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Master thesis

Lévy driven Queues: Convexity of the Correlation Function

by

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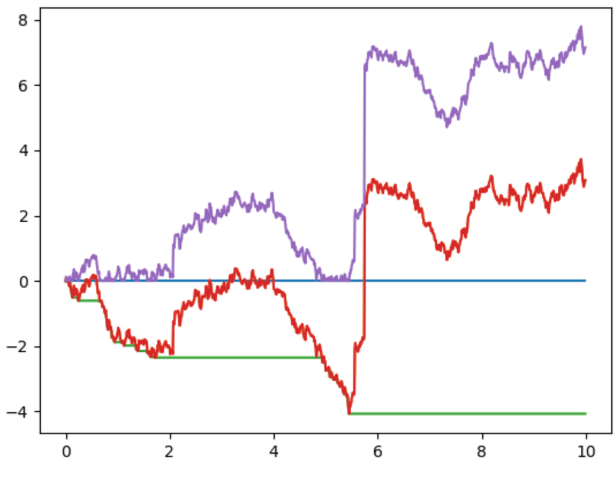

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Abstract

Let X_t be a Lévy process and $Q_t = X_t + \max(Q_0, -\inf_{0\leq s\leq t} X_s)$ a Lévy queue driven by that process. Assume there is a stationary distribution with a finite second moment. Let $r(t)$ be the correlation coefficient between Q_0 and Q_t in stationarity. It has been shown for spectrally positive an spectrally negative Lévy queues that $r(t)$ is positive, nonincreasing, and convex. This thesis reviews basic theory on Lévy queues and the stationary distributions of one-sided queues. It then generalizes the structural results on $r(t)$ from spectrally one-sided to general Lévy queues and finite buffer Lévy queues.

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1. Popular Summary

In this thesis we consider the theory on Lévy processes and Lévy queues. Lévy processes are often used in financial applications. Let for example X_t be the logarithm of a stock value at time t, assuming $X_0 = 0$. We make the simplifying assumption that for $s \leq t$ the increment $X_t - X_s$ is independent from $(X_u)_{0 \leq u \leq s}$ (independence), in other words, knowledge of the past and present does not influence the distribution of change in the future. We further assume that $\mathbb{P}(X_t - X_s \leq y) := p(t-s, y)$ for some function p, so the distribution of $X_t - X_s$ depends only on $t-s$ and is therefore the same as the distribution of $X_{t-s} - X_0 = X_{t-s}$ (stationarity). We show useful properties of these processes and fully characterize them in terms of their characteristic function $C(s) := \mathbb{E}e^{isX_t}$.

We then consider the Lévy queue Q_t, an object that appears as a limit of the workload in traditional queueing models. The Lévy queue starting in Q_0 driven by the Lévy process is then

$$Q_t := X_t + \max(Q_0, -\inf_{0 \leq s \leq t} X_t).$$

From this formula we see that Q_t goes up when X_t goes up (work arriving) and goes down when X_t goes down (work being done), unless $Q_t = 0$ (there is no work to be done, so X_t going down (work being done) has no effect). When $\mathbb{E}X_1 < 0$ the queue length Q_t has a tendency to stay 'close' to 0 and there is a stationary distribution Q^*:

$$\mathbb{P}(Q^* \leq y) := \lim_{t \to \infty} \mathbb{P}(Q_t \leq y).$$

This stationary distribution can be used to calculate long term performance measures of the Lévy queue and therefore approximate the performance of the traditional queues of which it is a limit. Furthermore, we show that Q^* has the same distribution as $\sup_{0 \leq t} X_t$, which is a quantity of interest in the stock value setting.
For Lévy processes with no discontinuities where X_t jumps up (called Spectrally Negative) we show that $\mathbb{P}(Q^* > y) = e^{-\beta_0 y}$ for some constant β_0. For Lévy processes with no discontinuities where X_t jumps down (called Spectrally Positive) we find a simple explicit expression for the Laplace transform $s \mapsto \mathbb{E}e^{-sQ^*}$ for any $s \geq 0$. These results can be used for proving many quantitative and qualitative properties (see e.g. [1]) of these types of Lévy queues.

One of these properties is that in both cases the correlation function

$$r_t := \frac{\mathbb{E}(Q_0 Q_t) - \mathbb{E}Q_0 \mathbb{E}Q_t}{\sqrt{Var(Q_0)Var(Q_t)}}$$

is nonnegative, nonincreasing and convex (rate of decrease is nonincreasing) in t when Q_0 has the same distribution as Q^*. In the general case (where the Lévy process can jump both up and down), there are no simple expressions for the distribution or Laplace transform of Q^*. This thesis sets up and applies a method to prove that r_t is still nonnegative, nonincreasing and convex in t.

2. Introduction

This introduction gives an overview of the topics of the thesis, which starts by giving the definitions and characterizations of Lévy processes (chapter 3) and queues (chapter 4). It shows results from the literature on the stationary distribution of spectrally one-sided Lévy queues (chapter 5). It then reformulates a known bound on the covariance of two random variables with given marginals (the rearrangement principle, chapter 6). Finally it uses that principle to generalize the structural results on the correlation function (positivity, monotonicity and convexity) from spectrally one-sided queues to the general case (chapter 7) and even for finite-buffer Lévy queues (chapters 8 and 9).

2.1. Lévy Processes

In this thesis we review the theory on Lévy processes. These are processes often used in financial applications. Let for example X_t be the logarithm of a stock value at time t, assuming $X_0 = 0$. We make the simplifying assumption that for $s \leq t$ the increment $X_t - X_s$ is independent from $(X_u)_{u \leq s}$. We further assume that $\mathbb{P}(X_t - X_s \leq y) := p(t-s, y)$, so the distribution of $X_t - X_s$ depends only on $t - s$ and is therefore the same as the distribution of $X_{t-s} - X_0 = X_{t-s}$. We show useful properties of these processes and fully characterize them in terms of their characteristic function $C(s) := \mathbb{E}e^{isX_t}$.

2.2. Lévy Queues

We then consider the Lévy queue (also called Lévy process reflected at 0), an object that appears as a limit of traditional queueing models. The Lévy queue starting in x fed by the Lévy process is then

$$Q_t := X_t + \max(Q_0, -\inf_{0 \leq s \leq t} X_t).$$

This is the solution to the Skorokhod problem $Q_t = Q_0 + X_t + L_t$ with L_t nondecreasing, $Q_t \geq 0$ and

$$\int_0^t Q_s dL_s = 0,$$

where the integral is to be seen as a Lesbesgue-Stieltjes integral, not a stochastic integral. The workload at time t of an M/G/1 queue is actually a Lévy queue, where $X_t = S_t - t$ and S_t is the total amount of work that has come in between times 0 and t,

$\arg\inf_{0\leq s\leq t} X_t$ is the last time the queue is empty (unless there is so much work at the beginning that the queue has never been empty).
When $\mathbb{E}X_1 < 0$ there is a stationary distribution Q^*. The stationary distribution can be used to calculate long term performance measures of the Lévy queue and therefore approximate the performance of the traditional queues of which it is a limit. Furthermore, we show that Q^* has the same distribution as $\sup_{0\leq t} X_t$, which is a quantity of interest in the stock value setting.

2.3. Spectrally One-Sided Lévy Queues

For Lévy processes with no discontinuities where X_t jumps up (Spectrally Negative) we show that $\mathbb{P}(Q^* > y) = e^{-\beta_0 y}$ for some constant β_0. We use a level crossing argument to show this is true. For finding the value of β_0, we use a Martingale.

For Lévy processes with no discontinuities where X_t jumps down (Spectrally Positive) we find an explicit expression for the Laplace transform $\mathbb{E}e^{-sQ^*}$ for any s. For the workload of M/G/1 queues, which can be seen as a dense subset of Spectrally Positive Lévy queues we use a traditional argument on the preemptive LIFO M/G/1 queue to get the Pollaczek-Khinchine formula. We also show how to find the Laplace transform for all Spectrally Positive Lévy queues using the Kella-Whitt martingale. As the proof that the Kella-Whitt martingale is a martingale is non-trivial, we also give an alternative argument using a simpler martingale.

These results can be used for proving many quantitative and qualitative properties (see e.g. [1]) of Spectrally Positive and Spectrally Negative Lévy queues. One of these properties concerns the correlation function

$$r_t := \frac{\mathbb{E}(Q_0 Q_t) - \mathbb{E}Q_0\mathbb{E}Q_t}{\sqrt{Var(Q_0)Var(Q_t)}},$$

which quantifies how strongly Q_t depends on Q_0. For approximations you can assume Q_0, Q_t to be independent when the correlation drops below some predetermined value. It has been shown that the correlation function for both spectrally positive and spectrally negative queues is nonnegative, nonincreasing and convex in t when Q_0 has the same distribution as Q^*. This has been shown for the M/G/1 queue in [8], for spectrally positive queues in [9], and for spectrally negative queues in [10]. In the general case (where the Lévy process can jump both up and down), there are no simple expressions for the distribution or Laplace transform of Q^*. Due to this fact, it has not been proven before that r_t is nonnegative, nonincreasing and convex in t.

2.4. Convexity of the Correlation Function in the General Case

We introduce the rearrangement principle, which establishes $\mathbb{E}Yf(X,Z) \leq \mathbb{E}Xf(X,Z)$ when X and Y are random variables with the same marginal distribution, f is nondecreasing in the first argument, and Z is independent from (X,Y). The principle has been around since 1951 [11], though we reformulate it to suit our needs. We apply this with $X = Q_s$, $Y = Q_0$ and $Z = (X_{s+v} - X_s)_{v \geq 0}$ to prove r_t is nonincreasing and convex (for nonnegativity we use $Y = Q'_0$ independent from the rest). This works since for $s \leq t \leq u$ we have that Q_t given Z is a nondecreasing function of Q_s and $Q_u - Q_t$ given Z is a nonincreasing function of Q_s. All we need to know abut the stationary distribution is that it exists. We observe that the same reasoning can be applied to the workload observed by the n-th arrival in a $G/G/1$ queue.

In the two final chapters, we consider the finite buffer Lévy queue, which is reflected at both 0 and $a > 0$. We use an explicit expression of Q_t in terms of Q_0, X to show that the key properties we use for the argument still hold (given X, Q_t is nondecreasing in Q_0 and $Q_t - Q_0$ nonincreasing in Q_0). We then apply the same reasoning as before. This shows the wide applicability of the rearrangement principle.

3. Lévy Processes

3.1. Introduction

Many sequences of random variables of interest are of the form

$$X_n := \sum_{i=1}^{n} Y_i,$$

where $Y_1, Y_2, ...$ are independently identically distributed. We then have for all $m < n$ that $X_0 = 0$, $X_n - X_m$ has the same distribution as X_{n-m}, and $X_n - X_m$ is independent from $X_0, ..., X_m$. In other words, X starts in 0 and has stationary independent increments.
Conversely, if a sequence X starts in 0 and has stationary independent increments, it is of the form $X_n := \sum_{i=1}^{n} Y_i$, where $Y_1, Y_2, ...$ are independently identically distributed.

In continuous time, there are processes with similar behavior, called Lévy processes. The sum definition is problematic to carry over to continuous time, but the increments characterization carries over nicely. A Lévy process is a stochastic process such that for all $s < t$ we have $X_0 = 0$, $X_t - X_s$ has the same distribution as X_{t-s}, and $X_t - X_s$ is independent from $(X_u)_{u \leq s}$. In other words, X starts in 0 and has stationary independent increments. Furthermore, we require X_t to be almost surely continuous from the right with left limits existing (a.s. càdlàg).

Effectively this means that for a sequence of timepoints $0 = t_0 \leq ... \leq t_n$ and a sequence of independent random variables Y_i, where Y_i has the same distribution as $X_{t_i - t_{i-1}}$, we have

$$X_{t_n} = \sum_{i=1}^{n} Y_i.$$

In this chapter we give some classical examples of Lévy processes, give a parametrization of all Lévy processes in terms of the characteristic function $\mathbb{E}e^{isX_t}$ and a way to construct any Lévy process from the classical examples.

3.2. Poisson Renewal Process

One classical example of a Lévy process is the Poisson renewal process. For a Poisson renewal process with parameter λ we take N_t^λ to have a Poisson distribution with parameter λt:

$$\mathbb{P}(N_t^\lambda = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}.$$

The characteristic function is then

$$C_{N_t^\lambda}(s) = \mathbb{E}e^{isN_t^\lambda} \tag{3.1}$$
$$= \sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}e^{isk} \tag{3.2}$$
$$= \sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t e^{is})^k}{k!}e^{is} \tag{3.3}$$
$$= e^{\lambda t e^{is} - \lambda t} \tag{3.4}$$
$$= e^{\lambda t (e^{is}-1)}. \tag{3.5}$$

From this characteristic function we can also see the well-known fact that the sum of two independent poisson random variables is again poisson:

$$C_{N_t^\lambda + N_u^{\lambda *}}(s) = \mathbb{E}e^{is(N_t^\lambda + N_u^{\lambda *})} \tag{3.6}$$
$$= \mathbb{E}e^{isN_t^\lambda}\mathbb{E}e^{isN_u^{\lambda *}} \tag{3.7}$$
$$= e^{\lambda t(e^{is}-1)}e^{\lambda u(e^{is}-1)} \tag{3.8}$$
$$= e^{\lambda (t+u)(e^{is}-1)}. \tag{3.9}$$

So we can indeed take N_{t+u}^λ to be $N_t^\lambda + N_u^{\lambda *}$ and get stationary independent increments.

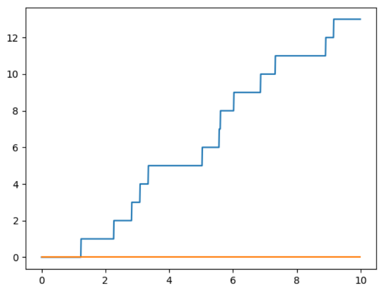

Figure 3.1.: A Poisson Renewal Process with $\lambda = 1$

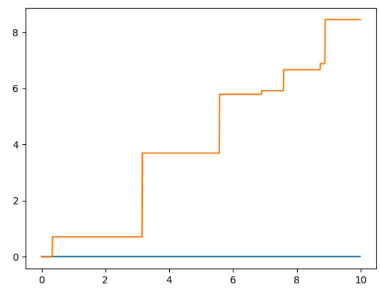

Figure 3.2.: A Compund Poisson Process with standard exponential jump sizes

The Poisson renewal process is a nondecreasing process that takes only integer values we take it to be right-continuous to satisfy the final Lévy condition. Note that the Poisson process is piecewise constant and jumps up with a jump size of 1. The probability of the first jump to be after time t is $\mathbb{P}(N_t^\lambda = 0) = e^{-\lambda t}$, so the time until the first jump has an exponential distribution with parameter λ. Due to independence and stationarity of the increments, the times between two consecutive jumps are iid exponential with parameter λ.

3.3. Compound Poisson Process

From the Poisson processes we can construct the compound Poisson processes, a class of Lévy processes that is dense in the class of all Lévy processes. We could take the jumps to have a different size and still get a Lévy process, we can take it even further and have random jump sizes. Let B_i be iid random variables that determine the jump sizes and N_t^λ be the number of jumps so far. Then we define the compound Poisson process:

$$P_t := \sum_{i=1}^{N_t^\lambda} B_i.$$

The characteristic function is then

$$C_{P_t}(s) = \mathbb{E}e^{isP_t} = \sum_{k=0}^{\infty} \mathbb{P}(N_t^\lambda = k)\mathbb{E}e^{is\sum_{i=1}^k B_i} = \sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\left(\mathbb{E}e^{isB_1}\right)^k = e^{\lambda t(\mathbb{E}e^{isB_1}-1)}.$$

Again we can see from the characteristic function that indeed the sum of independent variables $P_t + P'_u$ has the same distribution as P_{t+u}.

3.4. Brownian Motion

Another classic example of a Lévy process is Standard Brownian Motion. Standard Brownian Motion is usually defined as having the Lévy properties, with continuity instead of right-continuity and W_t being a normal distribution with mean 0 and variance t. Because of the stationary independent increments $(X_1, ..., X_{t_n})$ is a multivariate Gaussian with $Cov(X_{t_i}, X_{t_j}) = \min(t_i, t_j)$. The characteristic function is

$$\begin{aligned}
C_{W_t}(s) &:= \mathbb{E}e^{isW_t} && (3.10)\\
&= \mathbb{E}e^{is\sqrt{t}W_1} && (3.11)\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{is\sqrt{t}x - x^2/2}dx && (3.12)\\
&= e^{-s^2t/2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-is\sqrt{t})^2/2}dx && (3.13)\\
&= e^{-s^2t/2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx && (3.14)\\
&= e^{-s^2t/2}. && (3.15)
\end{aligned}$$

Here (3.13) is equal to (3.14) because $e^{-x^2/2}$ has a complex derivative and vanishes at infinity.

3.5. General Lévy Process

Using compound Poisson processes and Brownian motion we will construct all Lévy processes. Of course the deterministic ct is also a Lévy process. Any linear combination of independent Lévy processes is again a Lévy process, so combining the Lévy processes we have already seen we get

$$X_t = ct + \sigma W_t + \sum_{i=1}^{N_t^\lambda} B_i,$$

with characteristic function

$$C_{X_t}(s) = e^{t(cis - \sigma^2 s^2/2 + \lambda(-1+\mathbb{E}e^{isB_1}))}.$$

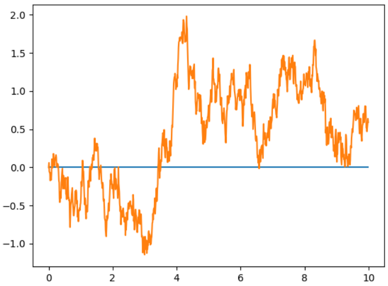

Figure 3.3.: A Brownian Motion Process with $\sigma^2 = 1$

This actually covers all Lévy processes with finite jump intensity; it has been shown (e.g. in [12]) that all Lévy processes have a characteristic function of the form $e^{t\xi(s)}$ with $\xi(s)$ given by the Lévy-Khinchine formula:

$$\xi(s) := isd - \sigma^2 s^2/2 + \int_{-\infty}^{\infty} e^{isx} - 1 - isx1_{|x|<1}\Pi(dx),$$

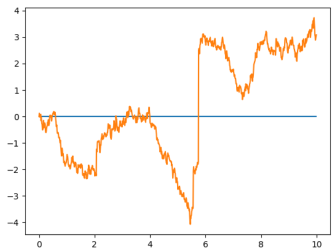

Figure 3.4.: A Lévy Process with $\sigma^2 = 1, c = -0.5, \lambda = 1$ and standard exponential jump sizes

where Π is a measure such that $\int_{-\infty}^{\infty} \min(x^2, 1)\Pi(dx) < \infty$ and $\Pi(\{0\}) = 0$. This measure gives the jump intensities; the time until the next jump with size $x \in A$ is exponential with parameter $\Pi(A)$. There are also Lévy processes with infinitely many jumps per time unit. This is when $\Pi(\mathbb{R}) = \infty$. The linear term in the integral is included so the integrand is of order $O(x^2)$ for small x. In the finite jump intensity case we have $\Pi(A) = \lambda\mathbb{P}(B_1 \in A)$ and $d = c + \lambda\mathbb{E}(B_1 1_{|B_1|<1})$. We can approximate X with infinite jump intensity by $X^{(n)}$ with finite jump intensity by taking $\Pi^{(n)}(A)$ to be $\Pi(\{x \in A : x \geq 1/n\})$. This way all $X^{(n)}$ have finite jump intensity and $X^{(n)} \to X$ in distribution as $n \to \infty$. We know this convergence holds because the characteristic functions converge.

3.6. Spectrally One-Sided Lévy Processes

When all jumps of X are in the positive direction ($\Pi(\mathbb{R}_{\leq 0}) = 0$) we call X spectrally positive (shorthand $X \in \mathcal{S}_+$). When all jumps of X are in the negative direction ($\Pi(\mathbb{R}_{\geq 0}) = 0$) we call X spectrally negative (shorthand $X \in \mathcal{S}_-$). When $X \in \mathcal{S}_+$

the only way it can decrease is in a continuous manner and when $X \in \mathcal{S}_-$ the only way it can increase is in a continuous manner. This makes $\inf_{0\leq s\leq t} X_s$ or $\sup_{0\leq s\leq t} X_s$ continuous, which makes analysis of the corresponding queues in the coming chapters easier. Furthermore, we can define real variants of the characteristic function that will be useful. When $X \in \mathcal{S}_+$ we define

$$\phi(\alpha) := \log \mathbb{E}e^{-\alpha X_1}$$

and when $X \in \mathcal{S}_-$ we define

$$\Phi(\beta) := \log \mathbb{E}e^{\beta X_1}.$$

Some straightforward calculation shows that

$$\phi(\alpha) = -\alpha d + \frac{1}{2}\alpha^2\sigma^2 + \int_{\mathbb{R}_{\geq 0}} (e^{-\alpha x} - 1 + \alpha x 1_{|x|<1})\Pi(dx),$$

$$\Phi(\beta) = \beta d + \frac{1}{2}\beta^2\sigma^2 + \int_{\mathbb{R}_{\leq 0}} (e^{\beta x} - 1 - \beta x 1_{|x|<1})\Pi(dx).$$

Both ϕ and Φ are convex. Furthermore, $\phi'(0) = -\mathbb{E}X_1$ and $\Phi'(0) = \mathbb{E}X_1$.
We assume $\mathbb{E}X_1 < 0$ as this will be the case in the processes that will be of interest in the coming chapters. We also assume X is not deterministic ($X_t = dt$). Then ϕ is increasing (to infinity) and therefore has an inverse ψ on $\mathbb{R}_{\geq 0}$.
On the other hand, Φ starts off decreasing and then increases to infinity, so there is a β_0 such that $\Phi(\beta_0) = 0$ from that point on Φ increases to infinity, so there is a function Ψ on $\mathbb{R}_{\geq 0}$ which is an inverse of Φ on $[\beta_0, \infty)$

4. Lévy Queues

4.1. Introduction

Lévy queue are stochastic processes that behave locally like Lévy processes when bigger than 0, but are not allowed to take negative values. These queues are to be interpreted as the amount of work left to be done in a system. Queues are nonnegative, so they have to be pushed up when X goes below 0, we can interpret this as having the capacity to do work (X going down) but no work to be done. When it has been pushed up it does not go back down again (relative to X). So we have $Q_t = Q_0 + X_t + L_t$, where L_t is nondecreasing and $L_0 = 0$. L_t can be seen as the amount of extra work that could have been done up to time t if it had been there from the beginning. We don't want L_t to be bigger than it needs to be, so if $L_s < L_t$ we there must be a $u \in (s, t]$ with $Q_u = 0$. This problem was first posed by Skorokhod [2, 3]. A classical example of a Lévy queue is the workload process of the M/G/1 queue. In this case we have $X_t = S_t - t$, where S_t is the total amount of work that has come in up to time t. In this case L_t is the time the queue has been empty up to time t.

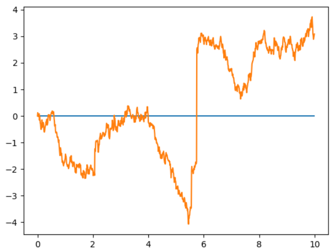

Figure 4.1.: A Lévy Process with $\sigma^2 = 1, c = -0.5, \lambda = 1$ and standard exponential jump sizes

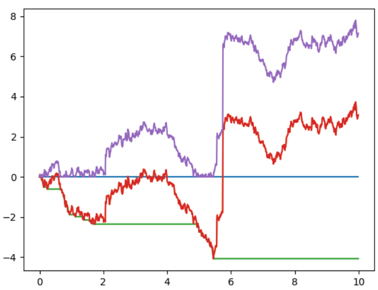

Figure 4.2.: The above Lévy process with it's infimum and the resulting queue

4.2. Skorokhod's Problem

Skorokhod's problem was as follows: we have a càdlàg function $f : [0, T] \to \mathbb{R}$ with $f(0) \geq 0$.

We want to find càdlàg $g, h : [0,T] \to \mathbb{R}_{\geq 0}$ such that $g(0) = f(0)$, $f + h = g$, h is nondecreasing and h only increases when $g = 0$, that is:

$$I_t := \int_0^t g(s)dh(s) = 0.$$

This integral must be seen as a Lebesgue-Stieltjes integral. This means that

$$\lim_{t\downarrow s} I_t - \lim_{t\uparrow s} I_t = g(s)\left(\lim_{t\downarrow s} h(t) - \lim_{t\uparrow s} h(t)\right).$$

Note that in this case $h(s) = \lim_{t\downarrow s} h(t)$ because h is right-continuous.

We have $-Q_0 - X_s \leq L_s \leq L_t$, so $-Q_0 - \inf_{0\leq s\leq t} X_s \leq L_t$. Suppose we make this an equality, we then have

$$Q_t = X_t + \max(Q_0, -\inf_{0\leq s\leq t} X_s),$$

which is nonnegative. It turns out that $L_t = \max(0, -Q_0 - \inf_{0\leq s\leq t} X_s)$ is actually the unique solution to this problem [4, prop.IX.2.2]. This gives

$$Q_t = X_t + \max(Q_0, -\inf_{0\leq s\leq t} X_s).$$

Q is also called X reflected at 0. The solution of the Skorokhod problem as a functional on càdlàg functions is:

$$h(t) = \Gamma_0[f](t) := f(t) + \left(\sup_{0\leq s\leq t} -f(s)\right)^+.$$

5. Stationary Distribution

We want to find the distribution of Q^* such that $Q_t \to Q^*$ in distribution as $t \to \infty$. This is called the stationary distribution and it is used to quantify the long term behaviour of a queue, such as the correlation function that we will ultimately prove to be convex. This stationary distribution exists if and only if $\mathbb{E}X_1 < 0$. To see this we extend time to the negative axis. Take $(X_{-t})_{t\geq 0}$ to be iid to $(-X_t)_{t\geq 0}$, then we still have stationary independent increments. Let $Q_0 = x$, we have

$$Q_t := X_t + \max(x, -\inf_{0\leq s\leq t} X_s) \tag{5.1}$$

$$= \max(x + X_t, \sup_{0\leq s\leq t}(X_t - X_s)) \tag{5.2}$$

$$= \max(x + X_t - X_0, \sup_{0\leq s\leq t}(X_t - X_s)). \tag{5.3}$$

So by stationarity Q_t given $Q_0 = x$ has the same distribution as Q_0 given $Q_{-t} = x$:

$$\max(x + X_0 - X_{-t}, \sup_{0\leq s\leq t}(X_0 - X_{s-t})) = \max(x - X_{-t}, \sup_{0\leq s\leq t}(-X_{s-t})).$$

This has the same distribution as

$$\max(x + X_t, \sup_{0\leq s\leq t}(X_{t-s})) = \max(x + X_t, \sup_{0\leq u\leq t}(X_u)).$$

This converges in distribution as t goes to infinity to $Q^* := \sup_{0\leq u<\infty} X_u$, which is a.s. finite, whenever $\mathbb{E}(X_1) < 0$ (see appendix for proof), else it tends to $+\infty$.

Suppose Q_0 has the same distribution as Q^*, then Q_t has the same distribution for all t. In the coming sections we find the (exponential)stationary distribution of a spectrally negative queue and a simple expression for the Laplace transform of a spectrally positive queue. There are many results about the stationary distribution in the general case. The study of the distribution of the all time supremum of a general Lévy process (which is equal to the stationary distribution) is part of Wiener-Hopf theory. [**?**] provides a brief introduction and [12, Chapter 6] a detailed description.

5.1. Spectrally Negative Queues

Suppose X is spectrally negative with $\mathbb{E}X_1 < 0$. We give an argument to show that the stationary distribution is exponential and find the parameter with a martingale. Let

$x, y > 0$. We consider the probability that

$$Q^* := \sup_{0 \le u < \infty} X_u \ge x + y.$$

We can condition on the event that $Q^* \ge x$. In this event there is a time τ_x that is the first time that X reaches x:

$$\tau_x := \inf_t X_t \ge x.$$

Since X can only increase in a continuous manner we have $X_{\tau_x} = x$ when $\tau_x < \infty$. So

$$\mathbb{P}(Q^* \ge x+y | Q^* \ge x) = \mathbb{P}(\sup_{0 \le u < \infty} X_u \ge X_{\tau_x} + y | Q^* \ge x) = \mathbb{P}(\sup_{0 \le u < \infty} X_u - X_{\tau_x} \ge y | \tau_x < \infty).$$

For $0 \le u < \tau_x$ we have $X_u < x$, so the above is equal to

$$\mathbb{P}(\sup_{\tau_x \le u < \infty} X_u - X_{\tau_x} \ge y | \tau_x < \infty) = \mathbb{P}(\sup_{0 \le u < \infty} X_u \ge y).$$

The last equality is due to stationary independent increments. Therefore

$$\mathbb{P}(Q^* \ge x + y) = \mathbb{P}(Q^* \ge x)\mathbb{P}(Q^* \ge y).$$

This means that Q^* has an exponential distribution.
The parameter of the exponential distribution can be found with a martingale trick since $0 = t\Phi(\beta_0) = \log \mathbb{E} e^{\beta_0 X_t}$ we have

$$\mathbb{E}(e^{\beta_0 X_{t+s}} | (X_u)_{0 \le u \le s}) = e^{\beta_0 X_s} \mathbb{E}(e^{\beta_0 (X_{t+s} - X_s)} | (X_u)_{0 \le u \le s}) = e^{\beta_0 X_s} \mathbb{E} e^{\beta_0 X_t} = e^{\beta_0 X_s},$$

so $e^{\beta_0 X_t}$ is a martingale. By stopping the martingale when it reaches level x we get

$$\mathbb{E}(e^{\beta_0 x} 1_{\tau_x < \infty}) = \mathbb{E}(e^{\beta_0 X_{\tau_x}} 1_{\tau_x < \infty}) = e^{\beta_0 X_0} = 1,$$

so $\mathbb{P}(\tau_x < \infty) = e^{-\beta_0 x}$.

5.2. The M/G/1 Queue

While Lévy queues appear as limits of traditional queues, they are not a strict generalization. However, the workload in any M/G/1 queue can be seen as a spectrally positive Lévy queue by considering the total amount of work in the system at time t. We have a server that processes jobs. The times between the arrival of one customer and the next are iid exponential with rate λ. The times it takes to process customers are iid with the same distribution as some nonnegative random variable B. Let Q_t be the amount of time it would take to process the work in the system at time t. Let $\rho := \lambda \mathbb{E}(B)$ be the fraction of time work needs to be done to keep up with the jobs coming in. If $\rho > 1$, then the amount of work to be done will go to infinity. If $\rho = 1$ the amount of work will return to 0 infinitely often but the time it takes between visits to 0 is infinite in expectation. The situation we are interested in is $\rho < 1$, so we assume this to be true (this is equivalent to $\mathbb{E}X_t < 0$). As t goes to infinity, Q_t will converge in distribution to some Q^*. We will find the Laplace transform of Q^*. This transform is given by the Pollaczek-Khinchine formula, a well known result from queueing theory. We will find it by constructing the stationary distribution. The result on the transform can then be extended to all Lévy processes that are spectrally positive.

Let $\mathbb{P}(R \geq x) := \frac{\mathbb{E}\max(0,B-x)}{\mathbb{E}B}$. R is said to have the residual lifetime distribution of B. It is the distribution of the time it would take to get to the end of the current job at a random time while processing an endless string of jobs. Let B_i be iid distributed according to B, $S_n = \sum_{i=1}^n B_i$ and let $1_D(x,y)$ be 1 if and only if there is no $S_n \in (x,y)$ then:

$$\mathbb{P}(R \geq x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T 1_D(t,t+x)dt.$$

We will prove that Q^* has the same distribution as

$$\sum_{i=1}^{G} R_i.$$

Here the R_i are iid distributed like R and G has the geometric distribution with parameter ρ: $\mathbb{P}(G \geq n) = \rho^n$. This suggests that there are G jobs in the system whose remaining work is iid distributed like R. This is true when the work is done in a specific order (the order the work is done in does not affect the total amount of work): preemptive last in first out. This means that we start working on any job the moment it arrives. If a job is being worked on and another job arrives, the work on the first job is interrupted until all other jobs arriving after it have been handled. It can be visualized as a stack of forms that have to be filled out. When a new form arrives it is put on top of the stack. You are always working on the top form and remove it when it is done. Let (j,y) be a pair of a nonnegative integer and a j-dimensional vector $(y_1,...,y_j)$. Then we want to prove

$$\pi(j,y) := \mathbb{P}(G = j, \forall i \leq j : R_i \geq y_i) = (1-\rho)\rho^j \prod_{i=1}^{j} \frac{\mathbb{E}\max(0,B-y_i)}{\mathbb{E}B}.$$

Since $\rho = \lambda\mathbb{E}B$, the right-hand side is equal to

$$(1-\rho)\lambda^j \prod_{i=1}^{j} \mathbb{E}\max(0,B-y_i).$$

We prove this by induction. Take $j=0$: the queue is empty the exact same proportion of time that no work is being done, which is $1-\rho$. Now suppose it is true for j, we want to find the proportion of time that

$$G = j+1, \forall i \leq j+1 : R_i \geq y_i.$$

The periods of time that this is true start with an arrival of a job while

$$G = j,\ \forall i \leq j:\ R_i \geq y_i,$$

this happens $\lambda\pi(j, y_1, ..., y_j)$ times per time unit. If the job size of the arrival is of size less then y_{j+1} the last condition is not fulfilled so this contributes 0 time. If the job size is bigger than y_{j+1} the full statement is true for a period of length $B - y_{j+1}$, so every arrival contrubutes on average $\mathbb{E}\max(0, B - y_{j+1})$ of time the statement is true. So the portion of time the statement is true is

$$\lambda\pi(j, y_1, ..., y_j)\mathbb{E}\max(0, B - y_{j+1}) = (1-\rho)\lambda^{j+1}\prod_{i=1}^{j+1}\mathbb{E}\max(0, B - y_i).$$

This proves the induction step. [5] used this line of reasoning in the more general case of determining the distribution at arrival times in $G/G/1$ queues.

We can also write this result in terms of the Laplace transforms:

$$\kappa(\alpha) := \mathbb{E}e^{-\alpha Q^*} = \sum_{j=0}^{\infty}(1-\rho)\rho^j\prod_{i=1}^{j}\mathbb{E}e^{-\alpha R_i} = \sum_{j=0}^{\infty}(1-\rho)\rho^j\left(\mathbb{E}e^{-\alpha R}\right)^j = \frac{1-\rho}{1-\rho\mathbb{E}e^{-\alpha R}}.$$

Furthermore, we can see that R has a probability density function $p_R(x) = \frac{1}{\mathbb{E}B}\mathbb{P}(B > x)$, so

$$\mathbb{E}e^{-\alpha R} = \frac{1}{\mathbb{E}B}\int_{(0,\infty)}\mathbb{P}(B > x)e^{-\alpha x}dx \tag{5.4}$$

$$= \frac{1}{\mathbb{E}B}\int_{(0,\infty)}\int_{(x,\infty)}e^{-\alpha x}dF_B(y)dx \tag{5.5}$$

$$= \frac{1}{\mathbb{E}B}\int_{(0,\infty)}\int_{(0,y)}e^{-\alpha x}dxdF_B(y) \tag{5.6}$$

$$= \frac{1}{\mathbb{E}B}\int_{(0,\infty)}\frac{1-e^{-\alpha y}}{\alpha}dF_B(y) \tag{5.7}$$

$$= \frac{1-e^{-\alpha B}}{\alpha\mathbb{E}B}. \tag{5.8}$$

Substituting this (and $\rho = \lambda\mathbb{E}B$) we get

$$\kappa(\alpha) = \frac{\alpha(1-\lambda\mathbb{E}B)}{\alpha+\lambda-\lambda\mathbb{E}e^{-\alpha B}} \tag{5.9}$$

$$= \frac{\alpha\phi'(0)}{\phi(\alpha)}. \tag{5.10}$$

(5.9) is a form of the Pollaczek-Khinchine formula. (5.10) is a simplification of this formula using the notation for spectrally positive Lévy processes. In the next section we show that this is valid for all spectrally positive Lévy processes.

5.3. Spectrally Positive Queues: Kella-Whitt Martingale

A popular way to show that the (5.10) holds for all spectrally positive Lévy queues use the so called Kella-Whitt martingale. See e.g. [1]. Let X be spectrally positive, then the Kella-Whitt martingale K_t [6] is a martingale (starting in 0):

$$K_t := \phi(\alpha)\int_0^t e^{-\alpha Q_s} ds + e^{-\alpha Q_0} - e^{-\alpha Q_t} - \alpha L_t.$$

Now take Q_0 to have the stationary distribution then

$$0 = \mathbb{E}K_1 = \phi(\alpha)\int_0^1 \mathbb{E}e^{-\alpha Q_s} ds + \mathbb{E}e^{-\alpha Q_0} - \mathbb{E}e^{-\alpha Q_1} - \mathbb{E}\alpha L_1 \tag{5.11}$$

$$= \phi(\alpha)\int_0^1 \mathbb{E}e^{-\alpha Q^*} ds + \mathbb{E}e^{-\alpha Q^*} - \mathbb{E}e^{-\alpha Q^*} - \mathbb{E}\alpha L_1 \tag{5.12}$$

$$= \phi(\alpha)\mathbb{E}e^{-\alpha Q^*} - \mathbb{E}\alpha L_1, \tag{5.13}$$

so

$$\mathbb{E}e^{-\alpha Q^*} = \frac{\mathbb{E}\alpha L_1}{\phi(\alpha)}$$

and

$$1 = \lim_{\alpha\downarrow 0} e^{-\alpha Q^*} = \lim_{\alpha\downarrow 0} \frac{\mathbb{E}\alpha L_1}{\phi(\alpha)} = \frac{\mathbb{E}L_1}{\phi'(0)},$$

so $\mathbb{E}L_1 = \phi'(0) = -\mathbb{E}X_1$.
The proof that Kella-Whitt martingale is a martingale is quite technical, so it is omitted here. The next section shows the same result with a much simpler martingale.

5.4. Spectrally Positive Queues: Alternative Martingale

We can also find the generalized Pollaczek-Khinchine formula by considering the infimum of X up to some exponential random time T. We take T to have parameter $\phi(\alpha)$, as this will make the formulas the simplest. Let $X \in \mathcal{S}_+$ with $\mathbb{E}X_1 < 0$. The following is a martingale:

$$M_t := e^{-\alpha X_t - \phi(\alpha)t}.$$

This is due to the fact that because of stationary independent increments we have for $0 \le s \le t$:

$$\mathbb{E}(e^{-\alpha(X_t)|(X_u)_{0\le u\le s}}) = e^{-\alpha X_s}\mathbb{E}e^{-\alpha(X_t - X_s)} = e^{\phi(\alpha)(t-s)}e^{-\alpha X_s}.$$

Let $\tau_x := \inf\{t : X_t \le -x\}$ be the first time the process goes below level $-x$, then τ is a stopping time and because X_t can only decrease in a continuous manner we have $X_{\tau_x} = -x$, so optional stopping gives us:

$$1 = \mathbb{E}M_0 = \mathbb{E}M_{\tau_x} = \mathbb{E}e^{-\alpha X_{\tau_x} - \phi(\alpha)\tau_x} = \mathbb{E}e^{\alpha x - \phi(\alpha)\tau_x} = e^{\alpha x}\mathbb{P}(T \ge \tau_x) = e^{\alpha x}\mathbb{P}(-\inf_{0\le s\le T} X_s \ge x).$$

This means that $L^0_T := -\inf_{0\leq s\leq T} X_s$ has an exponential distribution with parameter α. Given a deterministic Q_0, we have $L_T = Q_T - Q_0 - X_T = \max(0, L^0_T - Q_0)$, so

$$\mathbb{E}L_T = \max(0, L^0_T - Q_0) = \mathbb{P}(L^0_T \geq Q_0)\mathbb{E}(L^0_T - Q_0|L^0_T \geq Q_0) = e^{-\alpha Q_0}\frac{1}{\alpha},$$

so if we take Q_0 to have the stationary distribution we get $\mathbb{E}L_t = t\mathbb{E}L_1$ by stationarity. Furthermore, T is independent from $(L_t)_{t\geq 0}$, so $\mathbb{E}L_T = \mathbb{E}T\mathbb{E}L_1$. This gives us

$$\frac{\mathbb{E}L_1}{\phi(\alpha)} = \mathbb{E}L_1\mathbb{E}T = \mathbb{E}L_T = \mathbb{E}e^{-\alpha Q_0}\frac{1}{\alpha},$$

so

$$\mathbb{E}e^{-\alpha Q_0} = \frac{\alpha\mathbb{E}L_1}{\phi(\alpha)},$$

and by the same argument as the previous section $\mathbb{E}L_1 = \phi'(0) = -\mathbb{E}X_1$.

6. Rearrangement

6.1. Introduction

To prove convexity of the correlation function we will use the rearrangement principle laid out in this chapter. This principle basically says that given the marginal distributions, the expectation of the product is maximal when they are big at the same time. The basic idea is that for U uniform on $(0, 1)$ and F_X, F_Y the marginal distribution functions of random variables X, Y (not neccesarily independent) we have

$$\mathbb{E}(F_X^{-1}(U)F_Y^{-1}(1-U)) \leq \mathbb{E}(XY) \leq \mathbb{E}(F_X^{-1}(U)F_Y^{-1}(U)).$$

This was shown in [11]. The version we prove here is slightly different to fit into the proof of convexity, but the essence of the statement is the same. In the next section of this chapter we will review some basic tools and in the final chapter we will use these tools to formulate our version of the rearrangement principle.

6.2. Tools

6.2.1. Calculating Expectations

A well known technique for calculating expectations of nonnegative random variables is integrating tail probabilities. Let X be a nonnegative random variable with probability measure $\mathbb{P}_x$, then by switching order of integration we get:

$$\begin{aligned}
\int_{\mathbb{R}_{\geq 0}} \mathbb{P}(X > y)dy &= \int_{\mathbb{R}_{\geq 0}} \int_{(y,\infty)} d\mathbb{P}_X(x)dy && (6.1)\\
&= \int_{\mathbb{R}_{\geq 0}} \int_{[0,x)} dyd\mathbb{P}_X(x) && (6.2)\\
&= \int_{\mathbb{R}_{\geq 0}} xd\mathbb{P}_X(x) && (6.3)\\
&= \mathbb{E}(X). && (6.4)
\end{aligned}$$

We can do the same thing in two dimensions. Let X, Y be nonnegative random variables with probability measure $\mathbb{P}_{X,Y}$ and let λ^2 be the two-dimensional lesbesgue measure.

Then by switching order of integration we get:

$$\int_{\mathbb{R}^2_{\geq 0}} \mathbb{P}(X > z,\, Y > w) d\lambda^2(z,w) = \int_{\mathbb{R}^2_{\geq 0}} \int_{(z,\infty)\times(w,\infty)} d\mathbb{P}_{X,Y}(x,y) d\lambda^2(z,w) \tag{6.5}$$

$$= \int_{\mathbb{R}^2_{\geq 0}} \int_{[0,x)\times[0,y)} d\lambda^2(z,w) d\mathbb{P}_{X,Y}(x,y) \tag{6.6}$$

$$= \int_{\mathbb{R}^2_{\geq 0}} xy d\mathbb{P}_{X,Y}(x,y) \tag{6.7}$$

$$= \mathbb{E}(XY). \tag{6.8}$$

6.2.2. A trivial bound

Consider two events A, B then

$$\mathbb{P}(A \cap B) \leq \min(\mathbb{P}(A), \mathbb{P}(B)).$$

We will be using this bound along with the fact that given a nondecreasing function f and events $A := \{X > z\}$ and $B := \{f(X) > w\}$ we have either $A \subseteq B$ or $B \subseteq A$, so $\mathbb{P}(A \cap B) = \min(\mathbb{P}(A), \mathbb{P}(B))$.

6.2.3. Conditional Expectation

Suppose we have a function $f(a_1, a_2)$ in two arguments (which might themselves be any type of object) and two independent random variables (objects) R_1, R_2. We can then define $E_1(a_2) := \mathbb{E}f(R_1, a_2)$ (assuming this expectation exists for all a_2). The following identities hold whenever $\mathbb{E}f(R_1, R_2)$ exists:

$$\mathbb{E}(f(R_1, R_2)|R_2) = E_1(R_2)\, a.s.,$$

$$\mathbb{E}f(R_1, R_2) = \mathbb{E}\mathbb{E}(f(R_1, R_2)|R_2) = \mathbb{E}E_1(R_2).$$

6.3. The Result

Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be nondecreasing. Let X, Y be nonnegative random variables with the same marginal distribution. We then have

$$\begin{aligned}
\mathbb{E}(Yf(X)) &= \int_{\mathbb{R}^2_{\geq 0}} \mathbb{P}(Y > z, f(X) > w) d\lambda^2(z,w) && (6.9)\\
&\leq \int_{\mathbb{R}^2_{\geq 0}} \min\left(\mathbb{P}(Y > z), \mathbb{P}(f(X) > w)\right) d\lambda^2(z,w) && (6.10)\\
&= \int_{\mathbb{R}^2_{\geq 0}} \min\left(\mathbb{P}(X > z), \mathbb{P}(f(X) > w)\right) d\lambda^2(z,w) && (6.11)\\
&= \int_{\mathbb{R}^2_{\geq 0}} \mathbb{P}(X > z, f(X) > w) d\lambda^2(z,w) && (6.12)\\
&= \mathbb{E}(Xf(X)). && (6.13)
\end{aligned}$$

Now consider a random object Z independent from (X, Y) and a function $f(a_1, a_2)$ with $a_1 \in \mathbb{R}$ that is nondecreasing in the first argument. Define $E_x(a_2) := \mathbb{E}(Xf(X, a_2))$, $E_y(a_2) := \mathbb{E}(Yf(X, a_2))$. Then for all a_2 we have

$$E_y(a_2) \leq E_x(a_2),$$

so we get

$$\begin{aligned}
\mathbb{E}(Yf(X,Z)) &= \mathbb{E}E_y(Z) && (6.14)\\
&\leq \mathbb{E}E_x(Z) && (6.15)\\
&= \mathbb{E}(Xf(X,Z)). && (6.16)
\end{aligned}$$

This is the version of the rearrangement principle we will use in the next chapter to prove convexity of the correlation function.

7. Convexity of the Correlation Function

Throughout this chapter we use some shorthand notation. The increment of the Lévy process from time s to time $t \geq s$ is

$$X_{s,t} := X_t - X_s,$$

so $X_{s,t}$ has the same distribtuion as $X_{0,t-s} = X_{t-s}$ by stationarity and because of independence we have for any s: $(X_u)_{0 \leq u \leq s}$ independent from $(X_{s,u})_{s \leq u}$.
We also use the shorthand L_t^0 for the amount of reflection up to time t in a queue driven by X and starting in 0:

$$L_t^0 := -\inf_{0 \leq s \leq t} X_s.$$

This is convenient because it is independent from Q_0 and the actual amount of reflection until time t is simply $L_t = \max(0, L_t^0 - Q_0)$. Similarly we have the amount of reflection between time s and time $t \geq s$ when $Q_s = 0$

$$L_{s,t}^0 := -\inf_{s \leq u \leq t} X_{s,t},$$

which is independent from $Q_0, (X_u)_{0 \leq u \leq s}$ (and therefore $(Q_u)_{0 \leq u \leq s}$) and the actual amount of reflection between s and t is simply $L_t - L_s = \max(0, L_{s,t}^0 - Q_s)$.

7.1. Introduction

Let X be a Lévy process. We assume the initial workload Q_0 is independent from X and has the stationary distribution (the same distribution as $\sup_{0 \leq t} X_t$). The workload at time t is

$$Q_t = X_t + \max(Q_0, L_t^0).$$

Because Q_0 has the stationary distribution, we know that Q_t does as well. We want to show that the correlation function

$$r_t := \frac{\mathbb{E}(Q_0 Q_t) - \mathbb{E}Q_0 \mathbb{E}Q_t}{\sqrt{Var(Q_0) Var(Q_t)}}$$

is positive, nonincreasing and convex. Since the marginal distribution of Q_t is invariant, so are $\mathbb{E}Q_t$ and $Var(Q_t)$. It therefore suffices to show that $\mathbb{E}(Q_0 Q_t)$ is nonincreasing and convex and $\mathbb{E}(Q_0 Q_t) \geq \mathbb{E}Q_0 \mathbb{E}Q_t$. We will prove all three properties using the rearrangement principle.

7.2. Nonnegativity

Let Q_0^* be independent from Q_0, X. In this case we take

$$f(q,(x,l)) := x + \max(q,l).$$

It is clear that f is increasing in q. We use the following identity:

$$f(Q_0,(X_t,-\inf_{0\le s\le t} X_s)) = X_t + \max(Q_0, L_t^0) = Q_t.$$

We have (Q_0, Q_0^*) with identical marginal distributions independent from (X_t, L_t^0) so the rearrangement principle applies, so

$$\begin{aligned}
\mathbb{E}(Q_0Q_t) &= \mathbb{E}(Q_0 f(Q_0,(X_t,L_t^0))) && (7.1)\\
&\ge \mathbb{E}(Q_0^* f(Q_0,(X_t,L_t^0))) && (7.2)\\
&= \mathbb{E}(Q_0^*Q_t) = \mathbb{E}Q_0^*\mathbb{E}Q_t = \mathbb{E}Q_0\mathbb{E}Q_t. && (7.3)
\end{aligned}$$

7.3. Monontonicity

Again, we take

$$f(q,(x,l)) = x + \max(q,l),$$

however, this time we use

$$f(Q_s,(X_{s,t},L_{s,t}^0)) = X_{s,t} + \max(Q_s, L_{s,t}^0) = Q_t.$$

Again, f is nonincreasing in q and we have (Q_s, Q_0) with identical marginal distributions independent from $(X_{s,t}, L_{s,t}^0)$ so the rearrangement principle applies, so

$$\begin{aligned}
\mathbb{E}(Q_sQ_t) &= \mathbb{E}(Q_s f(Q_s,(X_{s,t},L_{s,t}^0))) && (7.4)\\
&\ge \mathbb{E}(Q_0 f(Q_s,(X_{s,t},L_{s,t}^0))) && (7.5)\\
&= \mathbb{E}(Q_0Q_t). && (7.6)
\end{aligned}$$

By stationarity we have

$$\mathbb{E}(Q_0Q_{t-s}) = \mathbb{E}(Q_sQ_t) \ge \mathbb{E}(Q_0Q_t).$$

This makes $\mathbb{E}(Q_0Q_t)$ nonincreasing.

7.4. Convexity

To show convexity we show for $0 \le s \le t \le u$ that increments over a time period of fixed length are at least as high as earlier increments of that length:

$$r_u - r_t \ge r_{u-s} - r_{t-s},$$

or equivalently

$$\mathbb{E}(Q_0(Q_u - Q_t)) \geq \mathbb{E}(Q_0(Q_{u-s} - Q_{t-s})) = \mathbb{E}(Q_s(Q_u - Q_t)).$$

We will use that Q_t is nondecreasing in Q_s and $Q_u - Q_t$ is nonincreasing in Q_t. We have

$$Q_t = X_{s,t} + \max(Q_s, L^0_{s,t}),$$

$$Q_u - Q_t = X_{t,u} + \max(Q_t, L^0_{t,u}) - Q_t = X_{t,u} + \max(0, L^0_{t,u} - Q_t).$$

We take

$$f(q_1, (x_1, l_1)) = x_1 + \max(q, l_1),$$
$$g(q_2, (x_2, l_2)) := x_2 + \max(0, l_2 - q_2)$$

and compose these to get

$$\begin{aligned} h(q_1, (x_1, l_1, x_2, l_2)) &= g(f(q_1, (x_1, l_1)), (x_2, l_2)) && (7.7)\\ &= x_2 + \max(0, l_2 - (x_1 + \max(q_1, l_1))). && (7.8) \end{aligned}$$

We can see that h is nonincreasing in q_1. Now substitute

$$\begin{aligned} h(Q_s, (X_{s,t}, L^0_{s,t}, X_{t,u}, L^0_{t,u})) &= g(f(Q_s, (X_{s,t}, L^0_{s,t})), (X_{t,u}, L^0_{t,u})) && (7.9)\\ &= g(Q_t, (X_{t,u}, L^0_{t,u})) && (7.10)\\ &= Q_u - Q_t. && (7.11) \end{aligned}$$

We have (Q_s, Q_0) with identical marginal distributions independent from $(X_{s,t}, L^0_{s,t}, X_{t,u}, L^0_{t,u})$ so the rearrangement principle applies (but since h is nonincreasing in q_1 instead of nondecreasing the inequality is reversed), so

$$\begin{aligned} \mathbb{E}(Q_s(Q_u - Q_t)) &= \mathbb{E}(Q_s h(Q_s, (X_{s,t}, L^0_{s,t}, X_{t,u}, L^0_{t,u}))) && (7.12)\\ &\leq \mathbb{E}(Q_0 h(Q_s, (X_{s,t}, L^0_{s,t}, X_{t,u}, L^0_{t,u}))) && (7.13)\\ &= \mathbb{E}(Q_0(Q_u - Q_t)). && (7.14) \end{aligned}$$

This concludes the proof.

7.5. Convexity for the G/G/1 Queue

The workload Q^a_n of the $G/G/1$ queue seen by the $n - th$ arrival satisfies Lindley's equation:

$$Q^a_{n+1} = \max(Q^a_n + B_n - A_n, 0),$$

where B_n, A_n are the size of the n-th job and the time between the n-th and $n+1$-th arrival respectively. Taking

$$X^a_n := \sum_{i=1}^{n} (B_{i-1} - A_{i-1}),$$

we get for $m \leq n$:

$$Q_n^a = X_n^a + \max(Q_m^a, -\inf_{m \leq k \leq n}(X_k^a - X_m^a)),$$

so the reasoning for the workload of a Lévy process also applies to the workload at arrival times of the $G/G/1$ queue, so the correlation function (sequence)

$$r_n^a := \frac{\mathbb{E}(Q_0^a Q_n^a) - \mathbb{E}Q_0^a \mathbb{E}Q_n^a}{\sqrt{Var(Q_0^a)Var(Q_n^a)}}$$

with Q_0^a having the stationary distribution is also nonnegative, nonincreasing and convex.

8. Two-Sided Skorokhod on Coupled Queues

8.1. Finite Buffer Lévy Queues

The Lévy analogue of a finite buffer queue is a Lévy process that is not only reflected in 0, but also in the level of the capacity a. In the two sided Skorokhod problem we are given a càdlàg $f : [0,T] \to \mathbb{R}$ with $f(0) \in [0,a]$ and we want to find càdlàg nondecreasing g_0, g_a with $g_0(0) = g_a(0) = 0$ such that $h := f + g_0 - g_a \in [0,a]$ and

$$\int_0^t f(s)dg_0(s) = \int_0^t (a - f(s))dg_a(s) = 0.$$

It has been shown in [7] that the unique solution to this problem is

$$h = \Gamma_{0,a}[f] := \Lambda_{0,a} \circ \Gamma_0[f],$$

where

$$(\Lambda_{0,a}f)(t) := f(t) - \sup_{0 \le s \le t} \min\left((f(s) - a)^+, \inf_{s \le u \le t} f(u)\right).$$

Suppose h is a solution on $[0,T]$ and $0 \le S < T$, then h is a solution for f on $[0,S]$ and for $f + g_0(S) - g_a(S)$ on $[S,T]$. In the coming sections we will use monotonicity arguments to show that, given X, Q_t is still nondecreasing in Q_0 and $Q_t - Q_0$ is still nonincreasing in Q_0.

8.2. Weak Preservation of Workload Order for Coupled Queues

Suppose we take $x_1 \le x_2$ and $f_i := x_i + X$ and $h_i = \Gamma_{0,a}[f_i]$. This represents two queues that have different initial workloads, that are driven by the same Lévy process. We want to prove that $h_1 \le h_2$. Let

$$\tau := \inf\{0 \le s \le T : h_1 \ge h_2\}.$$

If this infimum does not exist we have $h_1 \le h_2$ and we are done. Assume the infimum exists, then by càdlàg we have that the left and right limits are in opposite order:

$h_1(\tau^-) \leq h_2(\tau^-)$ and $h_1(\tau) \geq h_2(\tau)$. So if this is a continuity point for h_1, h_2, then $h_1(\tau) = h_2\tau$. If it is not a continuity point for both, then $h_1 - h_2$ jumps up. If $h_1 - h_2$ jumps up, then either $h_1(\tau) = 0$ or $h_2(\tau) = a$, both imply $h_1(\tau) \leq h_2(\tau)$, so in this case we also have $h_1(\tau) = h_2(\tau)$. We then have uniqueness on the rest of the interval, so $h_1(s) = h_2(s)$ for $\tau \leq s \leq T$. So on the whole interval we have $h_1 \leq h_2$.

Since we have $Q := \Gamma_{[0,a]}(Q_0 + X)$ this shows that Q_t given X is nondecreasing in Q_0.

8.3. Decreasing Differences

We want to show that $\Gamma_{0,a}[x + f](t) - \Gamma_{0,a}[x + f](0)$ is decreasing in x. We do this by showing both Γ_0 and $\Lambda_{0,a}$ bring ordered processes $f_1 \leq f_2$ closer together. Let $f_1 \leq f_2$, then

$$\Gamma_0 f_2(t) - f_2(t) = \left(\sup_{0\leq s\leq t} -f_2(s)\right)^+ \leq \left(\sup_{0\leq s\leq t} f_1(s)\right)^+ = \Gamma_0 f_1(t) - f_1(t),$$

so we have

$$\Gamma_0 f_2(t) - \Gamma_0 f_1(t) \leq f_2(t) - f_1(t).$$

Similarly, we have for $f_1 \leq f_2$:

$$\Lambda_{0,a} f_2(t) - f_2(t) = - \sup_{0\leq s\leq t} \min\left((f_2(s) - a)^+, \inf_{s\leq u\leq t} f_2(u)\right) \leq$$

$$- \sup_{0\leq s\leq t} \min\left((f_1(s) - a)^+, \inf_{s\leq u\leq t} f_1(u)\right) = \Lambda_{0,a} f_1(t) - f_1(t),$$

so we have

$$\Lambda_{0,a} f_2(t) - \Lambda_{0,a} f_1(t) \leq f_2(t) - f_1(t).$$

Let $x \leq y$ be two possible starting values, then $x + X \leq y + X$ (on the whole interval). We have $\Gamma_0(x + X) \leq \Gamma_0(y + X)$ as well, so:

$$\Gamma_{[0,a]}(y + X) - \Gamma_{[0,a]}(x + X) = \Lambda_{0,a}\Gamma_0(y + X) - \Lambda_{0,a}\Gamma_0(x + X) \leq$$
$$\Gamma_0(y + X) - \Gamma_0(x + X) \leq y + X - (x + X) = y - x$$

on the whole interval, so

$$\Gamma_{[0,a]}(y + X) - y \leq \Gamma_{[0,a]}(x + X) - x$$

on the whole interval, so specifically at time t. So $Q_t - Q_0$ given X is nonincreasing in Q_0. We can shift this property in time and also conclude that $Q_u - Q_t$ given $X_{t,u}$ is non-increasing in Q_t. We can combine this with the fact that Q_t given $X_{0,t}$ is nondecreasing in Q_0 (or Q_s with $s \leq t$) to get that $Q_u - Q_t$ given X is nonincreasing in Q_0 (or Q_s with $s \leq t$).

9. Convexity for Finite Buffer Lévy Queues

9.1. Introduction

The reasoning for nonnegativity, monotonicity, and convexity of the correlation function of a finite buffer queue with capacity a is the same as with the regular Lévy queue, though the details are different. The independent random object in the rearrangement principle will now be the stochastic process

$$v \mapsto X_{s+v} - X_s = X_{s,s+v}.$$

Let X be a Lévy process. We assume the initial workload Q_0 is independent from X and has the stationary distribution. The workload at time t is

$$Q_t = \Gamma_{0,a}[Q_0 + (X_v)_{v\geq 0}](t),$$

or when seen from the starting point at time $s \leq t$:

$$Q_t = \Gamma_{0,a}[Q_s + (X_{s+v} - X_s)_{v\geq 0}](t-s).$$

Because Q_0 has the stationary distribution, we know that Q_t does as well. We want to show that the correlation function

$$r_t := \frac{\mathbb{E}(Q_0 Q_t) - \mathbb{E}Q_0\mathbb{E}Q_t}{\sqrt{Var(Q_0)Var(Q_t)}}$$

is positive, nonincreasing and convex. Since the marginal distribution of Q_t is invariant, so are $\mathbb{E}Q_t$ and $Var(Q_t)$. It therefore suffices to show that $\mathbb{E}(Q_0 Q_t)$ is nonincreasing and convex and $\mathbb{E}(Q_0 Q_t) \geq \mathbb{E}Q_0\mathbb{E}Q_t$. We will prove all three properties using the rearrangement principle.

9.2. Nonnegativity

Let Q_0^* be independent from Q_0, X. In this case we take

$$f(q, (x_v)_{v\geq 0}) := \Gamma_{0,a}[q + (x_v)_{v\geq 0}](t),$$

we have seen that f is increasing in q. We use the following identity:

$$f(Q_0, (X_v)_{v\geq 0}) = \Gamma_{0,a}[Q_0 + (X_v)_{u\geq v}](t) = Q_t.$$

We have (Q_0, Q_0^*) with identical marginal distributions independent from $(X_v)_{v\geq 0})$ so the rearrangement principle applies, so

$$\mathbb{E}(Q_0 Q_t) = \mathbb{E}(Q_0 f(Q_0, (X_v)_{v\geq 0})) \tag{9.1}$$
$$\geq \mathbb{E}(Q_0^* f(Q_0, (X_v)_{v\geq 0}))) \tag{9.2}$$
$$= \mathbb{E}(Q_0^* Q_t) = \mathbb{E}Q_0^* \mathbb{E}Q_t = \mathbb{E}Q_0 \mathbb{E}Q_t. \tag{9.3}$$

9.3. Monotonicity

Again, we take

$$f(q, (x_v)_{v\geq 0}) := \Gamma_{0,a}[q + (x_v)_{v\geq 0}](t),$$

however, this time we use

$$f(Q_s, (X_{s,s+v})v \geq 0) = \Gamma_{0,a}[Q_s + (X_{s+v} - X_s)_{v\geq 0}](t-s) = Q_t.$$

Again, f is nonincreasing in q and we have (Q_s, Q_0) with identical marginal distributions independent from $(X_{s,s+v})_{v\geq 0}$ so the rearrangement principle applies, so

$$\mathbb{E}(Q_s Q_t) = \mathbb{E}(Q_s f(Q_s, (X_{s,s+v})_{v\geq 0})) \tag{9.4}$$
$$\geq \mathbb{E}(Q_0 f(Q_s, , (X_{s,s+v})_{v\geq 0})) \tag{9.5}$$
$$= \mathbb{E}(Q_0 Q_t). \tag{9.6}$$

By stationarity we have

$$\mathbb{E}(Q_0 Q_{t-s}) = \mathbb{E}(Q_s Q_t) \geq \mathbb{E}(Q_0 Q_t).$$

This makes $\mathbb{E}(Q_0 Q_t)$ nonincreasing.

9.4. Convexity

To show convexity we show for $0 \leq s \leq t \leq u$ that increments over a time period of fixed length are at least as high as earlier increments of that length:

$$r_u - r_t \geq r_{u-s} - r_{t-s},$$

or equivalently

$$\mathbb{E}(Q_0(Q_u - Q_t)) \geq \mathbb{E}(Q_0(Q_{u-s} - Q_{t-s})) = \mathbb{E}(Q_s(Q_u - Q_t)).$$

We will use that Q_t is nondecreasing in Q_s and $Q_u - Q_t$ is nonincreasing in Q_t. We have

$$Q_t = \Gamma_{0,a}[Q_s + (X_{s,s+v})_{v\geq 0}](t-s),$$

$$Q_u - Q_t = \Gamma_{0,a}[Q_s + (X_{t,t+v})_{v\geq 0}](u-t) - Q_t.$$

We take

$$f(q_1, (x^1_v)_{v\geq 0}) := \Gamma_{0,a}[q_1 + (x^1_v)_{v\geq 0}](t-s),$$
$$g(q_2, (x^2_v)_{v\geq 0}) := \Gamma_{0,a}[q_2 + (x^2_v)_{v\geq 0}](u-t),$$

and compose these to get

$$h(q_1, ((x^1_v)_{v\geq 0}, (x^2_v)_{v\geq 0})) = g(f(q_1, (x^1_v)_{v\geq 0}), (x^2_v)_{v\geq 0}). \tag{9.7}$$

We can see that h is nonincreasing in q_1 because g is nonincreasing in q_2 and f is nondecreasing in q_1. Now substitute

$$\begin{aligned} h(Q_s, ((X_{s,s+v})_{v\geq 0}, (X_{t,t+v})_{v\geq 0})) &= g(f(Q_s, (X_{s,s+v})_{v\geq 0}), (X_{t,t+v})_{v\geq 0}) && (9.8)\\ &= g(Q_t, (X_{t,t+v})_{v\geq 0}) && (9.9)\\ &= Q_u - Q_t. && (9.10)\end{aligned}$$

We have (Q_s, Q_0) with identical marginal distributions independent from $((X_{s,s+v})_{v\geq 0}, (X_{t,t+v})_{v\geq 0})$, so the rearrangement principle applies (but since h is nonincreasing in q_1 instead of nondecreasing the inequality is reversed), so

$$\begin{aligned} \mathbb{E}(Q_s(Q_u - Q_t)) &= \mathbb{E}(Q_s h(Q_s, ((X_{s,s+v})_{v\geq 0}, (X_{t,t+v})_{v\geq 0})) && (9.11)\\ &\leq \mathbb{E}(Q_0 h(Q_s, ((X_{s,s+v})_{v\geq 0}, (X_{t,t+v})_{v\geq 0})) && (9.12)\\ &= \mathbb{E}(Q_0(Q_u - Q_t)). && (9.13)\end{aligned}$$

This concludes the proof.

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A. Finiteness of the Supremum of a Lévy Process with Negative Expectation

Let $\mathbb{E}X_1 < 0$. We want to show that $\sup_{0\leq s} X_s < \infty$ a.s.. We will do this by finding a (random) time τ such that $X_t \leq 0$ for $t \geq \tau$. We then have

$$\sup_{0\leq s} X_s = \sup_{0\leq s\leq \tau} X_s < \infty,$$

since the supremum of any càdlàg function on a bounded interval is finite.

By the law of large numbers we have a.s. $X_n/n \to \mathbb{E}X_1$ for integer $n \to \infty$, so there is some N such that for all $n \geq N$ we have $X_n \leq n\mathbb{E}X_1/2$. So if the event $A_n := \{\sup_{n\leq s\leq n+1}(X_s - X_n) \geq -n\mathbb{E}X_1/2\}$ happens for only finitely many n, then there is some $N' \geq N$ such that A_n does not happen for $n \geq N'$, so for those n we have $\sup_{n\leq s\leq n+1} X_s < 0$ and therefore $\sup_{N'\leq s} X_s \leq 0$, therefore $\tau := N'$ has the desired property. We will show that $\mathbb{E}\sum_{n=1}^{\infty} 1_{A_n} < \infty$, which means $\mathbb{P}(\sum_{n=1}^{\infty} 1_{A_n} = \infty) = 0$ and then we are done.

We now consider the probability that $\sup_{n\leq s\leq n+1} X_s > 0$ for $n \geq N$ we have

$$\sup_{n\leq s\leq n+1} X_s \leq n\mathbb{E}X_1/2 + \sup_{n\leq s\leq n+1} (X_s - X_n).$$

The latter has the same distribution as $n\mathbb{E}X_1/2 + \sup_{0\leq s\leq 1} X_s$. To bound the probability of this being ≥ 0, we split X_t up in four parts $X_t = J_t^+ + J_t^- + M_t + dt$. For X_t to be at least $-n\mathbb{E}X_1/2$ we need at east one of the four parts to be at least $-n\mathbb{E}X_1/8$. The first part consists of all jumps of size at least 1 up, the second of all jumps of size at least 1 down and M_t the martingale that consists of the brownian motion part and the small jumps with compensation.

In terms of the characteristic functon we have

$$\frac{1}{t}\log \mathbb{E}e^{isX_t} = isd - \sigma^2 s^2/2 + \int_{-\infty}^{\infty} e^{isx} - 1 - isx1_{|x|<1}\Pi(dx), \tag{A.1}$$

$$\frac{1}{t}\log \mathbb{E}e^{isJ_t^+} = \int_{[1,\infty)} e^{isx} - 1 - isx1_{|x|<1}\Pi(dx), \tag{A.2}$$

$$\frac{1}{t}\log \mathbb{E}e^{isJ_t^-} = \int_{(-\infty,1]} e^{isx} - 1 - isx1_{|x|<1}\Pi(dx), \tag{A.3}$$

$$\frac{1}{t}\log \mathbb{E}e^{isM_t} = -\sigma^2 s^2/2 + \int_{(-1,1)} e^{isx} - 1 - isx1_{|x|<1}\Pi(dx), \tag{A.4}$$

$$\frac{1}{t}\log \mathbb{E}e^{isdt} = isd. \tag{A.5}$$

Let $C > 0$, we will find bounds on the expected number of times any of the parts are at least nC. We know J_t is nondecreasing, so the supremum is J_1^+, which has expectation $\int_{[1,\infty)} x\Pi(dx)$, which is finite, because else $\mathbb{E}X_1$ would not exist, this gives us

$$\sum_{n=1}^{\infty} \mathbb{P}(\sup_{0\le s\le 1} J_s^+ \ge nC) = \sum_{n=1}^{\infty} \mathbb{P}(J_1^+/C \ge n) \le \mathbb{E}J_1^+/C = \int_{[1,\infty)} x\Pi(dx)/C < \infty.$$

As J_t^- is nonincreasing the supremum is 0. M_t is a martingale, which makes M_t^2 a submartingale, so by Doob's martingale inequality we have

$$\mathbb{P}(\sup_{0\le s\le 1} M_s \ge nC) \le \mathbb{P}(\sup_{0\le s\le 1} M_s^2 \ge n^2C^2) \le \mathbb{E}M_1^2/(n^2C^2).$$

Summing over n then gives us

$$\sum_{n=1}^{\infty} \mathbb{P}(\sup_{0\le s\le 1} M_s \ge nC) \le \sum_{n=1}^{\infty} \mathbb{E}M_1^2/(n^2C^2) = \pi^2\mathbb{E}M_1^2/(6C^2) < \infty$$

finally $\sup_{0\le s\le 1} ds = \max(0,d) \ge nC$ for finitely many integers $n \ge 0$. Now let $C = -\mathbb{E}X_1/8$, then the expected number of times any of the four parts exceeds $nC = -n\mathbb{E}X_1/8$ is finite, so with probability 1 it only happens finitely many times, so there is some m such that for $n \ge m$ we have

$$\sup_{n\le s\le n+1} X_s = X_n + \sup_{n\le s\le n+1}(X_s - X_n) \le n\mathbb{E}X_1/2 - n\mathbb{E}X_1/8 - n\mathbb{E}X_1/8 - n\mathbb{E}X_1/8 - n\mathbb{E}X_1/8 = 0.$$

Now take $\tau = m$, then we have

$$\sup_{0\le s} X_s = \sup_{0\le s\le \tau} X_s < \infty.$$