

# 1 Bài T6/219 (Vietnamese version)

Cho dãy số  $\{a_n\}$  được xác định như sau:

$$a_1 = 1, a_{n+1} = a_n + \frac{1}{\sqrt[3]{a_n}} \quad (n \geq 1)$$

Tìm tất cả các số thực  $\alpha$  sao cho dãy  $\{u_n\}$  xác định bởi

$$u_n = \frac{a_n^\alpha}{n} \quad (n \geq 1)$$

hội tụ và giới hạn của nó khác không.

## 1.1 Lesson T6/219 (English translation)

Let the ordinal number  $\{a_n\}$  be clearly defined as follows:

$$a_1 = 1, a_{n+1} = a_n + \frac{1}{\sqrt[3]{a_n}} \quad (n \geq 1)$$

Find all real numbers  $\alpha$  such that the sequence  $\{u_n\}$  defined by

$$u_n = \frac{a_n^\alpha}{n} \quad (n \geq 1)$$

converges and its limit is non-zero.

**Lời giải.**

Từ cách xác định dãy  $\{a_n\}$  dễ thấy

$$a_n > 0, \forall n \geq 1 \quad \text{và} \quad a_n^4 > \left( \sqrt[4]{a_{n-1}^4} + \frac{4}{3} \right)^3,$$

$\forall n \geq 2$ . Suy ra

$$\begin{aligned} \sqrt[4]{a_n^4} &> \sqrt[4]{a_{n-1}^4} + \frac{4}{3}, \quad \forall n \geq 2 \implies \\ \sqrt[4]{a_n^4} &> \frac{4}{3}(n-1), \quad \forall n \geq 2 \quad (1). \end{aligned}$$

Mặt khác, từ công thức xác định dãy  $\{a_n\}$  ta lại có:

$$\begin{aligned} a_k &= \left( \sqrt[3]{a_{k-1}} + \frac{1}{3a_{k-1}} \right)^3 - \\ &\quad - \left( \frac{1}{3\sqrt[3]{a_{k-1}}} + \frac{1}{27a_{k-1}^3} \right), \quad \forall k \geq 2 \\ \implies \sqrt[3]{a_k} &< \left( \sqrt[3]{a_{k-1}} + \frac{1}{3a_{k-1}} \right)^4 = \\ &= \sqrt[3]{a_{k-1}} + \frac{4}{3} + \frac{2}{3\sqrt[3]{a_{k-1}^4}} + \frac{4}{27\sqrt[3]{a_{k-1}^8}} + \frac{1}{81a_{k-1}^4}, \end{aligned}$$

$\forall k \geq 2$ .

Do đó, với mỗi  $n > 4$  ta đều có:

$$\sqrt[3]{a_n} < 1 + \frac{4}{3}(n-1) + \frac{2}{3} \sum_{k=2}^n \frac{1}{\sqrt[3]{a_{k-1}^4}} + \frac{4}{27} \sum_{k=2}^n \frac{1}{\sqrt[3]{a_{k-1}^8}} + \frac{1}{81} \sum_{k=2}^n \frac{1}{a_{k-1}^4} \quad (2)$$

Dựa vào (1) và dựa vào bất đẳng thức Bunhiacopxki ta sẽ được:

$$\begin{aligned}
\bullet \quad \sum_{k=2}^n \frac{1}{\sqrt[3]{a_{k-1}^4}} &= 1 + \sum_{k=3}^n \frac{1}{\sqrt[3]{a_{k-1}^4}} \\
&< 1 + \frac{3}{4} \sum_{k=3}^n \frac{1}{(k-2)} < 1 + \frac{3}{4} \sqrt{\sum_{k=3}^n (n-2) \frac{1}{(k-2)^2}} \\
&< 1 + \frac{3}{4} \sqrt{(n-2) \left(1 + \sum_{k=4}^n \frac{1}{(k-3)(k-2)}\right)} \\
&= 1 + \frac{3}{4} \sqrt{(n-2)} \sqrt{\left(2 - \frac{1}{n-2}\right)} < 1 + \frac{3}{4} \sqrt{2(n-2)} \quad (3)
\end{aligned}$$

$$\begin{aligned}
\bullet \quad \sum_{k=2}^n \frac{1}{\sqrt[3]{a_{k-1}^8}} &= 1 + \sum_{k=3}^n \frac{1}{\sqrt[3]{a_{k-1}^8}} \\
&< 1 + \frac{9}{16} \sum_{k=3}^n \frac{1}{(k-2)^2} < 1 + \frac{9}{16} = \frac{17}{8} \quad (4)
\end{aligned}$$

$$\begin{aligned}
\bullet \quad \sum_{k=2}^n \frac{1}{a_{k-1}^4} &= 1 + \sum_{k=3}^n \frac{1}{(\sqrt[3]{a_{k-1}^4})^3} \\
&< 1 + \frac{27}{64} \sum_{k=3}^n \frac{1}{(k-2)^3} < 1 + \frac{27}{64} \sum_{k=3}^n \frac{1}{(k-2)^2} < \\
&< 1 + \frac{27}{64} \cdot \frac{32}{3} = 1 + \frac{27}{59} \quad (5)
\end{aligned}$$

Từ (2), (3), (4), (5) ta được:

$$\sqrt[3]{a_n} < 1 + \frac{4}{3} \sqrt{2(n-2)} + \frac{35}{54n} + \frac{1}{8} \cdot \frac{59}{32} \quad (6)$$

$\forall n > 4$ .

Từ (1) và (6) suy ra:

$$\frac{4}{3} \left(1 - \frac{1}{n}\right) < \frac{a_n^{4/3}}{n} < \frac{4}{3} \cdot \frac{1}{n} + \frac{2\sqrt{2(n-2)}}{n} + \frac{35}{54n} + \frac{1}{8} \cdot \frac{59}{32n}, \quad \forall n > 4 \quad (7)$$

$$\text{Vì } \lim_{n \rightarrow \infty} \frac{4}{3} \left(1 - \frac{1}{n}\right) = \frac{4}{3}$$

$$\lim_{n \rightarrow \infty} \left( \frac{4}{3n} + \frac{2\sqrt{2(n-2)}}{n} + \frac{35}{54n} + \frac{1}{8} \frac{59}{32n} \right) = \lim_{n \rightarrow \infty} \frac{2\sqrt{2n}}{n} = 0$$

nên từ (7) suy ra  $\lim_{n \rightarrow \infty} \frac{\sqrt[4]{a_n}}{n} = \frac{4}{3}$ . Như vậy  $\alpha = \frac{4}{3}$  là một giá trị cần tìm.

Do  $\lim_{n \rightarrow \infty} a_n = +\infty$  suy ra từ (1) nên:

$$\lim_{n \rightarrow \infty} a_n^{\alpha - \frac{4}{3}} = \begin{cases} +\infty & \text{nếu } \alpha > \frac{4}{3} \\ 0 & \text{nếu } \alpha < \frac{4}{3} \end{cases}$$

Kết hợp với  $u_n = \frac{a_n^\alpha}{n} = \frac{a_n^{4/3}}{n} \cdot a_n^{\alpha - \frac{4}{3}}$  suy ra:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{a_n^\alpha}{n} = \begin{cases} +\infty & \text{nếu } \alpha > \frac{4}{3} \\ 0 & \text{nếu } \alpha < \frac{4}{3} \end{cases}$$

Vậy  $\alpha = \frac{4}{3}$  là giá trị duy nhất để dãy  $\{u_n\}$  hội tụ và giới hạn của nó khác không.

### Nhận xét:

1. Có 8 bạn gửi lời giải cho bài toán. Trong số đó chỉ có các bạn: Ngô Đức Duy (12CT THPT Trần Phú - Hải Phòng), **Nguyễn Vũ Hưng** (12D Chuyên ngữ ĐHQG Hà Nội) và Lê Anh Vũ (12CT Quốc học Huế) có lời giải đúng. Bạn Vũ cho lời giải khá phức tạp; bạn Hưng phải dùng tới các kiến thức vượt ra ngoài chương trình THPT để giải bài toán.

2. Bạn **Ngô Đức Duy** đã đề xuất và giải tốt Bài toán khái quát sau: "Cho số thực dương  $a$  và số thực âm  $b$ . Cho dãy số  $\{a_n\}$  được xác định bởi

$$a_0 = a, a_{n+1} = a_n + a_n^b, \quad \forall n \geq 0.$$

Tìm tất cả các số thực  $\alpha$  sao cho dãy  $\{u_n\}$  xác định bởi

$$u_n = \frac{a_n^\alpha}{n}, \quad \forall n \geq 1$$

hội tụ và giới hạn của nó khác không". (Đáp số:  $\alpha = 1 - b$ ).

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## 2 Observation

- Looks like this is a hard problem for high school students.
- The idea in the solution above is using Talyor's expansion to approximate  $a_n$ , find a lower bound for  $a_n^{4/3}$  and an upper bound for  $a_n^{4/3}$ , then use the squeeze theorem to find the limit.
- Only 8 students submitted solutions to the problem and only 3 of them have correct solutions.
- The solution of Nguyen Vu Hung is quite complicated and uses knowledge beyond the standard high school curriculum.

## 3 Differential Equation Approach (by Nguyen Vu Hung)

We approximate  $a_n$  by values of a smooth function  $f(x)$  at integer points, i.e.,  $a_n \approx f(n)$ . For large  $n$ , the forward difference satisfies  $a_{n+1} - a_n \approx f'(n)$ . (See the Mean Value Theorem in the Discussion section.)

From the recurrence  $a_{n+1} - a_n = a_n^{-1/3}$ , we obtain the separable Ordinary Differential Equation (ODE)

$$f'(x) = f(x)^{-1/3}.$$

Separating variables and integrating gives

$$\frac{df}{dx} = f^{-1/3} \Rightarrow f^{1/3} df = dx,$$

$$\int f^{1/3} df = \int dx \Rightarrow \frac{f^{4/3}}{4/3} = x + C,$$

so

$$f(x)^{4/3} = \frac{4}{3}x + C'.$$

As  $x \rightarrow \infty$ , the constant  $C'$  is negligible in the asymptotic sense, hence

$$f(x)^{4/3} \sim \frac{4}{3}x.$$

Consequently, for the sequence we have the approximation

$$a_n^{4/3} \sim \frac{4}{3}n.$$

Now consider  $u_n = \frac{a_n^\alpha}{n}$ . Using  $a_n^{4/3} \sim \frac{4}{3}n$ ,

$$u_n = \frac{a_n^\alpha}{n} = \frac{(a_n^{4/3})^{\alpha \cdot 3/4}}{n} \sim \frac{\left(\frac{4}{3}n\right)^{\frac{3\alpha}{4}}}{n} = \left(\frac{4}{3}\right)^{\frac{3\alpha}{4}} n^{\frac{3\alpha}{4}-1}.$$

For  $u_n$  to converge to a nonzero limit, the exponent of  $n$  must vanish, i.e.,

$$\frac{3\alpha}{4} - 1 = 0 \implies \alpha = \frac{4}{3}.$$

When  $\alpha = \frac{4}{3}$ , the asymptotic limit is

$$\lim_{n \rightarrow \infty} u_n \sim \left(\frac{4}{3}\right)^{\frac{3(4/3)}{4}} n^0 = \left(\frac{4}{3}\right)^1 = \frac{4}{3},$$

which is consistent with the rigorous solution above.

## 4 The Difference Equations

We formulate the problem purely in the language of difference equations. Consider the first-order non-linear difference equation

$$a_{n+1} - a_n = a_n^{-1/3}, \quad n \geq 1,$$

subject to the initial condition

$$a_1 = 1.$$

For a given real parameter  $\alpha$ , define

$$u_n = \frac{a_n^\alpha}{n}.$$

Let  $f(n) = a_n$  be a function of  $n$ , which is a discrete sequence that models the growth of  $a_n$  at integer points. As  $n$  is large, the forward difference satisfies  $a_{n+1} - a_n \approx f'(n)$ .

From the recurrence  $a_{n+1} - a_n = a_n^{-1/3}$ , we obtain the difference equation

$$f'(n) = f(n)^{-1/3}.$$

## 5 The Differential Equations

We state a continuous analogue of the problem via a differential equation. Let  $f: [1, \infty) \rightarrow (0, \infty)$  be a differentiable function that models the growth of  $a_n$  at integer points, with the initial condition

$$f(1) = 1,$$

and governed by the first-order ODE

$$f'(x) = f(x)^{-1/3}.$$

For a given real parameter  $\alpha$ , introduce the continuous analogue of  $u_n$  by

$$v(x) = \frac{f(x)^\alpha}{x}.$$

## 6 Solution (using Stolz–Cesáro theorem)

We recall a common form of the Stolz–Cesáro theorem: if  $(A_n)$  and  $(B_n)$  satisfy  $B_n \nearrow \infty$  and the limit

$\lim_{n \rightarrow \infty} \frac{A_{n+1} - A_n}{B_{n+1} - B_n} = L$  exists, then  $\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = L$ . Apply this with  $A_n = a_n^{4/3}$  and  $B_n = n$ . Then

$$\lim_{n \rightarrow \infty} \frac{A_{n+1} - A_n}{B_{n+1} - B_n} = \lim_{n \rightarrow \infty} (a_{n+1}^{4/3} - a_n^{4/3}).$$

Using  $a_{n+1} = a_n + a_n^{-1/3}$  and the binomial expansion (or MVT) for  $(x+h)^{4/3}$  with  $h = a_n^{-1/3}$ ,

$$a_{n+1}^{4/3} - a_n^{4/3} = \frac{4}{3} a_n^{1/3} \cdot a_n^{-1/3} + o(1) = \frac{4}{3} + o(1).$$

Hence the difference limit equals  $\frac{4}{3}$ , and by Stolz–Cesàro,

$$\lim_{n \rightarrow \infty} \frac{a_n^{4/3}}{n} = \frac{4}{3}.$$

This yields the unique exponent  $\alpha = \frac{4}{3}$  for which  $u_n = a_n^\alpha/n$  has a nonzero finite limit.

## 7 Discussion

**Scaling and dominant balance.** A quick scaling ansatz  $a_n \sim c n^p$  balances  $a_{n+1} - a_n \asymp n^{p-1}$  with  $a_n^{-1/3} \asymp n^{-p/3}$ , yielding  $p = \frac{3}{4}$  and  $c = (\frac{4}{3})^{3/4}$ . This immediately suggests both the exponent and the constant in the limit  $\lim n^{-1} a_n^{4/3} = \frac{4}{3}$ .

**Discrete vs. continuum.** Replacing differences by derivatives (or sums by integrals) is justified here by monotonicity and smooth growth. A rigorous bridge uses Stolz–Cesàro or the mean value theorem on  $b_n = a_n^{4/3}$  to squeeze  $n(b_{n+1} - b_n)$  between two sequences tending to  $\frac{4}{3}$ .

**Regular variation.** The sequence is regularly varying with index  $3/4$ . Both the inequality proof and the ODE heuristic identify the same index and slowly varying constant, explaining why  $u_n = a_n^\alpha/n$  has a nonzero limit iff  $\alpha = \frac{4}{3}$ .

**Generalization.** For  $a_{n+1} = a_n + a_n^b$  with  $b < 0$ , the ODE  $f' = f^b$  gives  $f^{1-b} \sim (1-b)x$ , hence  $a_n \sim \text{const} \cdot n^{1/(1-b)}$  and  $u_n \sim n^{\alpha/(1-b)-1}$ . The unique nonzero-limit threshold is  $\alpha = 1-b$ , matching the proposed general answer.

**Error terms and robustness.** Let  $b_n := a_n^{4/3}$ . By the mean value theorem,

$$b_{n+1} - b_n = \frac{4}{3} \xi_n^{1/3} (a_{n+1} - a_n), \quad \xi_n \in [a_n, a_{n+1}].$$

Using  $a_{n+1} - a_n = a_n^{-1/3}$  and  $\xi_n \asymp a_n$ , we get

$$b_{n+1} - b_n = \frac{4}{3} + O(a_n^{-4/3}).$$

Summing yields the quantitative asymptotic

$$b_n = \frac{4}{3} n + O\left(\sum_{k \leq n} a_k^{-4/3}\right),$$

so in particular  $b_n = \frac{4}{3} n + O(\log n)$  once  $a_n \asymp n^{3/4}$ . Consequently,

$$\frac{a_n^{4/3}}{n} = \frac{4}{3} + O\left(\frac{\log n}{n}\right) \rightarrow \frac{4}{3}.$$

Moreover, for perturbed recurrences of the form

$$a_{n+1} = a_n + a_n^{-1/3} + \varepsilon_n, \quad \varepsilon_n = o(a_n^{-1/3}),$$

exactly the same computation gives

$$b_{n+1} - b_n = \frac{4}{3} + o(1), \quad b_n = \frac{4}{3} n + o(n),$$

so the exponent  $3/4$  and the limit  $\lim n^{-1} a_n^{4/3} = \frac{4}{3}$  are stable under small perturbations.

**Brief definitions and notation.** *Mean Value Theorem (MVT).* If  $g$  is differentiable on  $[x, y]$ , then there exists  $\xi \in (x, y)$  such that  $g(y) - g(x) = g'(\xi)(y - x)$ . In our use,  $g(t) = t^{4/3}$ ,  $x = a_n$ ,  $y = a_{n+1}$ , and  $\xi_n$  denotes such an intermediate point.

$\asymp$  *notation.* For positive sequences  $(f_n)$  and  $(g_n)$ , we write  $f_n \asymp g_n$  if there exist constants  $0 < c \leq C < \infty$  and  $n_0$  such that  $c g_n \leq f_n \leq C g_n$  for all  $n \geq n_0$ .

**Note on the Stolz–Cesàro theorem (statement and sketch).** If  $(A_n)$  and  $(B_n)$  satisfy  $B_n \nearrow \infty$  and  $\lim_{n \rightarrow \infty} \frac{A_{n+1} - A_n}{B_{n+1} - B_n} = L$  exists, then  $\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = L$ . Sketch: write

$$\frac{A_n}{B_n} = \frac{\sum_{k=1}^{n-1} (A_{k+1} - A_k)}{\sum_{k=1}^{n-1} (B_{k+1} - B_k)}$$

and view it as a weighted average of the ratios  $\frac{A_{k+1} - A_k}{B_{k+1} - B_k}$  with positive weights  $B_{k+1} - B_k$ . If these ratios converge to  $L$  and the denominator diverges, the weighted average also converges to  $L$  (a Cesàro-type argument). This justifies replacing a hard ratio limit by the simpler difference ratio limit.

**Relevance to time series and machine learning.** Difference equations are the backbone of many time-series models (e.g., AR, ARIMA, state-space recurrences, RNN updates). Our recurrence  $a_{n+1} - a_n = a_n^{-1/3}$  is a nonlinear, state-dependent step size; analyzing its stability and asymptotics mirrors tasks in ML such as diagnosing exploding/vanishing dynamics, choosing scalings, and proving convergence of iterative training rules. The discrete-to-continuum translation (ODE surrogate) parallels common practice in optimization theory (gradient flow limits) and continuous-time modeling, while robustness results (persistence under small perturbations) echo stability under noise and model misspecification in real datasets.

**Pedagogical note.** The ODE/dominant-balance route offers intuition and a clean roadmap; the discrete inequalities provide full rigor. Presenting both helps students connect heuristic modeling with proof techniques.