On the Convergence of Nonlinear Recurrence Sequences: $a_{n+1} = a_n + \frac{1}{\sqrt[3]{a_n}}$

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1 Problem T6/219 (1995)

Let the ordinal number $\{a_n\}$ be clearly defined as follows:

$$a_1 = 1, a_{n+1} = a_n + \frac{1}{\sqrt[3]{a_n}} \quad (n \geq 1)$$

Find all real numbers α such that the sequence $\{u_n\}$ defined by

$$u_n = \frac{a_n^\alpha}{n} \quad (n \geq 1)$$

converges and its limit is non-zero.

Solution

From the definition of the sequence $\{a_n\}$, we have

$$a_n > 0, \forall n \geq 1 \quad \text{and} \quad a_n^4 > \left(\sqrt[4]{a_{n-1}^4} + \frac{4}{3}\right)^3,$$

$\forall n \geq 2$. Hence,

$$\sqrt[4]{a_n^4} > \sqrt[4]{a_{n-1}^4} + \frac{4}{3}, \quad \forall n \geq 2 \implies$$

$$\sqrt[4]{a_n^4} > \frac{4}{3}(n-1), \quad \forall n \geq 2 \quad (1).$$

From the definition of the sequence $\{a_n\}$, we have

$$a_k = \left(\sqrt[3]{a_{k-1}} + \frac{1}{3a_{k-1}}\right)^3 - \left(\frac{1}{3\sqrt[3]{a_{k-1}}} + \frac{1}{27a_{k-1}^3}\right), \quad \forall k \geq 2$$

$$\implies \sqrt[3]{a_k}^4 < \left(\sqrt[3]{a_{k-1}} + \frac{1}{3a_{k-1}}\right)^4 =$$

$$= \sqrt[3]{a_{k-1}} + \frac{4}{3} + \frac{2}{3\sqrt[3]{a_{k-1}^4}} + \frac{4}{27\sqrt[3]{a_{k-1}^8}} + \frac{1}{81a_{k-1}^4},$$

$\forall k \geq 2$.

Hence, for each $n > 4$, we have:

$$\sqrt[3]{a_n} < 1 + \frac{4}{3}(n-1) + \frac{2}{3}\sum_{k=2}^{n} \frac{1}{\sqrt[3]{a_{k-1}^4}} + \frac{4}{27}\sum_{k=2}^{n} \frac{1}{\sqrt[3]{a_{k-1}^8}} + \frac{1}{81}\sum_{k=2}^{n} \frac{1}{a_{k-1}^4} \quad (2)$$

From (1) and the Cauchy-Schwarz inequality, we have:

- $\sum_{k=2}^{n} \frac{1}{\sqrt[3]{a_{k-1}^4}} = 1 + \sum_{k=3}^{n} \frac{1}{\sqrt[3]{a_{k-1}^4}}$

$$< 1 + \frac{3}{4}\sum_{k=3}^{n} \frac{1}{(k-2)} < 1 + \frac{3}{4}\sqrt{\sum_{k=3}^{n}(n-2)\frac{1}{(k-2)^2}}$$

$$< 1 + \frac{3}{4}\sqrt{(n-2)(1+\sum_{k=4}^{n}\frac{1}{(k-3)(k-2)})}$$

$$= 1 + \frac{3}{4}\sqrt{(n-2)}\sqrt{(2-\frac{1}{n-2})} < 1 + \frac{3}{4}\sqrt{2(n-2)} \quad (3)$$

- $\sum_{k=2}^{n} \frac{1}{\sqrt[3]{a_{k-1}^8}} = 1 + \sum_{k=3}^{n} \frac{1}{\sqrt[3]{a_{k-1}^2}}$

$$< 1 + \frac{9}{16}\sum_{k=3}^{n} \frac{1}{(k-2)^2} < 1 + \frac{9}{16} = \frac{17}{8} \quad (4)$$

- $\sum_{k=2}^{n} \frac{1}{a_{k-1}^4} = 1 + \sum_{k=3}^{n} \frac{1}{(\sqrt[3]{a_{k-1}^4})^3}$

$$< 1 + \frac{27}{64}\sum_{k=3}^{n} \frac{1}{(k-2)^3} < 1 + \frac{27}{64}\sum_{k=3}^{n} \frac{1}{(k-2)^2} <$$

$$< 1 + \frac{27}{64}\cdot\frac{32}{3} = 1 + \frac{27}{59} \quad (5)$$

From (2), (3), (4), (5), we have:

$$\sqrt[3]{a_n} < 1 + \frac{4}{3}\sqrt{2(n-2)} + \frac{35}{54n} + \frac{1}{8}\cdot\frac{59}{32} \quad (6)$$

$\forall n > 4$.

From (1) and (6), we have:

$$\frac{4}{3}\left(1-\frac{1}{n}\right) < \frac{a_n^{4/3}}{n} < \frac{4}{3}\cdot\frac{1}{n} + \frac{2\sqrt{2(n-2)}}{n} + \frac{35}{54n} + \frac{1}{8}\cdot\frac{59}{32n}, \quad \forall n > 4 \quad (7)$$

Since $\lim_{n\to\infty} \frac{4}{3}\left(1-\frac{1}{n}\right) = \frac{4}{3}$, we have:

$$\lim_{n\to\infty}\left(\frac{4}{3n} + \frac{2\sqrt{2(n-2)}}{n} + \frac{35}{54n} + \frac{1}{8}\frac{59}{32n}\right) = \lim_{n\to\infty}\frac{2\sqrt{2n}}{n} = 0$$

Hence, from (7), we have $\lim_{n\to\infty} \frac{\sqrt[4]{a_n}}{n} = \frac{4}{3}$. Hence, $\alpha = \frac{4}{3}$ is a value we need to find.

Since $\lim_{n\to\infty} a_n = +\infty$ from (1), we have:

$$\lim_{n\to\infty} a_n^{\alpha-\frac{4}{3}} = \begin{cases} +\infty & \text{nếu } \alpha > \frac{4}{3} \\ 0 & \text{nếu } \alpha < \frac{4}{3} \end{cases}$$

From $u_n = \frac{a_n^\alpha}{n} = \frac{a_n^{4/3}}{n}\cdot a_n^{\alpha-\frac{4}{3}}$, we have:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{a_n^\alpha}{n} = \begin{cases} +\infty & \text{nếu } \alpha > \frac{4}{3} \\ 0 & \text{nếu } \alpha < \frac{4}{3} \end{cases}$$

Hence, $\alpha = \frac{4}{3}$ is the only value that makes the sequence $\{u_n\}$ converge and its limit is non-zero.

Remarks

1. There are 8 students who submitted solutions to the problem. There are 8 students who submitted solutions to the problem. Among them, only the students: Ngô Đức Duy (12CT THPT Trần Phú - Hải Phòng), **Nguyễn Vũ Hưng** (12D Chuyên ngữ ĐHQG Hà Nội) and Lê Anh Vũ (12CT Quốc học Huế) have correct solutions. Student Vu has a complicated solution and Hung has to use knowledge beyond the standard high school curriculum to solve the problem.

2 Observation

- Looks like this is a hard problem for high school students.
- The idea in the solution above is using Talyor's expansion to approximate a_n, find a lower bound for $a_n^{4/3}$ and an upper bound for $a_n^{4/3}$, then use the squeeze theorem to find the limit.
- Only 8 students submitted solutions to the problem and only 3 of them have correct solutions.
- The solution of Nguyen Vu Hung is quite complicated and uses knowledge beyond the standard high school curriculum.

3 Differential Equation Approach (by Nguyen Vu Hung)

We approximate a_n by values of a smooth function $f(x)$ at integer points, i.e., $a_n \approx f(n)$. For large n, the forward difference satisfies $a_{n+1} - a_n \approx f'(n)$. (See the Mean Value Theorem in the Discussion section.)

From the recurrence $a_{n+1} - a_n = a_n^{-1/3}$, we obtain the separable Ordinary Differential Equation (ODE)

$$f'(x) = f(x)^{-1/3}.$$

Separating variables and integrating gives

$$\frac{df}{dx} = f^{-1/3} \quad \Rightarrow \quad f^{1/3}\,df = dx,$$

$$\int f^{1/3}\,df = \int dx \quad \Rightarrow \quad \frac{f^{4/3}}{4/3} = x + C,$$

so

$$f(x)^{4/3} = \frac{4}{3}x + C'.$$

As $x \to \infty$, the constant C' is negligible in the asymptotic sense, hence

$$f(x)^{4/3} \sim \frac{4}{3}x.$$

Consequently, for the sequence we have the approximation

$$a_n^{4/3} \sim \frac{4}{3}n.$$

Now consider $u_n = \frac{a_n^\alpha}{n}$. Using $a_n^{4/3} \sim \frac{4}{3}n$,

$$u_n = \frac{a_n^\alpha}{n} = \frac{\left(a_n^{4/3}\right)^{\alpha\cdot 3/4}}{n} \sim \frac{\left(\frac{4}{3}n\right)^{\frac{3\alpha}{4}}}{n} = \left(\frac{4}{3}\right)^{\frac{3\alpha}{4}} n^{\frac{3\alpha}{4}-1}.$$

For u_n to converge to a nonzero limit, the exponent of n must vanish, i.e.,

$$\frac{3\alpha}{4} - 1 = 0 \quad \Longrightarrow \quad \alpha = \frac{4}{3}.$$

When $\alpha = \frac{4}{3}$, the asymptotic limit is

$$\lim_{n\to\infty} u_n \sim \left(\frac{4}{3}\right)^{\frac{3(4/3)}{4}} n^0 = \left(\frac{4}{3}\right)^1 = \frac{4}{3},$$

which is consistent with the rigorous solution above.

4 The Difference Equations

We formulate the problem purely in the language of difference equations. Consider the first-order nonlinear difference equation

$$a_{n+1} - a_n = a_n^{-1/3}, \quad n \geq 1,$$

subject to the initial condition

$$a_1 = 1.$$

For a given real parameter α, define

$$u_n = \frac{a_n^{\alpha}}{n}.$$

Let $f(n) = a_n$ be a function of n, which is a discrete sequence that models the growth of a_n at integer points. As n is large, the forward difference satisfies $a_{n+1} - a_n \approx f'(n)$.

From the recurrence $a_{n+1} - a_n = a_n^{-1/3}$, we obtain the difference equation

$$f'(n) = f(n)^{-1/3}.$$

5 The Differential Equations

We state a continuous analogue of the problem via a differential equation. Let $f\colon [1, \infty) \to (0, \infty)$ be a differentiable function that models the growth of a_n at integer points, with the initial condition

$$f(1) = 1,$$

and governed by the first-order ODE

$$f'(x) = f(x)^{-1/3}.$$

For a given real parameter α, introduce the continuous analogue of u_n by

$$v(x) = \frac{f(x)^{\alpha}}{x}.$$

6 Solution (using Stolz–Cesáro theorem)

We recall a common form of the Stolz–Cesáro theorem: if (A_n) and (B_n) satisfy $B_n \nearrow \infty$ and the limit $\lim_{n\to\infty} \dfrac{A_{n+1} - A_n}{B_{n+1} - B_n} = L$ exists, then $\lim_{n\to\infty} \dfrac{A_n}{B_n} = L$. Apply this with $A_n = a_n^{4/3}$ and $B_n = n$. Then

$$\lim_{n\to\infty} \frac{A_{n+1} - A_n}{B_{n+1} - B_n} = \lim_{n\to\infty} \left(a_{n+1}^{4/3} - a_n^{4/3}\right).$$

Using $a_{n+1} = a_n + a_n^{-1/3}$ and the binomial expansion (or MVT) for $(x+h)^{4/3}$ with $h = a_n^{-1/3}$,

$$a_{n+1}^{4/3} - a_n^{4/3} = \frac{4}{3}\, a_n^{1/3} \cdot a_n^{-1/3} + o(1) = \frac{4}{3} + o(1).$$

Hence the difference limit equals $\frac{4}{3}$, and by Stolz–Cesáro,

$$\lim_{n\to\infty} \frac{a_n^{4/3}}{n} = \frac{4}{3}.$$

This yields the unique exponent $\alpha = \frac{4}{3}$ for which $u_n = a_n^{\alpha}/n$ has a nonzero finite limit.

7 Discussion

Scaling and dominant balance. A quick scaling ansatz $a_n \sim c\,n^p$ balances $a_{n+1} - a_n \asymp n^{p-1}$ with $a_n^{-1/3} \asymp n^{-p/3}$, yielding $p = \frac{3}{4}$ and $c = (\frac{4}{3})^{3/4}$. This immediately suggests both the exponent and the constant in the limit $\lim n^{-1} a_n^{4/3} = \frac{4}{3}$.

Discrete vs. continuum. Replacing differences by derivatives (or sums by integrals) is justified here by monotonicity and smooth growth. A rigorous bridge uses Stolz–Cesàro or the mean value theorem on $b_n = a_n^{4/3}$ to squeeze $n(b_{n+1} - b_n)$ between two sequences tending to $\frac{4}{3}$.

Regular variation. The sequence is regularly varying with index 3/4. Both the inequality proof and the ODE heuristic identify the same index and slowly varying constant, explaining why $u_n = a_n^\alpha/n$ has a nonzero limit iff $\alpha = \frac{4}{3}$.

Generalization. For $a_{n+1} = a_n + a_n^b$ with $b < 0$, the ODE $f' = f^b$ gives $f^{1-b} \sim (1-b)x$, hence $a_n \sim \text{const} \cdot n^{1/(1-b)}$ and $u_n \sim n^{\alpha/(1-b)-1}$. The unique nonzero-limit threshold is $\alpha = 1 - b$, matching the proposed general answer.

Error terms and robustness. Let $b_n := a_n^{4/3}$. By the mean value theorem,

$$b_{n+1} - b_n = \frac{4}{3}\,\xi_n^{1/3}\,(a_{n+1} - a_n), \quad \xi_n \in [a_n, a_{n+1}].$$

Using $a_{n+1} - a_n = a_n^{-1/3}$ and $\xi_n \asymp a_n$, we get

$$b_{n+1} - b_n = \frac{4}{3} + O\Big(a_n^{-4/3}\Big).$$

Summing yields the quantitative asymptotic

$$b_n = \frac{4}{3}\,n + O\left(\sum_{k \le n} a_k^{-4/3}\right),$$

so in particular $b_n = \frac{4}{3}n + O(\log n)$ once $a_n \asymp n^{3/4}$. Consequently,

$$\frac{a_n^{4/3}}{n} = \frac{4}{3} + O\Big(\frac{\log n}{n}\Big) \to \frac{4}{3}.$$

Moreover, for perturbed recurrences of the form

$$a_{n+1} = a_n + a_n^{-1/3} + \varepsilon_n, \qquad \varepsilon_n = o\Big(a_n^{-1/3}\Big),$$

exactly the same computation gives

$$b_{n+1} - b_n = \frac{4}{3} + o(1), \quad b_n = \frac{4}{3}\,n + o(n),$$

so the exponent 3/4 and the limit $\lim n^{-1} a_n^{4/3} = \frac{4}{3}$ are stable under small perturbations.

Brief definitions and notation. *Mean Value Theorem (MVT).* If g is differentiable on $[x, y]$, then there exists $\xi \in (x, y)$ such that $g(y) - g(x) = g'(\xi)(y - x)$. In our use, $g(t) = t^{4/3}$, $x = a_n$, $y = a_{n+1}$, and ξ_n denotes such an intermediate point.
$\asymp$ *notation.* For positive sequences (f_n) and (g_n), we write $f_n \asymp g_n$ if there exist constants $0 < c \le C < \infty$ and n_0 such that $c\,g_n \le f_n \le C\,g_n$ for all $n \ge n_0$.

Note on the Stolz–Cesáro theorem (statement and sketch). If (A_n) and (B_n) satisfy $B_n \nearrow \infty$ and $\lim\limits_{n\to\infty} \dfrac{A_{n+1} - A_n}{B_{n+1} - B_n} = L$ exists, then $\lim\limits_{n\to\infty} \dfrac{A_n}{B_n} = L$. Sketch: write

$$\frac{A_n}{B_n} = \frac{\sum_{k=1}^{n-1}(A_{k+1} - A_k)}{\sum_{k=1}^{n-1}(B_{k+1} - B_k)}$$

and view it as a weighted average of the ratios $\dfrac{A_{k+1} - A_k}{B_{k+1} - B_k}$ with positive weights $B_{k+1} - B_k$. If these ratios converge to L and the denominator diverges, the weighted average also converges to L (a Cesàro-type argument). This justifies replacing a hard ratio limit by the simpler difference ratio limit.

Pedagogical note. The ODE/dominant-balance route offers intuition and a clean roadmap; the discrete inequalities provide full rigor. Presenting both helps students connect heuristic modeling with proof techniques.

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Links:

- https://vuhung16au.github.io/
- https://github.com/vuhung16au/
- https://www.linkedin.com/in/nguyenvuhung/