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Notes on port-Hamiltonian systems

by

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Abstract

The theory of port-Hamiltonian systems provides a general framework for network modelling of physical systems. In this thesis, the focus lies on one specific application, namely the sloshing mechanic of a liquid in a mug while walking. The combination of the movement of the mug and the movement of the liquid represents a multiphysics system. This observation leads to the following research question: Can the combination of the liquid model and the mug model, taken from literature, be set in a port-Hamiltonian system form? The investigations show that unfortunately it is not possible to rewrite that model as a linear first order port-Hamiltonian system.

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Summary

Human beings sometimes spill liquid that is carried in an open mug. The spillage of the liquid often comes from the sloshing liquid that comes from the moving hand that holds the mug, or simply by walking around while holding the mug. In light of the increase in the use of robots as waiters in restaurants in the service industry, an accurate model describing the mug filled with liquid is necessary to avoid spillage for autonomous waiter robots.

One of the difficulties in modelling different kind of systems is the modelling of the interaction between these systems. In this case, the interaction between the liquid and the mug. Moving liquid has other physical characteristics than a solid mug. The different characteristics of the liquid and the mug can also be seen in their different mathematical models. Moving liquid can be described by the Navier-Stokes equations, an important set of nonlinear partial differential equations. The movement of a solid mug on the other hand can be described by relatively simple mechanical models, usually a set of ordinary differential equations.

There are at least two approaches to model the interactions of the mug and its liquid inside. The first is to model the mug and its liquid as a system using classical methods. In this thesis, a model for the mug and the liquid is used from the literature, that has been approached in the classical sense.

A second approach for modeling interactions between multiphysics systems, is to use a theory in which interactions are taking into account. Hamiltonian systems are very suited for modelling physical systems. The dynamics of the system is described by differential equations that involve the so-called Hamiltonian operator. The Hamiltonian represents the amount of energy in a system. An extension to Hamiltonian systems has appeared in the literature, namely port-Hamiltonian systems. The theory of port-Hamiltonian systems provides a general framework for network modelling of multiphysics systems. In port-Hamiltonian systems, the notion of energy is important for the interactions between multiphysics systems.

In this thesis, it is investigated if the combination of the liquid model and the model of the mug, taken from literature, can be set in a linear port-Hamiltonian system setting or not. The investigations show that it is not possible to rewrite that model as a linear first order port-Hamiltonian system. Given the relevance in the application of autonomous robotic waiters, it is recommended to model the mug and liquid directly in a port-Hamiltonian setting, presumably leading to a nonlinear port-Hamiltonian system.

Contents

1 Introduction

The theory of port-Hamiltonian systems provides a general framework for network modelling of multiphysics systems [1]. Port-Hamiltonian theory allows for the combination of different systems such as mechanical, thermal and electrical systems. Using a network approach, the subsystems can be connected, using ports, via the inputs and outputs of the system. This combination makes the usage of port-Hamiltonian models also very attractive in a systems and control setting for many applications. Since control systems interact with the controllable environment via these connections, many ideas of system and control theory can be expected to be set in a port-Hamiltonian setting, especially ideas that start from an interconnectivity point of view [2]. As the name implies, port-Hamiltonian theories also inherited many ideas and results from Hamiltonian theory. A classical Hamiltonian model of a physical system uses canonical coordinates to describe the system at any given point in time. The dynamics of the system are described by differential equations that involve the so-called Hamiltonian operator. The Hamiltonian represents the amount of energy in a system. In port-Hamiltonian systems, energy is given an important role, since the interconnection between multiphysics systems is energy driven [3]. An overview of the theory for port-Hamiltonian systems, including many references, can be found in [4].

Since the theory of port-Hamiltonian systems can also be applied to the combination of real-world and digital systems, the theory can also be applied to so-called cyber-physical systems [5]. In the modern days of digital systems, real-world application of the models requires numerical techniques [6].

It can be interesting to investigate if existing models of dynamical systems can be set in the port-Hamiltonian setting, such that by using the theory of port-Hamiltonian systems new results may be obtained that could perhaps not be obtained using the original model. Applying the theory of port-Hamiltonian systems may lead to insight in, for example, easy ways to stabilize a potentially unstable system using output feedback. In this thesis the focus lies on one specific application, namely the sloshing mechanic of a liquid in a mug while walking.

Sloshing can be a serious problem in satellites [7] as the sloshing can have serious effects on the stability of a satellite due to the interactions with the wall under micro-gravity conditions in space. Even if the liquid is confined to a closed space and spillage can not occur, avoiding sloshing altogether is a complex problem to solve [8]. Walking on Earth with a mug that is partially filled with a liquid can lead to sloshing and spillage. There have been earlier investigations on this subject, but in a recent paper [9] a model has been derived to mitigate the sloshing, and thus spillage. Some simplifications are made, such as that the mug is two-dimensional and placed on a smooth horizontal table that is forced to oscillate in one dimension via a spring connection.

The theory of port-Hamiltonian systems, as presented in [4], and the sloshing mug model presented in [9] are the starting points of this thesis. The combination of the mug and the liquid represents a multiphysics system. This observation leads to the following research question: Can the combination of the liquid model and the mug model be set in a linear port-Hamiltonian system form?

This research question is relevant as it is observed in [10] that there is an increase in the use of robots as waiters in restaurants in the service industry. Robots are cyber-physical systems. Setting the interaction of a liquid and the mug in a port-Hamiltonian context can be an important step towards autonomous robotic waiters that do not spill coffee, tea or beer on outdoor terraces that often have uneven floors.

The remainder of this thesis is as follows.

A few basic notations and definitions that will be used in later chapters, are introduced in Chapter 2.

Chapters 3 and 4 follow parts of the book *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, written by Birgit Jacob and Hans J. Zwart [4]. Chapter 3 serves as the finite dimensional introduction of Chapter 4, where the theory is extended to infinite dimensional spaces. In Chapter 3, basic system and control definitions like controllability and stabilizability are introduced, along with the beginning of port-Hamiltonian systems. An example is given to illustrate this theory.

Chapter 4 builds semigroup theory from the ground up, following [4]. Starting with its definition, along with the infinitesimal generator, the theory is set in a similar manner as the finite dimensional case. These operators are used to describe infinite dimensional port-Hamiltonian systems. These systems, along with their standard variant, are set in such a way they fit as a boundary control system. In order to use these boundary control systems, semigroups are used, in particular contraction semigroups. The chapter also has a few side tracks, such as exponential stability and group theory. Other references on semigroup and operator theory are [14] and [15].

In Chapter 5 it is investigated which types of partial differential equations can be rewritten as a port-Hamiltonian system. First order time derivatives, second order time derivatives and more complex systems are investigated. Some conditions on specific functions are derived for a system to be a port-Hamiltonian system.

In Chapter 6 the model in the paper *How to Mitigate Sloshing*, written by H. and J. R. Ockendon [9] is investigated. First, the model is derived in full detail. Next, an analytic solution is derived. This solution is then used to calculate so-called walking frequencies for which sloshing occurs. Using some of the results in Chapter 5, it is then investigated if the combination of the liquid model and the model of the mug can be rewritten as a port-Hamiltonian system. The difficulties encountered in this research are discussed.

2 Notations, Definitions and Spaces

This chapter will contain some notations, along with a few basic definitions and theory, that will be used later on. No proofs will be given. The more advanced definitions and theory will be provided in later chapters.

2.1 Notation

Transpose

For a real matrix A, the transpose of A is denoted by A^T. If A is complex, the conjugate transpose of A is denoted by A^*. The latter is sometimes referred to as Hermitian transpose.

Real and Complex space

As many of the finite dimensional theorems are valid on both the real and complex spaces, denote $\mathbb{K}$ as either $\mathbb{R}$ or $\mathbb{C}$, consistently.

2.2 Spaces

Hilbert Space

A space X is called a Hilbert space if it has an inner product $\langle \cdot, \cdot \rangle$, such that the associated norm defined by $\|x\| = \sqrt{\langle x, x \rangle}$ makes the space a complete metric space.

Bounded linear operators

For X_1 and X_2 Hilbert spaces, define

$$\mathcal{L}(X_1, X_2)$$

as the class of all bounded, linear operators from X_1 to X_2. If $X_1 = X_2$, this is instead denoted by $\mathcal{L}(X_1)$.

Lebesgue spaces

Let X be a Hilbert space, and Ω a closed subset of $\mathbb{R}$, then

$$L(\Omega;X) := \{f : \Omega \to X | \ \langle x, f(\cdot)\rangle \text{ is measurable for all } x \in X\},$$

and for $1 \leq p < \infty$

$$L^p(\Omega;X) := \{f \in L(\Omega;X) | \ \|f\|_p := \left(\int_\Omega \|f(t)\|_X^p dt\right)^{1/p} < \infty\},$$

and

$$L^\infty(\Omega;X) := \{f \in L(\Omega;X) | \ \|f\|_\infty := \sup_\Omega \text{ess} \|f(t)\|_X < \infty\}.$$

Square integrable

$$H^1([a,b];\mathbb{K}^n) := \{f(\zeta) \in L^2([a,b];\mathbb{K}^n) | \\ f \text{ is absolutely continuous and } \frac{df}{d\zeta} \in L^2([a,b];\mathbb{K}^n)\}.$$

Local

If a space has the subscript *loc*, it is assumed the input space consists of compact sets.

2.3 Definition

*-definite matrix

A square matrix A is said to be positive definite if $x^*Ax > 0$ for all column vectors x of appropriate size.
A square matrix A is said to be positive semi-definite if $x^*Ax \geq 0$ for all column vectors x of appropriate size.
Similarly, a square matrix A is said to be negative definite, negative semi-definite if $x^*Ax < 0$, respectively if $x^*Ax \leq 0$ for all column vectors x of appropriate size.

*-adjoint matrix

A matrix A is said to be self-adjoint if $A^* = A$.
A matrix A is said to be skew-adjoint if $A^* = -A$.

Operator norm

The norm of an operator A defined from a space X to another space, is defined as

$$\|A\|_X = \inf\{c \geq 0 | \ \|Ax\| \leq c\|x\| \ \forall x \in X\}.$$

Resolvent and Spectrum

Let X be a Banach space and let $A : D(A) \to X$ be a linear operator with domain $D(A) \subset X$. For any $\lambda \in \mathbb{C}$, define $A_\lambda := A - \lambda I$. λ is said to be a regular value if the inverse operator of A_λ, $(A - \lambda I)^{-1}$, exists and is a bounded linear operator. Then the resolvent set of A is defined by all the regular values of A,

$$\rho(A) := \{\lambda \in \mathbb{C} | \lambda \text{ is a regular value of } A\}.$$

The spectrum is defined as the complement of the resolvent set,

$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

By definition, for any $\lambda \in \sigma(A)$, the inverse of $(A - \lambda I)$ does not exist. In finite dimensional spaces, these values correspond exactly with the eigenvalues of the operator A. This does not hold for infinite dimensional spaces, the spectrum in this case can contain values that are not eigenvalues of A.

3 Finite dimensional

When talking about general finite dimensional systems in system theory, two different systems can be distinguished. Both work with an input, a box and an output. The box interacts with the input in some manner and gives the output. This can be as simple as addition or taking a single derivative, or more complex like describing an electrical network or the movement of a vibrating string over time.

The first type of system does nothing besides giving an output, the second system however uses the output as a form of feedback, and the output will be used as the new input. This is often used to control a system, which will be described in more detail later on. In terms of control theory, the second system is preferred. To illustrate this difference, see Figures 3.1 and 3.2.

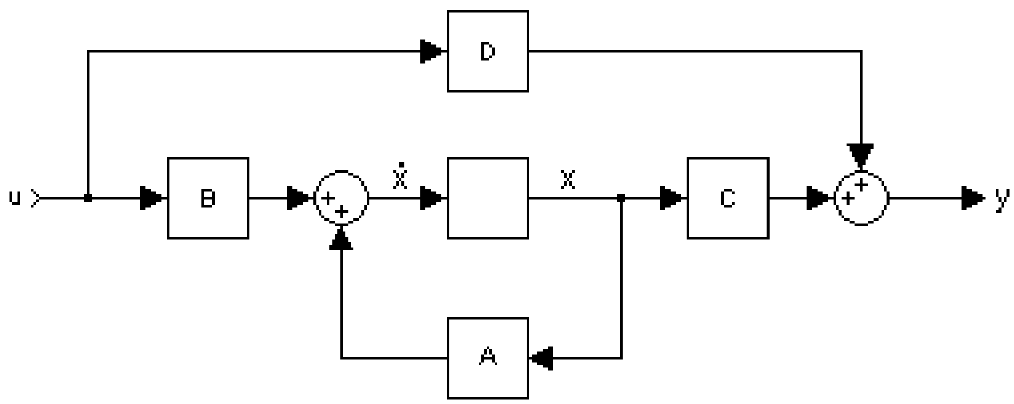

Figure 3.1: The first system, with input u and output y.

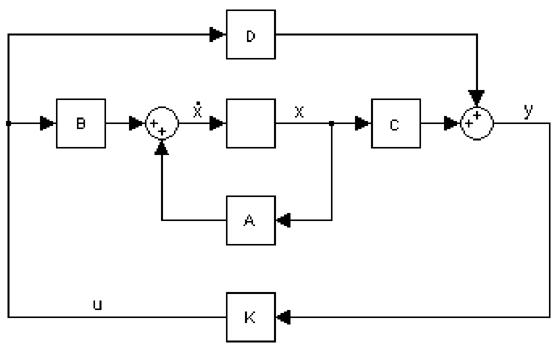

Figure 3.2: The second system, which is an extension of the first system via K.

As the main focus lies with infinite dimensional spaces, for which some of the following theory does not hold, and the finite dimensional systems serve merely as an introduction, most of the theory mentioned for finite dimensional spaces will not include a proof or sketch of a proof. Both the finite dimensional and the infinite dimensional case will follow the book [4]. See also [11] for finite dimensional continuous time systems, and [12] for discrete time systems.

3.1 Introduction

Using u for the input and y for the output, the first system can be described in the standard state space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3.1}$$

while the second system can be described in terms of the first system by the addition of a feedback loop using as input a linear function $u(t) = Fx(t)$ of the state, which is the so-called full observation case,

$$u(t) = Fx(t),$$

for the appropriate box and non-negative time t. Here $x, y \in \mathbb{K}^n$ are vectors and $A, C \in \mathbb{K}^{n\times n}$ and $B, D \in \mathbb{K}^{n\times m}$ and $F \in \mathbb{K}^{m\times n}$ are matrices. As the second system can be described in terms of the first system, the focus lies on the first system. In addition, it might be the case that the full state $x(t)$ is not available for use in a feedback controller (the partial observations case) in which case design of a controller may be based on inputs and outputs of the original system. This case will be discussed later on.

A first observation of the first system would be that the output y is not present in the differential equation of x, while the vector x is present in the equation for y. This means it is possible to solve the entire system for x first, and only then focus on solving the system for the output y. For this reason, it is useful for the moment to forget the output and focus mainly on the differential equation of the state space x itself. This thought process will also hold for the second system.

In order to solve the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t),$$

two things are needed. One can assume the information of the system that is modelled is known, in other words, the matrices A, B, C, D are known. In order to solve it, both an initial condition for $x(0)$ and the behaviour of u at every time t are needed. As u is the input of the system, it can be assumed this is known for all time. Hence in order to solve the system, it is only needed to specify an initial condition $x(0) = x_0$. Thus the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{3.2}$$

is obtained.

Suppose $u \in C([0,\infty);\mathbb{K}^m)$, then

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds \tag{3.3}$$

is an element of $C^1([0,\infty);\mathbb{K}^n)$. Furthermore

$$\begin{aligned}
\dot{x}(t) &= \frac{d}{dt}\left(e^{At}x(0)\right) + \frac{d}{dt}\left(\int_0^t e^{A(t-s)}Bu(s)ds\right)\\
&= Ae^{At}x(0) + e^{A(t-t)}Bu(t)\cdot\frac{d}{dt}t + \int_0^t \frac{d}{dt}e^{A(t-s)}Bu(s)ds\\
&= Ae^{At}x(0) + Bu(t) + \int_0^t Ae^{A(t-s)}Bu(s)ds\\
&= A\left(e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds\right) + Bu(t)\\
&= Ax(t) + Bu(t),
\end{aligned}$$

hence (3.3) is a solution to (3.2).

Conversely, suppose $u \in C([0,\infty);\mathbb{K}^m)$ and x is a solution of (3.2). Then

$$\begin{aligned}
\frac{d}{ds}\left(e^{A(t-s)}x(s)\right) &= e^{A(t-s)}\dot{x}(s) + (-A)e^{A(t-s)}x(s)\\
&= e^{A(t-s)}\left(Ax(s) + Bu(s)\right) - Ae^{A(t-s)}x(s)\\
&= e^{A(t-s)}Bu(s).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^t e^{A(t-s)}Bu(s)ds &= \int_0^t \frac{d}{ds}\left(e^{A(t-s)}x(s)\right)ds\\
&= e^{A(t-t)}x(t) - e^{A(t-0)}x(0)\\
&= x(t) - e^{At}x(0).
\end{aligned}$$

This implies

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds,$$

hence this solution is unique. The function (3.3) is a called the unique classical solution to (3.2). Thus for non-negative time, any continuous input leads to a unique classical solution for the system. If the input is not continuous, the classical solution will not suffice. In order to work with discontinuous functions, a different solution has to be introduced. These mild solutions do not consider all discontinuous input functions.

Suppose $u \in L^1_{loc}([0,\infty);\mathbb{K}^m)$, then the function $x \in C([0,\infty);\mathbb{K}^n)$ is called a mild solution of (3.2) if it satisfies

$$x(t) = x_0 + \int_0^t Ax(s) + Bu(s)ds.$$

3.2 Controllability

Besides the solution and mild solution for this system, there are two well-known properties that can be used to describe a system. The first is controllability. For the ease of notation, denote (3.2) by $\Sigma(A, B)$. Let $A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{n\times m}$. The system $\Sigma(A, B)$ is called controllable if for all $x_0, x_1 \in \mathbb{K}^n$ there exists a time $t_1 > 0$ and a function $u \in L^1((0, t_1); \mathbb{K}^m)$ such that the mild solution x of (3.2), given by (3.3), satisfies $x(t_1) = x_1$.

This means that every point can be moved to every other point in the space in finite time. If this is possible from the origin to every other point in space, it is called reachable instead. Hence controllability implies reachability. The notion of controllability is an important property to have in a system, since this implies that by choosing a specific input, the desired output of the system can be obtained. This is important in cases where the input of a system can not be exactly measured, and is merely an approximation of the actual input.

In order to check if a system is controllable, the controllability matrix $R(A, B)$ is defined by

$$R(A, B) := \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix},$$

which is an element of $\mathbb{K}^{n\times nm}$. One of the uses, and hence the name, lies in the rank of this matrix. If the rank of $R(A, B)$ equals n, the system $\Sigma(A, B)$ is controllable.

3.3 Stabilizability

The second property is called stabilizability. Let $A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{n\times m}$, then the system $\Sigma(A, B)$ is called stabilizable if for all $x_0 \in \mathbb{K}^n$ there exists a function $u \in L^1_{loc}((0, \infty); \mathbb{K}^m)$ such that the unique solution of (3.2) converges to zero as $t \to \infty$.

As controllability implies any point can be moved to any other point in the space, in particular, it can be moved towards zero. Thus controllability implies stabilizability.

In order to check for stabilizability, if one knows nothing about the controllability of the system, the pole placement problem can be used. This problem is stated as follows: Given $A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{n\times m}$ and complex numbers $\lambda_1, \dots, \lambda_n$, does there exist a matrix $F \in \mathbb{K}^{m\times n}$ such that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the matrix $A + BF$?

If this problem is solvable, the eigenvalues of $A + BF$ can be placed anywhere. This comes into play when the input is of the form $u(t) = Fx(t)$, which connects the discussion to the second system. Using this input, the differential equation changes into

$$\dot{x}(t) = Ax(t) + Bu(t) = (A + BF)\, x(t).$$

As the eigenvalues of $A+BF$ can be placed anywhere, the solution can be moved towards any point in space. Hence one of the reasons for using the pole placement problem, is that whenever it is solvable, the system $\Sigma(A, B)$ is controllable, which in turn implies stabilizability. Conversely, if the system is controllable, the pole placement problem is solvable.

It is also possible to change the definition of stability as follows: all of the possible solutions stay bounded and do not blow up over time, this is however not the definition that is used in this thesis. When talking about dynamical systems, this is often referred to as stable orbits.

A sufficient condition for the stability of a system, is that no eigenvalues of the matrix A lie on the imaginary axis. In particular, as most of the theory will be using a non-negative time, a sufficient condition is $Re(\lambda) < 0$ for all eigenvalues λ of the matrix. That is, all eigenvalues lie in the open left-half complex plane. For if this is not the case, solutions can not go to zero.

Checking if a system is stabilizable is in general hard or time consuming, even if it is known that a system is controllable, finding a desired feedback $u(t) = Fx(t)$ can be troublesome. It will however turn out to be quite easy for port-Hamiltonian systems, hence the reason for introducing these systems later on. Stabilizability is an important property for a system, as sometimes it is not possible to control all the components of a system, but if the system is stabilizable, at least these parts do not blow up.

3.4 Uniqueness

As can be seen in the upcoming example, the state space representation (3.1) is not unique. Multiple representations can be used to describe the same system. Suppose a system is represented for some A, B, C and D. Let $T \in \mathbb{K}^{n\times n}$ be an invertible matrix, and define a new representation by the transformation $\hat{x}(t) := T^{-1}x(t)$. This transforms (3.1) into

$$\begin{aligned}\dot{\hat{x}}(t) &= T^{-1}AT\hat{x}(t) + T^{-1}Bu(t),\\ y(t) &= CT\hat{x}(t) + Du(t),\end{aligned}$$

with initial condition $\hat{x}(0) = T^{-1}x_0$.

In order to describe the similarity between these two systems, let $A, \hat{A} \in \mathbb{K}^{n\times n}$ and $B, \hat{B} \in \mathbb{K}^{n\times m}$. Then the system $\Sigma(A, B)$ is called similar to $\Sigma(\hat{A}, \hat{B})$ if there exists an invertible matrix $T \in \mathbb{K}^{n\times n}$ such that $\hat{A} = T^{-1}AT$ and $\hat{B} = T^{-1}B$. If these two systems are similar, then

$$\begin{aligned}R(\hat{A}, \hat{B}) &= \begin{bmatrix}\hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B}\end{bmatrix}\\ &= \begin{bmatrix}T^{-1}B & T^{-1}ATT^{-1}B & T^{-1}ATT^{-1}ATT^{-1}B\end{bmatrix}\\ &= \begin{bmatrix}T^{-1}B & T^{-1}AB & T^{-1}AAB\end{bmatrix}\\ &= T^{-1}\begin{bmatrix}B & AB & A^2B\end{bmatrix}\\ &= T^{-1}R(A, B).\end{aligned}$$

As T is invertible, the rank of the two different controllability matrices is the same. Hence if one of the two systems is controllable, the other is as well.

Although there exist multiple representations of the same system, their properties do not depend on the representation itself, but on the system.

3.5 Port-Hamiltonian system

Having made a small introduction into more general system representations, it is time to do something more specific. In particular, a port-Hamiltonian system. A subclass of the systems in the form (3.1) is the class of port-Hamiltonian systems

$$\begin{aligned} \dot{x}(t) &= JHx(t) + Bu(t), \\ y(t) &= B^*Hx(t), \end{aligned} \tag{3.4}$$

where H is a positive definite matrix and J is a skew-adjoint matrix. Note that in this representation, the output does not directly depend on the input, $D = 0$, furthermore A is the form JH and $C = B^*H$ in the standard system representation. Since H is positive-definite, H is also self-adjoint. J is called the structure matrix and H is called the Hamiltonian density. The Hamiltonian associated to H is $\frac{1}{2}x^*Hx$. This can be seen as the kinetic energy of the system. This makes it natural to define the norm $\|\cdot\|_H$ on $\mathbb{K}^n$ as $\|x\|_H^2 := \frac{1}{2}x^*Hx$. Although there is mention of a Hamiltonian, the name port-Hamiltonian itself is best explained in the infinite dimensional case, see Section 4.6.

It is also possible to check if a port-Hamiltonian system is controllable and/or stabilizable. If $x(t)$ is a solution of a port-Hamiltonian system, then

$$\begin{aligned} \frac{d}{dt}\|x(t)\|_H^2 = \frac{d}{dt}\left(\frac{1}{2}x^*(t)Hx(t)\right) &= \frac{1}{2}x^*(t)H\dot{x}(t) + \frac{1}{2}\dot{x}^*(t)Hx(t) \\ &= \frac{1}{2}x^*(t)H\left(JHx(t) + Bu(t)\right) + \frac{1}{2}\left(JHx(t) + Bu(t)\right)^* Hx(t) \\ &= \frac{1}{2}\left(x^*(t)HJHx(t) + x^*(t)HBu(t) + x^*(t)H^*J^*Hx(t) + u^*(t)B^*Hx(t)\right) \\ &= \frac{1}{2}\left(x^*(t)HJHx(t) + y^*(t)u(t) - x^*(t)HJHx(t) + u^*(t)y(t)\right) \\ &= \frac{1}{2}\left(y^*(t)u(t) + u^*(t)y(t)\right) \\ &= Re\left(u^*(t)y(t)\right). \end{aligned}$$

Observe here that the change in energy over time depends only on the input and the output. If there is no input, $u \equiv 0$, the time derivative equals zero, hence $\|x(t)\|_H = \|x(0)\|_H$. Thus the energy of the system remains the same if there is no input. Because the norm does not change along the solutions in this case, the solution always has the same distance from the origin. Since the real part of an eigenvalue moves the solution either away or towards a fixed point, the real part of the eigenvalue has to be zero. The imaginary part of an eigenvalue allows the solution to spiral, which implies the norm of a solution does not have to change. Hence all eigenvalues of JH lie on the imaginary axis.

Now consider an input of the form $u(t) = -ky(t) = -kB^*Hx(t)$ for some $k > 0$, then

$$\frac{d}{dt}\|x(t)\|_H^2 = -k\frac{1}{2}\left(y^*(t)y(t) + y^*(t)y(t)\right) = -k\|y(t)\|^2,$$

which implies the norm of the solutions is non-increasing. Hence the energy of the system is non-increasing. As the solutions follows the equation

$$\dot{x}(t) = JHx(t) - kBB^*Hx(t) = (JH - kBB^*H)\,x(t),$$

the eigenvalues of $JH - kBB^*H$ lie all in the closed left half-plane. In order to prove that no eigenvalues lie on the imaginary axis, suppose by contradiction, that $\lambda \in i\mathbb{R}$ is an eigenvalue of $JH - kBB^*H$ with eigenvector $v \in \mathbb{C}^n$. Then

$$(JH - kBB^*H)\,v = \lambda v,$$

which implies both

$$\begin{aligned} v^*H\,(JH - kBB^*H)\,v &= v^*H\lambda v \\ v^*HJHv - kv^*HBB^*Hv &= \lambda v^*Hv, \end{aligned}$$

and

$$\begin{aligned} ((JH - kBB^*H)\,v)^*\,Hv &= (\lambda v)^*\,Hv \\ v^*H^*J^*Hv - kv^*H^*BB^*Hv &= \lambda^*v^*Hv \\ -v^*HJHv - kv^*HBB^*Hv &= -\lambda v^*Hv. \end{aligned}$$

Adding these two implications together results in

$$0 = kv^*H^*BB^*Hv = k\|B^*Hv\|^2.$$

As $k > 0$, this implies $B^*Hv = 0$. Hence

$$\lambda v = (JH - kBB^*H)\,v = JHv,$$

which in turn implies

$$\begin{aligned} (\lambda v)^* &= (JHv)^* \\ -\lambda v^* &= -v^*HJ. \end{aligned}$$

Combining this with $0 = B^*Hv$ results in

$$0 = \lambda\,(B^*Hv)^* = \lambda v^*HB = v^*HJHB.$$

This can be extended to

$$0 = v^*H(JH)^kB,$$

for all $k \in \mathbb{N}$. Hence by definition of the controllability matrix,

$$v^*H\,R(JH,B) = v^*H\begin{bmatrix} B & JHB & \dots & (JH)^{n-1}B \end{bmatrix} = 0.$$

If the system is controllable, then $v^*H = 0$, and hence $v = 0$. However, v was an eigenvector, which is a contradiction. Hence no eigenvalues lie on the imaginary axis.

This in turn implies all eigenvalues lie in the open left-half plane. Hence the system is stabilizable if it is controllable, with feedback $u(t) = -ky(t) = -kB^*Hx(t)$. Without stating the other half of the proof, a port-Hamiltonian system is controllable if and only if it is stabilizable.

This relation is the reason for using port-Hamiltonian systems. These systems are easily stabilized using the output feedback $u(t) = -ky(t)$, for any constant $k > 0$. Because these systems are stabilizable, they are also controllable. There is no need of going through any process of finding a feedback matrix F that has the desired properties, one only has to know what the output looks like.

3.6 Example

As an example for the finite dimensional theory, take the electrical network in Figure 3.3. The figure shows a closed system with a single voltage source V, two inductors, one on each side, and a single capacitor in the middle. Let L_1 and L_2 denote the inductance of each inductor and C the capacitance of the capacitor.

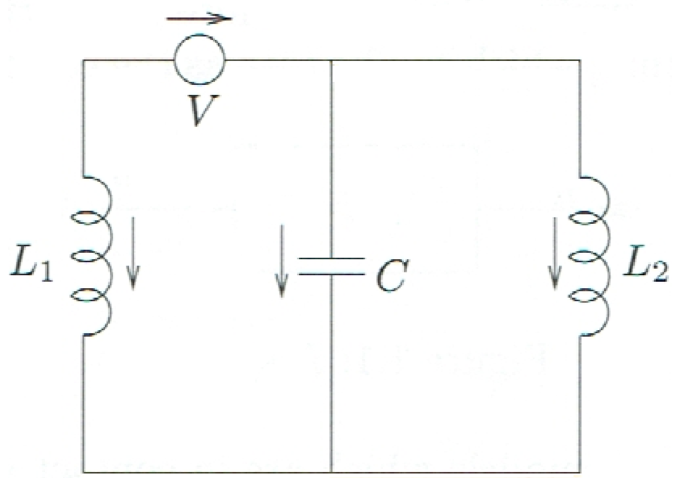

Figure 3.3: An electrical network.

This electrical network appears to be a port-based network. Each component of the system stores energy, and uses its ports to the other parts of the system to transport this energy. Hence this appears to be a candidate for a port-Hamiltonian system.

There are two relations which will be used to represent this electrical system in the standard form (3.1). The first relation links the voltage to the time derivative of the current of an inductor. The second relation links the current to the time derivative of the voltage of a capacitor. These relations are

$$
\begin{aligned}
L\frac{dI_L}{dt}(t) &= V_L(t),\\
C\frac{dV_C}{dt}(t) &= I_C(t).
\end{aligned}
$$

In order to convert the electrical network into a mathematical system, either the standard representation or a port-Hamiltonian system, requires a choice for both the input and the output. A logical input choice for u would be the voltage of the voltage source V. The output choice is less obvious, but suppose for the output y, the current I_{L_1} can be measured. Using above relations creates the following system of three differential equations to describe this network

$$
\begin{aligned}
L_1\frac{dI_{L_1}}{dt}(t) &= V_{L_1}(t) = V_C(t) + V(t),\\
L_2\frac{dI_{L_2}}{dt}(t) &= V_{L_2}(t) = V_C(t),\\
C\frac{dV_C}{dt}(t) &= I_C(t) = -I_{L_1}(t) - I_{L_2}(t).
\end{aligned}
$$

In order to write this as a system in the form of (3.1), a choice for the vector x is needed. One choice would be

$$
x(t) = \begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} = \begin{bmatrix} I_{L_1}(t)\\ I_{L_2}(t)\\ V_C(t)\end{bmatrix}.
$$

This vector can be rewritten using the three differential equations into

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} \frac{d}{dt}I_{L_1}(t)\\ \frac{d}{dt}I_{L_2}(t)\\ \frac{d}{dt}V_C(t)\end{bmatrix}\\
&= \begin{bmatrix} \frac{V_C}{L_1}(t) + \frac{V}{L_1}(t)\\ \frac{V_C}{L_2}(t)\\ -\frac{I_{L_1}}{C}(t) - \frac{I_{L_2}}{C}(t)\end{bmatrix}\\
&= \begin{bmatrix} \frac{x_3}{L_1}(t) + \frac{u}{L_1}(t)\\ \frac{x_3}{L_2}(t)\\ -\frac{x_1}{C}(t) - \frac{x_2}{C}(t)\end{bmatrix}\\
&= \begin{bmatrix} 0 & 0 & \frac{1}{L_1}\\ 0 & 0 & \frac{1}{L_2}\\ -\frac{1}{C} & -\frac{1}{C} & 0\end{bmatrix}\begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix} \frac{1}{L_1}\\ 0\\ 0\end{bmatrix} u(t).
\end{aligned}
$$

The equation for the output is then given by

$$
y(t) = I_{L_1} = \begin{bmatrix} 1 & 0 & 0\end{bmatrix}\begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}.
$$

Hence the system for this choice of x can be described by the matrices

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{L_1} \\ 0 & 0 & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix}, \ B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \ D = 0.$$

This representation is by no means unique, besides interchanging x_1 and x_3, and hence changing the matrices A, B and C, it is also possible to choose an entirely different state space vector. For example if the vector is changed to

$$\tilde{x}(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \end{bmatrix} = \begin{bmatrix} L_1 I_{L_1}(t) \\ L_2 I_{L_2}(t) \\ C V_C(t) \end{bmatrix}.$$

Then the output changes to

$$\tilde{y}(t) = I_{L_1} = \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \end{bmatrix}.$$

While taking the derivative gives the equation

$$\begin{aligned} \dot{\tilde{x}}(t) &= \begin{bmatrix} V_C(t) + V(t) \\ V_C(t) \\ -I_{L_1}(t) - I_{L_2}(t) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\tilde{x}_3}{C}(t) + u(t) \\ \frac{\tilde{x}_3}{C}(t) \\ -\frac{\tilde{x}_1}{L_1}(t) - \frac{\tilde{x}_2}{L_2}(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \frac{1}{C} \\ 0 & 0 & \frac{1}{C} \\ -\frac{1}{L_1} & -\frac{1}{L_2} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t). \end{aligned}$$

Hence the system for this vector can be described by the matrices

$$\tilde{A} = \begin{bmatrix} 0 & 0 & \frac{1}{C} \\ 0 & 0 & \frac{1}{C} \\ -\frac{1}{L_1} & -\frac{1}{L_2} & 0 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \end{bmatrix}, \ \tilde{D} = 0.$$

Comparing the two systems, the vectors are linked by

$$\tilde{x}(t) = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & C \end{bmatrix} x(t),$$

while the matrices are linked by

$$\tilde{A} = -A^T, \ \tilde{C} = B^T, \ \tilde{B} = C^T, \ \tilde{D} = D.$$

The next thing that can be checked for this system is controllability. Since both the controllability matrix

$$R(A,B) = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} & 0 & -\frac{1}{CL_1^2} \\ 0 & 0 & -\frac{1}{CL_1L_2} \\ 0 & -\frac{1}{CL_1} & 0 \end{bmatrix},$$

and the controllability matrix

$$R(\tilde{A},\tilde{B}) = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{CL_1} \\ 0 & 0 & -\frac{1}{CL_1} \\ 0 & -\frac{1}{L_1} & 0 \end{bmatrix},$$

have a rank of 3, they are full rank and the system is controllable. Because the system is controllable, it is also stabilizable.

From a mathematical standpoint, controllability for this system seems normal. But the implications for the real system are not that normal at all. Since controllability implies that from any setup of the system, the setup can be transitioned into a system where the capacitor is easily overloaded. However, the mathematical model does not include the option for overload in the calculations, and models the system further without problems.

The second to last thing that can be studied in this system is the before mentioned similarity. Two systems were created, $\Sigma(A,B)$ and $\Sigma(\tilde{A},\tilde{B})$. These systems are similar via the invertible matrix

$$T = \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \\ 0 & \frac{1}{L_2} & 0 \\ 0 & 0 & \frac{1}{C} \end{bmatrix}.$$

The final thing to do is to check if this system can be written as a port-Hamiltonian system. Converting the first representation of the general system into a port-Hamiltonian system of the form (3.4), requires a positive-definite H and a skew-adjoint matrix J such that

$$\begin{cases} JH = \begin{bmatrix} 0 & 0 & \frac{1}{L_1} \\ 0 & 0 & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix}, \\ \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}^T H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \end{cases}$$

One of such solutions is given by

$$
\begin{aligned}
J &= \begin{bmatrix} 0 & 0 & \frac{1}{CL_1} \\ 0 & 0 & \frac{1}{CL_2} \\ -\frac{1}{CL_1} & -\frac{1}{CL_2} & 0 \end{bmatrix}, \\
H &= \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & C \end{bmatrix}.
\end{aligned}
$$

Just like the general representation, the representation of a port-Hamiltonian system is also not unique. The port-Hamiltonian representation of the second system can be given by the solution of

$$
\begin{cases}
\tilde{J}\tilde{H} = \begin{bmatrix} 0 & 0 & \frac{1}{C} \\ 0 & 0 & \frac{1}{C} \\ -\frac{1}{L_1} & -\frac{1}{L_2} & 0 \end{bmatrix}, \\
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T H = \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \end{bmatrix}.
\end{cases}
$$

One of such solutions is given by

$$
\begin{aligned}
\tilde{J} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \\
\tilde{H} &= \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \\ 0 & \frac{1}{L_2} & 0 \\ 0 & 0 & \frac{1}{C} \end{bmatrix}.
\end{aligned}
$$

Since the matrices B and H are known in both port-Hamiltonian systems, the stabilizing feedback for each system can be given by

$$
\begin{aligned}
u(t) &= -kB^*Hx = -k\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x = -kI_{L_1}(t), \\
\tilde{u}(t) &= -k\tilde{B}^*\tilde{H}\tilde{x} = -k\begin{bmatrix} \frac{1}{L_1} & 0 & 0 \end{bmatrix} \tilde{x} = -kI_{L_1}(t),
\end{aligned}
$$

for all constants $k > 0$. Hence both port-Hamiltonian systems can be stabilized using the same feedback $u(t) = -kI_{L_1}(t)$. This is not strange, since both systems are based on the same original system with output $y(t) = I_{L_1}(t)$.

One observation to be made is the similarity between the invertible matrix T and the matrices H and $\tilde{H}$. Namely $T = \tilde{H}$ and $T^{-1} = H$. In other words, if the two systems are similar, then

$$\tilde{A} = T^{-1}AT = HA\tilde{H}.$$

This relation is not always true, as every positive-definite matrix is invertible, but the reverse is not true.

4 Infinite dimensional

After a brief introduction into finite dimensional spaces, it is time to start working on infinite dimensional spaces, still following the book [4]. It is emphasized that earlier works, for example [13], stay close to the formulation of control systems as introduced in Section 3.1, while in [4] the focus lies on the port-Hamiltonian formulation. To start off in infinite dimensional spaces, a new type of tool is needed to work with, because the usual framework for finite dimensions does not work well in infinite dimensions. Preferably, this new tool is an extension from the finite dimensional case. In its simplest form, it should at least represent a system of the same form as in (3.1). It turns out that the chosen tool indeed extends the finite dimensional port-Hamiltonian system to its infinite dimensional counterpart. This tool will be in the form of strongly continuous semigroups, or C_0-semigroups for short. For a more extended account of the theory of C_0-semigroups, see for example [14] or [15]. A C_0-semigroup is defined as follows.

4.1 C_0-semigroups

Let X be a Hilbert space. A family of operators $(T(t))_{t\geq 0}$ on X is called a strongly continuous semigroup if the following holds:

- $T(t) \in \mathcal{L}(X)\ \forall t \geq 0$;
- $T(0) = I$;
- $T(t+\tau) = T(t)T(\tau)\ \forall t, \tau \geq 0$;
- $\forall x_0 \in X,\ \|T(t)x_0 - x_0\| \to 0$ as $t \downarrow 0$.

The last condition means $t \mapsto T(t)$ is strongly continuous at zero. In terms of state space representation, the Hilbert space X is called the state space and its elements states. From now on, X will always, unless otherwise specified, refer to a Hilbert space. If no space is mentioned, it is assumed to be the Hilbert space X.

To illustrate this definition, first consider as an example the case where A is a bounded linear operator on X, and define $(T(t))_{t\geq 0}$ by the exponential

$$T(t) := e^{At} := \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}.$$

The first condition to be checked is if $(T(t))_{t\geq 0}$ is bounded.

$$\|T(t)\| = \left\| \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \left\| \frac{A^n t^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{\|A\|^n t^n}{n!} = e^{\|A\|t},$$

which is bounded for all non-negative time as long as the norm of the operator A is bounded. As a side note, the reverse is also true. The second condition is trivial, since

$$T(0) = \sum_{n=0}^{\infty} \frac{A^n 0^n}{n!} = e^{A0} = I.$$

The third condition requires

$$\begin{aligned}
T(t+\tau) &= \sum_{n=0}^{\infty} \frac{A^n (t+\tau)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \binom{n}{k} t^k \tau^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k \tau^{n-k} \\
&= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \sum_{n=k}^{\infty} \frac{A^{n-k} \tau^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \sum_{n=0}^{\infty} \frac{A^n \tau^n}{n!} \\
&= T(t)T(\tau).
\end{aligned}$$

And the final condition is true as well, since

$$\lim_{t \downarrow 0} \|T(t)x_0 - x_0\| = \lim_{t \downarrow 0} \|e^{At} x_0 - x_0\| = \|I x_0 - x_0\| = 0.$$

Hence $(T(t))t \geq 0$ satisfies all four conditions and is a C_0-semigroup when A is a bounded operator.

A second example of a strongly continuous semigroup is the shift operator defined by $(T(t)f)(\zeta) := f(\zeta + t)$ on $X = L^2([0,\infty), \mathbb{K}^n)$. The first property holds by definition of X. The second condition is true for

$$(T(0)f)(\zeta) = f(\zeta),$$

hence $T(0)$ is the identity.

$$\begin{aligned}
(T(t)T(\tau)f)(\zeta) &= (T(t)(T(\tau)f))(\zeta) \\
&= (T(t)f(\zeta+\tau)) \\
&= f(\zeta+\tau+t) \\
&= f(\zeta+t+\tau) \\
&= (T(t+\tau)f)(\zeta),
\end{aligned}$$

for all $t, \tau > 0$. The final property holds too, for

$$\lim_{t \downarrow 0} \|(T(t)f)(\zeta) - f(\zeta)\|^2 = \lim_{t \downarrow 0} \int_0^{\infty} (f(t+\zeta) - f(\zeta))^2 d\zeta = 0,$$

for all $f \in X$.

The terminology 'semi' is about the non-negative time usage. It is also possible to define a C_0-group. This is an extension of the semigroup, as this group uses negative time too. In other words, $(T(t))_{t\in\mathbb{R}}$ is called a strongly continuous group, or C_0-group if the following holds:

- $T(t) \in \mathcal{L}(X)\ \forall t \in \mathbb{R}$;
- $T(0) = I$;
- $T(t+\tau) = T(t)T(\tau)\ \forall t, \tau \in \mathbb{R}$;
- $\forall x_0 \in X,\ \|T(t)x_0 - x_0\| \to 0$ as $t \to 0$.

Since the exponential e^{At} for a bounded linear operator A is a semigroup, in order to check if it defines a group too, it is only necessary to check the properties for negative time, which it does. Intuitively it might be clear that if $(T(t))_{t\geq 0}$ defines a semigroup on non-negative time and non-positive time, it also defines a group. This theory shall be slightly expanded on. However, since a group can be created from two semigroups, and vice versa, the focus will lie mainly on semigroup theory, which can be extended to group theory.

Before more interesting examples will be discussed, first some properties of strongly continuous semigroups will be presented, and the concept of the infinitesimal generator will be discussed.

4.1.1 Properties

There are a few important and useful properties of a strongly continuous semigroup, one of the reasons for this choice of tool. For example the norm of the C_0-semigroup is bounded on all finite non-negative time intervals. Another one of these properties has to do with continuity. Namely, the mapping $t \mapsto T(t)$ is strongly continuous on the interval $[0, \infty)$. Using this property, a convergence result can be proven. Since $(T(t))_{t\geq 0}$ is strongly continuous, there exists $\tau, \epsilon > 0$ such that $\|T(s)x - x\| \leq \epsilon\ \forall s \in [0, \tau]$. Then for any $t \in [0, \tau]$,

$$\begin{aligned}\left\|\frac{1}{t}\int_0^t T(s)x ds - x\right\| = \left\|\frac{1}{t}\int_0^t T(s)x - x\, ds\right\| &\leq \frac{1}{t}\int_0^t \|T(s)x - x\| ds \\ &\leq \frac{1}{t}\int_0^t \epsilon ds = \epsilon.\end{aligned}$$

This leads to the following convergence result

$$\forall x \in X : \frac{1}{t}\int_0^t T(s)x ds \to x \text{ as } t \downarrow 0. \tag{4.1}$$

The next two properties are about the so called growth bound ω_0 of the semigroup.

$$\text{If } \omega_0 = \inf_{t>0}\left(\frac{1}{t}\log\|T(t)\|\right), \text{ then } \omega_0 = \lim_{t\to\infty}\left(\frac{1}{t}\log\|T(t)\|\right) < \infty.$$

To prove this, let $t_0 > 0$ be a fixed time and $M = \sup_{t\in[0,t_0]}\|T(t)\|$. Then, as the real numbers lie dense in the rational numbers, for all time $t \geq t_0$ there exists an $n \in \mathbb{N}$ such that $nt_0 \leq t < (n+1)t_0$. Hence

$$\begin{aligned}
\frac{\log\|T(t)\|}{t} &= \frac{\log\|T(t+nt_0-nt_0)\|}{t}\\
&= \frac{\log\|T(nt_0)T(t-nt_0)\|}{t}\\
&= \frac{\log\|T(t_0+t_0+\cdots+t_0)T(t-nt_0)\|}{t}\\
&= \frac{\log\|T(t_0)T(t_0)\ldots T(t_0)T(t-nt_0)\|}{t}\\
&= \frac{\log\|T(t_0)^nT(t-nt_0)\|}{t}\\
&\leq \frac{\log\|T(t_0)^n\|\|T(t-nt_0)\|}{t}\\
&= \frac{\log\|T(t_0)^n\|}{t} + \frac{\log\|T(t-nt_0)\|}{t}\\
&\leq \frac{n\log\|T(t_0)\|}{t} + \frac{\log\|T(t-nt_0)\|}{t}\\
&\leq \frac{\log\|T(t_0)\|}{t_0}\frac{nt_0}{t} + \frac{\log M}{t}.
\end{aligned}$$

Because $\frac{nt_0}{t} \leq 1$, if $\log\|T(t_0)\| \geq 0$

$$\frac{\log\|T(t_0)\|}{t_0}\frac{nt_0}{t} + \frac{\log M}{t} \leq \frac{\log\|T(t_0)\|}{t_0} + \frac{\log M}{t}.$$

If on the other hand $\log\|T(t_0)\| < 0$, as $nt_0 \leq t - t_0$,

$$\frac{\log\|T(t_0)\|}{t_0}\frac{nt_0}{t} + \frac{\log M}{t} \leq \frac{\log\|T(t_0)\|}{t_0}\frac{t-t_0}{t} + \frac{\log M}{t}.$$

These two inequalities imply together that

$$\limsup_{t\to\infty}\frac{\log\|T(t)\|}{t} \leq \frac{\log\|T(t_0)\|}{t_0} < \infty.$$

Since t_0 is arbitrary,

$$\limsup_{t\to\infty}\frac{\log\|T(t)\|}{t} \leq \inf_{t>0}\frac{\log\|T(t)\|}{t} \leq \liminf_{t\to\infty}\frac{\log\|T(t)\|}{t}.$$

This states the supremum is less than or equal to the infimum, hence

$$\omega_0 = \inf_{t>0} \frac{\log \|T(t)\|}{t} = \lim_{t\to\infty} \frac{\log \|T(t)\|}{t} < \infty,$$

as was what stated.

The final important property bounds the norm for all non-negative time. Let $\omega > \omega_0$, then there exists a $t_0 > 0$ such that $\frac{\log \|T(t)\|}{t} < \omega \ \forall t \geq t_0$. This implies $\|T(t)\| < e^{\omega t} \ \forall t \geq t_0$. If on the other hand $0 \leq t \leq t_0$, there exists a $M_0 > 1$ such that $\|T(t)\| \leq M_0$. Combining both results and defining

$$M_\omega := \begin{cases} M_0 & \omega \geq 0, \\ e^{-\omega t_0} M_0 & \omega < 0, \end{cases}$$

results in the following statement. For every $\omega > \omega_0$, there exists a constant M_ω such that for every $t \geq 0$

$$\|T(t)\| \leq M_\omega e^{\omega t}. \tag{4.2}$$

4.2 The infinitesimal generator

Continuing with a C_0-semigroup $(T(t))_{t\geq 0}$, if

$$\lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t}$$

exists, then x_0 is said to be in the domain of A, denoted by $x_0 \in D(A)$, where A is defined by

$$Ax_0 = \lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t} = \lim_{t\downarrow 0} \frac{T(t) - I}{t} x_0.$$

The linear operator A is called the infinitesimal generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$. Because the generator is defined by this limit, each C_0-semigroup has a unique infinitesimal generator. In terms of groups, the infinitesimal generator is instead defined as the two sided limit of time to zero.

Returning to the example of the exponential of a bounded linear operator, in that case

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{T(t)x_0 - x_0}{t} &= \lim_{t\downarrow 0} \frac{\sum_{n=0}^{\infty} \frac{A^n t^n}{n!} x_0 - x_0}{t} \\
&= \lim_{t\downarrow 0} \sum_{n=2}^{\infty} \frac{A^n t^n}{n!} \frac{x_0}{t} + \frac{A^1 t^1}{1!} \frac{x_0}{t} + \frac{A^0 t^0}{0!} \frac{x_0}{t} - \frac{x_0}{t} \\
&= \lim_{t\downarrow 0} \sum_{n=2}^{\infty} \frac{A^n t^n}{n!} \frac{x_0}{t} + Ax_0 + \frac{x_0}{t} - \frac{x_0}{t} \\
&= Ax_0.
\end{aligned}$$

Hence in that case the operator A is indeed the infinitesimal generator.

Not all infinitesimal generators need to be bounded operators. Take for example the shift operator $(T(t)f)(\zeta) = f(\zeta + t)$ from before. In this case

$$\lim_{t \downarrow 0} \frac{(T(t)f)(\zeta) - f(\zeta)}{t} = \lim_{t \downarrow 0} \frac{f(\zeta + t) - f(\zeta)}{t},$$

which is the definition of the right derivative of the function f with respect to the spatial variable. Hence the infinitesimal generator is exactly this right derivative, which is an unbounded operator.

The infinitesimal generator of a C_0-semigroup has some useful properties too, mostly related to the differentiation and integration of the strongly continuous semigroup. This means the generator is an operator on which standard calculations can be applied. However, the first useful property is not related to those and states,

$$\text{if } x_0 \in D(A), \text{ then } \forall t \geq 0 \;\; T(t)x_0 \in D(A). \tag{4.3}$$

This property might not seem useful, but will prove to be so in the end. In order to prove this assertion, consider the second property. Because the limit $s \downarrow 0$ of the first term of

$$T(t)\frac{T(s) - I}{s}x_0 = \frac{T(t+s) - T(t)}{s}x_0 = \frac{T(s) - I}{s}T(t)x_0$$

exists, the same limit also exists for the last term. By definition of the existence of this limit, $T(t)x_0 \in D(A)$ for all $t \geq 0$.

Furthermore, using the existence of this limit,

$$AT(t)x_0 = \lim_{s \downarrow 0} \frac{T(s) - I}{s}T(t)x_0 = \lim_{s \downarrow 0} T(t)\frac{T(s) - I}{s}x_0 = T(t)Ax_0.$$

On the other hand, for $t > s > 0$

$$\frac{T(s) - I}{s}T(t-s)x_0 = \frac{T(t-s) - T(t)}{-s}x_0 = T(t-s)\frac{T(s) - I}{s}x_0,$$

the limit $s \downarrow 0$ implies again the same equality. The first equality states the existence of the right derivative, while the second equality states the existence of the left derivative. Combining both results in

$$\frac{d}{dt}(T(t)x_0) = AT(t)x_0 = T(t)Ax_0 \;\; \forall x_0 \in D(A) \text{ and } t \geq 0. \tag{4.4}$$

Taking another derivative and using the same arguments gives by induction

$$\frac{d^n}{dt^n}(T(t)x_0) = A^n T(t)x_0 = T(t)A^n x_0 \;\; \forall x_0 \in D(A^n) \text{ and } t \geq 0.$$

For any $x_0 \in D(A)$, by integration of (4.4),

$$\int_0^t T(s)Ax_0 ds = \int_0^t \frac{d}{ds}(T(s)x_0)ds = T(t)x_0 - T(0)x_0 = T(t)x_0 - x_0.$$

Hence

$$T(t)x_0 - x_0 = (T(t) - I)x_0 = \int_0^t T(s)Ax_0 ds \ \ \forall x_0 \in D(A) \text{ and } t \geq 0.$$

On the other hand, again by integration of (4.4), using the other equality,

$$T(t)x_0 - x_0 = (T(t) - I)x_0 = A\int_0^t T(s)x_0 ds \ \ \forall x_0 \in D(A) \text{ and } t \geq 0.$$

The last property relating to the integral states

$$\int_0^t T(s)xds \in D(A) \ \ \forall x \in X.$$

In order to prove this, define $\rho = s + u$, then

$$\begin{aligned}
\frac{T(s) - I}{s}\int_0^t T(u)xdu =& \frac{1}{s}\int_0^t T(s)T(u)xdu - \frac{1}{s}\int_0^t T(u)xdu \\
=& \frac{1}{s}\int_0^t T(s+u)xdu - \frac{1}{s}\int_0^t T(u)xdu \\
=& \frac{1}{s}\int_s^{t+s} T(\rho)xd\rho - \frac{1}{s}\int_0^t T(u)xdu \\
=& \frac{1}{s}\int_s^t T(\rho)xd\rho + \frac{1}{s}\int_t^{t+s} T(\rho)xd\rho \\
&- \frac{1}{s}\int_0^s T(u)xdu - \frac{1}{s}\int_s^t T(u)xdu \\
=& \frac{1}{s}\left(\int_t^{t+s} T(\rho)xd\rho - \int_0^s T(u)xdu\right) \\
=& \frac{1}{s}\left(\int_0^s T(t+u)xdu - \int_0^s T(u)xdu\right) \\
=& \frac{1}{s}\left(\int_0^s T(u)(T(t) - I)xdu\right).
\end{aligned}$$

Taking the limit $s \downarrow 0$, and by (4.1),

$$\lim_{s\downarrow 0} \frac{T(s) - I}{s}\int_0^t T(u)xdu = \lim_{s\downarrow 0} \frac{1}{s}\left(\int_0^s T(u)(T(t) - I)xdu\right) = (T(t) - I)x \in D(A).$$

Hence by definition of the infinitesimal generator, $\int_0^t T(u)xdu \in D(A)$.

There are a few more properties of an infinitesimal generator A that are worth mentioning, without the proof: $D(A)$ lies dense in X, and A is a closed operator. As well as whenever $A_1 = A_2$ generate two C_0-semigroups $(T_1(t))_{t\geq 0}$ respectively $(T_2(t))_{t\geq 0}$, then $T_1(t) = T_2(t)$ $\forall t \geq 0$. The last thing to be mentioned with regards to the infinitesimal generator is regarding the resolvent. Let $Re(s) > 0$ and $\omega > \omega_0$, the growth bound, with $\|T(t)\| \leq Me^{\omega t}$. Then $s \in \rho(A)$ and for all $x \in X$:

- $(sI - A)^{-1}x = \int_0^\infty e^{-st}T(t)xdt$;
- $\|(sI - A)^{-1}\| \leq \frac{M}{Re(s)-\omega}$;
- For $\lambda, \mu \in \rho(A)$ the resolvent identity $(\lambda - \mu)(\lambda I - A)^{-1}(\mu I - A)^{-1} = (\mu I - A)^{-1} - (\lambda I - A)^{-1}$ holds;
- The mapping $s \mapsto (sI - A)^{-1}$ is analytic in $\rho(A)$.

4.3 Differential equation

The infinitesimal generator can be used to solve differential equations. In order to get a better understanding of how everything works, the input u will be negated in terms of (3.1). This leads to the following definition: A differentiable function $x : [0, \infty) \to X$ is called a classical solution of

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{4.5}$$

if for all $t \geq 0$, $x(t) \in D(A)$ and the equation is satisfied.

At this point the semigroup $(T(t))_{t\geq 0}$, alongside its infinitesimal generator A make their appearance to solve these type of abstract differential equation. If $x(t) := T(t)x_0$, then $x(0) = T(0)x_0 = x_0$, and $\dot{x}(t) = \frac{d}{dt}(T(t)x_0) = AT(t)x_0 = Ax(t)$, hence this function is a solution to (4.5). By (4.3) this means that whenever the initial point is in the domain of A, the entire trajectory is in the domain. If one has found a classical solution for some $x_0 \in D(A)$, the map $t \to T(t)x_0$ is uniquely determined. Hence the first property was useful after all.

Just like in the finite dimensional case, there is also a notion of a mild solution. A continuous function $x : [0, \infty) \to X$ is called a mild solution of (4.5) if $\int_0^t x(s)ds \in D(A)$, $x(0) = x_0$ and $x(t) - x(0) = A\int_0^t x(\tau)d\tau \quad \forall t \geq 0$.

4.4 Contraction semigroups

Since every C_0-semigroup has an infinitesimal generator, which is uniquely determined, it is also interesting to check which operators A generates a C_0-semigroup. A full characterization of which operators generate C_0-semigroups can be found in [14] and [15]. The book [4] confines itself to an important subclass of C_0-semigroups, namely the contraction semigroups. Let's start by defining this subset. A C_0-semigroup is called a contraction semigroup if $\|T(t)\| \leq 1 \ \forall t \geq 0$.

Not every semigroup is a contraction semigroup, but by (4.2), $\|\frac{1}{M_\omega}e^{-\omega t}T(t)\| \leq 1$, hence $(1/M_\omega e^{-\omega t}T(t))_{t\geq 0}$ is a contraction semigroup. Thus although not every C_0-semigroup is a contraction semigroup, it can be transformed into one.

One of the properties of a contraction semigroup is that whenever $t_2 \geq t_1$, $\|x(t_2)\| \leq \|x(t_1)\|$.

Contraction semigroups can be linked to dissipative operators. A linear operator $A : D(A) \subset X \to X$ is called dissipative if $Re\langle Ax, x\rangle \leq 0 \; \forall x \in D(A)$, or equivalently,

$$\|(\alpha I - A)x\| \geq \alpha\|x\| \; \forall x \in D(A), \; \alpha > 0. \tag{4.6}$$

In order to prove the equivalence,

$$\begin{aligned} \|(\alpha I - A)x\|\|x\| &\geq Re\langle(\alpha I - A)x, x\rangle = \alpha\langle x, x\rangle - Re\langle Ax, x\rangle \\ &= \alpha\|x\|^2 - Re\langle Ax, x\rangle \geq \alpha\|x\|^2, \end{aligned}$$

if $\alpha > 0$ and A is dissipative. Hence $\|(\alpha I - A)x\| \geq \alpha\|x\|$.
On the other hand,

$$\begin{aligned} \|(\alpha I - A)x\|^2 &= \|\alpha x - Ax\|^2 \\ &= \|\alpha x\|^2 - 2Re\langle Ax, \alpha x\rangle + \|Ax\|^2 \\ &= \alpha^2\|x\|^2 - 2\alpha Re\langle Ax, x\rangle + \|Ax\|^2, \end{aligned}$$

hence if $\|(\alpha I - A)x\| \geq \alpha\|x\|$,

$$\begin{aligned} 0 &\leq \|(\alpha I - A)x\|^2 - \alpha^2\|x\|^2 \\ &= \|Ax\|^2 - 2\alpha Re\langle Ax, x\rangle. \end{aligned}$$

This in turn implies

$$\begin{aligned} 0 &\leq \lim_{\alpha\to\infty} \frac{\|Ax\|^2 - 2\alpha Re\langle Ax, x\rangle}{\alpha} \\ &= \lim_{\alpha\to\infty} \frac{\|Ax\|^2}{\alpha} - 2Re\langle Ax, x\rangle \\ &= -2Re\langle Ax, x\rangle. \end{aligned}$$

Thus A is dissipative.

Having a relatively easy method of checking if an operator is dissipative, it is almost time to formulate a method of linking an operator to a contraction semigroup. However, in order to prove this link, there will be a need for two theorems. The first is called the Hille-Yosida Theorem. It states the following:

Theorem. (Hille-Yosida) (See [4].)
A necessary and sufficient condition for a closed, densely defined, linear operator A on a Hilbert space X to be the infinitesimal generator of a contraction semigroup is that $(0, \infty) \subset \rho(A)$ and

$$\|(\alpha I - A)^{-1}\| \leq \frac{1}{\alpha} \; \forall \alpha > 0.$$

The second theorem is from Lumer-Phillips. It states:

Theorem. (Lumer-Phillips) (See [4].)
Let A be a linear operator with domain $D(A)$ on a Hilbert space X. Then A is the

infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ *if and only if* A *is dissipative and the range of* $I - A$ *equals* X.

Although the second theorem already links an operator with a contraction semigroup, it is not always easy to check if $I - A$ is surjective. Using these two theorems, for the proof, the main method for checking if an operator generates a contraction semigroup can be formulated.

Theorem. (Lumer-Phillips) (See [4].)
A linear densely defined and closed operator A *is the infinitesimal generator of a contraction semigroup if and only if* A *and* A^* *are dissipative.*

Thus it is sufficient to study the operator and its adjoint, and check if these are dissipative for all elements in their respective domains.

In order to prove the last theorem, first suppose A generates a contraction semigroup. By Lumer-Phillips, A is dissipative, hence it is only needed to prove that A^* is dissipative as well. Since

$$\begin{aligned}\|(\alpha I - A^*)^{-1}\| &= \|\left((\alpha I - A)^*\right)^{-1}\| = \|(((\alpha I - A)^{-1})^*\| \\ &= \|(\alpha I - A)^{-1}\| \leq \frac{1}{\alpha},\end{aligned}$$

where the last step is true because of Hille-Yosida. Hence for all $x \in D(A^*)$

$$\begin{aligned}\|(\alpha I - A^*)^{-1}\| &\leq \frac{1}{\alpha}, \\ \|(\alpha I - A^*)\| &\geq \alpha, \\ \|(\alpha I - A^*)x\| &\geq \alpha\|x\|.\end{aligned}$$

By (4.6), A^* is dissipative. This proves one implication.

For the other implication, assume that A and A^* are dissipative. As per assumption A is also closed, the range of $\alpha I - A$ is closed for all $\alpha > 0$. This assertion can be proved using the dissipative property and the convergence of a Cauchy sequence. By contradiction of A not generating a contraction semigroup, assume there exists an $\alpha_0 > 0$ and a non-zero z such that z is orthogonal to the range of $\alpha_0 I - A$. Then for all $x \in D(A)$,

$$0 = \langle z, (\alpha_0 I - A)x\rangle = \langle \alpha_0 z, x\rangle - \langle z, Ax\rangle = \langle \alpha_0 z, x\rangle - \langle A^* z, x\rangle,$$

which implies $z \in D(A^*)$ and $A^* z = \alpha_0 z$. Combining this together with A^* being dissipative results in

$$0 \geq Re\langle A^* z, z\rangle = Re\langle \alpha_0 z, z\rangle = \alpha_0\langle z, z\rangle = \alpha_0\|z\|^2.$$

However, by assumption α_0 was positive, hence this is not possible. Thus there exists no such α_0 with an orthogonal element to $\alpha_0 I - A$. Thus for all $\alpha > 0$ the range of $\alpha_0 - I$ equals X. By Lumer-Phillips, A generates a contraction semigroup. This completes the proof. □

Since a linear densely defined and closed operator A is the infinitesimal generator of a contraction semigroup if and only if A and A^* are dissipative, for all $x \in D(A)$

$$Re\langle Ax, x\rangle \leq 0,$$

and for all $x \in D(A^*)$

$$Re\langle A^*x, x\rangle \leq 0,$$

must hold. Hence

$$\begin{aligned} Re\langle Ax, x\rangle + Re\langle A^*x, x\rangle &\leq 0, \\ Re\langle Ax, x\rangle + Re\langle x, Ax\rangle &\leq 0, \\ \langle Ax, x\rangle + \langle x, Ax\rangle &\leq 0. \end{aligned}$$

Something similar to the last inequality will return multiple times later on in a slightly different context.

4.5 C_0-groups

Coming back to group theory, there was mentioning of combining two semigroups into a group. This can be made precise by the following. A is the infinitesimal generator of a C_0-group $(T(t))_{t\in\mathbb{R}}$ if and only if A and $-A$ are infinitesimal generators of C_0-semigroups $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$ respectively.

If A generates a C_0-group $(T(t))_{t\in\mathbb{R}}$, then $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$ defined by

$$\begin{aligned} T_+(t) &:= T(t), \\ T_-(t) &:= T(-t), \end{aligned}$$

are C_0-semigroups with infinitesimal generators A and $-A$ respectively. If on the other hand A and $-A$ generate C_0-semigroups $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$ respectively, then A generates a C_0-group $(T(t))_{t\in\mathbb{R}}$ defined by

$$T(t) := \begin{cases} T_+(t) & t \geq 0, \\ T_-(-t) & t \leq 0. \end{cases}$$

An interesting concept arises when both A and $-A$ generate contraction semigroups, while combining them with above construction for a group. If this is the case, then for any $x_0 \in X$

$$\begin{aligned} \|x_0\| &= \|T(0)x_0\| = \|T(t-t)x_0\| = \|T(t)T(-t)x_0\| \\ &\leq \|T(t)\|\|T(-t)\|\|x_0\| = \|T_+(t)\|\|T_-(t)\|\|x_0\| \leq \|x_0\|. \end{aligned}$$

Since A and $-A$ generate contraction semigroups, the norm of their respective semigroups are lesser or equal than one. Thus by above inequality, $\|T(t)\| = \|T(-t)\| = 1$. Hence the norm of each operator in the group equals one. On the other hand, if the norm of the group equals one, then both $\|T_+(t)x_0\| = \|T(t)x_0\| = \|x_0\|$ and $\|T_-(t)x_0\| = \|T(-t)x_0\| = \|x_0\|$, which imply that both A and $-A$ generate contraction semigroups. Hence by combination: A and $-A$ generate a contraction semigroup if and only if A is the infinitesimal generator of a group with the property $\|T(t)x\| = \|x\|$. Groups with this property are often called unitary groups.

This gives some indication on how to check if a semigroup, or group, is a contraction semigroup and how those are linked to their infinitesimal generators. However, as mentioned before, C_0-groups can be constructed from C_0 semigroups, hence the focus lies with semigroup theory and almost no further notion will be made about groups.

4.6 Port-Hamiltonian system

Previously, an inner product was used to determine properties of semigroups and their infinitesimal generators. However, in the interest of port-Hamiltonian systems, not every inner product is as nice to work with. Along with a notion of an inner product, some more definitions need to be made.

Let $P_1 \in \mathbb{K}^{n\times n}$ be invertible and self-adjoint. Let $P_0 \in \mathbb{K}^{n\times n}$ be skew-adjoint. Let $H \in L^\infty([a,b];\mathbb{K}^{n\times n})$, such that H is self-adjoint and $mI \leq H(\zeta) \leq MI$ for almost every $\zeta \in [a,b]$ with constants $m, M > 0$. The Hilbert spaces are restricted to the form $X = L^2([a,b];\mathbb{K}^n)$, with the associated inner product

$$\langle f(\zeta), g(\zeta)\rangle = \frac{1}{2}\int_a^b g(\zeta)^* H(\zeta) f(\zeta) d\zeta.$$

Then the differential equation

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}\left(H(\zeta)x(\zeta,t)\right) + P_0\left(H(\zeta)x(\zeta,t)\right), \tag{4.7}$$

is called a linear, first order port-Hamiltonian system. In everything that follows, this is often referred to as just a port-Hamiltonian system, which should not be confused with the finite dimensional port-Hamiltonian systems. Every Hamiltonian system has a Hamiltonian associated to it, this one being $E : [a,b] \to \mathbb{K}$, given by

$$E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^* H(\zeta) x(\zeta,t) d\zeta.$$

Note that in more complex equations, it is inconvenient to write the time dependence, or even the spatial dependence in either H and/or x. In those cases either one, or both of these dependencies will be omitted, even though they still exist. As a second note, in terms of a system, there is no input function here. The differential equation that will be studied is of the form

$$\dot{x} = Ax,$$

which means that for the moment it is of interested to see if equation (4.7) can be rewritten in terms of a semigroup generator.

In some of the more simpler cases, P_0 can be set equal to zero. Since P_1 has to be invertible, it can never be zero. This leaves the equation

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}\left(H(\zeta)x(\zeta,t)\right),$$

which can be rewritten as

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial}{\partial \zeta}\left(H'(\zeta)x(\zeta,t)\right),$$

for the appropriate H'. However, this complicates things as the properties of P_1 and H are easier checked and satisfied than H'. Furthermore, there is only one representation of H', while there could exist multiple for the combination of P_1 and H. This can be seen in the first example of Section 4.10.

The linear first order port-Hamiltonian system on infinite dimensional spaces is an extension of the port-Hamiltonian systems on finite dimensional spaces. Negating the input and output, in finite dimensional spaces the system had the form

$$\dot{x} = JHx,$$

where J is skew-adjoint, and H is positive-definite. The new port-Hamiltonian system has the same form, where $J = P_1 \frac{\partial}{\partial \zeta} + P_0$. To see that this is an extension, it can be proven that this J is skew-adjoint. However this does require some assumptions. Namely an assumption on the boundary conditions. Note that the natural norm induced by the inner product for linear first order port-Hamiltonian systems is equivalent with the standard L^2-norm. This can be used in combination with

$$\begin{aligned}
\left\langle \frac{\partial}{\partial \zeta} f(\zeta,t), g(\zeta,t) \right\rangle &= \int_a^b \left(\frac{\partial}{\partial \zeta} f(\zeta,t)\right) \overline{g}(\zeta,t)\, d\zeta \\
&= \left[f(\zeta,t)g(\zeta,t)\right]_{\zeta=a}^{b} - \int_a^b f(\zeta,t)\left(\frac{\partial}{\partial \zeta}\overline{g}(\zeta,t)\right) d\zeta \\
&= 0 - \left\langle f(\zeta,t), \frac{\partial}{\partial \zeta} g(\zeta,t) \right\rangle,
\end{aligned}$$

provided the boundary conditions are such that $f(b,t)g(b,t) - f(a,t)g(a,t) = 0$. Hence P_0 and the derivative are skew-adjoint operators, while P_1 is a self-adjoint operator.

Thus

$$\begin{aligned}
\left\langle \left(P_1\frac{\partial}{\partial\zeta}+P_0\right)f(\zeta,t),g(\zeta,t)\right\rangle &= \left\langle P_1\frac{\partial}{\partial\zeta}f(\zeta,t),g(\zeta,t)\right\rangle + \langle P_0f(\zeta,t),g(\zeta,t)\rangle\\
&= \left\langle \frac{\partial}{\partial\zeta}f(\zeta,t),P_1g(\zeta,t)\right\rangle - \langle f(\zeta,t),P_0g(\zeta,t)\rangle\\
&= -\left\langle f(\zeta,t),\frac{\partial}{\partial\zeta}P_1g(\zeta,t)\right\rangle - \langle f(\zeta,t),P_0g(\zeta,t)\rangle\\
&= -\left\langle f(\zeta,t),P_1\frac{\partial}{\partial\zeta}g(\zeta,t)\right\rangle - \langle f(\zeta,t),P_0g(\zeta,t)\rangle\\
&= -\left\langle f(\zeta,t),\left(P_1\frac{\partial}{\partial\zeta}+P_0\right)g(\zeta,t)\right\rangle,
\end{aligned}$$

and the operator is indeed skew-adjoint. Hence the infinite dimensional system is an extension of the finite dimensional system, under some restrictions. The Hamiltonian, or energy, is also an extension of the finite dimensional case, since it has the same formula, except now x changes over the spatial variable, and hence needs to be integrated.

The associated Hamiltonian is denoted with an E, as it refers to the energy of the system. As the Hamiltonian, or the energy, of the system changes over time, it might be interesting to study how the energy changes. Hence

$$\begin{aligned}
\frac{dE}{dt} =& \frac{d}{dt}\frac{1}{2}\int_a^b x^*Hxd\zeta = \frac{1}{2}\int_a^b \frac{\partial}{\partial t}\left(x^*Hx\right)d\zeta = \frac{1}{2}\int_a^b \frac{\partial x^*}{\partial t}Hxd\zeta + \frac{1}{2}\int_a^b x^*H\frac{\partial x}{\partial t}d\zeta\\
=&\frac{1}{2}\int_a^b \left(P_1\frac{\partial}{\partial\zeta}\left(Hx\right)+P_0\left(Hx\right)\right)^* Hxd\zeta + \frac{1}{2}\int_a^b x^*H\left(P_1\frac{\partial}{\partial\zeta}\left(Hx\right)+P_0\left(Hx\right)\right)d\zeta\\
=&\frac{1}{2}\int_a^b \left(\frac{\partial}{\partial\zeta}\left(Hx\right)^*P_1^* + x^*H^*P_0^*\right)Hxd\zeta + \frac{1}{2}\int_a^b x^*H\left(P_1\frac{\partial}{\partial\zeta}\left(Hx\right)+P_0Hx\right)d\zeta\\
=&\frac{1}{2}\int_a^b \left(\frac{\partial}{\partial\zeta}\left(Hx\right)^*P_1 - x^*HP_0\right)Hxd\zeta + \frac{1}{2}\int_a^b x^*H\left(P_1\frac{\partial}{\partial\zeta}\left(Hx\right)+P_0Hx\right)d\zeta\\
=&\frac{1}{2}\int_a^b \left(\frac{\partial}{\partial\zeta}\left(Hx\right)^*P_1\right)Hxd\zeta + \frac{1}{2}\int_a^b x^*H\left(P_1\frac{\partial}{\partial\zeta}\left(Hx\right)\right)d\zeta\\
=&\frac{1}{2}\int_a^b \frac{\partial}{\partial\zeta}\left(Hx\right)^*P_1Hx + x^*HP_1\frac{\partial}{\partial\zeta}\left(Hx\right)d\zeta\\
=&\frac{1}{2}\int_a^b \frac{\partial}{\partial\zeta}\left(Hx\right)^*P_1Hx + x^*H^*P_1\frac{\partial}{\partial\zeta}\left(Hx\right)d\zeta\\
=&\frac{1}{2}\int_a^b \frac{\partial}{\partial\zeta}\left(\left(Hx\right)^*P_1Hx\right)d\zeta\\
=&\frac{1}{2}\left[\left(Hx\right)^*P_1Hx\right]_a^b.
\end{aligned}$$

This implies that the only changes in the energy happen at the boundary, not in the interior of the space. As was observed in Section 3.5, the changes in the energy in the finite dimensional case only happen via the input and output of the system. This is the reason for the name port-Hamiltonian systems. The Hamiltonian, which is often the energy of the system, changes only via the boundary, or, its ports to the outside world.

Writing (4.7) in terms of the previous state space representations results in the definition

$$A_0 x := P_1 \frac{\partial}{\partial \zeta}(Hx) + P_0(Hx),$$

with an associated domain given by $D(A_0) = \{x \in X | \; Hx \in H^1([a,b]; \mathbb{K}^n)\}$.

In order to complete this representation, some form of boundary conditions are needed. The boundary conditions are added to the domain of the operator A, as this is easier to work with. The boundary conditions will be expressed in terms of the boundary effort and the boundary flow, defined by

$$\begin{aligned} e_\partial &= \frac{1}{\sqrt{2}}\left((Hx)(b) + (Hx)(a)\right), \\ f_\partial &= \frac{1}{\sqrt{2}}\left(P_1(Hx)(b) - P_1(Hx)(a)\right). \end{aligned}$$

These boundary terms and the operator A_0 are linked by

$$\begin{aligned} 2Re\langle A_0 x, x\rangle_X =& \langle A_0 x, x\rangle_X + \langle x, A_0 x\rangle_X \\ =& \frac{1}{2}\int_a^b x^* H \left(P_1 \frac{\partial}{\partial \zeta}(Hx) + P_0(Hx)\right) d\zeta \\ &+ \frac{1}{2}\int_a^b \left(P_1 \frac{\partial}{\partial \zeta}(Hx) + P_0(Hx)\right)^* Hx \, d\zeta \\ =& \frac{1}{2}\int_a^b (Hx)^* \left(P_1 \frac{\partial}{\partial \zeta}(Hx)\right) + \left(\frac{\partial}{\partial \zeta}(Hx)\right)^* P_1 Hx \, d\zeta \\ &+ \frac{1}{2}\int_a^b (Hx)^* P_0(Hx) - (Hx)^* P_0(Hx) \, d\zeta \\ =& \frac{1}{2}\frac{\partial}{\partial \zeta}\left((Hx)^* P_1(Hx)\right) d\zeta \\ =& \frac{1}{2}\left((Hx)^*(b) P_1(Hx)(b) - (Hx)^*(a) P_1(Hx)(a)\right) \\ =& \frac{1}{2}(f_\partial^* e_\partial + e_\partial^* f_\partial). \end{aligned}$$

Note that these relations still hold for the definition of A in the upcoming theory.

In the definition of A_0, as well as the boundary effort and boundary flow, there are multiple uses of Hx, not an x solo. Hence it might be interesting to check if this is represented in the boundary conditions too. Suppose there exists an $\begin{bmatrix} u \\ y \end{bmatrix} \in \mathbb{K}^{2n}$, and the goal is finding an x_0 such that $\begin{bmatrix} (Hx_0)(b) \\ (Hx_0)(a) \end{bmatrix} = \begin{bmatrix} u \\ y \end{bmatrix}$. For $x_0 = H^{-1}\left(\frac{b-\zeta}{b-a} y + \frac{\zeta - a}{b-a} u\right)$, which is an element of $D(A_0)$,

$$\begin{bmatrix}(Hx_0)(b)\\(Hx_0)(a)\end{bmatrix} = \begin{bmatrix}\left(HH^{-1}\left(\frac{b-\zeta}{b-a}y+\frac{\zeta-a}{b-a}u\right)\right)(b)\\\left(HH^{-1}\left(\frac{b-\zeta}{b-a}y+\frac{\zeta-a}{b-a}u\right)\right)(a)\end{bmatrix}$$

$$= \begin{bmatrix}\frac{b-b}{b-a}y+\frac{b-a}{b-a}u\\\frac{b-a}{b-a}y+\frac{a-a}{b-a}u\end{bmatrix} = \begin{bmatrix}u\\y\end{bmatrix}.$$

Hence for any element of $\mathbb{K}^{2n}$, there exists an $x_0 \in D(A_0)$ to express the boundary in terms of Hx_0. Keeping this in mind, it might be more natural to describe the boundary conditions in terms of Hx and not solely in x.

For this reason, define the boundary conditions as

$$\tilde{W}_B\begin{bmatrix}H(b)x(b,t)\\H(a)x(a,t)\end{bmatrix} = 0, \quad t \geq 0,$$

for some $\tilde{W}_B$. This can be represented by the boundary effort and boundary flow. In order to do this, write

$$\begin{bmatrix}f_\partial\\e_\partial\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}P_1 & -P_1\\I & I\end{bmatrix}\begin{bmatrix}(Hx)(b)\\(Hx)(a)\end{bmatrix} = R_0\begin{bmatrix}(Hx)(b)\\(Hx)(a)\end{bmatrix},$$

for $R_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}P_1 & -P_1\\I & I\end{bmatrix}$.

As $\det(R_0) = \sqrt{2}\det(P_1)$, and P_1 is invertible, R_0 is invertible as well.

Hence another way of writing the boundary conditions is

$$W_B\begin{bmatrix}f_\partial(t)\\e_\partial(t)\end{bmatrix} = 0,$$

for $W_B = \tilde{W}_B R_0^{-1} \in \mathbb{K}^{n\times 2n}$.

Using these boundary conditions, the domain the operator A can be expressed in the form

$$Ax := P_1\frac{\partial}{\partial\zeta}(Hx) + P_0(Hx),$$

with the domain

$$D(A) = \{x \in L^2([a,b];\mathbb{K}^n) |\ Hx \in H^1([a,b];\mathbb{K}^n), W_B\begin{bmatrix}f_\partial\\e_\partial\end{bmatrix} = 0\}.$$

Note that the operator A is defined on a smaller domain than the operator A_0. Hence a linear first order port-Hamiltonian system is of the form

$$\dot{x} = Ax,$$

and the only thing that needs to be proven to fit in the previous context, is that the operator A is the infinitesimal generator of a contraction semigroup.

These boundary conditions can be used to check for a contraction semigroup. The proof is omitted, it can be found in [4].

Theorem.

For any A associated to a port-Hamiltonian system, and any W_B, or $\tilde{W}_B$, of rank n, the following are equivalent.

- A is the infinitesimal generator of a contraction semigroup on X.
- $Re\langle Ax, x\rangle_X \leq 0 \ \forall x \in D(A)$.
- $W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* \geq 0$.

Note that the implication from the first to second statement was already established, by the Lumer-Phillips theorem. However, previously the implication the other way around was not stated yet. Furthermore, the third statement links the boundary condition to a contraction semigroup.

Sometimes however it is not as easy as one might hope to check if a particular operator generates a contraction semigroup, but it might be easier to prove for another operator. Let Z be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $P \in \mathcal{L}(Z)$ be self-adjoint, with $P > \epsilon I$ for some $\epsilon > 0$. Define Z_P as the Hilbert space Z with inner product $\langle \cdot, \cdot \rangle_P := \langle \cdot, P \cdot \rangle$. Then the operator A with domain $D(A)$ generates a contraction semigroup on Z if and only if AP with domain $D(AP) = \{z \in Z | Pz \in D(A)\}$ generates a contraction semigroup on Z_P.

Since Z defines a Hilbert space, and the inner product on Z_P is defined through that inner product, and since P is self-adjoint with $P > \epsilon I$, Z_P does indeed define a Hilbert space. In order to prove the if and only if, because the inverse of a self-adjoint operator is also self-adjoint, and if $P > \epsilon I$, the inverse has a similar property. Hence by proving the if part, the only if part is proven too, and vice versa, for the appropriate P. To give a slight indication of the proof, assume A generates a contraction semigroup. Then A and A^* are both dissipative. Using the fact that P is self-adjoint and the construction of the inner product, for all elements of $D(AP)$, and for all elements of $D((AP)^*)$, the dissipative property holds for AP and $(AP)^*$. This implies AP does indeed generate a contraction semigroup with domain $D(AP)$.

Previously there was mention of a unitary group. Just like for a contraction group, there are equivalent statements for the unitary group. For any A associated to a port-Hamiltonian system, and any W_B or $\tilde{W}_B$ of rank n, the following are equivalent.

- A is the infinitesimal generator of a unitary group on X.
- $Re\langle Ax, x\rangle_X = 0 \ \forall x \in D(A)$.
- $W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* = 0$.

4.7 Stabilizability

After linking contraction semigroups to the boundary conditions, it is time to work a bit on the notion of stability for infinite dimensional systems. Specifically the notion of exponentially stable, this form of stability is often studied in semigroup theory. A C_0-semigroup $(T(t))_{t\geq 0}$ is called exponentially stable if there exist constants $M, \alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for $t \geq 0$.

This definition resembles (4.2). However, that property works with ω greater than the growth bound ω_0, while this definition works solely with the decay rate α. The supremum over all possible values of α is called the stability margin of the semigroup and equals minus the growth bound, $\alpha = -\omega_0$.

This definition can be extended for group theory. The only thing of interest here is to note that for unitary groups $\|T(t)x\| = \|x\|$, for all $x \in X$ and all $t \in \mathbb{R}$, and hence unitary groups can never be exponentially stable.

Since the solution of $\dot{x}(t) = Ax(t), x(0) = x_0$ is given by $x(t) = T(t)x_0$, if a semigroup is exponentially stable, then

$$\|x(t)\| = \|T(t)x_0\| \leq Me^{-\alpha t}\|x_0\|,$$

and the solution goes to zero exponentially fast as $t \to \infty$, hence the name.

One method of checking for exponential stability is via Datko's lemma. This states that a semigroup is exponentially stable if and only if

$$\int_0^\infty \|T(t)x\|^2 dt < \infty, \ \forall x \in X.$$

To prove this, if a semigroup is exponentially stable,

$$\begin{aligned}\int_0^\infty \|T(t)x\|^2 dt &\leq \int_0^\infty \|T(t)\|^2\|x\|^2 dt \\ &\leq \int_0^\infty M^2 e^{-2\alpha t}\|x\|^2 dt \\ &= \frac{M^2\|x\|^2}{2\alpha},\end{aligned}$$

which is finite for any x in a Hilbert space X.

On the other hand, if the integral is finite, by (4.2), there exists $M_\omega > 0$, where M_ω depends on $\omega > \omega_0$, such that $\|T(t)\| \leq M_\omega e^{\omega t}$ for all $t \geq 0$. Define for every $n \in \mathbb{N}$ the operator Q_n by $(Q_n x)(t) := \mathbb{1}_{[0,n]}T(t)x$. As $(T(t))_{t\geq 0}$ is a bounded linear operator, Q_n is also a bounded linear from X to $L^2([0,\infty); X)$ for every $n \in N$. Because $\int_0^\infty \|T(t)x\|^2 dt$ is finite for all $x \in X$, for all $x \in X$ the family $\{Q_n x, n \in \mathbb{N}\}$ is uniformly bounded in n, hence by the Uniform Boundedness Principle, $\|Q_n\| \leq \gamma$ for some $\gamma > 0$.

For $0 \leq t \leq 1$, $\|T(t)\| \leq M_\omega e^{\omega t} \leq M_\omega e^{\omega}$. On the other hand, for $t > 1$, by integration

$$\begin{aligned}
\frac{1 - e^{-2\omega t}}{2\omega}\|T(t)x\|^2 &= \int_0^t e^{-2\omega s} ds \|T(t)x\|^2 \\
&= \int_0^t e^{-2\omega s}\|T(t)x\|^2 ds \\
&\leq \int_0^t e^{-2\omega s}\|T(s)\|^2\|T(t-s)x\|^2 ds \\
&\leq \int_0^t e^{-2\omega s}\left(M_\omega e^{\omega s}\right)^2\|T(t-s)x\|^2 ds \\
&= M_\omega^2 \int_0^t \|T(t-s)x\|^2 ds \\
&= M_\omega^2 \int_0^t \|T(s)x\|^2 ds \\
&\leq M_\omega^2 \gamma^2 \|x\|^2.
\end{aligned}$$

Hence for all non-negative time, the norm of the operator T can be bounded. Thus there exists a $K > 0$ such that $\|T(t)\| \leq K$ for all $t \geq 0$.

Furthermore, once again by integration, for all $t > 0$,

$$\begin{aligned}
t\|T(t)x\|^2 &= \int_0^t \|T(t)x\|^2 ds \\
&\leq \int_0^t \|T(s)\|^2\|T(t-s)x\|^2 ds \\
&\leq \int_0^t K^2\|T(t-s)x\|^2 ds \\
&= K^2 \int_0^t \|T(s)x\|^2 ds \\
&\leq K^2\gamma^2\|x\|^2,
\end{aligned}$$

thus,

$$\|T(t)x\| \leq \frac{K\gamma}{\sqrt{t}}\|x\|.$$

Since K and γ are constants, for $t > K^2\gamma^2$, $\|T(t)\| < 1$. Hence $\inf_{t>0} \frac{1}{t}\log\|T(t)\| < 0$, thus the growth bound is negative, and by (4.2), there exists $M, \alpha > 0$ such that $\|T(t)\| \leq M^{-\alpha t}$ for all $t \geq 0$. Hence the semigroup is exponentially stable, what was to be proven.

Another method for checking exponential stability is via a so called Lyapunov equation. If A is the infinitesimal generator of a C_0-semigroup, the following statements are equivalent:

- $(T(t))_{t\geq 0}$ is exponentially stable.

- There exists a positive operator $P \in \mathcal{L}(X)$:
  $\langle Ax, Px \rangle + \langle Px, Ax \rangle = -\langle x, x \rangle \; \forall x \in D(A)$.

- There exists a positive operator $P \in \mathcal{L}(X)$:
  $\langle Ax, Px \rangle + \langle Px, Ax \rangle \leq -\langle x, x \rangle \; \forall x \in D(A)$.

In order to prove this, it will be proven that each statement implies the next statement. Since the second statement implies the third statement trivially, the focus lies on the other two implications. First comes the implication of exponentially stability to the second statement. Define the operator P by

$$\langle x_1, Px_2 \rangle = \int_0^\infty \langle T(t)x_1, T(t)x_2 \rangle dt,$$

for $x_1, x_2 \in D(A)$. If $(T(t))_{t\geq 0}$ is exponentially stable, this integral is finite and hence well defined. Using this operator and the exponential stability,

$$\begin{aligned}
|\langle x_1, Px_2 \rangle| &= \left| \int_0^\infty \langle T(t)x_1, T(t)x_2 \rangle dt \right| \leq \int_0^\infty |\langle T(t)x_1, T(t)x_2 \rangle| dt \\
&\leq \int_0^\infty \|T(t)x_1\| \|T(t)x_2\| dt \leq \int_0^\infty M^2 e^{-2\alpha t} \|x_1\| \|x_2\| dt \\
&= \frac{M^2}{2\alpha} \|x_1\| \|x_2\|,
\end{aligned}$$

hence $P \in \mathcal{L}(X)$. If

$$0 = \langle x, Px \rangle = \int_0^\infty \langle T(t)x, T(t)x \rangle dt = \int_0^\infty \|T(t)x\|^2 dt,$$

then $\|T(t)x\| = 0$ almost everywhere. This implies, using the strong continuity of the semigroup, that $x = 0$. Hence this is the only element for which $\langle x, Px \rangle = 0$, thus P is a positive operator. Furthermore,

$$\begin{aligned}
\langle Ax, Px \rangle + \langle Px, Ax \rangle &= \int_0^\infty \langle T(t)Ax, T(t)x \rangle + \langle T(t)x, T(t)Ax \rangle dt \\
&= \int_0^\infty \frac{d}{dt} \langle T(t)x, T(t)x \rangle dt \\
&= 0 - \langle T(0)x, T(0)x \rangle \\
&= -\langle x, x \rangle,
\end{aligned}$$

for all $x \in D(A)$. Hence the first statement implies the second statement.

Lastly, suppose the third statement is true. Then define the operator V by $V(x,t) := \langle PT(t)x, T(t)x \rangle$. Since P is a positive operator, V is non-negative. For any $x \in D(A)$, the derivative of $(T(t))_{t\geq 0}$ is well defined, hence the derivative of V is also well defined.

Hence

$$\begin{aligned}\frac{d}{dt}V(x,t) &= \langle PAT(t)x, T(t)x\rangle + \langle PT(t)x, AT(t)x\rangle \\ &= \langle AT(t)x, PT(t)x\rangle + \langle PT(t)x, AT(t)x\rangle \\ &\leq -\langle T(t)x, T(t)x\rangle \\ &= -\|T(t)x\|^2,\end{aligned}$$

by the third statement. Integration of this inequality results in

$$\begin{aligned}\int_0^t \frac{d}{ds}V(x,s)ds &\leq -\int_0^t \|T(s)x\|^2 ds \\ V(x,t) - V(x,0) &\leq -\int_0^t \|T(s)x\|^2 ds,\end{aligned}$$

which implies, using the non-negativity of V,

$$0 \leq V(x,t) \leq V(x,0) - \int_0^t \|T(s)x\|^2 ds.$$

Hence

$$\begin{aligned}\int_0^t \|T(s)x\|^2 ds &\leq V(x,0) \\ &= \langle PT(0)x, T(0)x\rangle \\ &= \langle Px, x\rangle\end{aligned}$$

for all $t \geq 0$ and $x \in D(A)$. Because $D(A)$ is dense in X, this inequality holds for all $x \in X$. Since P is a bounded operator, $\langle Px, x\rangle$ is finite, and by Datko's lemma, $(T(t))_{t\geq 0}$ is exponentially stable.

4.8 Spectral projection

There is one final method for checking exponential stability, which relies on the spectrum of an operator. However, this requires two definitions regarding invariance. If V is a subspace of the Hilbert space X, V is called $T(t)$-invariant if $T(t)V \subset V$ for all $t \geq 0$. Whenever there is no input for the system, $\dot{x} = Ax$, the solution is given by $x(t) = T(t)x_0$. Hence V is $T(t)$-invariant if the initial condition lies in V.

If A is the infinitesimal generator of a C_0-semigroup, the subspace V is A-invariant if $A(V \cap D(A)) \subset V$.

Suppose V is a closed subspace of X and $T(t)$-invariant, then for any $v \in V \cap D(A)$, $\frac{1}{t}(T(t) - I)v \in V$ for every $t > 0$. As V is closed, so is the limit. Furthermore, by definition of the infinitesimal generator,

$$\lim_{t\downarrow 0} \frac{T(t) - I}{t} v = Av.$$

Hence $Av \in V$. Thus V is also A-invariant.

Not all spaces have a non-trivial $T(t)$-invariant subspace. However, if the spectrum can be divided into two distinct regions, something can be done. Let $\sigma(A) = \sigma^+ \cup \sigma^-$, such that a rectifiable, closed, simple curve Γ can be drawn which encloses an open set containing σ^+ in its interior and σ^- in its exterior.

Define the operator P_Γ by

$$P_\Gamma x := \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} x d\lambda,$$

where Γ is traversed once counterclockwise. This operator is a projection. In order to prove that this is indeed a projection, the following holds. As $\lambda \to (\lambda I - A)^{-1}$ is uniformly bounded on Γ, P_Γ is a bounded linear operator on X. And for any $s \in \rho(A)$

$$\begin{aligned}
(sI - A)^{-1} P_\Gamma x &= (sI - A)^{-1} \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} x d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma (sI - A)^{-1}(\lambda I - A)^{-1} x d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1}(sI - A)^{-1} x d\lambda \\
&= P_\Gamma (sI - A)^{-1} x.
\end{aligned}$$

On the other hand, for $s \in \rho(A)$, outside Γ, using the resolvent identity and Cauchy's residue theorem,

$$\begin{aligned}
(sI - A)^{-1} P_\Gamma x &= \frac{1}{2\pi i}\int_\Gamma \frac{-(sI - A)^{-1} x}{s - \lambda} + \frac{(\lambda I - A)^{-1} x}{s - \lambda} d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{(\lambda I - A)^{-1} x}{s - \lambda} d\lambda.
\end{aligned}$$

If Γ' is another rectifiable, closed, simple curve enclosing Γ, then P_Γ is also given by

$$P_\Gamma x = \frac{1}{2\pi i}\int_{\Gamma'} (\lambda I - A)^{-1} x d\lambda.$$

Using all of the above,

$$\begin{aligned}
P_\Gamma P_\Gamma x &= \frac{1}{2\pi i}\int_{\Gamma'} (sI - A)^{-1} P_\Gamma x \, ds \\
&= \frac{1}{2\pi i}\int_{\Gamma'} \frac{1}{2\pi i}\int_\Gamma \frac{(\lambda I - A)^{-1} x}{s - \lambda} \, d\lambda \, ds \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{1}{2\pi i}\int_{\Gamma'} \frac{1}{s - \lambda} ds \, (\lambda I - A)^{-1} x \, d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} x \, d\lambda \\
&= P_\Gamma x,
\end{aligned}$$

hence the operator P_Γ is a projection.

This projection is called the spectral projection on σ^+, which induces a decomposition on the state space X as follows

$$\begin{aligned} X &= X^+ \oplus X^-, \\ X^+ &= P_\Gamma X, \\ X^- &= (I - P_\Gamma)X. \end{aligned}$$

This decomposition has a few properties, namely

- $\forall x \in D(A),\ P_\Gamma Ax = AP_\Gamma x$;
- $\forall s \in \rho(A),\ (sI - A)^{-1}P_\Gamma = P_\Gamma(sI - A)^{-1}$;
- Both X^+ and X^- are A-invariant;
- $\forall s \in \rho(A),\ (sI - A)^{-1}X^+ \subset X^+, (sI - A)^{-1}X^- \subset X^-$;
- $P_\Gamma X \subset X$, and $A^+ := A|_{X^+}$, is a bounded operator on X^+;
- $\sigma(A^+) = \sigma^+$ and $\sigma(A^-) = \sigma^-$, where $A^- := A|_{X^-}$;
- $\forall \lambda \in \rho(A),\ (\lambda I - A^+)^{-1} = (\lambda I - A)^{-1}|_{X^+}$ and $(\lambda I - A^-)^{-1} = (\lambda I - A)^{-1}|_{X^-}$;
- If σ^+ consists of only finitely many eigenvalues with finite order, then P_Γ projects onto the space of generalized eigenvectors of the enclosed eigenvalues. Thus

  $$\mathrm{ran} P_\Gamma = \sum_{\lambda_n \in \sigma^+} \ker(\lambda_n I - A)^{\nu(n)} = \sum_{\lambda_n \in \sigma^+} \ker(\lambda_n I - A^+)^{\nu(n)},$$

  where $\nu(n)$ is the order of λ_n;
- If $\sigma^+ = \{\lambda_n\}$, with λ_n an eigenvalue of multiplicity 1, then

  $$P_\Gamma z = \langle z, \psi_n \rangle \phi_n,$$

  where ϕ_n is the eigenvector of A corresponding to λ_n and ψ_n is an eigenvector of A^* corresponding to $\overline{\lambda_n}$ with $\langle \phi_n, \psi_n \rangle = 1$.

The operators A^+ and A^- are not only the restriction of A to their respective subspaces of X. They also generate a semigroup. If the spectrum of A is the union of σ^+ and σ^-, then X^+ and X^- are $T(t)$-invariant. Furthermore, $((T^+(t))_{t\geq 0}$ and $((T^-(t))_{t\geq 0}$, with $T^+(t) = T(t)|_{X^+}$ and $T^-(t) = T(t)|_{X^-}$, define C_0-semigroups on X^+ and X^- respectively. The infinitesimal generators of $(T^+(t))_{t\geq 0}$ and $(T^-(t))_{t\geq 0}$ are A^+ and A^- respectively.

All of these together give the final method of finding exponential stability. If the spectrum of A is the union of two parts, such that $\sigma^+ \subset \overline{\mathbb{C}_0^+} = \{s \in \mathbb{C} \mid Re(s) \geq 0\}$ and

$\sigma^- \subset \mathbb{C}_0^- = \{s \in \mathbb{C} \mid Re(s) < 0\}$, and a rectifiable, closed, simple curve Γ can be drawn such that it encloses an open set containing σ^+ in its interior and σ^- in its exterior. Furthermore, if

$$\sup_{s \in \mathbb{C}_0^+,\ s \text{ outside } \Gamma} \|(sI - A)^{-1}\| < \infty,$$

then the semigroup $(T^-(t))_{t \geq 0} := (T(t)|_{X^-})_{t \geq 0}$ is exponentially stable.

4.9 Stabilizability for port-Hamiltonian systems

In terms of port-Hamiltonian systems, there are two more methods for checking exponential stability. However, the port-Hamiltonian systems introduced as (4.7), had the property $H \in L^\infty([a,b]; \mathbb{K}^{n \times n})$, while for this theory it is preferable that H is continuously differentiable, $H \in C^1([a,b]; \mathbb{K}^{n \times n})$. For relative ease, assume that both

$$W_B \in \mathbb{K}^{n \times 2n} \text{ has rank } n,$$

and

$$W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* \geq 0.$$

Let A be defined by

$$Ax := P_1 \frac{\partial}{\partial \zeta}(Hx) + P_0(Hx),$$

with domain

$$D(A) = \{x \in L^2([a,b]; \mathbb{K}^n) \mid Hx \in H^1([a,b]; \mathbb{K}^n), W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0\},$$

then A generates a contraction semigroup.

In the finite dimensional case, port-Hamiltonian systems where easily stabilizable using the feedback $u(t) = -ky(t)$ for any constant $k > 0$. Since the linear first order port-Hamiltonian systems are an extension of these systems, it is expected to find something similar. The main difference between these two cases however, is that in the infinite dimensional setting, there is no specific output defined.

As mentioned before, the boundary conditions of a port-Hamiltonian system are best expressed in terms of Hx, not just in x. Since A generates a contraction semigroup, there exists constants $c, \tau > 0$ such that for every $x_0 \in D(A)$ the state trajectory $x(t) := T(t)x_0$ satisfies both

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau \|H(b)x(b,t)\|^2 dt \quad \text{and} \tag{4.8}$$

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau \|H(a)x(a,t)\|^2 dt. \tag{4.9}$$

The proof for these inequalities is quite long, but uses some clever tricks, hence a brief summary will be given. In light of the energy Hamiltonian, define $F : [a,b] \to \mathbb{R}$ by

$$F(\zeta) = \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^*(\zeta,t)H(\zeta)x(\zeta,t)dt,$$

for $\gamma > 0$ and $\tau > 2\gamma(b-a)$. Differentiating and using properties of P_1 and H, along with the assumption that γ is large enough such that both $P_1^{-1} + \gamma H$ and $-P_1^{-1} + \gamma H$ are coercive, results in

$$\frac{dF}{d\zeta}(\zeta) \geq -\int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^* \left(HP_0^* P_1^{-1} + P_1^{-1} P_0 H + \frac{dH}{d\zeta} \right) xdt.$$

Since P_1 and P_0 are constants and $\frac{dH}{d\zeta}(\zeta)$ is bounded, there exists a constant $\kappa > 0$ such that for all $\zeta \in [a,b]$

$$H(\zeta)P_0^* P_1^{-1} + P_1^{-1} H(\zeta) + \frac{dH}{d\zeta}(\zeta) \leq \kappa H(\zeta),$$

and hence

$$\frac{dF}{d\zeta}(\zeta) \geq -\kappa \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^* Hxdt = -\kappa F(\zeta).$$

Thus for all $\zeta_1 \in [a,b]$

$$\int_{\zeta_1}^{b} \frac{\frac{dF}{d\zeta}}{F}(\zeta)d\zeta \geq -\kappa \int_{\zeta_1}^{b} 1d\zeta,$$

or

$$\ln(F(b)) - \ln(F(\zeta_1)) \geq -\kappa(b-\zeta_1).$$

Hence

$$F(b) \geq F(\zeta_1)e^{-\kappa(b-\zeta_1)} \geq F(\zeta_1)e^{-\kappa(b-a)}$$

for all $\zeta_1 \in [a,b]$. Furthermore, because A generates a contraction semigroup and looking at the boundary of the integral,

$$\begin{aligned}\int_{\gamma(b-a)}^{\tau-\gamma(b-a)} \|x(t)\|_X^2 dt &\geq \int_{\gamma(b-a)}^{\tau-\gamma(b-a)} \|x(\tau-\gamma(b-a))\|_X^2 dt \\ &= \|x(\tau-\gamma(b-a))\|_X^2 \int_{\gamma(b-a)}^{\tau-\gamma(b-a)} 1dt \\ &= (\tau - 2\gamma(b-a))\|x(\tau-\gamma(b-a))\|_X^2.\end{aligned}$$

Combining all of the above results in

$$2(\tau - 2\gamma(b-a))\|x(\tau)\|_X^2 \leq m^{-1}(b-a)e^{\kappa(b-a)} \int_0^{\tau} \|H(b)x(b,t)\|^2 dt,$$

where m is such that $mI \leq H(\zeta)$, as per assumption. Hence for the appropriate constant c, and this choice of τ, the first inequality is proven. The other inequality can be proven analogously to the first equation, using

$$F(\zeta) = \int_{\gamma(\zeta-a)}^{\tau-\gamma(\zeta-a)} x^*(\zeta,t)H(\zeta)x(\zeta,t)dt.$$

Thus the norm of the trajectory can be bounded by the (integrated) norm of the boundary for a contraction semigroup. This bound can be used to prove exponential stability under some conditions, namely if either of the following two inequalities hold for some $k > 0$ and all $x_0 \in D(A)$,

$$\begin{aligned}
\langle Ax_0, x_0\rangle_X + \langle x_0, Ax_0\rangle_X &\leq -\kappa\|H(b)x_0(b)\|^2,\\
\langle Ax_0, x_0\rangle_X + \langle x_0, Ax_0\rangle_X &\leq -\kappa\|H(a)x_0(a)\|^2.
\end{aligned}$$

In order to prove this, suppose the first inequality holds, and x solves $\dot{x}(t) = Ax(t)$, then

$$\begin{aligned}
\frac{d\|x(t)\|_X^2}{dt} &= \frac{d\langle x(t), x(t)\rangle_X}{dt}\\
&= \langle Ax(t), x(t)\rangle_X + \langle x(t), Ax(t)\rangle\\
&\leq -\kappa\|H(b)x(b)\|^2.
\end{aligned}$$

Hence by integration

$$\begin{aligned}
\|x(\tau)\|_X^2 - \|x(0)\|_X^2 &= \int_0^\tau \frac{d\|x(t)\|_X^2}{dt}dt\\
&\leq \int_0^\tau -\kappa\|H(b)x(b,t)\|^2 dt.
\end{aligned}$$

Furthermore, using the reverse inequality of (4.8) and above, the bound becomes

$$\|x(\tau)\|_X^2 - \|x(0)\|_X^2 \leq -\frac{\kappa}{c}\|x(\tau)\|_X^2.$$

Note that if the second condition holds instead, the same inequality holds. And thus

$$\|x(\tau)\|_X^2 \leq \frac{c}{c+\kappa}\|x(0)\|_X^2.$$

This implies the semigroup generated by A has a norm strictly smaller than one. By definition this semigroup is exponentially stable.

As a last note, without a proof, if

$$W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* > 0,$$

the port-Hamiltonian system is also exponentially stable.

In the finite dimensional setting, the feedback was of the form $u(t) = -ky(t)$, and the energy changed only by changing u or y. In the infinite dimensional case the changes in energy appear only on the boundary, which depends on $H(\zeta)x(\zeta, t)$. Hence it is good to have a check for exponential stability that depends on the boundary conditions, either $H(a)x(a)$ or $H(b)x(b)$ alone, or combined. The first check for exponential stability for linear first order port-Hamiltonian systems appears to be in a similar form as for finite dimensional stabilizability, while the second check for exponential stability depends on both boundary conditions.

4.10 Example

In order to show some of the theory mentioned in this chapter, consider a vibrating string, which is attached to the wall on only one side, and some force u is applied on the free side. This force u, as the choice of notation also indicates, will act as the input to the system. See Figure 4.1. The movement of this string can be given by

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right), \tag{4.10}$$

for $\zeta \in [a, b]$. $w(\zeta, t)$ denotes the vertical displacement of the string at position ζ and time t. The function $T(\zeta)$ represents the Young's modulus of the string, not a C_0-semigroup. Finally, $\rho(\zeta)$ is the mass density of the string, which may vary along the length of the string. The associated Hamiltonian is given by the potential energy plus the kinetic energy,

$$E(t) = \frac{1}{2} \int_a^b \rho(\zeta) \left(\frac{\partial w}{\partial t}(\zeta, t) \right)^2 + T(\zeta) \left(\frac{\partial w}{\partial \zeta}(\zeta, t) \right)^2 d\zeta.$$

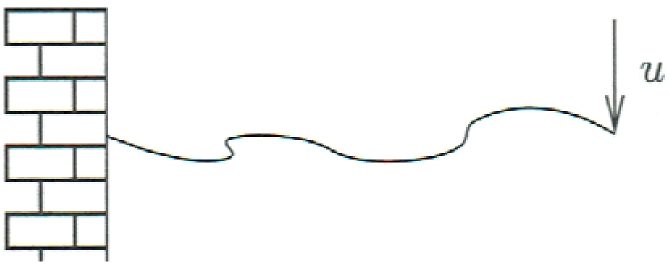

Figure 4.1: A vibrating string fixed on one side.

If (4.10) has a classical solution, then differentiating the Hamiltonian and using this solution, gives

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_a^b \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t)\frac{\partial^2 w}{\partial t^2}(\zeta,t) + T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\frac{\partial^2 w}{\partial\zeta\partial t}(\zeta,t)\, d\zeta \\
&= \int_a^b \frac{\partial w}{\partial t}(\zeta,t)\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right) + T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\frac{\partial^2 w}{\partial\zeta\partial t}(\zeta,t)\, d\zeta \\
&= \frac{\partial w}{\partial t}(\zeta,t)T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\bigg|_{\zeta=a}^{b} - \int_a^b T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\frac{\partial^2 w}{\partial\zeta\partial t}(\zeta,t)d\zeta \\
&\quad + \int_a^b T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\frac{\partial^2 w}{\partial\zeta\partial t}(\zeta,t)d\zeta \\
&=\frac{\partial w}{\partial t}(b,t)T(b)\frac{\partial w}{\partial \zeta}(b,t) - \frac{\partial w}{\partial t}(a,t)T(a)\frac{\partial w}{\partial \zeta}(a,t).
\end{aligned}$$

As was already established, the only change in energy happens on either side of the end of the string.

Since $w(\zeta,t)$ is the vertical displacement of of the string, $\frac{\partial w}{\partial t}$ is the velocity. Furthermore, because the string is fixed on the left wall, at $\zeta = a$, the velocity at that point equals zero. Since $T\frac{\partial w}{\partial \zeta}$ represents the force exerted on the string, a good choice for the input u is the force exerted on the string at the right hand side. This transforms the above derivative into

$$\frac{d}{dt}E(t) = \frac{\partial w}{\partial t}(b,t)u(t).$$

Looking back at Section 3.5, it was established that the time derivative of the energy was obtained by the output and the input. Hence it is now natural, to take the velocity at the right hand side, $\frac{\partial w}{\partial t}(b,t)$, as the value for the output, to resemble the equality found there.

For the state space, define $x_1 = \rho\frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$. These components are the momentum and strain respectively. These choices are logical for two reasons. The first reason is that these two parts together can describe the system. The second reason is that they both represent some form of energy, which can be used to describe the Hamiltonian. These definitions transform (4.10) into

$$\frac{\partial}{\partial t}x_1 = \frac{\partial}{\partial \zeta}(Tx_2),$$

and the following property holds

$$\frac{\partial}{\partial t}x_2 = \frac{\partial}{\partial t}\frac{\partial w}{\partial \zeta} = \frac{\partial}{\partial \zeta}\frac{\partial w}{\partial t} = \frac{\partial}{\partial \zeta}\left(\frac{1}{\rho}\rho\frac{\partial w}{\partial t}\right) = \frac{\partial}{\partial \zeta}\left(\frac{1}{\rho}x_1\right).$$

Combining these two together results in either

$$\begin{aligned}
\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{bmatrix}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right) \\
&= P_1\frac{\partial}{\partial \zeta}\left(H(\zeta)x(\zeta,t)\right),
\end{aligned}$$

or

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial \zeta}\left(\begin{bmatrix} 0 & T \\ \frac{1}{\rho(\zeta)} & 0(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} \right) \\
&= P_1' \frac{\partial}{\partial \zeta} \left(H'(\zeta) x(\zeta,t) \right),
\end{aligned}
$$

for the appropriate P_1, P_1' and H, H'. There exist more ways of writing the equation in this form, but those will all be a multiple of either of these two. The next step is checking if either of these two representations is actually a linear port-Hamiltonian system. Both P_1 and P_1' are self-adjoint and invertible. However, H' is not positive definite, so this choice is not a good one. On the other hand, there do exist $m, M > 0$ such that $mI \leq H \leq MI$. Furthermore, all other properties are satisfied for this solution. In this example, P_0 equals zero. As mentioned before, this is an example where it is better to study the equation with both P_1 and H, not combined, as multiple representations are possible, but not all give the desired result.

The Hamiltonian transforms into

$$
E(t) = \frac{1}{2} \int_a^b \begin{bmatrix} x_1(\zeta,t) & x_2(\zeta,t) \end{bmatrix} H(\zeta) \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} d\zeta,
$$

as was proven earlier. Using this state representation, the boundary flow and boundary effort are given by

$$
\begin{aligned}
f_\partial &= \frac{1}{\sqrt{2}} \begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b) - T(a)\frac{\partial w}{\partial \zeta}(a) \\ \frac{\partial w}{\partial t}(b) - \frac{\partial w}{\partial t}(a) \end{bmatrix}, \\
e_\partial &= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{\partial w}{\partial t}(b) + \frac{\partial w}{\partial t}(a) \\ T(b)\frac{\partial w}{\partial \zeta}(b) + T(a)\frac{\partial w}{\partial \zeta}(a) \end{bmatrix}.
\end{aligned}
$$

If there is no force applied at the free side, then both boundary conditions are set and

$$
\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b,t) \\ \frac{\partial w}{\partial t}(a) \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \\
&= W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix},
\end{aligned}
$$

for the appropriate W_B. The rank of W_B is 2, and as $W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* = 0$, the associated operator generates a contraction semigroup. However, this also implies the corresponding semigroup is a unitary group, which is not exponentially stable. This could have been obvious from the start, the energy of the system only changes on the boundary, and there is no dissipation of energy here. Hence this choice of boundary conditions does not imply a diminish in the systems energy. If one wants an exponentially stable semigroup for the vibrating string, something else has to be done with the boundary conditions to achieve stabilization.

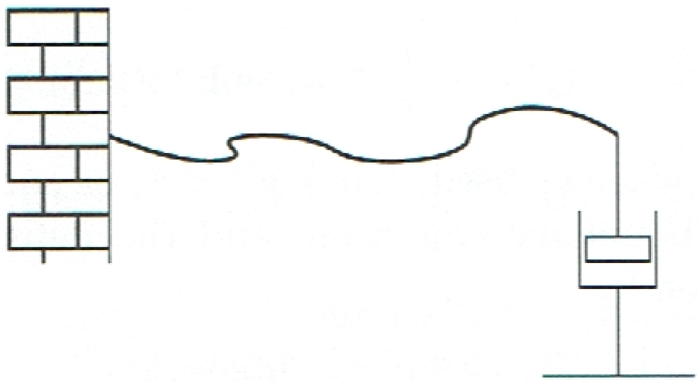

Figure 4.2: A vibrating string fixed on one side and a damper on the other side.

Suppose that instead of a free end on the right side of the string, there is a damper attached to the string, see Figure 4.2. Let the force of this damper, for some $k \geq 0$, be

$$T(b)\frac{\partial w}{\partial \zeta}(b,t) = -k\frac{\partial w}{\partial t}(b,t).$$

In the previous example it was established that the velocity at point b was a natural choice for the output. Hence this choice for a boundary condition is of the form $u(t) = -ky(t)$. This means that it is known beforehand that this should stabilize the system. This results into different boundary values, namely

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b,t) + k\frac{\partial w}{\partial t}(b) \\ \frac{\partial w}{\partial t}(a) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & k & k & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \\ &= \tilde{W}_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \end{aligned}$$

for the appropriate $\tilde{W}_B$. As

$$\tilde{W}_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tilde{W}_B^* = \begin{bmatrix} 2k & 0 \\ 0 & 0 \end{bmatrix},$$

the associated operator generates a contraction semigroup. This matrix however, is not positive definite, hence it is not possible to easily conclude this semigroup is also exponentially stable.

However, using the boundary conditions, both

$$\begin{aligned}\langle Ax, x\rangle_X + \langle x, Ax\rangle_X &= \frac{1}{2}(f_\partial^* e_\partial + e_\partial^* f_\partial)\\ &= \frac{\partial w}{\partial t}(b)T(b)\frac{\partial w}{\partial \zeta}(b) - \frac{\partial w}{\partial t}(a)T(a)\frac{\partial w}{\partial \zeta}(a)\\ &= -k\left(\frac{\partial w}{\partial t}(b)\right)^2,\end{aligned}$$

and by writing out the norm also

$$\begin{aligned}\|H(b)x(b)\|^2 &= \left\| \begin{bmatrix} \frac{1}{\rho(b)} & 0 \\ 0 & T(b)\end{bmatrix} \begin{bmatrix} \rho(b)\frac{\partial w}{\partial t}(b) \\ \frac{\partial w}{\partial \zeta}(b)\end{bmatrix}\right\|^2 = \left\| \begin{bmatrix} \frac{\partial w}{\partial t}(b) \\ T(b)\frac{\partial w}{\partial \zeta}(b)\end{bmatrix}\right\|^2\\ &= \left(\frac{\partial w}{\partial t}(b)\right)^2 + \left(T(b)\frac{\partial w}{\partial \zeta}(b)\right)^2\\ &= (k^2+1)\left(\frac{\partial w}{\partial t}(b)\right)^2,\end{aligned}$$

hence combining these two results in the inequality

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X \leq -\frac{k}{1+k^2}\|H(b)x(b)\|^2.$$

Because this inequality holds, the associated contraction semigroup is still exponentially stable. This time the boundary conditions included a damper, which constantly tried to reduce the energy in the system.

4.11 Extensions

There has been a mentioning of three different things in the infinite dimensional case, which has not been expanded on much. The first of these is the input. In the finite dimensional case, the theory easily included an input function, however, for the infinite dimensional case, the input has been mostly negated so far. The second thing has to do with the boundary conditions, which will be mentioned later. After that, the output will also be included in the boundary control systems.

4.11.1 Input

Most of the systems and theory so far has been for systems of the form (4.5). However, most systems in practice have an input. Previously this input was denoted by u. Suppose for the moment this input is denoted by $f \in C([0,\tau]; X)$, for $\tau > 0$, which gives the differential equation

$$\dot{x}(t) = Ax(t) + f(t), \quad x(0) = x_0, \tag{4.11}$$

for $t \geq 0$. Just as before, define a classical solution of (4.11) by the function $x : [0,\tau] \to X$ on $[0,\tau]$, if $x \in C^1([0,\tau];X)$, $x(t) \in D(A)$ for all $t \in [0,\tau]$ and it satisfies the differential equation. This function is called a classical solution on $[0,\infty)$ if it is a classical solution on $[0,\tau]$ for every $\tau \geq 0$.

If x is a classical solution, it is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds,$$

for $t \in [0,\tau]$. Furthermore, $Ax \in C([0,\tau];X)$, and the solution is the unique solution. If f is instead an element of $C^1([0,\tau];X)$, the function is also continuously differentiable on $[0,\tau]$.

Once again, there is also the notion of a mild solution. If instead $f \in L^1([0,\tau];X)$ and $x_0 \in X$, then the solution, which is still continuous, is called a mild solution instead.

Let U be the input space, which has to be a Hilbert space. If f is of the form Bu, with B a bounded linear operator, and $u \in U$, then the system representation is of the standard form. If A is the infinitesimal generator of a C_0-semigroup, and C and D are also bounded, then the mild solution is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds,$$

which results in the output equation, in standard representation

$$y(t) = CT(t)x_0 + C\int_0^t T(t-s)Bu(s)ds + Du(t).$$

On the other hand, if f is of the form Dx, a control input, for $D \in \mathcal{L}(X)$, then the differential equation for x changes into

$$\dot{x}(t) = (A+D)x(t).$$

If A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$, then $A+D$ is the infinitesimal generator of another C_0-semigroup, $(T_D(t))_{t\geq 0}$, and whenever $\|T(t)\| \leq Me^{\omega t}$, $\|T_D(t)\| \leq Me^{(\omega+M\|D\|)t}$. Furthermore, the operator satisfies the following two equations

$$T_D(t)x_0 = T(t)x_0 + \int_0^t T(t-s)DT_D(s)x_0 ds,$$
$$T_D(t)x_0 = T(t)x_0 + \int_0^t T_d(t-s)DT(s)x_0 ds.$$

If $B \in \mathcal{L}(U,X)$, and there exists an $F \in \mathcal{L}(X,U)$ such that $A+BF$ generates an exponentially stable C_0-semigroup, then the system $\Sigma(A,B)$ is called exponentially stabilizable.

One method for checking if a system is exponentially stable would be via the spectrum decomposition assumption at zero. This means there exists a rectifiable, simple, closed curve Γ, such that it encloses an open set containing σ^+ in its interior, and σ^- in its exterior, just as before. Using the same projection P_Γ, there is again a decomposition of the Hilbert space X into X^+ and X^-. As X can be decomposed, the same can be done for A, T and B. Define those as

$$A = \begin{bmatrix} A^+ & 0 \\ 0 & A^- \end{bmatrix},$$
$$T(t) = \begin{bmatrix} T^+(t) & 0 \\ 0 & T^-(t) \end{bmatrix},$$
$$B = \begin{bmatrix} B^+ \\ B^- \end{bmatrix},$$

with A^+ and A^- defined as before, which generate $T^+(t)$ and $T^-(t)$ respectively. Define $B^+ = P_\Gamma \in \mathcal{L}(U, X^+)$ and $B^- = (I - P_\Gamma)B \in \mathcal{L}(U, X^-)$. This decomposition implies that $\Sigma(A, B)$ is decomposed as $\Sigma(A^+, B^+)$ on X^+ and $\Sigma(A^-, B^-)$ on X^-.

Using this decomposition, $\Sigma(A, B)$ is exponentially stabilizable if and only if $\Sigma(A, B)$ satisfies the spectrum decomposition assumption at zero, X^+ is finite-dimensional, $(T^-(t))_{t\geq 0}$ is exponentially stable, and the finite-dimensional system $\Sigma(A^+, B^+)$ is controllable.

4.11.2 Boundary conditions

The before mentioned theory works great for systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

However, not every system can be represented in this form. For example the transport equation, given by

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t), && \zeta \in [0, 1], t \geq 0 \\ x(\zeta, 0) &= x_0(\zeta), && \zeta \in [0, 1], \\ x(1, t) &= u(t), && t \geq 0, \end{aligned}$$

can not be represented in the usual system, which implies none of the before mentioned theory can be used on this example.

Define

$$v(\zeta, t) := x(\zeta, t) - u(t),$$

which transforms the transport equation into

$$\begin{aligned} \frac{\partial v}{\partial t}(\zeta, t) &= \frac{\partial v}{\partial \zeta}(\zeta, t) - \dot{u}, && \zeta \in [0, 1], t \geq 0, \\ v(\zeta, 0) &= x_0(\zeta) - u(0), && \zeta \in [0, 1], \\ v(1, t) &= 0 && t \geq 0, \end{aligned}$$

which can be written as the differential equation of the standard form

$$\dot{v}(t) = Av(t) + B\tilde{u}(t),$$

with $\tilde{u} = \dot{u}$. This does imply that the derivative of the input needs to exist.

In abstract form, consider the system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \\ \mathfrak{B}x(t) &= u(t), \end{aligned} \tag{4.12}$$

with $x(0) = x_0$, $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$ is linear, the control function u takes values in the Hilbert space U, and the boundary operator $\mathfrak{B} : D(\mathfrak{B}) \subset X \to U$ is linear and satisfies $D(\mathfrak{A}) \subset D(\mathfrak{B})$.

In the transport equation example, the operators are given by $\mathfrak{A}x = \frac{dx}{d\zeta}$ and $\mathfrak{B}x = x(1)$.

This system is called a boundary control system if both the operator $A : D(A) \to X$, with $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and $Ax = \mathfrak{A}x$ for all $x \in D(A)$, is the infinitesimal generator of a C_0-semigroup. Furthermore, there has to exist an operator $B \in \mathcal{L}(U, X)$ such that for all $u \in U$, $Bu \in D(\mathfrak{A}), \mathfrak{A}B \in \mathcal{L}(U, X)$ and $\mathfrak{B}Bu = u$ for all $u \in U$.

The function $x : [0, \tau] \to X$ is called a classical solution of the boundary control system on $[0, \tau]$ if x is continuously differentiable, $x(t) \in D(\mathfrak{A})$ for all $t \in [0, \tau]$, and $x(t)$ satisfies the equations (4.12). The function $x : [0, \infty) \to X$ is called a classical solution of the boundary control system on $[0, \infty)$ if x is a classical solution on $[0, \tau]$ for all $\tau > 0$. Furthermore, the classical solution is unique.

For any boundary control system, define $v(t) := x(t) - Bu(t)$, then

$$\begin{aligned} \dot{v}(t) &= \dot{x}(t) - B\dot{u}(t) \\ &= \mathfrak{A}x(t) - B\dot{u}(t) \\ &= \mathfrak{A}\left(v(t) + Bu(t)\right) - B\dot{u}(t) \\ &= \mathfrak{A}v(t) + \mathfrak{A}Bu(t) - B\dot{u}(t) \\ &= Av(t) + \mathfrak{A}Bu(t) - B\dot{u}(t), \end{aligned}$$

for all $x \in D(A)$. Hence this $v(t)$ is a solution of

$$\dot{v}(t) = Av(t) + \mathfrak{A}Bu(t) - B\dot{u}, \quad v(0) = v_0. \tag{4.13}$$

Furthermore, if $u \in C^2([0, \tau]; U)$, and $v_0 = x_0 - Bu(0) \in D(A)$, the classical solutions of (4.12) and (4.13) are related by $v(t) = x(t) - Bu(t)$. Instead of stating $x_0 - Bu(0) \in D(A)$, it is equivalent to state $x_0 \in D(\mathfrak{A})$ and $\mathfrak{B}x_0 = u(0)$.

The mild solution of (4.13) is given by

$$v(t) = T(t)v_0 + \int_0^t T(t-s)\left(\mathfrak{A}Bu(s) - B\dot{u}(s)\right)\, ds$$

for all $v_0 \in X$ and all $u \in H^1([0, \tau]; U)$ and $\tau > 0$. Hence by the relation between x and v,

$$x(t) = T(t)(x_0 - Bu(0)) + \int_0^t T(t-s)(\mathfrak{A}Bu(s) - B\dot{u}(s))\, ds + Bu(t)$$

is called the mild solution of the boundary control system for all $x_0 \in X$ and all $u \in H^1([0, \tau]; U)$ and $\tau > 0$.

4.11.3 Output

The system of the form (4.12) can be extended to not only include an input, but also an output function. This system will be of the form

$$\begin{aligned}\dot{x}(t) &= \mathfrak{A}x(t),\\ \mathfrak{B}x(t) &= u(t),\\ \mathfrak{C}x(t) &= y(t),\end{aligned}$$

for output $y(t)$. In this system, $(\mathfrak{A}, \mathfrak{B})$ has to satisfy the conditions of the earlier system, and $\mathfrak{C}$ has to be a linear operator defined on $D(\mathfrak{A}) \mapsto Y$, for some Hilbert space Y.

In this case, if $u \in C^2([0,\tau]; U)$, $x_0 \in D(\mathfrak{A})$, and $\mathfrak{B}x_0 = u(0)$, then the classical solution is given by

$$\begin{aligned}x(t) &= T(t)(x_0 - Bu(0)) + \int_0^t T(t-s)(\mathfrak{A}Bu(s) - B\dot{u}(s))ds + Bu(t),\\ y(t) &= \mathfrak{C}T(t)(x_0 - Bu(0)) + \mathfrak{C}\int_0^t T(t-s)(\mathfrak{A}Bu(s) - B\dot{u}(s))ds + \mathfrak{C}Bu(t).\end{aligned}$$

4.11.4 Port-Hamiltonian system with boundary conditions

Instead of a general system, consider a port-Hamiltonian system, with the usual assumptions, of the following form

$$\begin{aligned}\frac{\partial x}{\partial t}(\zeta,t) &= P_1\frac{\partial}{\partial \zeta}(H(\zeta)x(\zeta,t)) + P_0(H(\zeta)x(\zeta,t)),\\ u(t) &= W_{B,1}\begin{bmatrix} f_\partial(t)\\ e_\partial(t)\end{bmatrix},\\ 0 &= W_{B,2}\begin{bmatrix} f_\partial(t)\\ e_\partial(t)\end{bmatrix}.\end{aligned}$$

Then define

$$\begin{aligned}\mathfrak{A}x &= P_1\frac{\partial}{\partial \zeta}(Hx) + P_0(Hx),\\ D(\mathfrak{A}) &= \left\{x \in L^2([a,b];\mathbb{K}^n) \middle| \ Hx \in H^1([a,b];\mathbb{K}^n), W_{B,2}\begin{bmatrix} f_\partial\\ e_\partial\end{bmatrix} = 0\right\},\\ \mathfrak{B} &= W_{B,1}\begin{bmatrix} f_\partial\\ e_\partial\end{bmatrix},\\ D(\mathfrak{B}) &= D(\mathfrak{A}).\end{aligned}$$

Using these definitions, the port-Hamiltonian system can be rewritten into a boundary control system of the form (4.12), if the operator

$$Ax = P_1\frac{\partial}{\partial \zeta}(H(\zeta)x(\zeta,t)) + P_0(H(\zeta)x(\zeta,t)),$$

with domain

$$D(A) = \left\{ x \in X \mid Hx \in H^1([a,b];\mathbb{K}^n), \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in \ker \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \right\},$$

generates a C_0-semigroup on X.

4.11.5 Example

In light of Figure 4.1, the previous example showed equation (4.10) could be transformed into a linear port-Hamiltonian system, by defining $x_1 = \rho \frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$, which gave

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} \right) \\ &= P_1 \frac{\partial}{\partial \zeta} \left(H(\zeta) x(\zeta,t) \right). \end{aligned}$$

Using this state representation, the boundary flow and boundary effort were given by

$$\begin{aligned} f_\partial &= \frac{1}{\sqrt{2}} \begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b) - T(a)\frac{\partial w}{\partial \zeta}(a) \\ \frac{\partial w}{\partial t}(b) - \frac{\partial w}{\partial t}(a) \end{bmatrix}, \\ e_\partial &= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{\partial w}{\partial t}(b) + \frac{\partial w}{\partial t}(a) \\ T(b)\frac{\partial w}{\partial \zeta}(b) + T(a)\frac{\partial w}{\partial \zeta}(a) \end{bmatrix}. \end{aligned}$$

In the previous example, the boundary conditions were $\frac{\partial w}{\partial t}(a,t) = 0$ and $T(b)\frac{\partial w}{\partial \zeta}(b,t) = 0$. Suppose instead of no force applied at the right hand side of the string, the force is given by the input u. This changes the boundary conditions in

$$\begin{aligned} 0 &= \frac{\partial w}{\partial t}(a,t), \\ u(t) &= T(b)\frac{\partial w}{\partial \zeta}(b,t). \end{aligned}$$

In order to convert this into a boundary control system, the first step is computing $W_{B,1}$ and $W_{B,2}$ such that

$$\begin{aligned} u(t) &= W_{B,1} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \\ 0 &= W_{B,2} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}. \end{aligned}$$

This implies

$$T(b)\frac{\partial w}{\partial \zeta}(b,t) = \frac{\sqrt{2}}{2}\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b) - T(a)\frac{\partial w}{\partial \zeta}(a) \\ \frac{\partial w}{\partial t}(b) - \frac{\partial w}{\partial t}(a) \\ \frac{\partial w}{\partial t}(b) + \frac{\partial w}{\partial t}(a) \\ T(b)\frac{\partial w}{\partial \zeta}(b) + T(a)\frac{\partial w}{\partial \zeta}(a)\end{bmatrix},$$

$$\frac{\partial w}{\partial t}(a,t) = \frac{\sqrt{2}}{2}\begin{bmatrix}0 & -1 & 1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b) - T(a)\frac{\partial w}{\partial \zeta}(a) \\ \frac{\partial w}{\partial t}(b) - \frac{\partial w}{\partial t}(a) \\ \frac{\partial w}{\partial t}(b) + \frac{\partial w}{\partial t}(a) \\ T(b)\frac{\partial w}{\partial \zeta}(b) + T(a)\frac{\partial w}{\partial \zeta}(a)\end{bmatrix}.$$

Hence

$$\begin{aligned} W_{B,1} &= \frac{\sqrt{2}}{2}\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}, \\ W_{B,2} &= \frac{\sqrt{2}}{2}\begin{bmatrix}0 & -1 & 1 & 0\end{bmatrix}. \end{aligned}$$

As $W_B = \begin{bmatrix} W_{B,1} \\ W_{B,2}\end{bmatrix}$, which is the same as in the previous example, it has already been shown that

$$\begin{bmatrix}0 \\ 0\end{bmatrix} = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix},$$

and thus $\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in \ker \begin{bmatrix} W_{B,1} \\ W_{B,2}\end{bmatrix}$. By definition,

$$\begin{aligned} \mathfrak{A}x &= P_1 \frac{\partial}{\partial \zeta}(Hx) + P_0(Hx), \\ D(\mathfrak{A}) &= \left\{ x \in L^2([a,b];\mathbb{K}^n) \mid\ Hx \in H^1([a,b];\mathbb{K}^n), W_{B,2}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\}, \\ \mathfrak{B} &= W_{B,1}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}, \\ D(\mathfrak{B}) &= D(\mathfrak{A}), \end{aligned}$$

is a boundary control system.

5 Classification

Ideally, the theory of port-Hamiltonian systems is applied to the upcoming sloshing mug example, and by applying the theory a linear first order port-Hamiltonian system with boundary conditions arises. But before this happens, let us first take a closer look at which type of general equations can be transformed into a linear first order port-Hamiltonian system. This will be useful as it gives insight in what to look for in an equation.

To refresh our memory,

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} \left(H(\zeta)x(\zeta, t)\right) + P_0 \left(H(\zeta)x(\zeta, t)\right), \tag{5.1}$$

is called a linear, first order port-Hamiltonian system if the following holds;

- $P_1 \in \mathbb{K}^{n\times n}$ is invertible and $P_1^* = P_1$,
- $P_0 \in \mathbb{K}^{n\times n}$ such that $P_0^* = -P_0$,
- $H \in L^\infty([a, b]; \mathbb{K}^{n\times n})$ such that $H^* = H$ and there exist constants $m, M > 0$ such that $mI \leq H \leq MI$ for almost every $\zeta \in [a, b]$.

If a differential equation can be written in this form, in light of the example of the sloshing mug, the next step is to check if the domain associated to the operator A, defined by

$$Ax = P_1 \frac{\partial}{\partial \zeta}(Hx) + P_0(Hx),$$

and

$$D(A) = \left\{ x \in X |\ Hx \in H^1([a, b]; \mathbb{K}^n), \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in \ker \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \right\},$$

generates a C_0-semigroup on X. Note that $W_{B,1}$ and $W_{B,2}$ are as previously defined in 4.11.4. A second note is that the focus lay with finding out if an operator generates a contraction semigroup, not a general C_0-semigroup. This second step is in order to check if the system satisfies the conditions to be classified as a boundary control system. However, let us first focus on classifying general differential equations as a linear first order port-Hamiltonian system or not.

Assume for any of the following, that $y = y(\zeta, t)$ depends on the spatial variable ζ and the time variable t, such that y itself is not a partial derivative with respect to the time nor space. Furthermore, any other functions, unless explicitly stated, may depend on ζ, but not on the time. Specifically, this means that for any of these functions, it is not some constant times y. Lastly, both y and other functions are not allowed to be identically zero, unless defined as such, and are bounded. This last condition helps with the boundedness on the matrix H. For readability purposes the dependencies are omitted. These conditions might seem restrictive from a mathematical standpoint, but since port-Hamiltonian systems often describe mechanical problems, and in these problems the physical functions are almost never identically zero or unbounded, these restrictions come natural in most port-Hamiltonian systems. In the examples of Chapters 3 and 4, these functions included the current and voltage of a source and capacitors, and the mass density and Young's modulus of a string.

Before attempting to write equations in the desired form, note that whenever a representation is found, it is not unique. When a system can be written in the form (5.1), it can be slightly altered into another equation that satisfies the same conditions. The simplest example is defining new P_0' and P_1' by multiplying P_0 and P_1 by a non-zero constant, while defining H' by dividing H by the same constant. Hence a new representation is found by P_0', P_1' and H'.

5.1 Change of basis

In the following few sections, certain equations will be successfully or unsuccessfully transformed into linear first order port-Hamiltonian systems. In the case of a successful transformation, it might be interesting to know if another port-Hamiltonian system can be found. The reverse holds for an unsuccessful transformation. Is it possible to conclude that it is impossible to transform that equation in a port-Hamiltonian system, or are there other options to explore.

Suppose an invertible $S \in \mathbb{K}^{n \times n}$ does not depend on ζ, and a port-Hamiltonian system of the form

$$\dot{x} = P_1 \frac{\partial}{\partial \zeta}(Hx) + P_0(Hx)$$

is known. Define a new vector $z \in \mathbb{K}^n$ by $z(\zeta, t) = Sx(\zeta, t)$. Then the already established system can be used to obtain

$$\begin{aligned}
\dot{z} &= S\dot{x} \\
&= SP_1 \frac{\partial}{\partial \zeta}(Hx) + SP_0(Hx) \\
&= SP_1 \frac{\partial}{\partial \zeta}(S^* S^{-*} H S^{-1} Sx) + SP_0(S^* S^{-*} H S^{-1} Sx) \\
&= (SP_1 S^*) \frac{\partial}{\partial \zeta} \big((S^{-*} H S^{-1})(Sx) \big) + (SP_0 S^*) \big((S^{-*} H S^{-1})(Sx) \big) \\
&= P_1' \frac{\partial}{\partial \zeta} H' z + P_0' H' z,
\end{aligned}$$

with

$$\begin{aligned}
P_0' &= SP_0S^*,\\
P_1' &= SP_1S^*,\\
H' &= S^{-*}HS^{-1}.
\end{aligned}$$

This new system is a linear first order port-Hamiltonian system, if P_0' is skew-adjoint, P_1' is self-adjoint, and H' is self-adjoint, and there exist constants m', M', such that $m'I \leq H' \leq M'I$. The first three conditions are easily checked, since

$$\begin{aligned}
P_0'^* = (SP_0S^*)^* = SP_0^*S^* = -P_0'^* &\iff P_0^* = -P_0,\\
P_1'^* = (SP_1S^*)^* = SP_1^*S^* = P_1'^* &\iff P_1^* = P_1,\\
H'^* = (S^{-*}HS^{-1})^* = S^{-*}H^*S^{-1} = H' &\iff H^* = H.
\end{aligned}$$

Furthermore, the constant and invertibility properties also depend only on the old system, not the choice of S. The last condition that needs to be checked is $m'I \leq H' = S^{-*}HS^{-1} \leq M'I$. Since S is invertible, S^{-1} has full rank. Furthermore, it is already known that $mI \leq H \leq MI$. By [16], these two conditions combined give the existence of $m', M' > 0$ such that $m'I \leq H' \leq M'I$.

Hence by using a transformation of the form $z = Sx$, the system in x is a linear first order port-Hamiltonian system if and only if the system in z is a linear first order port-Hamiltonian system, independent of the choice for the invertible constant square matrix S.

A second point of order are the boundary conditions. For the known system, the boundary effort can be rewritten to

$$\begin{aligned}
e_\partial &= \frac{1}{\sqrt{2}}\left[(Hx)(b) + (Hx)(a)\right]\\
&= \frac{1}{\sqrt{2}}\left[(S^*S^{-*}HS^{-1}Sx)(b) + (S^*S^{-*}HS^{-1}Sx)(a)\right]\\
&= \frac{1}{\sqrt{2}}\left[(S^*H'z)(b) + (S^*H'z)(a)\right]\\
&= S^*\frac{1}{\sqrt{2}}\left[(H'z)(b) + (H'z)(a)\right]\\
&= S^*e_\partial',
\end{aligned}$$

for the new system. Similarly for the boundary flow,

$$
\begin{aligned}
f_\partial &= \frac{1}{\sqrt{2}}\left[(P_1 H x)(b) - (P_1 H x)(a)\right] \\
&= \frac{1}{\sqrt{2}}\left[(S^{-1} S P_1 S^* S^{-*} H S^{-1} S x)(b) - (S^{-1} S P_1 S^* S^{-*} H S^{-1} S x)(a)\right] \\
&= \frac{1}{\sqrt{2}}\left[(S^{-1} P_1' H' z)(b) - (S^{-1} P_1' H' z)(a)\right] \\
&= S^{-1}\frac{1}{\sqrt{2}}\left[(P_1' H' z)(b) - (P_1' H' z)(a)\right] \\
&= S^{-1} f_\partial'.
\end{aligned}
$$

Hence the boundary conditions for these systems can be linked by

$$
0 = W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = W_B \begin{bmatrix} S^{-1} f_\partial' \\ S^* e_\partial' \end{bmatrix} = W_B \begin{bmatrix} S^{-1} & 0 \\ 0 & S^* \end{bmatrix} \begin{bmatrix} f_\partial' \\ e_\partial' \end{bmatrix} = W_B' \begin{bmatrix} f_\partial' \\ e_\partial' \end{bmatrix}.
$$

Furthermore,

$$
\begin{aligned}
W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* &= W_B \begin{bmatrix} S^{-1} & 0 \\ 0 & S^* \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^* \end{bmatrix}^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^* \end{bmatrix}^{-*} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^* \end{bmatrix}^{*} W_B^* \\
&= W_B' \begin{bmatrix} S^{-1} & 0 \\ 0 & S^* \end{bmatrix}^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^* \end{bmatrix}^{-*} W_B'^* \\
&= W_B' \begin{bmatrix} S & 0 \\ 0 & S^{-*} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} S^* & 0 \\ 0 & S^{-1} \end{bmatrix} W_B'^* \\
&= W_B' \begin{bmatrix} S & 0 \\ 0 & S^{-*} \end{bmatrix} \begin{bmatrix} 0 & S^{-1} \\ S^* & 0 \end{bmatrix} W_B'^* \\
&= W_B' \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B'^*,
\end{aligned}
$$

hence

$$
W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B * \geq 0 \iff W_B' \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B'^* \geq 0,
$$

and the operator associated to the original port-Hamiltonian system generates a contraction semigroup if and only if the operator associated to the new port-Hamiltonian system generates a contraction semigroup. This time, the choice of S does not matter, as long as the matrix is invertible.

When S is not a constant matrix, but is allowed to depend on the spatial variable, things get more complicated. Define $z(\zeta,t) = S(\zeta)x(\zeta,t)$, then it still holds that

$$
\dot{z} = S\dot{x} = SP_1 \frac{\partial}{\partial \zeta}(Hx) + SP_0(Hx).
$$

In order to rewrite x into z again, the same trick can be done to obtain

$$\dot{z} = SP_1 \frac{\partial}{\partial \zeta}(HS^{-1}z) + SP_0(HS^{-1}z).$$

Using the same definition for $H'(\zeta)$, but now with S depending on the spatial variable too, this equation can be transformed into

$$\dot{z} = SP_1 \frac{\partial}{\partial \zeta}(S^* H' z) + SP_0(S^* H' z).$$

Previously, the matrix S could be moved through the derivative, but this time this is not possible due to S depending on the spatial variable. Furthermore, the matrices SP_1, SP_0 and $S^* H'$ are not in the desired form for a linear first order port-Hamiltonian system. Hence something else has to be done in order to rewrite this in the desired form. Using the product rule,

$$\begin{aligned}
\dot{z} &= SP_1 \left(\frac{\partial}{\partial \zeta} S^* \right) (H'z) + SP_1 S^* \frac{\partial}{\partial \zeta}(H'z) + SP_0(S^* H' z) \\
&= SP_1 S^* \frac{\partial}{\partial \zeta}(H'z) + S \left(\left(P_1 \frac{\partial}{\partial \zeta} + P_0 \right) S^* \right) (H'z) \\
&= P_1' \frac{\partial}{\partial \zeta} H'z + P_0' H'z.
\end{aligned}$$

This time the equation does resemble a linear first order port-Hamiltonian system. However, both P_1' and P_0' now depend on the spatial variable, due to $S(\zeta)$. Furthermore, due to the derivative, P_0' is now an unbounded operator. On the other hand, some factors do appear to be as desired. P_1' is invertible and self-adjoint, P_0' is skew-adjoint, and H' is positive definite and bounded provided that $S(\zeta)$ stays bounded on the spatial interval. Hence in order to write this completely in the desired form, at least the spatial dependencies have to drop, which means S can not depend on the spatial variable.

Thus a new representation of a linear first order port-Hamiltonian system can be obtained from another representation of a linear first order port-Hamiltonian system via a constant matrix S. Using a non constant $S(\zeta)$, no new linear first order port-Hamiltonian system representation can be identified. Furthermore, one can not create a representation of the desired form, provided that it is impossible for the old representation to represent a linear first order port-Hamiltonian system.

5.2 First-order time derivative

In the next few sections, partial differential equations will be studied and an attempt will be made to write them into a port-Hamiltonian setting. Whether or not a boundary control system with specific input, output and boundary conditions can be rewritten as a linear first order port-Hamiltonian system or not, at the very least it has to have the desired partial differential equation form. Hence in the following sections potential input, output and boundary conditions will be neglected. That is, the only systems that will be discussed are of the form

$$\begin{aligned}\dot{x} &= A_0 x,\\ A_0 x &= P_1 \frac{\partial}{\partial \zeta}(Hx) + P_0(Hx).\end{aligned}$$

The second thing to note is the method for checking if there exist constants $m, M > 0$ such that $mI \leq H \leq MI$. It is possible for a positive definite matrix to contain non zero elements off the diagonal, or even negative values. However, since the matrix H also needs to be self-adjoint, and all self-adjoint matrices that have all positive eigenvalues are automatically positive definite, the easiest form for H to be in is only positive entries on the diagonal, and zero entries everywhere else. In this case the matrix H is self-adjoint, the eigenvalues are the diagonal entries, and hence positive, thus H is positive-definite. By assumption on the functions to be used, and the boundedness of the spatial interval, the entries on the diagonal will all be bounded, hence there exists $m, M > 0$ such that $mI \leq H(\zeta) \leq MI$ for almost every ζ in the spatial interval. Almost all the examples will turn out to be in this form. Hence the only requirement on those matrices is that each entry on the diagonal will be self-adjoint and positive. In this case those matrices will automatically turn out to be positive definite. If on the other hand it is preferable to obtain a matrix H that has non-zero elements off the diagonal, by the previous section, it is possible to create such a matrix.

The simplest equation for a first order time derivative that includes a spatial derivative is

$$\frac{\partial}{\partial t} y = a \frac{\partial}{\partial \zeta} by.$$

At first glance this appears to be already in the desired form, with $P_0 = 0$. However, this would imply $P_1 = a(\zeta)$, while P_1 is not allowed to depend on ζ. In order to avoid this, defining $x = \frac{1}{a}y$ moves the function a inside the spatial derivative. This transforms the equation into

$$\frac{\partial}{\partial t} x = \frac{\partial}{\partial \zeta} abx,$$

which has the desired form as long as $H = (ab)^* = ab > 0$. If $ab < 0$, let $P_1 = -1$ and $H = -ab$ instead.

Extending the equation to

$$\frac{\partial}{\partial t} y = a \frac{\partial}{\partial \zeta} by + cy,$$

this equation can be solved in the same manner by $x = \frac{1}{a}y$, which results in

$$\frac{\partial}{\partial t} x = \frac{\partial}{\partial \zeta} abx + cx.$$

Hence there is again a restriction on $(ab)^* = ab \neq 0$, but also by definition of P_0 and H, depending on the sign of ab, either $P_0 ab = c$ or $P_0 ab = -c$. As P_0 does not depend on ζ, this implies ab is a constant, P_0, times c, where this constant has the property $P_0^* = -P_0$. Since this is a one dimensional system, this property can only hold if P_0 is purely imaginary with no real part.

Slightly altering the equation in

$$\frac{\partial}{\partial t}y = a\frac{\partial}{\partial \zeta}by + c,$$

changes the problem dramatically. The equation has a part which does not depend on y, which means it is not solvable as is. In order to get rid of this function, it has to be included in the definition of x. However, this term does not appear in the other side, which implies at least a two-dimensional system is needed. Any definition of component x_i that includes a, also has to consist of at least $\frac{\partial}{\partial \zeta}dy$, as the partial derivative can not be left out. Hence the equation is reduced to some form of

$$\frac{\partial}{\partial t}x_1 = ax_2.$$

Sadly this equation can not be set in the form (5.1) with the correct properties on both P_0 and P_1.

Finally adding everything together for

$$\frac{\partial}{\partial t}y = a\frac{\partial}{\partial \zeta}by + cy + d,$$

has the same problem with a term that does not depend on y. Hence any one-dimensional system is doomed to fail. Thus in order to transform this into a linear first order port-Hamiltonian system, the constant term has to be included in a higher dimensional x. The simplest choice for this is $x_2 = ay + d$. Alongside with $x_1 = y$, the equation can be rewritten as

$$\frac{\partial}{\partial t}x_1 = a\frac{\partial}{\partial \zeta}bx_1 + x_2.$$

Unfortunately, there does not exist an equation for the time derivative of x_2, that does not involve a time derivative of x_1. Hence it is not possible to write the the equation using this x as a linear first order port-Hamiltonian system.

So far only equations with the order of the spatial derivative lower or equal to the order of the time derivative have been considered. Consider an equation where the order of the spatial derivative is strictly greater, for example

$$\frac{\partial}{\partial t}y = \frac{\partial^2}{\partial \zeta^2}y.$$

A one-dimensional system will not represent this equation as a port-Hamiltonian system, hence at least two dimensions are needed. Furthermore, the order of the spatial derivative has to be lowered by one in order to write it as a first order port-Hamiltonian system. This would potentially not be a problem in linear second order port-Hamiltonian systems.

In order to reduce this system properly, define $x_1 = y$ and $x_2 = \frac{\partial}{\partial \zeta} y$. This reduces the equation to

$$\frac{\partial}{\partial t} x_1 = \frac{\partial}{\partial \zeta} x_2,$$

and gives the first of two equations. The other equation is given by

$$\frac{\partial}{\partial t} x_2 = \frac{\partial}{\partial t} \frac{\partial}{\partial \zeta} y.$$

This can not be written in a form which includes x_1. Even if it would be possible to interchange the order of differentiation, that would give

$$\frac{\partial}{\partial t} x_2 = \frac{\partial^2}{\partial \zeta^2} x_2.$$

These two equations together do not yield a port-Hamiltonian system. Hence the heat equation can not be rewritten as a linear first order port-Hamiltonian system.

The heat equation is a well-known equation and is often used as an example in mathematics. Knowing that the heat equation can not be transformed into a linear first order port-Hamiltonian system, it might be interesting to know if it can be described by contraction semigroups. Consider the heat equation described by

$$\begin{aligned} \frac{\partial}{\partial t} y &= \frac{\partial^2}{\partial \zeta^2} y, \quad \text{for } 0 \leq \zeta \leq 1, \\ \frac{\partial}{\partial \zeta} y(0,t) &= 0 = \frac{\partial}{\partial \zeta} y(1,t), \end{aligned}$$

or the operator A defined by

$$Ax = \frac{\partial^2}{\partial \zeta^2} x,$$

on the domain

$$\begin{aligned} D(A) = \Big\{ & x \in L^2((0,1); \mathbb{K}^n) |\ x \text{ and } \frac{\partial}{\partial \zeta} x \text{ are absolutely continuous,} \\ & \frac{\partial^2}{\partial \zeta^2} x \in L^2((0,1); \mathbb{K}^n), \text{ and } \frac{\partial}{\partial \zeta} y(0,t) = 0 = \frac{\partial}{\partial \zeta} y(1,t) \Big\}. \end{aligned}$$

The first step in proving that the operator A is the infinitesimal generator of a contraction semigroup, is proving it is dissipative. The operator is indeed dissipative, since

$$\begin{aligned} 2Re\langle Ax, x \rangle &= \langle Ax, x \rangle + \langle x, Ax \rangle \\ &= \int_0^1 \left(\frac{\partial^2}{\partial \zeta^2} x \right) \overline{x} d\zeta + \int_0^1 x \overline{\frac{\partial^2}{\partial \zeta^2}} x d\zeta \\ &= \left(\overline{x} \frac{\partial}{\partial \zeta} x + x \overline{\frac{\partial}{\partial \zeta} x} \right) \Bigg|_{\zeta=0}^1 - 2 \int_0^1 \left(\frac{\partial}{\partial \zeta} x \right) \overline{\frac{\partial}{\partial \zeta}} x d\zeta \\ &= 0 - 2 \int_0^1 \left\| \frac{\partial}{\partial \zeta} x \right\|^2 d\zeta \leq 0, \end{aligned}$$

for all $x \in D(A)$. The next step is to prove the range of $(I - A)$ equals $L^2((0,1);\mathbb{K}^n)$, then Lumer-Phillips theorem can be used to conclude that the heat equation can be described using contraction semigroups. Let $y \in L^2((0,1);\mathbb{K}^n)$, and $x \in D(A)$, for a to be determined x. It has to hold that

$$\begin{aligned}(I - A)x &= f\\ x - \frac{\partial^2}{\partial\zeta^2}x &= f.\end{aligned}$$

Negating the time dependence, define x as

$$x(\zeta) := \frac{\cosh(\zeta)}{\sinh(1)}\int_0^1 \cosh(1-\tau)f(\tau)d\tau - \int_0^\zeta \sinh(\zeta-\tau)f(\tau)d\tau,$$

then by Leibniz integral rule

$$\begin{aligned}\frac{\partial}{\partial\zeta}x &= \frac{\sinh(\zeta)}{\sinh(1)}\int_0^1 \cosh(1-\tau)f(\tau)d\tau - \sinh(0)f(\zeta) - \int_0^\zeta \cosh(\zeta-t)f(\tau)d\tau\\ &= \frac{\sinh(\zeta)}{\sinh(1)}\int_0^1 \cosh(1-\tau)f(\tau)d\tau - \int_0^\zeta \cosh(\zeta-t)f(\tau)d\tau,\\ \frac{\partial^2}{\partial\zeta^2}x &= \frac{\cosh(\zeta)}{\sinh(1)}\int_0^1 \cosh(1-\tau)f(\tau)d\tau - \cosh(0)f(\zeta) - \int_0^\zeta \sinh(\zeta-t)f(\tau)d\tau\\ &= x(\zeta) - f(\zeta).\end{aligned}$$

Thus this choice of x, it has the property $(I - A)x = f$. By definition of the hyperbolic functions, f and the integral, $x, \frac{\partial^2}{\partial\zeta^2} \in L^2((0,1);\mathbb{K}^n)$, and x and $\frac{\partial}{\partial\zeta}x$ are absolutely continuous. Furthermore,

$$\begin{aligned}\frac{\partial}{\partial\zeta}x(0) &= \frac{\sinh(0)}{\sinh(1)}\int_0^1 \cosh(1-\tau)f(\tau)d\tau - \lim_{\zeta\downarrow 0}\int_0^\zeta \cosh(\zeta-t)f(\tau)d\tau = 0,\\ \frac{\partial}{\partial\zeta}x(1) &= \frac{\sinh(1)}{\sinh(1)}\int_0^1 \cosh(1-\tau)f(\tau)d\tau - \int_0^1 \cosh(\zeta-t)f(\tau)d\tau = 0,\end{aligned}$$

hence $x \in D(A)$, and the range of $(I - A)$ is closed. Thus the operator A generates a contraction semigroup. Even though the heat equation can not be written as a linear first order port-Hamiltonian system, it can still be represented by a contraction semigroup.

Continuing with linear port-Hamiltonian systems, if the order of the spatial is even higher, for example

$$\frac{\partial}{\partial t}y = \frac{\partial^3}{\partial\zeta^3}y,$$

the same approach can be made. However, now the order has to be reduced by two. Hence define $x_1 = y$ and $x_2 = \frac{\partial^2}{\partial\zeta^2}y$. The problem now arises that the time derivative of

x_2 can not be written as a first order spatial derivative or lower of x_1. In order to avoid this, define instead $x_1 = y$, $x_2 = \frac{\partial}{\partial\zeta}y$ and $x_3 = \frac{\partial^2}{\partial\zeta^2}y$. This results in the equations

$$\begin{aligned}
\frac{\partial}{\partial t}x_1 &= \frac{\partial}{\partial\zeta}x_3,\\
\frac{\partial}{\partial t}x_2 &= \frac{\partial}{\partial\zeta}x_1,\\
\frac{\partial}{\partial t}x_3 &= \frac{\partial}{\partial\zeta}x_2.
\end{aligned}$$

This is of the form

$$\frac{\partial}{\partial t}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\frac{\partial}{\partial\zeta}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}.$$

This looks promising, as H is already in the desired form. However, $P_1^* \neq P_1$. This matrix can be changed by interchanging some definitions of x_1, x_2 and x_3. Doing so gives the option of all available matrices constructed from the basis vectors. However, in Section 5.1, it has already been observed that this will not yield a new system that is a linear first order port-Hamiltonian system. This type of problem arises for all spatial derivatives which have a higher odd order.

On the other hand, if the higher order of the spatial derivative is even, for example

$$\frac{\partial}{\partial t}y = \frac{\partial^4}{\partial\zeta^4}y,$$

the system can be transformed using

$$\begin{aligned}
\frac{\partial}{\partial t}x_1 &= \frac{\partial}{\partial\zeta}x_4,\\
\frac{\partial}{\partial t}x_2 &= \frac{\partial}{\partial\zeta}x_1,\\
\frac{\partial}{\partial t}x_3 &= \frac{\partial}{\partial\zeta}x_2,\\
\frac{\partial}{\partial t}x_4 &= \frac{\partial}{\partial\zeta}x_3,
\end{aligned}$$

into

$$\frac{\partial}{\partial t}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\frac{\partial}{\partial\zeta}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}.$$

Where it was impossible to create a matrix P_1 that is self-adjoint and has a zero diagonal in the odd dimensional case, in the even case there exist matrices which are self-adjoint and have a zero diagonal. However, the current matrix is not of that form, and it is not possible to create such a self-adjoint matrix for this example, due to the nature of the dependencies of the components of x. Even by interchanging these definitions, the same problem occurs, namely P_1 can not be self-adjoint.

Hence it does not make much sense to try to identify equations which have a much higher order of spatial derivative than time derivative for a linear first order port-Hamiltonian system. These could potentially be identified using other orders of linear port-Hamiltonian systems.

Returning to the first example of higher order spatial derivatives, extending the equation to

$$\frac{\partial}{\partial t}y = a\frac{\partial}{\partial \zeta}b\frac{\partial}{\partial \zeta}cy + d,$$

which has a problem fitting as a port-Hamiltonian system due to the last term, or

$$\frac{\partial}{\partial t}y = a\frac{\partial}{\partial \zeta}b\frac{\partial}{\partial \zeta}cy + dy,$$

which also has a problem with the last term. The same holds for

$$\frac{\partial}{\partial t}y = a\frac{\partial}{\partial \zeta}b\frac{\partial}{\partial \zeta}cy + dy + e.$$

These problems can not be solved in two dimensions, and higher dimensions have a problem with getting an equation for the time derivative on the part that includes these last terms.

These problems diminish when another spatial derivative is involved, namely equations of the form

$$\frac{\partial}{\partial t}y = a\frac{\partial}{\partial \zeta}b\frac{\partial}{\partial \zeta}cy + d\frac{\partial}{\partial \zeta}ey.$$

If $c = e$, then $x_1 = \frac{1}{a}y$ and $x_2 = \frac{\partial}{\partial \zeta}cy$ results in

$$\begin{aligned}\frac{\partial}{\partial t}x_1 &= \frac{\partial}{\partial \zeta}bx_2 + \frac{d}{a}x_2,\\ \frac{\partial}{\partial t}x_2 &= \dots.\end{aligned}$$

This looks promising, as the first equation is in the desired form, while the second equation can not be written in a port-Hamiltonian setting. This does not imply that the equation can not be written as a port-Hamiltonian system. There are still options to explore.

A three dimensional system might provide a solution, however, any choice that involves the last part of the equation, leads to problems with either P_0 or P_1. The main problem with the last few equations, is that it is not possible to obtain an equation of the desired form for both components of x. This might be possible for higher order time derivatives.

5.3 Second-order time derivative

Things do change a little bit for second order time derivatives compared to first order time derivatives. The main difference is that one-dimensional systems will normally not work here. As one needs to reduce the order of the time derivative to a first order, the only way this can hold is when the other terms also have a first order time derivative. For example equations of the form

$$\frac{\partial^2}{\partial t^2}y = \frac{\partial}{\partial \zeta}\frac{\partial}{\partial t}y.$$

However these type of equations do normally not occur, hence these will not be further discussed, as one could simply redefine y to $x = \frac{\partial}{\partial t}y$, to obtain equations that were already discussed.

The simplest equation that includes at least one spatial derivative is of the form

$$\frac{\partial^2}{\partial t^2}y = a\frac{\partial}{\partial \zeta}by.$$

By defining $x_1 = \frac{1}{a}\frac{\partial}{\partial t}y$ and $x_2 = y$, two equations arise. Namely

$$\begin{aligned}\frac{\partial}{\partial t}x_1 &= \frac{\partial}{\partial \zeta}bx_2,\\ \frac{\partial}{\partial t}x_2 &= ax_1,\end{aligned}$$

which has problems with both P_0 and P_1. One point of order here, is the method for obtaining the second equation. The order of taking the partial derivatives was interchanged. This requires sufficient smoothness on the function y. From now on, assume that all the functions are sufficiently smooth to interchange the order of differentiation. The problems with P_0 and P_1 occur due to the first component depending on the second component only via a derivative, which is part of P_1, while the second component depends on the first component only without a derivative, which is part of P_0. Hence these two matrices can never be self-adjoint or skew-adjoint.

The problems of P_0 not being skew-adjoint and P_1 not being self-adjoint, still occurs for equations of the form

$$\frac{\partial^2}{\partial t^2}y = a\frac{\partial}{\partial \zeta}by + cy + d.$$

When a second order spatial derivative is involved, things turn out for the better. Equations of the form

$$\frac{\partial^2}{\partial t^2}y = a\frac{\partial}{\partial \zeta}b\frac{\partial}{\partial \zeta}cy,$$

can be transformed using $x_1 = \frac{1}{a}\frac{\partial}{\partial t}y$ and $x_2 = \frac{\partial}{\partial \zeta}cy$. This gives the following equations

$$\begin{aligned}\frac{\partial}{\partial t}x_1 &= \frac{\partial}{\partial \zeta}bx_2,\\ \frac{\partial}{\partial t}x_2 &= \frac{\partial}{\partial \zeta}acx_1.\end{aligned}$$

Hence the system can be written as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{bmatrix} ac & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where P_1 and H are already in the desired form, as long as $(ac)^* = ac > 0$ and $b^* = b > 0$. If on the other hand $(ac)^* = ac < 0$ and $b^* = b < 0$, it is possible to move the minus from H to P_1,

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{bmatrix} -ac & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and the conditions necessary for a port-Hamiltonian system are still satisfied. Lastly, if either ac or b is negative, but not both, this tactic does not work, and the equation can not be transformed into a linear first order port-Hamiltonian system.

The next equation is an extension of the previous equation, and can be approached in two different ways.

$$\frac{\partial^2}{\partial t^2} y = a\frac{\partial}{\partial \zeta} b \frac{\partial}{\partial \zeta} cy + d\frac{\partial}{\partial \zeta} ey.$$

The first approach requires $a = d$, for if this is not true, P_1 will depend on the spatial variable. By defining $x_1 = \frac{1}{a}\frac{\partial}{\partial t}y$ and $x_2 = b\frac{\partial}{\partial \zeta}cy + ey$, the equations

$$\begin{aligned} \frac{\partial}{\partial t} x_1 &= \frac{\partial}{\partial \zeta} x_2, \\ \frac{\partial}{\partial t} x_2 &= b\frac{\partial}{\partial \zeta} acx_1 + aey, \end{aligned}$$

arise. However, two problems occur now. The first is that P_1 will again depend on ζ, which can be avoided by assuming $b = e$, and not including this function in the definition of x_2. The second problem however can not be averted, which is that P_0 can not be skew-adjoint. In an attempt to avoid this problem, the equation has to be approached in three dimensions. Define in a similar manner, $x_1 = \frac{1}{a}\frac{\partial}{\partial t}y$, $x_2 = \frac{\partial}{\partial \zeta}cy$ and $x_3 = ey$. Again the P_1 will be depend on ζ, as long as $a \neq d$. But P_0 still has a problem with not being skew-adjoint. Changing $x_3 = \frac{\partial}{\partial \zeta}ey$ yields the same problem for P_0.

Hence any extension of the form

$$\frac{\partial^2}{\partial t^2} y = a\frac{\partial}{\partial \zeta} b \frac{\partial}{\partial \zeta} cy + d\frac{\partial}{\partial \zeta} ey + fy + g,$$

can not be written as a linear first order port-Hamiltonian system, as long as all the extra terms are non-zero.

Just like in the first order time derivative case, let us consider a case where the order of the spatial derivative is higher than the order of the time derivative. The most simple case is

$$\frac{\partial^2}{\partial t^2} y = \frac{\partial^3}{\partial \zeta^3} y.$$

In order to make this first order, at least two spatial derivatives have to be incorporated in x. If this is done in a two dimensional fashion, the equation is not transformed to a first order port-Hamiltonian system. Hence at least three dimensions are needed. Just as before, define $x_1 = \frac{\partial}{\partial t}y$, $x_2 = \frac{\partial^2}{\partial \zeta^2}y$ and $x_3 = \frac{\partial}{\partial \zeta}y$. This transforms in

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

And once again, P_1 can not be made self-adjoint in three dimensions. There exists no obvious method of writing this equations as a four dimensional system. Hence most of the standard equations for second order time derivatives have been evaluated.

5.4 Complex systems

Instead of the more standard equations from before, four more complex examples will be discussed, in more detail, here. The first three are all variants of the same basic system. The last example is about combining solutions or equations of the same form.

The basic equation is a coupled system, where both y and z are assumed to be sufficiently smooth, and depend on both time and space. The coupled system

$$\begin{cases} \frac{\partial^2}{\partial t^2}y = & a\frac{\partial}{\partial \zeta}b\frac{\partial}{\partial \zeta}cy + d\frac{\partial}{\partial \zeta}e\frac{\partial}{\partial \zeta}fz + g\frac{\partial}{\partial \zeta}hy + i\frac{\partial}{\partial \zeta}jz \\ \frac{\partial^2}{\partial t^2}z = & k\frac{\partial}{\partial \zeta}l\frac{\partial}{\partial \zeta}my + n\frac{\partial}{\partial \zeta}o\frac{\partial}{\partial \zeta}pz + q\frac{\partial}{\partial \zeta}ry + s\frac{\partial}{\partial \zeta}tz, \end{cases}$$

can also involve extra terms like uy or vw, however this results into failure of rewriting the system as a linear first order port-Hamiltonian system for the first example.

The first example involves trying to write this coupled system as a four dimensional system, two dimensions for each equation, in a basic manner. In order to do this, the amount of freedom needs to be reduced in order to attempt and write the system in the desired form. For this reason, assume $c = h = m = r$ and $f = j = p = t$, which reduces to coupled system to

$$\begin{cases} \frac{\partial^2}{\partial t^2}y = & a\frac{\partial}{\partial \zeta}b\frac{\partial}{\partial \zeta}cy + d\frac{\partial}{\partial \zeta}e\frac{\partial}{\partial \zeta}fz + g\frac{\partial}{\partial \zeta}cy + i\frac{\partial}{\partial \zeta}fz \\ \frac{\partial^2}{\partial t^2}z = & k\frac{\partial}{\partial \zeta}l\frac{\partial}{\partial \zeta}cy + n\frac{\partial}{\partial \zeta}o\frac{\partial}{\partial \zeta}fz + q\frac{\partial}{\partial \zeta}cy + s\frac{\partial}{\partial \zeta}fz. \end{cases}$$

Hence by defining $x_1 = \frac{\partial}{\partial t}y$, $x_2 = \frac{\partial}{\partial t}z$, $x_3 = \frac{\partial}{\partial \zeta}cy$ and $x_4 = \frac{\partial}{\partial \zeta}fz$, the time derivatives are given by

$$\begin{aligned} \frac{\partial}{\partial t}x_1 &= a\frac{\partial}{\partial \zeta}bx_3 + d\frac{\partial}{\partial \zeta}ex_4 + gx_3 + ix_4, \\ \frac{\partial}{\partial t}x_2 &= k\frac{\partial}{\partial \zeta}lx_3 + n\frac{\partial}{\partial \zeta}ox_4 + qx_3 + sx_4, \\ \frac{\partial}{\partial t}x_3 &= \frac{\partial}{\partial \zeta}cx_1, \\ \frac{\partial}{\partial t}x_4 &= \frac{\partial}{\partial \zeta}fx_2. \end{aligned}$$

This does imply the matrix P_1 will depend on ζ, which can be avoided by setting $a = d$ and $k = n$, and redefining $x_1 = \frac{1}{a}\frac{\partial}{\partial t}y$ and $x_2 = \frac{1}{k}\frac{\partial}{\partial t}z$. Furthermore, the matrix H will run into problems as long as $b \neq l$ and $c \neq o$. With the extra assumptions, these equations can be written as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{bmatrix} ac & 0 & 0 & 0 \\ 0 & fm & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{g}{a} & \frac{i}{a} \\ 0 & 0 & \frac{q}{k} & \frac{s}{k} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

The first thing to note is, if the second part of the equation is set to zero, $g = i = q = s = 0$, then H is in the desired form. If this is not the case, P_0 can not be skew-adjoint. The second observation is that P_1 is currently not self-adjoint. Hence x_1 can not depend on x_4, and x_2 can not depend on x_3, in the part where the partial derivative lives. This implies $k = d = 0$. Combining all of these together results in

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{bmatrix} ac & 0 & 0 & 0 \\ 0 & fm & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

This might look impressive, as the system has been written as a linear first order port-Hamiltonian system, but consider the cost of setting all of these functions equal to zero. The remaining coupled system is given by

$$\begin{cases} \frac{\partial^2}{\partial t^2}y = & a\frac{\partial}{\partial \zeta}b\frac{\partial}{\partial \zeta}cy \\ \frac{\partial^2}{\partial t^2}z = & n\frac{\partial}{\partial \zeta}o\frac{\partial}{\partial \zeta}fz, \end{cases}$$

which is not even a coupled system at all. This is a system with two separate equations, which was already solved, put together in a single system. Hence this simple choice of x does not result in an useful port-Hamiltonian system.

In the previous strategy, each component of x that represented a part of the coupled system, only represented one of the two functions of the coupled system. This ultimately resulted into an uncoupled system. In order to avoid this, this strategy involves at least one component that depends on both functions of the coupled system. This does require some assumptions before the analysis starts. Assume $a = i$, $b = q$, $c = r$ and $j = v$. This results into the coupled system

$$\begin{cases} \frac{\partial^2}{\partial t^2}y = & a\frac{\partial}{\partial \zeta}\left(b\frac{\partial}{\partial \zeta}cy + jz\right) + d\frac{\partial}{\partial \zeta}e\frac{\partial}{\partial \zeta}fz + g\frac{\partial}{\partial \zeta}hy \\ \frac{\partial^2}{\partial t^2}z = & k\frac{\partial}{\partial \zeta}l\frac{\partial}{\partial \zeta}my + s\frac{\partial}{\partial \zeta}tz + n\frac{\partial}{\partial \zeta}o\frac{\partial}{\partial \zeta}pz + \left(b\frac{\partial}{\partial \zeta}cy + jz\right). \end{cases}$$

With the upcoming choice for x, set $f = g = k = s = 0$. By defining $x_1 = \frac{1}{a}\frac{\partial}{\partial t}y$,

$x_2 = \frac{1}{n}\frac{\partial}{\partial t}z$, $x_3 = \frac{\partial}{\partial\zeta}cy + \frac{j}{b}z$ and $x_4 = \frac{\partial}{\partial\zeta}fz$, the system can be written as

$$
\begin{aligned}
\frac{\partial}{\partial t}x_1 &= \frac{\partial}{\partial\zeta}bx_3,\\
\frac{\partial}{\partial t}x_2 &= \frac{\partial}{\partial\zeta}ox_4 + \frac{b}{n}x_3,\\
\frac{\partial}{\partial t}x_3 &= \frac{\partial}{\partial\zeta}acx_1 + \frac{jn}{b}x_2,\\
\frac{\partial}{\partial t}x_4 &= \frac{\partial}{\partial\zeta}fnx_2,
\end{aligned}
$$

or

$$
\frac{\partial}{\partial t}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}\frac{\partial}{\partial\zeta}\begin{bmatrix}ac&0&0&0\\0&fn&0&0\\0&0&b&0\\0&0&0&o\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&\frac{b}{n}&0\\0&\frac{jn}{b}&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}.
$$

As long as $(ac)^* = ac > 0$, $(fn)^* = fn > 0$, $b^* = b > 0$ and $o^* = o > 0$, both P_1 and H are correct. All that remains is to construct a P_0 that does not depend on ζ and is skew-adjoint, such that

$$
\begin{bmatrix}0&0&0&0\\0&0&\frac{b}{n}&0\\0&\frac{jn}{b}&0&0\\0&0&0&0\end{bmatrix} = P_0\begin{bmatrix}ac&0&0&0\\0&fn&0&0\\0&0&b&0\\0&0&0&o\end{bmatrix}.
$$

Hence P_0 is of the form

$$
P_0 = \begin{bmatrix}0&0&0&0\\0&0&p_0&0\\0&p_1&0&0\\0&0&0&0\end{bmatrix}.
$$

As $p_0 b = \frac{b}{n}$, and P_0 may not depend on ζ, $p_0 = \frac{1}{n}$ can not depend on ζ, hence n can not depend on ζ. For p_1 there exist more choices, as long as $p_1 f = \frac{j}{b}$, with p_1 independent of ζ. This can either be accomplished by $f = \frac{j}{b}$, or none of those three functions are depending on ζ. Finally, P_0 being skew-adjoint implies $-n^* = \frac{bf}{j}$. Incorporating all assumptions, the coupled equation

$$
\begin{cases}
\frac{\partial^2}{\partial t^2}y = & a\frac{\partial}{\partial\zeta}\left(b\frac{\partial}{\partial\zeta}cy + jz\right)\\
\frac{\partial^2}{\partial t^2}z = & n\frac{\partial}{\partial\zeta}o\frac{\partial}{\partial\zeta}pz + \left(b\frac{\partial}{\partial\zeta}cy + jz\right),
\end{cases}
$$

can be written as a linear first order port-Hamiltonian system by

$$\frac{\partial}{\partial t}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}\frac{\partial}{\partial\zeta}\begin{bmatrix}ac&0&0&0\\0&fn&0&0\\0&0&b&0\\0&0&0&o\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$
$$+\begin{bmatrix}0&0&0&0\\0&0&\frac{1}{n}&0\\0&\frac{j}{bf}&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}ac&0&0&0\\0&fn&0&0\\0&0&b&0\\0&0&0&o\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}.$$

This time the coupled system stays as a coupled system for this choice of x. Once again, if all diagonal elements of H are negative instead, the minus sign can be moved into both P_0 and P_1, such that H contains only positive elements. This time however, it might also be possible that some elements of H are negative while others are positive. For example when both ac and b are negative, the minus sign can be moved into P_1, and P_1 stays self-adjoint. In this case, the minus sign must also be transferred into P_0. But since P_0H does not depend on ac, only a single element of the matrix receives a minus sign. This implies it is not skew-adjoint anymore. Instead, it has to hold that $n^* = \frac{bf}{j}$. The same argumentation can be made for both fn and o being negative instead.

However, there are still more options to explore, as there still were a lot of functions set to zero in the last attempt. Instead of only one component of x depending on both y and z, let a second component also depend on both y and z. The coupled system

$$\begin{cases}\frac{\partial^2}{\partial t^2}y = & a\frac{\partial}{\partial\zeta}b\frac{\partial}{\partial\zeta}cy + d\frac{\partial}{\partial\zeta}e\frac{\partial}{\partial\zeta}fz + g\frac{\partial}{\partial\zeta}hy + i\frac{\partial}{\partial\zeta}jz\\ \frac{\partial^2}{\partial t^2}z = & k\frac{\partial}{\partial\zeta}l\frac{\partial}{\partial\zeta}my + n\frac{\partial}{\partial\zeta}o\frac{\partial}{\partial\zeta}pz + q\frac{\partial}{\partial\zeta}ry + s\frac{\partial}{\partial\zeta}tz,\end{cases}$$

along with the assumptions $a = d = g = i$, $k = n = q = s$, $b = l$, $c = m$, $e = o$, $f = p$, $h = r$ and $j = t$, transforms the coupled system into

$$\begin{cases}\frac{\partial^2}{\partial t^2}y = & a\frac{\partial}{\partial\zeta}\left(b\frac{\partial}{\partial\zeta}cy + jz\right) + a\frac{\partial}{\partial\zeta}\left(e\frac{\partial}{\partial\zeta}fz + hy\right)\\ \frac{\partial^2}{\partial t^2}z = & k\frac{\partial}{\partial\zeta}\left(b\frac{\partial}{\partial\zeta}cy + jz\right) + k\frac{\partial}{\partial\zeta}\left(e\frac{\partial}{\partial\zeta}fz + hy\right).\end{cases}$$

Hence with $x_1 = \frac{1}{a}\frac{\partial}{\partial t}y$, $x_2 = \frac{1}{k}\frac{\partial}{\partial t}z$, $x_3 = \frac{\partial}{\partial\zeta}cy + \frac{j}{b}z$ and $x_4 = \frac{\partial}{\partial\zeta}fz + \frac{h}{c}y$, the coupled system can also be written as

$$\frac{\partial}{\partial t}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0&0&1&1\\0&0&1&1\\1&0&0&0\\0&1&0&0\end{bmatrix}\frac{\partial}{\partial\zeta}\begin{bmatrix}ac&0&0&0\\0&fk&0&0\\0&0&b&0\\0&0&0&c\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&\frac{jk}{b}&0&0\\\frac{ah}{c}&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}.$$

Once again, there are problems with creating a P_0 and P_1 with the correct properties. In order to avoid this, the system has to be reduced to an uncoupled system, just like the first attempt.

This does imply that for some choices of x the coupled system can be written as a linear first order port-Hamiltonian system, where the system stays coupled, while for others it uncouples. In these examples, a choice was made for x, and the right assumptions had to be made in order to fit those choices in the system. This has been mainly done to give some insight in the general problems that occur when trying to write a system as a port-Hamiltonian system. Normally, the components of x arise from the form of the equation, as no additional assumptions can be made on the system itself.

Finally, consider adding solutions, or more complex equations like

$$\frac{\partial^2}{\partial t^2}y = a\frac{\partial}{\partial \zeta}b\frac{\partial}{\partial \zeta}cy + d\frac{\partial}{\partial \zeta}e\frac{\partial}{\partial \zeta}fy,$$

with $a \neq d$, $b \neq e$ and $c \neq f$, such that the first part of the right side can be written in the form

$$P\frac{\partial}{\partial \zeta}Gx,$$

and the second part of the right side as

$$P'\frac{\partial}{\partial \zeta}G'x,$$

with P, P' self-adjoint, invertible and not dependent on ζ, and $mI \leq G \leq MI$ and $m'I \leq G' \leq M'I$. Each part on its own is a port-Hamiltonian system, but together they are not necessarily. For it has to hold that

$$P\frac{\partial}{\partial \zeta}Gx + P'\frac{\partial}{\partial \zeta}G'x = P_1\frac{\partial}{\partial \zeta}Hx,$$

with P_1 self-adjoint, invertible and not dependent on ζ, and $H > 0$ and bounded. This problem can easily be solved when $P = P'$, as then $P_1 = P$ and $H = G + G'$ is in the desired form, or when $G = G'$, as then $P_1 = P + P'$ and $H = G$ is in the desired form. The same holds when these matrices are a positive multiple of each other. Whenever this is not the case, things get a little more complicated.

To illustrate this, suppose x is two dimensional and real. Then because the matrices are self-adjoint, they are of the form

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2^* & p_3 \end{bmatrix},$$

$$P' = \begin{bmatrix} q_1 & q_2 \\ q_2^* & q_3 \end{bmatrix}.$$

Furthermore, the other two matrices are of the form

$$G = \begin{bmatrix} g_1 & g_2 \\ g_2^* & g_3 \end{bmatrix},$$

$$G' = \begin{bmatrix} h_1 & h_2 \\ h_2^* & h_3 \end{bmatrix}.$$

Hence

$$P\frac{\partial}{\partial\zeta}Gx + P'\frac{\partial}{\partial\zeta}G'x = \frac{\partial}{\partial\zeta}\begin{bmatrix} p_1g_1 + p_2g_2^* + q_1h_1 + q_2h_2^* & p_1g_2 + p_2g_3 + q_1h_2 + q_2h_3 \\ p_2^*g_1 + p_3g_2^* + q_2^*h_1 + q_3h_2^* & p_2^*g_2 + p_3g_3 + q_2^*h_2 + q_3h_3 \end{bmatrix}.$$

Since it is not possible to take any constants out of this new matrix H, and put them in front of the derivative in P_1, the matrix

$$H = \begin{bmatrix} p_1g_1 + p_2g_2^* + q_1h_1 + q_2h_2^* & p_1g_2 + p_2g_3 + q_1h_2 + q_2h_3 \\ p_2^*g_1 + p_3g_2^* + q_2^*h_1 + q_3h_2^* & p_2^*g_2 + p_3g_3 + q_2^*h_2 + q_3h_3 \end{bmatrix}$$

needs to be self-adjoint. This implies that

$$\begin{aligned}(p_1g_1 + p_2g_2^* + q_1h_1 + q_2h_2^*)^* &= p_1g_1 + p_2g_2^* + q_1h_1 + q_2h_2^*,\\ (p_1g_2 + p_2g_3 + q_1h_2 + q_2h_3)^* &= p_2^*g_1 + p_3g_2^* + q_2^*h_1 + q_3h_2^*,\\ (p_2^*g_2 + p_3g_3 + q_2^*h_2 + q_3h_3)^* &= p_2^*g_2 + p_3g_3 + q_2^*h_2 + q_3h_3.\end{aligned}$$

It is known that P, P', G and G' are self-adjoint. Hence the first and last equality hold if

$$\begin{aligned} p_2^* &= p_2,\\ g_2^* &= g_2,\\ q_2^* &= q_2,\\ h_2^* &= h_2.\end{aligned}$$

Note that these conditions imply all four matrices P, P', G and G' need to be real. The second equality reduces to

$$g_2(\zeta)(p_1 - p_3) + p_2(g_3(\zeta) - g_1(\zeta)) = q_2(h_1(\zeta) - h_3(\zeta)) + h_2(\zeta)(q_3 - q_1),$$

which is a restriction on the options for P, P', G and G'. Furthermore, the matrix H has to have to property $nI \leq H \leq NI$, for some constants $n, N > 0$. This property is not easily checked for general matrices H of this form. In higher dimensions, the self-adjoint property is still relatively easy to check, and provide conditions for when this holds. The positive definite ordering becomes even harder to describe for general matrices of higher order. This means that equations of the form

$$\frac{\partial^2}{\partial t^2}y = a\frac{\partial}{\partial\zeta}b\frac{\partial}{\partial\zeta}cy + d\frac{\partial}{\partial\zeta}e\frac{\partial}{\partial\zeta}fy,$$

where the functions themselves are not alike, are hard to describe in terms of linear first order port-Hamiltonian systems.

5.5 Boundary conditions

Given that a system can be written as a linear first order port-Hamiltonian system, the next step in boundary control systems, is to check if the associated operator generates a C_0-semigroup. As most of the theory presented before involves proving something is a contraction semigroup, which is automatically a C_0-semigroup, that can be proven instead. After obtaining the port-Hamiltonian system, the matrices P_0, P_1 and H are known. This implies the boundary effort and boundary flow can be obtained by

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (Hx)(b) \\ (Hx)(a) \end{bmatrix}.$$

In the best case scenario, the boundary conditions themselves are given in the form of the components of Hx, as in this case the boundary conditions can easily be rewritten in the form

$$W_{B,1} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = u,$$
$$W_{B,2} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0.$$

From here the matrix W_B can be obtained to calculate if $W_B \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_B^* \geq 0$, in order to conclude if the infinitesimal generator does generate a contraction semigroup.

In the simplest case, the dimension is one and real. Then $W_B = \begin{bmatrix} c & d \end{bmatrix}$, for some $c, d \in \mathbb{R}$. This has a rank of 1 if either c or d is non-zero. Furthermore,

$$W_B \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} W_B^T = 2cd \geq 0$$

if and only if $cd \geq 0$. If the space is complex instead, then

$$W_B \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} W_B^* = c\overline{d} + \overline{c}d = 2Re(c\overline{d}) \geq 0$$

if and only if $Re(c\overline{d}) \geq 0$. Hence in one dimension, the operator generates a contraction semigroup if either c or d is non-zero, and $Re(c\overline{d}) \geq 0$.

In two dimensions, things are much more complicated. The matrix itself has the form

$$W_B = \begin{bmatrix} c & d & e & f \\ g & h & i & j \end{bmatrix},$$

for $c, d, e, f, g, h, i, j \in \mathbb{R}$. This matrix has rank 2 if the rows are not a multiple of each other. Furthermore, by calculating

$$\begin{aligned}
W_B \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} W_B^T &= \begin{bmatrix} c & d & e & f \\ g & h & i & j \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & g \\ d & h \\ e & i \\ f & j \end{bmatrix} \\
&= \begin{bmatrix} c & d & e & f \\ g & h & i & j \end{bmatrix} \begin{bmatrix} e & i \\ f & j \\ c & g \\ d & h \end{bmatrix} \\
&= \begin{bmatrix} 2ce + 2df & ci + dj + eg + fh \\ ci + dj + eg + fh & 2gi + 2hj \end{bmatrix}.
\end{aligned}$$

The next step is computing when this matrix is positive semi-definite. This is a self-adjoint matrix, hence the only thing that needs to be checked is if the matrix has only positive eigenvalues. In the two dimensional case, the eigenvalues can be computed using the trace $Tr = 2(ce + df + gi + hj)$ and the determinant $Det = 4(ce + df)(gi + hj) - (ci + dj + eg + fh)^2$. Hence it is required for this matrix to be positive definite that

$$\frac{1}{2}\left(Tr \pm \sqrt{Tr^2 - 4Det}\right) > 0.$$

In this case the matrix is positive semi-definite, and the operator generates a contraction semigroup.

5.6 Conclusion

With all of the above systems and attempts in writing those as linear first order port-Hamiltonian systems, are there any conclusions that can be drawn from those calculations?

First and foremost, if one particular choice for x does not work, it does not mean the system can not be written as a port-Hamiltonian system. Sometimes a different choice does work. But whatever the choice is, the system has to have the correct properties. This implies that there can not be a lot of different functions in the system, as this results into problems with both creating x, as well as in H. The structure of H implies that each function before each individual component of x has to be the same.

In order to work with higher derivatives, a sufficient smoothness on each of the components of x has to be assumed. This is in order to change the order of taking the partial derivative.

The definitions of P_0 and P_1 state that the matrices are either skew-adjoint or self-adjoint. This has to be reflected in the choices of x. If one component of x depends on another component of x, the reverse has to hold too. It is however not enough to just depend on the other component, for it can not be that the first dependence is in P_0 and the reverse dependence is in P_1.

As P_1 can not depend on the spatial variable, if there is a function that depends on ζ before the partial derivative, it can be included in the component of x that entails the time derivative, to avoid any problems. If there exist multiple functions, they have to be a constant multiple of each other.

The more complex the system, the harder it will be to transform the system into a port-Hamiltonian setting, since the extra functions used, or components of x required, will result in less freedom to rewrite the system in the desired form.

The order of the time derivative can not differ too much from the order of the spatial derivative. As it might not be possible to write those equations as a linear first order port-Hamiltonian system. In the examples, a difference of one between the order of the time derivative and the spatial derivative already gave problems.

Some equations can not be written as a linear first order port-Hamiltonian system, but appear to be in the form of higher order port-Hamiltonian systems.

Sometimes the choices for the components of x come naturally by their physical interpretation. In the example of the vibrating string, the choices were based on the momentum and strain of the system, see Section 4.10. Such choices are natural ones to make for a physical system for which conservation of for example energy and impulse are important. This leads to a Hamiltonian system. For the equations described above, such a natural choice is not immediate or even possible. Generic equations do not necessarily model a physical system and a Hamiltonian is therefore not always implicitly defined.

6 Application

Walking with a mug filled with liquid requires some care to not spill anything. There are certain factors that influence how fast there can be spillage. Two of the more obvious factors are the walking speed and how full the mug is. There also exist possibilities of reducing the oscillations of the mug. To illustrate this, consider the following simplified model, following [9], a hand moves a two dimensional mug filled with a liquid horizontally via a spring. The oscillations of the hand represent walking, while the horizontal movement can be thought of the mug moving on a (friction-less) table. In order to avoid sloshing, the height of liquid has to be monitored at all times, and the mug is moved via the hand on the boundary. Hence this system can be seen as a boundary control system with the proper input and output. The idea is to first describe the model using physics, then solve any unknown equations analytically, and then represent the sloshing mechanism. After that, an attempt will be made to write the model as a linear first order port-Hamiltonian boundary control system.

6.1 Describing the model

In order to represent this idea as a mathematical model, the following parameters will be introduced. The mug, of mass M, has a width L and is filled with a liquid, of mass m, up to a height h. The oscillations on the surface of the liquid will be described later by the function $\eta(x, t)$. Here x represents the horizontal direction and t represents the time. Furthermore, y represents the vertical direction. The hand P oscillates horizontally with frequency ω and amplitude A, around the fixed mean position C. The spring PQ connects the hand to the mug, with spring constant λ. The origin O is placed such that the length of the spring PQ at rest is CO. Let $X(t)$ represent the position of the left wall of the mug, and hence $X(t) + L$ the position of the right wall of the mug. To illustrate this, see Figure 6.1.

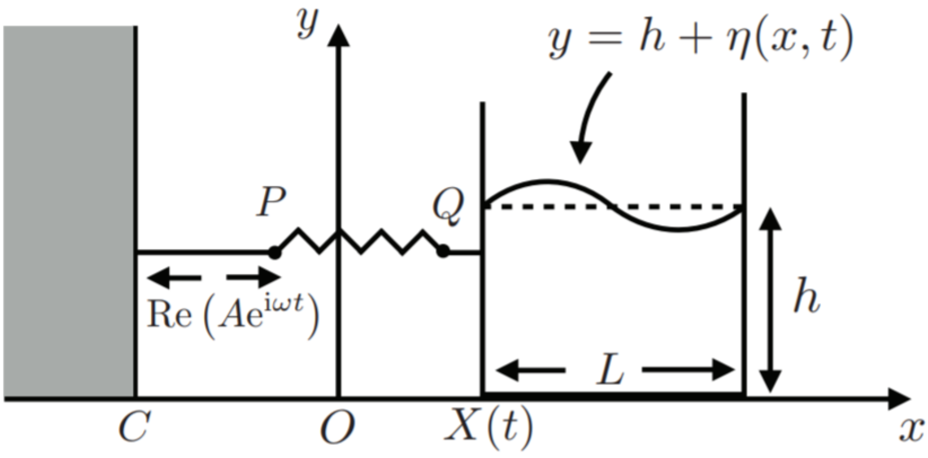

Figure 6.1: Model and parameters described in a figure.

Assume that the mug is perfectly rectangular, the liquid is incompressible and there is no motion perpendicular to the direction of the action of the spring. In particular, the mug stays on the table at all times, in the same manner and never tips over.

Finally, let $\phi(x, y, t)$ be the potential function. As this function describes a two dimensional function over time, it might be better to view the function as $\phi(x(t), y(t))$. If the liquid is initially at rest, the flow will always be irrotational, hence the velocity $u \in \mathbb{R}^2$, can be given in terms of the potential function by

$$u := \nabla\phi,$$

hence

$$u(x, y, t) = \begin{bmatrix} u_1(x, y, t) \\ u_2(x, y, t) \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix},$$

for all time t.

As the liquid is incompressible, and if there is conservation of mass due to no spillage, the density ρ is constant, and the continuity equation gives

$$\frac{\partial\rho}{\partial t} + \nabla(\rho u) = 0,$$
$$\frac{\partial\rho}{\partial t} + u\nabla\rho + \rho\nabla u = 0,$$

hence

$$\rho\nabla u = 0,$$
$$\rho\nabla^2\phi = 0,$$

thus

$$\nabla^2\phi = 0. \tag{6.1}$$

Three of the four boundary conditions, the boundaries where the liquid makes contact with the mug, are given by the fact that the normal velocity of the liquid and the velocity of the mug are the same. Hence

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \dot{X}(t) \text{ on } x = X(t),\\ \frac{\partial \phi}{\partial x} &= \dot{X}(t) \text{ on } x = X(t) + L,\end{aligned}$$

and

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = 0.$$

The final boundary condition is on the surface of the liquid, denoted by $y = h+\eta(x,t)$, for a yet to be determined function η. To formulate this boundary condition, two conditions are needed. As a particle on the surface stays on the surface, the first condition can be given by

$$\frac{d}{dt}(y - \eta(x,t)) = 0,$$

which can be rewritten as

$$\begin{aligned}\left(\frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial t}\frac{dt}{dt}\right)(y - \eta(x,t)) &= 0\\ \left(\frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial t}\right)(y - \eta(x,t)) &= 0\\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial y}u_2\right)(y - \eta(x,t)) &= 0\\ \left(\frac{\partial}{\partial t} + u\cdot\nabla\right)(y - \eta(x,t)) &= 0.\end{aligned}$$

The second condition comes from Bernoulli's equation for unsteady irrotational flow, with the atmospheric pressure p_a, and liquid pressure p, which states

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + g(y-h) = \frac{p_a}{\rho}.$$

On the surface of the liquid, $y = h+\eta(x,t)$, the atmospheric pressure equals the pressure of the liquid, hence

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta(x,t) = 0,$$

where g is the standard acceleration due to gravity.

In order to combine these two conditions, they are both linearized by ignoring any quadratic or higher order terms. Furthermore, a somewhat logical assumption might be that the oscillations on the surface of the liquid are much smaller than the height of the liquid in the mug, $\eta \ll h$.

Under this assumption, the first condition states

$$\left(\frac{\partial}{\partial t} + \nabla\phi \cdot \nabla\right)(y - \eta(x,t)) = 0$$
$$-\frac{\partial}{\partial t}\eta(x,t) + \nabla\phi \cdot \nabla y - \nabla\phi \cdot \nabla\eta(x,t) = 0$$
$$-\frac{\partial}{\partial t}\eta(x,t) + \frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\eta(x,t) - \frac{\partial\phi}{\partial t}\frac{\partial}{\partial t}\eta(x,t) = 0,$$

which, by ignoring higher order terms, results in

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial t}\eta(x,t)$$

for any particle on the surface.

Under the same assumption, the second condition states

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta(x,t) = 0,$$

on $y = h$, which by linearization results in

$$\frac{\partial\phi}{\partial t} + g\eta(x,t) = 0,$$

or

$$\eta(x,t) = -\frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{y=h}. \tag{6.2}$$

Finally combining these two conditions gives the boundary condition on the surface,

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0 \text{ on } y = h. \tag{6.3}$$

As a final, realistic, assumption, suppose the amplitude of the oscillations caused by the hand is small, and do not move the mug too much, then $X \ll L$. Along with the placement of the origin, this reformulates the first three boundary conditions as

$$\frac{\partial\phi}{\partial x} = \dot{X}(t) \text{ on } x = 0, L \tag{6.4}$$

and

$$\frac{\partial\phi}{\partial y} = 0 \text{ on } y = 0. \tag{6.5}$$

This assumption has drastic implications, as the boundary is now static, it does not depend on time nor placement of the mug.

From the linearization of Bernoulli's equation for unsteady irrotational flow, the pressure in the liquid can be given by

$$p(x,y,t) = p_a - \rho\frac{\partial\phi}{\partial t} - g\rho(y - h). \tag{6.6}$$

The last equation to describe this model, describes the movement of the left wall of the mug, and hence the entire mug. The forces exerting on that wall are the spring on the one side, and the liquid on the other side. The force of the spring can be obtained from Hooke's law in combination with the oscillations of the hand. As the mug consists of a single piece, the force of the liquid comes from both the left and right wall. Furthermore, the pressure of the liquid exerted on the walls is not equal at every height. The acceleration of the mug over time can be described using the position of the left wall of the mug. The total mass of the moving mug is the mass of the mug plus the mass of the liquid inside. Combining all of these forces, by Newton's first law,

$$(M+m)\ddot{X} = -\lambda\left(X - Re\left(Ae^{i\omega t}\right)\right) + \int_0^h p(L,y,t) - p(0,y,t)\,dy. \tag{6.7}$$

Up to this point, the paper [9] has been closely followed. However, the paper disregarded the mass of the liquid, while keeping the pressure, in the previous equation. This means that any of the following steps which follow up on this equation, use an extra m in comparison with the paper.

6.2 Solving the model

Having made a model, there are still some unknown parts in the model. A first step in solving this model would be to solve equation (6.1) subject to the boundary conditions (6.4)-(6.5). By separation of variables and omitting the time notation in the equations, if $\phi = f(x)g(y)$, then by (6.1),

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)}.$$

As each side depends on a different variable, for some constant $c \in \mathbb{R}$,

$$\begin{cases} f''(x) = cf(x), \\ g''(y) = cg(y). \end{cases}$$

There are three possibilities for c, positive, negative and zero. Any solution which has either f or g identically zero is not interesting to model, hence these type of solutions are ignored.

If $c = \theta^2$ is positive, then

$$\begin{cases} f(x) = ae^{\theta x} + be^{-\theta x}, \\ g(y) = \alpha\cos(\theta y) + \beta\sin(\theta y), \end{cases}$$

for constants $a, b, \alpha, \beta \in \mathbb{R}$. Filling these equations in the boundary conditions (6.4) gives for all y

$$f'(0)g(y) = \dot{X},$$
$$f'(L)g(y) = \dot{X},$$

hence

$$\begin{aligned} f'(0) &= f'(L) \\ \theta(a-b) &= \theta(ae^{\theta L} - be^{-\theta L}) \\ a &= -be^{-\theta L}. \end{aligned}$$

The other boundary condition (6.5) gives for all x

$$\begin{aligned} f(x)g'(0) &= 0 \\ f(x)\beta &= 0, \end{aligned}$$

hence $\beta = 0$. Thus a solution is given by

$$\begin{cases} f(x) = b(e^{-\theta x} - e^{\theta(x-L)}), \\ g(y) = \alpha \cos(\theta y). \end{cases}$$

However, there appears to be no periodicity in the horizontal direction of this solution, while this is expected due to the movement of the hand.

If $c = -\theta^2$ is negative, then

$$\begin{cases} f(x) = a\cos(\theta x) + b\sin(\theta x), \\ g(y) = \alpha e^{\theta y} + \beta e^{-\theta y}, \end{cases}$$

for constants $a, b, \alpha, \beta \in \mathbb{R}$. Filling these equations in the boundary conditions (6.4) gives for all y

$$\begin{aligned} f'(0)g(y) &= \dot{X}, \\ f'(L)g(y) &= \dot{X}, \end{aligned}$$

hence

$$\begin{aligned} f'(0) &= f'(L) \\ \theta b &= \theta(b\cos(\theta L) - a\sin(\theta L)) \\ b &= -a\cot\left(\frac{\theta L}{2}\right). \end{aligned}$$

The other boundary condition (6.5) gives for all x

$$\begin{aligned} f(x)g'(0) &= 0 \\ f(x)\theta(\alpha - \beta) &= 0, \end{aligned}$$

hence $\alpha = \beta$. Thus a solution is given by

$$\begin{cases} f(x) = -a, \\ g(y) = 2\cosh(\theta L). \end{cases}$$

However, there appears to be no periodicity at all in this solution, while this is expected due to the movement of the hand.

Lastly, if $c = 0$, then

$$\begin{cases} f(x) = ax + b, \\ g(y) = \alpha y + \beta, \end{cases}$$

for constants $a, b, \alpha, \beta \in \mathbb{R}$. Filling these equations in the boundary conditions (6.5) gives for all x

$$\begin{aligned} f(x)g'(0) &= 0 \\ f(x)\alpha &= 0, \end{aligned}$$

hence $\alpha = 0$. Using this fact and the other boundary condition (6.4), gives for all y

$$\begin{aligned} f'(0)g(y) &= \dot{X}, \\ f'(L)g(y) &= \dot{X}, \end{aligned}$$

hence

$$a\beta = \dot{X}.$$

Thus a solution is given by

$$\begin{cases} f(x) = ax + b, \\ g(y) = \beta. \end{cases}$$

This implies

$$\phi = a\beta x + b\beta = \dot{X}x + b\beta.$$

At first glance this might not seem like a periodic solution, but as the hand oscillates, and hence the mug moves, $\dot{X}$ will change signs over time.

It is possible to extend the last solution by adding homogeneous solutions to this solution, in other words, with boundary conditions

$$\frac{\partial \phi}{\partial x} = 0 \text{ on } x = 0, L$$

and

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = 0.$$

Using the same methods as before, $c > 0$ gives no useful solution, while for $c = -\theta^2 < 0$,

$$\begin{cases} f(x) = a\cos(\theta x) + b\sin(\theta x), \\ g(y) = \alpha e^{\theta y} + \beta e^{-\theta y}, \end{cases}$$

for constants $a, b, \alpha, \beta \in \mathbb{R}$. Filling these equations in the first boundary condition of (6.4) gives for all y

$$0 = f'(0)g(y) = \theta b g(y),$$

hence $b = 0$. Using this fact, the second boundary condition of (6.4) results in

$$0 = f'(L)g(y) = -a\theta\sin(\theta L)g(y),$$

hence either $a = 0$ or $\sin(\theta L) = 0$. As b is already zero, $a = 0$ would give $f \equiv 0$, which is not an interesting solution. On the other hand, $\sin(\theta L) = 0$ implies $\theta = \frac{n\pi}{L}$ for all $n \in \mathbb{N}$. The other boundary condition (6.5), gives for all x

$$0 = f(x)g'(0) = \theta f(x)(\alpha - \beta),$$

and hence $\alpha = \beta$. Thus a solution is given by

$$\begin{cases} f(x) = a\cos(\frac{n\pi x}{L}), \\ g(y) = 2\alpha\cosh(\frac{n\pi y}{L}). \end{cases}$$

This implies for all $n \in \mathbb{N}$

$$\phi = 2a\alpha\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right).$$

Lastly, for $c = 0$, the homogeneous solution is given by

$$\phi = b\beta.$$

This last solution can be used to cancel out the constant term of the general solution.

Combining the general solution with the homogeneous solutions gives a solution to equation (6.1) subject to the boundary conditions (6.4)-(6.5), by

$$\phi = \dot{X}x + \sum_{n=0}^{\infty} a_n(t)\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right). \tag{6.8}$$

A quick check shows

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

for all possible choices of a_n and all time t. This is according to the original equation (6.1).

As the hand moves the mug with liquid with a frequency ω, it is expected that the solution is also periodic with the same frequency. In order to include this, set $X = Re(X_0 e^{i\omega t})$ and $a_n = Re(\alpha_n e^{i\omega t})$. These α_n can be solved using the unused final boundary condition (6.3). Given

$$\begin{aligned}
\frac{\partial\phi}{\partial t} &= Re(-X_0\omega^2 e^{i\omega t})x + \sum_{n=0}^{\infty} Re(\alpha_n i\omega e^{i\omega t})\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right), \\
\frac{\partial^2\phi}{\partial t^2} &= Re(-X_0 i\omega^3 e^{i\omega t})x + \sum_{n=0}^{\infty} Re(-\alpha_n\omega^2 e^{i\omega t})\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right), \\
\frac{\partial\phi}{\partial y} &= \sum_{n=0}^{\infty} Re(\alpha_n e^{i\omega t})\cos\left(\frac{n\pi x}{L}\right)\frac{n\pi}{L}\sinh\left(\frac{n\pi y}{L}\right),
\end{aligned}$$

then by (6.3),

$$
\begin{aligned}
0 =& Re(-X_0 i\omega^3 e^{i\omega t})x + \sum_{n=0}^{\infty} Re(-\alpha_n \omega^2 e^{i\omega t}) \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi h}{L}\right) \\
&+ g \sum_{n=0}^{\infty} Re(\alpha_n e^{i\omega t}) \cos\left(\frac{n\pi x}{L}\right) \frac{n\pi}{L} \sinh\left(\frac{n\pi h}{L}\right) \\
=& Re(-X_0 i\omega^3 e^{i\omega t})x + \sum_{n=0}^{\infty} Re(\alpha_n e^{i\omega t}) \cos\left(\frac{n\pi x}{L}\right) \left[\frac{gn\pi}{L} \sinh\left(\frac{n\pi h}{L}\right) - \omega^2 \cosh\left(\frac{n\pi h}{L}\right)\right].
\end{aligned}
$$

Since X_0 is the initial position of the left wall, this is a real number. This implies that when $\alpha_n \in \mathbb{R}$, there does not exist a solution that holds for all $x \in \mathbb{R}$, and all non-negative time. The same argument holds when $\alpha_n \in \mathbb{C}$ with a non-zero real part. Hence any solution has $\alpha_n \in i\mathbb{R}$.

It turns out that solving this equation is best done by distinguishing between the zero and even and odd terms. This results in

$$
\begin{aligned}
0 =& Re(-X_0 i\omega^3 e^{i\omega t})x - Re(\alpha_0 \omega^2 e^{i\omega t}) \\
&+ \sum_{n=1}^{\infty} Re(\alpha_{2n} e^{i\omega t}) \cos\left(\frac{2n\pi x}{L}\right) \left[\frac{2gn\pi}{L} \sinh\left(\frac{2n\pi h}{L}\right) - \omega^2 \cosh\left(\frac{2n\pi h}{L}\right)\right] \\
&+ \sum_{n=0}^{\infty} Re(\alpha_{2n+1} e^{i\omega t}) \cos\left(\frac{(2n+1)\pi x}{L}\right) \left[\frac{g(2n+1)\pi}{L} \sinh\left(\frac{(2n+1)\pi h}{L}\right)\right. \\
&\left. - \omega^2 \cosh\left(\frac{(2n+1)\pi h}{L}\right)\right].
\end{aligned}
$$

With the introduction of

$$\theta_n^2 = \frac{2n\pi g}{L} \tanh\left(\frac{2n\pi h}{L}\right)$$

and

$$\omega_n^2 = \frac{(2n+1)\pi g}{L} \tanh\left(\frac{(2n+1)\pi h}{L}\right),$$

the equation transforms into

$$
\begin{aligned}
0 =& Re(-X_0 i\omega^3 e^{i\omega t})x - Re(\alpha_0 \omega^2 e^{i\omega t}) \\
&+ \sum_{n=1}^{\infty} Re(\alpha_{2n} e^{i\omega t}) \cos\left(\frac{2n\pi x}{L}\right) \cosh\left(\frac{2n\pi h}{L}\right) \left[\theta_n^2 - w^2\right] \\
&+ \sum_{n=0}^{\infty} Re(\alpha_{2n+1} e^{i\omega t}) \cos\left(\frac{(2n+1)\pi x}{L}\right) \cosh\left(\frac{(2n+1)\pi h}{L}\right) \left[\omega_n^2 - w^2\right].
\end{aligned}
$$

At this point, there is a freedom in the choice of α_n. In an effort to not juggle with the summation of both the even and odd terms, one choice would be to put $\alpha_{2n} = 0$ for

$n > 0$. This choice leaves

$$\begin{aligned} 0 =& Re(-X_0 i\omega^3 e^{i\omega t})x - Re(\alpha_0 \omega^2 e^{i\omega t}) \\ &+ \sum_{n=0}^{\infty} Re(\alpha_{2n+1} e^{i\omega t}) \cos\left(\frac{(2n+1)\pi x}{L}\right) \cosh\left(\frac{(2n+1)\pi h}{L}\right) \left[\omega_n^2 - w^2\right]. \end{aligned}$$

As this equality has to hold for all $x \in \mathbb{R}$, filling in $x = \frac{1}{2}L$ results in

$$0 = Re(-X_0 i\omega^3 e^{i\omega t})\frac{1}{2}L - Re(\alpha_0 \omega^2 e^{i\omega t}),$$

which implies $\alpha_0 = -\frac{1}{2} i\omega L X_0$. On the other hand, $x = 0$ results in

$$0 = Re\left(\frac{1}{2} i\omega^3 L X_0 e^{i\omega t}\right) + \sum_{n=0}^{\infty} Re(\alpha_{2n+1} e^{i\omega t}) \cosh\left(\frac{(2n+1)\pi h}{L}\right) \left[\omega_n^2 - w^2\right].$$

A solution to this can be given by

$$\alpha_{2n+1} = \frac{4 i\omega^3 L X_0}{(2n+1)^2 \pi^2 (\omega^2 - \omega_n^2) \cosh\left(\frac{(2n+1)\pi h}{L}\right)},$$

as filling this in results in

$$\begin{aligned} Re\left(\frac{1}{2} i\omega^3 L X_0 e^{i\omega t}\right) + \sum_{n=0}^{\infty} Re\left(\frac{4 i\omega^3 L X_0 (\omega_n^2 - \omega^2) \cosh\left(\frac{(2n+1)\pi h}{L}\right)}{(2n+1)^2 \pi^2 (\omega^2 - \omega_n^2) \cosh\left(\frac{(2n+1)\pi h}{L}\right)} e^{i\omega t}\right) &= \\ Re\left(\frac{1}{2} i\omega^3 L X_0 e^{i\omega t}\right) - \sum_{n=0}^{\infty} Re\left(\frac{4 i\omega^3 L X_0}{(2n+1)^2 \pi^2} e^{i\omega t}\right) &= \\ Re\left(\frac{1}{2} i\omega^3 L X_0 e^{i\omega t}\right) - Re\left(4 i\omega^3 L X_0 e^{i\omega t} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \pi^2}\right) &= \\ Re\left(\frac{1}{2} i\omega^3 L X_0 e^{i\omega t}\right) - Re\left(\frac{1}{2} i\omega^3 L X_0 e^{i\omega t}\right) &= 0. \end{aligned}$$

Another option to not juggle both the even and odd terms would be to put $\alpha_{2n+1} = 0$, which gives

$$\begin{aligned} 0 =& Re(-X_0 i\omega^3 e^{i\omega t})x - Re(\alpha_0 \omega^2 e^{i\omega t}) \\ &+ \sum_{n=1}^{\infty} Re(\alpha_{2n} e^{i\omega t}) \cos\left(\frac{2n\pi x}{L}\right) \cosh\left(\frac{2n\pi h}{L}\right) \left[\theta_n^2 - w^2\right]. \end{aligned}$$

Filling in $x = \frac{1}{4}L$ results in

$$0 = Re(-X_0 i\omega^3 e^{i\omega t})\frac{1}{4}L - Re(\alpha_0 \omega^2 e^{i\omega t}),$$

which implies $\alpha_0 = -\frac{1}{4}i\omega L X_0$. However, filling in $x = 0$ results in

$$0 = Re(-\frac{1}{4}i\omega^3 L X_0 e^{i\omega t}) + \sum_{n=1}^{\infty} Re(\alpha_{2n} e^{i\omega t}) \cosh\left(\frac{2n\pi h}{L}\right) \left[\theta_n^2 - w^2\right],$$

while $x = L$ results in

$$0 = Re(-\frac{3}{4}i\omega^3 L X_0 e^{i\omega t}) + \sum_{n=1}^{\infty} Re(\alpha_{2n} e^{i\omega t}) \cosh\left(\frac{2n\pi h}{L}\right) \left[\theta_n^2 - w^2\right].$$

These two conditions can not hold at the same time. Thus a solution where the odd terms are all zero is not possible.

Lastly, when α_{2n+1} are zero, and $\alpha_{2n}, n > 0$ and are zero, the remaining equation gives

$$0 = Re(-X_0 i\omega^3 e^{i\omega t})x - Re(\alpha_0 \omega^2 e^{i\omega t}),$$

which can not hold for all x.

To summarize, a solution of equation (6.8) under boundary condition (6.3) is given by

$$\begin{aligned}
\alpha_0 &= -\frac{1}{2}i\omega L X_0,\\
\alpha_{2n} &= 0 \text{ for } n > 0,\\
\alpha_{2n+1} &= \frac{4i\omega^3 L X_0}{(2n+1)^2\pi^2(\omega^2 - \omega_n^2)\cosh\left(\frac{(2n+1)\pi h}{L}\right)},\\
\omega_n^2 &= \frac{(2n+1)\pi g}{L}\tanh\left(\frac{(2n+1)\pi h}{L}\right).
\end{aligned}$$

Note that ω_n is strictly monotonically increasing in n.

The next step in solving the model is rewriting equation (6.7). From equation (6.6),

$$\begin{aligned}
\int_0^h p(L,y,t) - p(0,y,t)\,dy = &\int_0^h \rho\frac{\partial\phi}{\partial t}(0,y,t) - \rho\frac{\partial\phi}{\partial t}(L,y,t)\,dy\\
=&\rho\int_0^h \sum_{n=0}^{\infty} Re(\alpha_n i\omega e^{i\omega t})\cosh\left(\frac{n\pi y}{L}\right) - Re(-X_0\omega^2 e^{i\omega t})L\\
&- \sum_{n=0}^{\infty} Re(\alpha_n i\omega e^{i\omega t})(-1)^n \cosh\left(\frac{n\pi y}{L}\right)\,dy\\
=&\rho\int_0^h Re(X_0\omega^2 e^{i\omega t})L dy\\
&+ 2\rho\int_0^h \sum_{n=0}^{\infty} Re(\alpha_{2n+1} i\omega e^{i\omega t})\cosh\left(\frac{(2n+1)\pi y}{L}\right)\,dy\\
=&\rho L h Re(X_0\omega^2 e^{i\omega t})\\
&+ 2\rho\sum_{n=0}^{\infty} Re(\alpha_{2n+1} i\omega e^{i\omega t})\frac{L}{(2n+1)\pi}\sinh\left(\frac{(2n+1)\pi h}{L}\right).
\end{aligned}$$

Using this equality, along with the previously established relations for X and a_n, equation (6.7) transforms into

$$\begin{aligned}
-(M+m)Re(X_0\omega^2 e^{i\omega t}) =& -\lambda\left(Re(X_0 e^{i\omega t}) - Re(Ae^{i\omega t})\right) + \rho Lh Re(X_0\omega^2 e^{i\omega t}) \\
&+ \frac{2\rho L}{(2n+1)\pi}\sum_{n=0}^{\infty} Re(\alpha_{2n+1} i\omega e^{i\omega t})\sinh\left(\frac{(2n+1)\pi h}{L}\right) \\
=& -\lambda\left(Re(X_0 e^{i\omega t}) - Re(Ae^{i\omega t})\right) + \rho Lh Re(X_0\omega^2 e^{i\omega t}) \\
&- \frac{8\rho L^2 X_0\omega^4}{(2n+1)^3\pi^3}\sum_{n=0}^{\infty} Re\left(\frac{\tanh\left(\frac{(2n+1)\pi h}{L}\right)}{\omega^2-\omega_n^2} e^{i\omega t}\right) \\
=& -\lambda\left(Re(X_0 e^{i\omega t}) - Re(Ae^{i\omega t})\right) + \rho Lh Re(X_0\omega^2 e^{i\omega t}) \\
&- \frac{8\rho L^3 X_0}{(2n+1)^4\pi^4 g}\sum_{n=0}^{\infty} Re\left(\frac{\omega_n^2\omega^4}{\omega^2-\omega_n^2} e^{i\omega t}\right).
\end{aligned}$$

Using $\rho = \frac{m}{Lh}$, the above equation can be rewritten as

$$\Delta(\omega)X_0 := \left(\omega^2(M+2m) - \lambda + \frac{8mL^2}{gh}\sum_{n=0}^{\infty}\frac{\omega_n^2\omega^4}{(2n+1)^4\pi^4(\omega_n^2-\omega^2)}\right)X_0 = -\lambda A, \quad (6.9)$$

for the appropriate definition of $\Delta(\omega)$.

6.3 Sloshing

Equation (6.9) links the amplitude X_0 with the given spring constant λ, the frequency ω and amplitude A. This link can be used to calculate for which frequencies there occurs spillage in this model. For this, the equation that describes the sloshing has to be concretely computed. By (6.2),

$$\begin{aligned}
\eta(x,t) =& -\frac{1}{g}\frac{\partial\phi}{\partial t}(x,h,t) \\
=& \frac{1}{g}Re(X_0\omega^2 e^{i\omega t})x - \frac{1}{g}\sum_{n=0}^{\infty} Re(\alpha_n i\omega e^{i\omega t})\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi h}{L}\right) \\
=& \frac{1}{g}Re(X_0\omega^2 e^{i\omega t})x - \frac{1}{g}Re\left(\frac{1}{2}LX_0\omega^2 e^{i\omega t}\right) \\
&+ \frac{4\omega^4 LX_0}{g\pi^2}\sum_{n=0}^{\infty} Re\left(\frac{1}{(2n+1)^2(\omega^2-\omega_n^2)} e^{i\omega t}\right)\cos\left(\frac{(2n+1)\pi x}{L}\right).
\end{aligned}$$

Since this equation is periodic, and the interest lies in the height of this function, it is easiest to consider the function at the left wall, $x = 0$. This results in

$$\eta(0,t) = -\frac{1}{g}Re\left(\frac{1}{2}LX_0\omega^2 e^{i\omega t}\right) + \frac{4\omega^4 LX_0}{g\pi^2}\sum_{n=0}^{\infty} Re\left(\frac{1}{(2n+1)^2(\omega^2-\omega_n^2)} e^{i\omega t}\right),$$

or more concretely

$$\eta(0,t) = \left(-\frac{LX_0\omega^2}{2g} + \frac{4\omega^4 LX_0}{g\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2(\omega^2 - \omega_n^2)} \right) \cos(\omega t).$$

Note that the calculations here differ once more from the paper. The paper does not include the negative term before the summation. This term comes from everything associated with α_0. Hence any of the following computations that use η, will also differ from the paper by the term associated to α_0.

In order to describe the height of the sloshing, define $\|\eta\|$ as the modulus of the amplitude of $\eta(0,t)$. Hence by above equation,

$$\begin{aligned} \|\eta\| &= \left| -\frac{LX_0\omega^2}{2g} + \frac{4\omega^4 LX_0}{g\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2(\omega^2 - \omega_n^2)} \right| \\ &= \frac{L|X_0|\omega^2}{g} \left| -\frac{1}{2} + \frac{4\omega^2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2(\omega^2 - \omega_n^2)} \right|, \end{aligned}$$

and by (6.9),

$$\|\eta\| = \frac{\lambda L A\omega^2}{g|\Delta(\omega)|} \left| -\frac{1}{2} + \frac{4\omega^2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2(\omega^2 - \omega_n^2)} \right|.$$

Some walking frequencies will impact the sloshing of the liquid more than others. To check which frequencies ω impact the sloshing the most, consider $\|\eta\|$ as a function of ω. If everything is fixed but ω, at first glance the only possibilities for $\|\eta\| \to \infty$ are $\Delta(\omega) \to 0$ and $\omega \to \omega_n$ for all $n \in \mathbb{N}$. However, when $\omega \to \omega_n$, $\Delta(\omega)$ behaves as $O\left(\frac{1}{w_n^2 - w^2}\right)$, hence $\|\eta\|$ is finite. Thus in general the only thing of interest for spillage is for which ω,

$$\Delta(\omega) = \omega^2(M+2m) - \lambda + \frac{8mL^2}{gh} \sum_{n=0}^{\infty} \frac{\omega_n^2\omega^4}{(2n+1)^4\pi^4(\omega_n^2 - \omega^2)} \to 0.$$

With the introduction of the non-dimensional parameter $\Lambda = \frac{\lambda}{(M+2m)\omega_0^2}$, two cases can be distinguished. The first is a hard spring, for which $\Lambda \to \infty$. In this case, $\lambda \to \infty$, and the leading order of $\Delta(\omega)$ is λ. Hence $\Delta(\omega) \to -\lambda$, and by (6.9), $X_0 \to A$. This does mean that $\Delta(\omega)$ does not depend on ω, and in this case $\omega \to \omega_n$ results in $\|\eta\| \to \infty$.

The second case arises when the spring is soft, for which $\Lambda \ll 1$. In order to find for which ω $\Delta(\omega) \to 0$, the Taylor expansion up to the second order around the smallest root, ω_0 as it is monotone increasing in n, is used. As

$$\begin{aligned} \frac{d}{dt}\left(\omega^2(M+2m) - \lambda\right) &= 2\omega(M+2m), \\ \frac{d^2}{dt^2}\left(\omega^2(M+2m) - \lambda\right) &= 2(M+2m), \end{aligned}$$

and

$$\frac{d}{dt}\frac{8mL^2}{gh}\sum_{n=0}^{\infty}\frac{\omega_n^2\omega^4}{(2n+1)^4\pi^4(\omega_n^2-\omega^2)} = \frac{8mL^2}{gh}\sum_{n=0}^{\infty}\frac{2\omega_n^2\omega^3(2\omega_n^2-\omega^2)}{(2n+1)^4\pi^4(\omega_n^2-\omega^2)^2},$$
$$\frac{d^2}{dt^2}\frac{8mL^2}{gh}\sum_{n=0}^{\infty}\frac{\omega_n^2\omega^4}{(2n+1)^4\pi^4(\omega_n^2-\omega^2)} = \frac{8mL^2}{gh}\sum_{n=0}^{\infty}\frac{2\omega_n^2\omega^2(\omega^4-3\omega_n^2\omega^2+6\omega_n^4)}{(2n+1)^4\pi^4(\omega_n^2-\omega^2)^3}.$$

Hence

$$0=\Delta(\omega)\approx\Delta(0)+\frac{1}{2}\Delta'(0)\omega+\frac{1}{6}\Delta''(0)\omega^2=-\lambda+\frac{1}{6}2(M+2m)\omega^2.$$

By definition of the non-dimensional parameter, this can be transformed into

$$\begin{aligned}\lambda &= \frac{1}{3}(M+2m)\omega^2\\ \omega^2 &= 3\frac{\lambda}{M+2m}\\ &= 3\Lambda\omega_0^2.\end{aligned}$$

Since $\Lambda \ll 1$, the walking frequency, or the frequency of the hand movement, is much less than the lowest frequency which provides $\|\eta\|\to\infty$.

It is also interesting to study stabilization of the sloshing. As previously mentioned, $\omega\to\omega_0$ implies $\|\eta\|\to\infty$ for $\Lambda\to\infty$. However, the slightly altered equation

$$\|\eta\|-\frac{4LA\omega^4}{g\pi^2|\omega^2-\omega_0^2|}=\frac{\lambda LA\omega^2}{g|\Delta(\omega)|}\left|-\frac{1}{2}+\frac{4\omega^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\omega^2-\omega_n^2)}\right|-\frac{4LA\omega^4}{g\pi^2|\omega^2-\omega_0^2|},$$

behaves as

$$\frac{LA\omega^2}{g}\left|-\frac{1}{2}+\frac{4\omega^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\omega^2-\omega_n^2)}\right|-\frac{4LA\omega^4}{g\pi^2|\omega^2-\omega_0^2|},$$

as $\Delta(\omega)\to-\lambda$ for $\Lambda\to\infty$. Hence by the Cauchy-Schwarz inequality

$$\begin{aligned}\|\eta\|-\frac{4LA\omega_0^4}{g\pi^2|\omega^2-\omega_0^2|}\leq&\frac{LA\omega^2}{2g}+\frac{4LA\omega^4}{g\pi^2}\left|\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\omega^2-\omega_n^2)}\right|-\frac{4LA\omega^4}{g\pi^2|\omega^2-\omega_0^2|}\\ \leq&\frac{LA\omega^2}{2g}+\frac{4LA\omega^4}{g\pi^2}\left|\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2(\omega^2-\omega_n^2)}\right|\\ &+\frac{4LA\omega^4}{g\pi^2|\omega^2-\omega_0^2|}-\frac{4LA\omega^4}{g\pi^2|\omega^2-\omega_0^2|}.\end{aligned}$$

As $\omega\to\omega_0$, this equation behaves as

$$\frac{LA\omega_0^2}{2g}+\frac{4LA\omega_0^4}{g\pi^2}\left|\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2(\omega_0^2-\omega_n^2)}\right|,$$

which is finite by the strict monotonicity of ω_n. Hence as $\omega \to \omega_0$, $\|\eta\| - \frac{4LA\omega^4}{g\pi^2|\omega^2-\omega_0^2|}$ is finite, and there is no spillage.

It is also possible to study a different limit. By the same argumentation, as $\omega \to \omega_m$ for all $m \in \mathbb{N}$,

$$\|\eta\| - \frac{4LA\omega_m^4}{g\pi^2|\omega^2 - \omega_m^2|}$$

is finite. However, this is not true for all starting values of ω. If one stabilizes for ω_1 and starts at rest, $\omega = 0$, and then starts moving such that $\omega \to \omega_1$, it passes ω_0 and there will be spillage. Hence this stabilization only works for starting values of $(\omega_{m-1} + \epsilon, \omega_{m+1} - \epsilon)$, for $\epsilon > 0$.

On the other hand, stabilizing the sloshing if $\Lambda \ll 1$, is much harder. The smallest root of $\Delta(\omega) = 0$ was approached via the Taylor series by $\omega \to \sqrt{\Lambda}\omega_0$. By those same calculations, $\Delta(\omega) \to (M + 2m)(\omega^2 - \Lambda\omega_0)$ for this limit. Hence

$$\|\eta\| = \frac{\lambda LA\omega^2}{g|\Delta(\omega)|}\left|-\frac{1}{2} + \frac{4\omega^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\omega^2 - \omega_n^2)}\right|,$$

behaves in this limit as

$$\begin{aligned}\|\eta\| \to& \frac{\Lambda(M+2m)\omega_0^2 LA\Lambda\omega_0^2}{g|(M+2m)(\omega^2 - \Lambda\omega_0^2)|}\left|-\frac{1}{2} + \frac{4\Lambda\omega_0^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\Lambda\omega_0^2 - \omega_n^2)}\right| \\ =& \frac{\Lambda^2 LA\omega_0^4}{g|(\omega^2 - \Lambda\omega_0^2)|}\left|-\frac{1}{2} + \frac{4\Lambda\omega_0^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\Lambda\omega_0^2 - \omega_n^2)}\right|.\end{aligned}$$

There are two things to note here, as this equation can be split into two parts, the fraction and something in absolute value. As $\Lambda \ll 1$, the second part of the equation will never blow up when $\omega \to \sqrt{\Lambda}\omega_0$. The first part however, does blow up in this limit. Hence the only concern lies with the fraction in the first part of the equation. In order to compensate for this, the only option is by removing everything, or

$$\|\eta\| - \frac{\Lambda^2 LA\omega_0^4}{g|(\omega^2 - \Lambda\omega_0^2)|}\left|-\frac{1}{2} + \frac{4\Lambda\omega_0^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\Lambda\omega_0^2 - \omega_n^2)}\right| \to 0,$$

as $\omega \to \sqrt{\Lambda}\omega_0$.

This method does seem drastic in comparison with $\Lambda \to \infty$. In order to try something less drastic, let us take a step back. Since

$$\eta(0,t) = \frac{\lambda LA\omega^2}{g\Delta(\omega)}\left(\frac{1}{2} - \frac{4\omega^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\omega^2 - \omega_n^2)}\right)\cos(\omega t),$$

in the limit $\omega \to \sqrt{\Lambda}\omega_0$, this behaves as

$$\frac{\Lambda^2 LA\omega_0^4}{g(\omega^2 - \Lambda\omega_0^2)}\left(\frac{1}{2} - \frac{4\Lambda\omega_0^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\Lambda\omega_0^2 - \omega_n^2)}\right)\cos(\sqrt{\Lambda}t),$$

which goes to infinity. By the same reasoning as before, the second part is finite and the only part that blows up is the first part. Hence the only way to stabilize this, is again by steering the entire solution towards zero, which is the same result as before.

To conclude, for large Λ, the solution is stabilized by

$$\frac{4LA\omega_m^4}{g\pi^2|\omega^2-\omega_m^2|},$$

while for small Λ, the solution is stabilized by

$$\frac{\Lambda^2 LA\omega_0^4}{g|(\omega^2-\Lambda\omega_0^2)|}\left|-\frac{1}{2}+\frac{4\Lambda\omega_0^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(\Lambda\omega_0^2-\omega_n^2)}\right|.$$

In order to compare these, they can both be rewritten in terms of Λ as

$$\begin{aligned} O(1) &\quad \text{for } \Lambda\to\infty \\ O\left(\Lambda^2(|O(1)-O(\Lambda)|)\right) &\quad \text{for } \Lambda \ll 1. \end{aligned}$$

By defining the strength of the resonance of a frequency ω_m as $\lim_{\omega\to\omega_m}\|\eta\||\omega^2-\omega_m^2|$, the strength of different values of Λ can be compared. By this definition, there is a difference of at least order $O(\Lambda^2)$ in the strength of a resonance between a harder and softer spring in the lowest resonance frequency. This means that a softer spring is better for reducing the sloshing. Intuitively this might be clear, a hard spring merely extends the motions of the hand to the mug, while a softer spring moves with the oscillations and absorbs them partly.

6.4 Fitting the model

After creating a mathematical model of a hand moving a mug filled with liquid, and solving this model analytically, it is time to try and fit this model in a linear first order port-Hamiltonian setting. Since the hand moves the mug via the left wall of the mug, it is expected to transform the model into a port-Hamiltonian boundary control system. Whatever extra conditions that are needed for a port-Hamiltonian boundary control or regular system, a linear port-Hamiltonian differential equation is always needed. Hence let's list the major equations used in the model, in roughly the other they appeared in.

$$u = \nabla\phi, \tag{6.10}$$
$$\nabla^2\phi = 0, \tag{6.11}$$
$$\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)(y - \eta(x,t)) = 0, \tag{6.12}$$
$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0, \tag{6.13}$$
$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0, \text{ on } y = h, \tag{6.14}$$
$$\frac{\partial\phi}{\partial x} = \dot{X}(t), \text{ on } x = 0, L, \tag{6.15}$$
$$\frac{\partial\phi}{\partial y} = 0, \text{ on } y = 0, \tag{6.16}$$
$$\eta(x,t) = -\frac{1}{g}\frac{\partial\phi}{\partial t}, \tag{6.17}$$
$$(M+m)\ddot{X} = -\lambda(X - Re(Ae^{i\omega t})) + \int_0^h p(L,y,t) - p(0,y,t)\,dy, \tag{6.18}$$
$$\phi = \dot{X}x + \sum_{n=0}^{\infty} a_n(t)\cos\frac{n\pi x}{L}\cosh\frac{n\pi y}{L}. \tag{6.19}$$

Not all of these equations contain a time derivative. Equation (6.16) can not be written in the desired form, since it contains no time derivative. Although equations (6.10), (6.11) contain a ∇, these equations describe the potential and velocity of the two dimensional system over time. Hence these derivatives are not with respect to time, even though (6.11) describes the main equation of the model. Furthermore, equations (6.12) and (6.13) where linearized and combined to obtain the boundary condition (6.14). Hence those two equations serve no purpose at this time, even though they contain a time derivative. This elimination leaves fives equations, three of which, (6.15), (6.18) and (6.19), are not a partial derivative, but a derivative. The final two equations, (6.14) and (6.17), do contain a partial derivative.

Using (6.19), one extra equation can be obtained. By taking two time derivatives,

$$\frac{\partial^2\phi}{\partial t^2} = -\omega^2\phi. \tag{6.20}$$

In Chapter 5, no port-Hamiltonian representation was found for equations (6.14), (6.17) and (6.20). In a similar manner, the finite dimensional (6.15), (6.18) and (6.19), can not be transformed in a port-Hamiltonian system. Hence this model, whether finite or infinite dimensional, boundary control or not, can not be transformed in a linear first order port-Hamiltonian system.

Just because the system can not be transformed into a port-Hamiltonian system, does not mean it can not be transformed into a regular boundary control system. However, if any useful information is to be gained, some thought has to be put in what kind of equations have to be used. Preferably, since equation (6.11) describes the main dynamics, this equation is used. However this equation did not contain a time derivative. The next best equation to use would be the boundary condition on the surface of the liquid, equation (6.14). In this case, the equation describes the potential of the liquid in a single dimension, the x direction at height $y = h$. There are two boundary conditions, namely (6.15). In a way, these boundary conditions can be seen as the input function $v(t)$. By moving the hand, the mug is moved, which in turn can be given by $\dot{X}$. This results in the control system

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0, \text{ on } y = h,$$
$$\frac{\partial \phi}{\partial x} = \dot{X}(t), \text{ on } x = 0, L.$$

Using

$$z = \begin{bmatrix} \phi \\ \frac{\partial \phi}{\partial t} \end{bmatrix},$$

this system can be transformed in

$$\dot{z} = \mathfrak{A}z = \begin{bmatrix} 0 & 1 \\ Z & 0 \end{bmatrix} z,$$
$$v = \mathfrak{B}z = \begin{bmatrix} W & 0 \end{bmatrix} z,$$

where $Z\phi = -g\frac{\partial}{\partial y}\phi$, with $D(Z)$ contains all ϕ evaluated at $y = h$, and $W\phi = \frac{\partial}{\partial x}\phi$, with $D(W)$ contains all ϕ evaluated at $x = 0$ or $x = L$.

In order to prove that this control system is in fact a boundary control system, using the theory from Section 4.11.2, it has to at least hold that the operator A, defined on $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$, by $Az = \mathfrak{A}z$ for all $z \in D(A)$, generates a strongly continuous semigroup. Negating the fact that no domain has been properly defined yet, let's take a closer look at what has to be proven in order for an operator to be an infinitesimal generator. Since the theory lies with generating contraction semigroup, that has to be proven instead. For this, there were a few options, which can be divided into two groups. For the first part of the theorems, it has to at least hold that for the operator A,

$$\|(\alpha I - A)z\| \geq \alpha\|z\| \; \forall z \in D(A), \alpha > 0.$$

Hence

$$\|(\alpha I - A)z\| = \left\| \begin{bmatrix} \alpha & -1 \\ -Z & \alpha \end{bmatrix} z \right\|.$$

However, due to Z being an operator itself, this norm can not easily be bounded by the norm of z for all α. The other half of the theorems need that A is dissipative,

$$Re\langle Az, z\rangle \leq 0 \; \forall z \in D(A).$$

In order to use this inner product, define

$$\langle f, g\rangle = \int_0^L f^* g \, dx.$$

Assume for the moment that ϕ is real, and $w(x, y, t) \in D(A)$. Then

$$\begin{aligned}
Re\langle Aw, w\rangle &= \langle Aw, w\rangle \\
&= \int_0^L (Aw)^T w \, dx \\
&= \int_0^L \left(\begin{bmatrix} 0 & 1 \\ Z & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right)^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} dx \\
&= \int_0^L \begin{bmatrix} w_2 & Zw_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} dx \\
&= \int_0^L w_2 \left(w_1 + Zw_1 \right) dx \\
&= \int_0^L w_2 \left(w_1 - g\frac{\partial}{\partial y} w_1 \right) dx.
\end{aligned}$$

At this point, nothing can be done with the integral. The integral can not be changed using integration by parts, nor can be concluded that the integral is less or equal to zero for all $w \in D(A)$.

Hence it can not be concluded that A generates a contraction semi-group, and in turn a C_0-semigroup. It is however possible to consider another model. As it is still impossible to consider the potential function in the entire mug, consider only the left wall. Now the main equation is given by (6.15), and the boundary conditions are given by (6.14) and (6.16). Although this is still a one dimensional model, in contrast to the previous model, this one lies vertically instead of horizontally. The reason for this approach is that the main equation now has a partial derivative with respect to x, which gave problems in the integral in the previous attempt. However, in this case, the inner product is defined by

$$\langle f, g\rangle = \int_0^h f^* g \, dy.$$

And now the integral is with respect to y, but the equation itself is with respect to x. Following the same reasoning as before, it can not be concluded that A generates a contraction semigroup, and hence a C_0-semigroup.

Not using the main equation (6.11) results in nothing but problems. In a last desperate attempt to fit the model as a port-Hamiltonian boundary control system, consider the following. After the solution for ϕ was obtained, (6.19), the time was only present in X and a_n. These were rewritten, resulting in

$$\phi = Re(X_0 e^{i\omega t})x + \sum_{n=0}^{\infty} Re(\alpha_n e^{i\omega t}) \cos\frac{n\pi x}{L} \cosh\frac{n\pi y}{L}.$$

In turn, for the appropriate definition of $\phi_0(x,y)$, this can be further rewritten in

$$\phi(x,y,t) = Re\left(e^{i\omega t}\left(X_0 x + \sum_{n=0}^{\infty} \alpha_n \cos\frac{n\pi x}{L}\cosh\frac{n\pi y}{L}\right)\right) = Re\left(e^{i\omega t}\phi_0(x,y)\right).$$

Hence the model can also be described using ϕ_0. In which case equation (6.11) can be rewritten as

$$\frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial y^2} = 0. \tag{6.21}$$

At this point, by construction of ϕ_0 from ϕ, there is no time variable left. This means that the system can not be evaluated over time, which renders all of the theory useless. However, the x variable represents the horizontal space of the mug and attains finite values, while y represents the vertical space of the mug and may approach infinity. Hence it is possible to loosely view the y component as the time component in the system. The appropriate boundary conditions for (6.21) are

$$\begin{aligned}
&\frac{\partial \phi_0}{\partial x}(0,y) = \frac{\partial \phi_0}{\partial x}(L,y) = \dot{X}(t),\\
&\frac{\partial \phi_0}{\partial y}(x,0) = 0,\\
&g\frac{\partial \phi_0}{\partial y}(x,h) = \omega^2\phi_0(x,h).
\end{aligned}$$

The last equation is obtained using (6.20). The first equation however poses a problem, because it contains $\dot{X}(t)$, and the system should not depend on the time t. Hence replace this with some to be determined input function v. Equation (6.21) is of the form that has been studied already. In this case, $a = -1, b = 1, c = 1$, which are interchangeable. And in order to rewrite this as a linear first order port-Hamiltonian system, it had to hold that the sign of ac was equal to b. However this is never possible for any combination of changes of a, b, c. Hence the system is still not transformable into a port-Hamiltonian system.

It might still be possible to write this system as a general boundary control system. This time however, the derivative is with respect to the same variable as the integral in the computation of dissipativity. Using

$$z = \begin{bmatrix} -\frac{\partial \phi_0}{\partial y} \\ \frac{\partial \phi_0}{\partial x} \end{bmatrix},$$

(6.21) can be transformed into

$$\dot{z} = \mathfrak{A}z = \begin{bmatrix} 0 & W \\ -W & 0 \end{bmatrix} z,$$

where $W\phi_0 = \frac{\partial}{\partial x}\phi_0$. Whatever the extra conditions, boundary conditions and proper domains are, it must always hold that on some domain $D(A)$, the operator

$$Az = \begin{bmatrix} 0 & W \\ -W & 0 \end{bmatrix} z$$

generates a C_0-semigroup. Due to the nature of the matrix, it is again not feasible to use the Hille-Yosdia theorem. Furthermore, the analysis of proving dissipativity is analogous to the previous attempt, and due to W only taking a single derivative, using integration by parts does not prove non-positivity.

Hence even the attempt of removing time does not create a suitable environment for linear first order port-Hamiltonian systems.

It can be concluded that in the paper [9], the simplifications of removing an explicit time dependence, work great for obtaining a solution for the mitigating of sloshing. Unfortunately, these same simplifications do not allow this model to be rewritten as a linear first order port-Hamiltonian system. By stating the multiphysics mug liquid system directly in a port-Hamiltonian framework, a port-Hamiltonian system could potentially be obtained, but then it will be either nonlinear (if one involves the full Navier-Stokes equations) or perhaps a linear system, but it is not first order (and for which the theory still needs to be developed).

7 Conclusions and recommendations

In this thesis, port-Hamiltonian systems were discussed as a way to study multiphysics systems. Conditions were derived such that specific classes of partial differential equations can be rewritten as a linear first order port-Hamiltonian system. In the specific case where the order of the spatial derivative was not equal to the order of the time derivative, the partial differential equations could not be transformed into a linear first order port-Hamiltonian system. It is left for further research if some of these equations can be classified as a linear second order port-Hamiltonian system.

As an example of a multiphysics system, liquid sloshing in a mug was studied. A mathematical model, taken from the literature, was derived in full detail. An analytic solution was obtained and walking frequencies for which sloshing occurs were calculated. It was investigated in detail if the combination of the liquid model and the model for the mug could be rewritten as a linear first order port-Hamiltonian system. This proved to be impossible. The simplifications on the time component and linearization of the model did not improve this. Hence the model could also not be rewritten as a port-Hamiltonian boundary control system. It is left for further research if a different model of the mug and liquid, one where the potential function depends explicitly on time, or the boundary conditions are not linearized, does allow for a port-Hamiltonian system to arise.

Avoiding sloshing, and spilling the liquid, is important in the application of autonomous robotic waiters. Since an autonomous robotic waiter carrying a mug filled with liquid is a multiphysics system, and since the theory of port-Hamiltonian systems provides a general framework for network modelling of multiphysics systems, it is recommended to model the mug and liquid directly in a port-Hamiltonian setting. In a 3D setting, the system can then be modelled as a robot carrying a plate on which the mug is placed.

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