Pappus's hexagon theorem

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1 Overview

This document presents Pappus's hexagon theorem, one of the fundamental results in projective geometry. We explore its statement, historical context, and provide proofs using both affine transformations and the elegant connection to Pascal's Theorem. The document is designed for olympiad-level high school students and advanced mathematics enthusiasts who wish to understand this beautiful geometric result and its deep connections to projective geometry.

2 Problem Statements

2.1 Pappus's hexagon theorem

Theorem 1 (Pappus's hexagon theorem)**.** *Let A, B, C be three distinct points on a line g, and let a, b, c be three distinct points on another line h.*

Consider the intersection points:

$$X = (Ab) \cap (aB)$$
$$Y = (Ac) \cap (aC)$$
$$Z = (Bc) \cap (bC)$$

Then the points $X, Y,$ and Z are collinear. This common line is called the ***Pappus line****.*

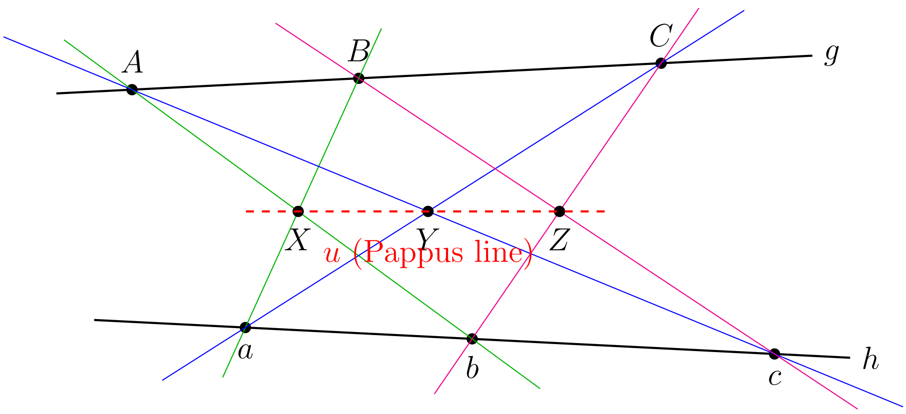


Figure 1: A diagram illustrating Pappus's hexagon theorem.

2.2 Special Case: Parallel Lines

When the lines g and h are parallel, Pappus's hexagon theorem still holds. In this case, the two lines never intersect, but the theorem remains valid in the affine plane. The Pappus line u will be parallel to both g and h in this configuration.

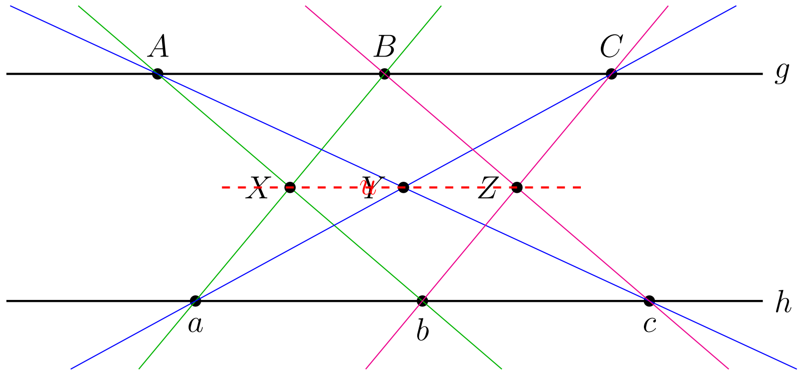


Figure 2: Pappus's hexagon theorem when lines g and h are parallel.

2.3 Special Case: Perpendicular Lines

When the lines g and h are perpendicular, Pappus's hexagon theorem provides a particularly elegant configuration. The perpendicular case can be transformed to the coordinate axes, making calculations more straightforward.

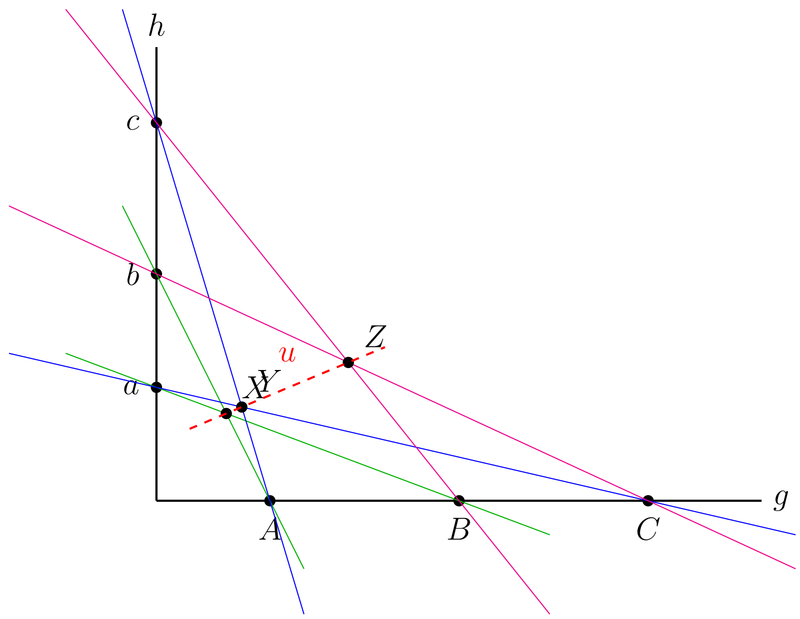


Figure 3: Pappus's hexagon theorem when lines g and h are perpendicular.

3 History of Pappus's hexagon theorem and Pappus

Pappus of Alexandria (c. 290–350 CE) was one of the last great mathematicians of antiquity. He lived in Alexandria, Egypt, during the later period of the Roman Empire. Pappus is best known for his work *Collection* (or *Synagoge*), a comprehensive compilation of Greek mathematical knowledge that preserved many results that would otherwise have been lost.

Pappus's hexagon theorem is one of the fundamental results in projective geometry. It was stated by Pappus in his *Collection*, though the modern formulation and proof techniques have evolved significantly since then.

The theorem holds a special place in the history of mathematics because:

- It is one of the earliest results in what we now call projective geometry

- It demonstrates the power of projective methods in solving geometric problems
- It connects to deeper results like Pascal's Theorem, showing the unity of geometric theory
- It remains a cornerstone result in modern algebraic geometry

Pappus's work influenced many later mathematicians, including Desargues, Pascal, and the founders of modern projective geometry in the 19th century.

4 Required Knowledge to Read This Document

Before diving into the solutions, readers should be familiar with the following concepts:

4.1 Linear Algebra

Basic understanding of vectors, linear transformations, and coordinate systems. Knowledge of matrix operations and their geometric interpretations will be helpful.

4.2 Linear Transformations

A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the origin and satisfies $T(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{w})$ for all vectors $\mathbf{v}, \mathbf{w}$ and scalars α, β. Linear transformations include rotations, reflections, scaling, and shearing.

4.3 Affine Transformations

An **affine transformation** is a composition of a linear transformation and a translation. Formally, $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where A is a matrix and $\mathbf{b}$ is a vector. Affine transformations preserve:

- Collinearity (points on a line remain on a line)
- Parallelism (parallel lines remain parallel)
- Ratios of distances along parallel lines

The key property we use is that affine transformations preserve collinearity, which allows us to simplify geometric problems by transforming them to more convenient coordinate systems.

4.4 Projective Geometry

Projective geometry extends Euclidean geometry by adding "points at infinity" and treating parallel lines as meeting at infinity. This unification allows many theorems to be stated more elegantly. The projective plane can be thought of as the Euclidean plane plus a "line at infinity."

In projective geometry, conic sections (circles, ellipses, parabolas, hyperbolas) are unified, and degenerate cases like two intersecting lines are also considered conics. This perspective is crucial for understanding the connection between Pappus's hexagon theorem and Pascal's Theorem.

5 Solutions

We present two approaches to proving Pappus's hexagon theorem: one using affine transformations (more computational) and one using Pascal's Theorem (more elegant and conceptual).
Note: Additional proofs of Pappus's hexagon theorem, including coordinate geometry proofs and other approaches, can be found on Wikipedia: https://en.wikipedia.org/wiki/Pappus's hexagon theorem.

5.1 Proof Using Affine Transformations

The strategy is to use an affine transformation to simplify the problem, then verify the result in the simplified case. Since affine transformations preserve collinearity, the result will hold in the general case.

5.1.1 The Big Idea

An affine transformation can "warp" the plane through translations, rotations, scaling, and shearing, but it preserves the property we care about: collinearity. If three points are collinear before transformation, they remain collinear after.

5.1.2 Case 1: Intersecting Lines

We consider the case where lines g and h intersect. By applying an affine transformation, we can map:

- Line g to the x-axis ($y = 0$)
- Line h to the y-axis ($x = 0$)
- The intersection point to the origin $(0, 0)$

After this transformation, we have:

- Points A, B, C on the x-axis: $A = (A_x, 0)$, $B = (B_x, 0)$, $C = (C_x, 0)$
- Points a, b, c on the y-axis: $a = (0, a_y)$, $b = (0, b_y)$, $c = (0, c_y)$

Using the intercept form of a line ($\frac{x}{x_{\text{int}}} + \frac{y}{y_{\text{int}}} = 1$), we can write equations for the lines:

- Line Ab: $\frac{x}{A_x} + \frac{y}{b_y} = 1$
- Line aB: $\frac{x}{B_x} + \frac{y}{a_y} = 1$

Solving these systems for each intersection point X, Y, Z and then verifying that they are collinear (by checking that the slopes $m_{XY} = m_{XZ}$) completes the proof. The algebra is tedious but straightforward, and the key insight is that the slopes turn out to be identical.

5.1.3 Case 2: Parallel Lines

When g and h are parallel, we can transform them to horizontal lines $y = 0$ and $y = 1$. A similar coordinate geometry argument shows that X, Y, Z are collinear, with the Pappus line being parallel to g and h.

Solution (Affine Transformation Proof)**.** *By applying an affine transformation, we reduce the general case to a coordinate geometry problem. In the simplified coordinate system (axes or parallel lines), direct calculation shows that the three intersection points X, Y, Z have the same slope, hence are collinear. Since affine transformations preserve collinearity, the result holds in the original configuration.*

5.2 Proof Using Pascal's Theorem

A more elegant proof recognizes that Pappus's hexagon theorem is a special case of Pascal's Theorem. This connection reveals the deeper structure of projective geometry.

5.2.1 The Connection

We view the two lines g and h as a **degenerate conic**—a conic section that has "degenerated" into two lines. In projective geometry, a pair of intersecting lines is considered a degenerate conic, defined by the equation $(ax+by+c)(dx+ey+f)=0$.

5.2.2 Constructing the Hexagon

We form a hexagon by alternating points from the two lines: $A-b-C-a-B-c$. This hexagon is inscribed in our degenerate conic (since all six vertices lie on $g \cup h$).

5.2.3 Applying Pascal's Theorem

Pascal's Theorem (discussed in detail in the next section) states that for any hexagon inscribed in a conic, the three intersection points of opposite sides are collinear. For our hexagon $A-b-C-a-B-c$:

- Opposite sides (Ab) and (aB) intersect at X
- Opposite sides (bC) and (Bc) intersect at Z
- Opposite sides (Ca) and (cA) intersect at Y

By Pascal's Theorem, X, Y, Z are collinear—exactly the statement of Pappus's hexagon theorem!

Solution (Pascal's Theorem Proof)**.** *Pappus's hexagon theorem follows immediately from Pascal's Theorem by recognizing that two lines form a degenerate conic. The hexagon $A-b-C-a-B-c$ inscribed in this degenerate conic has opposite sides intersecting at X, Y, Z, which must be collinear by Pascal's Theorem.*

6 Pascal's Theorem

6.1 Statement of Pascal's Theorem

Theorem 2 (Pascal's Theorem)**.** *Let A, B, C, D, E, F be six distinct points on a conic section (ellipse, parabola, or hyperbola). Consider the hexagon $A-B-C-D-E-F$ inscribed in the conic. Let the three pairs of opposite sides intersect at:*

$$X = (AB) \cap (DE)$$
$$Y = (BC) \cap (EF)$$
$$Z = (CD) \cap (FA)$$

*Then the points $X, Y,$ and Z are collinear. This line is called the **Pascal line**.*

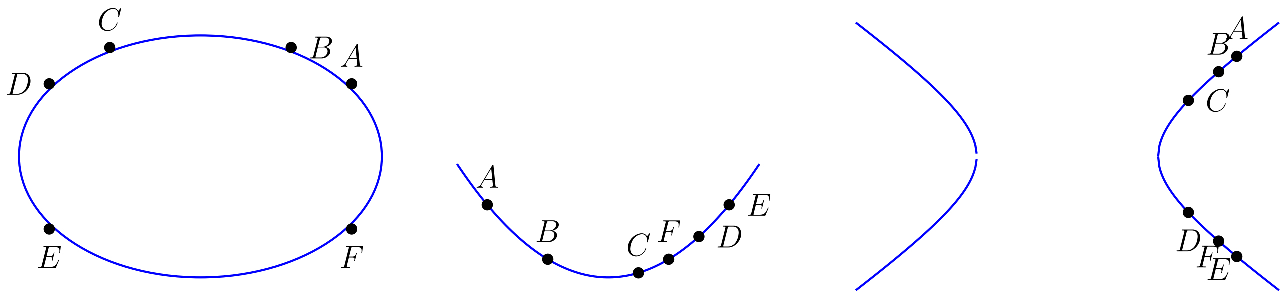
6.2 Note on Conic Sections

A **conic section** is a curve obtained by intersecting a cone with a plane. The three main types are:

- **Ellipse**: A closed curve (special case: circle)
- **Parabola**: An open curve with one branch
- **Hyperbola**: An open curve with two branches

In projective geometry, these are unified: all conics are projectively equivalent. Additionally, **degenerate conics**—pairs of lines (intersecting or parallel)—are also considered. This is why Pappus's hexagon theorem is a special case of Pascal's Theorem.

6.3 Illustrations of Pascal's Theorem



(a) Ellipse (b) Parabola (c) Hyperbola

Figure 4: Six points A, B, C, D, E, F on (a) ellipse, (b) parabola, and (c) hyperbola.

6.4 Formation of Conic Sections

Conic sections are curves formed by the intersection of a plane with a double cone (two cones joined at their apex). The type of conic section depends on the angle at which the plane intersects the cone.

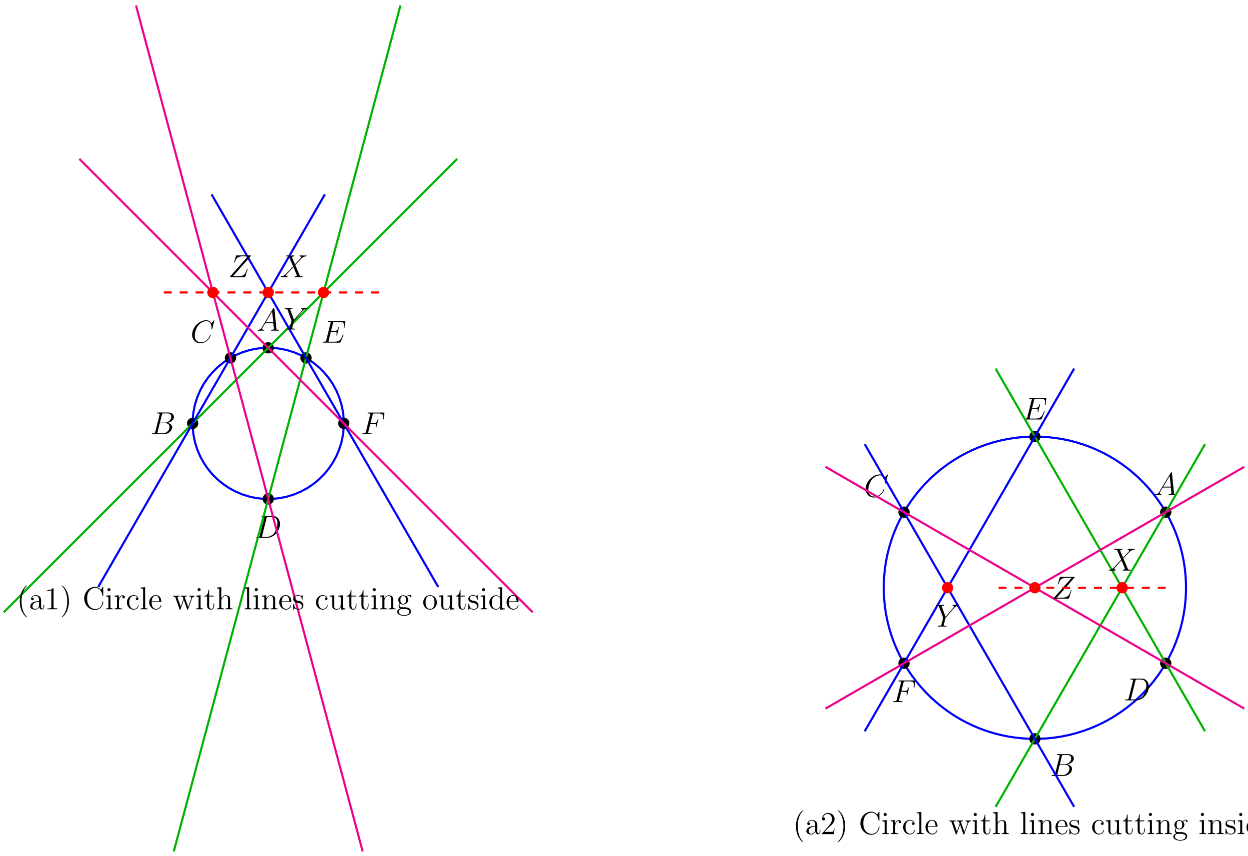
6.5 High-Level Proof of Pascal's Theorem

A complete proof of Pascal's Theorem requires advanced techniques from algebraic geometry or projective geometry. Here we outline the main ideas:

6.5.1 Projective Geometry Framework

We work in the projective plane, where:

- Every two distinct lines intersect (parallel lines meet at infinity)
- All conics are projectively equivalent
- The theorem can be stated purely in terms of incidence relations



(a1) Circle with lines cutting outside

(a2) Circle with lines cutting inside

Figure 5: Pascal's Theorem for a circle: (a1) when lines extend outside the circle, (a2) when lines intersect inside the circle. The intersection points are defined as $X = (AB) \cap (DE)$, $Y = (BC) \cap (EF)$, and $Z = (CD) \cap (FA)$. The red dashed line is the Pascal line.

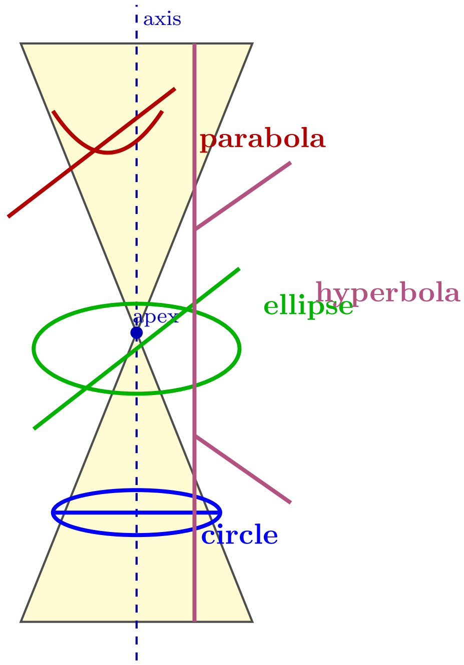


Figure 6: Formation of conic sections by intersecting a double cone with planes at different angles. A **circle** is formed by a horizontal plane (perpendicular to the axis), an **ellipse** by an angled plane (not parallel to the base and not through the apex), a **parabola** by a plane parallel to one side of the cone, and a **hyperbola** by a vertical plane cutting both cones.

6.5.2 Key Steps

1. **Projective Equivalence**: Since all conics are projectively equivalent, it suffices to prove the theorem for one conic (e.g., a circle).

2. **Coordinate Geometry Approach**: Using homogeneous coordinates, we can express the condition that six points lie on a conic as a determinant condition. The collinearity of X, Y, Z can then be verified algebraically.

3. **Bezout's Theorem**: This fundamental result in algebraic geometry states that two curves of degrees m and n intersect in at most mn points (counting multiplicity). This helps control the intersection structure.

4. **Cayley-Bacharach Theorem**: A more general result that implies Pascal's Theorem as a special case. It states that if two cubic curves intersect in nine points, then any cubic through eight of them must pass through the ninth.

6.5.3 The Elegant Insight

The most elegant modern proof uses the theory of cubic curves. The six points on the conic, together with the three intersection points X, Y, Z, can be related through cubic curves. The fact that the six points lie on a conic (a quadratic curve) forces a relationship that implies X, Y, Z are collinear.

6.5.4 Connection to Pappus

As we've seen, Pappus's hexagon theorem is the special case where the conic degenerates to two lines. This demonstrates the power of projective geometry: by working in the projective plane and considering degenerate cases, we unify many seemingly different theorems.

7 Future Works and Open Problems

Several directions for further exploration emerge from Pappus's hexagon theorem:

1. **Generalizations**: Pappus's hexagon theorem can be generalized to configurations with more points and lines. The study of such configurations leads to matroid theory and combinatorial geometry.

2. **Computational Geometry**: Efficient algorithms for verifying Pappus configurations and computing Pappus lines have applications in computer graphics and geometric modeling.

3. **Finite Projective Planes**: Pappus's hexagon theorem holds in finite projective planes as well. The study of finite geometries and their automorphism groups is an active area of research.

4. **Algebraic Proofs**: While we've presented geometric and projective proofs, purely algebraic proofs using coordinate geometry and elimination theory offer computational perspectives.

5. **Applications in Computer Vision**: Projective geometry, including results like Pappus's hexagon theorem, has applications in camera calibration and 3D reconstruction from images.

6. **Connection to Other Theorems**: Exploring relationships between Pappus's hexagon theorem, Desargues's Theorem, and other fundamental results in projective geometry reveals deep structural connections.

7. **Constructive Proofs**: Developing constructive (algorithmic) proofs that actually compute the Pappus line from given points has computational interest.

8. **Cayley–Bacharach Theorem**: This fundamental result in algebraic geometry states that if two cubic curves intersect in nine distinct points, then any cubic curve passing through eight of these points must also pass through the ninth. This theorem generalizes Pascal's Theorem and provides a deeper understanding of the intersection properties of curves in the projective plane. Exploring the connections between Pappus's hexagon theorem and the Cayley–Bacharach Theorem offers insights into the unified structure of projective geometry.

9. **Bézout's Theorem**: This cornerstone theorem in algebraic geometry asserts that two projective plane curves of degrees d_1 and d_2 without a common component intersect in exactly $d_1 \times d_2$ points, counted with multiplicity. Bézout's Theorem provides the theoretical foundation for understanding intersection multiplicities and is essential for proving many results in algebraic geometry, including those related to conic sections and their degenerations.

8 Conclusions

Pappus's hexagon theorem stands as a cornerstone of projective geometry, connecting classical Euclidean geometry to modern algebraic geometry. We have seen:

- The theorem's elegant statement about collinearity of intersection points

- Its historical significance dating back to Pappus of Alexandria
- Two proof approaches: the computational affine transformation method and the elegant connection to Pascal's Theorem
- The deep relationship between Pappus's hexagon theorem and Pascal's Theorem through the concept of degenerate conics
- The power of projective geometry in unifying seemingly different geometric phenomena

The theorem demonstrates that sometimes the most elegant proofs come from recognizing that a specific result is a special case of a more general theorem. By viewing two lines as a degenerate conic, Pappus's hexagon theorem becomes an immediate consequence of Pascal's Theorem, revealing the beautiful structure underlying projective geometry.

For olympiad students, mastering Pappus's hexagon theorem provides:

- A deeper understanding of projective geometry
- Experience with transformation techniques (affine transformations)
- Insight into how degenerate cases can illuminate general theorems
- A gateway to more advanced topics in algebraic geometry

Contact Information:

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GitHub: https://github.com/vuhung16au/

Repository: https://github.com/vuhung16au/math-olympiad-ml/tree/main/PappusTheorem