Theoretical Analysis of Trend Vanishing Moments for Directional Orthogonal Transforms

Shogo MURAMATSU, Dandan HAN, Tomoya KOBAYASHI and Hisakazu KIKUCHI

Dept. of Electrical and Electronic Eng., Niigata University, Japan

新潟大學

ABSTRACT

This work contributes to investigate theoretical properties of the trend vanishing moments (TVMs) which the authors have defined in a previous work and applied to the directional design of 2-D nonseparable GenLOT. Some significant properties of TVMs are shown theoretically and experimentally.

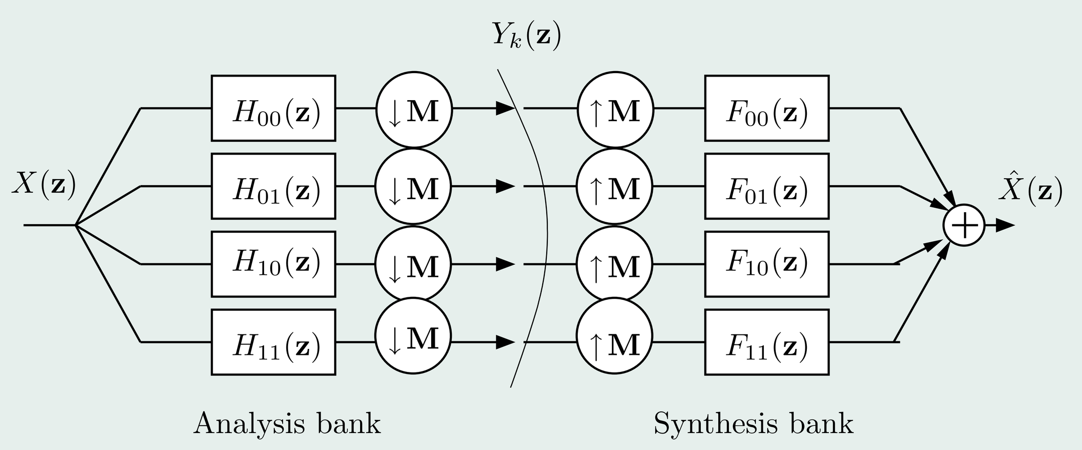
Key words– Multidimensional filter banks and wavelets, directional transforms, non-separable filter design, image coding

Introduction

- Recent development of image transforms involves non-separable ones for handling diagonal structures, such as Curvelets and Contourlets.
- We've also developed a novel class of 2-D GenLOT [ICIP2009,2010].
  - Attractive features include the symmetry, orthogonality and local variability of bases.
  - TVM was introduced instead of restrictive Directional VM (DVM).
- Main issue of this work is to prove the following relation w.r.t. TVM:

$Eq.$ (1)[Wavelet filters] $\Leftrightarrow$ $Eq.$ (2)[Scaling filter] $\Leftrightarrow$ $Eq.$ (3)[Polyphase matrix]

Definition of TVM



$$\mathbf{M} = \begin{pmatrix} M_y & 0 \\ 0 & M_x \end{pmatrix}$$

Definition (Trend Vanishing Moments of Order P)

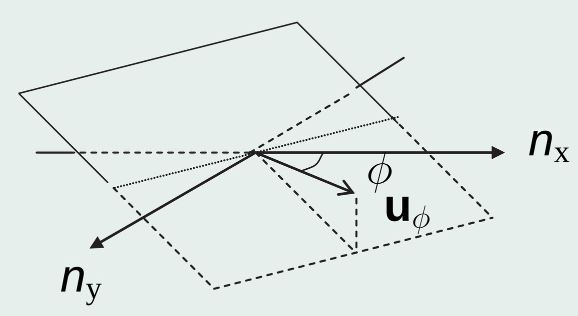
We say that a filter bank has P-order TVM along the direction $\mathbf{u}_\phi = (\sin\phi, \cos\phi)^T$ if trend moments $\mu_{k,\phi}^{(p)}$ of all wavelet filters up to $p = (P-1)$ vanishes, i.e.

$$0 = \mu_{k,\phi}^{(p)} = \sum_{\mathbf{n}\in\mathcal{Z}^2} h_k[\mathbf{n}] \sum_{q=0}^{p} \binom{p}{q} (n_y \sin\phi)^{p-q}(n_x\cos\phi)^q =$$

$$(-j)^p \sum_{q=0}^{p} \binom{p}{q} \sin^{p-q}\phi\cos^q\phi \frac{\partial^p}{\partial\omega_y^{p-q}\partial\omega_x^q} H_k\left(e^{j\omega^T}\right)\Bigg|_{\omega=\mathbf{0}} \quad (1)$$

for all $k = 1, 2, \cdots, M-1$ and $p = 0, 1, \cdots, P-1$, where $\mathbf{n} = [n_y, n_x]^T$ and $h_k[\mathbf{n}]$ is the impulse response of the k-th analysis filter $H_k(\mathbf{z})$, i.e. the k-th basis image for the paraunitary case.

- One-order TVM is identical to the classical one-order VM and holds the no-DC-leakage property.
- Two-order TVM annihilates one-order trend surfaces in the direction $\mathbf{u}_\phi$.



A trend surface proportional to $(n_y\sin\phi + n_x\cos\phi)$

Properties of TVM

- TVM condition is defined on wavelet filters.
- Orthogonality yields an identical condition on the scaling filter.

Theorem (TVM Condition for Scaling Filter)

For paraunitary filter banks with a decimation factor $\mathbf{M}$*, the condition in Eq. (2) holds if and only if Eq. (1) is satisfied.*

$$0 = \sum_{q=0}^{p} \binom{p}{q} \sin^{p-q}\phi\cos^q\phi \frac{\partial^p}{\partial\omega_y^{p-q}\partial\omega_x^q} H_0\left(e^{j\omega^T}\right)\Bigg|_{\omega=\omega_\ell} \quad (2)$$

for $p = 0, 1, \cdots, P-1$ *and all* $(M-1)$ *aliasing frequencies, i.e.* $\omega_\ell = 2\pi\mathbf{M}^{-T}\mathbf{k}_\ell$ *for* $\mathbf{k}_\ell \in \mathcal{N}\left(\mathbf{M}^T\right)\setminus\{\mathbf{o}\}$*, where* $\mathbf{k}_0 = \mathbf{o}$ *and* $\mathcal{N}(\mathbf{N}) = \left\{\mathbf{N}\mathbf{x} \in \mathcal{Z}^2 | \mathbf{x} \in [0,1)^2\right\}$.

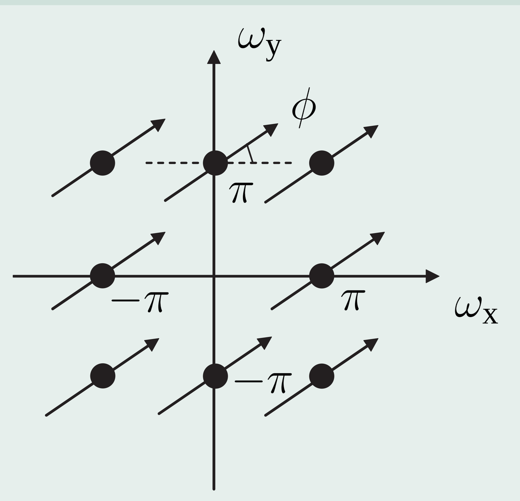


Illustration of TVM condition on a scaling filter $H_0(e^{\omega^T})$ for $M_y = M_x = 2$, where the dots shows the frequency points at which the response and derivatives in the direction ϕ become null.

- Further consideration through Theorem yields another representation.

Fact (Polyphase Matrix Representation of TVM Condition)

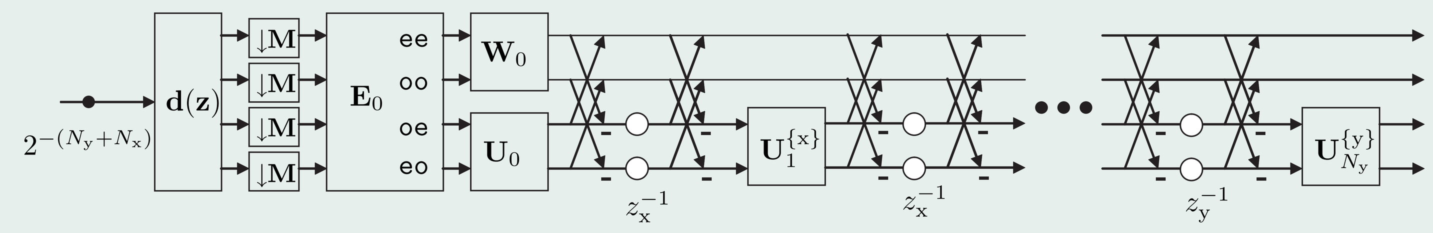
For an FIR paraunitary filter bank, the TVM condition of order P *in Eq. (1) is represented in terms of the polyphase matrix* $\mathbf{E}(\mathbf{z})$ *by*

$$c_p\mathbf{a}_M = \mathbf{m}_\phi^{(p)} = \sum_{q=0}^{p} \binom{p}{q} \sin^{p-q}\phi\cos^q\phi \frac{\partial^p}{\partial\omega_y^{p-q}\partial\omega_x^q} \mathbf{E}\left(\mathbf{z}^\mathbf{M}\right)\mathbf{d}(\mathbf{z})\Bigg|_{\mathbf{z}=\mathbf{1}} \quad (3)$$

for $p = 0, 1, \cdots, P-1$*, where* c_p *is an arbitrary constant,* $\mathbf{1} = (1, 1, \cdots, 1)^T$ *and* $\mathbf{a}_m$ *is the* $m \times 1$ *vector defined by* $\mathbf{a}_m = (1, 0, \cdots, 0)^T$.

Design and Simulation

- Let us verify the significance of the previous Fact. by appling it to a lattice structure of 2-D nonseparable GenLOT [ICIP2009]



- Orthonormality and linear-phase (symmetric) property are guaranteed structurally
- Polyphase matrix of order (N_y, N_x) is represented by

$$\mathbf{E}(z_y,\ z_x) = \prod_{n_y=1}^{N_y}\left\{\mathbf{R}_{n_y}^{\{y\}}\mathbf{Q}(z_y)\right\} \cdot \prod_{n_x=1}^{N_x}\left\{\mathbf{R}_{n_x}^{\{x\}}\mathbf{Q}(z_x)\right\} \cdot \mathbf{R}_0\mathbf{E}_0,$$

where $\mathbf{E}_0$ is the 2-D separable DCT, and $\mathbf{Q}(z_d) = \frac{1}{2}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I}\end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & z_d^{-1}\mathbf{I}\end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I}\end{pmatrix}$,

$\mathbf{R}_0 = \begin{pmatrix} \mathbf{W}_0 & \mathbf{O} \\ \mathbf{O} & \mathbf{U}_0\end{pmatrix}$, $\mathbf{R}_{n_d}^{\{d\}} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{U}_{n_d}^{\{d\}}\end{pmatrix}$.

- Matrices $\mathbf{W}_0$, $\mathbf{U}_0$ and $\mathbf{U}_{n_d}^{\{d\}}$ are orthonormal and controlled during design phase.
- From Fact, design procedures with TVMs can be derived. For 2×2-ch directional GenLOTs (DirLOTs) with two-order TVMs, we have

Step 1 Give a trend direction ϕ and set parameters as

- $d = \mathrm{x}$, $\mathbf{b} = -\frac{1}{2}\begin{pmatrix}\tan\phi \\ 1\end{pmatrix}$ and $(N_y, N_x) = (0, 2)$ for $\phi \in [\pi/4, 3\pi/4]$, or
- $d = \mathrm{y}$, $\mathbf{b} = -\frac{1}{2}\begin{pmatrix}1 \\ \cot\phi\end{pmatrix}$ and $(N_y, N_x) = (2, 0)$ for $\phi \in [-\pi/4, \pi/4]$,

Step 2 Calculate an angle λ as follows:

$$\lambda = (-1)^{s_0}\left\{\cos^{-1}\left(1 - \|\mathbf{b}\|^2/2\right) + \cos^{-1}\left(\|\mathbf{b}\|/2\right)\right\},$$

where $s_0 \in \{0, 1\}$.

Step 3 Impose parameter matrices $\mathbf{W}_0$, $\mathbf{U}_0$, $\mathbf{U}_1^{\{d\}}$ and $\mathbf{U}_2^{\{d\}}$ to be

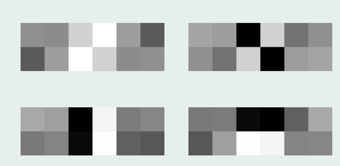
$$\mathbf{W}_0 = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{s_1}\end{pmatrix},\ \mathbf{U}_0 = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{s_2}\end{pmatrix}\begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda\end{pmatrix}\mathbf{P}[\mathbf{b}],$$

$$\mathbf{U}_1^{\{d\}} = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{s_3}\end{pmatrix}\mathbf{P}\left[-\mathbf{a}_2 - \mathbf{U}_0\mathbf{b}\right],\ \mathbf{U}_2^{\{d\}} = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{s_4}\end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

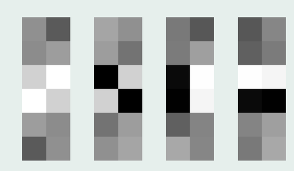
where $\mathbf{P}[\mathbf{x}]$ is a planer rotation or Householder matrix which maps vector $\mathbf{x}$ to vector $\mathbf{a}_2 = (1, 0)^T$, and $s_n \in \{0, 1\}$.

Step 4 Optimize parameters $s_n \in \{0, 1\}$ for $n = 0, 1, 2, 3, 4$ and θ for minimizing a given cost function.

- Design Examples



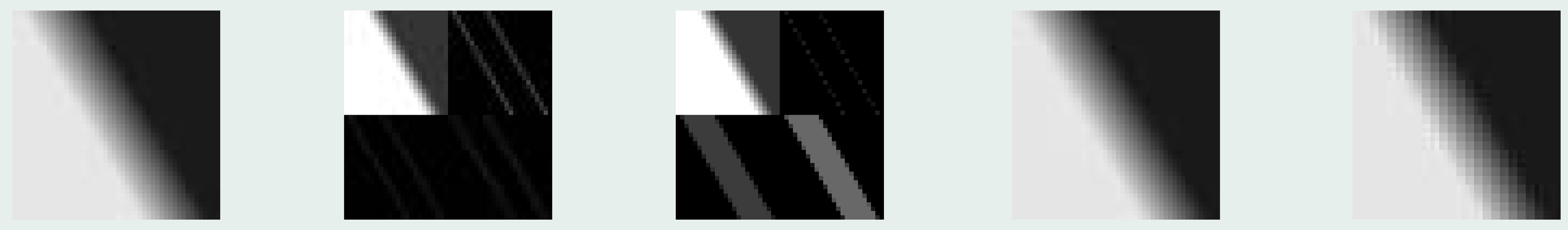
(a) $\phi = -\pi/6$, $(N_y, N_x) = (0, 2)$



(b) $\phi = 2\pi/3$, $(N_y, N_x) = (2, 0)$

Bases with two-order TVMs optimized for PB error & SB energy.

- Simulation Results



(c) Original (d) Subbands (DirLOT) (e) Subbands (Haar) (f) Reconstructed (DirLOT) PSNR=46.53dB (g) Reconstructed (Haar) PSNR=35.59dB

Simulation results of zonal coding for a ramp picture rotated by $-\pi/6$ (double precision grayscale of size 48×48).

Conclusions

- Theoretical properties of TVMs were investigated and then the mathematical meaning was discussed. Since the TVM condition imposes the moments point-wisely, the direction can flexibly be steered.
- Through simulations, the trend surface annihilation property was verified. The property is closely related to Laplace filters and attractive with the orthogonality when handling pictures in the transform domain.