infty\}$, respectively. We introduce the following constructions:*

(a) (shifted scale) For $\tau \in \mathbb{R}$, $X^\tau := \{X_{s+\tau}\}_{s\in S-\tau}$ is a Banach scale on $S - \tau = \{s - \tau : s \in S\}$ with bonding maps $\iota^X_{s+\tau,r+\tau}$, $s > r \in S - \tau$, and limit $(X_\infty, \{\iota^X_{\infty,s+\tau}\}_{s\in S-\tau})$.

(b) (product of scales) $X \times Y := \{X_s \times Y_s\}_{s\in S}$ is a Banach scale on S with bonding maps $\iota^X_{sr} \times \iota^Y_{sr}$, $s > r$, and limit $(X_\infty \times Y_\infty, \{\iota^X_{\infty s} \times \iota^Y_{\infty s}\}_{s\in S})$.

(c) (dual scale) Endow X_s^* with the operator norm, $s \in S$. If each X_s is a reflexive Banach space, then $X_{-\infty}$ is a Hausdorff LCS and $X^* := \{X^*_{-s}\}_{s\in -S}$ is a Banach scale on $-S = \{-s : s \in S\}$ with bonding maps $(\iota^X_{-r,-s})^* : X^*_{-s} \hookrightarrow X^*_{-r}$, $s > r \in -S$, and limit $(X^*_{-\infty}, \{(\iota^X_{-s,-\infty})^*\}_{s\in -S})$.

Proof. The crux of (a) is that a cone to X induces a cone to X^τ and vice-versa by shifting indices by τ. Similarly, (b) is proven by adjoining the universal cones X_∞ and Y_∞ to a cone $X_\infty \times Y_\infty$ to $X \times Y$ and proving universality by noting that if $(Z, \{f_s\})$ is a cone to $X \times Y$, then $(Z, \{\mathrm{pr}^X_s f_s\})$ and $(Z, \{\mathrm{pr}^Y_s f_s\})$ are cones to X and Y, respectively, where $\mathrm{pr}^X_s : X_s \times Y_s \mapsto X_s$ and $\mathrm{pr}^Y_s : X_s \times Y_s \mapsto Y_s$ are the canonical projections. Also, products of compact operators are compact.

As to (c), first note that $(\iota^X_{-r,-s})^*$ is compact for $s > r \in -S$ since the adjoint of a compact operator is compact [53, Theorem 4.19]. Secondly, the map $(\iota^X_{-r,-s})^*$ is injective since $\iota^X_{-r,-s}$ has dense range [53, Theorem 4.12]. Thirdly, since X_{-s} and X_{-r} are reflexive, the double adjoint $(\iota^X_{-r,-s})^{**} = \iota^X_{-r,-s} : X^{**}_{-r} \cong X_{-r} \mapsto X_{-s} \cong X^{**}_{-s}$ is injective, which again by [53, Theorem 4.12] implies that $(\iota^X_{-r,-s})^*$ has dense range. Consequently, by [18, Proposition 2], the limit $(X^*)_\infty$ of X^* is dense in each X^*_{-s}, $s \in -S$ and by [18, Proposition 3], the colimit $X_{-\infty}$ of X is Hausdorff. Note that [18] assumes the scale to be discrete, *i.e.*, $S = \mathbb{N}$, but we can reduce to that case, since S and $-S$ have countable cofinality.

Finally, to compute the limit of the dual scale, we follow the approach of [46]. We denote by $\mathrm{Cocone}(X, \mathbb{F}) = \{\{f_s \in X^*_s\}_{s\in S} : f_s = f_r\, \iota^X_{sr} \text{ for all } s > r \in S\} \subseteq \prod X^*_s$ the vector space of co-cones from X to $\mathbb{F}$. On the one hand, the map $X^*_{-\infty} \mapsto \mathrm{Cocone}(X, \mathbb{F}), h \mapsto \{h\, \iota^X_{s,-\infty}\}_{s\in S}$ is an isomorphism of vector spaces due to the universal property of $X_{-\infty}$. On the other hand, we have the explicit construction[2] of projective limits in (2.1), which when applied to X^* gives

$$(X^*)_\infty = \left\{ y = (y_s)_{s\in -S} \in \prod_{s\in -S} X^*_{-s} : (\iota^X_{-r,-s})^*(y_s) = y_r \text{ for all } s > r \in -S \right\}. \tag{2.2}$$

Using this construction, $\mathrm{Cocone}(X, \mathbb{F}) \mapsto (X^*)_\infty$, $(z_s)_{s\in S} \mapsto (z_{-s})_{s\in -S}$ is again an isomorphism of vector spaces. One can easily see that the composition of the two isomorphisms intertwines the canonical projections $(X^*)_\infty \mapsto X^*_{-s}$, $y \mapsto y_s$ with $(\iota^X_{-s,-\infty})^*$ for $s \in -S$. We make it a homeomorphism by pulling the (Frèchet) topology of $(X^*)_\infty$ back to $X^*_{-\infty}$. ◇

Remark 2.8. (a) Although the shifted scale X^τ is defined on $S - \tau$, in the sequel of this document, we will restrict this type of scales to $S \cap (S - \tau)$, since the surplus spaces are not needed for our purposes.

(b) For the product scale, if $\{X_s\}_{s\in S}$ and $\{Y_s\}_{s\in S'}$ are defined on different subsets $S, S' \subseteq \mathbb{R}$, one restricts both scales to the overlap $S \cap S'$ before applying the product construction.

[2]The construction in the category of vector spaces is identical to the one in LCS.

After having introduced natural operations on scales, our next goal is to study maps between them. In the roughest case, a map between scales can be a non-linear map between each layer. Eventually, we can add more structure to the map and require each layer to be linear or even continuous. This gives rise to the notion of morphisms and isomorphisms of scales.

Definition 2.9. Let $X = \{X_s\}_{s\in S}$, $Y = \{Y_s\}_{s\in S}$ and $Z = \{Z_s\}_{s\in S}$ be Banach scales on $S \subseteq \mathbb{R}$, and let $A \subseteq X$, $B \subseteq Y$ and $C \subseteq Z$ be systems of subsets.

(a) A map between the two systems of subsets A and B is a family of functions $f := \{f_s : A_s \mapsto B_s\}_{s\in S}$ which satisfies the compatibility requirement $f_r|_{A_s} = f_s$ for all $s > r \in S$ (as maps from A_s to B_r). Explicitly, this means that the diagram

$$\begin{tikzcd} A_s \arrow[r, "f_s"] \arrow[d, hook, "\iota^X_{sr}"] & B_s \arrow[d, hook, "\iota^Y_{sr}"] \\ A_r \arrow[r, "f_r"] & B_r \end{tikzcd}$$

is required to commute for all $s > r$. When the systems of subsets are clear from context, f is simply said to be a map of scales or a scale map. A map between the (full) Banach scales X and Y is simply defined as a map between the trivial systems of subsets $A = X$ and $B = Y$.

(b) Composition of maps of scales is defined layer-wise: if $f : A \mapsto B$ and $g : B \mapsto C$ are maps of scales, then $g \circ f : A \mapsto C$ defined by $(g \circ f)_s = g_s \circ f_s : A_s \mapsto C_s$, $s \in S$, is a map of scales. We define the identity map $\mathrm{id}_A : A \mapsto A$, $(\mathrm{id}_A)_s = \mathrm{id}_{A_s}$.

(c) A map of scales $f : A \mapsto B$ is called injective, surjective or bijective if each $f_s : A_s \mapsto B_s$ is injective, surjective or bijective, respectively. A bijective map of scales $f : A \mapsto B$ defines an inverse map $f^{-1} : B \mapsto A$, $(f^{-1})_s = {f_s}^{-1}$ with $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

(d) A map of scales $L : A \mapsto B$ is called linear if $A_s \subseteq X_s$ and $B_s \subseteq Y_s$ are linear subspaces and all $L_s : A_s \mapsto B_s$ are linear.

(e) A map of scales $f : A \mapsto B$ is called continuous or sc^0 if $f_s : A_s \mapsto B_s$ is continuous for all $s \in S$, where A_s and B_s inherit the topologies of X_s and Y_s, respectively. It is an isometry if each f_s is an isometry.

(f) A morphism between X and Y, also called an sc-operator, is an sc^0 linear map $T : X \mapsto Y$. An isomorphism between X and Y, or an sc-isomorphism, is a morphism $T : X \mapsto Y$ for which $T_s : X_s \xmapsto{\sim} Y_s$ is an isomorphism of vector spaces for all $s \in S$. It is an isometric isomorphism if, in addition, it is an isometry.

Remark 2.10. (a) If $T : X \xmapsto{\sim} Y$ is an isomorphism of scales as defined above, the open mapping theorem implies that $T^{-1} : Y \mapsto X$, defined by $(T^{-1})_s = T_s^{-1}$, is also a morphism of scales.

(b) A morphism $T : X \mapsto Y$ defines a cone to Y, $\{T_s\, \iota_{\infty s} : X_\infty \mapsto Y_s\}_{s\in S}$, which by the universal property of Y_∞ defines a unique continuous linear map $\lim_{s\in S} T_s := T_\infty : X_\infty \mapsto Y_\infty$ such that $\iota^Y_{\infty s}\, T_\infty = T_s\, \iota^X_{\infty s}$ for all $s \in S$. Similarly, there is a unique continuous linear map $\mathrm{colim}_{s\in S} T_s := T_{-\infty} : X_{-\infty} \mapsto Y_{-\infty}$ such that $T_{-\infty}\, \iota^X_{s,-\infty} = \iota^Y_{s,-\infty}\, T_s$ for all s. These are isomorphisms of topological vector spaces if T is an sc-isomorphism.

(c) If X and Y admit dual scales, a morphism $T : X \mapsto Y$ defines an adjoint morphism $T^* := \{T^*_{-s}\}_{s\in -S} : Y^* \mapsto X^*$.

(d) If $r := \inf_{\mathbb{R}} S \in S$, any subset $A_r \subseteq X_r$ defines a system of subsets by putting $A_s := A_r \cap X_s$, $s \geq r$, and this system is open if $A_r \subseteq X_r$ is open. Similar considerations hold for $B_r \subseteq Y_r$. A function $f_r : A_r \mapsto B_r$ with $f_r(A_s) \subseteq B_s$ for all $s \geq r$ then defines a unique map of scales $f : A \mapsto B$ by Lemma 2.3 which is linear if f is linear. Note that although f_r injective implies f injective, the same cannot be said about surjectivity. A counterexample is the inclusion $I : X|_{2\mathbb{N}_0} \mapsto X$, $I_k = \iota^X_{2k,k} : X_{2k} \hookrightarrow X_k$, $k \in \mathbb{N}_0$, for any proper scale X on $\mathbb{N}_0$.

(e) For $s \in S$, we can extend the norm $\|\cdot\|_s$ of Y_s to a map $\|\cdot\|_s : Y_{-\infty} \mapsto \overline{\mathbb{R}}$, where we simply set $\|y\|_s := \infty$ for $y \in Y_{-\infty} \setminus Y_s$. In this way, we can identify Y_s with $\{y \in Y_{-\infty} : \|y\|_s < \infty\}$ as vector spaces and define, equivalently to Definition 2.9(f), a morphism between scales to be a linear map $T : X_\infty \mapsto Y_{-\infty}$ such that $\|T\|_s := \sup_{x\in X_\infty, \|x\|_s \leq 1} \|T(x)\|_s < \infty$ for all $s \in S$. Indeed, such a map T extends uniquely to a linear continuous $T_s : X_s \mapsto Y_s$ for each $s \in S$ and this collection satisfies $T_r|_{X_s} = T_s$ $(s > r)$. Conversely, a scale morphism $\{T_s\}_{s\in S}$ induces $T = \iota^Y_{s,-\infty}\, T_s\, \iota^X_{\infty s}$, which is well defined independently of $s \in S$ and satisfies the corresponding condition. This alternative definition mirrors [43].

(f) If X and Y are Banach scales on different sets, say S and S', respectively, we restrict $A \subseteq X$ and $B \subseteq Y$ to $S \cap S'$, and define maps of scales $A \mapsto B$ simply as being maps of scales $A|_{S\cap S'} \mapsto B|_{S\cap S'}$.

(g) *(induced scales)* If $X = \{X_s\}_{s\in S}$ is now simply a collection of Banach spaces without bonding maps *a priori*, $Y = \{Y_s\}_{s\in S}$ is a Banach scale, and we are given continuous linear isomorphisms $\Psi_s : X_s \overset{\sim}{\longmapsto} Y_s$ for $s \in S$, then there is a unique structure of a Banach scale on X such that $\Psi = \{\Psi_s\}_{s\in S}$ is an isomorphism of scales. This structure is obtained by pulling back the bonding maps of Y, and the required properties for a Banach scale on X are directly derived from the corresponding properties on Y. For instance, the bonding maps $\iota^X_{sr} = \Psi_r^{-1}\, \iota^Y_{sr}\, \Psi_s$ are compact, $s > r$, since ι^Y_{sr} is compact, and Ψ_s and Ψ_r^{-1} are continuous [14, Chapter VI, Proposition 3.5].

We conclude this section with some basic definitions of linear symplectic geometry for Banach spaces and scales. A (strong) symplectic form on a real Banach space X is a continuous skew-symmetric bilinear form $\omega : X \times X \mapsto \mathbb{R}$ such that the induced map $\iota_\omega : X \mapsto X^*$, $w \mapsto \omega(\cdot, w)$ is an isomorphism of locally convex spaces. The pair

(X, ω) is then called a symplectic Banach space. A symplectic Banach space is always reflexive, since $-(\iota_\omega^{-1})^* \circ \iota_\omega : X \xrightarrow{\sim} X^{**}$ is the canonical injection. Accordingly, if $X = \{X_s\}_{s\in S}$ is a Banach scale over the reals and on an index set $S \subseteq \mathbb{R}$ with $S = -S$, a symplectic structure on X is a skew-symmetric collection $\omega = \{\omega_s\}_{s\in S}$ of continuous and bilinear forms $\omega_s : X_s \times X_{-s} \mapsto \mathbb{R}$, $s \in S$, which induce an isomorphism of scales $\iota_\omega : X \xrightarrow{\sim} X^*$, $X_s \ni w \mapsto \omega_{-s}(\cdot, w) \in X^*_{-s}$. In this context, skew-symmetry means $\omega_s(v, w) = -\omega_{-s}(w, v)$ for all $v \in X_s$, $w \in X_{-s}$ and $s \in S$. Due to continuity, it is enough to check this condition for smooth vectors $v, w \in X_\infty$. Also, note that the existence of the dual scale X^* is an immediate consequence of the individual isomorphisms in ι_ω since, similarly as for the single Banach space, $-((\iota_\omega)_{-s}^{-1})^* \circ (\iota_\omega)_s : X_s \xrightarrow{\sim} X_s^{**}$ is the canonical injection. The pair (X, ω) is called a symplectic Banach scale.

Remark 2.11. By using the intrinsic identification of a symplectic Banach scale with its dual, we can extend single-sided Banach scales which are isomorphic to a given symplectic scale. Indeed, let (X, ω) be a symplectic Banach scale on $S = -S \subsetneq \mathbb{R}$, and let Y be a Banach scale on $S_{\geq 0} := S \cap \mathbb{R}_{\geq 0}$ with isomorphism of scales $\Psi : Y \xrightarrow{\sim} X|_{S_{\geq 0}}$. Define also $S_{>0} := S \cap \mathbb{R}_{>0}$. The adjoint of Ψ and ι_ω induce an isomorphism $(\underline{\Psi}^{-1})^* : (Y|_{S_{>0}})^* \xrightarrow{\sim} (X|_{S_{>0}})^* \overset{\iota_\omega}{\underset{\sim}{\leftarrow}} X|_{-S_{>0}}$. Define $Y_s := Y^*_{-s}$ and $\Psi_s := (\Psi_{-s}^{-1})^* : Y_s \xrightarrow{\sim} X_s$ for $s \in -S_{>0}$. As in Remark 2.10(g), we obtain bonding maps for the extended collection $Y = \{Y_s\}_{s\in S}$ making it into a Banach scale with extended isomorphism $\underline{\Psi} = \{\Psi_s\}_{s\in S} : Y \xrightarrow{\sim} X$. In particular, by setting $Y = X|_{S_{\geq 0}}$ and $\Psi = \mathrm{id}_X$, we see that a symplectic Banach scale (X, ω) is completely determined by its one-sided structure $X|_{S_{\geq 0}}$.

If (X, ω) and (Y, η) are symplectic Banach scales on $S = -S \subseteq \mathbb{R}$ and $S' \subseteq S$ is a subset, a morphism of scales $T : X|_{S'} \mapsto Y|_{S'}$ always induces a symplectic adjoint

$$T^{\omega,\eta} : Y|_{-S'} \overset{\iota_\eta}{\underset{\sim}{\mapsto}} (Y|_{S'})^* \xrightarrow{T^*} (X|_{S'})^* \overset{\iota_\omega}{\underset{\sim}{\leftarrow}} X|_{-S'} \tag{2.3}$$

uniquely defined by the relation $\eta(Tv, w) = \omega(v, T^{\omega,\eta} w)$ for all $v \in X_s$, $w \in Y_{-s}$ and $s \in S'$. If $S' \cap (-S') \neq \emptyset$, such a morphism T is called symplectic if $\eta(Tv, Tw) = \omega(v, w)$ for all $v \in X_s$, $w \in X_{-s}$ and $s \in S' \cap (-S')$. Again due to continuity, it is enough to check this for $v, w \in X_\infty \subseteq (X|_{S'})_\infty$. It is easy to see that a morphism T is symplectic if and only if $T^{\omega,\eta} \circ T = \mathrm{id}_{X|_{S'\cap(-S')}}$. Note that since $(T^{\omega,\eta})^{\eta,\omega} = T$, $T^{\omega,\eta}$ is symplectic if and only if $T \circ T^{\omega,\eta} = \mathrm{id}_{Y|_{S'\cap(-S')}}$. Also, both $T^{\omega,\eta} \circ T = \mathrm{id}_X$ and $T \circ T^{\omega,\eta} = \mathrm{id}_Y$ hold on $S' \cap (-S')$ if and only if T is an isomorphism of scales on $S' \cap (-S')$ which is symplectic. In that case, $T^{-1} = T^{\omega,\eta}$ on $S' \cap (-S')$ and we call T a linear symplectomorphism of scales.

2.2 Hilbert Scales

A Hilbert scale is a Banach scale $\{X_s\}_{s\in S}$, where each X_s is required to be a Hilbert space. This important special case of Banach scales arises naturally in symplectic geometry: starting from a complex separable Hilbert space X, we can define a linear symplectic form $X \times X \mapsto \mathbb{R}$ and a scale structure $\underline{X} = \{X_s\}_{s\in\mathbb{R}}$ with center $X_0 \cong X$. To differentiate between Hilbert spaces and scales and to avoid ambiguity, we will underline Hilbert scales and scale maps in this section.

We start with a prototypical example of a Hilbert scale, $\underline{l^2_\nu}$, which will characterize all Hilbert scales we will be dealing with. This is a scale on $\mathbb{R}$ with center $l^2 = \{x \in \mathbb{F}^{\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty\}$, and where we can identify spaces on the one side of the scale with the duals of their symmetric counterparts. Using this prototype we build, for every separable Hilbert space X, a scale $\underline{X}$ by pulling back the scale structure of $\underline{l^2_\nu}$ using the isometric isomorphism arising from a Hilbert basis $\{\phi_k\}_{k\in\mathbb{Z}}$.

Proposition 2.12. *Let $\nu \in \mathbb{R}^{\mathbb{Z}}_{>0}$ be a sequence with $\nu_n \to \infty$ as $|n| \to \infty$, define*

$$l^{2,s}_\nu := l^{2,s} := \left\{ x \in \mathbb{F}^{\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |x_n|^2 \nu_n^{2s} < \infty \right\} \tag{2.4}$$

for $s \in \mathbb{R}$ and endow this vector space with the (real or hermitian) inner product

$$\langle x, y\rangle_s := \sum_{n\in\mathbb{Z}} x_n \overline{y_n}\, \nu_n^{2s} . \tag{2.5}$$

Then each $l^{2,s}$ is a Hilbert space and $\underline{l^2_\nu} := \{l^{2,s}\}_{s\in\mathbb{R}}$ is a Hilbert scale on $\mathbb{R}$ with limit $l^{2,\infty} := \cap_{s\in\mathbb{R}} l^{2,s}$ and colimit $l^{2,-\infty} := \cup_{s\in\mathbb{R}} l^{2,s}$. Furthermore, the collection $F := \mathrm{Sp}\{\delta_k : k \in \mathbb{Z}\} \subseteq l^{2,\infty}$ of finite linear combinations of the standard basis $\delta_k(n) = \delta_{kn}$ is dense in each $l^{2,s}$, $s \in \mathbb{R} \cup \{\infty\}$.

Proof. Each $l^{2,s}$, $s \in \mathbb{R}$, is a weighted l^2 space, and consequently a Hilbert space. It is also clear that $l^{2,s} \subseteq l^{2,r}$ for $s > r \in \mathbb{R}$, hence $\underline{l^2_\nu} = \{l^{2,s}\}_{s\in\mathbb{R}}$ is a descending filtration with mentioned limit and colimit. Let $s \in \mathbb{R} \cup \{\infty\}$ and define, for $k \in \mathbb{N}$, the projection $p^k_s : l^{2,s} \mapsto F$, $x \mapsto x \cdot \mathbb{1}_{\{-k+1,-k+2,\dots,k-1\}}$, where the product $\cdot$ is pointwise and $\mathbb{1}_{(\cdot)} \in \{0,1\}^{\mathbb{Z}}$ is the indicator function. Then F is dense in $l^{2,s}$: if $s < \infty$, for $x \in l^{2,s}$, $\|x - p^k_s(x)\|_s \to 0$ as $k \to \infty$, and if $s = \infty$, $\|x - p^k_s(x)\|_{s'} \to 0$ for all $s' \in \mathbb{R}$. This implies, in particular, that $l^{2,\infty}$ is dense in $l^{2,s}$ for each $s \in \mathbb{R}$. Furthermore, for $\mathbb{R} \ni s > r$, let $\iota_{sr} : l^{2,s} \hookrightarrow l^{2,r}$ and $\iota_{\infty r} : l^{2,\infty} \hookrightarrow l^{2,r}$ be the inclusions and define $p^k_{sr} := \iota_{\infty r}\, p^k_s : l^{2,s} \mapsto l^{2,r}$, $k \in \mathbb{N}$. Fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $\nu_n^{2(r-s)} \le \epsilon$ for all $|n| \ge N$. For $x \in l^{2,s}$ with $\|x\|_s \le 1$, we have that whenever $k \ge N$,

$$\|(\iota_{sr} - p^k_{sr})(x)\|^2_r = \sum_{|n|\ge k} |x_n|^2 \nu_n^{2s} \nu_n^{2(r-s)} \le \epsilon \|x\|^2_s \le \epsilon . \tag{2.6}$$

From this, we conclude that $\|\iota_{sr} - p^k_{sr}\|_{\mathrm{B}(l^{2,s},l^{2,r})} \to 0$. Seen that p^k_{sr} is a finite rank operator for each $k \in \mathbb{N}$ and that the compact operators are a closed subset of $\mathrm{B}(l^{2,s}, l^{2,r})$, we conclude that ι_{sr} is compact. $\diamond$

Lemma 2.13. *Let $\nu \in \mathbb{R}^{\mathbb{Z}}_{>0}$ and $\underline{l^2_\nu}$ be as in Proposition 2.12. Define complex conjugation $\overline{(\cdot)} : l^{2,s} \mapsto l^{2,s}$ pointwise[3]. Then*

[3] This is a conjugate linear isomorphism which squares to the identity when $\mathbb{F} = \mathbb{C}$ and simply the identity map when $\mathbb{F} = \mathbb{R}$.

(a) For $s \in \mathbb{R}$, we have an ($\mathbb{F}$-)bilinear continuous pairing $l^{2,s} \times l^{2,-s} \mapsto \mathbb{F}$, $(x, y) \mapsto \langle x, \overline{y} \rangle_0 = \sum_{n \in \mathbb{Z}} x_n y_n$.

(b) The induced map $l^{2,-s} \mapsto (l^{2,s})^*$, $y \mapsto \langle \cdot, \overline{y} \rangle_0$ is an isometric isomorphism with inverse $D \mapsto \{D(\delta_n)\}_{n \in \mathbb{Z}}$ for each $s \in \mathbb{R}$. It induces an isometric isomorphism of scales $\underline{l^2_\nu} \xrightarrow{\sim} (\underline{l^2_\nu})^*$.

(c) We have an isomorphism of vector spaces $l^{2,-\infty} \xrightarrow{\sim} (l^{2,\infty})^*$ given by the same formula (and same inverse).

Proof. For the first statement, note that for $x \in l^{2,s}$ and $y \in l^{2,-s}$, Hölder's inequality gives

$$|\langle x, \overline{y} \rangle_0| \leq \sum_{n \in \mathbb{Z}} \big(|x_n| \nu_n^s\big)\big(|y_n| \nu_n^{-s}\big) \leq \|x\|_s \|y\|_{-s}\,, \tag{2.7}$$

so that $\langle \cdot, \overline{\cdot} \rangle_0$ is defined and continuous (it is clearly bilinear). As far as the second statement is concerned, by the above, with $y \in l^{2,-s}$, $\|\langle \cdot, \overline{y} \rangle_0\| \leq \|y\|_{-s}$. But also with $x_n := \overline{y_n} \nu_n^{-2s}$, we have $x = \{x_n\}_{n \in \mathbb{Z}} \in l^{2,s}$, $\|x\|_s = \|y\|_{-s}$ and $|\langle x, \overline{y} \rangle_0| = \|y\|_{-s}^2$. We thus conclude that $y \mapsto \langle \cdot, \overline{y} \rangle_0$ is an isometry. Now let $D = \langle \cdot, x \rangle_s \in (l^{2,s})^*$ for $x \in l^{2,s}$. Then $D(\delta_n) = \overline{x_n} \nu_n^{2s}$ and $\{\overline{x_n} \nu_n^{2s}\}_{n \in \mathbb{Z}} \in l^{2,-s}$, hence we have a well-defined candidate inverse map. The fact that it is indeed an inverse comes from the fact that $\mathrm{Sp}\{\delta_k : k \in \mathbb{Z}\} \subseteq l^{2,s}$ is dense (Proposition 2.12). Finally, since Hilbert spaces are reflexive, $(\underline{l^2_\nu})^*$ exists and the scale isomorphism is a direct consequence of the former isomorphisms.

To prove the last statement, we first note that the direct map is well defined, since $(l^{2,s})^* \subseteq (l^{2,\infty})^*$ for all $s \in \mathbb{R}$. For the inverse map, we use the fact that since $l^{2,\infty}$ is a limit of the Banach spaces $l^{2,s}$, any $D \in (l^{2,\infty})^*$ factors through some $l^{2,s}$ [29, Theorem 5.1.1]. Choose $D \in (l^{2,\infty})^*$ thus arbitrarily and let $s \in \mathbb{R}$ and $D_s \in (l^{2,s})^*$ be such that $D = D_s \, \iota_{\infty s}$. Then $D(\delta_n) = D_s(\delta_n) = y_n$, where $y \in l^{2,-s} \subseteq l^{2,-\infty}$ is the result of the (inverse) isomorphism in (b) applied to D_s. It follows that the candidate inverse is well-defined, and again we invoke Proposition 2.12 to complete the proof. ◇

Note that even if we choose $\mathbb{F} = \mathbb{C}$ in Proposition 2.12, the complex Hilbert spaces $l^{2,s}_{\mathbb{C}} = l^{2,s}$ can still be regarded as a *real* Hilbert spaces $(l^{2,s}_{\mathbb{C}})_{\mathbb{R}}$ by restricting scalar multiplication to the reals and using the inner product $\mathrm{Re}\{\langle \cdot, \cdot \rangle_s\}$. It is not difficult to see that $\{(l^{2,s}_{\mathbb{C}})_{\mathbb{R}}\}_{s \in \mathbb{R}}$ is still a Hilbert scale (over $\mathbb{R}$). The following corollary, which is needed to handle real symplectic forms on complex Hilbert spaces, extends Lemma 2.13 to this scale.

Corollary 2.14. *In the setting of Lemma* 2.13 *with $\mathbb{F} = \mathbb{C}$, let $l^{2,s}_{\mathbb{C}} = l^{2,s}_{\nu,\mathbb{C}}$ be Eq.* (2.4)*. Define $\tilde{\nu} \in \mathbb{R}^{\mathbb{Z}}_{>0}$ by $\tilde{\nu}_{2n} = \tilde{\nu}_{2n+1} := \nu_n$, $n \in \mathbb{Z}$, and let $l^{2,s}_{\mathbb{R}} = l^{2,s}_{\tilde{\nu},\mathbb{R}}$ be Eq.* (2.4) *with $\mathbb{F} = \mathbb{R}$ instead of $\mathbb{C}$ and $\tilde{\nu}$ instead of ν. Then we have a continuous pairing*

$$(l^{2,s}_{\mathbb{C}})_{\mathbb{R}} \times (l^{2,-s}_{\mathbb{C}})_{\mathbb{R}} \mapsto \mathbb{R},\ (x, y) \mapsto \mathrm{Re}\{\langle x, y \rangle_0\} = \mathrm{Re}\left\{\sum_{n \in \mathbb{Z}} x_n \overline{y_n}\right\} \tag{2.8}$$

which induces an isometric isomorphism.

Proof. By choosing the orthonormal basis of $(l^2_{\mathbb{C}})_{\mathbb{R}}$ given by $\tilde{\delta}_{2k} = \delta_k$, $\tilde{\delta}_{2k+1} = \mathrm{i}\delta_k$, $k \in \mathbb{Z}$, we obtain ($\mathbb{R}$-linear) isometric isomorphisms $(l^{2,s}_{\mathbb{C}})_{\mathbb{R}} \overset{\sim}{\longmapsto} l^{2,s}_{\mathbb{R}}$, $x \mapsto (\ldots, x^{\mathrm{R}}_{-1}, x^{\mathrm{I}}_{-1}, x^{\mathrm{R}}_0, x^{\mathrm{I}}_0, x^{\mathrm{R}}_1, x^{\mathrm{I}}_1, \ldots)$, $s \in \mathbb{R}$, where $x_n = x^{\mathrm{R}}_n + \mathrm{i}x^{\mathrm{I}}_n$ is the real-imaginary decomposition. Composing this map with the pairing of Lemma 2.13(a) delivers the desired pairing. $\diamond$

Now let X be a separable infinite-dimensional Hilbert space with orthonormal basis $\{\phi_k\}_{k\in\mathbb{Z}} \subseteq X$. This basis induces an isometric isomorphism $\Phi : X \overset{\sim}{\longmapsto} l^2, v \mapsto \{\langle v, \phi_k\rangle\}_{k\in\mathbb{Z}}$. Let also $\nu \in \mathbb{R}^{\mathbb{Z}}_{>0}$ be as in the above discussion. From X, we construct a Hilbert scale $\underline{X} = \{X_s\}_{s\in\mathbb{R}}$ with center $X_0 \cong X$ as follows. Restrict Φ to an isomorphism of vector spaces $\Phi_\infty := \Phi|_{X_\infty} : X_\infty \overset{\sim}{\longmapsto} l^{2,\infty}$, with $X_\infty := \Phi^{-1}(l^{2,\infty}) \subseteq X$, and pull the limit topology of $l^{2,\infty}$ back to X_∞. Subsequently, define the isomorphism $\Phi_{-\infty} : (X_\infty)^* \cong (l^{2,\infty})^* \overset{\sim}{\longmapsto} l^{2,-\infty}, D \mapsto \{D(\phi_n)\}_{n\in\mathbb{Z}}$ using the map of Lemma 2.13(c). Then, in a similar fashion, we can restrict $\Phi_{-\infty}$ to isomorphisms $\Phi_s := \Phi_{-\infty}|_{X_s} : X_s \overset{\sim}{\longmapsto} l^{2,s}$, with $X_s := \Phi^{-1}_{-\infty}(l^{2,s})$, and pull the inner product of $l^{2,s}$ back to X_s, $s \in \mathbb{R}$. By construction, $\underline{X} := \{X_s\}_{s\in\mathbb{R}}$ is a Hilbert scale and $\underline{\Phi} := \{\Phi_s\}_{s\in\mathbb{R}} : \underline{X} \overset{\sim}{\longmapsto} \underline{l^2_\nu}$ is an isometric isomorphism of scales. Consequently, all properties of $\underline{l^2_\nu}$ carry directly over to $\underline{X}$. The following proposition reveals some of these properties.

Proposition 2.15. *Let X be a separable infinite-dimensional Hilbert space with orthonormal basis $\{\phi_k\}_{k\in\mathbb{Z}}$ and corresponding isometric isomorphism $\Phi : X \overset{\sim}{\longmapsto} l^2$. Furthermore, let $\nu \in \mathbb{R}^{\mathbb{Z}}_{>0}$ as in Proposition 2.12, $\underline{X} = \{X_s\}_{s\in\mathbb{R}}$ be the corresponding induced Hilbert scale with isometric isomorphism of scales $\underline{\Phi} : \underline{X} \overset{\sim}{\longmapsto} \underline{l^2_\nu}$. To ease notation, pull complex conjugation back to X_s, i.e., define $\overline{(\cdot)} : X_s \mapsto X_s$, $v \mapsto \Phi_s^{-1}(\overline{\Phi_s v})$, $s \in \mathbb{R}$. Then the following holds:*

(a) For $s \geq 0$, we can make the identification

$$X_s \cong \Phi^{-1}(l^{2,s}) = \{v \in X : \|\Phi(v)\|_s < \infty\}\,. \tag{2.9}$$

(b) The limit of $\underline{X}$ is $\cap X_s \cong X_\infty$ as topological vector spaces ($D \overset{\sim}{\longmapsto} \sum_{n\in\mathbb{Z}} D(\phi_n)\phi_n$) and the colimit is $X_{-\infty} := \cup X_s = X^*_\infty$.

(c) We have a continuous pairing $X_s \times X_{-s} \mapsto \mathbb{F}$, $(v, w) \mapsto \langle v, \overline{w}\rangle_0 = \langle \Phi_s v, \overline{\Phi_{-s} w}\rangle_0$. This map induces an isometric isomorphism $X_{-s} \overset{\sim}{\longmapsto} (X_s)^*$ for each $s \in \mathbb{R}$, and hence an isometric isomorphism of scales $\underline{X} \overset{\sim}{\longmapsto} \underline{X}^*$, $w \mapsto \langle \cdot, \overline{w}\rangle_0$.

Proof. The proof boils down to composing obvious maps. For $s \geq 0$, $\Phi_s : X_s \overset{\sim}{\longmapsto} l^{2,s}$ and $\Phi|_{\Phi^{-1}(l^{2,s})} : \Phi^{-1}(l^{2,s}) \overset{\sim}{\longmapsto} l^{2,s}$ are vector space isomorphisms which are isometries by construction, provided we pull the inner product of $l^{2,s}$ back to the corresponding spaces. The isomorphism for the limit is obtained using the limit map $\lim_{s\in\mathbb{R}} \Phi_s : \cap_{s\in\mathbb{R}} X_s \overset{\sim}{\longmapsto} l^{2,\infty}$ and $\Phi_\infty : X_\infty \overset{\sim}{\longmapsto} l^{2,\infty}$. For the remaining statements, one can use Lemma 2.13. $\diamond$

Example 2.16. For the separable complex Hilbert space

$$L^2(S^1, \mathbb{C}) = \left\{u : S^1 \mapsto \mathbb{C} : u \text{ measurable and } \int_{S^1} |u(x)|^2 \,\mathrm{d}x < \infty\right\} \tag{2.10}$$

with Fourier basis $\left\{x \mapsto \frac{e^{ikx}}{\sqrt{2\pi}} : k \in \mathbb{Z}\right\}$, inner product $\langle u, v\rangle = \int_{S^1} u(x)\overline{v(x)}\,dx$, and sequence $\nu_n := (1+n^2)^{1/2}$, $n \in \mathbb{Z}$, the induced Hilbert scale is the scale of Levi-Sobolev spaces [29]

$$W^{s,2}(S^1,\mathbb{C}) = \left\{D \in C^\infty(S^1,\mathbb{C})^* : \sum_{n\in\mathbb{Z}} |D(x \mapsto (2\pi)^{-1/2}e^{inx})|^2(1+n^2)^s < \infty\right\}, \quad (2.11)$$

$s \in \mathbb{R}$, with the smooth functions $C^\infty(S^1,\mathbb{C}) = \lim_{k\in\mathbb{N}} C^k(S^1,\mathbb{C})$ as limit and the distributions $C^\infty(S^1,\mathbb{C})^*$ as colimit. For $s \geq 0$, these spaces are simply

$$W^{s,2}(S^1,\mathbb{C}) \cong \left\{u \in L^2(S^1,\mathbb{C}) : \sum_{n\in\mathbb{Z}} |\hat{u}_n|^2(1+n^2)^s < \infty\right\}, \quad (2.12)$$

where $\hat{u}_n = \frac{1}{\sqrt{2\pi}}\int_{S^1} u(x)e^{-inx}\,dx$ is the n^{th} Fourier coefficient of u and where we identify L^2 functions with the subspace $W^{0,2} \subseteq (C^\infty)^*$ of distributions by sending $u \in L^2$ to $(\varphi \mapsto \sum_{n\in\mathbb{Z}} \hat{u}_n\hat{\varphi}_n = \int_{S^1} u(x)\varphi(-x)\,dx) \in W^{0,2}$.

Remark 2.17. Similarly to Remark 2.11, a single-sided Hilbert scale which is isomorphic to $\underline{X}$ extends to a double-sided scale, but now using the isometric isomorphism of Proposition 2.15(c) induced by the inner product of X instead of a symplectic structure. Clearly, the extended isomorphism is isometric if and only if the original isomorphism is so. Again, $\underline{X}$ is completely determined by its one-sided structure $\underline{X}|_{\mathbb{R}_{\geq 0}}$.

Regarding linear symplectic geometry on Hilbert spaces, if $(X, \langle\cdot,\cdot\rangle)$ is a complex Hilbert space, it can be given the structure of a real Hilbert space $X_\mathbb{R} = X$ with inner product $\langle\cdot,\cdot\rangle_\mathbb{R} := \mathrm{Re}\{\langle\cdot,\cdot\rangle\}$ in a way similar to the discussion prior to Corollary 2.14. With this structure, $\omega = -\mathrm{Im}\{\langle\cdot,\cdot\rangle\}$ is a symplectic form on $X_\mathbb{R}$, denoted the standard symplectic form. It is compatible with the inner product $\langle\cdot,\cdot\rangle_\mathbb{R}$ and the complex structure $\mathrm{i} : X \mapsto X$, $v \mapsto \mathrm{i}v$, in the sense that $\omega = \langle \mathrm{i}\cdot,\cdot\rangle_\mathbb{R}$.

In the same way, if X is a separable complex Hilbert space and $\underline{X} = \{X_s\}_{s\in\mathbb{R}}$ is the construction of Proposition 2.15, we can regard each X_s as a real Hilbert space $(X_s)_\mathbb{R}$. Thereby, we obtain isometric ($\mathbb{R}$-linear) isomorphisms $\Phi_s : (X_s)_\mathbb{R} \xrightarrow{\sim} (l^{2,s}_\mathbb{C})_\mathbb{R}$, $s \in \mathbb{R}$, which give rise to a scale structure (over $\mathbb{R}$) on $\{(X_s)_\mathbb{R}\}_{s\in\mathbb{R}}$. By abuse of notation, we also denote this restriction-of-scalars scale by $\underline{X}$. By Corollary 2.14, the map $\langle\cdot,\cdot\rangle_{0,\mathbb{R}} := \mathrm{Re}\{\langle\cdot,\cdot\rangle_0\} : (X_s)_\mathbb{R} \times (X_{-s})_\mathbb{R} \mapsto \mathbb{R}$, $(v,w) \mapsto \mathrm{Re}\{\langle\Phi_s v, \Phi_{-s} w\rangle_0\}$ induces an isomorphism of scales, hence $\omega = \langle \mathrm{i}\cdot,\cdot\rangle_{0,\mathbb{R}} = -\mathrm{Im}\{\langle\cdot,\cdot\rangle_0\}$ furnishes $\underline{X}$ with a symplectic structure, denoted the standard symplectic structure on the scale $\underline{X}$ induced by X.

Example 2.18. For $X = l^2_\mathbb{C}$, the previous construction endows $\{(l^{2,s}_\mathbb{C})_\mathbb{R}\}_{s\in\mathbb{R}}$ with the symplectic structure

$$l^{2,s}_\mathbb{C} \times l^{2,-s}_\mathbb{C} \mapsto \mathbb{R},\ (x,y) \mapsto -\mathrm{Im}\left\{\sum_{n\in\mathbb{Z}} x_n\overline{y_n}\right\} \quad (2.13)$$

which pulls back for $N \in \mathbb{N}$ via the embedding $\mathbb{C}^{2N-1} \hookrightarrow l^{2,\infty}_\mathbb{C}$

$$x = (x_{-N+1}, x_{-N+2}, \ldots, x_{N-1}) \mapsto \begin{cases} x_n & \text{if } |n| < N \\ 0 & \text{otherwise} \end{cases} \quad (2.14)$$

to the standard symplectic form $\frac{\mathrm{i}}{2}\sum_{n=-N+1}^{N-1}\mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}$ on $\mathbb{C}^{2N-1}$. For Example 2.16, one pre-composes (2.13) with the Fourier transform $W^{s,2} \overset{\sim}{\longmapsto} l^{2,s}_{\mathbb{C}}$, $D \mapsto \big\{D\big(x \mapsto \frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}}\big)\big\}_{n\in\mathbb{Z}}$.

Remark 2.19. In view of Proposition 2.15(c) and the previous discussion, we see that symplectic forms on the real Hilbert scale $\underline{X}$ derived from a separable real or complex Hilbert space X are in 1-1 correspondence with automorphisms $J : \underline{X} \overset{\sim}{\longmapsto} \underline{X}$ satisfying $\langle J_\infty(v), w\rangle_{0,\mathbb{R}} = -\langle v, J_\infty(w)\rangle_{0,\mathbb{R}}$ for all $v, w \in X_\infty$. Such automorphisms are called anti self-adjoint in [43], seen that they equivalently satisfy $J = -J^* : \underline{X}^* \cong \underline{X} \mapsto \underline{X} \cong \underline{X}^*$. In this formulation, the standard symplectic structure on the scale $\underline{X}$ derived from a separable complex Hilbert space X is identified with its complex structure, *i.e.*, multiplication by $\mathrm{i} : \underline{X} \overset{\sim}{\longmapsto} \underline{X}$.

2.3 Scale Calculus

In this section, we introduce the notion of sc-smoothness by Hofer, Wysocki and Zehnder [32,35,36,38]. Although the original theory concerns Banach scales on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, later on when defining partial sc-derivatives, we will need to work with scales on a finite index-set $N + 1 := \{0, 1, \ldots, N\}$, $N \geq 1$. This need comes from the fact that partial differentiation works by fixing one of the coordinates, and when $f : X \times Y \mapsto Z$ is a scale map and $y \in Y_m$ is a fixed element, the map $f(\cdot, y)$ is only a scale map on layers $\{0, 1, \ldots, m\}$. Defining partial sc-differentiation is not only important for the convenience of more easily dealing with scale maps defined on a product scale, but it also plays a central role in expressing flow equations on scales. The latter will be done in Chapter 3. In the following presentation, we will thus slightly generalize the theory of Hofer et al. to include the possibility for finitely-indexed scales. For notational convenience, the statement "scale on ∞" will mean a scale on $\mathbb{N}_0$ and, as usual, $\infty + k = \infty$ for $k \in \mathbb{Z}$.

To put the new notion by Hofer et al. in context, we recall the classical notion of smoothness between Banach spaces. Let X and Y be real Banach spaces and let $U \subseteq X$ be open. A function $f : U \mapsto Y$ is said to be Frèchet differentiable if there exists a function $\mathrm{d}f : U \mapsto \mathrm{B}(X, Y)$ with

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - \mathrm{d}f(x)\cdot h\|_Y}{\|h\|_X} = 0\,, \tag{2.15}$$

for all $x \in U$, where $h \in X$ is sent to 0 in X. As usual, Frèchet differentiable functions are continuous. Note that the space $\mathrm{B}(X, Y)$ has the structure of a Banach space by using the operator norm on linear continuous maps $X \mapsto Y$, so that one can iterate the concept using $\mathrm{d}f$. Hence, the function f is said to be C^1 if it is differentiable and $\mathrm{d}f$ is C^0 (continuous) and we define, recursively, f to be C^{k+1} if $\mathrm{d}f$ is C^k, $k \geq 1$. The function f is then said to be C^∞ or smooth if it is C^k for all $k \in \mathbb{N}$. For $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$, $m, n \in \mathbb{N}$, this notion recovers standard multivariate calculus.

Before proceeding with the introduction of the new notion, recall that if X is a Banach scale on $N + 1$, $N \in \mathbb{N} \cup \{\infty\}$, and $U_0 \subseteq X_0$ is an open subset, U_0 induces an open system of subsets $U \subseteq X$ given by $U_k := U_0 \cap X_k$, $k \in N+1$. If Y is another Banach scale on $N+1$, a map of scales $f : U \mapsto Y$ can be seen as a map $f_0 : U_0 \mapsto Y_0$ such that $f(U_k) \subseteq Y_k$ for all $k \in N+1$ (cf. Remark 2.10(d)). We use this identification in what follows, always regarding scales and maps as their zeroth layers: $X \equiv X_0$, $U \equiv U_0$, $(f : U \mapsto Y) \equiv (f_0 : U_0 \mapsto Y_0)$, and so on. Finally, we recall that for $k \in N+1$, $U^k = \{U_{i+k}\}_{i \in N+1-k}$ is the k-shifted system of subsets. Clearly, it is an open system of subsets of the shifted scale $X^k = \{X_{i+k}\}_{i \in N+1-k}$ which is induced by $U_k = (U^k)_0$.

Definition 2.20. Let X and Y be real Banach scales on $N+1$, $N \in \mathbb{N} \cup \{\infty\}$, and let $U \subseteq X$ be open.

(a) The tangent bundle of U is defined as $\mathrm{T}U := U^1 \times X = \{U_{m+1} \times X_m\}_{m \in N}$. It is an open system of subsets $\mathrm{T}U \subseteq \mathrm{T}X = X^1 \times X$ induced by $U_1 \times X_0$.

(b) An sc^0 map $f : U \mapsto Y$ is said to be sc^1 if there exists an sc^0 map $\mathrm{D}f : \mathrm{T}U \mapsto Y$ which is linear in the second argument and such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - \mathrm{D}f(x,h)\|_{Y_0}}{\|h\|_{X_1}} = 0 \tag{2.16}$$

for all $x \in U_1$, where $h \in X_1$ is sent to 0 in X_1. We use the notation $\mathrm{D}_x f := \mathrm{D}f(x, \cdot) \in \mathrm{B}(X_m, Y_m)$ for $x \in U_{m+1}$, $m \in N$.

(c) For an sc^1 map $f : U \mapsto Y$, we define its tangent map $\mathrm{T}f : \mathrm{T}U \mapsto \mathrm{T}Y$, $(x,v) \mapsto (f(x), \mathrm{D}f(x,v))$, which is clearly sc^0.

(d) For $k \in N$, f is recursively defined to be sc^{k+1} if $\mathrm{T}f$ is sc^k. In that case, with $\mathrm{T}^{k+1}U := \mathrm{T}^k(\mathrm{T}U)$ and $\mathrm{T}^{k+1}Y := \mathrm{T}^k(\mathrm{T}Y)$, the $(k+1)^{\mathrm{st}}$ tangent map of f is defined as $\mathrm{T}^{k+1}f := \mathrm{T}^k(\mathrm{T}f) : \mathrm{T}^{k+1}U \mapsto \mathrm{T}^{k+1}Y$.

(e) The map f is said to be sc^∞ or sc-smooth if it is sc^k for every $k \in N+1$. Clearly, for N finite, an sc-smooth map is simply an sc^N map.

(f) For $U \subseteq X$ and $V \subseteq Y$ open and $k \in \{1, 2, \dots, N, \infty\}$, a bijective scale map $f : U \mapsto V$ is an sc^k-diffeomorphism if both $f : U \mapsto V \subseteq Y$ and $f^{-1} : V \mapsto U \subseteq X$ are sc^k. For $k = 0$, we adopt the terminology sc^0-homeomorphism instead.

Remark 2.21. (a) Re-interpreting Equation (2.16), we see that in fact we require $f|_{U_1} : U_1 \mapsto Y_0$ to be Frèchet differentiable with derivative $(\mathrm{D}_x f)|_{X_1} \in \mathrm{B}(X_1, Y_0)$ for all $x \in U_1$. As mentioned in the following Proposition 2.22, the definition of an sc^1 map actually implies that $f|_{U_1}$ is C^1.

(b) If we endow each $\mathrm{B}(X_m, Y_m)$ with the compact-open topology, we can re-interpret the sc^0 condition on $\mathrm{D}f$. We have that $\mathrm{D}f$ is sc^0 if and only if $U_{m+1} \mapsto \mathrm{B}(X_m, Y_m)$, $x \mapsto \mathrm{D}_x f$ is continuous for all $m \in N$.

(c) The tangent map $\mathrm{T}f$ is defined even if f is only assumed to be scale-differentiable: to define the latter, simply replace "sc^0 map $\mathrm{D}f$" by "scale map $\mathrm{D}f$" in Definition 2.20(b). In that case, $\mathrm{T}f$ is sc^0 if and only if $\mathrm{D}f$ is so. Actually, Corollary 2.28 below will prove that $\mathrm{T}f$ is sc^k if and only if $\mathrm{D}f$ is so, $k \in N$.

(d) We emphasize that both Df and $\mathrm{T}f$ are defined on scales with cardinality one less than the cardinality of the original scales X and Y. Hence, for scales on $N+1$, a map can be at most sc^N. Of course this is only relevant when N is finite.

Note that due to the density of X_1 in X_0, if the sc-derivative $\mathrm{D}f$ exists it is unique, and therefore the sc^k conditions are local, meaning that $f : U \mapsto Y$ is sc^k if and only if for each $x \in U$, there exists an open neighbourhood $V(x) \subseteq U$ such that $f|_V : V \mapsto Y$ is sc^k. For notational convenience later on, we define $\mathrm{sc}^k(U, Y) := \{\mathrm{sc}^k \text{ maps } U \mapsto Y\}$ for $k \in N+1$.

Hofer et al. examine several properties of sc-smoothness and derive results relating this new notion with classical Frèchet differentiability [32,35,36]. Here, we state an equivalent formulation of the sc^1 condition (Definition 2.20(b)) and present two essential results which make sc-smoothness compatible with scale shifting and composition of maps. The latter result is especially relevant when introducing an sc-smooth structure on a topological space so that differentiation behaves well while changing coordinate charts (Section 2.4).

Proposition 2.22 (in [36, Proposition 2.1]). *Let X and Y be real Banach scales on $N+1$, $N \in \mathbb{N} \cup \{\infty\}$, and $U \subseteq X$ be open. Then an sc^0 map $f : U \mapsto Y$ is sc^1 if and only if for $0 \le m < N$, the following holds:*

(a) $f|_{U_{m+1}} : U_{m+1} \mapsto Y_m$ is C^1.

(b) For $x \in U_{m+1}$, $\mathrm{d}(f|_{U_{m+1}})(x) \in \mathrm{B}(X_{m+1}, Y_m)$ extends to a continuous operator $\bar{\mathrm{d}}(f|_{U_{m+1}})(x) \in \mathrm{B}(X_m, Y_m)$. Equivalently, $\mathrm{d}(f|_{U_{m+1}})(x) : X_{m+1} \subseteq X_m \mapsto Y_m$ is continuous when X_{m+1} inherits the topology of X_m.

(c) the map $(\mathrm{D}f)_m : U_{m+1} \times X_m \mapsto Y_m$, $(x, \xi) \mapsto \bar{\mathrm{d}}(f|_{U_{m+1}})(x) \cdot \xi$ is continuous.

If these conditions hold, the sc-derivative $\mathrm{D}f : U^1 \times X \mapsto Y$ is simply $\{(\mathrm{D}f)_m\}_{m \in N}$.

Remark 2.23. If an sc^0 map $f : U \mapsto Y$ is such that each $f|_{U_m} : U_m \mapsto Y_m$ is C^1, $0 \le m < N$ then, since the bonding maps $U_{m+1} \hookrightarrow U_m$ are restrictions of linear continuous maps, we see that the conditions of Proposition 2.22 are fulfilled, where the extension $\bar{\mathrm{d}}(f|_{U_{m+1}})(x)$ is simply the original derivative $\mathrm{d}(f|_{U_m} : U_m \mapsto Y_m)(x)$. In particular, we thus observe that scale maps which are C^1 on each layer are sc^1.

Proposition 2.24 (in [36, Proposition 2.2]). *Let X and Y be real Banach scales on $N+2$, $N \in \mathbb{N} \cup \{\infty\}$, $U \subseteq X$ be open, and $1 \le k \le N$. If $f : U \mapsto Y$ is sc^k, then $f|_{U_1} : U^1 \mapsto Y^1$ is sc^k and $\mathrm{D}(f|_{U_1}) = (\mathrm{D}f)|_{\mathrm{T}U_1} : \mathrm{T}U^1 \mapsto Y^1$.*

Proposition 2.25 (Chain Rule [32, Theorem 2.16])**.** *Let X, Y and Z be real Banach scales on $N+1$, $N \in \mathbb{N}\cup\{\infty\}$, $U \subseteq X$ and $V \subseteq Y$ be open, and $1 \leq k \leq N$. Assume we are given two* sc^k *maps $f : U \mapsto Y$ and $g : V \mapsto Z$ with $f(U) \subseteq V$. Then $g \circ f : U \mapsto Z$ is* sc^k *and*

$$\mathrm{T}^m(g \circ f) = (\mathrm{T}^m g) \circ (\mathrm{T}^m f) \tag{2.17}$$

for all $1 \leq m \leq k$.

Proof. The proposition was proved in [32] for $k = 1$. For $k > 1$, induction using the definition of an sc^k map gives the result. ◇

As an application of these results, we derive basic properties of scale differentiation and prove sc smoothness of some template maps.

Lemma 2.26. *Let X, Y, Z and W be real Banach scales on $N+1$, $N \in \mathbb{N}\cup\{\infty\}$, $U \subseteq X$ and $V \subseteq Y$ be open, and $1 \leq k \leq N$. If $f : U \mapsto Z$ and $g : V \mapsto W$ are* sc^k*, then $f \times g : U \times V \mapsto Z \times W$ is* sc^k *and $\mathrm{D}(f \times g) = \mathrm{D}f \times \mathrm{D}g : \mathrm{T}U \times \mathrm{T}V \cong \mathrm{T}(U \times V) \mapsto Z \times W$. Consequently, $\mathrm{T}(f \times g) = \mathrm{T}f \times \mathrm{T}g : \mathrm{T}U \times \mathrm{T}V \cong \mathrm{T}(U \times V) \mapsto \mathrm{T}(Z \times W) \cong \mathrm{T}Z \times \mathrm{T}W$.*

Proof. For $(x, y) \in X_1 \times Y_1$, we have

$$\begin{aligned}\frac{\|f \times g((x,y)+(t,s)) - f \times g(x,y) - \mathrm{D}f \times \mathrm{D}g((x,y),(t,s))\|_{Z_0 \times W_0}}{\|(t,s)\|_{X_1 \times Y_1}} \leq \\ \frac{\|f(x+t) - f(x) - \mathrm{D}f(x,t)\|_{Z_0}}{\|t\|_{X_1}} + \frac{\|g(y+s) - g(y) - \mathrm{D}g(y,s)\|_{W_0}}{\|s\|_{Y_1}} \to 0 \end{aligned} \tag{2.18}$$

as $t \to 0$ in X_1 and $s \to 0$ in Y_1. Furthermore, the tangent map is $\mathrm{T}(f \times g) = \mathrm{T}f \times \mathrm{T}g$ which is sc^0. Finally, by induction, if the statement is assumed for some $1 \leq k < N$ and f and g are sc^{k+1}, then $\mathrm{T}(f \times g) = \mathrm{T}f \times \mathrm{T}g$ is sc^k by the induction hypothesis, hence $f \times g$ is sc^{k+1}. This inductive method of proving smoothness is called *bootstrapping.* ◇

Lemma 2.27. *Let X, Y, Z and W be real Banach scales on $N+1$, $N \in \mathbb{N}\cup\{\infty\}$, $U \subseteq X$ and $V \subseteq Y$ be open, and $1 \leq k \leq N$.*

(a) The inclusion $\iota : U \mapsto X$ is sc^N *and $\mathrm{D}_x\iota = \mathrm{id}_{X_0}$ for all $x \in U_1$.*

(b) For $y \in Y_N$, $\mathrm{const}_y : U \mapsto Y$, $x \mapsto y$ is sc^N *and $\mathrm{D}_x(\mathrm{const}_y) = 0$ for all $x \in U_1$.*

(c) An sc^0*-operator $L : X \mapsto Y$ is sc^N and $\mathrm{D}_x L = L \in \mathrm{B}(X_0, Y_0)$ for all $x \in X_1$.*

(d) An sc^0 *bilinear map $B : X \times Y \mapsto Z$ is sc^N and $\mathrm{D}B((x,y),(\xi,\eta)) = B(\xi, y) + B(x, \eta)$ for all $(x, y) \in X_1 \times Y_1$ and $(\xi, \eta) \in X_0 \times Y_0$.*

(e) (Coordinate Restriction) If $f : U \times V \mapsto Z$ is sc^k *and $y \in V_N$, then $f(\cdot, y) : U \mapsto Z$ is* sc^k *and $\mathrm{D}[f(\cdot, y)] = \mathrm{D}f(\cdot, y, \cdot, 0) : \mathrm{T}U \mapsto Z$.*

(f) (Diagonal Product) If $f : U \mapsto Y$ and $g : U \mapsto Z$ are sc^k*, then $f \times_{\mathrm{diag}} g : U \mapsto Y \times Z$, $x \mapsto (f(x), g(x))$ is* sc^k *and $\mathrm{D}_x(f \times_{\mathrm{diag}} g) = \mathrm{D}_x f \times_{\mathrm{diag}} \mathrm{D}_x g$ for all $x \in U_1$.*

(g) *(Linearity of the* sc*-Derivative)* $\mathrm{sc}^k(U,Y)$ *is a linear subspace of* $\mathrm{sc}^0(U,Y)$ *and* $\mathrm{D} : \mathrm{sc}^1(U,Y) \mapsto \mathrm{sc}^0(\mathrm{T}U,Y)$, $f \mapsto \mathrm{D}f$ *is linear. Hence,* $\mathrm{T} : \mathrm{sc}^k(U,Y) \mapsto \mathrm{sc}^{k-1}(\mathrm{T}U,\mathrm{T}Y)$ *is linear.*

(h) *(Leibniz Rule) If* $f : U \mapsto Y$ *and* $g : U \mapsto Z$ *are* sc^k *and* $B = \cdot : Y \times Z \mapsto W$ *is* sc^0 *bilinear, then* $f \cdot g : U \mapsto W$, $x \mapsto f(x) \cdot g(x)$ *is* sc^k *and* $\mathrm{D}(f \cdot g)(x,\xi) = \mathrm{D}f(x,\xi) \cdot g(x) + f(x) \cdot \mathrm{D}g(x,\xi)$ *for all* $x \in U_1$ *and* $\xi \in X_0$.

Proof. Beginning with (c), we have $\|L(x+h) - L(x) - \mathrm{D}L(x,h)\|_{Y_0} = 0$ for all $x, h \in X_1$ by linearity, hence the Frèchet condition is satisfied. The tangent map is given by $\mathrm{T}L = L|_{X_1} \times L : X^1 \times X \mapsto Y^1 \times Y$, which is sc^0. To prove smoothness, we again use bootstrapping: if L is sc^k for $k < N$, then $\mathrm{T}L$ is also sc^k by Lemma 2.26 and Proposition 2.24, whence L is sc^{k+1}. In other words, $\mathrm{T}L$ is as smooth as L is. One can prove (a) and (b) using the same methodology.

For (f), we note that $f \times_{\mathrm{diag}} g = (f \times g) \circ \mathrm{diag}_X|_U$, where $\mathrm{diag}_X : X \mapsto X \times X$, $x \mapsto (x,x)$ is the (sc^0 linear) diagonal map, and use the chain rule (Proposition 2.25), Lemma 2.26 and Lemma 2.27(a),(c). Similarly, regarding (g), $\mathrm{sc}^k(U,Y)$ is an (additive) subgroup of $\mathrm{sc}^0(U,Y)$ and the derivative is additive, since for sc^k maps $f, g : U \mapsto Y$, $f + g$ factors as $a \circ (f \times_{\mathrm{diag}} g)$, where $a : Y \times Y \mapsto Y$ is the (sc^0 linear) vector space addition. Regarding (e), it is clear that $y \in V_N$ induces a scale map $f(\cdot, y)$, since $V_N \subseteq V_m$ for $0 \le m < N+1$ and f is a scale map. This map factors as $f \circ (\mathrm{id}_U \times \mathrm{const}_y) \circ \mathrm{diag}_X|_U$.

The Frèchet condition of (d) is satisfied, since for some $K > 0$

$$\frac{\|B(x+t, y+s) - B(x,y) - B(t,y) - B(x,s)\|_{Z_0}}{\|(t,y)\|_{X_1 \times Y_1}} \le K \left(\frac{1}{\|t\|_{X_1}} + \frac{1}{\|s\|_{Y_1}} \right)^{-1} \to 0 \tag{2.19}$$

as $t \to 0$ in X_1 and $s \to 0$ in Y_1. Again, $\mathrm{T}B$ is as smooth as B is, since it is a composition of above proven constructions derived from B. The Leibniz rule is an easy corollary stemming from $f \cdot g = B \circ (f \times_{\mathrm{diag}} g)$ and the homogeneity of (g) is a corollary thereof with $B : \mathbb{F} \times Y \mapsto Y$ given by the scalar multiplication of Y and the maps $\mathrm{const}_\alpha : U \mapsto \mathbb{F}$ for $\alpha \in \mathbb{F}$ and $f : U \mapsto Y$. ◇

Corollary 2.28. *Let* X *and* Y *be real Banach scales on* $N+1$, $N \in \mathbb{N} \cup \{\infty\}$, $U \subseteq X$ *be open, and* $f : U \mapsto Y$ *be an* sc^1 *map. Then for* $k \in N$, $\mathrm{T}f$ *is* sc^k *if and only* $\mathrm{D}f$ *is* sc^k.

Proof. We have $\mathrm{D}f = \mathrm{pr}_2^{\mathrm{T}Y} \circ \mathrm{T}f : \mathrm{T}U \mapsto Y$ and $\mathrm{T}f = ((f|_{U_1} \circ \mathrm{pr}_1^{\mathrm{T}U}) \times \mathrm{D}f) \circ \mathrm{diag}_{\mathrm{T}U} : \mathrm{T}U \mapsto \mathrm{T}Y$, where $\mathrm{pr}_2^{\mathrm{T}Y} : Y^1 \times Y \mapsto Y$ and $\mathrm{pr}_1^{\mathrm{T}U} : U^1 \times X \mapsto U^1$ are the canonical projections. ◇

As the coordinate restriction property of Lemma 2.27(e) indicates, if $f : U \times V \mapsto Z$ is an sc^1 map, where the scales are on $N+1$, then $f(\cdot, y) : U|_{m+2} \mapsto Z$ is an sc^1 map for every $y \in V_{m+1}$, $0 \le m < N$. Differentiating these maps in the scale sense, we obtain an sc^0 map $\frac{\partial f}{\partial x} : U^1 \times V^1 \times X \mapsto Z$ given by $\frac{\partial f}{\partial x}(x,y,\xi) := \mathrm{D}[f(\cdot,y)](x,\xi) = \mathrm{D}f(x,y,\xi,0) \in Z_m$ for $x \in U_{m+1}$, $y \in V_{m+1}$ and $\xi \in X_m$. Similarly, $f(x,\cdot) : V|_{m+2} \mapsto Z$ is sc^1 for $x \in U_{m+1}$

and we obtain an sc^0 map $\frac{\partial f}{\partial y} : U^1 \times V^1 \times Y \mapsto Z$. We label these maps as the partial sc-derivatives of f with respect to x and y, respectively.

An interesting question to ask is whether the converse of the last paragraph holds. Given an sc^0 map $f : U \times V \mapsto Z$ where $f(\cdot, y)$ and $f(x, \cdot)$ are all sc^1, we wish to investigate whether we can expect f to be (jointly) sc^1. As above, we can define partial sc-differentiation and answer the question positively, provided that the partial sc-derivatives are *jointly* continuous. Without this extra condition the claim is false, since it fails even in finite dimensions [19, Chapter II, Example 4].

Proposition 2.29. *Let X, Y and Z be real Banach scales on $N+1$, $N \in \mathbb{N} \cup \{\infty\}$, $U \subseteq X$ and $V \subseteq Y$ be open, and $f : U \times V \mapsto Z$ be an sc^0 map.*

(a) If $f(\cdot, y) : U|_{m+2} \mapsto Z$ is sc^1 for every $y \in V_{m+1}$, $0 \le m < N$, then $\frac{\partial f}{\partial x} : U^1 \times V^1 \times X \mapsto Z$ defined by $\frac{\partial f}{\partial x}(x, y, \xi) := \mathrm{D}[f(\cdot, y)](x, \xi) \in Z_m$ for $x \in U_{m+1}$, $y \in V_{m+1}$ and $\xi \in X_m$, is a scale map. It is linear in the third argument and $\frac{\partial f}{\partial x}(\cdot, y, \cdot)$ is sc^0 for all $y \in V_{m+1}$. Similarly, if $f(x, \cdot) : V|_{m+2} \mapsto Z$ is sc^1 for every $x \in U_{m+1}$, we have a well-defined scale map $\frac{\partial f}{\partial y} : U^1 \times V^1 \times Y \mapsto Z$. In general, these maps need not be (jointly) sc^0, even in the finite-dimensional case.

(b) For $0 \le k < N$, f is sc^{k+1} if and only if $f(\cdot, y)$ and $f(x, \cdot)$ are sc^1 for every $x \in U_{m+1}$ and $y \in V_{m+1}$, $0 \le m < N$, and the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are sc^k. If this holds, then the total sc-derivative $\mathrm{D}f : U^1 \times V^1 \times X \times Y \mapsto Z$ is given by $\mathrm{D}f(x, y, \xi, \eta) = \frac{\partial f}{\partial x}(x, y, \xi) + \frac{\partial f}{\partial y}(x, y, \eta)$.

Proof. Assume that the hypothesis of (a) holds and let $1 \le n < N$, $0 \le m < n$ and $y \in V_{n+1}$. Since f is a scale map, we have $f(\cdot, y \in V_{n+1}) = f(\cdot, y \in V_{m+1}) : U|_{m+2} \mapsto Z$ and the sc-derivatives of the two functions coincide on their common scale domain. But also $\mathrm{D}[f(\cdot, y \in V_{n+1}) : U|_{n+2} \mapsto Z]$ is a scale map and we obtain for $x \in U_{n+1}$ and $\xi \in X_n$

$$\begin{aligned} Z_m \ni \mathrm{D}[f(\cdot, y \in V_{m+1}) : U|_{m+2} \mapsto Z](x \in U_{m+1}, \xi \in X_m) = \\ Z_m \ni \mathrm{D}[f(\cdot, y \in V_{n+1}) : U|_{n+2} \mapsto Z](x \in U_{m+1}, \xi \in X_m) = \\ \mathrm{D}[f(\cdot, y \in V_{n+1}) : U|_{n+2} \mapsto Z](x \in U_{n+1}, \xi \in X_n) \in Z_n \subseteq Z_m\,, \end{aligned} \tag{2.20}$$

which means that $\frac{\partial f}{\partial x}$ is a well-defined scale map.

As to (b), the "only if" part is clear from the discussion previous to the proposition. The formula for the total sc-derivative is then a direct consequence of the linearity of $\mathrm{D}f(x, y, \cdot) : X \times Y \mapsto Z$ for $x \in U_1$ and $y \in V_1$. The "if" part is based on [62, Proposition 4.14]. Assume the right-hand side of the claimed equivalence now and define the candidate derivative $\mathrm{D}f$ as in the postulated formula. It is clear that $\mathrm{D}f$ (hence also $\mathrm{T}f$) is sc^k if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are so, so that only the Frèchet condition of sc-differentiation remains to be shown. For $x \in U_1$, $y \in V_1$ and $t \in X_1$ and $s \in Y_1$ small enough, we have

$$\frac{\|f(x+t,y+s) - f(x,y) - \frac{\partial f}{\partial x}(x,y,t) - \frac{\partial f}{\partial y}(x,y,s)\|_{Z_0}}{\|t\|_{X_1} + \|s\|_{Y_1}} \leq$$
$$\frac{\|f(x+t,y+s) - f(x,y+s) - \frac{\partial f}{\partial x}(x,y+s,t)\|_{Z_0}}{\|t\|_{X_1}} + \frac{\|\frac{\partial f}{\partial x}(x,y+s,t) - \frac{\partial f}{\partial x}(x,y,t)\|_{Z_0}}{\|t\|_{X_1}} +$$
$$\frac{\|f(x,y+s) - f(x,y) - \frac{\partial f}{\partial y}(x,y,s)\|_{Z_0}}{\|s\|_{Y_1}}. \tag{2.21}$$

Since by Proposition 2.22(a), $f(\cdot, y+s)$ is C^1 for all s small, we may rewrite the first term of (2.21) and conclude

$$\frac{\| \int_0^1 \left[\frac{\partial f}{\partial x}(x+\alpha t, y+s,t) - \frac{\partial f}{\partial x}(x,y+s,t)\right] \mathrm{d}\alpha\|_{Z_0}}{\|t\|_{X_1}} \leq$$
$$\sup_{\alpha\in[0,1]} \left\| \frac{\partial f}{\partial x}(x+\alpha t, y+s,\cdot)|_{X_1} - \frac{\partial f}{\partial x}(x,y+s,\cdot)|_{X_1} \right\|_{\mathrm{B}(X_1,Z_0)} \to 0 \tag{2.22}$$

as $(t,s) \to 0$ in $X_1 \times Y_1$ since, due to the same argument as in [35, Lemma 2.6], the map $U_1 \times V_1 \mapsto \mathrm{B}(X_1, Z_0)\,, (x,y) \mapsto \frac{\partial f}{\partial x}(x,y,\cdot)|_{X_1}$ is continuous. For the second term of (2.21) a similar argument applies, and the third term vanishes as $s \to 0$ by definition of the sc-derivative of $f(x,\cdot)$. $\diamond$

As a last examination of the properties of sc-smoothness, we now turn our attention to the special case of sc-calculus when the domain scale $X = \mathbb{R}$ is the real one-dimensional constant scale. This kind of maps will be used when dealing with sc-smooth flows in Chapter 3. To begin with, note that for a real Banach scale Y on $N+1$, $N \in \mathbb{N} \cup \{\infty\}$, and $A \subseteq \mathbb{R}$, scale (sc^0) maps $A \mapsto Y$ are in 1-1 correspondence to plain one-layer (continuous) maps $A \mapsto Y_N$, since all elements of A are smooth. The correspondence is obtained by sending a scale (sc^0) map $f : A \mapsto Y$ to its limit map $\lim_{k\in N+1} f_k : A \mapsto Y_N$. Furthermore, for a general Banach scale X' on $N+1$ and $U' \subseteq X'$ open, scale maps $g : U' \times \mathbb{R} \mapsto Y$ which are linear in the second argument are in 1-1 correspondence with scale maps $\bar{g} : U' \mapsto Y$ by sending g to $\bar{g} = g(\cdot, 1)$ and g is sc^k if and only if $\bar{g}$ is so, $k \geq 0$. This allows us to simplify the treatment of the sc-derivative of sc^1 maps $f : U \subseteq \mathbb{R} \mapsto Y_N$ by analysing $\mathrm{D}f(\cdot, 1) : U \mapsto Y_{N-1}$ instead of $\mathrm{D}f$. The following proposition, which ends the section, relates classical smoothness with sc-smoothness of maps $f : U \subseteq \mathbb{R} \mapsto Y_N$.

Proposition 2.30. *Let Y be a real Banach scale on $N+1$, $N \in \mathbb{N} \cup \{\infty\}$, $U \subseteq \mathbb{R}$ be open, and $k \in N+1$. Then a scale map $f : U \subseteq \mathbb{R} \mapsto Y_N$ is sc^k if and only if $f : U \subseteq \mathbb{R} \mapsto Y_{N-l}$ is C^l (in the classical sense[4]) for $0 \leq l \leq k$. If this holds and*

[4] For the case $N = \infty$, $Y_N = Y_\infty$ is a Frèchet space. One can still talk about continuous differentiability of maps $f : U \subseteq \mathbb{R} \mapsto Y_\infty$ by, as usual, requiring the existence of a continuous map $\frac{\mathrm{d}f}{\mathrm{d}t} : U \mapsto Y_\infty$ with $\frac{f(t+h)-f(t)}{h} \to \frac{\mathrm{d}f}{\mathrm{d}t}(t)$ in Y_∞ as $h \to 0$, for all $t \in U$. f is recursively defined to be C^{k+1} if $\frac{\mathrm{d}f}{\mathrm{d}t}$ is C^k, $k \geq 1$.

$k \geq 1$, the sc*-derivative* $\mathrm{D}f(\cdot, 1) : U \mapsto Y_{N-1}$ *and the classical derivative* $\frac{\mathrm{d}f}{\mathrm{d}t} : U \mapsto Y_{N-1}$ *coincide. In particular, if* $N = \infty$*,* $f : U \subseteq \mathbb{R} \mapsto Y_\infty$ *is* sc^∞ *if and only if it is* C^∞*.*

Proof. The case $k = 0$ is clear. For $k = 1$, by Proposition 2.22, if f is sc^0, it is sc^1 if and only if $f : U \mapsto Y_m$ is C^1 for $0 \leq m < N$. If this holds, the sc-derivative is simply the classical derivative $\frac{\mathrm{d}f}{\mathrm{d}t} : U \mapsto Y_{N-1}$ upon the above described identifications. In turn, all $f : U \mapsto Y_m$ being C^1 is equivalent to $f : U \mapsto Y_{N-1}$ being C^1, since the bonding maps $\iota^Y_{N-1,m}$ are linear continuous, $m \in N - 1$. Note that this equivalence is still true when $N = \infty$, since the topology of Y_∞ is generated by the norms of Y_m, $m \geq 0$. Finally, inductively for $1 \leq k < N$ and f in sc^1, f in sc^{k+1} is equivalent to $\frac{\mathrm{d}f}{\mathrm{d}t} : U \mapsto Y_{N-1}$ in sc^k, which is in turn equivalent to $f : U \mapsto Y_{N-1-l}$ in C^{l+1} for $0 \leq l \leq k$ by the induction hypothesis. ◇

2.4 Scale Manifolds

In a way similar to standard differential geometry [45,63], we can generalize sc-smoothness to topological spaces which are locally modeled on a real Banach scale. We begin by introducing the notion of an sc^∞-manifold [32,36] and show subsequently that every sc^∞-manifold gives rise to a filtration induced by the local scale structure. Next, we define the tangent scale at each point of a filtration subspace: a finitely-indexed Banach scale for which each choice of a coordinate chart gives rise to an isomorphism to the local model. Finally, we define tangent bundles, sc-smooth maps between sc^∞-manifolds and corresponding tangent maps.

Definition 2.31. Let M be a Hausdorff space and X be a real Banach scale on $\mathbb{N}_0$.

(a) A coordinate chart is a pair (U, ϕ), where $U \subseteq M$ is an open subset and $\phi : U \xrightarrow{\sim} \phi(U) \subseteq X_0$ is a homeomorphism onto an open subset $\phi(U)$ of X_0. U is called the coordinate domain of (U, ϕ), whereas ϕ is called the coordinate map.

(b) Two coordinate charts (U, ϕ) and (U', ψ) are called sc^∞-compatible whenever $U \cap U' = \emptyset$ or the transition map $\psi \circ \phi^{-1} : \phi(U \cap U') \mapsto \psi(U \cap U')$ is an sc^∞-diffeomorphism.

(c) An sc^∞-atlas is a collection $\mathcal{A} = \{(U_a, \phi_a)\}_{a \in A}$ of coordinate charts such that $\cup_{a \in A} U_a = M$ and (U_a, ϕ_a) is sc^∞-compatible with (U_b, ϕ_b) for all $a, b \in A$.

(d) An sc^∞-atlas $\mathcal{A}$ is called maximal whenever for all sc^∞-atlases $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B}$, we have $\mathcal{B} = \mathcal{A}$.

(e) An sc-smooth structure for M is a maximal sc^∞-atlas $\mathcal{A}$ of charts from open subsets of M onto open subsets of X_0 as above. The pair $(M, \mathcal{A})$ is called an sc^∞-manifold (locally) modeled on X. Usually, $\mathcal{A}$ is suppressed from notation.

Remark 2.32. (a) Replacing ∞ by k, one has the definition of an sc^k-manifold, $k \in \mathbb{N}_0$. Note that sc^k-manifolds are Banach manifolds as well.

(b) With the same proof as in the finite-dimensional case, an sc^∞-atlas $\mathcal{A}$ admits a unique maximal atlas $\overline{\mathcal{A}}$ with $\mathcal{A} \subseteq \overline{\mathcal{A}}$ [45, Proposition 1.17(a)]. Hence, for defining an sc^∞-manifold, we only need to specify some (eventually non-maximal) atlas.

(c) It follows from the definition of an sc^∞-manifold that a basis for the topology of M is given by sets of the form $\phi^{-1}(V)$, where (U, ϕ) is a coordinate chart and $V \subseteq \phi(U)$ is open.

(d) If M is a set instead of a topological space, one can modify the definition requiring, for each coordinate chart (U, ϕ), the coordinate domain U only to be a subset of M, ϕ to be a bijection onto an open subset $\phi(U)$ of X_0, and $\phi(U \cap U') \subseteq \phi(U)$ to be open for all coordinate domains $U' \subseteq M$, while maintaining the sc^∞ compatibility condition. The basis in (c) then generates a unique topology on M such that the coordinate domains are open in M and the coordinate maps are homeomorphisms. Consequently, if M endowed with this topology is Hausdorff, the coordinate charts induce an sc-smooth structure on M.

(e) It is possible to extend the definition to allow for the Banach scale X to be dependent on the coordinate chart, giving triples (U, ϕ, X) as trivializations. It is not difficult to see that, for a given Banach scale X_{ref}, the set of points $p \in M$ for which there exists a chart $(U(p), \phi, X)$ around p with $X \cong X_{\mathrm{ref}}$ (as Banach scales) is clopen. By restricting our attention to each connected component of M separately, we may thus assume that $X = X_{\mathrm{ref}}$ is fixed for all charts (U, ϕ, X).

(f) If one needs additional properties for the sc^∞-manifolds, it is possible to strengthen the topological requirements. For instance, if one is interested in partitions of unity, one should require M to be metrizable: by Smirnov's metrization theorem, a locally metrizable space is paracompact Hausdorff if and only if it is metrizable [50, Theorem 42.1 and Theorem 41.4]. An even stronger assumption is to require M (and X) to be separable metrizable. This assumption facilitates constructive proofs, limits the cardinality of the spaces involved and is equivalent to the conditions regular Hausdorff and second-countable [50, Theorem 32.2, Exercise 30.5(a), Theorem 34.1 and Theorem 30.3(b)]. In this thesis, we only assume sc^∞-manifolds to be Hausdorff, since this property is easily shown to be inherited by each member of the filtration induced by an sc^∞-manifold M (Lemma 2.33).

If M is an sc^∞-manifold, (U, ϕ) is a coordinate chart and we define $V := \phi(U) \subseteq X_0$, the open system of subsets induced by V, $V_m = V \cap X_m$, $m \in \mathbb{N}_0$, pulls back to a descending filtration on U, $U_m := \phi^{-1}(V_m)$, $m \in \mathbb{N}_0$. An sc^∞-atlas $\mathcal{A} = \{(U_a, \phi_a)\}_{a \in A}$ for the sc-smooth structure of M then induces a global filtration $M_m := \cup_{a \in A}(U_a)_m$, $m \in \mathbb{N}_0$. Since the transition maps of $\mathcal{A}$ are scale maps, we have $U_m \cap U' = U \cap U'_m = U_m \cap U'_m$ and the filtration is independent of the chosen atlas. For a generic $U \subseteq M$ open, we may then set $U_m := U \cap M_m$, $m \in \mathbb{N}_0$, and this definition is consistent with the case that U is a coordinate domain.

Each coordinate chart $\phi : U \overset{\sim}{\longmapsto} V \subseteq X_0$ of M restricts to bijections $\phi_m := \phi|_{U_m} : U_m \overset{\sim}{\longmapsto} V_m$. We claim that for an sc^∞-atlas $\mathcal{A}$ as above and fixed $m > 0$, the bijections

$\{(\phi_a)_m\}_{a\in A}$, induce an sc-smooth structure on M_m with local model X^m. It is then clear that the filtration of M_m is given by $(M_m)_l = M_{m+l}$, $l \in \mathbb{N}_0$. In analogy to the linear case of a Banach scale, we define the shifted filtration $M^m := \{M_{m+l}\}_{l\in\mathbb{N}_0}$ and the limit $M_\infty := \cap_{m\in\mathbb{N}} M_m$ with limit topology.

Lemma 2.33. *Let M be an sc^∞-manifold, $\mathcal{A} = \{(U_a, \phi_a)\}_{a\in A}$ be an sc^∞-atlas for the* sc*-smooth structure of M and $M_m = \cup_{a\in A}(U_a)_m$, $m \in \mathbb{N}_0$. Then, for each $m \in \mathbb{N}_0$, M_m is an sc^∞-manifold with local model X^m and coordinate charts $(\phi_a)_m : (U_a)_m \xrightarrow{\sim} (V_a)_m$, $a \in A$. Furthermore, the inclusions $M_m \hookrightarrow M_l$ are continuous for all $m > l \in \mathbb{N}_0$.*

Proof. Only the case $m > 0$ is new. We aim to prove the statement by using Remark 2.32(d) for the set M_m, the bijections $(\phi_a)_m$, $a \in A$, and the local model X^m. First, let $\mathcal{A} \ni \phi : U \xrightarrow{\sim} V \subseteq X_0$ be a coordinate chart with induced bijections $\phi_m : U_m \xrightarrow{\sim} V_m$, $m \in \mathbb{N}_0$, and $U' \in \mathcal{A}$ be a further coordinate domain. One has $\phi_m(U_m \cap U') = \phi(U \cap U') \cap V_m$ and since $U_m \cap U' = U_m \cap U'_m$, the continuity of the inclusion map $V_m \hookrightarrow V$ implies that $\phi_m(U_m \cap U'_m)$ is open in V_m. Hence, the bijections $(\phi_a)_m$ induce a topology on M_m with basis given by sets of the form $(\phi_a)_m^{-1}(W)$ with $W \subseteq (V_a)_m$ open[5] and $a \in A$.

To see that the inclusions $M_m \hookrightarrow M_l$ are continuous for $m > l \in \mathbb{N}_0$, let $(U, \phi) \in \mathcal{A}$, $W \subseteq V_l$ be open, and $x \in \phi_l^{-1}(W) \cap M_m$. Then $x \in U_m$, hence $x \in \phi_m^{-1}(W \cap V_m) \subseteq \phi_l^{-1}(W) \cap M_m$. Again using the continuity of the inclusion $V_m \hookrightarrow V_l$, we conclude that x is an inner point. Now, the fact that $M_m \hookrightarrow M_0$ is continuous precisely means that the topology on M_m is finer than the subspace topology induced by M_0. The Hausdorff property of M_m then follows directly from the fact that $M = M_0$ is Hausdorff. ◇

To complement the basic definitions, we give simple constructions of sc^∞-manifolds which work the same way as in the finite-dimensional case and, subsequently, we introduce the pivotal example of an sc^∞-manifold which will be used when discussing Hamiltonian flows in Chapter 3. The proof of the constructions is simple and left to the reader.

Lemma 2.34. *Let X and Y be real Banach scales on $\mathbb{N}_0$, M and N be sc^∞-manifolds locally modeled on X and Y, and $\mathcal{A} = \{(U_a, \phi_a)\}_{a\in A}$ and $\mathcal{B} = \{(\Omega_b, \psi_b)\}_{b\in B}$ be atlases for M and N, respectively.*

- *(a) X is by itself an sc^∞-manifold: the single chart $\mathrm{id}_{X_0} : X_0 \mapsto X_0$ defines an* sc*-smooth structure on X_0 with local (global!) model X and filtration $(X_0)_m = \iota_{m0}(X_m) \cong X_m$, where $\iota_{m0} : X_m \hookrightarrow X_0$ is the bonding map. Clearly, the* sc*-smooth structure of $(X_0)_m$ is the one obtained by applying this construction to X^m.*
- *(b) An open subset $U \subseteq M$ has an* sc*-smooth structure given by the charts $\phi_a|_{U\cap U_a} : U \cap U_a \xrightarrow{\sim} \phi_a(U \cap U_a)$, $a \in A$, local model X and filtration $U_m = U \cap M_m$, $m \in \mathbb{N}_0$.*
- *(c) The product $M \times N$ has an* sc*-smooth structure given by the charts $\phi_a \times \psi_b : U_a \times \Omega_b \xrightarrow{\sim} \phi_a(U_a) \times \psi_b(\Omega_b)$, $(a, b) \in A \times B$, local model $X \times Y$ and filtration $(M \times N)_m = M_m \times N_m$, $m \in \mathbb{N}_0$.*

[5] we stress the fact that $(V_a)_m$ inherits the (finer) topology of X_m and not the one of $V_a \subseteq X_0$.

Example 2.35. If X is a separable Hilbert space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\underline{X}$ is its induced scale (restricted to $\mathbb{N}_0$), then the projectivization of $X = X_0$, $M := \mathrm{P}(X) = {}^{X \setminus \{0\}}\!\big/_{\mathbb{F}^*}$, is an sc$^\infty$-manifold modeled on $\underline{X}$. To prove this, first note that the isometric isomorphism $X \xrightarrow{\sim} l^2_{\mathbb{F}}$ descends to a homeomorphism $\mathrm{P}(X) \xrightarrow{\sim} \mathrm{P}(l^2_{\mathbb{F}})$, hence $X = l^2_{\mathbb{F}}$ without loss of generality. Similarly to the finite-dimensional case of $\mathrm{P}(\mathbb{F}^d) = \mathbb{F}\mathrm{P}^{d-1}$, we can define $U_a := \{[x] \in \mathrm{P}(X) : x_a \neq 0\} \subseteq \mathrm{P}(X)$ open, $a \in \mathbb{Z}$, and coordinate charts $\phi_a : U_a \xrightarrow{\sim} X_0$ given by

$$\phi_a([x])_n = \frac{1}{x_a} \cdot \begin{cases} x_n & \text{if } n < a \\ x_{n+1} & \text{if } n \geq a\,. \end{cases} \tag{2.23}$$

These charts are easily seen to be homeomorphisms and the corresponding transition maps are sc-smooth.

To obtain a natural filtration on $\mathrm{P}(X)$, we need to impose a slight condition on the defining sequences. Specifically, if the sequences $\{\frac{\nu_n}{\nu_{n+1}}\}_{n\in\mathbb{Z}}$ and $\{\frac{\nu_{n+1}}{\nu_n}\}_{n\in\mathbb{Z}}$ are bounded, we can define a compatible atlas for $\mathrm{P}(X_m)$ given by (2.23), but as maps $\{[x] \in \mathrm{P}(X_m) : x_a \neq 0\} \xrightarrow{\sim} X_m$. In that case, we have a homeomorphism $M_m \xrightarrow{\sim} \mathrm{P}(X_m)$, $[x]_{\mathrm{P}(X_0)} \mapsto [x]_{\mathrm{P}(X_m)}$, which is actually an sc$^\infty$-diffeomorphism (see Definition 2.38 further on).

Concluding this example, we remark that the topology of $\mathrm{P}(X)$ has quite desirable properties. On the one hand, $\mathrm{P}(X)$ is metrizable: with the unit sphere $\mathrm{S}(X) = \{x \in X : \|x\| = 1\}$ and the usual homeomorphism $\mathrm{P}(X) \xrightarrow{\sim} {}^{\mathrm{S}(X)}\!\big/_{S^1}$, $[x] \mapsto [\frac{x}{\|x\|}]$, a compatible metric is $d([x],[y]) = \inf_{\lambda \in S^1} \|\lambda x - y\|$ for $x, y \in \mathrm{S}(X)$. On the other hand, since continuous images of separable spaces are separable [50, Exercise 30.11] and $X \setminus \{0\}$ is separable, $\mathrm{P}(X) = {}^{X \setminus \{0\}}\!\big/_{\mathbb{F}^*}$ is also separable.

As Lemma 2.33 and the preceding discussion show, an sc$^\infty$-manifold can be reinterpreted as a filtration of topological spaces which is, locally, levelwise homeomorphic to an open subset of the local model, and where the transition maps are sc-smooth. It is then natural to extend the definitions of Section 2.1 involving scale maps to this context. A function $f : M \mapsto N$ between two sc$^\infty$-manifolds M and N modeled on X and Y, respectively, is said to be a scale map if $f(M_m) \subseteq N_m$ for each $m \in \mathbb{N}_0$. In other words, we require f to restrict to maps $f_m := f|_{M_m} : M_m \mapsto N_m$. A scale map $f : M \mapsto N$ is called sc^0 if all f_m are continuous. It is an sc^0-homeomorphism if, additionally, it is bijective and $f^{-1} : N \mapsto M$ is sc^0.

Following our developments of Section 2.3, one also expects to define sc^k maps ($k > 0$) in the manifold context. For this, we need the concept of tangent scales and bundles on sc$^\infty$-manifolds.

Definition 2.36. Let M be an sc$^\infty$-manifold modeled on a real Banach scale X on $\mathbb{N}_0$ and let $p \in M_{m+1}$, $m \in \mathbb{N}_0$.

(a) For pairs $((U,\phi),v)$, where $(p \in U, \phi)$ is a coordinate chart of M around p and $v \in X_m$, define $((U,\phi),v)$ and $((U',\psi),w)$ to be equivalent if $\mathrm{D}(\psi\,\phi^{-1})(\phi(p),v) = w$. Define the m^{th}-layer tangent space $(\mathrm{T}_pM)_m$ of M at p to be the corresponding quotient space.

(b) Endow $(\mathrm{T}_pM)_m$ with the structure of a Banachable (completely normable) space by pulling back the vector space operations and topology of X_m via the well-defined bijection $(\mathrm{D}_p\phi)_m : (\mathrm{T}_pM)_m \overset{\sim}{\longmapsto} X_m$, $[(U', \psi), w] \mapsto \mathrm{D}(\phi\,\psi^{-1})(\psi(p), w)$ for a given coordinate chart $(U(p), \phi)$.

(c) Since $p \in M_{m+1} \subseteq M_m \subseteq \ldots \subseteq M_1$, we have well-defined k^{th}-layer tangent spaces $(\mathrm{T}_pM)_k$, $0 \leq k \leq m$. The maps $(\mathrm{D}_p\phi)_k$, $k \in m+1$, induce the structure of a Banachable scale on $\mathrm{T}_pM := \{(\mathrm{T}_pM)_k\}_{k\in\{0,1,\ldots,m\}}$ with bonding maps $(\mathrm{T}_pM)_k \hookrightarrow (\mathrm{T}_pM)_l$, $[(U, \phi), v] \mapsto [(U, \phi), \iota_{kl}(v)]$, $k > l \in m+1$, and with this structure, $\mathrm{D}_p\phi := \{(\mathrm{D}_p\phi)_k\}_{k\in m+1} : \mathrm{T}_pM \overset{\sim}{\longmapsto} X|_{m+1}$ is an isomorphism of scales. The scale T_pM is called the tangent scale of M at p.

One can easily verify that the structures introduced in Definition 2.36(b),(c) are independent of the chosen coordinate chart (U, ϕ). We emphasize the fact that, as in the finite-dimensional case, there is no preferred norm on $(\mathrm{T}_pM)_m$, hence the denomination of Banach*able* spaces $(\mathrm{T}_pM)_m$ and scale T_pM, where we only refer to their vector topology. Norms on T_pM only become defined when choosing coordinates and two coordinate charts induce equivalent norms. Note also that when $p \in M_\infty$, we obtain a tangent scale T_pM on $\mathbb{N}_0$ since $M_\infty \subseteq M_{m+1}$ for all $m \in \mathbb{N}_0$.

The constructions of Lemma 2.34 have the tangent scales expected from finite dimensions, as the following lemma shows.

Lemma 2.37. *Let X and Y be real Banach scales on $\mathbb{N}_0$, M and N be sc^∞-manifolds locally modeled on X and Y, respectively, and let also $m \in \mathbb{N}_0$. We have canonical isomorphisms of scales as follows.*

(a) The tangent scale of the sc^∞-manifold X at $x \in X_{m+1}$ is $\mathrm{T}_xX \cong X|_{m+1}$.

(b) The tangent scale of an open subset $U \subseteq M$ at $p \in U_{m+1}$ is $\mathrm{T}_pU \cong \mathrm{T}_pM$.

(c) The tangent scale of $M \times N$ at $(p, q) \in (M \times N)_{m+1}$ is $\mathrm{T}_{(p,q)}(M \times N) \cong \mathrm{T}_pM \times \mathrm{T}_qN$.

Proof. The construction of the isomorphisms is straightforward. For example, for $p \in M_{m+1}$ and $q \in N_{m+1}$, coordinate charts $(U(p), \phi)$ and $(\Omega(q), \psi)$ of M and N, respectively, give rise to isomorphisms of scales $\mathrm{D}_p\phi : \mathrm{T}_pM \overset{\sim}{\longmapsto} X|_{m+1}$, $\mathrm{D}_q\psi : \mathrm{T}_qN \overset{\sim}{\longmapsto} Y|_{m+1}$ and $\mathrm{D}_{(p,q)}(\phi \times \psi) : \mathrm{T}_{(p,q)}(M \times N) \overset{\sim}{\longmapsto} (X \times Y)|_{m+1}$, which combine to the isomorphism $(\mathrm{D}_p\phi \times \mathrm{D}_q\psi)^{-1} \circ \mathrm{D}_{(p,q)}(\phi \times \psi) : \mathrm{T}_{(p,q)}(M \times N) \overset{\sim}{\longmapsto} \mathrm{T}_pM \times \mathrm{T}_qN$, $[(U \times \Omega, \phi \times \psi), (v, w)] \mapsto ([(U, \phi), v], [(\Omega, \psi), w])$. This isomorphism is easily seen to be independent of the chosen coordinate charts. $\diamond$

By varying p, we construct the tangent bundle of M as $\mathrm{T}M := \cup_{p\in M_1}\{p\} \times (\mathrm{T}_pM)_0$ with the canonical projection $\pi_{\mathrm{T}M} : \mathrm{T}M \mapsto M_1$, $(p, v) \mapsto p$. If for $U \subseteq M$ open we set $\mathrm{T}U := \pi_{\mathrm{T}M}^{-1}(U_1)$, each coordinate chart $\phi : U \overset{\sim}{\longmapsto} V \subseteq X_0$ of M induces a bijection $\mathrm{T}\phi : \mathrm{T}U \overset{\sim}{\longmapsto} V_1 \times X_0$, $(p, v) \mapsto (\phi(p), \mathrm{D}_p\phi(v))$. Moreover, if (U', ψ) is an additional coordinate chart, then $\mathrm{T}\phi(\mathrm{T}U \cap \mathrm{T}U') = \phi(U \cap U')_1 \times X_0$ is open in $V_1 \times X_0$, and since $\psi \circ \phi^{-1} : \phi(U \cap U') \overset{\sim}{\longmapsto} \psi(U \cap U')$ is sc-smooth, the transition map $\mathrm{T}\psi \circ (\mathrm{T}\phi)^{-1} : \phi(U \cap U')^1 \times X \overset{\sim}{\longmapsto} \psi(U \cap U')^1 \times X$ is sc-smooth as well. Consequently, an sc^∞-atlas

$\{(U_a, \phi_a)\}_{a\in A}$ for M gives rise to an sc$^\infty$-atlas $\{(\mathrm{T}U_a, \mathrm{T}\phi_a)\}_{a\in A}$ for $\mathrm{T}M$, whence an sc-smooth structure with local model $X^1 \times X$. It is easy to see that the induced filtration is simply $(\mathrm{T}M)_m = \cup_{p\in M_{m+1}}\{p\} \times (\mathrm{T}_pM)_m$, where $(\mathrm{T}_pM)_m$ is considered as a subspace of $(\mathrm{T}_pM)_0$ via the corresponding bonding map. Moreover, the fiber $\pi^{-1}_{\mathrm{T}M}(p)$ over $p \in M_{m+1}$ is $\{p\} \times (\mathrm{T}_pM)_0 \cong (\mathrm{T}_pM)_0$, where we recover the scale structure of the tangent scale as $(\mathrm{T}_pM)_m \subseteq (\mathrm{T}_pM)_{m-1} \subseteq \ldots \subseteq (\mathrm{T}_pM)_0$.

Tangent scales and bundles allow us to formalize sc-smoothness on sc$^\infty$-manifolds by working locally, as the following definition shows.

Definition 2.38. Let M and N be sc$^\infty$-manifolds modeled on X and Y, respectively, $f : M \mapsto N$ be an sc^0 map, and let $k \in \mathbb{N} \cup \{\infty\}$.

(a) The map f is said to be sc^k if for each $p \in M$ there are charts $(U(p), \phi)$ of M and $(\Omega(f(p)), \psi)$ of N such that $f(U) \subseteq \Omega$ and $\psi\, f|_U\, \phi^{-1} : \phi(U) \mapsto \psi(\Omega) \subseteq Y$ is sc^k.

(b) If f is sc^k we define, for each $p \in M_{m+1}$, $m \in \mathbb{N}_0$, the derivative of f at p to be the scale morphism $\mathrm{D}_pf : \mathrm{T}_pM \mapsto \mathrm{T}_{f(p)}N$ given by the diagram

$$\begin{tikzcd} \mathrm{T}_pM \arrow[r, "\mathrm{D}_pf"] \arrow[d, "\mathrm{D}_p\phi"', "\sim"] & \mathrm{T}_{f(p)}N \arrow[d, "\sim"', "\mathrm{D}_{f(p)}\psi"] \\ X|_{m+1} \arrow[r, "\mathrm{D}_{\phi(p)}(\psi\, f|_U\, \phi^{-1})"] & Y|_{m+1} \end{tikzcd},$$

where the lower row is the sc-derivative of Section 2.3. We also define the tangent map $\mathrm{T}f : \mathrm{T}M \mapsto \mathrm{T}N$, $(p, v) \mapsto (f(p), \mathrm{D}_pf(v))$.

(c) The map f is said to be an sc^k immersion (submersion) if it is sc^k and D_pf is injective (surjective) for all $p \in M_{m+1}$ and $m \in \mathbb{N}_0$.

(d) The map f is said to be an sc^k-diffeomorphism if it is sc^k, bijective and $f^{-1} : N \mapsto M$ is sc^k.

As expected, since the transition maps of M and N are sc$^\infty$-diffeomorphisms and since the chain rule of Proposition 2.25 holds, Definition 2.38(a),(b) is independent of the choice of charts (U, ϕ) and (Ω, ψ) satisfying $f(U) \subseteq \Omega$. Also, one could have dropped the umbrella assumption that f is sc^0, since this follows directly from part (a). Some easy consequences of this definition are summarized in the following lemmas.

Lemma 2.39 (Chain Rule)**.** *If M, N and P are* sc$^\infty$*-manifolds and $f : M \mapsto N$ and $g : N \mapsto P$ are* sc^k, *$k \in \mathbb{N}\cup\{\infty\}$, then $g\circ f : M \mapsto P$ is* sc^k *with $\mathrm{D}_p(g\circ f) = \mathrm{D}_{f(p)}g\circ \mathrm{D}_pf : \mathrm{T}_pM \mapsto \mathrm{T}_{g\circ f(p)}P$ for every $p \in M_{m+1}$ and $m \in \mathbb{N}_0$, hence $\mathrm{T}(g \circ f) = \mathrm{T}g \circ \mathrm{T}f : \mathrm{T}M \mapsto \mathrm{T}P$.*

Proof. Apply the chain rule for Banach scales in Proposition 2.25. ◇

Lemma 2.40. *Let M and N be* sc$^\infty$*-manifolds modeled on X and Y, respectively.*

(a) *A coordinate chart* $\varphi : U \overset{\sim}{\longmapsto} V \subseteq X_0$ *of* M *is an* sc^∞*-diffeomorphism.*

(b) *The inclusion maps* $M^k \hookrightarrow M^l$ *are injective* sc*-smooth immersions,* $k > l \in \mathbb{N}_0$.

(c) *The projection* $\pi_{\mathrm{T}M} : \mathrm{T}M \mapsto M^1$ *is a surjective* sc*-smooth submersion.*

(d) *For an* sc^1 *map* $f : M \mapsto N$*,* f *is* sc^k *if and only if* $\mathrm{T}f : \mathrm{T}M \mapsto \mathrm{T}N$ *is* sc^{k-1}*,* $k \in \mathbb{N} \cup \{\infty\}$.

Proof. Part (a) essentially holds by definition: for every $p \in U$, we can use trivialize U using φ itself and V using id_V. In turn, the maps of (b) are locally given by the inclusions $V^k \hookrightarrow V^l$, where $\phi : U \overset{\sim}{\longmapsto} V \subseteq X_0$ is a coordinate chart, and the latter have sc-derivative $X^k \hookrightarrow X^l$ at every point in V^{k+1}. Similarly, the tangent bundle projection is locally given by the projection $V^1 \times X \mapsto V^1$. Finally, the tangent map of an sc^1 map f is given around $p \in M$ by $V^1 \times X \mapsto W^1 \times Y$, $(x, v) \mapsto (\psi\, f|_U\, \phi^{-1}(x), \mathrm{D}(\psi\, f|_U\, \phi^{-1})(x, v))$, where $\phi : U(p) \overset{\sim}{\longmapsto} V \subseteq X_0$ and $\psi : \Omega(f(p)) \overset{\sim}{\longmapsto} W \subseteq Y_0$ are charts with $f(U) \subseteq \Omega$. $\diamond$

Similarly to Section 2.3, one can introduce partial differentiation by generalizing scale manifolds to local models on a finitely-indexed Banach scale and defining, for sc^∞-manifolds M, N and P, an sc-smooth map $f : M \times N \mapsto P$ and $(p, q) \in (M \times N)_{m+1}$, the partial derivative at (p, q) with respect to the first component to be $\frac{\partial f}{\partial p}(p, q) := \mathrm{D}_p[f(\cdot, q)] : \mathrm{T}_p M \mapsto \mathrm{T}_{f(p,q)} P$. For this procedure, the mentioned generalization of scale manifolds is needed, since the local models of the manifolds in the map $f(\cdot, q) : M|_{m+2} \mapsto P|_{m+2}$ are finitely-indexed. A simpler definition which delivers the same result is

$$\frac{\partial f}{\partial p}(p, q) := \mathrm{D}_{(p,q)} f(\cdot, 0) : \mathrm{T}_p M \mapsto \mathrm{T}_{f(p,q)} P\,, \tag{2.24}$$

where we use Lemma 2.37(c). Naturally, a similar formula holds for differentiation with respect to the second component. We also remark that when $M = \mathbb{R}$, we regard $\frac{\partial f}{\partial p}(p, q)$ as an element of $(\mathrm{T}_{f(p,q)} P)_m$ by noting $\mathrm{T}_p \mathbb{R} \cong \mathbb{R}$ and applying 1 to the above map of scales.

As a final addendum to this section, we define sections of the tangent bundle in a similar manner as in the finite-dimensional case.

Definition 2.41. An sc^k section of the tangent bundle of an sc^∞-manifold M, $k \in \mathbb{N}_0 \cup \{\infty\}$, is an sc^k map $s : M^1 \mapsto \mathrm{T}M$ with $\pi_{\mathrm{T}M} \circ s = \mathrm{id}_{M^1}$.

3 Hamiltonian Partial Differential Equations

In this chapter, we generalize the concepts of Hamiltonian vector fields and flows in finite-dimensional symplectic geometry to the sc-calculus framework. We first introduce the relevant concepts in the linear case of a Banach scale while being guided by the prototypical example of the free Schrödinger equation. Eventually, we arrive at the conclusion that an extension of sc-calculus is needed to handle Hamiltonian maps: strong sc-smoothness. After an extensive motivation, we define this new concept and show that it is invariant under pre-composition with sc-smooth symplectomorphisms. This property makes strong sc-smoothness amenable to being used with sc$^\infty$-manifolds.

Subsequently, we introduce symplectic sc$^\infty$-manifolds by restricting the coordinate charts so that the transition maps are symplectic. Once a symplectic sc$^\infty$-manifold is given, it is possible to extend the tangent structure of the manifold to support a symplectic form on each tangent scale. After outlining the necessary backbone, we generalize strong sc-smoothness, Hamiltonian vector fields and flows to this non-linear case, illustrating its use with the free Schrödinger equation on a projective Hilbert space.

3.1 Flows on Banach scales

We begin defining sc-smooth vector fields and (global) flows on Banach scales and use these to formalize the free Schrödinger equation. Subsequently, we discuss existence and uniqueness of this type of flows. This is a very hard question to pursue in general, and most references are careful when it comes to general well-posedness of Hamiltonian PDEs, either assuming it in some form [1, 5, 43] or deducing it for specific Hamiltonian PDEs and under specific assumptions (*e.g.*, [8, 10, 12, 51, 52, 56, 60]). In this thesis, we refrain from a detailed research into this question and show by means of a counter-example that, in general, sc$^\infty$-flows need not exist (even locally). We then provide sufficient conditions, albeit rather strong, which ensure that two sc$^\infty$-flows for the same vector field coincide.

Let X be a real Banach scale on $\mathbb{N}_0$. We define an autonomous sc-smooth vector field on X to be an sc$^\infty$ map $V : X^1 \mapsto X$. We use the shifted scale X^1 for the domain, since the vector fields we are interested in Hamiltonian PDEs are densely defined (*e.g.*, see the prototypical Example 3.1 further on). In the following, all vector fields will be understood to be autonomous. V is said to be complete, or to have a global flow, if there exists an sc^∞ map $\varphi : \mathbb{R} \times X \mapsto X$ such that

$$\frac{\partial \varphi}{\partial t}(t, v) = V \circ \varphi(t, v)\,, \qquad \varphi(0, v) = v \tag{3.1}$$

for all $t \in \mathbb{R}$ and $v \in X_{m+1}$, $m \in \mathbb{N}_0$, where $\frac{\partial \varphi}{\partial t} : \mathbb{R} \times X^1 \mapsto X$ is the partial derivative of φ in the scale sense. Due to the uniqueness property of Lemma 2.3, it is sufficient that the equality holds for $v \in X_1$. Clearly, if a global flow φ exists, then the initial value problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = V \circ u : \mathbb{R} \mapsto X_k\,, \qquad u(0) = u_0 \tag{3.2}$$

for an sc^{k+1} unknown $u : \mathbb{R} \mapsto X_{k+1}$ and initial condition $u_0 \in X_{k+1}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, has solution given by $t \mapsto \varphi(t, u_0)$. Also, the vector field can be recovered as $V = \frac{\partial \varphi}{\partial t}(0, \cdot)$. As usual, we notate $\varphi_t := \varphi(t, \cdot) : X \mapsto X$ for $t \in \mathbb{R}$.

Example 3.1. The free Schrödinger equation

$$\mathrm{i}u_t = -\Delta u \tag{3.3}$$

for $u : \mathbb{R} \times S^1 \mapsto \mathbb{C}$, $(t, x) \mapsto u(t, x)$, where $\Delta = (\cdot)_{xx}$ is the Laplacian operator, can be rewritten in evolution form by taking the (double-spaced) Levi-Sobolev scale $X_k = W^{2k,2}(S^1, \mathbb{C})$, $k \in \mathbb{N}_0$, and defining the vector field $V : X^1 \mapsto X$, $u \mapsto \mathrm{i}\Delta u$. Here, X is seen as a real scale and the Laplacian $\Delta : X_{k+1} \mapsto X_k$ is taken in the weak sense, corresponding to the Fourier multiplier $[n \mapsto (\mathrm{i}n)^2] \in \mathbb{C}^{\mathbb{Z}}$. The evolution equation then simply reads $\dot{u} = V(u)$, $u(0) = u_0$ for $u \in \mathrm{sc}^1(\mathbb{R}, X|_{\{0,1\}})$ and $u_0 \in X_1$.

We claim that the vector field V is complete with sc-smooth flow given by

$$\varphi : \mathbb{R} \times X \mapsto X,\ (t, u) \mapsto \mathrm{e}^{\mathrm{i}t\Delta} u\,, \tag{3.4}$$

where $\mathrm{e}^{\mathrm{i}t\Delta}$ is the bounded linear X_k operator with Fourier multiplier $\mathrm{e}^{-\mathrm{i}tn^2}$. To see that φ is sc^0 to begin with, identify the scale X with $\{l^{2,2k}_{\mathbb{C}}\}_{k \in \mathbb{N}_0}$ using the Fourier series. For $x \in l^{2,2k}$ fixed and $N \in \mathbb{N}$, the function $\mathbb{R} \mapsto \mathbb{C}^{2N-1} \subseteq l^{2,2k}$, $t \mapsto \{\mathrm{e}^{-\mathrm{i}tn^2} x_n\}_{|n|<N}$ is continuous and

$$\sup_{t \in \mathbb{R}} \|\{\mathrm{e}^{-\mathrm{i}tn^2} x_n\}_{|n|<N} - \varphi(t, x)\|_{l^{2,2k}} = \sup_{t \in \mathbb{R}} \left(\sum_{|n| \geq N} |\mathrm{e}^{-\mathrm{i}tn^2} x_n|^2 (1 + n^2)^{2k} \right)^{\frac{1}{2}} \to 0 \tag{3.5}$$

as $N \to \infty$, from where $\varphi(\cdot, u) : \mathbb{R} \mapsto X_k$ is continuous for all $u \in X_k$ by the uniform limit theorem. To prove joint continuity of $\varphi : \mathbb{R} \times X_k \mapsto X_k$, just note that φ is linear in the second argument, that

$$\|\varphi(t, u) - \varphi(t_0, u_0)\|_{X_k} \leq \|\varphi(t, \cdot)\|_{\mathrm{B}(X_k)} \|u - u_0\|_{X_k} + \|\varphi(t, u_0) - \varphi(t_0, u_0)\|_{X_k} \tag{3.6}$$

for $t, t_0 \in \mathbb{R}$ and $u, u_0 \in X_k$, and that $\varphi(t, \cdot)$ is uniformly bounded in $\mathrm{B}(X_k)$ (by 1).

For the sc-smoothness claim, we use Proposition 2.29 and Proposition 2.30. It is clear that $\varphi(t,\cdot)$ is sc^1 for all $t \in \mathbb{R}$ since it is sc^0 linear, and $\varphi(\cdot,u) : \mathbb{R} \mapsto X_m$ is C^1 for $u \in X_{m+1}$, $m \geq 0$, with derivative $\frac{\mathrm{d}\varphi(\cdot,u)}{\mathrm{d}t}(t) = \mathrm{i}\Delta\varphi(t,u)$. The partial derivatives $\frac{\partial\varphi}{\partial t}$ and $\frac{\partial\varphi}{\partial u}$ are sc^0 and give rise to the sc-derivative

$$\mathrm{D}\varphi : \mathbb{R} \times X^1 \times \mathbb{R} \times X \mapsto X,\ (t,u,h,\xi) \mapsto \varphi(t,\xi) + h\,\mathrm{i}\Delta\varphi(t,u)\,, \tag{3.7}$$

which is as smooth as φ is, since $\mathrm{i}\Delta = V : X^1 \mapsto X$ is sc^0 linear. From (3.4) and (3.7), it is clear that (3.1) holds.

In general, an sc-smooth flow of a vector field need not exist, even locally. A simple counter-example is given by the equation $\mathrm{i}u_t = u_x$, corresponding to the vector field $V = -\mathrm{i}(\cdot)_x : X^1 \mapsto X$ on the Levi-Sobolev scale $X_k = W^{k,2}(S^1,\mathbb{C})$, $k \in \mathbb{N}_0$. Indeed, if a flow would exist, then for every $u_0 \in X_1$, we could find a continuous solution $u : \mathbb{R} \mapsto X_1$ of (3.2) which is C^1 into X_0. Extracting the n^{th} Fourier series coefficient we obtain, for each $n \in \mathbb{Z}$, a C^1 map $\mathbb{R} \mapsto \mathbb{C}, t \mapsto \widehat{u(t)}_n$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{u(t)}_n = -\mathrm{i}(\mathrm{i}n)\,\widehat{u(t)}_n\,, \tag{3.8}$$

whence $\widehat{u(t)}_n = (\widehat{u_0})_n \mathrm{e}^{nt}$. Clearly, for example $(\widehat{u_0})_n = 1/(1+n^2)$ gives $u_0 \in X_1$ but not even $u(t) \in X_{-\infty}$ for any $t \in \mathbb{R} \setminus \{0\}$. A counter-example for a Hamiltonian PDE is given by the “bad” Boussinesq equation [16]

$$u_{tt} = u_{xx} + \mu\, u_{xxxx} + (u^2)_{xx} \tag{3.9}$$

for $u : \mathbb{R} \times [0,1] \mapsto \mathbb{R}$, $(t,x) \mapsto u(t,x)$, where $\mu > 0$ is a parameter. As de la Llave notes, this equation “is not well posed in any reasonable space and it is not hard to find analytic initial conditions for which the solution is not defined (in almost any weak sense) in any interval of time” [16].

When an sc^∞ flow φ of a vector field V does exist, it is then the question whether this flow is unique, or in other words, whether two flows φ and ψ which satisfy (3.1) for the same V coincide. It is known that uniqueness breaks down when the solutions of (3.2) are allowed to be weak enough [13, 55, 58] and a positive answer without restrictions on V should, in principle, not be expected. Unfortunately, the cited examples cannot be easily described in the sc-framework using an sc-smooth vector field. The construction of a counter-example for uniqueness is thus left as future work.

In this thesis, instead of looking for uniqueness for the general case of an sc-smooth vector field, we restrict our attention to scales induced by a separable Hilbert space and vector fields which act component-wise in the frequency domain (*e.g.*, Fourier multipliers). We give a positive answer by analysing each Fourier coefficient of the solution of (3.2), thereby reducing the problem to the finite-dimensional case. At this point we note that this assumption is rather restrictive. For example, non-linear polynomial terms in the vector field induce self-convolutions in the frequency domain instead of point-wise multiplication. Nevertheless, the assumption is still suitable for our prototypical Example 3.1.

Proposition 3.2. *Let $\underline{X}$ be the real Hilbert scale induced by a separable Hilbert space X over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with respect to some orthonormal basis and sequence ν. Restrict this scale to $\mathbb{N}_0$. Additionally, let $V : \underline{X}^1 \mapsto \underline{X}$ be a vector field which acts component-wise in the frequency domain, i.e., with the identification induced by the orthonormal basis $\widehat{(\cdot)} : \underline{X} \overset{\sim}{\longmapsto} \underline{l_\nu^2}$, $V(x) = \{V_n(x_n)\}_{n\in\mathbb{Z}}$ for functions $V_n : \mathbb{F} \mapsto \mathbb{F}$, $n \in \mathbb{Z}$. If each V_n is locally Lipschitz and an sc^1 solution $u : \mathbb{R} \mapsto X_1$ of* (3.2) *exists with initial condition $u_0 \in X_1$, then it is unique.*

Proof. Let $u, v : \mathbb{R} \mapsto X_1 \in \mathrm{sc}^1$ be two solutions of (3.2), both with the same initial condition $u_0 \in X_1$. By taking the n^{th} (generalized) Fourier coefficient, the map $t \mapsto \widehat{u(t)}_n : \mathbb{R} \mapsto \mathbb{F}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{u(t)}_n = V_n(\widehat{u(t)}_n) \qquad\qquad \widehat{u(0)}_n = (\widehat{u_0})_n \tag{3.10}$$

and the same holds for $t \mapsto \widehat{v(t)}_n$ with the same initial condition. Since V_n is locally Lipschitz, finite-dimensional ODE theory [59] dictates that $\widehat{u(t)}_n = \widehat{v(t)}_n$, hence $u = v$. ◇

Remark 3.3. Naturally, we can slightly generalize Proposition 3.2 by assuming V to act in a "block-wise diagonal" manner, *i.e.*, $V(x) = \{V_i(x_{N_i})\}_{i\in I}$ for some disjoint collection $\{N_i \subseteq \mathbb{Z} : i \in I\}$ with all N_i finite and $\cup N_i = \mathbb{Z}$, where $V_i : \mathbb{F}^{N_i} \mapsto \mathbb{F}^{N_i}$ are all locally Lipschitz, $i \in I$. Here, for $N \subseteq \mathbb{Z}$, x_N simply means $\{x_n\}_{n\in N} \in \mathbb{F}^N$.

Corollary 3.4. *Under the assumptions of Proposition* 3.2, *if a flow of V exists, it is unique.*

Proof. Let $\varphi, \psi : \mathbb{R} \times \underline{X} \mapsto \underline{X}$ satisfy (3.1) for the same vector field V. Then for each $v_0 \in X_1$, (3.2) is satisfied for both $\varphi(\cdot, v_0)$ and $\psi(\cdot, v_0)$ and the initial condition v_0 coincides. From Proposition 3.2, we conclude that $\varphi|_{\mathbb{R}\times X_1} = \psi|_{\mathbb{R}\times X_1}$. Since $\mathbb{R} \times X_1$ is dense in $\mathbb{R} \times X_0$ and $\varphi_0, \psi_0 : \mathbb{R} \times X_0 \mapsto X_0$ are continuous, the corollary follows. ◇

Lemma 3.5. *If an sc^∞ flow φ of a vector field V is unique, it satisfies the group law*

$$\varphi_{t+s} = \varphi_t \circ \varphi_s : X \mapsto X \tag{3.11}$$

for all $t, s \in \mathbb{R}$.

Proof. Similarly to Corollary 3.4, one proves that this holds as maps $X_1 \mapsto X_1$ in the same way as the finite-dimensional case, subsequently using the density of X_1 in X_0. ◇

3.2 Hamiltonian Flows on Symplectic Scales

After having introduced flows and vector fields in the sc-framework, we are ready to discuss the central part of this chapter. Our aim is to define, in a meaningful way, what it means for an sc-smooth vector field and flow to be Hamiltonian. To do so, we need to introduce a new notion of smoothness for Hamiltonian functions. With this new notion, a smooth real-valued Hamiltonian function generates an sc-smooth vector field by means of a symplectic structure and corresponding symplectic-gradient relation, similarly to the finite-dimensional case. We derive a chain rule for this notion which is valid while pre-composing the Hamiltonian function with sc^∞-symplectomorphisms and which enables its usage with sc^∞-manifolds in the following section.

We start with a symplectic Banach scale (X, ω) on $\mathbb{Z}$. As in Example 3.1, in the case of Hamiltonian PDEs, we can only expect Hamiltonian functions to be densely defined and, as such, we need to work with scale maps $h : X^1 \mapsto \mathbb{R}$. To motivate the need of a new smoothness concept, analyse the usual ω-gradient relation used pointwise to obtain the vector field V_h from the Hamiltonian h

$$-\,\mathrm{D}h = \omega(\cdot, V_h)\,. \tag{3.12}$$

In the scale framework, we wish to obtain a map $(V_h)_m : X_{m+1} \mapsto X_m$ for each $m \geq 0$. Since ω pairs X_{-m} with X_m, the derivative $\mathrm{D}_x h$ should be an element of X^*_{-m} for each $x \in X_{m+1}$. Hence, the "new derivative" should induce a map $(\mathrm{D}h)_m : X_{m+1} \times X_{-m} \mapsto \mathbb{R}$, $(x, \xi) \mapsto \mathrm{D}_x h \cdot \xi$ for each $m \geq 0$ which is linear in the second argument.

In principle, given a Hamiltonian function $h : X^1 \mapsto \mathbb{R}$, it would be possible to use the theory by Hofer of Section 2.3 to define an sc-derivative $(\mathrm{D}h)_m : X_{m+1} \times X_m \mapsto \mathbb{R}$, $m \geq 0$, since the condition of an sc^1 map in Definition 2.20(b) does not need the map to be defined on the zeroth layer. It is not difficult to double-check the proofs in [35, 36] and see that the theory carries over *mutatis mutandis* for these densely-defined maps. Nevertheless, comparing this derivative with the desired form of the last paragraph, we see that test vectors are taken from X_m instead of X_{-m}. Since the former space is smaller (remember that $m \geq 0$), the Hofer sc^1 requirement is not strong enough to obtain a scale map $V_h : X^1 \mapsto X$. Indeed, the only case where the test spaces match is $m = 0$, and with the original sc-theory, relation (3.12) only provides a vector field $V_h : X_1 \mapsto X_0$ without any scale structure *a priori*. In contrast, in our concept, we allow the smoothness of test vectors to decrease as the smoothness of the differentiation point increases, thereby obtaining a scale structure on V_h.

For densely-defined maps, we shall refer to the marginally modified Hofer sc-smoothness concept as densely-defined sc-smoothness, and to our alternative as *strongly* densely-defined sc-smoothness[1]. For the latter, "densely-defined" will be frequently omitted from the terminology, seen that this is the only kind of strong sc-smoothness this thesis deals with. For clarity, we reproduce the definition of densely-defined sc-smoothness and, subsequently, we introduce the new smoothness concept. In the following, the restriction of a Banach scale X on $\mathbb{Z}$ to $\mathbb{N}_0$ will be denoted by $X_{\geq 0}$.

Definition 3.6. Let X be a real Banach scale on $\mathbb{N}_0$, and let $U \subseteq X$ be open. An sc^0 map $h : U^1 \mapsto \mathbb{R}$ is said to be densely-defined sc^1 if there exists an sc^0 map $\mathrm{D}h : U^1 \times X \mapsto \mathbb{R}$

[1] This concept is disjoint from the definition of a strong sc^k map in [38, Remark 1.3]: the latter is simply a map which is C^k on each layer.

which is linear in the second argument and such that for all $x \in U_1$

$$\lim_{t\to 0} \frac{|h(x+t) - h(x) - \mathrm{D}h(x,t)|}{\|t\|_{X_1}} = 0 \tag{3.13}$$

as $t \to 0 \in X_1$. We use the notation $\mathrm{D}_x h := \mathrm{D}h(x,\cdot) \in X_m^*$ for $x \in U_{m+1}$, $m \in \mathbb{N}_0$. The map h is densely-defined sc^{k+1}, $k \in \mathbb{N} \cup \{\infty\}$ if it is densely-defined sc^1 and $\mathrm{D}h : U^1 \times X \mapsto \mathbb{R}$ is sc^k (in the usual sense).

Definition 3.7. Let X be a real reflexive Banach scale on $\mathbb{Z}$, $U \subseteq X_{\geq 0}$ open and $h : U^1 \mapsto \mathbb{R}$ be an sc^0 map.

(a) The map h is called strongly densely-defined sc^1, or simply strongly sc^1, if there exists an sc^0 map $\mathrm{D}h : U^1 \mapsto X^*$ such that for all $x \in U_1$

$$\frac{|h(x+t) - h(x) - \mathrm{D}h(x) \cdot t|}{\|t\|_{X_1}} \to 0 \tag{3.14}$$

as $t \to 0$ in X_1. We use the notation $D_x h := \mathrm{D}h(x) \in X_{-m}^*$ for $x \in U_{m+1}$, $m \in \mathbb{N}_0$.

(b) The map h is called strongly (densely-defined) sc^{k+1} if it is strongly sc^1 and $\mathrm{D}h : U^1 \mapsto X^*$ is sc^k in the original Hofer sense, $k \in \mathbb{N} \cup \{\infty\}$.

Remark 3.8. (a) The reader might notice that for the definition of strongly sc^1 maps, instead of requiring $\mathrm{D}h$ as above to be sc^0, it would be more natural and compatible with the Hofer sc^1 condition to require each map $U_{m+1} \times X_{-m} \mapsto \mathbb{R}$, $(x,\xi) \mapsto \mathrm{D}_x h \cdot \xi$ to be continuous, $m \geq 0$. This weaker condition would suffice to prove the Frèchet condition (3.14) in a chain rule scenario, but not the continuity of the derivative of the composed map (cf. Remark 3.14).

(b) If $h : U^1 \mapsto \mathbb{R}$ is strongly sc^1, then the one-layer map $h : U_1 \mapsto \mathbb{R}$ is C^1, since the inclusion $X_0^* \subseteq X_1^*$ is continuous. Also, by using the bonding (inclusion) maps $X_m \hookrightarrow X_{-m}$ for $m \geq 1$, the derivative $\mathrm{D}h : U^1 \mapsto X^*$ of a strongly sc^1 map h induces an sc^0 map $U^1 \times X \mapsto \mathbb{R}$, $(x,\xi) \mapsto \mathrm{D}_x h \cdot \xi$. Taking (3.14) into account, we see that strongly sc^1 maps h are densely-defined sc^1. Proposition 3.11 expands on the relations between different smoothness concepts.

In the same way as in Hofer scale calculus, one can prove that the derivative of a strongly sc^1 map is unique, and that for a strongly sc^k map $h : U^1 \mapsto \mathbb{R}$ and $V \subseteq U$ open, $h|_{V_1} : V^1 \mapsto \mathbb{R}$ is still strongly sc^k with $\mathrm{D}(h|_{V_1}) = (\mathrm{D}h)|_{V_1} : V^1 \mapsto X^*$. From this, one proves locality of the strong sc^k conditions, meaning that $h : U^1 \mapsto \mathbb{R}$ is strongly sc^k if and only if for each $x \in U_1$, there exists an open neighbourhood $V(x) \subseteq U$ such that $h|_{V_1} : V^1 \mapsto Y$ is strongly sc^k.

As announced in the motivation of strong sc-smoothness, if (X,ω) is a symplectic Banach scale on $\mathbb{Z}$, a strongly sc^1 map $h : X^1_{\geq 0} \mapsto \mathbb{R}$ induces an sc^0 vector field $V_h : X^1_{\geq 0} \mapsto X$ which is uniquely defined by the ω-gradient relation (3.12), where the derivative D is in the strong sense. Since ι_ω is an isomorphism of scales, V_h is sc^k if and only if h is strongly sc^{k+1}, $k \in \mathbb{N}_0 \cup \{\infty\}$. This leads to the following definition we have been working towards.

Definition 3.9. Let (X, ω) be a symplectic Banach scale on $\mathbb{Z}$. An sc-smooth vector field $V : X^1_{\geq 0} \mapsto X$ is said to be Hamiltonian if there exists a strongly sc-smooth map $h : X^1_{\geq 0} \mapsto \mathbb{R}$ such that

$$-\,\mathrm{D}h = \omega(\cdot, V) \tag{3.15}$$

holds pointwise. If a Hamiltonian sc-smooth vector field V has a global flow $\varphi : \mathbb{R} \times X_{\geq 0} \mapsto X_{\geq 0}$, then the flow φ is said to be Hamiltonian.

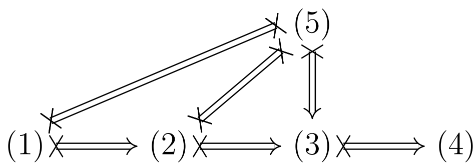
Example 3.10. For the Banach scale $X_k = W^{2k,2}(S^1, \mathbb{C})$, $k \in \mathbb{Z}$, with its standard symplectic structure[2] $\omega = \langle \mathrm{i}\cdot, \cdot \rangle_0$, the vector field $V = \mathrm{i}\Delta$ and flow $\varphi_t = \mathrm{e}^{\mathrm{i}t\Delta}$ of Example 3.1 are Hamiltonian. Indeed, consider $h : X^1_{\geq 0} \mapsto \mathbb{R}$ given by

$$h(u) = \frac{\|u_x\|_0^2}{2} = \frac{1}{2} \int_{S^1} |u_x(a)|^2 \,\mathrm{d}a\,, \tag{3.16}$$

where $(\cdot)_x : X^1 \mapsto X$ is the weak differentiation operator (Fourier multiplier $[n \mapsto \mathrm{i}n] \in \mathbb{C}^{\mathbb{Z}}$). The Frèchet condition (3.14) is satisfied with $\mathrm{D}_u h \cdot \xi = \langle u_x, \xi_x \rangle_0 = \int_{S^1} u_x(a) \cdot \xi_x(a) \,\mathrm{d}a$ for $u, \xi \in X_1$, and by integration by parts, this map extends to an sc^0 map $\mathrm{D}h : X^1 \mapsto X^*$, $u \mapsto -\langle u_{xx}, \cdot \rangle_0$. Since $\mathrm{D}h$ happens to be linear, we conclude that h is strongly sc-smooth. For $u \in X_{m+1}$, $m \geq 0$, we then have $\omega(\cdot, \mathrm{i}\Delta u) = \langle \cdot, u_{xx} \rangle_0 = -\mathrm{D}_u h$ for $u \in X_{m+1}$, $m \geq 0$.

The following proposition clarifies the relationships between the several smoothness concepts used so far. Note in the proposition that a scale map $h : U \subseteq X_{\geq 0} \mapsto \mathbb{R}$ which is C^k on each layer satisfies (1), and that a map which satisfies (4) is C^k on each layer as a map $h : U^k \mapsto \mathbb{R}$.

Proposition 3.11. *Let X be a real reflexive Banach scale on $\mathbb{Z}$, $U \subseteq X_{\geq 0}$ open and $h : U \mapsto \mathbb{R}$ be an sc^0 map. We have the following implication diagram, where $A \Longrightarrow B$ means "A implies B" and $A \not\Longrightarrow B$ means "A does in general not imply B".*



where, for $k \in \mathbb{N}$, we label:
(1) $h : U_m \mapsto \mathbb{R}$ is C^{m+1} for $m \in \{0, 1, \ldots, k-1\}$;
(2) $h : U \mapsto \mathbb{R}$ is sc^k;
(3) $h : U^1 \mapsto \mathbb{R}$ is densely-defined sc^k;
(4) $h : U_m \mapsto \mathbb{R}$ is C^m for $m \in \{1, 2, \ldots, k\}$;
(5) $h : U^1 \mapsto \mathbb{R}$ is strongly densely-defined sc^k.

[2] recall that X is seen as a real scale, and the inner product should be interpreted as the *real-valued* inner product of X_0.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are proved in [36, Proposition 2.4] and [36, Proposition 2.3], respectively. (2) $\Rightarrow$ (3) is direct from the definition. To prove (5) $\Rightarrow$ (3), note that we have an sc^0 bilinear evaluation map $\mathrm{ev} : X^* \times X_{\geq 0} \mapsto \mathbb{R}$, $(T \in X^*_{-m}, x \in X_m) \mapsto T|_{X_m}(x)$, $m \geq 0$. Consequently, the sc^{k-1} derivative $\mathrm{D}h : U^1 \mapsto X^*$ induces an sc^{k-1} derivative $\mathrm{ev} \circ (\mathrm{D}h \times \mathrm{id}_{X_{\geq 0}}) : U^1 \times X \mapsto \mathbb{R}$ which satisfies the Frèchet condition by hypothesis.

To prove the counter-implications, four counter-examples suffice. Firstly, with $X_s = l^{2,s}_{\mathbb{R}}$, define $h : X|_{\mathbb{N}_0} \mapsto \mathbb{R}$, $x \mapsto \langle x, y\rangle_0$ for some $y \in X_0 \setminus X_{1/2}$. Then $h : X_m \mapsto \mathbb{R}$ is C^∞ for all $m \in \mathbb{N}_0$, and from $\mathrm{d}h(x) \cdot \xi = \langle \xi, y\rangle_0$, $x, \xi \in X_0$, it is easy to see that $\mathrm{d}h(x) : X_0 \subseteq X_{-1} \mapsto \mathbb{R}$ is not continuous for $x \in X_2$. This proves (1) $\not\Rightarrow$ (5).

The second counter-example is similar and proves (4) $\not\Rightarrow$ (3). Define $h : X|^1_{\mathbb{N}_0} \mapsto \mathbb{R}$, $x \mapsto \frac{1}{2}\|x\|_1^2$ for the same scale X. We have $h : X_m \mapsto \mathbb{R}$ is C^∞ for all $m \geq 1$ but the derivative $\mathrm{d}h(x) = \langle x, \cdot\rangle_1 : X_1 \mapsto \mathbb{R}$ cannot be extended to a continuous linear map $X_0 \mapsto \mathbb{R}$ if we take $x \in X_1 \setminus X_{3/2}$.

Thirdly, to prove (5) $\not\Rightarrow$ (2), let now $X_s = W^{2s,2}(S^1, \mathbb{C})$. The Hamiltonian for the free Schrödinger equation in Example 3.10 is densely-defined sc-smooth but cannot be extended to a map $h : X|_{\mathbb{N}_0} \mapsto \mathbb{R}$ satisfying (2), since it is not even continuous with respect to the topology of X_0.

Last but not least, we prove (2) $\not\Rightarrow$ (1). For the same scale X, let $h : \mathbb{R} \times X|_{\mathbb{N}_0} \mapsto \mathbb{R}$, $(t, u) \mapsto \langle \mathrm{e}^{\mathrm{i}t\Delta}u, v\rangle_{0,\mathbb{R}}$, where $v \in X_0 \setminus X_{1/2}$. This map is sc-smooth but its zeroth layer $h_0 : \mathbb{R} \times X_0 \mapsto \mathbb{R}$ is not C^1. Indeed, if that was the case, the sc-derivative $\mathrm{D}h : \mathbb{R} \times X^1 \times \mathbb{R} \times X \mapsto \mathbb{R}$ would be such that $(\mathrm{D}h)_0(0, \cdot, 1, 0) : X_1 \subseteq X_0 \mapsto \mathbb{R}$, $u \mapsto \langle \mathrm{i}\Delta u, v\rangle_0$ is continuous, which is not the case. The remaining counter-implications are a consequence of these four. $\diamond$

Definition 3.7 introduced the concept of strong sc-smoothness on a real Banach scale X on $\mathbb{Z}$. To generalize this concept to sc^∞-manifolds later on, we need it to be invariant under sc-smooth coordinate changes. For $U \subseteq X_{\geq 0}$ and $V \subseteq Y_{\geq 0}$ open, a strongly sc^k map $h : V^1 \mapsto \mathbb{R}$, $k \in \mathbb{N} \cup \{\infty\}$, and an sc^∞-diffeomorphism $f : U \xrightarrow{\sim} V$, it is then the question whether the composition $h \circ f : U^1 \mapsto \mathbb{R}$ is also strongly sc^k. To answer this question positively we need, for each element $x \in X_{m+1}$, $m \geq 0$, to map $\mathrm{D}_{f(x)}h \in Y^*_{-m}$ via $\mathrm{D}_x f$ to an element $\mathrm{D}_x(h \circ f) \in X^*_{-m}$ to be defined. If $\mathrm{D}_x f$ existed as a continuous linear map $X_{-m} \mapsto Y_{-m}$, we could take its adjoint, setting $\mathrm{D}_x(h \circ f) := \mathrm{D}_{f(x)}h \circ \mathrm{D}_x f$ as usual. Nevertheless, f only defines a scale map $\mathrm{D}_x f : X|_{m+1} \mapsto Y|_{m+1}$ for the non-negative indices $m + 1 = \{0, 1, \dots, m\}$.

To solve the problem raised above, we need to extend $\mathrm{D}_x f$ to a scale morphism on $\{-m, -m+1, \dots, m\}$ for $x \in X_{m+1}$. If we assume that we have symplectic structures ω and η on X and Y, respectively, then, by the discussion at the end Section 2.1, $(\mathrm{D}_x f)^{-1} : Y|_{m+1} \mapsto X|_{m+1}$ induces a scale morphism $((\mathrm{D}_x f)^{-1})^{\eta,\omega} : X|_{\{-m,-m+1,\dots,0\}} \mapsto Y|_{\{-m,-m+1,\dots,0\}}$ by using the symplectic structures to identify X and Y with their duals. In order that the two morphisms $\mathrm{D}_x f$ on $\{0, 1, \dots, m\}$ and $((\mathrm{D}_x f)^{-1})^{\eta,\omega}$ on $\{-m, -m+1, \dots, 0\}$ glue together to a scale map on $\{-m, -m+1, \dots, m\}$, we need them to coincide on the zeroth layer, *i.e.*, $(\mathrm{D}_x f)_0 = ((\mathrm{D}_x f)^{-1})^{\eta,\omega}_0 : X_0 \mapsto Y_0$. This condition precisely means that $\mathrm{D}_x f$ should be a linear symplectomorphism of scales for all $x \in X_{m+1}$ and

$m \in \mathbb{N}_0$. Of course, it is enough to require this condition for $m = 0$. The following definition and proposition solidify this discussion.

Definition 3.12. Let (X, ω) and (Y, η) be symplectic Banach scales on $\mathbb{Z}$ and $U \subseteq X_{\geq 0}$ be open. An sc-smooth map $f : U \subseteq X_{\geq 0} \mapsto Y_{\geq 0}$ is called symplectic whenever $\mathrm{D}_x f : X_0 \mapsto Y_0$ is symplectic for all $x \in U_1$, that is,

$$\eta(\mathrm{D}_x f \cdot v, \mathrm{D}_x f \cdot w) = \omega(v, w) \tag{3.17}$$

for all $v, w \in X_0$ and $x \in U_1$ (or equivalently, $v, w \in X_\infty$ and $x \in U_\infty$). It is an sc^∞-symplectomorphism if, in addition, there exists $V \subseteq Y_{\geq 0}$ open with $f(U) = V$ and $f : U \mapsto V$ is an sc^∞-diffeomorphism.

Proposition 3.13. *(Chain rule for strong* sc*-maps) Let (X, ω) and (Y, η) be symplectic Banach scales on $\mathbb{Z}$, $U \subseteq X_{\geq 0}$ and $V \subseteq Y_{\geq 0}$ be open, $h : V^1 \mapsto \mathbb{R}$ be an* sc^0 *map and $f : U \overset{\sim}{\longmapsto} V$ be an sc^∞-symplectomorphism. Then for $k \in \mathbb{N} \cup \{\infty\}$, h is strongly sc^k if and only if $h \circ f : U^1 \mapsto \mathbb{R}$ is so. If this is the case, the chain rule*

$$\mathrm{D}_x(h \circ f) = \mathrm{D}_{f(x)} h \circ ((\mathrm{D}_x f)^{-1})^{\eta,\omega} \tag{3.18}$$

holds for all $x \in U_{m+1}$, $m \in \mathbb{N}_0$.

Proof. Clearly, it suffices to prove the "only if" part, since f^{-1} is also an sc^∞-symplectomorphism. Starting with the regularity of the candidate derivative (3.18), apply the adjoint construction $((\cdot)^{\eta,\omega})^*$ pointwise to $\mathrm{D}(f^{-1}) : V^1 \times Y \mapsto X$ to obtain a scale map $(\mathrm{D}(f^{-1})^{\eta,\omega})^* : V^1 \times Y^* \mapsto X^*$ given by the diagram

$$\begin{tikzcd}
V^1 \times Y \arrow[r, "\mathrm{D}(f^{-1})"] \arrow[d, "\mathrm{id}_{V^1} \times \iota_\eta"', "\sim"] & X \arrow[d, "\iota_\omega", "\sim"'] \\
V^1 \times Y^* \arrow[r, "(\mathrm{D}(f^{-1})^{\eta,\omega})^*"] & X^*
\end{tikzcd} \tag{3.19}$$

Since f and the vertical maps in the diagram are sc^∞-diffeomorphisms, it follows that $(\mathrm{D}(f^{-1})^{\eta,\omega})^*$ is sc-smooth. Now simply note that by the chain rule for f of Proposition 2.25, we have $\mathrm{D}(h \circ f) = (\mathrm{D}(f^{-1})^{\eta,\omega})^* \circ (\mathrm{id}_{V^1} \times \mathrm{D}h) \circ \mathrm{diag}_{V^1} \circ f|_{U_1}$, and as such, $\mathrm{D}(h \circ f)$ is as smooth as $\mathrm{D}h$ is.

To prove the Frèchet condition (3.14), we first note that since $h : V_1 \mapsto \mathbb{R}$ is C^1, the fundamental theorem of calculus together with the hypothesis that f is symplectic gives for $x \in U_1$ and $t \in X_1$ small

$$\begin{aligned}
&h(f(x+t)) - h(f(x)) - \mathrm{D}h(f(x)) \circ (\mathrm{D}f(x)^{-1})^{\eta,\omega} \cdot t \\
&= \int_0^1 \mathrm{D}h(af(x+t) + (1-a)f(x)) \cdot (f(x+t) - f(x)) \,\mathrm{d}a - \mathrm{D}h(f(x)) \circ \mathrm{D}f(x) \cdot t
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&= \int_0^1 \mathrm{D}h(af(x+t) + (1-a)f(x)) \cdot (f(x+t) - f(x) - \mathrm{D}f(x) \cdot t) \,\mathrm{d}a \\
&\quad + \int_0^1 [\mathrm{D}h(af(x+t) + (1-a)f(x)) - \mathrm{D}h(f(x))] \circ \mathrm{D}f(x) \cdot t \,\mathrm{d}a \,.
\end{aligned} \tag{3.21}$$

The remainder of the proof is similar to the original proof for sc^1 maps [32, Theorem 2.16]. The integrand of first term in (3.21) divided by $\|t\|_{X_1}$ converges to 0 uniformly in $a \in [0, 1]$ as $t \to 0$ in X_1 due to the sc-differentiability of f and the continuity of $\mathrm{D}h : V_1 \mapsto Y_0^*$. In turn, the second integrand term divided by $\|t\|_{X_1}$ converges uniformly to 0 due to the compactness of $\{\mathrm{D}_x f \cdot \frac{t}{\|t\|_1} : t \in X_1 \setminus \{0\}\} \subseteq Y_0$ and again the continuity of $\mathrm{D}h$. $\diamond$

Remark 3.14. As hinted in Remark 3.8(a), we could have loosened the sc^0-continuity of $\mathrm{D}h$ to the requirement that $(\mathrm{D}h)_m : V_{m+1} \times Y_{-m} \mapsto \mathbb{R}, (x, \xi) \mapsto \mathrm{D}_x h \cdot \xi$ be continuous for each $m \geq 0$, and the last paragraph of this proof would still hold as in the original proof of Hofer. Nevertheless, this weaker requirement would need the continuity of $\mathrm{D}(f^{-1})_m : V_{m+1} \mapsto \mathrm{B}(Y_m, X_m)$ with respect to a stronger topology on $\mathrm{B}(Y_m, X_m)$ than the compact-open topology to prove the continuity of $(\mathrm{D}(h \circ f))_m : U_{m+1} \times X_{-m} \mapsto \mathbb{R}$. The issue here is that for this alternative definition, we would need to endow each space in the Banach scales X^* and Y^* with the compact-open topology, with the consequence that the vertical maps in (3.19) would not be levelwise homeomorphisms anymore.

3.3 Hamiltonian Flows on Symplectic Scale Manifolds

The aim of this section is to generalize the concepts of Section 3.2 to the case of an sc^∞-manifold. To accomplish this task we need to introduce several new structures, such as an extension of the tangent scales to the negative indices, cotangent bundles and strong sc-smooth maps on manifolds. It turns out that the crucial requirement to enable this is that the transition maps of the sc^∞-manifold are symplectic. This condition gives rise to the concept of a symplectic sc^∞-manifold, where we can define the new objects appealing to the local model by means of a coordinate chart. Due to the assumption on the transition maps, the result is independent of the coordinate chart used to define the structure. Once the desired structures are formed, we obtain an elegant, direct and natural generalization of Hamiltonian vector fields and flows for the case of sc^∞-manifolds.

We begin by defining symplectic sc-smooth manifolds. The definition is similar to the definition of general sc-smooth manifolds, but we require the transition maps to be symplectic.

Definition 3.15. Let (X, ω) be a symplectic Banach scale on $\mathbb{Z}$ and let M be an sc^∞-manifold locally modeled on $X_{\geq 0} = X|_{\mathbb{N}_0}$.

(a) Two coordinate charts (U, ϕ) and (U', ψ) of M are said to be symplectically compatible if the transition map $\psi \circ \phi^{-1} : \phi(U \cap U') \mapsto \psi(U \cap U')$ is an sc^∞-symplectomorphism of (X, ω).

(b) A symplectic atlas for M is an atlas $\mathcal{A} = \{(U_a, \phi_a)\}_{a \in A}$ for M such that (U_a, ϕ_a) and (U_b, ϕ_b) are symplectically compatible for all $a, b \in A$.

(c) If a symplectic atlas $\mathcal{A}$ for M exists, it is contained in a unique maximal symplectic atlas $\bar{\mathcal{A}}$. The pair $(M, \bar{\mathcal{A}})$ is then said to be a symplectic sc^∞-manifold locally modeled on (X,ω) and $\bar{\mathcal{A}}$ is its symplectic sc-smooth structure. Usually, the latter is suppressed from notation.

Unless otherwise stated, we always take coordinate charts of a symplectic sc^∞-manifold from its symplectic sc-smooth structure, whence the transition maps are always assumed to be symplectic.

Example 3.16. The projectivization of a complex separable Hilbert space X of Example 2.35 is a symplectic sc^∞-manifold if we endow the induced scale $\underline{X}$ with its standard symplectic structure. Again $X = l^2_{\mathbb{C}}$ without loss of generality, since the isometric isomorphism $\underline{X} \overset{\sim}{\longmapsto} l^2_{\mathbb{C}}$ is symplectic (by definition). With U_a as in Example 2.35, $a \in \mathbb{Z}$, and being $B = \{x \in X_0 : \|x\|_0 < 1\}$ the unit ball of $X = X_0$, we can define a symplectic atlas $\{(U_a, \psi_a)\}_{a\in\mathbb{Z}}$ with $\psi_a : U_a \overset{\sim}{\longmapsto} B \subseteq X_0$ given by

$$\psi_a([x])_n = \frac{|x_a|}{x_a\|x\|_0} \cdot \begin{cases} x_n & \text{if } n < a \\ x_{n+1} & \text{if } n \geq a\,. \end{cases} \tag{3.22}$$

For a symplectic sc^∞-manifold M modeled on (X,ω) and $p \in M_{m+1}$, $m \in \mathbb{N}_0 \cup \{\infty\}$, choose a coordinate chart (U,ϕ) of M around p. By following the procedure of Remark 2.11 with the induced isomorphism of scales $\mathrm{D}_p\phi : \mathrm{T}_pM \overset{\sim}{\longmapsto} X|_{m+1}$, we can extend the tangent scale T_pM to a scale on $\{-m, -m+1, \ldots, m\}$ (on $\mathbb{Z}$ if $m = \infty$). Recall that this is done by declaring $(\mathrm{T}_pM)_k := (\mathrm{T}_pM)^*_{-k}$ and $(\mathrm{D}_p\phi)_k := ((\mathrm{D}_p\phi)^{-1}_{-k})^* : (\mathrm{T}_pM)_k \overset{\sim}{\longmapsto} X^*_{-k} \overset{\iota_\omega}{\underset{\sim}{\longleftarrow}} X_k$ for $-m \leq k < 0$, subsequently pulling back the bonding maps of $X|_{\{-m,-m+1,\ldots,0\}}$. Due to the invariance of the transition maps of M under ω, the identification map $(\iota_{\omega_p})_0 : (\mathrm{T}_pM)_0 \overset{\sim}{\longmapsto} (\mathrm{T}_pM)^*_0$ defined by the diagram

$$\begin{tikzcd} (\mathrm{T}_pM)_0 \arrow[r, "(\iota_{\omega_p})_0", "\sim"'] \arrow[d, "(\mathrm{D}_p\phi)_0"', "\sim"] & (\mathrm{T}_pM)^*_0 \arrow[d, "\sim"', "((\mathrm{D}_p\phi)^{-1}_0)^*"] \\ X_0 \arrow[r, "(\iota_\omega)_0", "\sim"'] & X^*_0 \end{tikzcd} \tag{3.23}$$

is independent of the chosen chart (U,ϕ). Consequently, the bonding maps of the extended scale are independent of the chosen chart as well. This independence will allow us to pull the local symplectic structure back to T_pM further on. Note also that since X is a reflexive scale, T_pM admits a dual scale T^*_pM, which is also a scale on $\{-m, -m+1, \ldots, m\}$.

A structure which follows from extended tangent scales is the cotangent bundle. Once this structure is defined, we can introduce sc^k sections in the usual manner.

Proposition 3.17. *Let M be a symplectic sc^∞-manifold locally modeled on (X,ω). Then, the cotangent bundle*

$$\mathrm{T}^*M := \cup_{p\in M_1}\{p\} \times (\mathrm{T}_pM)^*_0 \tag{3.24}$$

is an sc^∞-manifold locally modeled on $X^1_{\geq 0} \times X^*$. Its induced filtration is $(\mathrm{T}^*M)_m = \cup_{p\in M_{m+1}}\{p\} \times (\mathrm{T}_pM)^*_{-m}$, $m \in \mathbb{N}_0$, where $(\mathrm{T}_pM)^*_{-m} \subseteq (\mathrm{T}_pM)^*_0$ via the adjoint of the *bonding map* $(\mathrm{T}_pM)_0 \hookrightarrow (\mathrm{T}_pM)_{-m}$. *Also, the bundle projection* $\pi_{\mathrm{T}^*M} : \mathrm{T}^*M \mapsto M^1$, $(p,\alpha) \mapsto p$ *is a surjective* sc*-smooth submersion and the fiber* $\pi^{-1}_{\mathrm{T}^*M}(p)$ *over* $p \in M_{m+1}$ *is* $\{p\} \times (\mathrm{T}_pM)^*_0 \cong (\mathrm{T}_pM)^*_0$ *with the scale structure of* $(\mathrm{T}^*_pM)_{\geq 0}$*:* $(\mathrm{T}_pM)^*_{-m} \subseteq (\mathrm{T}_pM)^*_{-m+1} \subseteq \ldots \subseteq (\mathrm{T}_pM)^*_0$.

Proof. The methodology is similar to the construction of the tangent bundle. Letting $\pi : \mathrm{T}^*M \mapsto M_1$ be solely a map of sets in the first place, we define $\mathrm{T}^*U := \pi^{-1}_{\mathrm{T}^*M}(U_1)$ for $U \subseteq M$ open. A coordinate chart (U,ϕ) of M induces a bijection $\mathrm{T}^*\phi : \mathrm{T}^*U \mapsto V_1 \times X^*_0$, $(p,\alpha) \mapsto (\phi(p), ((\mathrm{D}_p\phi)^{-1}_0)^* \cdot \alpha)$. If (U',ψ) is an additional coordinate chart we have, on the one hand, $(\mathrm{D}_x(\psi\phi^{-1})^{\omega,\omega}_0)^* = (\mathrm{D}_x(\psi\phi^{-1})^{-1}_0)^* : X^*_0 \xrightarrow{\sim} X^*_0$ for all $x \in \phi(U \cap U')_1$ since $\psi\circ\phi^{-1} : \phi(U\cap U') \xrightarrow{\sim} \psi(U\cap U')$ is an sc$^\infty$-symplectomorphism. On the other hand, using a diagram similar to (3.19), we see that $(\mathrm{D}(\psi\phi^{-1})^{\omega,\omega})^* : \phi(U \cap U')^1 \times X^* \mapsto X^*$ is sc-smooth, where the operations are taken pointwise. Together, these conditions imply that the induced transition map $\mathrm{T}^*\psi \circ (\mathrm{T}^*\phi)^{-1} : \phi(U \cap U')^1 \times X^* \xrightarrow{\sim} \psi(U \cap U')^1 \times X^*$ preserves scales and is sc-smooth. The remaining statements are easily verified. $\diamond$

Definition 3.18. An sc^k section of the cotangent bundle of a symplectic sc$^\infty$-manifold M, $k \in \mathbb{N}_0 \cup \{\infty\}$, is an sc^k map $s : M^1 \mapsto \mathrm{T}^*M$ with $\pi_{\mathrm{T}^*M} \circ s = \mathrm{id}_{M^1}$.

As suggested above, extended tangent scales and cotangent bundles enable us to pull the symplectic form of the local model back to the symplectic sc$^\infty$-manifold. We do this by letting the diagram in (3.23) operate on the entire scale. For a symplectic sc$^\infty$-manifold M locally modeled on (X,ω) and $p \in M_{m+1}$, $m \in \mathbb{N}_0 \cup \{\infty\}$, define thus, using a coordinate chart $(U(p),\phi)$ of M, an isomorphism of scales $\iota_{\omega_p} : \mathrm{T}_pM \xrightarrow{\sim} \mathrm{T}^*_pM$ by the diagram

$$\begin{array}{ccc} \mathrm{T}_pM & \xrightarrow[\sim]{\iota_{\omega_p}} & \mathrm{T}^*_pM \\ {\scriptstyle \mathrm{D}_p\phi}\downarrow\wr & & \wr\downarrow{\scriptstyle ((\mathrm{D}_p\phi)^{-1})^*} \\ X & \xrightarrow[\sim]{\iota_\omega} & X^* \end{array},$$

where X is restricted to $\{-m,-m+1,\ldots,m\}$. From this diagram, we obtain corresponding skew-symmetric bilinear maps $\omega_p := (v,w) \mapsto \iota_{\omega_p}(w)\cdot v : (\mathrm{T}_pM)_{-k} \times (\mathrm{T}_pM)_k \mapsto \mathbb{R}$, $k \in \{-m,-m+1,\ldots,m\}$. Again, this is a coordinate-free definition. These isomorphisms collect to an sc$^\infty$-diffeomorphism $\iota_\omega : \mathrm{T}M \xrightarrow{\sim} \mathrm{T}^*M$, $(p,v) \mapsto (p, \iota_{\omega_p}(v))$ which maps fibers of $\mathrm{T}M$ to fibers of T^*M and is linear on each fiber, or in other words, it is an isomorphism of sc-smooth vector bundles[3].

The final coordinate-free structure that we introduce on symplectic sc$^\infty$-manifolds are strongly densely-defined sc^k maps. This is a simple generalization of the concept in Section 3.2 and its coordinate independence is a direct consequence of Proposition 3.13 and of the locality of the strong sc^k conditions. The proof of well-definedness is a simple manipulation of the concepts introduced so far and will be omitted.

Definition 3.19. Let M be a symplectic sc$^\infty$-manifold locally modeled on (X,ω).

[3]For simplicity and lack of necessity thereof in this document, we do not define the category of sc-smooth vector bundles in their full generality, only noting that $\mathrm{T}M$ and T^*M belong to it.

(a) An sc^0 map $h : M^1 \mapsto \mathbb{R}$ is said to be strongly (densely-defined) sc^k, $k \in \mathbb{N} \cup \{\infty\}$, if for all $p \in M_1$ there exists a coordinate chart $\phi : U(p) \overset{\sim}{\longmapsto} V \subseteq X_0$ of M such that $h \circ \phi^{-1} : V^1 \mapsto \mathbb{R}$ is strongly densely-defined sc^k.

(b) For a strongly sc^k map $h : M^1 \mapsto \mathbb{R}$, we define at each $p \in M_{m+1}$, $m \in \mathbb{N}_0$, the m^{th} level derivative of h at p to be $(\mathrm{D}_p h)_m := \mathrm{D}_{\phi(p)}(h \circ \phi^{-1}) \circ (\mathrm{D}_p \phi)_{-m} \in (\mathrm{T}_p M)^*_{-m}$.

(c) Varying p above, we obtain the derivative of h: an sc^{k-1} section of the cotangent bundle $\mathrm{D}h : M^1 \mapsto \mathrm{T}^*M$, $M_{m+1} \ni p \mapsto (p, (\mathrm{D}_p h)_m) \in (\mathrm{T}^*M)_m$, $m \in \mathbb{N}_0$.

The technical work carried out above drastically facilitates the task of generalizing Hamiltonian vector fields and flows to symplectic sc^∞-manifolds. Indeed, this is now a question of seamlessly combining the toolkit developed in this document. First, for an sc^∞-manifold M (not compulsorily symplectic), we define an autonomous sc-smooth vector field $V : M^1 \mapsto \mathrm{T}M$ simply to be an sc-smooth section of the tangent bundle as in Definition 2.41.

As seen at the end of Section 2.4, sc-smooth maps $\varphi : \mathbb{R} \times M \mapsto M$ define a partial derivative $\frac{\partial \varphi}{\partial t}(t,p) \in (\mathrm{T}_{\varphi(t,p)} M)_m$ for $t \in \mathbb{R}$ and $p \in M_{m+1}$, $m \in \mathbb{N}_0$. Varying t and p, we obtain an sc-smooth map $\frac{\partial \varphi}{\partial t} : \mathbb{R} \times M^1 \mapsto \mathrm{T}M, (t,p) \mapsto (\varphi(t,p), \frac{\partial \varphi}{\partial t}(t,p))$, which is locally the diagonal product of φ in local coordinates and its partial derivative as a map of scales, as used in Section 3.1. An sc-smooth vector field $V : M^1 \mapsto \mathrm{T}M$ is then said to have a global flow if there exists an sc-smooth map $\varphi : \mathbb{R} \times M \mapsto M$ such that

$$\frac{\partial \varphi}{\partial t}(t,p) = V \circ \varphi(t,p) \qquad\qquad \varphi(0,p) = p \tag{3.25}$$

for all $t \in \mathbb{R}$ and $p \in M_{m+1}$, $m \in \mathbb{N}_0$. As in the scale case, one only needs to check this condition for $m = 0$ and we can further recover the vector field as $V = \frac{\partial \varphi}{\partial t}(0, \cdot)$.

For a symplectic sc^∞-manifold M, a strongly sc-smooth map $h : M^1 \mapsto \mathbb{R}$ gives rise to an sc-smooth vector field $V_h : M^1 \mapsto \mathrm{T}M$ uniquely defined by the relation

$$-\,\mathrm{D}h = \iota_\omega \circ V_h : M^1 \mapsto \mathrm{T}^*M\,, \tag{3.26}$$

since $\iota_\omega : \mathrm{T}M \overset{\sim}{\longmapsto} \mathrm{T}^*M$ is an sc-smooth vector bundle isomorphism (as described above). This equation reads pointwise as $-(\mathrm{D}_p h)_m = \omega_p(\cdot, V_h(p))$ for $p \in M_{m+1}$, $m \in \mathbb{N}_0$. The definition of a Hamiltonian vector field is now apparent.

Definition 3.20. Let M be a symplectic sc^∞-manifold and $\iota_\omega : \mathrm{T}M \overset{\sim}{\longmapsto} \mathrm{T}^*M$ be the induced sc-smooth vector bundle isomorphism. An sc-smooth vector field $V : M^1 \mapsto \mathrm{T}M$ is said to be Hamiltonian if there exists a strongly sc-smooth map $h : M^1 \mapsto \mathbb{R}$ such that

$$-\,\mathrm{D}h = \iota_\omega \circ V : M^1 \mapsto \mathrm{T}^*M\,. \tag{3.27}$$

If a Hamiltonian vector field has a global flow $\varphi : \mathbb{R} \times M \mapsto M$, the flow is said to be Hamiltonian.

Example 3.21. The flow of the free Schrödinger equation in Example 3.1 on the Banach scale $X_{\geq 0}$, with $X = \{W^{2k,2}(S^1, \mathbb{C})\}_{k\in\mathbb{Z}}$, descends to the projectivization $M = \mathrm{P}(X_0)$, and the descended flow map $\bar{\varphi} : \mathbb{R} \times M \mapsto M$ is easily seen to be sc-smooth. We claim that this flow is Hamiltonian. In the first place, with the coordinates of Example 3.16, the vector field $\bar{V} = \frac{\partial\bar{\varphi}}{\partial t}(0, \cdot)$ is given by $\bar{V}_a := \mathrm{pr}_2^{\mathrm{T}B}\, \mathrm{T}(\psi_a)\, \bar{V}\, \psi_a^{-1} : B^1 \mapsto X$, $u \mapsto \mathrm{i}\sigma_a(u)$, $a \in \mathbb{Z}$, where $\mathrm{pr}_2^{\mathrm{T}B} : B^1 \times X \mapsto X$ is the canonical projection and $\sigma_a : X^1 \mapsto X$ is the sc^0 Fourier multiplier with coefficients

$$(\widehat{\sigma_a})_n = \begin{cases} a^2 - n^2 & \text{if } n < a \\ a^2 - (n+1)^2 & \text{if } n \geq a\,. \end{cases} \tag{3.28}$$

In the second place, we can define the Hamiltonian function

$$\bar{h} : M^1 \mapsto \mathbb{R},\ u \mapsto \frac{1}{2}\frac{\|u_x\|_0^2}{\|u\|_0^2}\,. \tag{3.29}$$

This is a densely-defined strongly sc-smooth map with derivative locally given by $\mathrm{D}(\bar{h}_a) : B^1 \mapsto X^*$, $x \mapsto -\langle\sigma_a(x), \cdot\rangle_0$, where $\bar{h}_a := \bar{h} \circ \psi_a^{-1}$. Finally, we conclude that if $\omega = \langle \mathrm{i}\cdot, \cdot\rangle_0$ is the standard symplectic structure on X, $\bar{h}$ generates $\bar{V}$.

Remark 3.22. By analogy with the finite-dimensional case, Example 3.21 can be interpreted in the trend of symplectic reduction. Consider the standard action of S^1 on X given by pointwise multiplication. Defining the (S^1-invariant) momentum map $\mu : X_{\geq 0} \mapsto \mathbb{R}$

$$\mu(x) = \frac{1}{2}(1 - \|x\|_0^2)\,, \tag{3.30}$$

we see that the action is free on $\mu^{-1}(0) = \mathrm{S}(X_0)$ and that the projective Hilbert space is

$$M \cong \mu^{-1}(0)\big/ S^1,\ [u] \mapsto \left[\frac{u}{\|u\|_0}\right]\,. \tag{3.31}$$

The Hamiltonian $h : X^1_{\geq 0} \mapsto \mathbb{R}$ of Example 3.10 is S^1-invariant and descends to the Hamiltonian $\bar{h} : M^1 \mapsto \mathbb{R}$ of Example 3.21 under this identification. Similarly, the flow $\varphi : \mathbb{R} \times X_{\geq 0} \mapsto X_{\geq 0}$ of Example 3.1 descends to $\bar{\varphi} : \mathbb{R} \times M \mapsto M$ and, as seen in Example 3.21, $\bar{\varphi}$ is generated by $\bar{h}$.

4 Conclusion

This thesis introduced symplectic scale manifolds and corresponding Hamiltonian vector fields and flows. As backbone, we used the scale smoothness and manifold theory of Hofer-Wysocki-Zehnder's work on polyfolds. To cope with the regularity requirements of Hamiltonian functions, we defined the concept of strong scale smoothness which refines the original concept by Hofer. The new concept is invariant upon pre-composition with scale smooth symplectomorphisms, hence it fits perfectly into Hofer's framework. We further defined the cotangent bundle and bundle isomorphism of the symplectic structure, concepts which follow directly from the definition of a symplectic scale manifold. From the introduced concepts, we generalized the classical symplectic gradient relation from finite dimensions. We developed and verified the concepts at the hand of the free Schrödinger equation.

To solidify the theory presented in this thesis, further validation with additional Hamiltonian PDEs is needed. Indeed, the free Schrödinger equation is linear and completely described using the Fourier series, while Hamiltonian PDEs of interest are non-linear. In any case, for most common Hamiltonian PDEs, the vector field is given by pointwise products of derivatives. Since Sobolev spaces are algebras under pointwise multiplication if their order is high enough, we expect it to be possible to cast vector fields of interest into sc-smooth maps for an adequate Sobolev scale. Naturally, as lightly touched upon in Section 3.1, even if the vector field is sc-smooth and generated by a Hamiltonian function, whether the flow exists remains a rather involved question.

An interesting direction for further research would be to push the boundaries of what we can generalize from finite-dimensional symplectic geometry. For example, one could investigate symplectic, (co-)isotropic, Lagrangian sc-smooth submanifolds and normal form theorems. In particular, we have defined symplectic sc$^\infty$-manifolds on the basis of the transition maps. It would be interesting to prove a version of Darboux's theorem asserting the equivalence of our definition with, in some sense, the closedness of the induced symplectic form.

A particularly relevant achievement in finite dimensions which would be important to generalize in this context is symplectic reduction. Our example of reduced space — the projectivization of L^2 — is obtained using an everywhere defined momentum map $\mu : L^2 \mapsto \mathbb{R}$ which is smooth on each layer of the scale induced by L^2. The result is a manifold which is itself a smooth Banach manifold where the transition maps are smooth on each layer. A natural question which arises is whether this can be done more generally, allowing the momentum map only to be densely-defined and/or having rougher smoothness properties, and what manifold structure the reduced space would acquire. Also, one would like to construct relevant examples of Hamiltonian manifolds where the reduced space does get a rougher structure than a smooth Banach manifold on each layer (*e.g.*, an sc-smooth manifold).

Finally, it could be enlightening to use this theory in symplectic topology, for example to study non-squeezing of symplectic sc-smooth maps or to count fixed points of Hamiltonian sc^∞-diffeomorphisms (*i.e.*, to study variants of Arnold's conjecture in this context). The last years have seen an increase of interest in the generalization of these problems from finite to infinite dimensions, with solutions ranging from finite-dimensional approximations to non-standard analysis. More on this topic can be found in [1, 23, 24] and the references therein.

The present work contributes by introducing notions of symplectic geometry for scale manifolds geared towards the formalization of Hamiltonian PDEs. We hope that this work will inspire mathematicians and physicists on their way to better understand the geometrical aspects of these, and to grasp the concept of symplectic geometry in infinite dimensions.

Bibliography

[1] A. Abbondandolo and P. Majer. A non-squeezing theorem for convex symplectic images of the Hilbert ball. *Calc. Var. and PDE*, 54:1469–1506, Oct. 2015.

[2] R. Abraham and J. E. Marsden. *Foundations of Mechanics*. Addison-Wesley, 2nd edition, 1987.

[3] N. Alon, J. Bourgain, A. Connes, M. Gromov, and V. Milman, editors. *Problems in Hamiltonian PDE's*, pages 32–56. Birkhäuser, Basel, 2010.

[4] H. Amann. *Linear and quasilinear parabolic problems. Volume I: Abstract linear theory*. Monographs in Mathematics. Birkhäuser, Basel, 1995.

[5] Z. Ammari and Q. Liard. On the uniqueness of probability measure solutions to Liouville's equation of Hamiltonian PDEs. *ArXiv e-prints*, Feb. 2016. `arXiv:1602.06716`.

[6] A. Ashtekar and T. A. Schilling. Geometrical formulation of quantum mechanics. *ArXiv e-prints*, Jun. 1997. `arXiv:gr-qc/9706069`.

[7] J. C. Baez and A. E. Hoffnung. Convenient categories of smooth spaces. *ArXiv e-prints*, Jul. 2008.

[8] J. L. Bona and R. L. Sachs. Global existence of smooth solutions and stability of solitary waves fora generalized Boussinesq equation. *Commun. Math. Phys.*, 118(1):15–29, Mar. 1988.

[9] R. Bonic. Some properties of Hilbert scales. *Proc. Amer. Math. Soc.*, 18:1000–1003, 1967.

[10] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part I: Schrödinger equations. *Geometric and Functional Analysis*, 3(2):107–156, 1993.

[11] O. Caps. *Evolution equations in scales of Banach spaces*. Teubner, 2002.

[12] R. Castelli, M. Gameiro, and J.-P. Lessard. Rigorous numerics for ill-posed PDEs: periodic orbits in the Boussinesq equation. *ArXiv e-prints*, Sep. 2015. `arXiv:1509.08648`.

[13] M. Christ. Nonuniqueness of weak solutions of the nonlinear Schrödinger equation. *ArXiv e-prints*, Mar. 2005. `arXiv:math/0503366`.

[14] J. B. Conway. *A course in functional analysis.* Springer Verlag, 2nd edition, 1990.

[15] W. Craig. Transformation theory of Hamiltonian PDE and the problem of water waves. *Hamiltonian Dynamical Systems and Applications*, pages 67–83, 2008.

[16] R. de la Llave. A smooth center manifold theorem which applies to some ill-posed partial differential equations with unbounded nonlinearities. *J. Dyn. Diff. Equations*, 21(3):371–415, 2009.

[17] V. Domínguez, N. Heuer, and F.-J. Sayas. Hilbert scales and Sobolev spaces defined by associated Legendre functions. *Journal of Computational and Applied Mathematics*, 235(12):3481–3501, 2011.

[18] E. Dubinsky. Projective and inductive limits of Banach spaces. *ICM Studia Mathematica*, XLII(3):259–263, 1972.

[19] C. H. Edwards Jr. *Advanced calculus of several variables.* Dover Publications, unabridged, corrected edition, 1994.

[20] H. Egger. Semiiterative regularization in Hilbert scales. Technical report, Johann Radon Institute for Computational and Applied Mathematics, Dec. 2004.

[21] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems.* Springer Netherlands, 2000.

[22] O. Fabert. Transversality problems in symplectic field theory and a new Fredholm theory. *ArXiv e-prints*, Mar. 2010. `arXiv:1003.0651`.

[23] O. Fabert. Infinite-dimensional symplectic non-squeezing using non-standard analysis. *ArXiv e-prints*, Jan. 2015. `arXiv:1501.05905`.

[24] O. Fabert. Floer theory for Hamiltonian PDE using model theory. *ArXiv e-prints*, Mar. 2017. `arXiv:1507.00482v2`.

[25] K. Floret. Some aspects of the theory of locally convex inductive limits. In K.-D. Bierstedt and B. Fuchssteiner, editors, *Functional Analysis: Surveys and Recent Results II*, volume 38 of *Mathematics Studies*, pages 205–238. North Holland, 1980.

[26] R. H. Fox. On topologies for function spaces. *Bull. Amer. Math. Soc.*, 51(6):429–432, 1945.

[27] M. Friesen. Linear evolution equations in scales of Banach spaces. *ArXiv e-prints*, Aug. 2016. `arXiv:1608.03138`.

[28] P. Garrett. Basic Categorial Constructions. URL: `http://www-users.math.umn.edu/~garrett/m/fun/Notes/06_categories.pdf`, Nov. 2010.

[29] P. Garrett. Functions on circles: Fourier series, I. URL: `http://www-users.math.umn.edu/~garrett/m/fun/notes_2012-13/04_blevi_sobolev.pdf`, Apr. 2013.

[30] A. Gerstenberger. A version of scale calculus and the associated Fredholm theory. *ArXiv e-prints*, Feb. 2016. `arXiv:1602.07108`.

[31] H. Goldstein, C. Poole, and J. Safko. *Classical Mechanics*. Addison Wesley, 3rd edition, 2002.

[32] H. Hofer, K. Wysocki, and E. Zehnder. A general Fredholm theory I: A splicing-based differential geometry. *ArXiv Mathematics e-prints*, Dec. 2006. `arXiv:math/0612604`.

[33] H. Hofer, K. Wysocki, and E. Zehnder. A general Fredholm theory II: Implicit function theorems. *ArXiv e-prints*, May 2007. `arXiv:0705.1310`.

[34] H. Hofer, K. Wysocki, and E. Zehnder. A general Fredholm theory III: Fredholm functors and polyfolds. *ArXiv e-prints*, Oct. 2008. `arXiv:0810.0736`.

[35] H. Hofer, K. Wysocki, and E. Zehnder. Lectures on polyfolds and applications I: Basic concepts and illustrations. URL: `http://www.cims.nyu.edu/~hofer/polyfolds/script1.pdf`, Dec. 2008.

[36] H. Hofer, K. Wysocki, and E. Zehnder. Sc-Smoothness, retractions and new models for smooth spaces. *ArXiv e-prints*, Feb. 2010. `arXiv:1002.3381`.

[37] H. Hofer, K. Wysocki, and E. Zehnder. Applications of polyfold theory I: The polyfolds of Gromov-Witten theory. *ArXiv e-prints*, Jul. 2011. `arXiv:1107.2097`.

[38] H. Hofer, K. Wysocki, and E. Zehnder. Polyfold and Fredholm Theory. *ArXiv e-prints*, Jul. 2017. `arXiv:1707.08941`.

[39] B. Hofmann and P. Mathé. Tikhonov regularization with oversmoothing penalty for non-linear ill-posed problems in Hilbert scales. *ArXiv e-prints*, May 2017. `arXiv:1705.03289`.

[40] P. Iglesias-Zemmour. *Diffeology*. American Mathematical Society, 2013.

[41] S. G. Krein and Y. I. Petunin. Scales of Banach spaces. *Russian Mathematical Surveys*, 21(2):85–159, 1966.

[42] A. Kriegl and P. W. Michor. *The convenient setting of global analysis*. American Mathematical Society, 1997.

[43] S. B. Kuksin. *Analysis of Hamiltonian PDEs*. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, Sept. 2000.

[44] S. Lang. *Introduction to differentiable manifolds*. Springer Verlag, 2nd edition, 2002.

[45] J. M. Lee. *Introduction to smooth manifolds*. Springer Verlag, 2nd edition, 2012.

[46] T. Leinster. *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Jul. 2014.

[47] G. Z. Liu. *Evolution equations and scales of Banach spaces*. PhD thesis, Technical University Eindhoven, Eindhoven, The Netherlands, Jan. 1989.

[48] D. McDuff and D. A. Salamon. *Introduction to symplectic topology*. Oxford University Press, New York, 2nd edition, 1998.

[49] P. Miller. Lecture 4: The Korteweg-de Vries Equation (Continuation), 2006. URL: `http://math.arizona.edu/~mcl/Miller/MillerLecture04.pdf`.

[50] J. R. Munkres. *Topology*. Prentice-Hall, Inc., 2nd edition, 2000.

[51] J. Murphy and F. Pusateri. Almost global existence for cubic nonlinear Schrödinger equations in one space dimension. *ArXiv e-prints*, May 2016. `arXiv:1605.03247`.

[52] T. Ozawa and J. Zhai. Global existence of small classical solutions to nonlinear Schrödinger equations. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 25(2):303–311, Jan. 2007.

[53] W. Rudin. *Functional Analysis*. McGraw-Hill, Inc., 2nd edition, 1991.

[54] H. H. Schaefer. *Topological Vector Spaces*. Graduate Texts in Mathematics 3. Springer Verlag, 1971.

[55] S. Sivaji-Ganesh. Lecture notes on partial differential equations. URL: `http://www.math.iitb.ac.in/~siva/ma51515/chapter2C.pdf`, 2015.

[56] A. Sjöberg. On the Korteweg-de Vries equation: Existence and uniqueness. *Math. Analysis and Applications*, 29(3):569–579, Mar. 1970.

[57] A. Stacey. Comparative Smootheology. *ArXiv e-prints*, Feb. 2008. `arXiv:0802.2225`.

[58] T. Tao. Global existence and uniqueness results for weak solutions of the focusing mass-critical non-linear Schrödinger equation. *ArXiv e-prints*, Jul. 2008. `arXiv:0807.2676`.

[59] G. Teschl. *Ordinary differential equations and dynamical systems*. American Mathematical Society, 2012.

[60] S. Wang and G. Chen. Small amplitude solutions of the generalized IMBq equation. *J. Math. Anal. Appl.*, 274(2):846–866, Oct. 2002.

[61] K. Wehrheim. Fredholm notions in scale calculus and Hamiltonian Floer theory. *ArXiv e-prints*, Sep. 2012. `arXiv:1209.4040`.

[62] E. Zeidler. *Nonlinear functional analysis and its applications I: Fixed-point theorems*. Springer Verlag, 1986.

[63] E. Zeidler. *Nonlinear functional analysis and its applications IV: Applications to mathematical physics*. Springer Verlag, corrected second printing edition, 1997.