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Hausdorff moment problem: Reconstruction of distributions☆

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Abstract

The problem of approximation of the moment-determinate cumulative distribution function (cdf) from its moments is studied. This method of recovering an unknown distribution is natural in certain incomplete models like multiplicative-censoring or biased sampling when the moments of unobserved distributions are related in a simple way to the moments of an observed distribution. In this article some properties of the proposed construction are derived. The uniform and L_1-rates of convergence of the approximated cdf to the target distribution are obtained.

Keywords: Hausdorff moment problem; Moment-recovered distribution; L_1-rate of approximation; Uniform rate of approximation

1. Introduction

Let us consider the so-called probabilistic (Hamburger) moment problem: For a given sequence $\mu = \{1 = \mu_0, \mu_1, \ldots\}$ of real numbers, define a probability distribution F on the real line $(-\infty, \infty)$, such that μ_j represents the jth moment of F. When the support of F is the positive real line, then the problem is known as a Stieltjes moment problem. The case when distribution F has a finite support, say, $\mathrm{supp}\{F\} = [0, T]$ with $T < \infty$, is known as a Hausdorff moment problem. There are two important questions related to the moment problem:

(a) Is F uniquely determined by the sequence of moments μ?
(b) How should this uniquely defined distribution F be reconstructed from μ?

If there is a positive answer to the question (a), we say that the distribution F is moment-determinate (M-determinate), otherwise it is M-indeterminate. There are many articles investigating the conditions (for example, Carleman's and Krein's conditions), under which the distributions are M-determinate or M-indeterminate, see, Akhiezer (1965), Feller (1971), Lin (1997), and Stoyanov (2000) among others. We will study question (b) in the Hausdorff case.

The reconstruction of a distribution from its moments is very helpful, for instance, when the target distribution has no closed form although the corresponding moments are easily calculated. One such example represents the distribution of a finite weighted sum of independent chi-squared r.v.'s, each with degree of freedom, say 1.

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The estimation of distributions via moments is natural in certain incomplete models, e.g., mixtures, multiplicative-censoring or biased sampling, when the moments of unobserved distributions are related in a simple way to the moments of an observed distribution. In all such cases one can estimate the target distribution via the estimation of the transformed moments of observed distributions. See, for example, Mnatsakanov and Klaassen (2003) and Mnatsakanov and Ruymgaart (2005), where the estimation problem of an unknown mixing cdf in the Poisson mixture model and, the estimation of a cdf in the biased sampling model have been studied, respectively.

Several inversion formulas were obtained by inverting the moment generating function and Laplace transform (Shohat and Tamarkin, 1943; Feller, 1971; Chauveau et al., 1994; Tagliani and Velasquez, 2003) in the case of the Stieltjes moment problem. Lindsay et al. (2000) proposed to approximate the M-determinate cdf F by a finite mixture of gamma distributions. The number of components in a mixture is chosen to match several assigned moments of F. It was proved that the corresponding approximant is the cdf itself. The disadvantage of the approximation is that their procedure requires calculations of high order Hankel determinants. It is worth noting that according to ill-conditioning of the Hankel matrices this and the so-called Maximum Entropy (*ME*) method (see Kevasan and Kapur, 1992) are not useful when the number of assigned moments is large. To avoid the numerical instability, Novi Inverardi and Tagliani (2003) proposed to use the fractional moments in *ME* construction.

In the Hausdorff case, Feller (1971) constructed the approximants of F via high-order differences of the μ_j, $j = 0, 1, \ldots$, and proved its weak convergence. Our construction, $\mathcal{K}_{\alpha,T}^{-1}\mu$ (see (2) in Section 2), also depends directly on the moment sequence μ. It can be applied in the Stieltjes case as well, assuming $T = T(\alpha) \to \infty$, as $\alpha \to \infty$. This case requires a more in depth analysis due to instability problems and will be considered in a separate article.

The aim of the present work is to investigate the properties of the approximants $\mathcal{K}_{\alpha,T}^{-1}\mu$ and to derive the rate of approximation as $\alpha \to \infty$ and T is fixed.

The paper is organized as follows. We introduce some notations and assumptions in Section 2. Then we show how to recover the distributions of two different types of convolutions, the distribution in some biased models, and the distribution of the transformation $\phi(X)$ of X via moments (see, Theorem 1 and Corollary 1) in Section 3. In Theorem 2 the uniform and L_1-rates of convergence of reconstructed cdfs are obtained. Finally, in Section 4 we consider two simple examples. The comparison of the graphs of the target cdfs with corresponding moment-recovered counterparts are presented (see Figs. 1 and 2).

2. Some notations and assumptions

Suppose that cdf F has a finite support, $\mathrm{supp}\{F\} = [0, T]$, $T < \infty$. Let us denote the ordinary moments of F by

$$\mu_{j,F} = \int t^j \mathrm{d}F(t) = (\mathcal{K}F)(j), \quad j \in \mathbb{N} = \{0, 1, \ldots\}, \tag{1}$$

and assume that the moment sequence $\mu_F = (\mu_{0,F}, \mu_{1,F}, \ldots)$ determines F uniquely, i.e., F is M-determinate.

An approximate inverse of the operator $\mathcal{K}$ from (1), constructed according to

$$(\mathcal{K}_{\alpha,T}^{-1}\mu_F)(x) = \sum_{k=0}^{[\alpha\frac{x}{T}]} \sum_{j=k}^{\alpha} \binom{\alpha}{j}\binom{j}{k} \frac{(-1)^{j-k}}{T^j}\mu_{j,F}, \quad 0 \le x \le T, \alpha \in \mathbb{N}, \tag{2}$$

is such that $\mathcal{K}_{\alpha,T}^{-1}\mathcal{K}F \to_w F$, as $\alpha \to \infty$ (see, Mnatsakanov and Ruymgaart (2003), where $T = 1$). Here $\to_w$ denotes the weak convergence of cdfs, i.e. convergence at each continuity point of the limiting cdf. The success of the inversion formula (2) hinges on the convergence

$$B_\alpha(u, v) = \sum_{k=0}^{[\alpha v]} \binom{\alpha}{k} u^k (1-u)^{\alpha-k} \to \begin{cases} 1, & u < v \\ 0, & u > v, \end{cases} \tag{3}$$

as $\alpha \to \infty$. This result is immediate from a suitable interpretation of the left hand side as a sum of binomial probabilities.

For any moment sequence $\nu = \{\nu_j, j \in \mathbb{N}\}$, let us denote by F_ν the corresponding moment-recovered function via operator $\mathcal{K}_{\alpha,T}^{-1}$, as $\alpha \to \infty$ and $\mu_F = \nu$ in (2), i.e.

$$F_\nu = \lim_{\alpha\to\infty} F_{\alpha,\nu} = \lim_{\alpha\to\infty} \mathcal{K}_{\alpha,T}^{-1}\nu,$$

say, in the sense of weak convergence. Below, in Theorem 2, we assume that F is absolutely continuous with respect to the Lebesgue measure and $f(x) = \mathrm{d}F(x)/\mathrm{d}x$. Denote the *supremum* norm of a function $\phi : [0, T] \to \mathbb{R}$ by $\|\phi\| = \sup_{0\le t\le T} |\phi(x)|$ and the L_1-norm on $([0, T], \mathrm{d}x)$ by $\|\phi\|_{L_1} = \int_0^T |\phi(x)|\mathrm{d}x$, respectively. Let

$$\beta(u, a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1}(1-u)^{b-1}, \quad 0 < u < 1, \tag{4}$$

be a pdf of the *Beta* (a, b) distribution with the parameters $a = [\alpha \frac{x}{T}] + 1$ and $b = \alpha - [\alpha \frac{x}{T}]$.

3. Asymptotic properties of $F_{\alpha,\nu}$

In this Section we present some asymptotic properties of moment-recovered cdf $F_{\alpha,\nu} = \mathcal{K}_{\alpha,T}^{-1}\nu$ defined in Section 2. The uniform and L_1-rates of convergence of $F_{\alpha,\nu}$ to F are derived as well.

Let us introduce two different convolutions of cdfs F_1 and F_2. Namely, denote the convolution with respect to the multiplication operation by

$$F_1 \otimes F_2(x) = \int F_1(x/\tau)\mathrm{d}F_2(\tau), \quad 0 \leqslant x \leqslant T^2,$$

while the convolution with respect to the addition operation is denoted by

$$F_1 \star F_2(x) = \int F_1(x-\tau)\mathrm{d}F_2(\tau), \quad 0 \leqslant x \leqslant 2T.$$

The following notations will be useful as well: if $\nu_1 = \{\mu_{j,F_1}, j \in \mathbb{N}\}$ and $\nu_2 = \{\mu_{j,F_2}, j \in \mathbb{N}\}$ represent two moment sequences, then let us use the following notations: $\nu_1 \odot \nu_2 = \{\mu_{j,F_1} \times \mu_{j,F_2}, j \in \mathbb{N}\}$ and $\nu_1 \oplus \nu_2 = \{\bar{\nu}_j, j \in \mathbb{N}\}$ where

$$\bar{\nu}_j = \sum_{m=0}^{j} \binom{j}{m} \mu_{m,F_1} \times \mu_{j-m,F_2}. \tag{5}$$

One can easily verify that the moment sequences of $F_1 \otimes F_2$ and $F_1 \star F_2$ are $\nu_1 \odot \nu_2$ and $\nu_1 \oplus \nu_2$, respectively. Also the reverse statements are true, see, (i) and (ii) in the following statement.

Theorem 1. (i) *If* $\nu = \nu_1 \odot \nu_2$*, then*

$$F_{\alpha,\nu} \to_w F_\nu, \quad \text{as } \alpha \to \infty, \tag{6}$$

with $F_\nu = F_1 \otimes F_2$.

(ii) *If* $\nu = \nu_1 \oplus \nu_2$*, then* (6) *holds with* $F_\nu = F_1 \star F_2$.

(iii) *If for some* $a > 0$ *and* $b \ge 0$, $\nu = \{\bar{\nu}_j = \mu_{aj+b,F}/\mu_{b,F}, j \in \mathbb{N}\}$*, then* (6) *holds with*

$$F_\nu(x) = \frac{1}{\mu_{b,F}} \int_0^{x^{1/a}} t^b \mathrm{d}F(t), \quad x \in [0, T^a]. \tag{7}$$

(iv) *If* $\nu = \{\bar{\nu}_j, j \in \mathbb{N}\}$*, with*

$$\bar{\nu}_j = \int [\phi(t)]^j \mathrm{d}F(t), \tag{8}$$

for some continuous and increasing function $\phi : [0, T] \to [0, T']$*, then* (6) *holds with* $F_\nu(x) = F(\phi^{-1}(x)), x \in [0, T']$.

Proof of Theorem 1. Note that cdf $F_1 \otimes F_2$ is defined on $[0, T_0]$, with $T_0 = T^2$. Therefore, to prove (i), one can use (3) with the arguments of $B_\alpha(\cdot, \cdot)$ equal to tu/T_0 and x/T_0, respectively. Indeed, with $\nu = \nu_1 \odot \nu_2$, we have

$$
\begin{aligned}
F_{\alpha,\nu}(x) &= (\mathcal{K}^{-1}_{\alpha,T_0}\nu)(x) = \iint \sum_{k=0}^{[\alpha \frac{x}{T_0}]} \sum_{j=k}^{\alpha} \binom{\alpha}{j}\binom{j}{k} \frac{(-1)^{j-k}}{T_0^j} t^j u^j \mathrm{d}F_1(t)\mathrm{d}F_2(u) \\
&= \iint \sum_{k=0}^{[\alpha \frac{x}{T_0}]} \binom{\alpha}{k} \left(\frac{tu}{T_0}\right)^k \sum_{m=0}^{\alpha-k} \binom{\alpha-k}{m} \left(-\frac{tu}{T_0}\right)^m \mathrm{d}F_1(t)\mathrm{d}F_2(u) \\
&= \iint B_\alpha\left(\frac{tu}{T_0}, \frac{x}{T_0}\right) \mathrm{d}F_1(t)\mathrm{d}F_2(u) \to \iint 1_{[0,x)}(tu)\mathrm{d}F_1(t)\mathrm{d}F_2(u) \\
&= \int F_1(x/u)\mathrm{d}F_2(u) = F_1 \otimes F_2(x).
\end{aligned} \tag{9}
$$

By a similar argument used in (9), we can show (ii)–(iv) very easily. For example, to show (ii), note that replacing in (5) the moments μ_{m,F_1} and μ_{j-m,F_2} by corresponding integrals with respect to F_1 and F_2 we obtain

$$
\bar{\nu}_j = \iint (t+u)^j \mathrm{d}F_1(t)\mathrm{d}F_2(u).
$$

Since $F_1 \star F_2$ is defined on $[0, T_1]$, $T_1 = 2T$, again using (3) with the arguments of $B_\alpha(\cdot,\cdot)$ equal to $(t+u)/T_1$ and x/T_1, respectively, we derive (ii). □

Corollary 1. (i) *If* $\nu = \{\bar{\nu}_j,\ j \in \mathbb{N}\}$ *with* $\bar{\nu}_j$ *defined according to* (8) *and* $F^*_{\alpha,\nu}(x) = F_{\alpha,\nu}(\phi(x)),\ x \in [0,T]$, *then*

$$
F^*_{\alpha,\nu} \to_w F, \ \text{as}\ \alpha \to \infty. \tag{10}
$$

(ii) *If* $\nu = \{\mu_{aj,F},\ j \in \mathbb{N}\}$ *for some positive* a, *then* (10) *holds with* $F^*_{\alpha,\nu}(x) = F_{\alpha,\nu}(x^a),\ x \in [0,T]$.

(iii) *If* $\nu = \beta_1\nu_1 + \beta_2\nu_2$ *with* $\beta_1 + \beta_2 = 1,\ \beta_1, \beta_2 > 0$ *and* $\nu_1 = \{a_1^j,\ j \in \mathbb{N}\}$ *and* $\nu_2 = \{a_2^j,\ j \in \mathbb{N}\}$ *for some* $0 < a_1 < a_2 < T$, *then* (6) *holds with* $F_\nu(x) = \beta_1 1_{[a_1,T]}(x) + \beta_2 1_{[a_2,T]}(x)$.

(iv) *If* $\nu = \{a^j \mu_{j,F},\ j \in \mathbb{N}\}$ *for some positive* a, *then* (6) *holds with* $F_\nu(x) = F(x/a),\ x \in [0, aT]$.

Proof of Corollary 1. The transformation $x \mapsto \phi(x)$ and the statement (iv) from Theorem 1 yield Corollary 1(i). The statement (ii) is a special case of (i) since $\mu_{aj,F} = \int \phi(t)^j \mathrm{d}F(t)$ with $\phi(t) = t^a$. Since the distribution with the moments $\mu_j = a^j,\ j \in \mathbb{N}$ is degenerated at a, the linearity of the inverse transformation $\mathcal{K}^{-1}_{\alpha,T}$ defined in (2) yields Corollary 1(iii), while Theorem 1(i) yields Corollary 1(iv), respectively. □

Remark 1. If $a = 1$ in Theorem 1(iii), then cdf F_ν from (7) represents the biased sampling model with the weight function $w(t) = t^b$.

Remark 2. The construction $F^*_{\alpha,\nu}$ from Corollary 1(ii) can be very helpful when only the finite number of integer moments up to, say, $m^* < \infty$ are available. In this case one can use the fractional moments $\mu^*_j = \mu_{aj,F}$ with $a = m^*/\alpha$ and $j = 0, 1, \ldots, \alpha$, and reconstruct F by means of $F^*_{\alpha,\nu}$ where $\nu = \{\mu^*_j,\ j = 0, 1, \ldots, \alpha\}$. The application of the fractional moments in the problem of recovering pdf f based on the *ME* approach has been considered by Novi Inverardi et al. (2003) (see also the bibliography in the later article).

From Corollary 1(iii) it follows that one can reconstruct (or estimate via the empirical moments) a discrete cdf. Actually, for smooth cdf F the convergence in (6) is uniform. Since the proofs of statements in Theorem 1(i)–(iv) are similar to each other, let us consider here only the case with $\nu = \mu_F$.

Theorem 2. *Let* $\nu = \{\mu_{j,F},\ j \in \mathbb{N}\}$. *If* f' *is bounded on* $[0,T]$ *and* $\alpha \to \infty$, *then*

$$
\|F_{\alpha,\nu} - F\| \le \frac{T}{\alpha+1}\left\{\|f\| + \frac{T}{2}\|f'\|\right\} + o\left(\frac{1}{\alpha}\right), \tag{11}
$$

$$
\|F_{\alpha,\nu} - F\|_{L_1} \le \frac{T}{\alpha+1}\left\{1 + \frac{T^2}{12}\|f'\|\right\} + o\left(\frac{1}{\alpha}\right). \tag{12}
$$

Proof of Theorem 2. From (1) and (2) we have

$$
\begin{aligned}
F_{\alpha,\nu}(x) &= (\mathcal{K}_{\alpha,T}^{-1}\mathcal{K}F)(x) = \int \sum_{k=0}^{[\alpha \frac{x}{T}]} \sum_{j=k}^{\alpha} \binom{\alpha}{j}\binom{j}{k}\left(\frac{t}{T}\right)^k \left(-\frac{t}{T}\right)^{j-k} \mathrm{d}F(t) \\
&= \int \sum_{k=0}^{[\alpha \frac{x}{T}]} \binom{\alpha}{k}\left(\frac{t}{T}\right)^k \sum_{m=0}^{\alpha-k}\binom{\alpha-k}{m}\left(-\frac{t}{T}\right)^m \mathrm{d}F(t) \qquad (13)\\
&= \int B_\alpha\left(\frac{t}{T}, \frac{x}{T}\right)\mathrm{d}F(t).
\end{aligned}
$$

Now, integration by parts in the last equation of (13) enable us to write

$$F_{\alpha,\nu}(x) - F(x) = \int_0^1 [F(uT) - F(x)]\beta(u, a, b)\mathrm{d}u, \qquad (14)$$

where

$$F(uT) - F(x) = f(x)T\left(u - \frac{x}{T}\right) + T^2 \int_{\frac{x}{T}}^{u} \mathrm{d}\tau \int_{\frac{x}{T}}^{\tau} f'(yT)\mathrm{d}y. \qquad (15)$$

Note also that the mean and variance of *Beta* (a, b) distribution defined in (4) are such that

$$\eta_\alpha := \int_0^1 u\beta(u, a, b)\mathrm{d}u = \frac{[\alpha \frac{x}{T}] + 1}{\alpha + 1}, \qquad (16)$$

$$\sigma_\alpha^2 := \int_0^1 (u - \eta_\alpha)^2 \beta(u, a, b)\mathrm{d}u = \frac{([\alpha \frac{x}{T}] + 1)(\alpha - [\alpha \frac{x}{T}])}{(\alpha + 1)^2(\alpha + 2)}, \qquad (17)$$

and

$$\left|\eta_\alpha - \frac{x}{T}\right| \le \frac{1}{\alpha + 1}\max\left(\frac{x}{T}, 1 - \frac{x}{T}\right). \qquad (18)$$

Hence, the combination of (14)–(18) yields (11):

$$
\begin{aligned}
\|F_{\alpha,\nu} - F\| &\le \sup_{0\le x\le T}\left|f(x)T\int_0^1\left(u - \frac{x}{T}\right)\beta(u, a, b)\mathrm{d}u\right| \\
&\quad + \sup_{0\le x\le T}\left|\int_0^1 \beta(u, a, b)\mathrm{d}u\left[T^2\int_{\frac{x}{T}}^{u}\mathrm{d}\tau\int_{\frac{x}{T}}^{\tau} f'(yT)\mathrm{d}y\right]\right| \\
&\le T\|f\|\sup_{0\le x\le T}\left|\eta_\alpha - \frac{x}{T}\right| + \|f'\|T^2 \sup_{0\le x\le T}\int_0^1 \frac{1}{2}\left(u - \frac{x}{T}\right)^2\beta(u, a, b)\mathrm{d}u \\
&\le \frac{T}{\alpha + 1}\|f\| + \frac{T^2}{2(\alpha + 1)}\|f'\| + \frac{T^2}{2(\alpha + 1)^2}\|f'\|.
\end{aligned}
$$

The proof of (12) is based on Eqs. (14)–(18) again and is omitted here. □

4. Examples

Example 1. Let us recover the distribution F with assigned moments $\nu_j = 9/(j + 3)^2$, $j \in \mathbb{N}$. Note that $\nu_j = 1/(j/3 + 1)^2 = \nu_{aj,G}$, where $a = 1/3$, $\nu_G = \mu \odot \mu$, and $\mu = \{1/(j + 1), j \in \mathbb{N}\}$. Since the uniform cdf $U[0, 1]$ is M-determinate with the moment sequence $\mu = \{1/(j + 1), j \in \mathbb{N}\}$, we conclude from Theorem 1, see (i), that $G(x) = U[0, 1] \otimes U[0, 1](x) = x - x\log(x),\ 0 \le x \le 1$. Now taking $a = 1/3$, and $b = 0$ in Theorem 1(iii), we derive $F(x) = G(x^3) = x^3 - x^3\log(x^3),\ 0 \le x \le 1$. We conducted the computations of moment-recovered cdf $F_{\alpha,\nu}$ when $\alpha = 30$. See Fig. 1 with F, the dashed curve, and $F_{\alpha,\nu}$, the step function, respectively.

Example 2. Let us approximate the cdf A_2 of the sum of two independent $Beta(1/2, 1)$ random variables having the sequence of moments: $\mu_{j,F_k} = 1/(2j+1)$, $j \in \mathbb{N}$, $k = 1, 2$. Based on Theorem 1(ii), we can use the approximation $F_{\alpha,\nu}$ with $\nu = \{\bar{\nu}_j, j \in \mathbb{N}\}$ defined according to (5) and $\mu_{j,F_k} = 1/(2j+1)$. On Fig. 2 we plotted

$$A_2(x) = \begin{cases} \dfrac{\pi x}{4}, & 0 \le x \le 1 \\ \sqrt{x-1} + \dfrac{x}{2}\left(2\sin^{-1}\sqrt{\dfrac{1}{x}} - \dfrac{\pi}{2}\right), & 1 < x \le 2, \end{cases}$$

the dashed curve, and its moment-recovered cdf $F_{\alpha,\nu}$, the step function, respectively. Here we took again $\alpha = 30$.

Remark 3. By taking parameter $\alpha = 100$, one can verify that in the Examples 1 and 2 the approximants $F_{\alpha,\nu}$ almost coincide with the corresponding cdfs F and A_2, respectively.

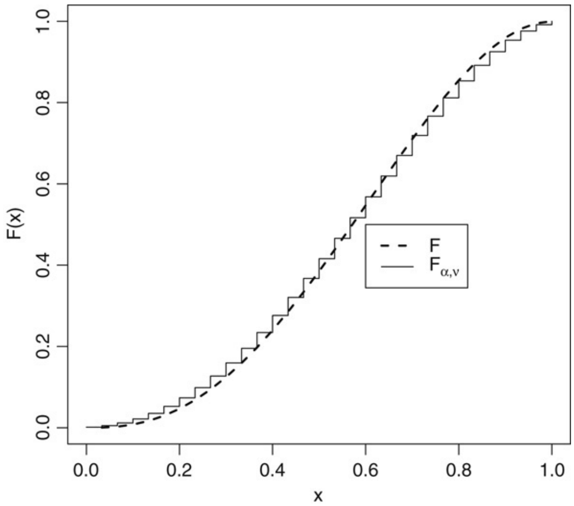

Fig. 1. Approx. of F by $F_{\alpha,\nu}$.

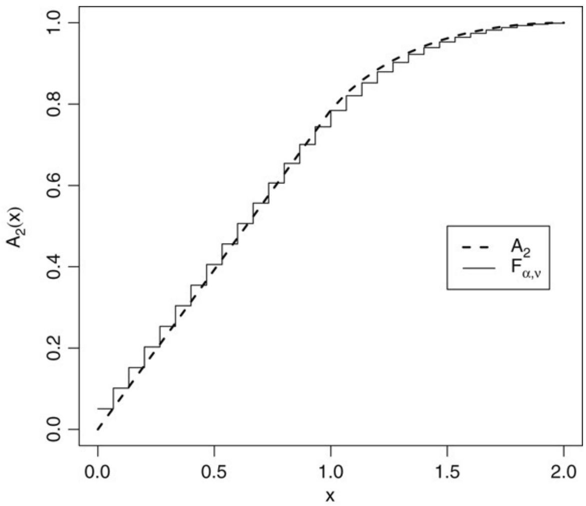

Fig. 2. Approx. of A_2 by $F_{\alpha,\nu}$.

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