MSc Mathematics

Track: Algebra and Geometry

Master thesis

Homology of hyperplane sections

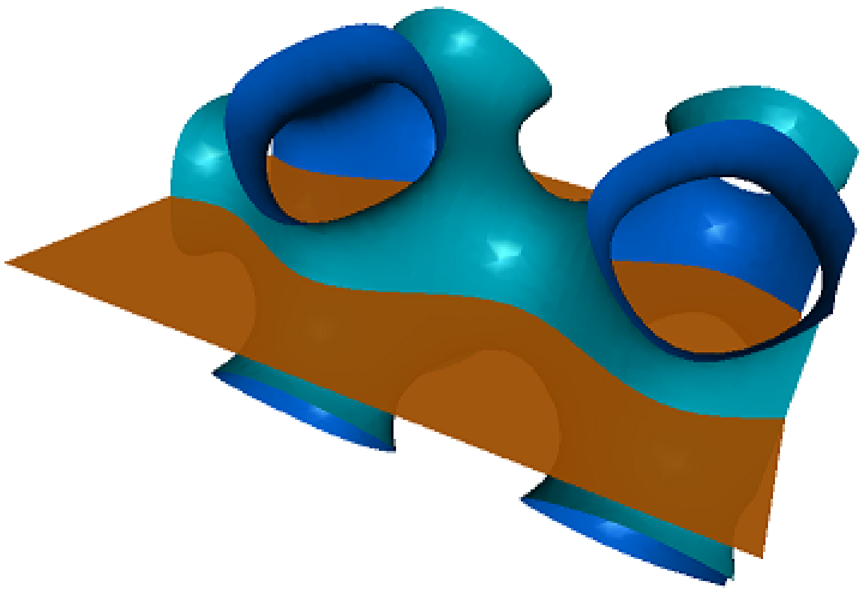
by

Francisco Hernández Iglesias

June 15, 2018

Supervisor: dr. Mingmin Shen

Second examiner: dr. Federica Pasquotto



Department of Mathematics

Faculty of Sciences

VU UNIVERSITY AMSTERDAM

Abstract

The purpose of this thesis is to study the homology of transversal hyperplane sections X_b of a complex projective non-singular variety X through Lefschetz's geometric arguments.

The most important result is Lefschetz's theorem on the homology of hyperplane sections: "The inclusion $X_b \hookrightarrow X$ induces isomorphisms in homology $H_q(X_b) \xrightarrow{\sim} H_q(X)$ for all $q \leq n-2$ and an epimorphism $H_{n-1}(X_b) \twoheadrightarrow H_{n-1}(X)$, where $n = \dim X$". Combined with Poincaré duality, it effectively computes all homology groups of X_b except for the middle one, provided the homology groups of X are known.

The rest of the thesis studies the middle homology $H_{n-1}(X_b)$. Two different approaches are taken: firstly the study of the module of vanishing cycles, which represents the extra structure of $H_{n-1}(X_b)$ with respect to $H_{n-1}(X)$, and secondly the definition of a monodromy action on $H_{n-1}(X_b)$.

Title: Homology of hyperplane sections
Author: Francisco Hernández Iglesias, f.hernandeziglesias@outlook.com, 2593811
Supervisor: dr. Mingmin Shen
Second examiner: dr. Federica Pasquotto
Date: June 15, 2018

Department of Mathematics
VU University Amsterdam
de Boelelaan 1081, 1081 HV Amsterdam
`http://www.math.vu.nl/`

Contents

1. Introduction

1.1. Popular summary

What is topology? The fundamental idea of topology is to generalize the notion of distance. Thus, in a space endowed with a topology it can be distinguished whether two points are "far" or "close". Topology is the most basic mathematical structure in which the notions of continuity, compactness and connectedness can be rigorously defined.

What is algebraic topology? Algebraic topology assigns algebraic invariants to topological spaces to obtain information about them. This thesis works with some algebraic invariants called the *homology groups*. For n a non-negative integer, the n-th homology group of a topological space is the group of n-simplexes (i.e., n-dimensional triangles) wrapping the space such that they cannot be collapsed continuously to an $(n-1)$-simplex. For example, the second homology group would be the group of triangles wrapping the space and meeting in a line such that they cannot be collapsed to that line. All the algebraic topology requirements to understand this thesis can be found in [7].

What is algebraic geometry? Algebraic geometry is the study of algebraic varieties. Roughly speaking, an algebraic variety is the geometric locus given by the solution to a system of polynomial equations. Therefore, studying the intrinsic geometric properties of such loci will translate to an improved understanding of the properties of systems of polynomial equations. All the algebraic geometry requirements to understand this thesis can be found in [8].

What is this thesis about? Assume the homology groups of a certain complex algebraic variety, i.e., the locus determined by a system of polynomial equations with complex coefficients, are known. The goal of this thesis is to study the homology groups of its transversal hyperplane sections, i.e., the locus determined by the same system of equations together with an additional linear equation. In Chapter 2 all homology groups except for the middle one are computed thanks to Lefschetz's theorem on the homology of hyperplane sections. Chapters 3 and 4 analyze in depth the structure of the unknown middle homology group.

1.2. Scientific introduction

Computing the homology groups of complex projective varieties is one of the most basic problems in differential topology. Not only is this interesting from a purely topological point of view, but also from an algebraic one, since topological and algebraic invariants are related. A well-known example of such relations is the Hodge decomposition.

This thesis aims to answer the following question: given a smooth, irreducible, projective variety $X \subset \mathbb{P}^N$, and a transversal hyperplane section X_b, describe its homology groups $H_q(X_b)$. For this purpose, the theoretical machinery developed by Klaus Lamotke in [1] will be the main tool. It is historically important to acknowledge that Lamotke's article is not an exposition of genuine concepts, but rather a new explanation using modern homology theory of the geometric ideas originally introduced by Solomon Lefschetz in 1924, when he published the proof of his famous hyperplane theorem. Moreover, Lamotke's approach to translate Lefschetz's concepts into homology theory is not the only one. For instance, Aldo Andreotti and Theodore Frankel prove Lefschetz's hyperplane theorem in [2] resorting to real Morse theory (see [6]), which is arguably a more straightforward and insightful proof than the one presented here. Nevertheless, the tools they employ are not easily adaptable to effectively study the vanishing cycles or derive the Picard-Lefschetz's formula, hence the preference for Lamotke's methods in this thesis.

The central concept presented in [1] is the *modification*, a blow-up of X along a codimension 2 subvariety X' resulting from the intersection of X with an $(N-2)$-dimensional projective subspace $A \subset \mathbb{P}^N$. If Y is the modification of X, there exists a surjective morphism $f : Y \to \mathbb{P}^1$ with the following property: the fibre over a point $Y_b = f^{-1}(\{b\})$ is isomorphic to a section X_b of X given by an hyperplane containing A. Therefore, the study of smooth fibres of f is equivalent to the study of smooth hyperplane sections of X.

It can be shown that the modification Y is irreducible and non-singular, and the map f has a finite number of non-degenerate critical points, outside of which is a fibration. The study of the topology of spaces endowed with such holomorphic functions is known as Picard-Lefschetz theory, the holomorphic analogue of Morse theory. As references for many computations done with Picard-Lefschetz theory in Chapters 2 and especially 4, refer to [3] (in French), [4] and [5]. In those references it can be appreciated that Picard-Lefschetz theory is a very powerful tool that goes beyond the scope of the topics discussed in this thesis.

The underlying idea behind most of the proofs in this text is to perform the computations in the modification Y, where an explicit holomorphic coordinate description of f in a neighbourhood around the critical points is given by Morse Lemma for complex manifolds. Firstly, the local computations are done using these descriptions. Secondly, global conclusions for Y are drawn: roughly speaking, only critical points contribute to homology. Finally, the results found for Y and its fibres Y_b are translated back into X and its hyperplane sections X_b.

Chapter 2 aims for the first application of Picard-Lefschetz theory, namely the proof of Lefschetz's theorem on the homology of hyperplane sections: "$H_q(X, X_b) = 0$ for all $q \leq n-1$, where $n = \dim X$". If the homology groups of X are known, this theorem and Poincaré duality can compute all homology groups of X_b except $H_{n-1}(X_b)$. Several applications of this result are explored, among which the theorem on the homology of hypersurface sections stands out: "$H_q(X, X_F) = 0$ for all $q \leq n-1$, where $X_F \subset X$ is a smooth transversal hypersurface section of X". Its most important corollary gives a recipe to compute all homology groups of smooth complete intersections except for the middle one. It is also interesting to observe that the corresponding result for relative cohomology also holds, via the universal coefficient formula: "$H^q(X, X_b) = 0$ for all $q \leq n-1$".

As a consequence of Lefschetz's hyperplane theorem, the inclusion $X_b \hookrightarrow X$ induces a surjective homomorphism $H_{n-1}(X_b) \twoheadrightarrow H_{n-1}(X)$ with non-trivial kernel, in general. This kernel $V \in H_{n-1}(X_b)$ is the *module of vanishing cycles*, and its study is done in Chapter 3. The generators of V reflect the local behaviour of f around its critical points, and are called the vanishing cycles $\delta_1, \dots, \delta_r$. Analogously, the image I^* of the injective map $H^{n-1}(X) \hookrightarrow H^{n-1}(X_b)$ is called the *module of invariant cocycles*, and its Poincaré dual $I \subset H_{n-1}(X_b)$ is the *module of invariant cycles.* The two main results from this chapter are a description of I as the orthogonal complement of V with respect to the Kronecker pairing, and the rank formula for the middle homology: "rk $H_{n-1}(X_b) = \text{rk } V + \text{rk } I$".

In Chapter 4, a monodromy action of some fundamental group π on the homology groups of the fibres $H_q(Y_b)$ (and therefore on $H_q(X_b)$) is defined and studied. Here $\pi = \pi_1(\mathbb{P}^1 \setminus \{t_1, \dots, t_r\})$, where $t_1, \dots, t_r$ are the critical values of f. The definition of such an action is purely topological, as it can be seen in Appendix B. In the first place, the generators and relations of π are computed, namely it is generated by the *elementary paths* $w_1, \dots, w_r$, i.e., loops encircling the critical values. Computing the action of the elementary paths yields the Picard-Lefschetz's formula: "w_i acts trivially on $H_q(Y_b)$ for $q \neq n-1$. The action on an element $x \in H_{n-1}(Y_b)$ is given by $(w_i)_*(x) = x + (-1)^{n(n+1)/2}\langle x, \delta_i \rangle \delta_i$, where δ_i is the corresponding vanishing cycle and $\langle \cdot, \cdot \rangle$ denotes the Kronecker pairing". A very immediate conclusion is that the module of invariant cycles I consists precisely of the invariant elements under the action of π, since they are the orthogonal complement of the vanishing cycles with respect to the Kronecker pairing.

Further questions closely related to what is covered here are the Hard Lefschetz theorem, the monodromy theorem (see [1]) and an analogous result for relative homotopy of hyperplane sections (see [6]).

2. Lefschetz's Theorem

2.1. Dual variety

Let $X \subset \mathbb{P}^N$ be a closed irreducible subvariety of dimension n which may have singularities. Let $X_e \subset X$ denote its simple (non-singular) points, which form a non-empty open subset. Consider the variety

$$V'_X = \{(x, y) \in \mathbb{P}^N \times \check{\mathbb{P}}^N | x \in X_e,\ H_y \text{ is tangent to } X \text{ at } x\},$$

where $H_y \subset \mathbb{P}^N$ is the hyperplane corresponding to $y \in \check{\mathbb{P}}^N$. Let $V_X = \mathrm{cl}(V'_X)$ be its closure, called the *tangent hyperplane bundle* of X. It comes with two projection maps $\pi_1 : V_X \to X$ and $\pi_2 : V_X \to \check{\mathbb{P}}^N$.

Definition 2.1.1. *The dual variety of X is $\check{X} = \pi_2(V_X)$.*

Lemma 2.1.2. *$\check{X}$ is a closed irreducible subvariety of $\check{\mathbb{P}}^N$ of at most $(N-1)$-dimensions.*

Proof. It is enough to show V_X is irreducible and $(N-1)$-dimensional. Consider the first projection $\pi_1 : V'_X \to X_e$, which fibres V'_X locally trivially. The fibres are $(N-n-1)$-dimensional projective subspaces of $\check{\mathbb{P}}^N$, so V'_X is obtained by gluing varieties of the form $U_i \times \check{\mathbb{P}}^{N-n-1}$ along the n-dimensional, irreducible variety $X = \bigcup_i U_i$. Therefore, V'_X is irreducible and $(N-1)$-dimensional, and so is its closure, V_X. □

Remark.

- *In general $\check{X}$ is singular, even if X is not.*
- *If X is smooth, $V'_X = V_X$, and $\check{X} \subset \check{\mathbb{P}}^N$ consists of the hyperplanes $H \subset \mathbb{P}^N$ which are tangent to X.*

The relation $\check{\check{X}} = X$ is given by the *duality theorem*:

Theorem 2.1.3 (Duality)**.** *The tangent hyperplane bundles of X and $\check{X}$ coincide, i.e., $V_{\check{X}} = V_X$.*

Proof. Let $\check{X}_e$ denote the simple points of $\check{X}$. Let

$$U = \{(c, b) \in V_X | c \in X_e,\ b \in \check{X}_e,\ \pi_2 \text{ has maximal rank at } (c, b)\},$$

where the maximal rank is $\dim \check{X}$. The set U is open and non-empty. The first step is to show $U \subset V_{\check{X}}$, for which another bundle has to be introduced. Let

$$W = \{(x, y) \in \mathbb{P}^N \times \check{\mathbb{P}}^N | x \in X \cap H_y\}$$

be the bundle of all hyperplane sections of X and let $p_1 : W \to X$ and $p_2 : W \to \check{\mathbb{P}}^N$ be its projections. Around a point $(c, b) \in W$, it is possible to give an explicit trivialization as in (2.2.2), hence p_1 defines a locally trivial fibration whose fibres are hyperplanes of $\check{\mathbb{P}}^N$, namely $p_1^{-1}(\{c\}) = \{c\} \times {}_cH$, where ${}_cH \subset \check{\mathbb{P}}^N$ is the hyperplane of $\check{\mathbb{P}}^N$ corresponding to $c \in \mathbb{P}^N$. Therefore, $W \subset \mathbb{P}^N \times \check{\mathbb{P}}^N$ is closed, smooth, irreducible, and $\dim W = N + n - 1$ by an analogous argument to that used in Lemma 2.1.2.

Observation: Let $W_e = p_1^{-1}(X_e)$ be the simple points of W. Given $(c, b) \in W_e$, p_2 has maximal rank at (c, b) if and only if H_b intersects X transversally or, equivalently, if and only if $(c, b) \notin V'_X$.

Let $(c, b) \in U$, then $\{c\} \times {}_cH \subset W$ and $T_{(c,b)}(\{c\} \times {}_cH) \subset T_{(c,b)}W$, where T_a denotes the tangent space at a. Therefore,

$$(dp_2)(T_{(c,b)}(\{c\} \times {}_cH)) \subset (dp_2)(T_{(c,b)}W),$$

where dp_2 denotes the differential of p_2. Since $\{c\} \times {}_cH$ is mapped isomorphically to ${}_cH$ under p_2,

$$(dp_2)(T_{(c,b)}(\{c\} \times {}_cH)) = T_b({}_cH),$$

which in particular shows $\operatorname{rank}(p_2|_{\{c\}\times {}_cH}, (c, b)) = N - 1$. By the previous observation, $\operatorname{rank}(p_2, (c, b)) \leq N - 1$. In conclusion $N - 1 = \operatorname{rank}(p_2|_{\{c\}\times {}_cH}, (c, b)) \leq \operatorname{rank}(p_2, (c, b)) \leq N - 1$, so $\operatorname{rank}(p_2, (c, b)) = N - 1$ and

$$(dp_2)(T_{(c,b)}W) = T_b({}_cH).$$

On the other hand, $V_X \subset W$, so $T_{(c,b)}V_X \subset T_{(c,b)}W$ and

$$(d\pi_2)(T_{(c,b)}V_X) = (dp_2)(T_{(c,b)}V_X) \subset (dp_2)(T_{(c,b)}W) = T_b({}_cH).$$

Since $(c, b) \in U$, $b \in \check{X}$ and π_2 has rank $\dim \check{X}$ at (c, b). Therefore,

$$T_b\check{X} = (d\pi_2)(T_{(c,b)}V_X) \subset T_b({}_cH), \tag{2.1.1}$$

which means ${}_cH$ is tangent to $\check{X}$ at b, so $(c, b) \in V_{\check{X}}$ and $U \subset V_{\check{X}}$. V_X is irreducible, so the open subset $U \subset V_X$ is irreducible and contained in the irreducible $V_{\check{X}}$, which satisfies $\dim V_X = \dim V_{\check{X}}$. This can only happen if $V_X = V_{\check{X}}$. $\square$

Definition 2.1.4. *Class of X. If $\dim \check{X} \leq N-2$, assign class$(X) = 0$. If $\dim \check{X} = N-1$, $\check{X}$ is an hypersurface of degree r, assign class$(X) = r$.*

2.2. Modification

From here on, assume X is smooth.

Let $A \subset \mathbb{P}^N$ be a fixed $(N-2)$-dimensional projective subspace. The set of hyperplanes in $\mathbb{P}^N$ containing A is called a *pencil* and A is called the *axis*. A pencil of hyperplanes defines a line in the dual projective space $G \subset \check{\mathbb{P}}^N$, hence the pencil is denoted by $\{H_t\}_{t\in G}$.

Denote by $X_t = X \cap H_t$ the *hyperplane sections* of X so that $X = \bigcup_{t\in G} X_t$. The *exceptional subset* is $X' = X \cap A$.

Definition 2.2.1. *The modification of X with respect to the pencil $\{H_t\}_{t\in G}$ is the variety*

$$Y = \{(x,t) \in X \times G | x \in H_t\}$$

The modification comes with two projections, $p : Y \to X$ and $f : Y \to G$. Note $Y' = p^{-1}(X') = X' \times G \cong X' \times \mathbb{P}^1$ and $p : Y \setminus Y' \cong X \setminus X'$ is an isomorphism. Therefore, $p : Y \to X$ is the blow-up of X along X'. The fibres of f are isomorphic to the hyperplane sections $Y_t = f^{-1}(t) \cong X_t$.

Let $G \subset \check{\mathbb{P}}^N$ be such that G intersects $\check{X}$ transversally and avoids the singular set, in particular $|G \cap \check{X}| = r = \text{class}(X)$. The following lemma will set the necessary preconditions to use holomorphic Morse theory on $f : Y \to G$.

Lemma 2.2.2.

1. *A intersects X transversally, and therefore X' and Y' are non-singular and have $n-2$ and $n-1$ dimensions, respectively.*
2. *Y is irreducible and non-singular.*
3. *f has r critical values, and no two critical points lie in the same fibre of f.*
4. *The critical points of f are non-degenerate*

Proof. Recall the bundle of all hyperplane sections $W = \{(x,y) \in \mathbb{P}^N \times \check{\mathbb{P}}^N | x \in X \cap H_y\}$ and its projections p_1, p_2 as in Theorem 2.1.3. Note $Y = p_2^{-1}(G)$ and $f = p_2|_Y$.

Case 1: $r = 0$. In this case, $G \cap \check{X} = \emptyset$, so all hyperplanes of $\{H_t\}_{t\in G}$ intersect X transversally, and so does the axis A. Note all points of G are regular values of p_2, hence f has no critical points. Smoothness of $Y = p_2^{-1}(G)$ follows from the transversality remark below and Proposition 2.2.3. To show Y is irreducible note that X is irreducible, and so is the open $X \setminus X'$. Therefore $Y \setminus Y' \cong X \setminus X'$ is irreducible, and its closure $\text{cl}(Y \setminus Y') \subset Y$ is an irreducible component of Y. If there is another irreducible component, it must be contained in Y'. But the local dimension of Y is n at every point, so every irreducible component of Y is n-dimensional and cannot be contained in the $(n-1)$-dimensional subset Y'. Therefore, $Y = \text{cl}(Y \setminus Y')$ is irreducible.

Case 2: $r > 0$. For $b \in G \setminus \check{X}$ the argument above holds in a small enough neighbourhood $b \in U \subset G$: The hyperplane H_b intersects X transversally, $p_2^{-1}(U) \subset Y$ is non-singular and f has no critical values in U.

Let $b \in G \cap \check{X} \subset \check{X}_e$. $V'_{\check{X}}$ is mapped isomorphically onto $\check{X}_e$, hence there exists a unique $c \in X$ such that $(c,b) \in V = V_X = V_{\check{X}}$. Note in the case $r > 0$, $\dim \check{X} = \dim {}_cH = N-1$, therefore the relation (2.1.1) shown in Theorem 2.1.3 becomes

$$T_b({}_cH) = (dp_2)(T_{(c,b)}W) = (d\pi_2)(T_{(c,b)}V) = T_b\check{X}. \qquad (2.2.1)$$

Transversality: The projection $p_2 : W \to \check{\mathbb{P}}^N$ is transversal to G, i.e., if $b \in G$ and $(c, b) \in W$, $T_b\check{\mathbb{P}}^N$ is spanned by $(dp_2)(T_{(c,d)}W)$ and T_bG. If p_2 has maximal rank N at (c, b), then $(dp_2)(T_{(c,d)}W) = T_b\check{\mathbb{P}}^N$. Note this is the case for all $(c, b) \in W$ with $b \in G$ for $r = 0$. Otherwise, $(c, b) \in V$ by the remark in the proof of Theorem 2.1.3, hence $(dp_2)(T_{(c,b)}W) = T_b\check{X}$ by (2.2.1). But G and $\check{X}$ intersect transversally at b, so $T_b\check{X} \cap T_bG = 0$ as subspaces of $T_b\check{\mathbb{P}}^N$. Note $\dim T_b\check{X} = N - 1$, $\dim T_bG = 1$ and $\dim T_b\check{\mathbb{P}}^N = N$, so $T_b\check{\mathbb{P}}^N = T_b\check{X} \oplus T_bG = (dp_2)(T_{(c,b)}W) \oplus T_bG$, so p_2 is transversal to G.

1. Suppose A does not intersect X transversally. Then there exists a hyperplane $H_b \in \{H_t\}_{t\in G}$ tangent to X at $c \in A$, hence $(c, b) \in V_X = V$. On the other hand, $c \in A \subset H_b \Leftrightarrow b \in G \subset {}_cH$. Since G intersects $\check{X}$ transversally at b, so does ${}_cH$ by (2.2.1), hence $(c, b) \notin V_{\check{X}} = V$, contradiction. In conclusion, A intersects X transversally.

2. Smoothness follows from Proposition 2.2.3 and transversality of p_2. Irreducibility is proved the same way as in the case $r = 0$.

3. For $(c, b) \in Y$:

$$(df)(T_{(c,b)}Y) = (dp_2)(T_{(c,b)}W) \cap T_bG.$$

If $b \in G \setminus \check{X}$, then $(dp_2)(T_{(c,b)}W) = T_b\check{\mathbb{P}}^N$, so $(df)(T_{(c,b)}Y) = T_bG$, f has maximal rank 1 at b and b is not a critical point. If $b \in G \cap \check{X}$, $(c, b) \in V$, so $(dp_2)(T_{(c,b)}W) = T_b\check{X}$, hence $(dp_2)(T_{(c,b)}W) \cap T_bG = 0$, since $\check{X}$ intersects G transversally at b. Therefore, f has rank 0 at b and (c, b) is a critical point of f. Note $\pi_2^{-1}(\{b\}) = \{(c, b)\}$, since π_2 maps $V'_{\check{X}}$ isomorphically onto $\check{X}_e \supset G \cap \check{X}$, therefore no two critical points lie on the same fibre.

4. Let $(c, b) \in Y$ be a critical point of f, so $(c, b) \in V$. Choose projective coordinates $x = (x_0 : \cdots : x_N) \in \mathbb{P}^N$, $y = (y_0 : \cdots : y_N) \in \check{\mathbb{P}}^N$ such that $c = (1 : 0 : \cdots : 0)$, $b = (0 : \cdots : 0 : 1)$ and $G = \{y_1 = \cdots = y_{N-1} = 0\} \subset \check{\mathbb{P}}^N$. Choose the following trivialization of $p_1 : W \to X$ over $U = \{x \in X | x_0 \neq 0\}$:

$$\begin{array}{rcl} U \times \check{\mathbb{P}}^{N-1} & \longrightarrow & p_1^{-1}(U) \\ (x, z) & \longrightarrow & \left(x, \left(-\sum_{i=1}^{N} x_i z_i : x_0 z_1 : \cdots : x_0 z_N\right)\right), \end{array} \qquad (2.2.2)$$

where $z = (z_1 : \cdots : z_N) \in \check{\mathbb{P}}^{N-1}$. Let $(t_1, \ldots, t_n)$ be local holomorphic coordinates of X around c, and consider the affine coordinates $\zeta_1 = \frac{z_1}{z_N}, \ldots, \zeta_{N-1} = \frac{z_{N-1}}{z_N}$, which yield $(t_1, \ldots, t_n, \zeta_1, \ldots, \zeta_{N-1})$ as coordinates in a neighbourhood of $(c, b) \in W$. Consider the affine coordinates $\eta_0 = \frac{y_0}{y_N}$, ..., $\eta_{N-1} = \frac{y_{N-1}}{y_N}$ for $U' = \{y \in \check{\mathbb{P}}^N : y_N \neq 0\} \subset \check{\mathbb{P}}^N$ so that $b = 0 \in U'$. Thus, in a neighbourhood of $(c, b) \in W$, the projection $p_2 : W \to \check{\mathbb{P}}^N$ takes the following form:

$$\eta_0 = g(t, \zeta), \qquad \eta_1 = \zeta_1, \ \ldots, \ \eta_{N-1} = \zeta_{N-1},$$

where $t = (t_1, \ldots, t_n)$, $\zeta = (\zeta_1, \ldots, \zeta_{N-1})$ and $g(t, \zeta)$ is some holomorphic function such that $t \to g(t, 0)$ gives a coordinate description of $f : Y \to G$ in a neighbourhood of (c, b). The Jacobian is given by

$$J(p_2) = \begin{pmatrix} \frac{\partial g}{\partial t_1} & \ldots & \frac{\partial g}{\partial t_n} & * & * & \ldots & * \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

which fails to have maximal rank if and only if

$$\frac{\partial g}{\partial t_1} = \cdots = \frac{\partial g}{\partial t_n} = 0. \tag{2.2.3}$$

Note (2.2.3) are precisely the defining equations of V. The equations of $c \in \check{\mathbb{P}}^N$ are

$$\eta_1 = \cdots = \eta_{N-1} = 0.$$

Since $\{(c, b)\} = V \cap p_2^{-1}(\{c\})$, this point will be determined by the equations

$$\frac{\partial g}{\partial t_1} = \cdots = \frac{\partial g}{\partial t_n} = \zeta_1 = \cdots = \zeta_{N-1} = 0,$$

which are exactly $N + n - 1 = \operatorname{codim}(W, \{(c, b)\}) = \dim W$ equations. Therefore, the Jacobian of its defining equations

$$\begin{pmatrix} \frac{\partial^2 g}{\partial t_1^2} & \ldots & \frac{\partial^2 g}{\partial t_1 t_n} & * & * & \ldots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial^2 g}{\partial t_n t_1} & \ldots & \frac{\partial^2 g}{\partial t_n^2} & * & * & \ldots & * \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

has rank $N + n - 1$ at (c, b), which implies that the submatrix

$$H = \begin{pmatrix} \frac{\partial^2 g}{\partial t_1^2} & \ldots & \frac{\partial^2 g}{\partial t_1 t_n} \\ \vdots & & \vdots \\ \frac{\partial^2 g}{\partial t_n t_1} & \ldots & \frac{\partial^2 g}{\partial t_n^2} \end{pmatrix}$$

has rank n at (c, b). Observe H is the Hessian matrix of f, so the Hessian of f has maximal rank n at the critical point (c, b). In other words, (c, b) is non-degenerate.

□

Proposition 2.2.3. *Let $p : W \to Z$ be a dominant morphism between smooth varieties and let $G \subset Z$ be a smooth subvariety. If p is transversal to G, i.e., if for every $b \in G$, $\tilde{b} \in p^{-1}(\{b\})$ there is a surjection*

$$T_{\tilde{b}}W \longrightarrow N_{G/Z,b} = T_bZ/T_bG,$$

then $Y = p^{-1}(G)$ is smooth.

Proof. Let $b \in G$ and $\tilde{b} \in p^{-1}(\{b\}) \subset Y$. Locally in a neighbourhood of b, G is defined by $t_1 = t_2 = \cdots = t_r = 0$, where $r = \operatorname{codim}(Z, G)$. Therefore, Y is locally defined by $p^*t_1 = p^*t_2 = \cdots = p^*t_r = 0$ in a neighbourhood of $\tilde{b}$. Y is smooth at $\tilde{b}$ if and only if the 1-forms $p^*dt_1, \ldots, p^*dt_r$ are linearly independent at $\tilde{b}$, which happens if and only if the map

$$\left(I_{G\subset Z}/I^2_{G\subset Z}\right)_b \xrightarrow{p^*} \Omega^1_{W,\tilde{b}}$$

is injective, where $I_{G\subset Z}$ denotes the ideal sheaf defining G. Finally, note the map above being injective is equivalent to its dual being surjective

$$T_{W,\tilde{b}} \longrightarrow N_{G/Z,b},$$

hence Y is smooth at $\tilde{b}$ and the variety Y is smooth. □

2.3. Holomorphic Morse theory

Singular homology with coefficients in an arbitrary principal ideal domain R will be considered in this section.

Decompose the projective line G into two closed hemispheres D_+ and D_- such that the critical values of $f : Y \to G$ are contained in $\operatorname{int}(D_+)$. Let

$$S^1 = D_+ \cap D_-, \qquad Y_\pm = f^{-1}(D_\pm), \qquad Y_0 = f^{-1}(S^1)$$

and choose a base point $b \in S^1$.

The goal of this section is to prove the following critical lemma using holomorphic Morse theory on $f : Y \to G$, which is a Morse function because of Lemma 2.2.2.

Lemma 2.3.1.

$$H_q(Y_+, Y_b) = 0, \ \forall q \neq n$$

and $H_n(Y_+, Y_b)$ is free of rank $r = class(X)$.

Choose a holomorphic coordinate t identifying D_+ with the closed unit disc in $\mathbb{C}$ such that b corresponds to 1. Let $t_1, \ldots, t_r \in \operatorname{int}(D_+)$ denote the critical values corresponding to the critical points $x_1, \ldots, x_r \in Y_+$, and choose $\rho > 0$ such that the discs $D_i = \{t \in D : |t - t_i| \leq \rho\}$ are disjoint and contained in D_+. Let $T_i = f^{-1}(D_i)$ and $F_i = f^{-1}(\{t_i + \rho\})$.

Let l_i be $\mathcal{C}^\infty$-embedded intervals from b to $t_i+\rho$ such that $l = \bigcup_{i=1}^r l_i$ can be contracted within itself to $\{b\}$ and D_+ can be contracted to $k = l \cup \bigcup_{i=1}^r D_i$.

Lemma 2.3.2. *The fibre Y_b is a strong deformation retract of $L = f^{-1}(l)$ and $K = f^{-1}(k)$ is strong deformation retract of Y_+.*

Proof. By Theorem A.1.2, $f : Y_+ \setminus f^{-1}(\{t_1, \ldots, t_r\}) \longrightarrow D_+ \setminus \{t_1, \ldots, t_r\}$ defines a locally trivial fibre bundle. Since $l \subset D_+ \setminus \{t_1, \ldots, t_r\}$, $f : L \to l$ is a subbundle. By the homotopy lifting theorem, the contraction from l to $\{b\}$ can be lifted so that Y_b becomes a strong deformation retract of L. In the same way, the contraction of $D_+ \setminus \{t_1, \ldots, t_r\}$ to $l \cup \bigcup_{i=1}^{r}(D_i \setminus t_i)$ can be lifted so that $L \cup \bigcup_{i=1}^{r}(T_i \setminus f^{-1}(t_i))$ becomes a strong deformation retract of $Y_+ \setminus f^{-1}(\{t_1, \ldots, t_r\})$. Since the t_i are interior points of k, the singular fibres can be filled in so that K is a strong deformation retract of Y_+. $\square$

Lemma 2.3.3. *The inclusions induce isomorphisms in homology*

$$\bigoplus_{i=1}^{r} H_q(T_i, F_i) \longrightarrow H_q(K, L) \longrightarrow H_q(Y_+, L) \longleftarrow H_q(Y_+, Y_b)$$

for all q.

Proof. The two last isomorphisms are a consequence of Lemma 2.3.2. For the first one, note that the inclusion $(\bigcup_{i=1}^{r} D_i, \bigcup_{i=1}^{r}\{t_i + \rho\}) \to (k, l)$ is an excision, since for all i, l_i can be seen as a deformation retract of $l_i' = l_i \cup D_i'$, with $D_i' \subset D_i$ a contractible subset such that $x_i \notin D_i'$ and $t_i + \rho \in D_i'$. Thus, $(k, l) \to (k, l')$ for $l' = \bigcup_{i=1}^{r} l_i$ is an excision, and it is left to show that $(\bigcup_{i=1}^{r} D_i, \bigcup_{i=1}^{r}\{t_i + \rho\}) \to (k, l')$ is an excision. Taking the complement $(k \setminus Z, l' \setminus Z) = (\bigcup_{i=1}^{r} D_i, \bigcup_{i=1}^{r} D_i')$ where $Z = l \setminus (\bigcup_{i=1}^{r}\{t_i + \rho\})$ defines the excision $(\bigcup_{i=1}^{r} D_i, \bigcup_{i=1}^{r} D_i') \to (k, l')$, since $\overline{Z} = l \subset \operatorname{int}(l')$. Finally, note $(\bigcup_{i=1}^{r} D_i, \bigcup_{i=1}\{t_i + \rho\}) \to (\bigcup_{i=1}^{r} D_i, \bigcup_{i=1} D_i')$ is an excision, since D_i' were chosen to be contractible to $\{t_i + \rho\}$. In conclusion, the inclusions of pairs

$$\left(\bigcup_{i=1}^{r} D_i, \bigcup_{i=1}^{r}\{t_i + \rho\}\right) \to \left(\bigcup_{i=1}^{r} D_i, \bigcup_{i=1}^{r} D_i'\right) \to (k, l') \leftarrow (k, l)$$

are all excisions and so is the inclusion $(\bigcup_{i=1}^{r} D_i, \bigcup_{i=1}^{r}\{t_i + \rho\}) \to (k, l)$. Since all the critical values of f in (k, l) are in $(\bigcup_{i=1}^{r} D_i, \bigcup_{i=1}^{r}\{t_i + \rho\})$, this excision can be lifted to $(\bigcup_{i=1}^{r} T_i, \bigcup_{i=1}^{r} F_i) \to (K, L)$, which proves the result. $\square$

As a consequence of Lemma 2.3.3, the study of $H_q(Y_+, Y_b)$ has been reduced to the study of $H_q(T_i, F_i)$, i.e., the study of the relative homology of fibres above small neighbourhoods of the critical values with respect to a regular fibre. Fix i and let $(z_1, \ldots, z_n)$ be holomorphic coordinates of Y in a neighbourhood B of x_i such that the coordinate description of $f|_B$ is

$$f(z) = t_i + z_1^2 + \cdots + z_n^2.$$

The existence of such a B is guaranteed by Morse Lemma A.1.1. Without loss of generality choose B to be a ball centered at x_i:

$$B = \{z \in \mathbb{C}^n : \|z\| = |z_1|^2 + \cdots + |z_n|^2 \leq \epsilon^2\}.$$

After fixing such an ϵ, shrink the radius of D_i so that $\rho < \epsilon^2$. Let

$$T = T_i \cap B, \qquad T' = T \cap \partial B, \qquad F = F_i \cap B, \qquad F' = F \cap \partial B.$$

Lemma 2.3.4. *$F_i \setminus int(B)$ is a strong deformation retract of $T_i \setminus int(B)$ and F' is a strong deformation retract of T'.*

Proof. In this proof, consider Y and its subsets as real manifolds. Note $x_i \in \mathrm{int}(B)$, hence f has maximal rank 2 on $T_i \setminus \mathrm{int}(B)$. In particular, its restriction to the partial boundary T' also has rank 2 everywhere. Thus, by Theorem A.1.3, this defines a locally trivial fibration with explicit trivialization the fibre preserving diffeomorphism

$$(T_i \setminus \mathrm{int}(B), \partial T) \cong (F_i \setminus \mathrm{int}(B), \partial F) \times D_i.$$

The result follows immediately because D_i is contractible to $t_i + \rho$. □

Lemma 2.3.5. *The inclusion $(T, F) \to (T_i, F_i)$ induces isomorphisms in the homology groups.*

Proof. By Lemma 2.3.4, the inclusions $i_1 : (T, F) = (T, F' \cup F) \to (T, T' \cup F)$ and $i_2 : (T_i, F_i) = (T_i, \overline{F_i \setminus B} \cup F_i) \to (T_i, \overline{T_i \setminus B} \cup F_i)$ induce isomorphisms in homology. Note the inclusion $j_2 : (T, T' \cup F) \to (T_i, \overline{T_i \setminus B} \cup F_i)$ is an excision: consider the diagram of inclusions

$$\begin{tikzcd}
(T \setminus T', F \setminus F') \arrow[r, "k_1"] \arrow[d, "k_2"] & (T_i, \overline{T_i \setminus B} \cup F_i) \\
(T, T' \cup F). \arrow[ur, "j_2"] &
\end{tikzcd}$$

The inclusion k_1 is an excision, since $(T_i \setminus Z, (\overline{T_i \setminus B} \cup F_i) \setminus Z)$, with $Z = \overline{T_i \setminus B}$ closed, and the same is true for k_2, since T' is closed. Therefore, j_2 must be an excision. Since $j_1 : (T, F) \to (T_i, F_i)$ is such that $j_2 i_1 = i_2 j_1$, the induced map in homology can be written as $j_{1*} = i_{2*}^{-1} j_{2*} i_{1*}$, which is the composition of three isomorphisms, hence an isomorphism. □

Finally, it is left to study the homology groups $H_q(T, F)$.

Proposition 2.3.6. *The connecting homomorphism*

$$\partial_* : H_q(T, F) \to H_{q-1}(F)$$

is an isomorphism for all $q \geq 2$ and $H_0(T, F) = 0$. Moreover, if $n \geq 2$, $H_1(T, F) = 0$.

Proof. Consider the explicit coordinate description

$$T = \{z \in \mathbb{C}^n | |z_1|^2 + \cdots + |z_n|^2 \leq \epsilon^2,\ |z_1^2 + \cdots + z_n^2| \leq \rho\}.$$

Let $w = (w_1, \dots, w_n) \in T$ and $t \in [0,1]$. Then

$$|t^2 w_1^2 + \cdots + t^2 w_n^2| = t^2 |x_1^2 + \cdots + x_n^2| \le t^2 \rho \le \rho,$$
$$|tx_1|^2 + \cdots + |tx_n|^2 = t^2(|x_1|^2 + \cdots + |x_n|^2) \le t^2 \epsilon^2 \le \epsilon^2,$$

so $T \subset \mathbb{C}^n$ is a star domain with center 0, hence contractible. Therefore, $H_q(T) = 0$ for all $q \ge 1$ and $H_0(T) \cong \mathbb{Z}$, so ∂_* is indeed an isomorphism for $q > 1$. Finally, consider the exact sequence

$$0 \to H_1(T,F) \to H_0(F) \to H_0(T) \to H_0(T,F) \to 0. \tag{2.3.1}$$

By the next Proposition 2.3.7, F has the homotopy type of an $(n-1)$-dimensional sphere. If $n = 1$, $F \simeq S^0$, hence $H_0(F) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus, $H_0(F) \to H_0(T)$ is surjective, so $H_0(T,F) = \operatorname{coker}(H_0(F) \to H_0(T)) = 0$. If $n \ge 2$, F is path-connected and hence $H_0(F) \to H_0(T) \cong \mathbb{Z}$ is an isomorphism, so $H_1(T,F) = H_0(T,F) = 0$. $\square$

The next goal is to compute the homology groups of F.

Proposition 2.3.7. *F has the homotopy type of the $(n-1)$-dimensional sphere.*

Proof. Consider the coordinate description of F

$$F = \{z \in T | z_1^2 + \cdots + z_n^2 = \rho\}$$

and the subspace of the tangent bundle of S^{n-1} consisting of vectors with norm less or equal than 1:

$$Q = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n | \|u\| = 1, \|v\| \le 1, \langle u, v\rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in the Euclidean space. There is a diffeomorphism between F and Q defined in the following way:

- Decompose $z \in F$ in its real and imaginary parts: $z = x + iy$.
- Note that

$$\begin{aligned} z_1^2 + \cdots + z_n^2 &= (x_1 + iy_1)^2 + \cdots + (x_n + iy_n)^2 \\ &= x_1^2 + \cdots + x_n^2 - y_1^2 - \cdots - y_n^2 + 2i(x_1 y_1 + \cdots + x_n y_n) \\ &= \|x\|^2 - \|y\|^2 + 2i\langle x, y\rangle = \rho, \end{aligned}$$

  so it is possible to rewrite F as

$$F = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | \|x\|^2 + \|y\|^2 \le \epsilon^2,\ \|x\|^2 - \|y\|^2 = \rho,\ \langle x, y\rangle = 0\}.$$

- Define the diffeomorphism and its inverse in the following way

$$\begin{array}{rcl} F & \longleftrightarrow & Q \\ (x,y) & \longmapsto & (u,v) = \left(\dfrac{x}{\|x\|}, \dfrac{y}{\sigma}\right) \\ (x,y) = ((\sqrt{\sigma^2\|v\|^2 + \rho})u, \sigma v) & \longleftarrow & (u,v), \end{array}$$

  where $\sigma = \sqrt{\frac{\epsilon^2 - \rho}{2}}$.

Note the above diffeomorphism maps the real part of F

$$S^{n-1} = \{z \in F | z = \Re(z)\}$$

to the zero section $Q_0 = \{(u, 0) \in Q\}$ of Q, which is a strong deformation retract of Q. Therefore, S^{n-1} is a strong deformation retract of F and they have the same homotopy type. $\square$

Corollary 2.3.8. *$H_q(T, F) = 0$ for $q \neq n$ and $H_n(T, F)$ is free of rank 1 generated by an orientation*

$$\Delta = \{z \in T | z = \Re(z)\}$$

of the real n-dimensional disk.

Proof. First, consider the case $n = 1$. Recall the sequence (2.3.1):

$$0 \to H_1(T, F) \to H_0(F) \to H_0(T) \to 0,$$

where $H_0(T, F) = 0$ by Proposition 2.3.6. By Proposition 2.3.7, F has the homotopy type of S^0, so $H_0(F) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus, $H_1(T, F) = \ker(H_0(F) \to H_0(T)) \cong \mathbb{Z}$ is of rank 1. Finally, for $q \geq 2$,

$$H_q(T, F) \cong H_{q-1}(F) = 0,$$

which shows the result. If $n \geq 2$, by Proposition 2.3.7, the homology groups of F are

$$H_q(F) \cong \begin{cases} \mathbb{Z}, & q = 0, n-1 \\ 0, & \text{else} \end{cases}$$

and $H_{n-1}(F)$ is generated by an orientation δ of $S^{n-1} \subset F$. By proposition 2.3.6, $H_0(T, F) = H_1(T, F) = 0$, and there are isomorphisms for $q \geq 2$:

$$H_q(T, F) \cong H_{q-1}(F) \cong \begin{cases} \mathbb{Z}, & q = n \\ 0, & \text{else} \end{cases}$$

Note $H_n(T, F)$ is generated by $[\Delta] = (\partial_*)^{-1}(\delta)$, which is an orientation of the real n-dimensional disk. $\square$

Finally, it is possible to prove Lemma 2.3.1.

Lemma 2.3.1.

$$H_q(Y_+, Y_b) = 0, \ \forall q \neq n$$

and $H_n(Y_+, Y_b)$ is free of rank $r = class(X)$.

Proof. By Lemma 2.3.3, $H_q(Y_+, Y_b) \cong \bigoplus_{i=1}^r H_q(T_i, F_i)$ for all q. By Lemma 2.3.5 and Corollary 2.3.8, $H_q(T_i, F_i) = 0$ for all $i = 1, \ldots, r$ and $q \neq n$ and $H_n(T_i, F_i) \cong \mathbb{Z}$. Therefore,

$$H_q(Y_+, Y_b) \cong \begin{cases} \mathbb{Z}^{\oplus r}, & q = n \\ 0, & \text{else} \end{cases}$$

$\square$

2.4. Homology of hyperplane sections

The main goal of this section is to compute the relative homology groups $H_q(X, X_b)$. For this, it is necessary to relate $H_q(X, X_b)$ to the already known $H_q(Y_+, Y_b)$.

Lemma 2.4.1. *There are isomorphisms*

$$H_q(Y, Y_+) \cong H_{q-2}(X_b), \tag{2.4.1}$$
$$H_q(Y, Y_+ \cup Y') \cong H_{q-2}(X_b, X'), \tag{2.4.2}$$

for all q.

Proof. For (2.4.1), consider the diagram

$$H_q(Y, Y_+) \longleftarrow H_q(Y_-, Y_0) \cong H_q(X_b \times (D_-, S^1)) \longrightarrow H_{q-2}(X_b).$$

The map induced by the inclusion $H_q(Y_-, Y_0) \to H_q(Y, Y_+)$ is an isomorphism as a consequence of Theorem A.2.2, since $Y \setminus Y_+ = Y_- \setminus Y_0 = \operatorname{int}(Y_-)$. Since f has no critical points in Y_-, by Theorem A.1.2, there is a diffeomorphism $Y_- \cong X_b \times D_-$ whose restriction to Y_0 is $Y_0 \cong X_b \times S^1$, which induces the isomorphism $H_q(Y_-, Y_0) \cong H_q(X_b \times (D_-, S^1))$. Finally, by the Künneth Theorem A.2.1,

$$H_q(X_b \times (D_-, S^1)) \cong \bigoplus_{i+j=q} H_i(X_b) \otimes H_j(D_-, S^1).$$

Since $H_j(D_-, S^1) = 0$ for $j \neq 2$ and $H_2(D_-, S^1) \cong R$,

$$H_q(X_b \times (D_-, S^1)) \cong H_{q-2}(X_b) \otimes_R R \cong H_{q-2}(X_b),$$

which proves (2.4.1).

For (2.4.2), the proof is analogous. $Y \setminus (Y_+ \cup Y') = Y_- \setminus (Y_0 \cup Y'_-)$, where $Y'_- = Y' \cap Y_-$, so by Theorem A.2.2, $H_q(Y_-, Y_0 \cup Y'_-) \cong H_q(Y, Y_+ \cup Y')$. Note $Y' = X' \times G$, so $f|'_Y$ is the trivial fibration and $Y'_- = X' \times D_-$. Combining this with the existing the diffeomorphism $Y_- \cong X_b \times D_-$ given by Theorem A.1.2 applied to $f|_{Y_-}$ yields the diffeomorphism of pairs $(Y_-, Y_0 \cup Y'_-) \cong (X_b \times D_-, X_b \times S^1 \cup X' \times D_-) = (X_b, X') \times (D_-, S^1)$. Therefore, there is an isomorphism $H_q(Y_-, Y_0 \cup Y'_-) \cong H_q((X_b, X') \times (D_-, S^1))$. Finally, by Künneth

$$\begin{aligned} H_q((X_b, X') \times (D_-, S^1)) &\cong \bigoplus_{i+j=q} H_i(X_b, X') \otimes H_j(D_-, S^1) \\ &\cong H_{q-2}(X_b, X') \otimes H_2(D_-, S^1) \cong H_{q-2}(X_b, X'), \end{aligned}$$

which proves (2.4.2). $\square$

Lemma 2.4.2. *There are isomorphisms*

$$L : H_{q+1}(Y, Y_b) \cong H_{q-1}(X_b). \tag{2.4.3}$$
$$L' : H_{q+1}(X, X_b) \cong H_{q-1}(X_b, X'), \tag{2.4.4}$$

for all $q \neq n-1, n$.

Proof. For (2.4.3), consider the homology sequence of the triple (Y, Y_+, Y_b)

$$\cdots \to H_{q+1}(Y_+, Y_b) \to H_{q+1}(Y, Y_b) \xrightarrow{L} H_{q-1}(X_b) \to H_q(Y_+, Y_b) \to \dots \qquad (2.4.5)$$

where $H_{q+1}(Y, Y_+)$ has been replaced by $H_{q-1}(X_b)$ using the isomorphism (2.4.1). By Lemma 2.3.1, $H_q(Y_+, Y_b) = 0$ for all $q \neq n$, so the sequence above yields the isomorphisms $L : H_{q+1}(Y, Y_b) \cong H_q(X_b)$ for all $q \neq n-1, n$.

For (2.4.4), note:

1. $H_q(Y, Y_+ \cup Y') \cong H_{q-2}(X_b, X')$ by (2.4.2).

2. $p_* : H_q(Y, Y_b \cup Y') \to H_q(X, X_b)$ is an isomorphism by Theorem A.2.2.

3. $H_q(Y_+ \cup Y', Y_b \cup Y') \cong H_q(Y_+, Y_b)$. To prove this, consider the chain of inclusions $(Y_+, Y_b) \hookrightarrow (Y_+, Y_b \cup Y'_+) \hookrightarrow (Y_+ \cup Y', Y_b \cup Y')$, where $Y'_+ = Y' \cap Y_+$. Note $Y_b = X_b \times \{b\}$ is a deformation retract of $Y_b \cup Y'_+ = X_b \times \{b\} \cup X' \times D_+$, so the first inclusion induces isomorphisms in homology. The second one is also an excision by Theorem A.2.2.

Consider the long exact sequence of the triple $(Y, Y_+ \cup Y', Y_b \cup Y')$:

$$\cdots \to H_{q+2}(Y_+, Y_b) \xrightarrow{p_*} H_{q+2}(X, X_b) \xrightarrow{L'} H_q(X_b, X') \to H_{q+1}(Y_+, Y_b) \to \dots \qquad (2.4.6)$$

where the pertinent replacements have been made according to the points 1, 2 and 3. By Lemma 2.3.1, $H_q(Y_+, Y_b) = 0$ for all $q \neq n$, so $L' : H_{q+1}(X, X_b) \to H_{q-1}(X_b, X')$ is an isomorphism for all $q \neq n-1, n$. ☐

The first application of these results gives information about X_b as a variety.

Proposition 2.4.3. *If $n = \dim X \geq 2$, the generic hyperplane section X_b is non-singular and irreducible.*

Proof. The hyperplane section is taken to be generic, hence $b \notin \check{X}$ and X_b is non-singular. Therefore, irreducibility is equivalent to connectedness. Since $n \geq 2$, (2.4.3) gives $H_0(Y, Y_b) = H_1(Y, Y_b) = 0$. Consider the long exact sequence of (Y, Y_b):

$$0 = H_1(Y, Y_b) \to H_0(Y_b) \to H_0(Y) \to H_0(Y, Y_b) = 0$$

so $H_0(Y_b) \cong H_0(Y)$, i.e., Y_b and Y have the same number of connected components. But Y is connected by Lemma 2.2.2, hence Y_b is connected and so is $X_b \cong X_b \times \{b\} = Y_b$. ☐

From this result it is possible to prove Lefschetz's famous

Theorem 2.4.4 (Homology of hyperplane sections)**.** *Let $b \in G$ be a regular value of f. Then*

$$H_q(X, X_b) = 0, \ \forall q \leq n-1.$$

Proof. Proceed by induction on $n = \dim X$. For $n = 1$, the statement is trivial. Suppose it holds for $n-1 \geq 1$. Then $n \geq 2$ and by Proposition 2.4.3, X_b is an $(n-1)$-dimensional, non-singular, irreducible closed subvariety of $H_b \cong \mathbb{P}^{N-1}$, and $X' = A \cap X$ is a transversal hyperplane section of X_b. By induction hypothesis, $H_q(X_b, X') = 0$ for all $q \leq n-2$. Thus, by the isomorphisms (2.4.4), if $q+1 \leq n-1$,

$$L : H_{q+1}(X, X_b) \cong H_{q-1}(X_b, X') = 0,$$

which proves the result. □

Corollary 2.4.5. *The inclusion $X_b \hookrightarrow X$ induces isomorphisms in homology $H_q(X_b) \xrightarrow{\sim} H_q(X)$ for $q \leq n-2$ and an epimorphism $H_{n-1}(X_b) \twoheadrightarrow H_{n-1}(X)$.*

Proof. Consider the long exact sequence of the pair (X, X_b):

$$\cdots \to H_{q+1}(X, X_b) \to H_q(X_b) \to H_q(X) \to H_q(X, X_b) \to \ldots$$

If $q \leq n-1$, $H_q(X, X_b) = 0$ by Theorem 2.4.4, so $H_q(X_b) \to H_q(X)$ is surjective. If $q \leq n-2$, $H_{q+1}(X, X_b) = 0$, so $H_q(X_b) \to H_q(X)$ is an isomorphism. □

Applying the universal coefficient formula to Theorem 2.4.4 yields

Corollary 2.4.6 (Cohomology of hyperplane sections)**.** *Let $b \in G$ be a regular value of f. Then*

$$H^q(X, X_b) = 0, \ \forall q \leq n-1.$$

2.5. Further applications

In this section, several applications of Lefschetz's theorem on hyperplane sections and the theory developed to prove it will be discussed. Firstly, the topological Euler characteristics can be computed using (2.4.3) in the following way. By Theorem A.2.1, there is an isomorphism

$$\begin{aligned} H_q(Y') &\cong H_q(X' \times G) \\ &\cong H_q(X') \otimes H_0(G) \oplus H_{q-2}(X') \times H_2(G) \\ &\cong H_q(X') \oplus H_{q-2}(X'). \end{aligned} \tag{2.5.1}$$

Thus, there is a canonical homomorphism obtained by composition of the maps:

$$\kappa : H_{q-2}(X') \to H_q(X') \oplus H_{q-2}(X') \xrightarrow{\sim} H_q(Y') \to H_q(Y).$$

Lemma 2.5.1. *The sequence*

$$0 \to H_{q-2}(X') \xrightarrow{\kappa} H_q(Y) \xrightarrow{p_*} H_q(X) \to 0$$

is split exact for all q.

Proof. First note p_* is epimorphic. Given $x \in H_q(X)$, let $u \in H^{2n-q}(X)$ be its Poincaré dual, i.e., $x = u \frown [X]$, where $[X] \in H_{2n}(X)$ is the orientation class. Then $p^*(u) \frown [Y] \in H_q(Y)$ is such that $p_*(p^*(u) \frown [Y]) = u \frown p_*[Y] = u \frown [X] = x$. Consider the long exact sequences of the pairs (Y, Y') and (X, X'):

$$\begin{tikzcd}
H_{q+1}(Y) \arrow[r, "s_*"] \arrow[d, "p_*"] & H_{q+1}(Y,Y') \arrow[r, "\partial_*"] \arrow[d, "p'_*"] & H_q(X') \oplus H_{q-2}(X') \arrow[r, "i_*"] \arrow[d, "\mathrm{pr}"] & H_q(Y) \arrow[r, "s_*"] \arrow[d, "p_*"] & H_q(Y,Y') \arrow[d, "p'_*"] \\
H_{q+1}(X) \arrow[r, "s_*"] & H_{q+1}(X,X') \arrow[r, "\partial_*"] & H_q(X') \arrow[r, "i_*"] & H_q(X) \arrow[r, "s_*"] & H_q(X,X')
\end{tikzcd}$$

where $H_q(Y')$ has been replaced by $H_q(X') \oplus H_{q-2}(X')$ using the isomorphism (2.5.1). Note the maps p'_* are isomorphisms as a consequence of Theorem A.2.2. The remainder of the proof is done by diagram chasing:

- The map κ is injective. Let $x \in \ker \kappa \subset H_{q-2}(X') \subset H_q(X') \oplus H_{q-2}(X')$. Since $i_*|_{H_{q-2}(X')} = \kappa$, $x \in \ker i_* = \operatorname{im} \partial_*$, so there exists $y \in H_{q+1}(Y, Y')$ such that $x = \partial_*(y)$. Then $\partial_*(p'_*(y)) = \mathrm{pr}(\partial_*(y)) = \mathrm{pr}(x) = 0$, so $p'_*(y) \in \ker \partial_* = \operatorname{im} s_*$, so $\exists z \in H_{q+1}(X)$ such that $s_*(z) = p'_*(y)$. Take $t \in H_{q+1}(Y)$ with $p_*(t) = z$. Then $s_*(t) = ((p'_*)^{-1} \circ s_* \circ p_*)(t) = y$, so finally $x = \partial_*(y) = (\partial_* \circ s_*)(z) = 0$.

- $p_* \circ \kappa = 0$. Let $x \in H_{q-2}(X')$. Then $(p_* \circ \kappa)(x) = (p_* \circ i_*)(x) = (i_* \circ \mathrm{pr})(x) = 0$.

- $\ker p_* \subset \operatorname{im} \kappa$. Let $x \in \ker p_* \subset H_q(Y)$. Then $s_*(x) = ((p'_*)^{-1} \circ s_* \circ p_*)(x) = 0$, so $x \in \ker s_* = \operatorname{im} i_*$, i.e., there exists $y \in H_q(X') \oplus H_{q-2}(X')$ such that $i_*(y) = x$. Note $(i_* \circ \mathrm{pr})(y) = (p_* \circ i_*)(y) = p_*(x) = 0$, so $\mathrm{pr}(y) \in \ker i_* = \operatorname{im} \partial_*$, so $\exists z \in H_{q+1}(X, X')$ such that $\partial_*(z) = \mathrm{pr}(y)$. Let $t = (p'_*)^{-1}(z)$. Then $\mathrm{pr}(y - \partial_*(t)) = \mathrm{pr}(y) - (\mathrm{pr} \circ \partial_*)(t) = \mathrm{pr}(y) - (\partial_* \circ p'_*)(t) = \mathrm{pr}(y) - \partial_*(z) = \mathrm{pr}(y) - \mathrm{pr}(y) = 0$, i.e., $y - \partial_*(t) \in H_{q-2}(X')$. Applying κ: $\kappa(y - \partial_*(t)) = i_*(y - \partial_*(t)) = i_*(y) - (i_* \circ \partial_*)(t) = i_*(y) - 0 = i_*(y) = x$, so $x \in \operatorname{im} \kappa$.

The sequence splits because the correspondence $u \frown [X] \mapsto p^*(u) \frown [Y]$ defines a section of p_*. □

Proposition 2.5.2.

$$e(X) = 2e(X_b) - e(X') + (-1)^n r,$$

where e denotes the topological Euler characteristic and $r = \mathit{class}(X)$.

Proof. It follows immediately from previous results. By Lemma 2.5.1,

$$e(Y) = e(X) + e(X').$$

By the long exact sequence (2.4.5),

$$e(Y) - e(Y_b) = e(Y, Y_b) = e(X_b) - e(Y_+, Y_b).$$

Note $Y_b \cong X_b$, so $e(Y_b) = e(X_b)$ and $e(Y_+, Y_b) = (-1)^{n+1}r$ as a consequence of Lemma 2.3.1. Therefore,

$$e(Y) = 2e(X_b) + (-1)^n r$$

and

$$e(X) = 2e(X_b) - e(X') + (-1)^n r.$$

□

The following corollary of Theorem 2.4.4 allows to compute the homologies of hypersurface sections. A particularly relevant case are complete intersections.

Corollary 2.5.3 (Homology of hypersurface sections)**.** *Let $F \subset \mathbb{P}^N$ be a hypersurface such that all points of $F \cap X$ are simple points of F and F intersects X transversally. Then*

$$H_q(X, X \cap F) = 0, \ \forall q \leq n-1$$

Proof. Let $d = \deg F$. Consider the Veronese map of degree d:

$$\begin{aligned} \nu_d : \mathbb{P}^N &\longrightarrow \mathbb{P}^M \\ [x_0 : \cdots : x_N] &\longmapsto [\mu_0 : \cdots : \mu_M], \end{aligned}$$

where $M = \binom{N+d}{d} - 1$ and $\mu_0, \ldots, \mu_M$ are the total degree d monomials in $x_0, \ldots, x_N$. The Veronese map satisfies the following properties:

- There is a one-to-one correspondence:

$$\begin{aligned} \{\text{Hypersurfaces of degree } d \text{ in } \mathbb{P}^N\} &\longleftrightarrow \{\text{Hyperplanes in } \mathbb{P}^M\} \\ F = \left\{ \sum_{j=0}^{M} a_j \mu_j = 0 \right\} &\longleftrightarrow H_F = \left\{ \sum_{j=0}^{M} a_j z_j = 0 \right\} \end{aligned}$$

- It is a regular embedding, meaning $\nu_d(\mathbb{P}^N) \cong \mathbb{P}^N$. In particular, $\nu_d(X)$ is a smooth n-dimensional variety isomorphic to X.

- It preserves transversal intersections: F intersects X transversally if and only if H_F intersects $\nu_d(X)$ transversally.

Therefore, for $q \leq n-1$:

$$H_q(X, X \cap F) \cong H_q(\nu_d(X), \nu_d(X) \cap H_F) = 0$$

by Theorem 2.4.4. □

Corollary 2.5.4. *Let $Y \subset \mathbb{P}^N$ be an n-dimensional smooth complete intersection. Then*

$$H_q(\mathbb{P}^N, Y) = 0, \ \forall q \leq n.$$

Proof. Write $Y = F_1 \cap \cdots \cap F_s$, with $s = \operatorname{codim}(\mathbb{P}^N, Y) = N - n$. Apply the previous corollary to $X = \mathbb{P}^N$ and $F = F_1$, then to $X = F_1$ and $F = F_2$, then to $X = F_1 \cap F_2$ and $F = F_3$ and so on. □

3. Vanishing cycles

Corollary 2.4.5 gives isomorphisms $H_q(X_b) \xrightarrow{\sim} H_q(X)$ for $q \leq n-2$ and an epimorphism for $q = n-1$. It is of particular interest to analyze the middle homology $H_{n-1}(X_b)$, whose extra structure with respect to $H_{n-1}(X)$ arises from the fact that the free abelian group $H_n(Y_+, Y_b)$ is not 0 in general.

Definition 3.0.1. *Let $\partial_* : H_n(Y_+, Y_b) \to H_{n-1}(Y_b) \cong H_{n-1}(X_b)$ be the connecting homomorphism in the homology sequence of (Y_+, Y_b). Its image*

$$V = \partial_*(H_n(Y_+, Y_b)) \subset H_{n-1}(X_b)$$

is called the module of vanishing cycles.

Proposition 3.0.2.

$$rk\ H_{n-1}(X_b) = rk\ V + rk\ H_{n-1}(X)$$

Proof. Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
H_n(Y_+, Y_b) \arrow[r, "\partial_*"] \arrow[d, "p_1"] & H_{n-1}(Y_b) \arrow[r, "i_*"] \arrow[d, "p_2"] & H_{n-1}(Y_+) \arrow[r] \arrow[d, "p_3"] & 0 \\
H_n(X, X_b) \arrow[r, "\partial_*"] & H_{n-1}(X_b) \arrow[r, "i_*"] & H_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

where p_1, p_2 and p_3 are the maps induced by $p : Y \to X$. Note p_2 is an isomorphism, since $X_b \cong Y_b$. Recall the long exact sequence (2.4.6):

$$\cdots \to H_n(Y_+, Y_b) \xrightarrow{p_1} H_n(X, X_b) \to H_{n-2}(X_b, X') \to \ldots$$

Since $H_{n-2}(X_b, X') = 0$ by Theorem 2.4.4, p_1 is surjective. By the Five Lemma, p_3 is an isomorphism. Therefore,

$$\begin{aligned}
V &= \mathrm{im}\,(p_2 \circ \partial_*) = \mathrm{im}\,(\partial_* \circ p_1) \\
&= \mathrm{im}\,(\partial_* : H_n(X, X_b) \to H_{n-1}(X_b)) \\
&= \ker(i_* : H_{n-1}(X_b) \to H_{n-1}(X)),
\end{aligned}$$

where $\mathrm{im}\,(\partial_* \circ p_1) = \mathrm{im}\ \partial_*$ by surjectivity of p_1. Finally,

$$\mathrm{rk}\ H_{n-1}(X_b) = \mathrm{rk} \ker i_* + \mathrm{rk\ im}\ i_* = \mathrm{rk}\ V + \mathrm{rk}\ H_{n-1}(X).$$

$\square$

Remark. *Since p_2 and p_3 are isomorphisms,*

$$V = \ker(i_* : H_{n-1}(X_b) \to H_{n-1}(X)) \cong \ker(i_* : H_{n-1}(Y_b) \to H_{n-1}(Y_+)),$$

so V will be seen as a submodule of $H_{n-1}(X_b)$ or $H_{n-1}(Y_b)$ depending on the context, without changing the notation.

It is possible to translate the commutative diagram in the proof of Proposition 3.0.2 into cohomology:

$$\begin{tikzcd}
H^n(Y_+, Y_b) & H^{n-1}(Y_b) \arrow[l, "\delta^*"'] & H^{n-1}(Y_+) \arrow[l, "i^*"'] & 0 \arrow[l] \\
H^n(X, X_b) \arrow[u, "p^1"] & H^{n-1}(X_b) \arrow[l, "\delta^*"] \arrow[u, "p^2"] & H^{n-1}(X) \arrow[l, "i^*"] \arrow[u, "p^3"] & 0 \arrow[l]
\end{tikzcd}$$

where p^1 is a monomorphism and p^2 and p^3 are isomorphisms.

Definition 3.0.3 (Invariant cycles and invariant cocycles)**.**

- *The kernel*

  $$I^* = \ker(\delta^* : H^{n-1}(X_b) \to H^n(X, X_b)) = im(i^* : H^{n-1}(X) \to H^{n-1}(X_b))$$

  is called the module of invariant cocycles

- *The module of invariant cycles is the Poincaré dual of I^*, namely*

  $$I = \{u \frown [X_b] | u \in I^*\} \subset H_{n-1}(X_b)$$

  where $[X_b] \in H_{2n-2}(X)$ is the fundamental class of the hyperplane section X_b.

Remark. *As with the vanishing cycles, the invariant (co)cycles will be seen indistinctly as a submodule of $H_{n-1}(X_b)$ $(H^{n-1}(X_b))$ or $H_{n-1}(Y_b)$ $(H^{n-1}(Y_b))$, depending on the context.*

Let $i_! : H_{n+1}(X) \to H_{n-1}(X_b)$ be the transfer homomorphism as in Definition A.3.2, i.e.,

$$i_! : H_{n+1}(X) \xrightarrow[D_X]{\sim} H^{n-1}(X) \xrightarrow[i^*]{} H^{n-1}(X_b) \xrightarrow[D_{X_b}^{-1}]{\sim} H_{n-1}(X_b).$$

Thus,

$$\begin{aligned}
\mathrm{im}(i_!) &= \mathrm{im}(D_{X_b}^{-1} \circ i^* \circ D_X) = \mathrm{im}(D_{X_b}^{-1} \circ i^*) \\
&= \{D_{X_b}^{-1}(u) | u \in \mathrm{im}\ i^*\} = \{u \frown [X_b] | u \in I^*\} = I.
\end{aligned}$$

i^* is injective by Corollary 2.4.6, hence $i_!$ is injective, and

$$\mathrm{rk}\ I = \mathrm{rk}\ H_{n+1}(X) = \mathrm{rk}\ H_{n-1}(X),$$

where the last equality comes from Poincaré duality and the universal coefficient formula. Combining this last equality with Proposition 3.0.2 yields the rank formula:

Corollary 3.0.4.

$$rk\ H_{n-1}(X_b) = rk\ I + rk\ V.$$

It is sometimes useful to describe the invariant cycles as the orthogonal complement of the vanishing cycles in $H_{n-1}(X_b)$ with respect to certain intersection form. Let $\langle \cdot, \cdot \rangle_K : H^q(X_b) \times H_q(X_b) \to R$ denote the Kronecker pairing between cohomology and homology.

Proposition 3.0.5. *Let $\langle \cdot, \cdot \rangle : H_{n-1}(X_b) \times H_{n-1}(X_b) \to R$ be the intersection form defined by*

$$\langle x, y \rangle = \langle D(x), y \rangle_K, \tag{3.0.1}$$

where $D(x) \in H^{n-1}(X_b)$ is the unique element such that $D(x) \frown [X_b] = x$. Then

$$I = \{y \in H_{n-1}(X_b) | \langle y, x \rangle = 0,\ \forall x \in V\}.$$

Proof. For the proof it is convenient to consider I^* and V as subspaces of $H^{n-1}(Y_b)$ and $H_{n-1}(Y_b)$, i.e., $I^* = \ker(\delta * : H^{n-1}(Y_b) \to H^n(Y_+, Y_b))$ and $V = \mathrm{im}(\partial_* : H_n(Y_+, Y_b) \to H_{n-1}(Y_b))$. Let $\alpha \in I^*$, $y \in V$. Then $\exists z \in H_n(Y_+, Y_b)$ such that $y = \partial_*(z)$. Then

$$\langle \alpha, y \rangle_K = \langle \alpha, \partial_*(z) \rangle_K = \alpha(\partial_*(z)) = (\delta^* \alpha)(z) = 0,$$

so $\langle \alpha, y \rangle_K = 0$ for all $y \in V$. Conversely, let $\alpha \in H^{n-1}(Y_b)$ be such that $\langle \alpha, y \rangle_K = 0$ for all $y \in V$. Then

$$0 = \langle \alpha, y \rangle_K = \langle \alpha, \partial_*(z) \rangle_K = \alpha(\partial_*(z)) = (\delta^*(\alpha))(z)$$

for all $z \in H_n(Y_+, Y_b)$. Since $H^n(Y_+, Y_b) \cong \mathrm{Hom}(H_n(Y_+, Y_b), R)$ by the universal coefficient formula noting that $H_{n-1}(Y_+, Y_b) = 0$, $\delta^*(\alpha) = 0$, so $\alpha \in \ker(\delta^*) = I^*$. Therefore,

$$I^* = \{\alpha \in H^{n-1}(Y_b) | \langle \alpha, x \rangle_K = 0,\ \forall x \in V\}.$$

Seeing V and I^* as subspaces of $H_{n-1}(X_b)$ and $H^{n-1}(X_b)$, respectively:

$$I^* = \{\alpha \in H^{n-1}(X_b) | \langle \alpha, x \rangle_K = 0,\ \forall x \in V\}.$$

Taking Poincaré duals:

$$I = \{y \in H_{n-1}(X_b) | \langle y, x \rangle = 0,\ \forall x \in V\}.$$

□

4. The Picard-Lefschetz formula

4.1. Elementary paths

Recall the setting of section 2.3: $f : Y \to G$ is a holomorphic mapping between the n-dimensional compact complex manifold Y and the projective line G such that all its critical points $x_1, \dots, x_r$ are non-degenerate and no two lie in the same fibre. Remove the critical values $t_1, \dots, t_r$ from G

$$G^* = G \setminus \{t_1, \dots, t_r\},$$

and the singular fibres from Y

$$Y^* = Y \setminus f^{-1}(\{t_1, \dots, t_r\}).$$

The goal of this section is to study the action of $\pi_1(G^*, b)$ on $H_q(Y_b)$ (and therefore on $H_q(X_b)$) for all q. Firstly, it is necessary to describe $\pi_1(G^*, b)$ with generators and relations, since the action of any element in $\pi_1(G^*, b)$ will be determined by the action of its generators. Let $t_\nu \in \{t_1, \dots, t_r\}$ be a critical value, and let t be a local coordinate of G in a neighbourhood of t_ν. Let $\rho > 0$ be such that the disk D_ν with center t_ν and radius ρ is such that $t_\mu \in D_\nu \Leftrightarrow \mu = \nu$, i.e., only contains the critical value t_ν. Let l_ν be a path in G^* from b to $t_\nu + \rho$, and let

$$\omega_\nu(s) = t_v + \rho e^{2\pi i s}, \ 0 \leq s \leq 1.$$

Definition 4.1.1 (Elementary path)**.** *The path*

$$w_\nu = l_\nu^{-1} \cdot \omega_\nu \cdot l_\nu$$

is called an elementary path encircling t_ν.

Note G^* is a 2-sphere with r points removed. Thus, its fundamental group is generated by the elementary paths encircling the missing points. By choosing an appropriate ordering of the critical values t_i and fitting paths l_i, the fundamental group takes the form

$$\pi_1(G^*, b) = \langle [w_1], \dots, [w_r] \rangle / \{[w_1] \cdot [w_2] \cdots [w_r]\}.$$

In other words, $\pi_1(G^*, b)$ is generated by the elementary paths and there is only one relation: $[w_1] \cdot [w_2] \cdots [w_r] = 1$.

4.2. Extensions around a critical point

Appendix B contains a compilation of the topological results that will be used in the following sections. Let T, T', F, F' be as defined in section 2.3. The map $f : T \to D = \{t \in \mathbb{C} | |t| \leq \rho\}$ given by $f(z) = z_1^2 + \cdots + z_n^2$ restricts to a fibre bundle $f|_{f^{-1}(D^*)} : f^{-1}(D^*) \to D^*$ with typical fibre F, where $D^* = D \setminus \{0\}$. Consider the subspace $T' \subset T$. Then

- $f : (f^{-1}(D^*), T') \to (D^*, D^*)$ is a fibre bundle of pairs, as a consequence of Theorem A.1.3.
- $F' = F \cap T'$ is a strong deformation retract of T' by Lemma 2.3.4.

Note this is the relative situation considered in appendix B.2. Let ω be the loop

$$\begin{aligned} \omega : I &\longrightarrow D^* \\ t &\longmapsto \rho e^{2\pi i t} \end{aligned}$$

and consider the relative extensions

$$\tau_\omega : H_{q-1}(F, F') \to H_q(T, F)$$

as defined in appendix B.2. Note for $q \neq n$, $\tau_\omega = 0$ because $H_q(T, F) = 0$. Therefore, the only non-trivial extension is

$$\tau_\omega : H_{n-1}(F, F') \to H_n(T, F).$$

Let $[\Delta] \in H_n(T, F)$ be a generator as in Corollary 2.3.8, and let $s = \partial_*[\Delta] = [S^{n-1}] \in H_{n-1}(F)$ be the corresponding orientation of the $(n-1)$-dimensional sphere generating $H_{n-1}(F)$.

Lemma 4.2.1. *Let $c \in H_{n-1}(F, F')$ be such that $\langle c, s \rangle = 1$. Then*

$$\tau_\omega(c) = -(-1)^{n(n-1)/2}[\Delta].$$

Proof. Before proceeding with the proof, some clarifications have to be made. How is the intersection pairing $\langle \cdot, \cdot \rangle : H_{n-1}(F, F') \times H_{n-1}(F) \to \mathbb{Z}$ defined? Why does such a $c \in H_{n-1}(F, F')$ exist?

Structure of $H_{n-1}(F, F')$: Recall F is diffeomorphic to the disc bundle $Q = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n | \; \|u\| = 1, \|v\| \leq 1, \langle u, v \rangle = 0\}$ of S^{n-1}. Under this diffeomorphism, F' corresponds to the sphere bundle $Q' = \{(u, v) \in Q | \; \|v\| = 1\}$. There are two generators for the cycles in Q: an orientation of the $(n-1)$-sphere, $[S^{n-1}]$, and an orientation of the n-disc, $[D^n]$. The boundary of $[D^n]$ is contained in Q', so $[D^n] \neq 0 \in H_{n-1}(F, F')$. On the other hand, $[S^{n-1}]$ can be moved to a cycle contained in Q' and $a \in \mathbb{Z}$ copies of D^n so $[S^{n-1}] = a[D^n] \in H_{n-1}(F, F')$. In conclusion, an orientation of the n-disc $[D^n]$ generates $H_{n-1}(F, F')$.

Intersection form: If the choice of orientation $d \in H_{n-1}(F, F')$ of the disc D^n agrees with the chosen orientation $s = [S^{n-1}] \in H_{n-1}(F)$ of S^{n-1}, then $\langle d, s\rangle = 1$, otherwise $\langle d, s\rangle = -1$. Since d generates $H_{n-1}(F, F')$ and s generates $H_{n-1}(F)$, this declaration can be extended to a non-degenerate intersection form $\langle\cdot,\cdot\rangle : H_{n-1}(F, F') \times H_{n-1}(F) \to \mathbb{Z}$. Denote by c the orientation of the disc such that $\langle c, s\rangle = 1$. The form is non-degenerate, hence any element of $H_{n-1}(F, F')$ is completely determined by its intersection number with s, so $x = \langle x, s\rangle \cdot c$ for all $x \in H_{n-1}(F, F')$.

Since $[\Delta]$ generates $H_n(T, F)$, it is clear that $\tau_\omega(c) = \gamma[\Delta]$ for some $\gamma \in \mathbb{Z}$. It remains to prove $\gamma = -(-1)^{n(n-1)/2}$. Consider the diagram

$$\begin{tikzcd}
H_n(F\times I, \partial(F\times I)) \arrow[r, "W_*"] \arrow[d, "\partial_*"] & H_n(T, T'\cup F) \arrow[d, "\partial_*"] & H_n(T,F) \arrow[l, "\mathrm{incl}_*"'] \arrow[d, "\partial_*"] & \\
H_{n-1}(\partial(F\times I)) \arrow[r, "W_*"] \arrow[d, "\cong"] & H_{n-1}(T'\cup F) \arrow[r, "R_*"] & H_{n-1}(F) \arrow[r, "(\Re)_*"] & H_{n-1}(S^{n-1}) \arrow[d, "\cong"] \\
H_{n-1}(\partial(Q\times I)) \arrow[rrr, "g_*"] & & & H_{n-1}(Q_0) \arrow[d, equal] \\
H_{n-1}(\partial(C\times I)) \arrow[u, "\mathrm{incl}_*"] \arrow[rrr, "g_*"] & & & H_{n-1}(Q_0)
\end{tikzcd}$$

where

- Q and Q_0 and the diffeomorphism $F \to Q$ are defined as in Proposition 2.3.7.
- $Q' = \{(u, v) \in Q | \, \|v\| = 1\}$.
- $W : F \times I \to T$ comes from the global trivialization of $\omega^* f^{-1}(D^*) \to I$, where $\omega^* f^{-1}(D^*) = \{(t, z) \in I \times f^{-1}(D^*)| \, \omega(t) = f(z)\}$ is the pull-back of f by ω (see Appendix B for more details). Its explicit form is given below.
- $\Re : F \to S^{n-1}$ is the real part.
- $C = \{e_1 + iv | \, v \in \mathbb{R}^n, \|v\| \le 1, v \perp e_1\} \subset Q$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$.
- $g : \partial(Q \times I) = Q' \times I \cup Q \times \partial I \to Q_0$ is given by $(u + iv, t) \mapsto \Re(e^{i\pi t}(u + iv))$.
- $R : T' \cup F \to F$ is given by $R|_F = \mathrm{id}_F$ and $R|_{T'}$ is the composition of the map $R' : T' \to Q$ defined below and the diffeomorphism $Q \to F$.

The map $W : F \times I \to T$ is given by $(z, t) \mapsto e^{\pi it} z$. Two conditions have to be checked:

- $f(W(z, t)) = f(e^{\pi it} z) = e^{2\pi it} f(z) = \rho e^{2\pi it} = \omega(t)$, so $W(z, t) \in f^{-1}(\{\omega(t)\})$ for all $t \in I$, $z \in F$.
- $W_t : F \cong f^{-1}(\{\omega(t)\})$ is an isomorphism for all t, since W_t^{-1} given by $z \mapsto e^{-\pi it} z$ defines its inverse.

Note there are other possible choices for the map W, but the homotopy class of ω determines W up to homotopy relative to ∂I, so any such W would induce the same maps in homology.

The map $R' : T' \to Q$ is defined as follows. Let $z \in T'$ and write $f(z) = re^{2\pi i\varphi}$ in polar coordinates. Let $z' = e^{-\pi i\varphi} z$, and let $z' = x' + iy'$ be its decomposition into real and imaginary parts. Define

$$\begin{aligned} R' : T' &\longrightarrow Q \\ z &\longmapsto R'(z) = e^{\pi i\varphi}\left(\frac{x'}{\|x'\|} + i\frac{y'}{\|y'\|}\right), \end{aligned}$$

were the points of Q are denoted by $u + iv$.

Commutativity of the subdiagrams: The top left diagram commutes by naturality of the long exact sequence in homology. The top right diagram commutes because $F \xrightarrow{\text{incl}} T' \cup F \xrightarrow{R} F$ is the identity, i.e. R is a retraction (and hence both R_* and $(\text{incl})_*$ are isomorphisms). Commutativity of the bottom diagram is obvious. It is left to check that the middle diagram commutes. Let $(z, t) \in \partial(F \times I)$, and let $z = x + iy$ be its decomposition into real and imaginary parts. Let $\Psi : F \xrightarrow{\sim} Q$ be the diffeomorphism above and let $u + iv \in Q$ be the image of z under Ψ. Then

$$(\Re \circ R' \circ W)(z, t) = (\Re \circ R')(e^{i\pi t} z)$$

Since $f(e^{i\pi t} z) = e^{2i\pi t} f(z) = \rho e^{2i\pi t}$, pick $z' = e^{-i\pi t} e^{i\pi t} z = z$. Then

$$\begin{aligned} R'(e^{i\pi t} z) &= e^{i\pi t}\left(\frac{x'}{\|x'\|} + i\frac{y'}{\|y'\|}\right) = e^{i\pi t}\left(\frac{x}{\|x\|} + i\frac{y}{\|y\|}\right) = e^{i\pi t} R'(z) \\ &= e^{i\pi t}(\Psi \circ R)(z) = e^{i\pi t}\Psi(z) = e^{i\pi t}(u + iv), \end{aligned}$$

where $R(z) = z$ since R is a retraction and $z \in F$. Taking the real part

$$(\Re \circ R' \circ W)(z, t) = \Re\left(e^{i\pi t}(u + iv)\right) = g(u + iv, t),$$

which shows commutativity.

Take $c \times \mathfrak{i} \in H_n(F \times I, \partial(F \times I))$. This is mapped to $\tau_w(c) = \gamma[\Delta] \in H_n(T, F)$. Note $\partial_* : H_n(T, F) \to H_{n-1}(F)$ is an isomorphism by Proposition 2.3.6, and both $(\Re)_* : H_{n-1}(F) \to H_{n-1}(S^{n-1})$ and $H_{n-1}(S^{n-1}) \to H_{n-1}(Q_0)$ are isomorphisms by Proposition 2.3.7. Therefore, $[\Delta]$ must be mapped to $[Q_0]$, and the image of $c \times \mathfrak{i}$ in $H_{n-1}(Q_0)$ is $\gamma[Q_0]$. By commutativity, $c \times \mathfrak{i}$ is mapped to $\gamma[Q_0]$ under the maps $H_n(F \times I, \partial(F \times I)) \xrightarrow{\partial_*} H_{n-1}(\partial(F \times I)) \xrightarrow{\sim} H_{n-1}(\partial(Q \times I)) \xrightarrow{g_*} H_{n-1}(Q_0)$. Thus, γ will be given by the orientation of $\partial(C \times I) \cong S^{n-1}$ determined by $c \times \mathfrak{i}$ and the mapping degree of g.

Orientation of $\partial(C \times I)$: Let $(v_2, \dots, v_n) \in C$ be a coordinate system around e_1. Choosing a positively oriented coordinate system $(u_2, \dots, u_n)$ of Q_0 yields a coordinate system $(v_2, \dots, v_n, u_2, \dots, u_n)$ of F around e_1. Since $\langle c, s\rangle = 1$, the orientation of $(v_2, \dots, v_n)$ differs from the orientation determined by $c \times \mathfrak{i}$ by the same factor as the orientation of $(v_2, \dots, v_n, u_2, \dots, u_n)$ differs from the canonical orientation of F, which is given by any complex coordinate system, in particular: $(u_2 + iv_2, \dots, u_n + iv_n)$. This yields the positively oriented real system $(u_2, v_2, \dots, u_n, v_n)$, whose orientation differs from the one of $(v_2, \dots, v_n, u_2, \dots, u_n)$ by the sign of the corresponding permutation, $(-1)^{n(n-1)/2}$.

Degree of g: Consider the point $(e_1 + ie_2, 1/2) \in \partial C \times I \subset \partial(C \times I)$, which is the only inverse image of the point $-e_2 \in Q_0$. Thus, the mapping degree of g can be computed as the local mapping degree of g at $(e_1 + ie_2, 1/2)$. The orientation of C given by $(v_2, \dots, v_n)$ together with the canonical orientation of I yields an orientation of $\partial(C \times I)$ for which the coordinate system $(v_3, \dots, v_n, t)$ around $(e_1 + ie_2, 1/2)$ is positively oriented. Choose a positively oriented coordinate system $(u_1, u_3, \dots, u_n)$ around $-e_2 \in Q_0$. With respect to these coordinates, g is given by

$$g(v_3, \dots, v_n, t) = (\cos \pi t, -\sin \pi t \cdot v_3, \dots, -\sin \pi t \cdot v_n).$$

Consider the Jacobian

$$Jg = \begin{pmatrix} 0 & 0 & \dots & 0 & -\pi \sin \pi t \\ -\sin \pi t & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -\sin \pi t & 0 \end{pmatrix}.$$

Evaluated at $(e_1 + ie_2, 1/2)$:

$$Jg\,(e_1 + ie_2, 1/2) = \begin{pmatrix} 0 & 0 & \dots & 0 & -\pi \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix},$$

which is negative. Therefore, with respect to the chosen orientations the degree of g is -1.

Combining the computation of the orientation of $\partial(C \times I)$ determined by $c \times \mathfrak{i}$ with the degree of g yields $\gamma = -(-1)^{n(n-1)/2}$, which finishes the proof. $\square$

Let T_ν and D_ν be as defined in section 2.3, i.e., D_ν the disc of radius $\rho > 0$ with center the critical value t_ν and $T_\nu = f^{-1}(D_\nu)$. Let $D_\nu^* = D_\nu \setminus \{t_\nu\}$ and let $F_\nu = f^{-1}(\{t_\nu + \rho\})$. Recall there exists a ball of radius $\epsilon > 0$ with center the critical point $x_\nu \in Y$ such that $T = T_\nu \cap B$ and $F = F_\nu \cap B$. Let $[\Delta_\nu] \in H_n(T_\nu, F_\nu)$ be a generator and let $s = \partial_*[\Delta_\nu] \in H_{n-1}(F_\nu)$. Let ω_ν be the path encircling t_ν as in section 4.1.

Lemma 4.2.2. *The extension along the path ω_ν is given by*

$$\begin{array}{rcl} \tau_{\omega_\nu} : H_{n-1}(F_\nu) & \longrightarrow & H_n(T_\nu, F_\nu) \\ x & \longrightarrow & -(-1)^{n(n-1)/2}\langle x, s\rangle \cdot [\Delta_\nu] \end{array}$$

Proof. Note $F_\nu \setminus \mathrm{int}(B)$ is a strong deformation retract of F_ν by Lemma 2.3.4, so the conditions stated in appendix B.2 are satisfied and it is possible to define the relative extension $\tau_{\omega_\nu} : H_{n-1}(F_\nu, F_\nu \setminus \mathrm{int}(B)) \to H_{n-1}(T_\nu, F_\nu)$. Let $r : (F_\nu, \emptyset) \to (F_\nu, F_\nu \setminus \mathrm{int}(B))$ denote the inclusion. The commutative diagrams

$$\begin{tikzcd} (T_\nu, \emptyset) \arrow[r] \arrow[d, "f"] & (T_\nu, F_\nu \setminus \mathrm{int}(B)) \arrow[d, "f"] \\ D_\nu \arrow[r, equal] & D_\nu \end{tikzcd} \qquad \begin{tikzcd} (T, T') \arrow[r] \arrow[d, "f"] & (T_\nu, F_\nu \setminus \mathrm{int}(B)) \arrow[d, "f"] \\ D_\nu \arrow[r, equal] & D_\nu \end{tikzcd}$$

induce by Proposition B.1.5 two commutative diagrams

$$\begin{tikzcd} H_{n-1}(F_\nu) \arrow[r, "r_*"] \arrow[d, "\tau_{\omega_\nu}"] & H_{n-1}(F_\nu, F_\nu \setminus \mathrm{int}(B)) \arrow[d, "\tau_{\omega_\nu}"] \\ H_n(T_\nu, F_\nu) \arrow[r, equal] & H_n(T_\nu, F_\nu). \end{tikzcd} \qquad \begin{tikzcd} H_{n-1}(F, F') \arrow[r] \arrow[d, "\tau_{\omega_\nu}"] & H_{n-1}(F_\nu, F_\nu \setminus \mathrm{int}(B)) \arrow[d, "\tau_{\omega_\nu}"] \\ H_n(T, F) \arrow[r] & H_n(T_\nu, F_\nu). \end{tikzcd}$$

Putting both together:

$$\begin{tikzcd} H_{n-1}(F_\nu) \arrow[r, "r_*"] \arrow[dr, "\tau_{\omega_\nu}"] & H_{n-1}(F_\nu, F_\nu \setminus \mathrm{int}(B)) \arrow[d, "\tau_{\omega_\nu}"] & H_{n-1}(F, F') \arrow[l, "\sim"'] \arrow[d, "\tau_{\omega_\nu}"] \\ & H_n(T_\nu, F_\nu) & H_n(T, F). \arrow[l, "\sim"'] \end{tikzcd}$$

Note $F \setminus F' = F_\nu \cap B \setminus F \cap \partial B = F \cap \mathrm{int}(B) = F_\nu \setminus (F_\nu \setminus \mathrm{int}(B))$, so by Theorem A.2.2, $H_{n-1}(F, F') \to H_{n-1}(F_\nu, F_\nu \setminus \mathrm{int}(B))$ is an isomorphism. Also, by Lemma 2.3.5, $H_n(T, F) \to H_n(T_\nu, F_\nu)$ is an isomorphism and maps the free generator $[\Delta]$ to the free generator $[\Delta_\nu]$. The result follows by noting that the homomorphism in the top row maps $x \in H_{n-1}(F_\nu)$ to $\langle x, s\rangle \cdot c \in H_{n-1}(F, F')$, since c generates $H_{n-1}(F, F')$ and $y = \langle y, s\rangle \cdot c$ for all $y \in H_{n-1}(F, F')$ (see "Structure of $H_{n-1}(F, F')$" and "Intersection form" in the proof of Lemma 4.2.1). By Lemma 4.2.1, τ_{ω_ν} maps this element to $-(-1)^{n(n-1)/2}\langle x, s\rangle \cdot [\Delta] \in H_n(T, F)$, and so to $-(-1)^{n(n-1)/2}\langle x, s\rangle \cdot [\Delta_\nu] \in H_n(T_\nu, F_\nu)$. By commutativity,

$$\tau_{\omega_\nu}(x) = -(-1)^{n(n-1)/2}\langle x, s\rangle \cdot [\Delta_\nu],$$

which proves the result. $\square$

4.3. Extensions along elementary paths

The final step is computing the extensions along the elementary paths w_i. For that, consider the chain of homomorphisms

$$H_n(T, F) \to H_n(T_i, F_i) \to H_n(Y_+, L) \to H_n(Y_+, Y_b),$$

where $H_n(T, F) \to H_n(T_i, F_i)$ is an isomorphism by Lemma 2.3.5, and $H_n(T_i, F_i) \to H_n(Y_+, L)$ and $H_n(Y_+, L) \to H_n(Y_+, Y_b)$ are a monomorphism and an isomorphism, respectively, by Lemma 2.3.3. The generator $[\Delta] \in H_n(T, F)$ maps to a generator $\Delta_i \in H_n(Y_+, Y_b)$ and $\Delta_1, \ldots, \Delta_r$ generate $H_n(Y_+, Y_b)$ freely.

Definition 4.3.1. *1. $\Delta_i \in H_n(Y_+, Y_b)$ is called a thimble.*

2. Let $\partial_* : H_n(Y_+, Y_b) \to H_{n-1}(Y_b)$ be the connecting homomorphism. $\delta_i = \partial_*(\Delta_i)$ is called a vanishing cycle.

Lemma 4.3.2. *The extension along the elementary path w_i is given by*

$$\begin{aligned} \tau_{w_i} : H_{n-1}(Y_b) &\longrightarrow H_n(Y_+, Y_b) \\ x &\longmapsto -(-1)^{n(n-1)/2} \langle x, \delta_i \rangle \cdot \Delta_i. \end{aligned}$$

Proof. Consider the diagram

$$\begin{tikzcd} H_{n-1}(Y_b) \arrow[rr, "(l_i)_*"] \arrow[d, "\tau_{w_i}"] \arrow[dr, "\tau_{w_i}"] & & H_{n-1}(F_i) \arrow[d, "\tau_{\omega_i}"] \arrow[dl, "\tau_{\omega_i}"'] \\ H_n(Y_+, Y_b) \arrow[r, "\sim"] & H_n(Y_+, L) & H_n(T_i, F_i). \arrow[l] \end{tikzcd}$$

where l_i was the path from b to $t_i + \rho$. Lower triangles commute by Proposition B.1.5 applied to the diagrams

$$\begin{tikzcd} Y_+ \arrow[r] \arrow[d, "f"] & Y_+ \arrow[d, "f"] \\ D_+ \arrow[r] & D_+ \end{tikzcd} \qquad \begin{tikzcd} T_i \arrow[r] \arrow[d, "f"] & Y_+ \arrow[d, "f"] \\ D_i \arrow[r] & D_+. \end{tikzcd}$$

With respect to the upper triangle, recall $w_i = l_i^{-1} \cdot \omega_i \cdot l_i$. Therefore, by Proposition B.1.7,

$$\tau_{w_i} = \tau_{l_i^{-1} \cdot \omega_i \cdot l_i} = \tau_{l_i^{-1}} \circ \omega_i^* \circ l_i^* + \tau_{\omega_i} \circ l_i^* + \tau_{l_i}$$

Note the image of l_i lies in L, so by Proposition B.1.4, $\tau_{l_i^{-1}} = \tau_{l_i} = 0$. Therefore,

$$\tau_{w_i} = \tau_{l_i^{-1} \cdot \omega_i \cdot l_i} = \tau_{\omega_i} \circ l_i^*$$

and the diagram commutes. Finally, $x \in H_{n-1}(Y_b)$ is mapped to $\tilde{x} \in H_{n-1}(F_i)$, which by Lemma 4.2.2 is mapped to $-(-1)^{n(n-1)/2}\langle \tilde{x}, \tilde{\delta_i}\rangle \cdot [\Delta_i]$. Since l_i^* is an isomorphism which preserves intersection numbers, $\langle \tilde{x}, \tilde{\delta_i}\rangle = \langle x, \delta_i\rangle$. The bottom homomorphism $H_n(T_i, F_i) \to H_n(Y_+, Y_b) \to H_n(Y_+, Y_b)$ maps $[\Delta_i]$ to the thimble Δ_i, so by commutativity

$$\tau_{w_i}(x) = -(-1)^{n(n-1)/2}\langle x, \delta_i\rangle \cdot \Delta_i,$$

which proves the result. □

Theorem 4.3.3 (Picard-Lefschetz formula). *For $q \neq n-1$, the fundamental group $\pi_1(G^*, b)$ acts trivially on $H_q(Y_b)$. For $q = n-1$, the action of the elementary path w_i is given by*

$$(w_i)_*(x) = x + (-1)^{n(n+1)/2}\langle x, \delta_i\rangle\delta_i, \quad x \in H_{n-1}(Y_b).$$

Proof. By Proposition B.1.6, $w \in \pi_1(G^*, b)$ acts as

$$w_*(x) = x + (-1)^q \partial_* \tau_w(x).$$

Since $\tau_w : H_q(Y_b) \to H_{q+1}(Y_+, Y_b)$ and $H_{q+1}(Y_+, Y_b) = 0$ for all $q \neq n-1$ by Lemma 2.3.1, the action is trivial for $q \neq n-1$. For $q = n-1$, let w_i be an elementary path. By Lemma 4.3.2,

$$\tau_{w_i}(x) = -(-1)^{n(n-1)/2}\langle x, \delta_i\rangle \cdot \Delta_i.$$

Thus,

$$\begin{aligned}(w_i)_*(x) &= x + (-1)^{n-1}\partial_*\tau_{w_i}(x) = x + (-1)^{n-1}\partial_*(-(-1)^{n(n-1)/2}\langle x, \delta_i\rangle \cdot \Delta_i) \\ &= x + (-1)^{n-1+n(n-1)/2}\langle x, \delta_i\rangle \cdot \partial_*(\Delta_i) = x + (-1)^{n(n+1)/2}\langle x, \delta_i\rangle \cdot \delta_i.\end{aligned}$$

□

The most immediate consequence of Theorem 4.3.3 is a characterization of the module of invariant cycles I in terms of the action of $\pi_1(G^*, b)$.

Proposition 4.3.4. *The module $I \subset H_{n-1}(Y_b)$ of invariant cycles is exactly the submodule of $H_{n-1}(Y_b)$ which is invariant under the action of $\pi_1(G^*, b)$.*

Proof. Recall the I is the orthogonal complement of the module of vanishing cycles V with respect to the intersection form $\langle \cdot, \cdot \rangle$:

$$I = \{y \in H_{n-1}(Y_b) | \langle y, x\rangle = 0, \ \forall x \in V\}.$$

Since V is generated by the vanishing cycles $\delta_1, \ldots, \delta_r$, this definition is equivalent to

$$I = \{y \in H_{n-1}(Y_b) | \langle y, \delta_i\rangle, \forall i = 1, \ldots, r\}.$$

Since $\pi_1(G^*, b)$ is generated by the elementary paths $w_1, \ldots, w_r$, an element $x \in H_{n-1}(Y_b)$ is invariant under the action of $\pi_1(G^*, b)$ if and only if $w_i(x) = x$ for all $i = 1, \ldots, r$. Recall the Picard-Lefschetz formula 4.3.3:

$$(w_i)_*(x) = x + (-1)^{n(n+1)/2}\langle x, \delta_i\rangle\delta_i.$$

Thus, $(w_i)_*(x) = x \Leftrightarrow \langle x, \delta_i\rangle = 0$. In other words, x is invariant under the action of $\pi_1(G^*, b)$ if and only if $\langle x, \delta_i\rangle = 0$ for all i, i.e., $x \in I$. □

Bibliography

[1] Klaus Lamotke, *The topology of complex projective varieties after S. Lefschetz*, Pergamon Press - Topolgy, **Vol. 20**, pp. 15-51, 1981.

[2] Aldo Andreotti and Theodore Frankel, *The Lefschetz theorem on hyperplane sections*, Annals of Mathematics, **Vol. 69**, pp. 714-717, 1959.

[3] P. Deligne and N.M.Katz, *Groupes de monodromie en géométrie algébrique*, Springer - SGA 7.II, **340**, 1973.

[4] Paul Seidel, *Fukaya Categories and Picard-Lefschetz Theory*, European Mathematical Society, 2008.

[5] V.A.Vassiliev, *Applied Picard-Lefschetz Theory*, American Mathematical Society - Mathematical Surveys and Monographs, **Vol. 97**, 2002.

[6] John Milnor, *Morse Theory*, Princeton University Press - Annals of Mathematical Studies, **No. 51**, 1963.

[7] Glen E. Bredon, *Topology and Geometry*, Springer Verlag - Graduate texts in mathematics, **139**, 1993.

[8] Robin Hartshorne, *Algebraic Geometry*, Springer Verlag - Graduate texts in mathematics, **52**, 1977.

A. Preliminaries

A.1. Differential geometry

Theorem A.1.1 (Morse Lemma)**.** *Let $\mathcal{M}$ be a complex manifold of dimension n and let $f : \mathcal{M} \to \mathbb{C}$ be a holomorphic function. If $x \in \mathcal{M}$ is a non-degenerate critical point of f, there exists a complex chart (U, φ) with holomorphic coordinates $\varphi = (z_1, \ldots, z_n)$ such that*

$$f(z) = f(x) + z_1^2 + \cdots + z_n^2$$

for all $z \in U$. In particular, (U, φ) is centered at x.

Theorem A.1.2 (Ehresmann's fibration)**.** *Let $f : M \to N$ be a smooth map between manifolds. If*

- *f is a surjective submersion*
- *f is proper*

then f defines a locally trivial fibration.

Theorem A.1.3 (Ehresmann's fibration for manifolds with boundary)**.** *Let $f : (M, \partial M) \to (N, \partial N)$ be a smooth map between manifolds with boundary. If*

- *f is a surjective submersion*
- *f is proper*
- *$f|_{\partial M}$ also satisfies the above properties*

then f defines a locally trivial fibration.

A.2. Topology

Theorem A.2.1 (Künneth)**.** *Let X and Y be topological spaces, R a principal ideal domain and let $k \in \mathbb{Z}_{\geq 0}$. Then the sequence*

$$0 \to \bigoplus_{i+j=k} H_i(X;R) \otimes_R H_j(Y;R) \to H_k(X \times Y;R) \to \bigoplus_{i+j=k-1} Tor_1^R(H_i(X;R), H_j(Y;R)) \to 0$$

is exact. More generally, after replacing X and Y with pairs (X, X') and (Y, Y'), the formula still holds true for relative homology.[1]

[1]The notation $(X, X') \times (Y, Y')$ denotes $(X \times Y, X \times Y' \cup X' \times Y)$.

Theorem A.2.2. *Let $f : (X, A) \to (Y, B)$ be a continuous map between pairs of compact Euclidean neighbourhood retracts such that $f : X \setminus A \to Y \setminus B$ is a homeomorphism. Then f induces an isomorphism*

$$f_* : H_n(X, A) \to H_n(Y, B)$$

of the relative singular homology.

A.3. Poincaré duality

Theorem A.3.1 (Poincaré duality)**.** *Let M be a closed oriented n-manifold, and let $k \in \mathbb{Z}$. Then the map*

$$\begin{array}{rcl} H^k(M; \mathbb{Z}) & \longrightarrow & H_{n-k}(M; \mathbb{Z}) \\ \alpha & \longmapsto & \alpha \frown [M], \end{array}$$

where $[M] \in H_n(M; \mathbb{Z})$ is the fundamental class is an isomorphism.

Definition A.3.2 (Transfer homomorphism)**.** *Let D_M denote the inverse of the Poincaré duality isomorphism given as in Theorem A.3.1, i.e., $D_M(\alpha) \frown [M] = \alpha$. Let $f : N \to M$ be a smooth map between compact manifolds of dimensions n and m, respectively. The maps*

$$\begin{array}{rcl} f^! : H^{n-p}(N; \mathbb{Z}) & \longrightarrow & H^{m-p}(M; \mathbb{Z}) \\ f_! : H_{m-p}(M; \mathbb{Z}) & \longrightarrow & H_{n-p}(N; \mathbb{Z}) \end{array}$$

given by

$$\begin{aligned} f^! &= D_M \circ f_* \circ D_N^{-1} \\ f_! &= D_N^{-1} \circ f^* \circ D_M \end{aligned}$$

are called the transfer homomorphisms.

B. Topology of Lefschetz's fibrations

B.1. Extensions along paths

The goal of this appendix is to provide the necessary topological tools to define and compute the action of $\pi_1(G^*, b)$ on the homology groups. Let $f : A \to B$ be a continuous map and let $B^* \subset B$ be a subspace such that $f|_E : E \to B^*$, where $E = f^{-1}(B^*)$, is a locally trivial fibre bundle. Let $F_y = f^{-1}(\{y\})$ denote the fibre over y. Let $w : I = [0, 1] \to B^*$ be a path from a to b. Let w^*E be the pullback of f by w:

$$w^*E = \{(t, e) \in I \times E | w(t) = f(e)\}.$$

The first projection $\mathrm{pr}_1 : w^*E \to I$ realizes w^*E as a fibre bundle over I whose fibre over $t \in I$ is $\mathrm{pr}_1^{-1}(\{t\}) = \{e \in E | f(e) = w(t)\} = F_{w(t)}$.

Proposition B.1.1. *The fibre bundle $pr_1 : w^*E \to I$ is trivial.*

Consider the map $W : F_a \times I \xrightarrow{\varphi} w^*E \xrightarrow{\mathrm{pr}_2} E \hookrightarrow A$, where $\varphi : F_a \times I \to w^*E$ is a trivialization. The map W has the following properties:

- $f \circ W(x, t) = w(t)$ and $W(x, 0) = x$ for all $x \in F_a$, $t \in I$.
- For a fixed $t \in I$, $W_t : F_a \to F_{w(t)}$ is a homeomorphism.
- For L with $F_a \cup F_b \subset L \subset A$, W is a map between pairs

$$W : F_a \times (I, \partial I) \to (A, L)$$

In other words, W is a homotopy from the identity on F_a to a homeomorphism $W_1 : F_a \cong F_b$. If w' is a path from a to b homotopic to w, then W' is homotopic to W relative to $F_a \times \partial I$ and W'_1 is isotopic to W_1. Therefore, there are well-defined maps in homology

$$w_* = (W_1)_* : H_*(F_a) \xrightarrow{\sim} H_*(F_b)$$
$$W_* : H_*(F_a \times (I, \partial I)) \to H_*(A, L)$$

which only depend on the homotopy class of w.

Definition B.1.2. *If $w : I \to B^*$ is a loop at a, $w_* : H_*(F_a) \xrightarrow{\sim} H_*(F_a)$ is called the algebraic monodromy along w and $W_1 : F_a \cong F_a$ a geometric monodromy.*

Definition B.1.3. *Let* $\mathfrak{i} \in H_1(I, \partial I)$ *denote the canonical generator. The map*

$$\tau_w : H_q(F_a) \xrightarrow{(-)\times \mathfrak{i}} H_{q+1}(F_a \times (I, \partial I)) \xrightarrow{W_*} H_{q+1}(A, L)$$

is called the extension along w.

Note it only depends on the homotopy class of w because W_* does. Presented below are more relevant properties of the maps τ_w:

Proposition B.1.4. *If* $L \supset f^{-1}(w(I))$, $t_w = 0$.

Proposition B.1.5. τ_w *is natural in* $f : A \to B$, *i.e., a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d, "f"] & A_1 \arrow[d, "f_1"] \\ B \arrow[r, "\varphi"] & B_1 \end{tikzcd}$$

such that $\varphi(B^*) \subset B_1^*$ *and* $\Phi(L) \subset L_1$ *induces a commutative diagram*

$$\begin{tikzcd} H_q(F_a) \arrow[r, "(\Phi_a)_*"] \arrow[d, "\tau_w"] & H_q(F_{1,\varphi(a)}) \arrow[d, "\tau_{\varphi \circ w}"] \\ H_{q+1}(A, L) \arrow[r, "(\Phi)_*"] & H_{q+1}(A_1, L_1) \end{tikzcd}$$

Proposition B.1.6. *Let* $\partial_* : H_{q+1}(A, L) \to H_q(L)$ *denote the connecting homomorphism. Let* $x, w_*(x) \in H_q(L)$ *denote the images of* $x \in H_q(F_a)$ *and* $w_*(x) \in H_q(F_b)$, *respectively, under the maps induced by the inclusions* $F_a \hookrightarrow L$ *and* $F_b \hookrightarrow L$. *Then*

$$(-1)^q \partial_* \tau_w(x) = w_*(x) - x, \ \forall x \in H_q(F_a).$$

Proposition B.1.7. *Let* w *be a path from* a *to* b, *and let* v *be a path from* b *to* c. *Let* $L \supset F_a \cup F_b \cup F_c$. *Then*

$$\tau_{v \cdot w} = \tau_v \circ w_* + \tau_w, \qquad (v \cdot w)_* = v_* \circ w_*.$$

B.2. Relative extensions

Let $A' \subset A$ be a subspace, and let $E' = E \cap A'$ and $F'_y = F_y \cap A'$. Assume that:

- f fibres the pair (E, E') locally trivially over B^*.
- F'_a is a strong deformation retract of A'.

Then W can be seen as a map of pairs:

$$W : (F_a, F'_a) \times (I, \partial I) = (F_a \times I, F_a \times \partial I \cup F'_a \times I) \to (A, L \cup A').$$

Definition B.2.1. *The map*

$$\tau_w : H_q(F, F'_a) \xrightarrow{(-)\times \mathfrak{i}} H_{q+1}((F_a, F'_a) \times (I, \partial I)) \xrightarrow{W_*} H_{q+1}(A, L \cup A')$$

is called the relative extension along w.

Proposition B.2.2. *Propositions B.1.4-B.1.7 also hold for relative extensions.*