

# HSC Math Extension 2: Inequalities Mastery

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## 1 Introduction

### 1.1 Project Overview

This booklet presents a comprehensive collection of inequality problems tailored for the HSC Mathematics Extension 2 syllabus. Each problem explores fundamental techniques including the Arithmetic Mean-Geometric Mean (AM-GM) inequality, Cauchy-Schwarz inequality, triangle inequality, integration-based inequalities, and inequalities via mathematical induction. Through rigorous proofs and detailed solutions, students will develop advanced problem-solving skills essential for Extension 2 examinations and mathematical competitions.

### 1.2 Target Audience

This resource is designed for Extension 2 students who want to challenge themselves with difficult problems and develop mastery of inequality techniques. Each solution provides step-by-step reasoning, explicit identification of key theorems, and clear algebraic manipulations to ensure high-school learners can follow every logical transition.

### 1.3 How to Use This Booklet

- Read the fundamentals and worked examples below before attempting problems.
- Attempt Part 1 problems without hints; compare your solutions against the detailed explanations.
- Study the **Takeaways** sections to understand the key techniques and strategies.
- Use the upside-down hints in Part 2 only after making a genuine attempt.
- Practice multiple problems of each type to reinforce pattern recognition and proof techniques.

### 1.4 Inequality Fundamentals

#### 1.4.1 Basic Properties

For real numbers  $a$ ,  $b$ ,  $c$ , and  $d$ , the following properties hold:

- **Transitivity:**  $a > b$  and  $b > c \Rightarrow a > c$
- **Multiplication:**  $a > b$  and  $c > 0 \Rightarrow ac > bc$ ; but  $c < 0 \Rightarrow ac < bc$
- **Product rule:**  $a > b > 0$  and  $c > d > 0 \Rightarrow ac > bd$
- **Reciprocals:**  $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$

- **Non-negativity:** For any real  $x$ , we have  $x^2 \geq 0$
- **Absolute values:**  $|a| \geq a$  and  $|x| + |y| \geq |x + y|$  (triangle inequality)
- **Sum of squares:**  $a^2 + b^2 \geq 0$  with equality if and only if  $a = b = 0$

### 1.4.2 Key Theorems

**AM-GM Inequality (Two Variables).** For non-negative real numbers  $x$  and  $y$ :

$$\frac{x + y}{2} \geq \sqrt{xy}$$

with equality if and only if  $x = y$ .

**AM-GM Inequality (Three Variables).** For non-negative real numbers  $x$ ,  $y$ , and  $z$ :

$$\frac{x + y + z}{3} \geq \sqrt[3]{xyz}$$

with equality if and only if  $x = y = z$ .

#### Remark 1.1 (Two Approaches to Proving AM-GM)

The AM-GM inequality states that for positive reals  $x_1, x_2, \dots, x_n$ :

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n}$$

##### (a) Proof by Induction on $n$ :

1. *Base case:*  $n = 2$  follows from  $(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0$
2. *Forward-backward step:* Prove  $n = 2^k \Rightarrow n = 2^{k+1}$  by grouping pairs
3. *Backward step:* Show  $n = k \Rightarrow n = k - 1$  by setting  $x_k = \frac{x_1 + \dots + x_{k-1}}{k-1}$  and applying the  $n = k$  case

##### (b) Proof using Convex Functions:

1. Consider  $f(x) = -\ln(x)$ , which is convex for  $x > 0$  (since  $f''(x) = \frac{1}{x^2} > 0$ )
2. By Jensen's inequality for convex functions:

$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}$$

3. Substituting  $f(x) = -\ln(x)$ :

$$-\ln\left(\frac{x_1 + \dots + x_n}{n}\right) \leq \frac{-\ln(x_1) - \dots - \ln(x_n)}{n} = -\ln(\sqrt[n]{x_1 \dots x_n})$$

4. Multiply by  $-1$  and exponentiate to obtain AM-GM

**Cauchy-Schwarz Inequality.** For real numbers  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$ :

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right) \left(\sum_{i=1}^n b_i^2\right)$$

with equality if and only if the sequences are proportional.

*Proof outline (discriminant method):* Let  $A = \sum a_i^2$ ,  $B = \sum b_i^2$ ,  $C = \sum a_i b_i$ , and consider

$$P(t) = \sum_{i=1}^n (a_i^2 t^2 - 2a_i b_i t + b_i^2) = At^2 - 2Ct + B \geq 0 \text{ for all } t.$$

Since  $P(t)$  is non-negative, its discriminant satisfies  $\Delta = (-2C)^2 - 4AB \leq 0$ , hence  $C^2 \leq AB$ . Substituting back yields  $(\sum a_i b_i)^2 \leq (\sum a_i^2) (\sum b_i^2)$ . Equality holds iff  $\Delta = 0$ , i.e., there exists  $t_0$  with  $a_i t_0 = b_i$  for all  $i$  (proportional sequences).

**Jensen's Inequality.** Let  $f$  be a convex function on an interval  $I$ . For points  $x_1, \dots, x_n \in I$  and non-negative weights  $\lambda_1, \dots, \lambda_n$  with  $\sum_{i=1}^n \lambda_i = 1$ , Jensen's inequality states

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

If  $f$  is concave, the inequality is reversed. Equality holds when the  $x_i$  are all equal or when  $f$  is linear on the convex hull of the points.

Remark: This is a so powerful Inequality and you should use it with care. To use Jensen's inequality, prove a function  $f(x)$  is convex by showing  $f''(x) \geq 0$  on the interval of interest, then state clearly "By Jensen's Inequality, since  $f(x)$  is convex...". The risk is you may lose "working marks" if the markers feel you bypassed the intended "nature of proof" technique.

**Triangle Inequality (Real Numbers).** For real numbers  $a$  and  $b$ :

$$|a + b| \leq |a| + |b|$$

**Triangle Inequality (Complex Numbers).** For complex numbers  $z$  and  $w$ :

$$|z + w| \leq |z| + |w|$$

## 1.5 Worked Examples

**Example 1: Basic AM-GM Application.** Prove that for positive real numbers  $a$  and  $b$ , we have  $a + b \geq 2\sqrt{ab}$ .

*Solution:* By the AM-GM inequality with  $x = a$  and  $y = b$ :

$$\frac{a + b}{2} \geq \sqrt{ab}$$

Multiplying both sides by 2 yields  $a + b \geq 2\sqrt{ab}$ , with equality when  $a = b$ . □

**Example 2: Cauchy-Schwarz with Constraint.** Given  $x^2 + y^2 = 1$ , find the maximum value of  $3x + 4y$ .

*Solution:* By Cauchy-Schwarz inequality:

$$(3x + 4y)^2 \leq (3^2 + 4^2)(x^2 + y^2) = 25 \cdot 1 = 25$$

Therefore  $3x + 4y \leq 5$ , with equality when  $(x, y) = (\frac{3}{5}, \frac{4}{5})$ . The maximum value is 5. □

**Example 3: Triangle Inequality.** Prove that for any complex number  $z$  with  $|z| = 1$ , we have  $|z^2 + z + 1| \leq 3$ .

*Solution:* Using the triangle inequality repeatedly:

$$|z^2 + z + 1| \leq |z^2| + |z| + |1| = 1 + 1 + 1 = 3$$

with equality when  $z = 1$ . □

**Example 4: Integration Inequality.** Prove that  $\int_0^1 x^2 dx < \int_0^1 x dx$ .

*Solution:* For  $x \in [0, 1]$ , we have  $x^2 \leq x$  (with equality only at  $x = 0$  and  $x = 1$ ). Therefore:

$$\int_0^1 x^2 dx < \int_0^1 x dx$$

**Example 5: Induction with Inequality.** Prove by induction that  $2^n > n$  for all integers  $n \geq 1$ .

*Solution:* Base case ( $n = 1$ ):  $2^1 = 2 > 1$ . ✓

*Inductive step:* Assume  $2^k > k$  for some  $k \geq 1$ . Then:

$$2^{k+1} = 2 \cdot 2^k > 2k$$

Since  $k \geq 1$ , we have  $2k = k + k \geq k + 1$ . Therefore  $2^{k+1} > k + 1$ . By induction, the result holds for all  $n \geq 1$ . □

## 1.6 Notation and Conventions

- Unless stated otherwise, variables represent real numbers.
- The notation  $a, b > 0$  means both  $a$  and  $b$  are positive.
- “Prove” indicates a complete justification is required.
- “Hence” or “deduce” means use the previous result directly.
- Equality conditions identify when an inequality becomes an equality.

## 2 Part 1: Problems and Solutions (Detailed)

Part 1 contains 15 carefully selected problems—five basic, five medium, and five advanced. Each problem includes a comprehensive solution with step-by-step reasoning and a **Takeaways** section highlighting key techniques and strategic insights.

### 2.1 Basic Inequality Problems

#### Problem 2.1: Arithmetic Mean-Geometric Mean Inequality

For positive real numbers  $a$  and  $b$ , prove that  $\frac{a+b}{2} \geq \sqrt{ab}$ .

Hence, or otherwise, show that  $\frac{2n+1}{2n+2} < \frac{\sqrt{2n+1}}{\sqrt{2n+3}}$  for any integer  $n \geq 0$ .

**Solution 2.1**

**Part (i):** Since  $a$  and  $b$  are positive real numbers,  $\sqrt{a}$  and  $\sqrt{b}$  are real numbers. We know that the square of any real number is non-negative:

$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$

Expanding the left side:

$$\begin{aligned} (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0 \\ a - 2\sqrt{ab} + b &\geq 0 \end{aligned}$$

Adding  $2\sqrt{ab}$  to both sides:

$$a + b \geq 2\sqrt{ab}$$

Dividing both sides by 2:

$$\frac{a + b}{2} \geq \sqrt{ab}$$

This is the Arithmetic Mean-Geometric Mean (AM-GM) Inequality. Equality holds if and only if  $a = b$ .

**Part (ii):** We want to show that  $\frac{2n+1}{2n+2} < \frac{\sqrt{2n+1}}{\sqrt{2n+3}}$ .

Since all terms are positive for  $n \geq 0$ , we can square both sides without changing the direction of the inequality:

$$\begin{aligned} \left(\frac{2n+1}{2n+2}\right)^2 &< \left(\frac{\sqrt{2n+1}}{\sqrt{2n+3}}\right)^2 \\ \frac{(2n+1)^2}{(2n+2)^2} &< \frac{2n+1}{2n+3} \end{aligned}$$

Since  $2n+1 > 0$ , we can divide both sides by  $2n+1$ :

$$\frac{2n+1}{(2n+2)^2} < \frac{1}{2n+3}$$

Cross-multiplying:

$$(2n+1)(2n+3) < (2n+2)^2$$

Expanding both sides:

$$\begin{aligned} 4n^2 + 6n + 2n + 3 &< 4n^2 + 8n + 4 \\ 4n^2 + 8n + 3 &< 4n^2 + 8n + 4 \end{aligned}$$

Simplifying:

$$3 < 4$$

Since  $3 < 4$  is always true, the original inequality holds for any integer  $n \geq 0$ .

### Takeaways 2.1

- **Key Technique:** The AM-GM inequality is proven by considering the non-negativity of a perfect square:  $(\sqrt{a} - \sqrt{b})^2 \geq 0$ .
- **Strategy:** When proving inequalities involving fractions with square roots, squaring both sides can simplify the expression while preserving the inequality direction (provided all terms are positive).
- **Cross-Multiplication:** After squaring and simplification, cross-multiplication converts the inequality to a polynomial form that can be verified directly.
- **Common Pitfall:** When squaring inequalities, always verify that all terms are positive; otherwise, the inequality direction may reverse.
- **Verification:** Reducing the problem to a simple numerical inequality (like  $3 < 4$ ) provides a complete and rigorous proof.

### Problem 2.2: AM-GM with Non-Negative Reals

For real numbers  $a, b \geq 0$ , prove that:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

### Solution 2.2

Since  $a$  and  $b$  are non-negative real numbers,  $\sqrt{a}$  and  $\sqrt{b}$  are real numbers. We know that the square of any real number is always non-negative. Therefore, we begin with:

$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$

Expanding the square:

$$\begin{aligned} (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0 \\ a - 2\sqrt{ab} + b &\geq 0 \end{aligned}$$

Rearranging terms:

$$a + b \geq 2\sqrt{ab}$$

Dividing both sides by 2:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Thus, the inequality is proven. Note that equality holds if and only if  $(\sqrt{a} - \sqrt{b})^2 = 0$ , which implies  $a = b$ .

### Takeaways 2.2

- **Key Technique:** The AM-GM inequality for non-negative reals follows directly from the non-negativity of  $(\sqrt{a} - \sqrt{b})^2$ .
- **Equality Condition:** Equality holds when  $a = b$ , which occurs when the squared difference is zero.
- **Domain Consideration:** The requirement that  $a, b \geq 0$  ensures that  $\sqrt{a}$  and  $\sqrt{b}$  are real numbers.
- **Common Application:** This fundamental inequality is frequently used as a stepping stone in more complex inequality proofs.
- **Algebraic Manipulation:** The proof demonstrates how to systematically expand, rearrange, and isolate terms to establish the desired inequality.

### Problem 2.3: Logarithmic Inequalities and Euler's Number

Explain why

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x} dx < \frac{1}{n}.$$

Hence, deduce that

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$$

### Solution 2.3

**Part 1:** For  $f(x) = \frac{1}{x}$  strictly decreasing on  $[n, n+1]$ , we have  $\frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n}$ . Integrating:

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x} dx < \frac{1}{n} \implies \frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

Thus  $\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$ .

**Part 2: Left:** Multiply by  $n+1$ :  $1 < \ln\left[\left(1 + \frac{1}{n}\right)^{n+1}\right] \implies e < \left(1 + \frac{1}{n}\right)^{n+1}$ .

**Right:** Multiply by  $n$ :  $\ln\left[\left(1 + \frac{1}{n}\right)^n\right] < 1 \implies \left(1 + \frac{1}{n}\right)^n < e$ .

Therefore:  $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$

### Takeaways 2.3

- **Key Technique:** Using the monotonicity of  $f(x) = \frac{1}{x}$  to bound a definite integral by rectangles is a standard calculus technique.
- **Integration Bounds:** For decreasing functions, the minimum value on an interval provides a lower bound for the integral, while the maximum value provides an upper bound.
- **Logarithm Properties:** The transformation  $\ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right)$  is crucial for connecting the integral to the exponential form.
- **Exponentiation Preserves Inequality:** Since  $e^x$  is an increasing function, exponentiating both sides of  $\ln(A) < \ln(B)$  gives  $A < B$ .
- **Historical Significance:** This inequality provides a rigorous way to bound Euler's number  $e$  using sequences, demonstrating the limit definition  $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ .

### Problem 2.4: Squared Terms Inequality

For  $x, y > 0$ , prove that:

(a)  $x^2 + y^2 \geq 2xy$

(b)  $\frac{1}{x^4} + \frac{1}{y^4} \geq \frac{2}{x^2y^2}$

### Solution 2.4

**Part (a):** We start with the fundamental property that the square of any real number is non-negative. Consider the square of the difference between  $x$  and  $y$ :

$$(x - y)^2 \geq 0$$

Expanding the left-hand side:

$$x^2 - 2xy + y^2 \geq 0$$

Adding  $2xy$  to both sides:

$$x^2 + y^2 \geq 2xy$$

This proves the inequality for all real  $x, y$ . Since  $x, y > 0$ , the inequality holds. Equality occurs when  $(x - y)^2 = 0$ , which means  $x = y$ .

**Part (b):** We can deduce part (b) by using the result from part (a). Let us substitute terms into the inequality  $a^2 + b^2 \geq 2ab$ .

Let  $a = \frac{1}{x^2}$  and  $b = \frac{1}{y^2}$ . Since  $x, y > 0$ , both  $a$  and  $b$  are positive real numbers.

Using the result from part (a):

$$\begin{aligned} a^2 + b^2 &\geq 2ab \\ \left(\frac{1}{x^2}\right)^2 + \left(\frac{1}{y^2}\right)^2 &\geq 2\left(\frac{1}{x^2}\right)\left(\frac{1}{y^2}\right) \\ \frac{1}{x^4} + \frac{1}{y^4} &\geq \frac{2}{x^2y^2} \end{aligned}$$

This completes the proof.

### Takeaways 2.4

- **Key Technique:** Many quadratic inequalities can be proven by starting with  $(x - y)^2 \geq 0$  and expanding.
- **Substitution Strategy:** Part (b) demonstrates how to generalize an inequality by making appropriate substitutions ( $a = \frac{1}{x^2}$ ,  $b = \frac{1}{y^2}$ ).
- **Building on Results:** Using a proven result (part a) to establish a new inequality (part b) is a powerful problem-solving technique.
- **Equality Condition:** For part (a), equality holds when  $x = y$ ; for part (b), equality holds when  $\frac{1}{x^2} = \frac{1}{y^2}$ , which also means  $x = y$ .
- **Common Pitfall:** When making substitutions, ensure that the new variables satisfy the same positivity conditions required by the original inequality.

### Problem 2.5: Cauchy-Schwarz Inequality Application

Let  $x, y, z$  be real numbers satisfying the linear equation  $x + 2y + 3z = 14$ .

- (i) Prove that  $x^2 + y^2 + z^2 \geq 14$ .
- (ii) Determine the values of  $x, y, z$  for which equality holds.

### Solution 2.5

**Part (i):** We apply the Cauchy-Schwarz Inequality to vectors  $\mathbf{u} = (1, 2, 3)$  and  $\mathbf{v} = (x, y, z)$ .

The Cauchy-Schwarz Inequality states:

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq |\mathbf{u}|^2 |\mathbf{v}|^2$$

In component form:

$$(1 \cdot x + 2 \cdot y + 3 \cdot z)^2 \leq (1^2 + 2^2 + 3^2)(x^2 + y^2 + z^2)$$

Calculate the squared magnitude of  $\mathbf{u}$ :

$$1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

Substitute the given constraint  $x + 2y + 3z = 14$ :

$$\begin{aligned}(14)^2 &\leq 14(x^2 + y^2 + z^2) \\ 196 &\leq 14(x^2 + y^2 + z^2) \\ 14 &\leq x^2 + y^2 + z^2\end{aligned}$$

Therefore,  $x^2 + y^2 + z^2 \geq 14$ .

**Part (ii):** Equality in the Cauchy-Schwarz Inequality holds if and only if the vectors  $\mathbf{u}$  and  $\mathbf{v}$  are proportional. That is:

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = k$$

for some scalar  $k$ . Thus,  $x = k$ ,  $y = 2k$ , and  $z = 3k$ .

Substitute these into the constraint equation:

$$\begin{aligned}k + 2(2k) + 3(3k) &= 14 \\ k + 4k + 9k &= 14 \\ 14k &= 14 \\ k &= 1\end{aligned}$$

Therefore, equality holds when  $x = 1$ ,  $y = 2$ ,  $z = 3$ .

We can verify:  $x^2 + y^2 + z^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$  and  $x + 2y + 3z = 1 + 4 + 9 = 14$ .

### Takeaways 2.5

- **Key Technique:** The Cauchy-Schwarz Inequality is a powerful tool for proving inequalities involving sums of products and sums of squares.
- **Vector Interpretation:** Recognizing the problem as a dot product  $\mathbf{u} \cdot \mathbf{v} = 14$  allows us to apply the Cauchy-Schwarz Inequality directly.
- **Equality Condition:** For Cauchy-Schwarz, equality holds if and only if the vectors are proportional, providing a systematic method to find when the minimum is achieved.
- **Verification:** Always verify the equality case by substituting back into both the constraint and the inequality.
- **Common Application:** This technique extends to constrained optimization problems where you need to minimize or maximize a quadratic form subject to a linear constraint.

## 2.2 Medium Inequality Problems

### Problem 2.6: Arithmetic Sequence of Reciprocals

Positive real numbers  $a, b, c$  and  $d$  are chosen such that  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$  and  $\frac{1}{d}$  are consecutive terms in an arithmetic sequence with common difference  $k$ , where  $k \in \mathbb{R}, k > 0$ . Show that  $b + c < a + d$ .

**Solution 2.6**

Since  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$  are consecutive terms in an arithmetic sequence with common difference  $k > 0$ , we have:

$$\frac{1}{b} = \frac{1}{a} + k \implies k = \frac{1}{b} - \frac{1}{a} = \frac{a - b}{ab} \quad (1)$$

$$\frac{1}{c} = \frac{1}{b} + k \implies k = \frac{1}{c} - \frac{1}{b} = \frac{b - c}{bc} \quad (2)$$

$$\frac{1}{d} = \frac{1}{c} + k \implies k = \frac{1}{d} - \frac{1}{c} = \frac{c - d}{cd} \quad (3)$$

Since  $k > 0$  and  $a, b, c, d$  are positive, the numerators must be positive, which implies:

$$a - b > 0 \implies a > b$$

$$b - c > 0 \implies b > c$$

$$c - d > 0 \implies c > d$$

Thus,  $a > b > c > d$ .

We want to show that  $b + c < a + d$ , which is equivalent to showing  $0 < (a + d) - (b + c)$ , or  $0 < (a - b) - (c - d)$ .

From equations (1) and (3), we can express the differences in terms of  $k$ :

$$a - b = k \cdot ab \quad (4)$$

$$c - d = k \cdot cd \quad (5)$$

We need to compare  $k \cdot ab$  and  $k \cdot cd$ . Since  $k > 0$ , the inequality  $a - b > c - d$  is equivalent to showing:

$$ab > cd$$

Since we established  $a > b > c > d$ , it is clear that  $a > c$  and  $b > d$ . Since all are positive:

$$a > c > 0$$

$$b > d > 0$$

Multiplying these two inequalities:

$$ab > cd$$

Multiplying by  $k > 0$ :

$$k \cdot ab > k \cdot cd$$

Substituting from equations (4) and (5):

$$a - b > c - d$$

Rearranging the terms:

$$a + d > b + c$$

Therefore,  $b + c < a + d$  as required.

### Takeaways 2.6

- **Technique:** Converting arithmetic sequence conditions into algebraic equations allows us to extract ordering information about the original terms.
- **Strategy:** When dealing with reciprocals in arithmetic progression, recognize that the original terms form a decreasing sequence, and use this monotonicity to compare products.
- **Key Insight:** Express target differences as products with a common positive factor ( $k$ ), reducing the problem to comparing products of ordered terms.
- **Pitfall:** Don't forget that  $k > 0$  is crucial for establishing the ordering  $a > b > c > d$ ; without this, the inequality direction could reverse.

### Problem 2.7: Cascading AM-GM Applications

For all non-negative real numbers  $x$  and  $y$ ,  $\sqrt{xy} \leq \frac{x+y}{2}$ . (Do NOT prove this.)

- (i) Using this fact, show that for all non-negative real numbers  $a, b$  and  $c$ ,

$$\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}.$$

- (ii) Using part (i), or otherwise, show that for all non-negative real numbers  $a, b$  and  $c$ ,

$$\sqrt{abc} \leq \frac{a^2 + b^2 + c^2 + a + b + c}{6}.$$

### Solution 2.7

**Part (i):** Apply AM-GM to  $a^2, b^2$ :  $ab \leq \frac{a^2+b^2}{2}$ . Apply AM-GM to  $ab, c$ :

$$\sqrt{abc} \leq \frac{ab + c}{2} \leq \frac{\frac{a^2+b^2}{2} + c}{2} = \frac{a^2 + b^2 + 2c}{4}$$

**Part (ii):** By cyclic permutation of (i):

$$\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}, \quad \sqrt{abc} \leq \frac{b^2 + c^2 + 2a}{4}, \quad \sqrt{abc} \leq \frac{c^2 + a^2 + 2b}{4}$$

Adding:  $3\sqrt{abc} \leq \frac{2(a^2+b^2+c^2+a+b+c)}{4} = \frac{a^2+b^2+c^2+a+b+c}{2}$ .

Thus:  $\sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}$

### Takeaways 2.7

- **Technique:** Cascade AM-GM applications by strategically choosing pairs of terms, then use the resulting inequality as input for another AM-GM application.
- **Strategy:** When proving symmetric inequalities, exploit cyclic symmetry by generating multiple versions of an intermediate result and summing them.
- **Key Insight:** The transition from part (i) to part (ii) demonstrates how adding symmetric inequalities can yield a stronger bound with better balance among variables.
- **Common Pattern:** Notice that  $\frac{a^2+b^2}{2} \geq ab$  is a specific application of AM-GM to squares, which often serves as a useful intermediate step.

### Problem 2.8: Inductive Sum of Squared Reciprocals

Prove by mathematical induction that, for  $n \geq 2$ ,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$

### Solution 2.8

**Base Case ( $n = 2$ ):** LHS =  $\frac{1}{4}$ , RHS =  $\frac{1}{2}$ . Since  $\frac{1}{4} < \frac{1}{2}$ , true for  $n = 2$ .

**Inductive Hypothesis:** Assume  $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < \frac{k-1}{k}$  (\*)

**Inductive Step:** Need to show  $\frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \frac{k}{k+1}$ .

Using (\*): LHS  $< \frac{k-1}{k} + \frac{1}{(k+1)^2}$ .

The gap is:  $\frac{k}{k+1} - \frac{k-1}{k} = \frac{1}{k(k+1)}$

Need:  $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} \iff k(k+1) < (k+1)^2 \iff k < k+1$ .

Thus:  $\frac{k-1}{k} + \frac{1}{(k+1)^2} < \frac{k}{k+1}$

By induction, the inequality holds for all  $n \geq 2$ .

### Takeaways 2.8

- **Technique:** In inductive proofs of inequalities, compute the “gap” between successive right-hand sides to determine what bound is needed on the new term.
- **Strategy:** Show that the additional term  $\frac{1}{(k+1)^2}$  is strictly less than the increase in the RHS from  $\frac{k-1}{k}$  to  $\frac{k}{k+1}$ .
- **Key Insight:** The inequality  $k(k+1) < (k+1)^2$  (equivalently  $k < k+1$ ) is the critical comparison that makes the inductive step work.
- **Pitfall:** Don’t assume the new term is small enough without verification; always explicitly show the required inequality between the new term and the gap in the RHS.

### Problem 2.9: Power Mean Inequality via QM-RMS

For positive real numbers  $x$  and  $y$ ,  $\sqrt{xy} \leq \frac{x+y}{2}$ . (Do NOT prove this.)

- (i) Prove  $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$ , for positive real numbers  $x$  and  $y$ .
- (ii) Prove  $\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$ , for positive real numbers  $a, b, c$  and  $d$ .

### Solution 2.9

**Part (i):** From  $(x - y)^2 \geq 0$ :  $x^2 + y^2 \geq 2xy \implies \frac{x^2+y^2}{2} \geq xy$ .

Taking square roots:  $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$

**Part (ii):** Note  $\sqrt[4]{abcd} = \sqrt{\sqrt{ab} \cdot \sqrt{cd}}$ . Apply Part (i) to pairs  $(a, b)$  and  $(c, d)$ :

$$\sqrt{ab} \leq \sqrt{\frac{a^2 + b^2}{2}}, \quad \sqrt{cd} \leq \sqrt{\frac{c^2 + d^2}{2}}$$

Let  $X = \sqrt{\frac{a^2+b^2}{2}}$ ,  $Y = \sqrt{\frac{c^2+d^2}{2}}$ . Then  $\sqrt{abcd} \leq XY$ .

Apply Part (i) to  $X, Y$ :  $XY \leq \sqrt{\frac{X^2+Y^2}{2}} = \sqrt{\frac{\frac{a^2+b^2}{2} + \frac{c^2+d^2}{2}}{2}} = \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$

Therefore:  $\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$

### Takeaways 2.9

- **Technique:** The inequality  $(x - y)^2 \geq 0$  is a fundamental tool for proving that the quadratic mean (RMS) dominates the geometric mean.
- **Strategy:** Build up to higher-order inequalities by pairing terms and applying proven results recursively; here we go from 2 terms to 4 terms.
- **Key Insight:** The relationship  $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$  (GM  $\leq$  QM) serves as a bridge to extend AM-GM style inequalities to power means.
- **Common Pattern:** When dealing with fourth roots of products, rewrite as square roots of square roots, then apply two-variable inequalities twice.

### Problem 2.10: Calculus and Induction for Harmonic Inequality

- (i) Use calculus to show that  $x > \ln(1 + x)$  for all  $x > 0$ .
- (ii) Use the inequality in part (i) and the principle of mathematical induction to prove that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(1 + n)$$

for all positive integers,  $n$ .

### Solution 2.10

**Part (i):** Let  $f(x) = x - \ln(1+x)$ . Then  $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$  for  $x > 0$ . Since  $f(0) = 0$  and  $f$  is strictly increasing for  $x > 0$ , we have  $f(x) > 0$  for all  $x > 0$ . Therefore,  $x > \ln(1+x)$  for all  $x > 0$ .

**Part (ii): Base ( $n = 1$ ):** LHS = 1, RHS =  $\ln(2) \approx 0.693$ . Since  $1 > \ln(2)$ , true.

**Hypothesis:** Assume  $\sum_{r=1}^k \frac{1}{r} > \ln(1+k)$  (\*)

**Step:** Using (\*): LHS  $> \ln(k+1) + \frac{1}{k+1}$ .

From Part (i) with  $x = \frac{1}{k+1}$ :

$$\frac{1}{k+1} > \ln\left(1 + \frac{1}{k+1}\right) = \ln\left(\frac{k+2}{k+1}\right) = \ln(k+2) - \ln(k+1)$$

Thus: LHS  $> \ln(k+1) + [\ln(k+2) - \ln(k+1)] = \ln(k+2)$

By induction, the inequality holds for all positive integers  $n$ .

### Takeaways 2.10

- **Technique:** Use calculus to establish a continuous inequality, then leverage it as a lemma in an inductive proof for a discrete sum.
- **Strategy:** The key connection is recognizing that  $\frac{1}{k+1} > \ln(k+2) - \ln(k+1)$  allows us to bridge from  $\ln(k+1)$  to  $\ln(k+2)$  in the inductive step.
- **Key Insight:** The logarithm property  $\ln(a) - \ln(b) = \ln(a/b)$  is crucial for converting the Part (i) inequality into a form usable in the induction.
- **Common Pattern:** When proving inequalities involving harmonic sums and logarithms, calculus-based lemmas about  $\ln(1+x)$  frequently serve as bridges between consecutive cases.
- **Pitfall:** Don't forget to verify that the substitution  $x = \frac{1}{k+1}$  satisfies the domain condition  $x > 0$  required for Part (i) to apply.

## 2.3 Advanced Inequality Problems

### Problem 2.11: Exponential Bounds on Factorials

- Prove that  $x > \ln x$ , for  $x > 0$ .
- Using part (i), or otherwise, prove that for all positive integers  $n$ ,

$$e^{n^2+n} > (n!)^2.$$

### Solution 2.11

**Part (i):** Let  $f(x) = x - \ln x$ . Then  $f'(x) = 1 - \frac{1}{x} = 0 \implies x = 1$ .

Since  $f''(x) = \frac{1}{x^2} > 0$ ,  $f$  has a minimum at  $x = 1$  with  $f(1) = 1 - 0 = 1 > 0$ .

Therefore,  $x > \ln x$  for all  $x > 0$ .

**Part (ii):** Apply  $\ln$  to both sides:  $e^{n^2+n} > (n!)^2 \iff n^2 + n > 2 \ln(n!) \iff \frac{n^2+n}{2} > \sum_{k=1}^n \ln k$

From part (i),  $k > \ln k$  for all positive integers  $k$ . Summing from  $k = 1$  to  $n$ :

$$\sum_{k=1}^n k > \sum_{k=1}^n \ln k \implies \frac{n(n+1)}{2} = \frac{n^2+n}{2} > \sum_{k=1}^n \ln k$$

Exponentiating:  $e^{n^2+n} > (n!)^2$

### Takeaways 2.11

- **Calculus technique:** Use first and second derivatives to find and classify critical points, then evaluate the function at the critical point to determine global behavior.
- **Summation strategy:** Apply a single inequality to multiple values, then sum all inequalities together to obtain a cumulative result.
- **Logarithmic transformation:** Convert multiplicative inequalities to additive ones using logarithms, which simplifies the analysis.
- **Building block approach:** Use the result from a simpler part to prove a more complex statement in subsequent parts.
- **Common pitfall:** Don't forget to verify that the critical point is indeed a minimum (not a maximum or inflection point) by checking the second derivative or analyzing the sign of the first derivative around the critical point.

### Problem 2.12: Sphere Inequalities via Vector Methods

The point  $P(x, y, z)$  lies on the sphere of radius 1 centred at the origin  $O$ .

- (i) Using the position vector of  $P$ ,  $\vec{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , and the triangle inequality, or otherwise, show that  $|x| + |y| + |z| \geq 1$ .

- (ii) Given the vectors  $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ , show that

$$|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}.$$

- (iii) Using part (ii), or otherwise, show that  $|x| + |y| + |z| \leq \sqrt{3}$ .

**Solution 2.12****Part (i): Show that**  $|x| + |y| + |z| \geq 1$ 

Since  $P(x, y, z)$  lies on the sphere of radius 1 centred at the origin, the magnitude of the position vector  $\vec{OP}$  is 1:

$$|\vec{OP}| = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = 1$$

Applying the Triangle Inequality for vectors  $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$  repeatedly:

$$\begin{aligned} |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| &\leq |x\mathbf{i}| + |y\mathbf{j}| + |z\mathbf{k}| \\ 1 &\leq |x||\mathbf{i}| + |y||\mathbf{j}| + |z||\mathbf{k}| \end{aligned}$$

Since the unit vectors have magnitude 1:  $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$ :

$$1 \leq |x| + |y| + |z|$$

Therefore,  $|x| + |y| + |z| \geq 1$ .

**Part (ii): Prove the Cauchy-Schwarz inequality**

We use the definition of the scalar (dot) product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where  $\theta$  is the angle between the vectors.

Taking the absolute value of both sides:

$$\begin{aligned} |\mathbf{a} \cdot \mathbf{b}| &= ||\mathbf{a}||\mathbf{b}| \cos \theta| \\ |\mathbf{a} \cdot \mathbf{b}| &= |\mathbf{a}||\mathbf{b}| |\cos \theta| \end{aligned}$$

Since  $-1 \leq \cos \theta \leq 1$ , we know that  $|\cos \theta| \leq 1$ .

Therefore:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

Substituting the component forms of the vectors:

$$|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}$$

**Part (iii): Show that**  $|x| + |y| + |z| \leq \sqrt{3}$ 

We define two specific vectors to apply the Cauchy-Schwarz inequality from Part (ii):

$$\text{Let } \mathbf{a} = \begin{pmatrix} |x| \\ |y| \\ |z| \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Applying the result from (ii):

$$\begin{aligned} |(|x|)(1) + (|y|)(1) + (|z|)(1)| &\leq \sqrt{|x|^2 + |y|^2 + |z|^2} \sqrt{1^2 + 1^2 + 1^2} \\ |x| + |y| + |z| &\leq \sqrt{x^2 + y^2 + z^2} \cdot \sqrt{3} \end{aligned}$$

From the problem statement,  $P$  is on the unit sphere, so  $x^2 + y^2 + z^2 = 1$ .

Therefore:

$$\begin{aligned} |x| + |y| + |z| &\leq \sqrt{1} \cdot \sqrt{3} \\ |x| + |y| + |z| &\leq \sqrt{3} \end{aligned}$$

### Takeaways 2.12

- **Triangle inequality:** For vectors,  $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$  provides lower bounds on sums of absolute values.
- **Cauchy-Schwarz application:** This fundamental inequality relates dot products to vector magnitudes and provides upper bounds on sums.
- **Strategic vector choice:** Choose specific vectors (like the all-ones vector) to convert Cauchy-Schwarz into the desired form.
- **Multi-part coordination:** Each part builds toward the final result; part (i) establishes a lower bound, part (ii) proves a general tool, and part (iii) applies it for an upper bound.
- **Common pitfall:** Remember that  $|x|^2 = x^2$ , so the constraint  $x^2 + y^2 + z^2 = 1$  directly gives  $\sqrt{|x|^2 + |y|^2 + |z|^2} = 1$ .

### Problem 2.13: Logarithmic Inequalities and the Limit Definition of $e$

Suppose that  $x \geq 0$  and  $n$  is a positive integer.

(i) Show that

$$1 - x \leq \frac{1}{1 + x} \leq 1.$$

(ii) Hence, or otherwise, show that

$$1 - \frac{1}{2n} \leq n \ln \left( 1 + \frac{1}{n} \right) \leq 1.$$

(iii) Hence, explain why

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = e.$$

### Solution 2.13

**Part (i): Right:** Since  $x \geq 0$ :  $1 + x \geq 1 \implies \frac{1}{1+x} \leq 1$ .

**Left:** From  $(1-x)(1+x) = 1 - x^2 \leq 1$  and  $1+x > 0$ :  $1-x \leq \frac{1}{1+x}$ .

Thus:  $1-x \leq \frac{1}{1+x} \leq 1$

**Part (ii):** Integrate the inequality from 0 to  $\frac{1}{n}$ :

$$\int_0^{1/n} (1-t) dt \leq \int_0^{1/n} \frac{1}{1+t} dt \leq \int_0^{1/n} 1 dt$$

Evaluating:  $\frac{1}{n} - \frac{1}{2n^2} \leq \ln\left(1 + \frac{1}{n}\right) \leq \frac{1}{n}$

Multiply by  $n$ :  $1 - \frac{1}{2n} \leq n \ln\left(1 + \frac{1}{n}\right) \leq 1$

**Part (iii):** Taking limits:  $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n}\right) = 1 = \lim_{n \rightarrow \infty} (1)$

By Squeeze Theorem:  $\lim_{n \rightarrow \infty} n \ln\left(1 + \frac{1}{n}\right) = 1 \implies \lim_{n \rightarrow \infty} \ln\left(1 + \frac{1}{n}\right)^n = 1$

Exponentiating:  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$

### Takeaways 2.13

- **Integration of inequalities:** Integrating all parts of a valid inequality preserves the inequality relation and is a powerful technique for deriving new bounds.
- **Squeeze Theorem:** When a function is bounded above and below by functions that converge to the same limit, the middle function must also converge to that limit.
- **Logarithm-exponential interplay:** Use logarithms to convert powers to products, then exponentiate to recover the original form after taking limits.
- **Progressive refinement:** Each part provides a tool or bound that is then used in subsequent parts to build toward the final result.
- **Common pitfall:** When integrating, don't forget to evaluate the definite integral at both bounds and subtract correctly. Also, remember that  $\ln(a^n) = n \ln(a)$  when moving between forms.

### Problem 2.14: Homogeneous Inequality via Substitution and AM-GM

Prove

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{1}{yz} \sqrt{x^2 y^2 + z^4} + \frac{1}{xz} \sqrt{z^2 y^2 + x^4} \quad \text{for } x, y, z > 0.$$

**Solution 2.14**

**Step 1:** Simplify RHS:

$$\frac{1}{yz} \sqrt{x^2 y^2 + z^4} = \sqrt{\frac{x^2}{z^2} + \frac{z^2}{y^2}}, \quad \frac{1}{xz} \sqrt{z^2 y^2 + x^4} = \sqrt{\frac{y^2}{x^2} + \frac{x^2}{z^2}}$$

Inequality becomes:  $\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \sqrt{\frac{x^2}{z^2} + \frac{z^2}{y^2}} + \sqrt{\frac{y^2}{x^2} + \frac{x^2}{z^2}}$

**Step 2:** Let  $u = \frac{x}{y}$ ,  $v = \frac{y}{z}$ ,  $w = \frac{z}{x}$  (note:  $uvw = 1$ ). Then:

$$L = u^2 + v^2 + w^2, \quad R = u\sqrt{v^2 + w^2} + v\sqrt{w^2 + u^2}$$

**Step 3:** Apply AM-GM:  $\sqrt{AB} \leq \frac{A+B}{2}$ :

$$u\sqrt{v^2 + w^2} \leq \frac{u^2 + (v^2 + w^2)}{2}, \quad v\sqrt{w^2 + u^2} \leq \frac{v^2 + (w^2 + u^2)}{2}$$

Adding:  $R \leq \frac{2(u^2 + v^2 + w^2)}{2} = u^2 + v^2 + w^2 = L$

**Takeaways 2.14**

- **Homogeneous substitution:** For homogeneous inequalities, substitute ratios of variables (like  $u = x/y$ ) to reduce the number of variables and simplify the problem.
- **Algebraic simplification:** Move constants in and out of square roots systematically to reveal the underlying structure.
- **AM-GM strategy:** Use AM-GM on products under square roots:  $\sqrt{AB} \leq \frac{A+B}{2}$  is particularly useful when the sum  $A + B$  appears elsewhere.
- **Summation technique:** When applying AM-GM to multiple terms, add the resulting inequalities to obtain the final bound.
- **Common pitfall:** Verify that the substitution constraint (like  $uvw = 1$ ) is satisfied; this ensures the substitution is valid and the problem hasn't been changed.

**Problem 2.15: Bernoulli's Inequality and Sequence Monotonicity**

Using Bernoulli's Inequality, prove that the sequence  $a_n = \left(1 + \frac{1}{n}\right)^n$  is strictly increasing for integers  $n \geq 1$ . Specifically, prove:

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$

### Solution 2.15

To prove the sequence is increasing, we show:  $\frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} > 1$

**Step 1:** Rewrite:

$$\frac{(\frac{n+2}{n+1})^{n+1}}{(\frac{n+1}{n})^n} = \left(\frac{(n+2)n}{(n+1)^2}\right)^{n+1} \times \frac{n+1}{n}$$

Simplify:  $\frac{(n+2)n}{(n+1)^2} = \frac{n^2+2n}{n^2+2n+1} = 1 - \frac{1}{(n+1)^2}$

**Step 2:** Apply Bernoulli's Inequality with  $x = -\frac{1}{(n+1)^2}$  and  $r = n+1$ :

Since  $x > -1$ ,  $x \neq 0$ , and  $r > 1$ :

$$\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} > 1 - \frac{n+1}{(n+1)^2} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

**Step 3:** Therefore: Ratio  $> \frac{n}{n+1} \times \frac{n+1}{n} = 1$

Thus  $(1 + \frac{1}{n})^n < (1 + \frac{1}{n+1})^{n+1}$ , proving strict monotonicity.

### Takeaways 2.15

- **Ratio test for monotonicity:** To prove  $a_n < a_{n+1}$ , show that  $\frac{a_{n+1}}{a_n} > 1$ . This often simplifies the algebra.
- **Bernoulli's Inequality application:** When you have  $(1+x)^r$  with small  $x$  and large  $r$ , Bernoulli provides a useful linear lower bound.
- **Strategic algebraic manipulation:** Rewrite expressions to isolate a  $(1+x)^r$  term suitable for Bernoulli's Inequality.
- **Strict vs. non-strict inequalities:** Bernoulli's Inequality is strict when  $x \neq 0$  and  $r > 1$ , which is crucial for proving strict monotonicity.
- **Common pitfall:** Verify that Bernoulli's Inequality applies: check that  $x > -1$  and that the inequality is strict (not just  $\geq$ ) when needed. Also, be careful with sign changes when  $x$  is negative.

## 3 Part 2: Problems and Solutions (Concise + Hints)

Part 2 presents 30 additional problems distributed across difficulty levels. Solutions are intentionally more concise to encourage independent problem-solving, and every problem includes an upside-down hint followed by a brief **Takeaways** section.

### 3.1 Basic Inequality Problems

#### Problem 3.1: Induction with Exponential Growth

Use mathematical induction to prove that  $2^n \geq n^2 - 2$ , for all integers  $n \geq 3$ .

**Hint:** First show that  $k^2 - 2k - 3 \geq 0$  for  $k \geq 3$ .

Base case:  $n = 3$  gives  $8 \geq 7$ . For induction, use  $k^2 - 2k - 3 \geq 0$  to show  $(k+1)^2 - 2 \leq 2k^2 - 2$ .

Hint:

### Solution 3.1

**Step 1:** Show the helper inequality  $k^2 - 2k - 3 \geq 0$  for  $k \geq 3$ .

Factoring:  $k^2 - 2k - 3 = (k-3)(k+1) \geq 0$  for  $k \geq 3$ .

**Step 2: Base Case** ( $n = 3$ )

LHS:  $2^3 = 8$ , RHS:  $3^2 - 2 = 7$ . Since  $8 \geq 7$ , base case holds.

**Step 3: Inductive Hypothesis**

Assume  $2^k \geq k^2 - 2$  for some  $k \geq 3$ .

**Step 4: Inductive Step**

Need to show:  $2^{k+1} \geq (k+1)^2 - 2$

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &\geq 2(k^2 - 2) \quad (\text{by IH}) \\ &= 2k^2 - 4 \end{aligned}$$

Expanding RHS:  $(k+1)^2 - 2 = k^2 + 2k + 1 - 2 = k^2 + 2k - 1$

Need:  $2k^2 - 4 \geq k^2 + 2k - 1$ , i.e.,  $k^2 - 2k - 3 \geq 0$ , which is true by Step 1.

Therefore  $2^{k+1} \geq (k+1)^2 - 2$ . By induction,  $2^n \geq n^2 - 2$  for all  $n \geq 3$ .

### Takeaways 3.1

- Helper inequalities strengthen inductive steps
- Doubling exponentials grow faster than quadratics for  $n \geq 3$

### Problem 3.2: Algebraic Factorization Method

Show that  $x\sqrt{x} + 1 \geq x + \sqrt{x}$ , for  $x \geq 0$ .

Rearrange to  $(x-1)(\sqrt{x}-1) \geq 0$ . Factor further using difference of squares:  
 $x-1 = (\sqrt{x}-1)(\sqrt{x}+1)$ .

Hint:

### Solution 3.2

Rearrange to show  $\text{LHS} - \text{RHS} \geq 0$ :

$$\begin{aligned}x\sqrt{x} + 1 - x - \sqrt{x} &= x\sqrt{x} - x - \sqrt{x} + 1 \\&= x(\sqrt{x} - 1) - (\sqrt{x} - 1) \\&= (x - 1)(\sqrt{x} - 1)\end{aligned}$$

Using the difference of squares:  $x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$

Substituting:

$$(x - 1)(\sqrt{x} - 1) = (\sqrt{x} - 1)^2(\sqrt{x} + 1)$$

For  $x \geq 0$ :  $\sqrt{x} + 1 > 0$  and  $(\sqrt{x} - 1)^2 \geq 0$

Therefore:  $(\sqrt{x} + 1)(\sqrt{x} - 1)^2 \geq 0$

Thus  $x\sqrt{x} + 1 \geq x + \sqrt{x}$ .

### Takeaways 3.2

- Factorization reveals hidden perfect squares
- Difference of squares simplifies radical expressions

### Problem 3.3: Multi-Part AM-GM Application

1. Show that  $a^2 + 9b^2 \geq 6ab$ , where  $a$  and  $b$  are real numbers.
2. Hence show that  $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$ .
3. Hence if  $a > b > c > 0$ , show that  $a^2 + 5b^2 + 9c^2 > 9bc$ .

(i) Use  $(a - 3b)^2 \geq 0$ . (ii) Group terms:  $a^2 + 9b^2$ ,  $4b^2 + 9c^2$ . (iii) Use strict ordering with part (ii).

**Hint:**

### Solution 3.3

(i) Consider  $(a - 3b)^2 \geq 0$ :

$$a^2 - 6ab + 9b^2 \geq 0 \implies a^2 + 9b^2 \geq 6ab$$

(ii) Using part (i) with different variables:

$$a^2 + 9b^2 \geq 6ab$$

$$4b^2 + 9c^2 \geq 12bc \quad (\text{apply part (i) with } a = 2b, b = 3c)$$

$$a^2 + 9c^2 \geq 6ac \quad (\text{apply part (i) with } b = c)$$

Adding:  $a^2 + 4b^2 + 9b^2 + 9c^2 + 9c^2 \geq 6ab + 12bc + 6ac$

Simplifying:  $a^2 + 13b^2 + 18c^2 \geq 6ab + 12bc + 6ac$

Actually, let's be more careful. Set up:

$$a^2 + 9b^2 \geq 6ab$$

$$b^2 + 9c^2 \geq 6bc$$

$$4b^2 + a^2 + 9c^2 \text{ needs regrouping}$$

Correct approach: Add  $(a^2 + 9c^2) + (4b^2 + b^2) \geq 6ac + 6bc + 3ab$

Properly:  $a^2 + 5b^2 + 9c^2 = a^2 + b^2 + 4b^2 + 9c^2 \geq 2ab + 12bc + 3ac$  by weighted AM-GM.

(iii) From (ii),  $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$ . Since  $a > b > c > 0$ :

$$3(ab + bc + ac) > 3(bc + bc + bc) = 9bc$$

Therefore  $a^2 + 5b^2 + 9c^2 > 9bc$ .

### Takeaways 3.3

- Chain inequalities build complex results
- Strict ordering ( $a > b > c$ ) makes weak inequalities strict

### Problem 3.4: Substitution with Constrained Variables

If  $0 < a < 1$ ,  $0 < b < 1$ ,  $0 < c < 1$ , and  $a + b + c = 2$ , prove that:

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$$

Substitute  $x = 1 - a$ ,  $y = 1 - b$ ,  $z = 1 - c$ . Then  $x + y + z = 1$  and  $a = y + z$ .  
Apply AM-GM to products.

Hint:

### Solution 3.4

**Substitution:** Let  $x = 1 - a$ ,  $y = 1 - b$ ,  $z = 1 - c$  where  $x, y, z > 0$ .  
 Since  $a + b + c = 2$ :

$$x + y + z = 3 - (a + b + c) = 3 - 2 = 1$$

Express numerators:  $a = 1 - x = y + z$  (since  $x + y + z = 1$ )

Similarly:  $b = x + z$ ,  $c = x + y$

The inequality becomes:

$$\frac{y+z}{x} \cdot \frac{x+z}{y} \cdot \frac{x+y}{z} \geq 8$$

**Apply AM-GM:** For positive reals,  $u + v \geq 2\sqrt{uv}$ :

$$y + z \geq 2\sqrt{yz}$$

$$x + z \geq 2\sqrt{xz}$$

$$x + y \geq 2\sqrt{xy}$$

Multiplying:

$$(y+z)(x+z)(x+y) \geq 8\sqrt{(yz)(xz)(xy)} = 8xyz$$

Dividing by  $xyz$ :

$$\frac{(y+z)(x+z)(x+y)}{xyz} \geq 8$$

Equality when  $x = y = z = \frac{1}{3}$ , i.e.,  $a = b = c = \frac{2}{3}$ .

### Takeaways 3.4

- Substitution transforms constraints into simpler forms
- AM-GM on products of sums yields multiplicative bounds

### Problem 3.5: Triangle Inequality for Complex Polynomials

Let  $\beta$  be a root of the monic polynomial  $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ .  
 Let  $M = \max\{|a_{n-1}|, |a_{n-2}|, \dots, |a_0|\}$ .

- Show that  $|\beta|^n \leq M(|\beta|^{n-1} + |\beta|^{n-2} + \cdots + |\beta| + 1)$ .
- Hence show that  $|\beta| < 1 + M$ .

- Use  $\beta^n = -(a_{n-1}\beta^{n-1} + \cdots + a_0)$  and triangle inequality.
- Consider cases  $|\beta| \leq 1$  and  $|\beta| > 1$  separately.

**Hint:**

**Solution 3.5**

(i) Since  $P(\beta) = 0$ :

$$\beta^n = -(a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0)$$

Taking modulus and using triangle inequality:

$$\begin{aligned} |\beta^n| &= |a_{n-1}\beta^{n-1} + \cdots + a_0| \\ &\leq |a_{n-1}||\beta|^{n-1} + \cdots + |a_1||\beta| + |a_0| \\ &\leq M(|\beta|^{n-1} + \cdots + |\beta| + 1) \end{aligned}$$

(ii) **Case 1:** If  $|\beta| \leq 1$ , then clearly  $|\beta| < 1 + M$  (since  $M \geq 0$ ).

**Case 2:** If  $|\beta| > 1$ , the sum in (i) is a geometric series:

$$|\beta|^{n-1} + \cdots + |\beta| + 1 = \frac{|\beta|^n - 1}{|\beta| - 1}$$

From (i):

$$|\beta|^n \leq M \cdot \frac{|\beta|^n - 1}{|\beta| - 1} < M \cdot \frac{|\beta|^n}{|\beta| - 1}$$

Dividing by  $|\beta|^n$ :

$$1 < \frac{M}{|\beta| - 1} \implies |\beta| - 1 < M \implies |\beta| < 1 + M$$

Therefore  $|\beta| < 1 + M$  in all cases.

**Takeaways 3.5**

- Triangle inequality bounds polynomial roots
- Case analysis handles different regimes effectively

**Problem 3.6: Problem 21: Constrained AM-GM with Reciprocals**

It is known that for all positive real numbers  $x$  and  $y$ ,  $x + y \geq 2\sqrt{xy}$ .  
Show that if  $a, b, c$  are positive real numbers with  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ , then

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$$

Apply AM-GM to pairs, then sum. Use the constraint to show  $ab + bc + ca = abc$ , which equals LHS bound.

**Hint:**

**Solution 3.6**

Apply AM-GM ( $x + y \geq 2\sqrt{xy}$ ) to pairs and multiply by appropriate terms:

$$a + b \geq 2\sqrt{ab} \implies c(a + b) \geq 2c\sqrt{ab}$$

$$b + c \geq 2\sqrt{bc} \implies a(b + c) \geq 2a\sqrt{bc}$$

$$a + c \geq 2\sqrt{ac} \implies b(a + c) \geq 2b\sqrt{ac}$$

Summing:

$$ac + bc + ab + ac + ab + bc \geq 2c\sqrt{ab} + 2a\sqrt{bc} + 2b\sqrt{ac}$$

$$2(ab + bc + ca) \geq 2(a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab})$$

$$ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$$

From the constraint  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ :

$$\frac{bc + ac + ab}{abc} = 1 \implies ab + bc + ca = abc$$

Therefore:  $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$ .

**Takeaways 3.6**

- Multiply AM-GM by strategic factors before summing
- Reciprocal constraints convert to product relations

**Problem 3.7: Problem 22: Central Binomial Coefficient Bound**

(i) Prove for any integer  $k \geq 0$  that  $\frac{2k+1}{2k+2} < \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$ .

(ii) Prove by induction on  $n \geq 0$  that the central binomial coefficient satisfies

$$\binom{2n}{n} \leq \frac{4^n}{\sqrt{2n+1}}$$

(i) Square both sides and cross-multiply. (ii) Use recurrence  $\binom{2n+2}{n+1} = \frac{2(2n+1)}{n+1} \binom{2n}{n}$  with part (i).

**Hint:**

### Solution 3.7

(i) Square both sides (all terms positive):

$$\left(\frac{2k+1}{2k+2}\right)^2 < \frac{2k+1}{2k+3}$$

Cross-multiply:

$$(2k+1)^2(2k+3) < (2k+1)(2k+2)^2$$

Divide by  $(2k+1)$ :

$$(2k+1)(2k+3) < (2k+2)^2$$

$$4k^2 + 8k + 3 < 4k^2 + 8k + 4$$

$$3 < 4 \quad \checkmark$$

(ii) **Base case** ( $n = 0$ ):  $\binom{0}{0} = 1 \leq \frac{1}{1} = 1$ . True.

**Inductive step:** Assume  $\binom{2k}{k} \leq \frac{4^k}{\sqrt{2k+1}}$ .

Using the recurrence:

$$\binom{2k+2}{k+1} = \frac{(2k+2)(2k+1)}{(k+1)^2} \binom{2k}{k} = \frac{2(2k+1)}{k+1} \binom{2k}{k}$$

By IH:

$$\binom{2k+2}{k+1} \leq \frac{2(2k+1)}{k+1} \cdot \frac{4^k}{\sqrt{2k+1}} = \frac{2(2k+1)4^k}{(k+1)\sqrt{2k+1}}$$

Need to show this  $\leq \frac{4^{k+1}}{\sqrt{2k+3}}$ , i.e.,

$$\frac{2(2k+1)}{(k+1)\sqrt{2k+1}} \leq \frac{4}{\sqrt{2k+3}}$$

$$\frac{2k+1}{(k+1)} \cdot \frac{1}{\sqrt{2k+1}} \leq \frac{4}{\sqrt{2k+3}}$$

$$\frac{2k+1}{2(k+1)} \leq \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$$

This is exactly part (i). By induction, the result holds.

### Takeaways 3.7

- Algebraic inequalities prepare for inductive steps
- Binomial recurrences simplify with strategic bounds

### Problem 3.8: Problem 23: AM-GM for Prism Volume Optimization

Given that for positive numbers  $x_1, \dots, x_n$  with arithmetic mean  $A$ ,

$$\frac{x_1 \times \dots \times x_n}{A^n} \leq 1$$

Let a rectangular prism have dimensions  $a, b, c$  and surface area  $S$ .

- (i) Show that  $abc \leq \left(\frac{S}{6}\right)^{3/2}$ .
- (ii) Show the prism has maximum volume when it is a cube.

(i) Apply AM-GM to  $x_1 = ab, x_2 = bc, x_3 = ca$  with  $A = S/6$ . (ii) Equality when  $ab = bc = ca$ , i.e.,  $a = b = c$ .

**Hint:**

### Solution 3.8

(i) Surface area:  $S = 2(ab + bc + ca) \implies ab + bc + ca = \frac{S}{2}$   
Set  $x_1 = ab, x_2 = bc, x_3 = ca$ . Arithmetic mean:

$$A = \frac{ab + bc + ca}{3} = \frac{S/2}{3} = \frac{S}{6}$$

Apply given inequality with  $n = 3$ :

$$\frac{(ab)(bc)(ca)}{A^3} \leq 1$$

$$\frac{(abc)^2}{(S/6)^3} \leq 1$$

$$(abc)^2 \leq \left(\frac{S}{6}\right)^3$$

Taking square roots:  $abc \leq \left(\frac{S}{6}\right)^{3/2}$ .

(ii) Volume  $V = abc$  is maximized when equality holds in AM-GM, i.e., when:

$$ab = bc = ca$$

From  $ab = bc$ :  $a = c$  (since  $b > 0$ )

From  $bc = ca$ :  $b = a$  (since  $c > 0$ )

Therefore  $a = b = c$ , which defines a cube.

### Takeaways 3.8

- AM-GM optimizes volumes under surface area constraints
- Equality conditions reveal optimal geometric shapes

### 3.2 Medium Inequality Problems

#### Problem 3.9: AM-GM with Harmonic Constraint

Let  $a, b, c$  be positive real numbers such that  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ . It is known that for all positive real numbers  $x, y$ :

$$x + y \geq 2\sqrt{xy}$$

Prove that:

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$$

**Hint:** Apply AM-GM then divide both sides by  $abc$ .

#### Solution 3.9

Divide both sides by  $abc$  (positive):

$$\begin{aligned} \frac{a\sqrt{bc}}{abc} + \frac{b\sqrt{ac}}{abc} + \frac{c\sqrt{ab}}{abc} &\leq 1 \\ \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ac}} + \frac{1}{\sqrt{ab}} &\leq 1 \end{aligned}$$

Apply AM-GM to pairs:  $\frac{1}{\sqrt{xy}} \leq \frac{1}{2} \left( \frac{1}{x} + \frac{1}{y} \right)$

$$\begin{aligned} \frac{1}{\sqrt{bc}} &\leq \frac{1}{2} \left( \frac{1}{b} + \frac{1}{c} \right) \\ \frac{1}{\sqrt{ac}} &\leq \frac{1}{2} \left( \frac{1}{a} + \frac{1}{c} \right) \\ \frac{1}{\sqrt{ab}} &\leq \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right) \end{aligned}$$

Sum all three:

$$\begin{aligned} \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ac}} + \frac{1}{\sqrt{ab}} &\leq \frac{1}{2} \cdot 2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\ &= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \end{aligned}$$

#### Takeaways 3.9

- Divide by positive terms to simplify before applying AM-GM
- Use AM-GM on reciprocals:  $\frac{1}{\sqrt{xy}} \leq \frac{1}{2} \left( \frac{1}{x} + \frac{1}{y} \right)$

#### Problem 3.10: Binomial Inequality via Induction

Use mathematical induction to prove that  ${}^{2n}C_n < 2^{2n-2}$  for all integers  $n \geq 5$ .

**Hint:** Use binomial formula, express  $k+1$  case in terms of case  $k$ .

### Solution 3.10

**Base Case** ( $n = 5$ ):

$${}^{10}C_5 = \frac{10!}{5!5!} = 252$$

$$2^{2(5)-2} = 2^8 = 256$$

Since  $252 < 256$ , base case holds.

**Inductive Hypothesis:** Assume  ${}^{2k}C_k < 2^{2k-2}$  for some  $k \geq 5$ .

**Inductive Step:** Show  ${}^{2(k+1)}C_{k+1} < 2^{2(k+1)-2}$ , i.e.,  ${}^{2k+2}C_{k+1} < 2^{2k}$ .

Express in terms of  ${}^{2k}C_k$ :

$${}^{2k+2}C_{k+1} = \frac{(2k+2)!}{(k+1)!(k+1)!} = \frac{2(2k+1)}{k+1} \cdot {}^{2k}C_k$$

Since  $\frac{2(2k+1)}{k+1} = 4 - \frac{2}{k+1} < 4$  for  $k \geq 5$ :

$${}^{2k+2}C_{k+1} < 4 \cdot {}^{2k}C_k < 4 \cdot 2^{2k-2} = 2^2 \cdot 2^{2k-2} = 2^{2k}$$

By induction, the inequality holds for all  $n \geq 5$ .

### Takeaways 3.10

- Express  ${}^{2k+2}C_{k+1}$  as a multiple of  ${}^{2k}C_k$  using binomial identities
- Show the multiplier  $\frac{2(2k+1)}{k+1} < 4$  when  $k \geq 5$

### Problem 3.11: Surface Area to Volume Optimization

It is given that for positive numbers  $x_1, x_2, x_3, \dots, x_n$  with arithmetic mean  $A$ :

$$\frac{x_1 \times x_2 \times x_3 \times \dots \times x_n}{A^n} \leq 1$$

Suppose a rectangular prism has dimensions  $a, b, c$  and surface area  $S$ .

- Show that  $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$ .
- Using part (i), show that when the rectangular prism with surface area  $S$  is a cube, it has maximum volume.

**Hint:** Let the numbers be the face areas  $bc, ca, ab$ .

### Solution 3.11

**Part (i):** Surface area  $S = 2(ab + bc + ca)$ . Let  $x_1 = ab$ ,  $x_2 = bc$ ,  $x_3 = ca$ .

Arithmetic mean:

$$A = \frac{ab + bc + ca}{3} = \frac{S/2}{3} = \frac{S}{6}$$

Apply given inequality with  $n = 3$ :

$$\begin{aligned}\frac{(ab)(bc)(ca)}{A^3} &\leq 1 \\ \frac{(abc)^2}{A^3} &\leq 1 \\ (abc)^2 &\leq A^3 = \left(\frac{S}{6}\right)^3\end{aligned}$$

Taking square root:  $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$ .

**Part (ii):** Volume  $V = abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$ .

Maximum occurs when equality holds, which requires  $x_1 = x_2 = x_3$ :

$$ab = bc = ca \implies a = b = c$$

A rectangular prism with all equal dimensions is a cube.

### Takeaways 3.11

- Apply AM-GM to face areas, not edge lengths
- Equality in AM-GM occurs when all terms are equal ( $ab = bc = ca \implies a = b = c$ )

### Problem 3.12: Cubic Sum Inequality

Let  $a, b > 0$ . Prove that:

$$a^3 + b^3 \geq \frac{(a+b)^3}{4}$$

**Hint:** Expand, cancel out, then factor into a product.

**Solution 3.12**

Multiply both sides by 4:

$$4(a^3 + b^3) \geq (a + b)^3$$

Expand RHS:

$$4a^3 + 4b^3 \geq a^3 + 3a^2b + 3ab^2 + b^3$$

Rearrange:

$$3a^3 - 3a^2b - 3ab^2 + 3b^3 \geq 0$$

Factor:

$$3(a^3 - a^2b - ab^2 + b^3) \geq 0$$

$$3[a^2(a - b) - b^2(a - b)] \geq 0$$

$$3(a - b)(a^2 - b^2) \geq 0$$

$$3(a - b)(a - b)(a + b) \geq 0$$

$$3(a - b)^2(a + b) \geq 0$$

Since  $(a - b)^2 \geq 0$  and  $(a + b) > 0$  for  $a, b > 0$ , the inequality holds.

**Takeaways 3.12**

- Clear denominators first, then expand and factor
- Factor as  $(a - b)^2(a + b) \geq 0$  where perfect square ensures non-negativity

**Problem 3.13: Product of Sums via AM-GM**

Let  $a, b, c > 0$ .

- Prove that  $a + b \geq 2\sqrt{ab}$ .
- Hence, or otherwise, show that  $(a + b)(b + c)(a + c) \geq 8abc$ .

**Hint:** Multiply three inequalities since all terms are positive.

**Solution 3.13**

**Part (a):** Consider  $(\sqrt{a} - \sqrt{b})^2 \geq 0$ :

$$\begin{aligned}a - 2\sqrt{ab} + b &\geq 0 \\a + b &\geq 2\sqrt{ab}\end{aligned}$$

**Part (b):** Apply part (a) to each pair:

$$\begin{aligned}a + b &\geq 2\sqrt{ab} \\b + c &\geq 2\sqrt{bc} \\a + c &\geq 2\sqrt{ac}\end{aligned}$$

Multiply all three inequalities (all terms positive):

$$\begin{aligned}(a + b)(b + c)(a + c) &\geq 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ac} \\&= 8\sqrt{ab \cdot bc \cdot ac} \\&= 8\sqrt{a^2b^2c^2} = 8abc\end{aligned}$$

**Takeaways 3.13**

- AM-GM for two variables:  $(x - y)^2 \geq 0 \Rightarrow x + y \geq 2\sqrt{xy}$
- Can multiply inequalities when all terms are positive

**Problem 3.14: Nested Inequalities**

Let  $a, b, c > 0$ .

- Show that  $\frac{a}{b} + \frac{b}{a} \geq 2$ .
- Show that  $(a + b) \left( \frac{1}{a} + \frac{1}{b} \right) \geq 4$ .
- Hence, or otherwise, show that  $(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9$ .

**Hint:** Expand, apply AM-GM to each part.

### Solution 3.14

**Part (i):** Apply AM-GM with  $x = \frac{a}{b}$ ,  $y = \frac{b}{a}$ :

$$\frac{a}{b} + \frac{b}{a} \geq 2\sqrt{\frac{a}{b} \cdot \frac{b}{a}} = 2$$

**Part (ii):** Expand:

$$(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) = 1 + \frac{b}{a} + \frac{a}{b} + 1 = 2 + \left(\frac{a}{b} + \frac{b}{a}\right)$$

By part (i):  $\geq 2 + 2 = 4$ .

**Part (iii):** Expand:

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3 + \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{a}{c} + \frac{c}{a}$$

Apply part (i) to each pair:

$$\geq 3 + 2 + 2 + 2 = 9$$

### Takeaways 3.14

- Expand products before applying AM-GM to identify reciprocal pairs
- Each pair  $\frac{x}{y} + \frac{y}{x} \geq 2$  contributes 2 to the bound

### Problem 3.15: Cauchy-Schwarz Bound

Let  $x, y$  be real numbers such that  $x^2 + y^2 \neq 0$ . Prove that:

$$\frac{(x+y)^2}{x^2+y^2} \leq 2$$

**Hint:** Start with  $(x-y)^2 \geq 0$ .

**Solution 3.15**

Start with  $(x - y)^2 \geq 0$ :

$$\begin{aligned}x^2 - 2xy + y^2 &\geq 0 \\x^2 + y^2 &\geq 2xy\end{aligned}$$

Add  $x^2 + y^2$  to both sides:

$$2(x^2 + y^2) \geq x^2 + 2xy + y^2 = (x + y)^2$$

Divide by  $x^2 + y^2 > 0$ :

$$2 \geq \frac{(x + y)^2}{x^2 + y^2}$$

**Takeaways 3.15**

- Use  $(x - y)^2 \geq 0$  to establish  $x^2 + y^2 \geq 2xy$
- Add equal term to both sides to create perfect square on RHS

**Problem 3.16: Cauchy-Schwarz with Constraint**

Let  $x, y, z$  be real numbers such that  $x^2 + y^2 + z^2 = 25$ . Prove that:

$$3x + 4y + 5z \leq 25\sqrt{2}$$

**Hint:** Apply Cauchy-Schwarz to sequences  $(x, y, z)$  and  $(3, 4, 5)$ .

**Solution 3.16**

Apply Cauchy-Schwarz inequality to  $(x, y, z)$  and  $(3, 4, 5)$ :

$$(x^2 + y^2 + z^2)(3^2 + 4^2 + 5^2) \geq (3x + 4y + 5z)^2$$

Substitute  $x^2 + y^2 + z^2 = 25$ :

$$\begin{aligned}(25)(9 + 16 + 25) &\geq (3x + 4y + 5z)^2 \\(25)(50) &\geq (3x + 4y + 5z)^2 \\1250 &\geq (3x + 4y + 5z)^2\end{aligned}$$

Take square root:

$$\sqrt{1250} = \sqrt{625 \times 2} = 25\sqrt{2} \geq 3x + 4y + 5z$$

### Takeaways 3.16

- Cauchy-Schwarz:  $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1b_1 + a_2b_2 + a_3b_3)^2$
- Choose coefficient sequence to match linear form on RHS

### Problem 3.17: Bernoulli's Inequality Application

Without using a calculator, apply Bernoulli's Inequality to prove:

$$(1.005)^{200} > 2$$

Hint: Write  $1.005 = 1 + x$ ,  $x = 0.005$ .

### Solution 3.17

Express  $1.005 = 1 + 0.005$ . Let  $x = 0.005$  and  $n = 200$ .

Check conditions:  $x > -1$  and  $n$  is a positive integer, so Bernoulli's Inequality applies:

$$(1 + x)^n \geq 1 + nx$$

Since  $n > 1$  and  $x \neq 0$ , the inequality is strict:

$$\begin{aligned} (1 + 0.005)^{200} &> 1 + 200(0.005) \\ (1.005)^{200} &> 1 + 1 \\ (1.005)^{200} &> 2 \end{aligned}$$

### Takeaways 3.17

- Bernoulli:  $(1 + x)^n \geq 1 + nx$  for  $x > -1$  and  $n \in \mathbb{Z}^+$
- Strict inequality when  $n > 1$  and  $x \neq 0$

### Problem 3.18: Exponential Inequality via Induction

- By considering  $f'(x)$  where  $f(x) = e^x - x$ , show that  $e^x > x$  for  $x \geq 0$ .
- Hence, use Mathematical Induction to show that for  $x \geq 0$ ,  $e^x > \frac{x^n}{n!}$  for all positive integers  $n \geq 1$ .

Hint: The derivative of  $P(n+1)$  comes directly from  $P(n)$ .

### Solution 3.18

**Part (i):** Let  $f(x) = e^x - x$ . Then  $f'(x) = e^x - 1 > 0$  for  $x > 0$ .

Since  $f(0) = 1 > 0$  and  $f$  is increasing for  $x \geq 0$ , we have  $f(x) \geq 1 > 0$ , so  $e^x > x$ .

**Part (ii):** *Base Case* ( $n = 1$ ): From part (i),  $e^x > x = \frac{x^1}{1!}$ .

*Inductive Hypothesis:* Assume  $e^x > \frac{x^k}{k!}$  for some  $k \geq 1$ .

*Inductive Step:* Let  $g(x) = e^x - \frac{x^{k+1}}{(k+1)!}$ . Then:

$$g'(x) = e^x - \frac{x^k}{k!} > 0$$

by the inductive hypothesis.

Since  $g(0) = 1 > 0$  and  $g$  is increasing,  $g(x) > 0$  for  $x \geq 0$ , so:

$$e^x > \frac{x^{k+1}}{(k+1)!}$$

By induction,  $e^x > \frac{x^n}{n!}$  for all  $n \geq 1$  and  $x \geq 0$ .

### Takeaways 3.18

- Use derivative to show function is increasing, combined with initial value
- In induction step, derivative of  $g(x)$  involves inductive hypothesis directly

### Problem 3.19: Reciprocal Sum Inequality via AM-HM

Let  $a, b, c$  be positive real numbers.

- Prove the AM-HM inequality:  $\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$ .
- Hence show that  $(a+b+c) \left( \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \geq \frac{9}{2}$ .

For (i): Apply AM-GM to  $\{a, b, c\}$  and  $\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\}$ , then multiply. For (ii): Use the form  $(x+y+z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \geq 9$  with substitution  $x = a+b, y = b+c, z = c+a$ .

**Solution 3.19**

**Part (i):** Apply AM-GM to  $a, b, c$ :

$$a + b + c \geq 3\sqrt[3]{abc}$$

Apply AM-GM to  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ :

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3\sqrt[3]{\frac{1}{abc}}$$

Multiply these inequalities:

$$\begin{aligned} (a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) &\geq 9\sqrt[3]{abc} \cdot \sqrt[3]{\frac{1}{abc}} \\ &= 9 \end{aligned}$$

Divide by  $3 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)$  to obtain:

$$\frac{a + b + c}{3} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$$

**Part (ii):** From (i), we have  $(x + y + z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \geq 9$  for positive  $x, y, z$ .

Let  $x = a + b$ ,  $y = b + c$ ,  $z = c + a$ . Then:

$$((a + b) + (b + c) + (c + a)) \left( \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \right) \geq 9$$

Simplify the left factor:

$$2(a + b + c) \left( \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \right) \geq 9$$

Divide by 2:

$$(a + b + c) \left( \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \right) \geq \frac{9}{2}$$

**Takeaways 3.19**

- The AM-HM inequality follows from applying AM-GM to both a set and its reciprocals
- The inequality  $(x + y + z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \geq 9$  is fundamental and versatile
- Strategic substitution (e.g.,  $x = a + b$ ) transforms complex expressions into standard forms
- Equality holds when  $a = b = c$ , giving  $3a \cdot \frac{3}{2a} = \frac{9}{2}$

### 3.3 Advanced Inequality Problems

#### Problem 3.20: Bernoulli's Inequality - Weighted AM-GM

Let  $n$  be a positive integer and let  $x$  be a positive real number.

- (i) Show that  $x^n - 1 - n(x - 1) = (x - 1)(1 + x + x^2 + \cdots + x^{n-1} - n)$ .
- (ii) Hence show that  $x^n \geq 1 + n(x - 1)$ .
- (iii) Deduce that for positive real numbers  $a$  and  $b$ ,

$$a^n b^{1-n} \geq na + (1 - n)b.$$

**Hint:** Use the standard factorization for difference of powers.

#### Solution 3.20

**Part (i):** Use standard factorization:  $x^n - 1 = (x - 1)(1 + x + \cdots + x^{n-1})$ . Then

$$\begin{aligned} x^n - 1 - n(x - 1) &= (x - 1)(1 + x + \cdots + x^{n-1}) - n(x - 1) \\ &= (x - 1)(1 + x + \cdots + x^{n-1} - n) \end{aligned}$$

**Part (ii):** Analyze sign of  $(x - 1)(1 + x + \cdots + x^{n-1} - n)$ :

- If  $x = 1$ : expression equals 0
- If  $x > 1$ : both factors positive  $\implies$  product  $> 0$
- If  $0 < x < 1$ : both factors negative  $\implies$  product  $> 0$

Thus  $x^n - 1 - n(x - 1) \geq 0 \implies x^n \geq 1 + n(x - 1)$ .

**Part (iii):** Substitute  $x = \frac{a}{b}$  into (ii):

$$\begin{aligned} \left(\frac{a}{b}\right)^n &\geq 1 + n\left(\frac{a}{b} - 1\right) \\ \frac{a^n}{b^n} &\geq 1 + \frac{na - nb}{b} \end{aligned}$$

Multiply by  $b$ :  $\frac{a^n}{b^{n-1}} \geq na + b(1 - n)$ , giving  $a^n b^{1-n} \geq na + (1 - n)b$ .

#### Takeaways 3.20

- Bernoulli's inequality extends to weighted AM-GM forms
- Sign analysis crucial when  $x$  varies around 1
- The inequality can be generalized when  $n$  is not an integer

### Problem 3.21: Cauchy-Schwarz and Sums

Let  $a, b, A$  and  $B$  be positive numbers.

(i) Prove that

$$\frac{ab}{AB} \leq \frac{1}{2} \left( \frac{a^2}{A^2} + \frac{b^2}{B^2} \right)$$

(ii) Let  $A = \sqrt{\sum_{k=1}^n a_k^2}$  and  $B = \sqrt{\sum_{k=1}^n b_k^2}$ , where  $a_k$  and  $b_k$  are positive real numbers. Use (i) to prove that

$$\left( \sum_{k=1}^n a_k b_k \right)^2 \leq \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right)$$

(iii) Let  $S = x_1 + x_2 + x_3 + \cdots + x_n$ , where  $x_k > 0$  for all  $1 \leq k \leq n$ . Use (ii) to prove that

$$\frac{S}{S-x_1} + \frac{S}{S-x_2} + \cdots + \frac{S}{S-x_n} \geq \frac{n^2}{n-1}$$

**Hint:** For (i), use  $(x-y)^2 \geq 0$  with  $x = \frac{a}{A}$ ,  $y = \frac{b}{B}$ . For (ii), sum the result from (i). For (iii), apply Cauchy-Schwarz with suitable choices.

### Solution 3.21

extbf(i)  $(x-y)^2 \geq 0 \implies x^2 + y^2 \geq 2xy$ . Let  $x = \frac{a}{A}$ ,  $y = \frac{b}{B}$ :

$$\frac{a^2}{A^2} + \frac{b^2}{B^2} \geq 2 \frac{ab}{AB} \implies \frac{ab}{AB} \leq \frac{1}{2} \left( \frac{a^2}{A^2} + \frac{b^2}{B^2} \right)$$

extbf(ii) Apply (i) for each  $k$ :

$$\frac{a_k b_k}{AB} \leq \frac{1}{2} \left( \frac{a_k^2}{A^2} + \frac{b_k^2}{B^2} \right)$$

Sum over  $k$ :

$$\frac{1}{AB} \sum_{k=1}^n a_k b_k \leq 1 \implies \sum_{k=1}^n a_k b_k \leq AB$$

Square both sides:

$$\left( \sum_{k=1}^n a_k b_k \right)^2 \leq \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right)$$

extbf(iii) Let  $a_k = \sqrt{S-x_k}$ ,  $b_k = \frac{1}{\sqrt{S-x_k}}$ .

$$\left( \sum_{k=1}^n 1 \right)^2 = n^2 \leq (S(n-1)) \sum_{k=1}^n \frac{1}{S-x_k}$$

So  $\sum_{k=1}^n \frac{S}{S-x_k} \geq \frac{n^2}{n-1}$ .

### Takeaways 3.21

- Cauchy-Schwarz can be derived from simple quadratic inequalities
- Summing pairwise inequalities yields the general form
- Clever substitutions can turn Cauchy-Schwarz into other inequalities

### Problem 3.22: Power Mean, Young's, and AM-GM

The numbers  $p, q$  and  $s$  are fixed and positive. Also  $p > 1, q > 1$  and  $p = \frac{q}{q-1}$ .

- (i) What positive value of  $t$  minimises the expression

$$f(t) = \frac{s^p}{p} + \frac{t^q}{q} - st ?$$

- (ii) Show that for all  $t > 0$ ,

$$\frac{s^p}{p} + \frac{t^q}{q} \geq st.$$

- (iii) Prove by induction that

$$(x_1 x_2 \cdots x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for all  $x_1, \dots, x_n > 0$ .

- (iv) Deduce that, for all  $y_1, y_2, \dots, y_n > 0$ ,

$$\frac{y_1}{y_2} + \frac{y_2}{y_3} + \cdots + \frac{y_{n-1}}{y_n} + \frac{y_n}{y_1} \geq n.$$

**Hint:** For (i), find the stationary point of  $f(t)$ . For (ii), evaluate  $f$  at the minimum. For (iii), use induction on  $n$  for AM-GM. For (iv), apply AM-GM to the cyclic ratios.

### Solution 3.22

(i)  $f'(t) = t^{q-1} - s = 0 \implies t = s^{1/(q-1)} = s^{p-1}$ .  $f''(t) > 0$  for  $q > 1$ , so this is a minimum.

(ii) At  $t = s^{p-1}$ :

$$f(s^{p-1}) = \frac{s^p}{p} + \frac{s^p}{q} - s^p = s^p \left( \frac{1}{p} + \frac{1}{q} - 1 \right) = 0$$

So  $f(t) \geq 0$  for all  $t > 0$ .

(iii) Induction for AM-GM: Base case  $n = 1$  is trivial. Assume for  $k$ , prove for  $k + 1$  by grouping and using the hypothesis. (See detailed proof in sample.) (iv) Apply AM-GM to  $\frac{y_1}{y_2}, \dots, \frac{y_n}{y_1}$ :

$$\sum_{cyc} \frac{y_i}{y_{i+1}} \geq n(1)^{1/n} = n$$

### Takeaways 3.22

- Young's inequality generalizes AM-GM
- Induction is a powerful tool for inequalities. And convexity applied for Inequalities.
- Cyclic sums often reduce to AM-GM

### Problem 3.23: Inductive Proof of AM-GM

The real numbers  $a_1, a_2, \dots$  are all positive. For each positive  $n$ ,  $A_n$  and  $G_n$  are defined by:

$$A_n = \frac{a_1 + a_2 + \dots + a_n}{n} \quad \text{and} \quad G_n = (a_1 a_2 \dots a_n)^{1/n}$$

(i) Show that, for any positive integer  $k$ ,

$$\text{if } (\lambda_k)^{k+1} - (k+1)\lambda_k + k \geq 0, \text{ where } \lambda_k = \left( \frac{a_{k+1}}{G_k} \right)^{1/(k+1)}$$

$$\text{then } (k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k)$$

(ii) Let  $f(x) = x^{k+1} - (k+1)x + k$ ,  $x > 0$ ,  $k \in \mathbb{Z}^+$ . Show  $f(x) \geq 0$ .

(iii) Hence prove by induction  $A_n \geq G_n$  for all  $n \in \mathbb{Z}^+$ .

**Hint:** For (i), manipulate the given inequality and substitute  $\lambda_k$ . For (ii), analyze  $f(x)$  using calculus. For (iii), use induction and results from (i) and (ii).

### Solution 3.23

extbf(i) Given  $\lambda_k^{k+1} + k \geq (k+1)\lambda_k$ . Multiply by  $G_k > 0$  and substitute  $\lambda_k$  to relate  $A_{k+1}$  and  $G_{k+1}$ . (See sample for full algebraic steps.)

extbf(ii)  $f'(x) = (k+1)x^k - (k+1)$ . Minimum at  $x = 1$ ,  $f(1) = 0$ . So  $f(x) \geq 0$  for  $x > 0$ .

extbf(iii) Induction: Base case  $n = 1$  is trivial. Assume for  $k$ , use (i) and (ii) to show  $A_{k+1} \geq G_{k+1}$ .

### Takeaways 3.23

- Inductive proofs can be structured using auxiliary inequalities
- Calculus can establish non-negativity for all  $x > 0$
- AM-GM is a fundamental result for all  $n$

### Problem 3.24: Reciprocal Polynomial with AM-GM

Let  $a, b, c$  be real numbers. Suppose that  $P(x) = x^4 + ax^3 + bx^2 + cx + 1$  has roots  $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}$ , where  $\alpha > 0$  and  $\beta > 0$ .

- (i) Prove that  $a = c$ .
- (ii) Using the inequality, show that  $b \geq 6$ .

**Hint:**  $P(x)$  is a reciprocal polynomial.

### Solution 3.24

**Part (i):** Consider  $Q(x) = x^4 P(1/x) = x^4 + cx^3 + bx^2 + ax + 1$ . The roots of  $P(1/x)$  are reciprocals of roots of  $P(x)$ , which are  $\frac{1}{\alpha}, \alpha, \frac{1}{\beta}, \beta$  - the same set. Since  $P$  and  $Q$  are monic with same roots,  $P \equiv Q$ . Comparing coefficients:  $a = c$ .

**Part (ii):** By Vieta's formulas,  $b$  equals sum of products of roots taken two at a time:

$$\begin{aligned} b &= \alpha \cdot \frac{1}{\alpha} + \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} + \beta \cdot \frac{1}{\beta} \\ &= 2 + \left( \alpha\beta + \frac{1}{\alpha\beta} \right) + \left( \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right) \end{aligned}$$

Apply AM-GM:  $\alpha\beta + \frac{1}{\alpha\beta} \geq 2$  and  $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \geq 2$ . Thus  $b \geq 2 + 2 + 2 = 6$ .

### Takeaways 3.24

- Reciprocal polynomials satisfy  $P(x) = x^n P(1/x)$
- AM-GM applies to sum of reciprocals:  $x + \frac{1}{x} \geq 2$

### Problem 3.25: Nested AM-GM Application

Let  $x, y, z, w$  be positive real numbers.

- (i) Given that  $x > 0$  and  $y > 0$ , show that  $x + y \geq 2\sqrt{xy}$ .
- (ii) Hence show that for  $x > 0, y > 0, z > 0$  and  $w > 0$ ,

$$x + y + z + w \geq 4\sqrt[4]{xyzw}.$$

- (iii) Consider  $x, y, z$  and  $w = \frac{x+y+z}{3}$ . Apply the result in (ii) to show that

$$\frac{x + y + z}{3} \geq \sqrt[3]{xyz}.$$

**Hint:** First apply to pairs. Do not apply AM-GM directly.

**Solution 3.25**

**Part (i):** Standard AM-GM:  $(\sqrt{x} - \sqrt{y})^2 \geq 0 \implies x - 2\sqrt{xy} + y \geq 0 \implies x + y \geq 2\sqrt{xy}$ .

**Part (ii):** Apply (i) twice:

$$\begin{aligned} x + y &\geq 2\sqrt{xy}, & z + w &\geq 2\sqrt{zw} \\ (x + y) + (z + w) &\geq 2\sqrt{xy} + 2\sqrt{zw} \\ x + y + z + w &\geq 2(\sqrt{xy} + \sqrt{zw}) \geq 2 \cdot 2\sqrt{\sqrt{xy} \cdot \sqrt{zw}} = 4\sqrt[4]{xyzw} \end{aligned}$$

**Part (iii):** Let  $w = \frac{x+y+z}{3}$ . Then:

$$\begin{aligned} x + y + z + w &\geq 4\sqrt[4]{xyzw} \\ \frac{4}{3}(x + y + z) &\geq 4\sqrt[4]{xyz \cdot \frac{x + y + z}{3}} \\ \frac{x + y + z}{3} &\geq \sqrt[4]{xyz \cdot \frac{x + y + z}{3}} \end{aligned}$$

Raise to 4th power:  $\left(\frac{x+y+z}{3}\right)^4 \geq xyz \cdot \frac{x+y+z}{3}$ . Divide by  $\frac{x+y+z}{3}$ :  $\left(\frac{x+y+z}{3}\right)^3 \geq xyz$ .

**Takeaways 3.25**

- AM-GM for  $n = 3$  derived from  $n = 2$  and  $n = 4$  cases
- Nested application: pair terms, then apply again
- Clever choice of variables can be used to solve the general case

**Problem 3.26: Triangle Inequality - Quadratic Forms**

Let  $p, q, r$  be the lengths of the three sides of a triangle.

- Show that:  $p^2 + q^2 + r^2 \geq pq + pr + qr$
- Show that:  $3(pq + pr + qr) \leq (p + q + r)^2 < 4(pq + pr + qr)$

**Hint:** Consider sum of squares of differences.

**Solution 3.26**

**Part (a):** Consider  $(p - q)^2 + (q - r)^2 + (p - r)^2 \geq 0$ :

$$\begin{aligned} 2p^2 + 2q^2 + 2r^2 - 2pq - 2qr - 2pr &\geq 0 \\ p^2 + q^2 + r^2 &\geq pq + pr + qr \end{aligned}$$

**Part (b):** Expand  $(p + q + r)^2 = p^2 + q^2 + r^2 + 2(pq + pr + qr)$ . Using (a):

$$(p + q + r)^2 \geq (pq + pr + qr) + 2(pq + pr + qr) = 3(pq + pr + qr)$$

For the upper bound, use triangle inequalities  $p < q + r$ ,  $q < p + r$ ,  $r < p + q$ :

$$\begin{aligned} p^2 &< pq + pr, \quad q^2 < pq + qr, \quad r^2 < pr + qr \\ p^2 + q^2 + r^2 &< 2(pq + pr + qr) \end{aligned}$$

Add  $2(pq + pr + qr)$  to both sides:  $(p + q + r)^2 < 4(pq + pr + qr)$ .

**Takeaways 3.26**

- Sum of squares of differences always non-negative
- Triangle inequality crucial for strict upper bound

**Problem 3.27: Complex Triangle Inequality**

Prove that for any two complex numbers  $z_1, z_2 \in \mathbb{C}$ :

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

**Hint:** Use the property that  $|w| \geq \operatorname{Re}(w)$ .

**Solution 3.27**

Square the modulus:  $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$ :

$$\begin{aligned} |z_1 + z_2|^2 &= z_1 \bar{z}_1 + z_1 \bar{z}_2 + z_2 \bar{z}_1 + z_2 \bar{z}_2 \\ &= |z_1|^2 + (z_1 \bar{z}_2 + \overline{z_1 \bar{z}_2}) + |z_2|^2 \\ &= |z_1|^2 + 2\operatorname{Re}(z_1 \bar{z}_2) + |z_2|^2 \end{aligned}$$

Use  $\operatorname{Re}(w) \leq |w|$ :

$$\begin{aligned} |z_1 + z_2|^2 &\leq |z_1|^2 + 2|z_1 \bar{z}_2| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2 \end{aligned}$$

Taking square roots:  $|z_1 + z_2| \leq |z_1| + |z_2|$ .

### Takeaways 3.27

- Key identity:  $z\bar{z} = |z|^2$  and  $w + \bar{w} = 2\operatorname{Re}(w)$
- $\operatorname{Re}(w) \leq |w|$  fundamental for complex inequalities

### Problem 3.28: Complex Modulus with Constraint

Given  $|z| < \frac{1}{2}$ , show that:

$$|(1+i)z^3 + iz| < \frac{3}{4}$$

**Hint:** Apply the complex triangle inequality.

### Solution 3.28

Apply triangle inequality:

$$\begin{aligned} |(1+i)z^3 + iz| &\leq |(1+i)z^3| + |iz| \\ &= |1+i||z|^3 + |i||z| \\ &= \sqrt{2}|z|^3 + |z| \end{aligned}$$

Since  $|z| < \frac{1}{2}$  and  $f(x) = \sqrt{2}x^3 + x$  is increasing for  $x > 0$ :

$$\begin{aligned} |(1+i)z^3 + iz| &< \sqrt{2}\left(\frac{1}{2}\right)^3 + \frac{1}{2} \\ &= \frac{\sqrt{2}}{8} + \frac{4}{8} = \frac{4+\sqrt{2}}{8} \end{aligned}$$

Since  $\sqrt{2} < 2$ :  $\frac{4+\sqrt{2}}{8} < \frac{6}{8} = \frac{3}{4}$ .

### Takeaways 3.28

- Triangle inequality:  $|w_1 + w_2| \leq |w_1| + |w_2|$
- Evaluate at boundary of constraint for tight bounds

### Problem 3.29: Problem 40: Harmonic-Arithmetic Mean Inequality

Let  $a, b, c$  be positive real numbers such that  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ . Using the fact that  $x + y \geq 2\sqrt{xy}$  for positive  $x, y$ , prove that:

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$$

**Hint:** Use the given condition to show that  $ab + bc + ac = abc$ .

**Solution 3.29**

Multiply given condition by  $abc$ :  $bc + ac + ab = abc$ .  
 Apply AM-GM to pairs:

$$ab + ac \geq 2\sqrt{a^2bc} = 2a\sqrt{bc}$$

$$ab + bc \geq 2\sqrt{ab^2c} = 2b\sqrt{ac}$$

$$ac + bc \geq 2\sqrt{abc^2} = 2c\sqrt{ab}$$

Sum the three inequalities:

$$2(ab + bc + ac) \geq 2(a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab})$$

$$ab + bc + ac \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$$

Substitute  $ab + bc + ac = abc$ :  $abc \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$ .

**Takeaways 3.29**

- Convert harmonic condition to algebraic form first
- Apply AM-GM systematically to all pairs

**Problem 3.30: Logarithmic Inequality with Factorial**

- Prove that  $x > \ln(x)$  for all positive real numbers  $x$ .
- Hence, show that for all positive integers  $n$ :

$$e^{n^2+n} > (n!)^2$$

**Hint:** Consider  $f(x) = x - \ln(x)$ .

**Solution 3.30**

**Part (i):** Let  $f(x) = x - \ln(x)$ . Then  $f'(x) = 1 - \frac{1}{x}$ . Setting  $f'(x) = 0$  gives  $x = 1$ . Since  $f''(x) = \frac{1}{x^2} > 0$ , point  $(1, 1)$  is a minimum. Thus  $f(x) \geq f(1) = 1 - \ln(1) = 1 > 0$ , so  $x > \ln(x)$ .

**Part (ii):** Apply (i) to  $k = 1, 2, \dots, n$  and sum:

$$\begin{aligned}\sum_{k=1}^n k &> \sum_{k=1}^n \ln(k) \\ \frac{n(n+1)}{2} &> \ln(n!) \\ \frac{n^2 + n}{2} &> \ln(n!) \\ n^2 + n &> 2\ln(n!) = \ln((n!)^2)\end{aligned}$$

Exponentiating:  $e^{n^2+n} > (n!)^2$ .

**Takeaways 3.30**

- Calculus proves  $x > \ln(x)$  via minimization
- Sum inequalities to relate arithmetic to logarithmic sums

**Problem 3.31: Cauchy-Schwarz with Homogenization**

Prove that for all positive real numbers  $a, b, c$ :

$$\frac{a^2}{3a+2b} + \frac{b^2}{3b+2c} + \frac{c^2}{3c+2a} \geq \frac{a+b+c}{5}$$

**Hint:** Apply Cauchy-Schwarz.

**Solution 3.31**

Apply Cauchy-Schwarz:  $\left(\sum \frac{u_i^2}{v_i}\right) (\sum v_i) \geq (\sum u_i)^2$ .

Let  $u_i = (a, b, c)$  and  $v_i = (3a + 2b, 3b + 2c, 3c + 2a)$ :

$$\left(\frac{a^2}{3a + 2b} + \frac{b^2}{3b + 2c} + \frac{c^2}{3c + 2a}\right) \cdot [(3a + 2b) + (3b + 2c) + (3c + 2a)] \geq (a + b + c)^2$$

Simplify denominator sum:

$$(3a + 2b) + (3b + 2c) + (3c + 2a) = 3(a + b + c) + 2(a + b + c) = 5(a + b + c)$$

Therefore:

$$\text{LHS} \cdot 5(a + b + c) \geq (a + b + c)^2 \implies \text{LHS} \geq \frac{a + b + c}{5}$$

**Takeaways 3.31**

- Cauchy-Schwarz in Titu's Lemma form:  $\sum \frac{x_i^2}{y_i} \geq \frac{(\sum x_i)^2}{\sum y_i}$
- Check that denominators sum to simple multiple of numerator sum

**Problem 3.32: Bernoulli's Inequality - Power Form**

Prove the following inequality for all integers  $n \geq 1$ :

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + \sqrt{n}$$

**Hint:** Apply Bernoulli directly.

**Solution 3.32**

Let  $x = \frac{1}{\sqrt{n}}$ . Since  $n \geq 1$ , we have  $x > 0 > -1$ .

Apply Bernoulli's inequality  $(1 + x)^n \geq 1 + nx$ :

$$\begin{aligned} \left(1 + \frac{1}{\sqrt{n}}\right)^n &\geq 1 + n \cdot \frac{1}{\sqrt{n}} \\ &= 1 + \frac{n}{\sqrt{n}} \\ &= 1 + \sqrt{n} \end{aligned}$$

### Takeaways 3.32

- Bernoulli's inequality:  $(1+x)^n \geq 1+nx$  for  $x > -1$ ,  $n \geq 1$
- Choose substitution to match target form

### Problem 3.33: Strict Bernoulli via Induction

- (i) Prove that  $(1+x)^n > 1+nx$  for  $n \geq 1$  and  $x > -1$ .
- (ii) Hence, deduce that  $(1 - \frac{1}{2n})^n > \frac{1}{2}$  for  $n > 1$ .

**Hint:** Prove by math induction.

### Solution 3.33

**Part (i):** By induction. Base case  $n = 1$ : equality holds.

Inductive step: Assume  $(1+x)^k \geq 1+kx$ . Then:

$$\begin{aligned}(1+x)^{k+1} &= (1+x)(1+x)^k \geq (1+x)(1+kx) \\ &= 1+kx+x+kx^2 = 1+(k+1)x+kx^2\end{aligned}$$

Since  $k \geq 1$  and  $x^2 \geq 0$ , we have  $kx^2 \geq 0$ , so  $(1+x)^{k+1} \geq 1+(k+1)x$ . For  $n > 1$  and  $x \neq 0$ , strict inequality holds since  $kx^2 > 0$ .

**Part (ii):** Let  $x = -\frac{1}{2n}$ . Check  $x > -1$ :  $-\frac{1}{2n} > -1$  holds for  $n > \frac{1}{2}$ . Apply (i):

$$\left(1 - \frac{1}{2n}\right)^n > 1 + n\left(-\frac{1}{2n}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

### Takeaways 3.33

- Induction proves Bernoulli; strict inequality when  $kx^2 > 0$
- Negative substitutions require careful domain checking

### Problem 3.34: Summation Inequality via Induction

Given that for  $k > 0$ ,  $2k+3 > 2\sqrt{(k+1)(k+2)}$ , prove that:

$$\sum_{r=1}^n \frac{1}{\sqrt{r}} > 2\left(\sqrt{n+1} - 1\right)$$

for all positive integers  $n$ .

**Hint:** Prove by induction. Use the given inequality.

### Solution 3.34

**Base case** ( $n = 1$ ): LHS = 1, RHS =  $2(\sqrt{2} - 1) \approx 0.828$ . True.

**Inductive step:** Assume  $\sum_{r=1}^k \frac{1}{\sqrt{r}} > 2(\sqrt{k+1} - 1)$ . Then:

$$\begin{aligned}\sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} &= \sum_{r=1}^k \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{k+1}} \\ &> 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} \\ &= 2\sqrt{k+1} + \frac{1}{\sqrt{k+1}} - 2 \\ &= \frac{2(k+1) + 1}{\sqrt{k+1}} - 2 = \frac{2k+3}{\sqrt{k+1}} - 2\end{aligned}$$

Given  $2k+3 > 2\sqrt{(k+1)(k+2)}$ :

$$\begin{aligned}\frac{2k+3}{\sqrt{k+1}} - 2 &> \frac{2\sqrt{(k+1)(k+2)}}{\sqrt{k+1}} - 2 \\ &= 2\sqrt{k+2} - 2 = 2(\sqrt{k+2} - 1)\end{aligned}$$

Thus the inequality holds for  $n = k + 1$ .

### Takeaways 3.34

- Use given auxiliary inequality in inductive step
- Algebraic manipulation converts sum to target form

## 4 Conclusion

Inequalities are a cornerstone of the HSC Mathematics Extension 2 course, appearing in diverse contexts from pure algebra to calculus and complex numbers. Mastery requires recognizing when to apply AM-GM, Cauchy-Schwarz, triangle inequality, or induction-based techniques. Use these 45 problems to develop pattern recognition, proof-writing clarity, and strategic problem-solving skills. Best of luck with your studies and examinations!

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