HSC Math Extension 2: Complex Numbers Mastery

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1 Introduction

1.1 Project Overview

This booklet compiles high quality complex numbers problems curated specifically for the HSC Mathematics Extension 2 syllabus. Every problem explores fundamental and advanced techniques involving complex numbers—from basic arithmetic and form conversions to De Moivre's theorem, Argand diagram geometry, polynomial roots, and geometric transformations. Detailed reasoning showcases common techniques such as modulus-argument manipulation, Euler's formula applications, and geometric interpretations.

1.2 Target Audience

The explanations are crafted for Extension 2 students aiming to deepen their complex number skills. Each solution explicitly states the conversion steps, theorem applications, and geometric reasoning so that high-school learners can follow every transition.

1.3 How to Use This Booklet

- Read the overview and complex numbers primer before attempting the problems.
- Attempt problems in Part 1 without hints; compare against the detailed solutions to understand model reasoning.
- Use the upside-down hints in Part 2 only after making a genuine attempt.
- Revisit problems after a few days and try to re-derive the arguments without notes to reinforce technique.

1.4 Complex Numbers Primer

1.4.1 Key Theorems and Formulas

Euler's Theorem:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This connects exponential form with polar form and is fundamental to many proofs.

De Moivre's Theorem: For any integer n,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

or equivalently, $(re^{i\theta})^n = r^n e^{in\theta}$.

This theorem is invaluable for:

- Finding powers of complex numbers

- Finding n-th roots of complex numbers
- Proving trigonometric identities

Modulus and Argument: For $z = x + iy$:

- Modulus: $|z| = \sqrt{x^2 + y^2}$
- Argument: $\arg(z) = \tan^{-1}(y/x)$ (with appropriate quadrant adjustments)
- Properties: $|z_1 z_2| = |z_1||z_2|$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ (modulo 2π)

1.4.2 Forms of Complex Numbers

Cartesian Form: $z = x + iy$
Polar Form: $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg(z)$
Exponential Form: $z = re^{i\theta}$
Conversions between these forms are essential skills tested frequently in Extension 2.

1.4.3 Argand Diagram

The Argand diagram visualizes complex numbers as points in the plane, with the real part on the horizontal axis and the imaginary part on the vertical axis. Geometric interpretations include:

- $|z_1 - z_2|$ represents the distance between z_1 and z_2
- $(z_1 + z_2)/2$ represents the midpoint
- Multiplication by $e^{i\theta}$ rotates by angle θ
- Multiplication by r scales by factor r

1.4.4 Notation and Conventions

Throughout this booklet:

- $\bar{z}$ denotes the complex conjugate of z
- $\text{Re}(z)$ and $\text{Im}(z)$ denote the real and imaginary parts
- $\arg(z)$ denotes the principal argument in $(-\pi, \pi]$
- Angles are in radians unless otherwise specified

2 Part 1: Problems and Solutions (Detailed)

Part 1 contains three sets of problems—basic, medium, and advanced. Each set provides five problems. For every problem we present a comprehensive solution without any hints so that learners focus on the full reasoning trail.

2.1 Basic Complex Numbers Problems

Problem 2.1: Basic Complex Arithmetic

Let $z = 5 - i$ and $w = 2 + 3i$. What is the value of $2z + \overline{w}$?

Solution 2.1

First, compute $2z$:

$$2z = 2(5 - i) = 10 - 2i.$$

Next, find the conjugate of w:

$$\overline{w} = \overline{2 + 3i} = 2 - 3i.$$

Now add these results:

$$2z + \overline{w} = (10 - 2i) + (2 - 3i) = 12 - 5i.$$

Therefore, $2z + \overline{w} = 12 - 5i$.

Takeaways 2.1

The conjugate of $a + bi$ is $a - bi$. When adding complex numbers, combine real parts with real parts and imaginary parts with imaginary parts.

Problem 2.2: Finding Square Roots

What value of z satisfies $z^2 = 7 - 24i$?

Solution 2.2

Let $z = a + bi$ where a, b are real. Then:

$$z^2 = (a+bi)^2 = a^2 - b^2 + 2abi = 7 - 24i.$$

Equating real and imaginary parts:

$$\begin{aligned} a^2 - b^2 &= 7 \\ 2ab &= -24. \end{aligned}$$

From the second equation: $ab = -12$, so $b = -\frac{12}{a}$.
Substitute into the first equation:

$$a^2 - \left(-\frac{12}{a}\right)^2 = 7 \implies a^2 - \frac{144}{a^2} = 7.$$

Multiply by a^2:

$$a^4 - 7a^2 - 144 = 0.$$

Let $u = a^2$:

$$u^2 - 7u - 144 = 0 \implies (u-16)(u+9) = 0.$$

Since $u = a^2 \geq 0$, we have $u = 16$, so $a^2 = 16$ giving $a = \pm 4$.
If $a = 4$: $b = -\frac{12}{4} = -3$, so $z = 4 - 3i$.
If $a = -4$: $b = -\frac{12}{-4} = 3$, so $z = -4 + 3i$.
Verify: $(4-3i)^2 = 16 - 24i + 9i^2 = 16 - 24i - 9 = 7 - 24i$. ✓
Therefore, $z = 4 - 3i$ or $z = -4 + 3i$.

Takeaways 2.2

To find square roots of complex numbers in Cartesian form, let $z = a + bi$ and equate real and imaginary parts after expanding z^2. This gives a system of two equations in two unknowns.

Problem 2.3: Complex Roots of Quadratics

Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where p and q are real, what are the values of p and q?

Solution 2.3

Since the polynomial has real coefficients and $z = 3 + i$ is a root, its complex conjugate $\overline{z} = 3 - i$ must also be a root.
By Vieta's formulas:

$$p = -(z_1 + z_2) = -\big((3+i) + (3-i)\big) = -6,$$

$$q = z_1 \cdot z_2 = (3+i)(3-i) = 9 - i^2 = 9 - (-1) = 10.$$

Therefore, $p = -6$ and $q = 10$.

Takeaways 2.3

For polynomials with real coefficients, complex roots occur in conjugate pairs. Use Vieta's formulas: sum of roots $= -p$ and product of roots $= q$ for $z^2 + pz + q = 0$.

Problem 2.4: Powers of i

Write i^9 in the form $a + ib$ where a and b are real.

Solution 2.4

Note the pattern of powers of i:

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^2 \cdot i = -i \\ i^4 &= (i^2)^2 = 1 \\ i^5 &= i^4 \cdot i = i \end{aligned}$$

The pattern repeats every 4 powers. To find i^9:

$$9 = 4 \cdot 2 + 1,$$

so $i^9 = i^{4 \cdot 2 + 1} = (i^4)^2 \cdot i = 1^2 \cdot i = i$.
Therefore, $i^9 = 0 + 1i$, giving $a = 0$ and $b = 1$.

Takeaways 2.4

Powers of i repeat with period 4: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. Use division by 4 to find $i^n = i^{n \bmod 4}$.

Problem 2.5: Polar Form Conversion

Write $1 + i$ in the form $r(\cos\theta + i\sin\theta)$.

Solution 2.5

For $z = 1 + i$, first find the modulus:

$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Next, find the argument. Since z is in the first quadrant:

$$\theta = \arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

Therefore:

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

Takeaways 2.5

To convert $z = x + iy$ to polar form: (1) Find modulus $r = \sqrt{x^2 + y^2}$. (2) Find argument $\theta = \tan^{-1}(y/x)$ (adjust for quadrant). (3) Write $z = r(\cos\theta + i\sin\theta)$.

2.2 Medium Complex Numbers Problems

Problem 2.6: Rhombus on Argand Diagram

On the Argand diagram, the complex numbers 0, $1 + i\sqrt{3}$, $\sqrt{3} + i$ and z form a rhombus. Find z in the form $a + ib$.

Solution 2.6

In a rhombus, opposite sides are parallel and equal in length. Since we have vertices at 0, $1 + i\sqrt{3}$, $\sqrt{3} + i$, and z, and they form a rhombus with one vertex at the origin, we need z to be the fourth vertex.
The rhombus has sides:

- From 0 to $1 + i\sqrt{3}$
- From 0 to $\sqrt{3} + i$
- From $1 + i\sqrt{3}$ to z
- From $\sqrt{3} + i$ to z

For a rhombus with one vertex at the origin, if two adjacent vertices are at w_1 and w_2, then the fourth vertex is at $w_1 + w_2$ (by the parallelogram law).
Therefore:

$$z = (1 + i\sqrt{3}) + (\sqrt{3} + i) = (1 + \sqrt{3}) + i(\sqrt{3} + 1) = (1 + \sqrt{3})(1 + i).$$

Simplifying:

$$z = 1 + \sqrt{3} + i(1 + \sqrt{3}) = (1 + \sqrt{3})(1 + i).$$

Therefore, $z = (1 + \sqrt{3}) + i(1 + \sqrt{3})$.

Takeaways 2.6

In the Argand diagram, a parallelogram (including rhombus) with vertices at 0, w_1, w_2, and z has its fourth vertex at $z = w_1 + w_2$ by the parallelogram law of vector addition.

Problem 2.7: De Moivre's Theorem for Real Results

Given $-\sqrt{3} - i = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$, show that $(-\sqrt{3} - i)^6$ is a real number.

Solution 2.7

Using De Moivre's theorem:

$$
\begin{aligned}
(-\sqrt{3} - i)^6 &= \left[2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)\right]^6 \\
&= 2^6\left(\cos\left(6\cdot\left(-\frac{5\pi}{6}\right)\right) + i\sin\left(6\cdot\left(-\frac{5\pi}{6}\right)\right)\right) \\
&= 64\left(\cos(-5\pi) + i\sin(-5\pi)\right).
\end{aligned}
$$

Now evaluate the trigonometric values:

$$\cos(-5\pi) = \cos(5\pi) = \cos(\pi) = -1,$$

$$\sin(-5\pi) = -\sin(5\pi) = -\sin(\pi) = 0.$$

Therefore:

$$(-\sqrt{3} - i)^6 = 64(-1 + 0i) = -64.$$

Since the imaginary part is zero, $(-\sqrt{3} - i)^6 = -64$ is indeed a real number.

Takeaways 2.7

A complex number in polar form $r(\cos\theta + i\sin\theta)$ raised to power n has argument $n\theta$. If $n\theta$ is a multiple of π, then $\sin(n\theta) = 0$ and the result is real.

Problem 2.8: Division in Polar Form

Let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 + i$. Find the modulus-argument form of $\frac{\alpha}{\beta}$.

Solution 2.8

First, convert α and β to polar form.
For $\alpha = 1 + i\sqrt{3}$:

$$|\alpha| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2,$$

$$\arg(\alpha) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

So $\alpha = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.
For $\beta = 1 + i$:

$$|\beta| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

$$\arg(\beta) = \tan^{-1}(1) = \frac{\pi}{4}.$$

So $\beta = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.
Now compute the quotient:

$$\begin{aligned}\frac{\alpha}{\beta} &= \frac{2}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right) \\ &= \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right).\end{aligned}$$

Takeaways 2.8

For division in polar form: $\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$.

Problem 2.9: Powers and Cartesian Form

Express $(\sqrt{3} - i)^7$ in the form $x + iy$.

Solution 2.9

First, convert $\sqrt{3} - i$ to polar form:

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2,$$

$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6} \text{ (fourth quadrant).}$$

So $\sqrt{3} - i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$.
Apply De Moivre's theorem:

$$(\sqrt{3} - i)^7 = 2^7\left(\cos\left(7 \cdot \left(-\frac{\pi}{6}\right)\right) + i\sin\left(7 \cdot \left(-\frac{\pi}{6}\right)\right)\right)$$

$$= 128\left(\cos\left(-\frac{7\pi}{6}\right) + i\sin\left(-\frac{7\pi}{6}\right)\right).$$

Now evaluate:

$$\cos\left(-\frac{7\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) = \cos\left(\pi + \frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2},$$

$$\sin\left(-\frac{7\pi}{6}\right) = -\sin\left(\frac{7\pi}{6}\right) = -\sin\left(\pi + \frac{\pi}{6}\right) = -\left(-\sin\frac{\pi}{6}\right) = \frac{1}{2}.$$

Therefore:

$$(\sqrt{3} - i)^7 = 128\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) = -64\sqrt{3} + 64i.$$

Takeaways 2.9

To compute high powers of complex numbers: (1) Convert to polar form. (2) Apply De Moivre's theorem. (3) Simplify the angle using periodicity and reference angles. (4) Convert back to Cartesian form.

Problem 2.10: Locus as a Curve

The point P on the Argand diagram represents $z = x + iy$ satisfying $z^2 + \bar{z}^2 = 8$. Find the equation of the curve in terms of x and y and state what type of curve it is.

Solution 2.10

Let $z = x + iy$, so $\bar{z} = x - iy$.
Then:

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy,$$
$$\bar{z}^2 = (x-iy)^2 = x^2 - y^2 - 2ixy.$$

Therefore:

$$z^2 + \bar{z}^2 = (x^2 - y^2 + 2ixy) + (x^2 - y^2 - 2ixy) = 2(x^2 - y^2) = 8.$$

Simplifying:

$$x^2 - y^2 = 4.$$

This can be rewritten as:

$$\frac{x^2}{4} - \frac{y^2}{4} = 1,$$

which is a rectangular hyperbola with center at the origin, with branches along the x-axis at $x = \pm 2$.

Takeaways 2.10

To find the locus of points satisfying a complex equation: (1) Substitute $z = x + iy$ and $\bar{z} = x - iy$. (2) Expand and simplify. (3) Equate real and imaginary parts. (4) Identify the curve type.

2.3 Advanced Complex Numbers Problems

Problem 2.11: Geometric Proof with Complex Numbers

Points A and B represent complex numbers z and w on a circle centered at O. Point C represents $z + w$ and also lies on the circle. Show geometrically that $\angle AOB = \frac{2\pi}{3}$.

Solution 2.11

Given that $|z| = |w| = |z + w| = r$ (the radius of the circle), we need to find $\angle AOB$. Since $OACB$ forms a parallelogram (by vector addition), and we know:

- $OA = |z| = r$
- $OB = |w| = r$
- $OC = |z + w| = r$
- $AC = |w| = r$ (opposite side of parallelogram)
- $BC = |z| = r$ (opposite side of parallelogram)

All sides of the rhombus $OACB$ have length r. Moreover, diagonal OC also has length r.
Consider triangle OAC: all three sides have length r (since $OA = AC = OC = r$), so it is equilateral. Therefore, $\angle AOC = \frac{\pi}{3}$.
Similarly, triangle OBC has $OB = BC = OC = r$, so it is also equilateral. Therefore, $\angle BOC = \frac{\pi}{3}$.
Thus:

$$\angle AOB = \angle AOC + \angle BOC = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}.$$

Takeaways 2.11

When complex numbers are represented geometrically, vector addition corresponds to the parallelogram law. A rhombus with all sides equal to its diagonal consists of two equilateral triangles.

Problem 2.12: Complex Division and Real Parts

Let $z = 2(\cos\theta + i\sin\theta)$. Show that the real part of $\frac{1}{1-z}$ is $\frac{1-2\cos\theta}{5-4\cos\theta}$.

Solution 2.12

First, compute $1 - z$:

$$1 - z = 1 - 2(\cos\theta + i\sin\theta) = (1 - 2\cos\theta) - 2i\sin\theta.$$

To find $\frac{1}{1-z}$, multiply numerator and denominator by the conjugate of $1 - z$:

$$\begin{aligned}
\frac{1}{1-z} &= \frac{1}{(1-2\cos\theta) - 2i\sin\theta} \cdot \frac{(1-2\cos\theta) + 2i\sin\theta}{(1-2\cos\theta) + 2i\sin\theta} \\
&= \frac{(1-2\cos\theta) + 2i\sin\theta}{(1-2\cos\theta)^2 + (2\sin\theta)^2}.
\end{aligned}$$

Compute the denominator:

$$\begin{aligned}
(1-2\cos\theta)^2 + 4\sin^2\theta &= 1 - 4\cos\theta + 4\cos^2\theta + 4\sin^2\theta \\
&= 1 - 4\cos\theta + 4(\cos^2\theta + \sin^2\theta) \\
&= 1 - 4\cos\theta + 4 \\
&= 5 - 4\cos\theta.
\end{aligned}$$

Therefore:

$$\frac{1}{1-z} = \frac{(1-2\cos\theta) + 2i\sin\theta}{5 - 4\cos\theta}.$$

The real part is:

$$\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1 - 2\cos\theta}{5 - 4\cos\theta}.$$

Takeaways 2.12

To find the real part of a complex fraction, rationalize by multiplying by the conjugate of the denominator. Use the identity $\cos^2\theta + \sin^2\theta = 1$ to simplify.

Problem 2.13: Finding Complex Roots of Polynomials

Two zeros of $P(x) = x^4 - 12x^3 + 59x^2 - 138x + 130$ are $a + ib$ and $a + 2ib$ where a, b are real and $b > 0$. Find a and b.

Solution 2.13

Since $P(x)$ has real coefficients, complex roots occur in conjugate pairs. If $a+ib$ is a root, then $a-ib$ is also a root. Similarly, if $a+2ib$ is a root, then $a-2ib$ is also a root. Therefore, the four roots are: $a+ib$, $a-ib$, $a+2ib$, $a-2ib$.
By Vieta's formulas, the sum of roots equals the negative of the coefficient of x^3 divided by the leading coefficient:

$$(a+ib)+(a-ib)+(a+2ib)+(a-2ib)=4a=12,$$

so $a=3$.
The product of roots equals the constant term:

$$[(a+ib)(a-ib)][(a+2ib)(a-2ib)]=(a^2+b^2)(a^2+4b^2)=130.$$

Substituting $a=3$:

$$(9+b^2)(9+4b^2)=130.$$

Expand:

$$81+36b^2+9b^2+4b^4=130,$$
$$4b^4+45b^2+81=130,$$
$$4b^4+45b^2-49=0.$$

Let $u=b^2$:

$$4u^2+45u-49=0.$$

Using the quadratic formula:

$$u=\frac{-45\pm\sqrt{45^2+4\cdot 4\cdot 49}}{8}=\frac{-45\pm\sqrt{2025+784}}{8}=\frac{-45\pm\sqrt{2809}}{8}=\frac{-45\pm 53}{8}.$$

Since $u=b^2\geq 0$: $u=\frac{-45+53}{8}=\frac{8}{8}=1$.
Therefore, $b^2=1$, so $b=1$ (since $b>0$).
Thus, $a=3$ and $b=1$.

Takeaways 2.13

For polynomials with real coefficients, complex roots come in conjugate pairs. Use Vieta's formulas (sum and product of roots) along with conjugate pairing to set up equations for unknown parameters.

Problem 2.14: Trigonometric Identity via Complex Numbers

Show that $\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$.

Solution 2.14

By De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta.$$

Expand the left side using the binomial theorem:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^4 &= \sum_{k=0}^{4}\binom{4}{k}\cos^{4-k}\theta(i\sin\theta)^k \\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta \\
&\quad + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta \\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta \\
&\quad - 4i\cos\theta\sin^3\theta + \sin^4\theta \\
&= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) \\
&\quad + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta).
\end{aligned}$$

Equating real parts:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Takeaways 2.14

De Moivre's theorem combined with binomial expansion provides a systematic way to derive multiple-angle formulas. Equate real and imaginary parts to obtain separate identities for $\cos(n\theta)$ and $\sin(n\theta)$.

Problem 2.15: Conjugate Pairs and De Moivre

Show that $(1+i)^n + (1-i)^n = 2(\sqrt{2})^n \cos\frac{n\pi}{4}$.

Solution 2.15

First, convert $1+i$ and $1-i$ to polar form.
For $1+i$:

$$|1+i| = \sqrt{2}, \quad \arg(1+i) = \frac{\pi}{4},$$

so $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\pi/4}$.
For $1-i$:

$$|1-i| = \sqrt{2}, \quad \arg(1-i) = -\frac{\pi}{4},$$

so $1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}e^{-i\pi/4}$.
By De Moivre's theorem:

$$(1+i)^n = (\sqrt{2})^n e^{in\pi/4} = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right),$$

$$(1-i)^n = (\sqrt{2})^n e^{-in\pi/4} = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right).$$

Adding:

$$\begin{aligned}(1+i)^n + (1-i)^n &= (\sqrt{2})^n\left[\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + \left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)\right] \\ &= (\sqrt{2})^n \cdot 2\cos\frac{n\pi}{4} \\ &= 2(\sqrt{2})^n\cos\frac{n\pi}{4}.\end{aligned}$$

Takeaways 2.15

For conjugate pairs z and $\bar{z}$ in polar form, $z^n + \bar{z}^n = 2|z|^n\cos(n\arg(z))$ because the imaginary parts cancel. This is useful for simplifying sums involving conjugate powers.

3 Part 2: Problems and Solutions (Concise + Hints)

Part 2 revisits three new sets of problems. Solutions are intentionally briefer to encourage student ownership, and every problem includes an upside-down hint.

3.1 Basic Complex Numbers Problems

Problem 3.1: Multiplication by i as Rotation

The complex number z is shown on the Argand diagram. Which of the following best represents iz?

Hint: Multiplication by i rotates a complex number by 90° counterclockwise about the origin while preserving its modulus.

Solution 3.1: Sketch

If $z = a+bi$, then $iz = i(a+bi) = ai+bi^2 = -b+ai$. This represents a $90°$ counterclockwise rotation. On the diagram, iz would be at position $(-b, a)$ if z is at (a, b).

Problem 3.2: Quadrant Determination

The Argand diagram shows z in the first quadrant and w in the second quadrant. Which complex number could lie in the 3rd quadrant: $-w$, $2iz$, $\bar{z}$, or $w - z$?

Hint: Check the signs of the real and imaginary parts of each option. Third quadrant means both parts are negative.

Solution 3.2: Sketch

If z is in Q1: $\mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0$. If w is in Q2: $\mathrm{Re}(w) < 0, \mathrm{Im}(w) > 0$. Then $w - z$ has $\mathrm{Re}(w - z) = \mathrm{Re}(w) - \mathrm{Re}(z) < 0$ and $\mathrm{Im}(w - z) = \mathrm{Im}(w) - \mathrm{Im}(z)$ which could be negative if $\mathrm{Im}(z) > \mathrm{Im}(w)$. Answer: (D) $w - z$.

Problem 3.3: Doubling the Argument

A complex number z is on the unit circle. Which diagram best shows $\frac{z^2}{|z|}$?

Hint: If $z = e^{i\theta}$, then $z^2 = e^{i2\theta}$ and $|z| = 1$.

Solution 3.3: Sketch

Since $|z| = 1$, we have $\frac{z^2}{|z|} = z^2$. By De Moivre's theorem, z^2 has the same modulus but double the argument. If z is at angle θ, then z^2 is at angle 2θ.

Problem 3.4: Polynomial with Complex Zero

Which polynomial could have $2+i$ as a zero, given that k is real: (A) $x^3 - 4x^2 + kx$, (B) $x^3 - 4x^2 + kx + 5$, (C) $x^3 - 5x^2 + kx$, (D) $x^3 - 5x^2 + kx + 5$?

Hint: If the polynomial has real coefficients and $2+i$ is a root, then $2-i$ must also be a root. The sum of these roots is 4.

Solution 3.4: Sketch

Roots $2+i$ and $2-i$ sum to 4. Let the third root be r. Then sum of all roots $= 4+r$. From the coefficient of x^2: $-(4+r) = -4$ or -5. If -4, then $r = 0$, giving option (A) or (B). Check: $(x)(x^2 - 4x + 5) = x^3 - 4x^2 + 5x$, but we need $+5$ constant. Answer: (B).

Problem 3.5: Squaring Complex Numbers

Let $z = 3 + i$ and $w = 2 - 5i$. Find z^2 in the form $x + iy$.

Hint: Use $(a + bi)^2 = a^2 - b^2 + 2abi$.

Solution 3.5: Sketch

$z^2 = (3 + i)^2 = 9 + 6i + i^2 = 9 + 6i - 1 = 8 + 6i$.

Problem 3.6: Conjugate Subtraction

Let $z = 4 + i$ and $w = \bar{z}$. Find $w - z$ in the form $x + iy$.

Hint: If $z = 4 + i$, then $\bar{z} = 4 - i$.

Solution 3.6: Sketch

$w = \bar{z} = 4 - i$. Then $w - z = (4 - i) - (4 + i) = -2i$.

Problem 3.7: Division by Complex Number

Let $z = 2 + i$. Find $\frac{4}{z}$ in the form $x + iy$.

Hint: Multiply numerator and denominator by $\bar{z} = 2 - i$.

Solution 3.7: Sketch

$\frac{4}{2+i} = \frac{4(2-i)}{(2+i)(2-i)} = \frac{8-4i}{4-i^2} = \frac{8-4i}{5} = \frac{8}{5} - \frac{4}{5}i$.

Problem 3.8: Complex Multiplication

Let $z = 3 + i$ and $w = 1 - i$. Find zw in the form $x + iy$.

Hint: Use FOIL: $(a + bi)(c + di) = ac - bd + (ad + bc)i$.

Solution 3.8: Sketch

$zw = (3 + i)(1 - i) = 3 - 3i + i - i^2 = 3 - 2i + 1 = 4 - 2i$.

Problem 3.9: Another Division

Let $w = 1 - i$. Find $\frac{6}{w}$ in the form $x + iy$.

Hint: Multiply by conjugate: $\bar{w} = 1 + i$.

Solution 3.9: Sketch

$\frac{6}{1-i} = \frac{6(1+i)}{(1-i)(1+i)} = \frac{6+6i}{1-i^2} = \frac{6+6i}{2} = 3 + 3i.$

Problem 3.10: Dividing by Conjugate

Let $z = 4 + i$ and $w = \bar{z}$. Find $\frac{z}{w}$ in the form $x + iy$.

Hint: $w = 4 - i$, so $\frac{z}{w} = \frac{4+i}{4-i}$.

Solution 3.10: Sketch

$\frac{4+i}{4-i} = \frac{(4+i)(4+i)}{(4-i)(4+i)} = \frac{16+8i+i^2}{16-i^2} = \frac{15+8i}{17} = \frac{15}{17} + \frac{8}{17}i.$

Problem 3.11: Modulus from Polar Form

Let $z = \frac{1}{2}(\cos\theta + i\sin\theta)$. Find $|z|$.

Hint: For $z = r(\cos\theta + i\sin\theta)$, we have $|z| = r$.

Solution 3.11: Sketch

$|z| = \frac{1}{2}.$

Problem 3.12: Division in Cartesian Form

Express $\frac{3-i}{2+i}$ in the form $x + iy$.

Hint: Multiply by $\frac{2-i}{2-i}$.

Solution 3.12: Sketch

$\frac{3-i}{2+i} = \frac{(3-i)(2-i)}{(2+i)(2-i)} = \frac{6-3i-2i+i^2}{4-i^2} = \frac{5-5i}{5} = 1 - i.$

Problem 3.13: Addition with Conjugate

Let $z = 1 + 3i$ and $w = 2 - i$. Find $z + \bar{w}$.

Hint: $\bar{w} = 2 + i$.

Solution 3.13: Sketch

$z + \bar{w} = (1 + 3i) + (2 + i) = 3 + 4i$.

Problem 3.14: Product with Conjugate

Evaluate $w\bar{z}$ given $w = -1 + 4i$ and $z = 2 - i$.

Hint: $\bar{z} = 2 + i$.

Solution 3.14: Sketch

$w\bar{z} = (-1 + 4i)(2 + i) = -2 - i + 8i + 4i^2 = -2 + 7i - 4 = -6 + 7i$.

Problem 3.15: Polar Form of Given Number

Let $w = 1 + i\sqrt{3}$. Express w in modulus-argument form.

Hint: $|w| = \sqrt{1 + 3} = 2$, and $\arg(w) = \tan^{-1}(\sqrt{3}/1) = \pi/3$.

Solution 3.15: Sketch

$w = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

Problem 3.16: Showing a Power is Real

Let $z = \sqrt{3} - i$. Show that z^6 is real.

Hint: Convert to polar form: $z = 2(\cos(-\pi/6) + i\sin(-\pi/6))$.

Solution 3.16: Sketch

$z^6 = 2^6(\cos(-\pi) + i\sin(-\pi)) = 64(-1 + 0i) = -64$, which is real.

Problem 3.17: Principal Argument

Evaluate $\text{Arg}(z)$ for $z = -\sqrt{3} + i$.

Hint: z is in the second quadrant. Use $\tan\theta = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}$.

Solution 3.17: Sketch

Reference angle is $\pi/6$. Since z is in Q2, $\text{Arg}(z) = \pi - \pi/6 = \frac{5\pi}{6}$.

3.2 Medium Complex Numbers Problems

Problem 3.18: Region with Real and Argument Constraints

Let R be the region in the complex plane defined by $1 < \text{Re}(z) \leq 3$ and $\frac{\pi}{6} \leq \text{Arg}(z) < \frac{\pi}{3}$. Sketch this region.

Hint: The region is bounded by two vertical lines ($x = 1$ and $x = 3$) and two rays from the origin at angles $\pi/6$ and $\pi/3$.

Solution 3.18: Sketch

Draw vertical lines at $x = 1$ (dashed, excluded) and $x = 3$ (solid, included). Draw rays from origin at angles 30° (solid) and 60° (dashed). The region is the intersection in the first quadrant.

Problem 3.19: Statement Analysis

Which statement about complex numbers is true? (Multiple options about arguments, exponential form, etc.)

Hint: Check each statement. Remember that $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ only modulo 2π.

Solution 3.19: Sketch

(A) False: arctan doesn't account for quadrant. (B) False: principal argument addition requires modulo 2π adjustment. (C) False: $\theta_1 = \theta_2 + 2\pi k$ for integer k. (D) False: same issue as (A). The correct answer depends on careful reading; typically (C) is closest if equality means modulo 2π.

Problem 3.20: Roots of Unity Application

Suppose that $x + \frac{1}{x} = -1$. What is the value of $x^{2016} + \frac{1}{x^{2016}}$?

Hint: Solve $x^2 + x + 1 = 0$ to find $x = e^{i2\pi/3}$ or $x = e^{-i2\pi/3}$. These are cube roots of unity.

Solution 3.20: Sketch

$x^2 + x + 1 = 0$ gives $x = e^{\pm i2\pi/3}$. Since $x^3 = 1$, we have $x^{2016} = x^{3\cdot 672} = 1$. Thus $x^{2016} + \frac{1}{x^{2016}} = 1 + 1 = 2$.

Problem 3.21: Shading a Region

On an Argand diagram, shade the region where $0 \leq \mathrm{Re}(z) \leq 2$ and $|z - (1 - i)| \leq 2$ both hold.

Hint: The first inequality is a vertical strip. The second is a disk centered at $(1, -1)$ with radius 2.

Solution 3.21: Sketch

Draw vertical lines at $x = 0$ and $x = 2$. Draw circle centered at $(1, -1)$ with radius 2. Shade the intersection.

Problem 3.22: Modulus-Argument Division

Given $\frac{1+\sqrt{3}i}{1+i} = \frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}i$, express $\frac{1+i\sqrt{3}}{1+i}$ in modulus-argument form by converting both numerator and denominator.

Hint:$|1 + i\sqrt{3}| = 2$, $\arg(1 + i\sqrt{3}) = \pi/3$; $|1 + i| = \sqrt{2}$, $\arg(1 + i) = \pi/4$.

Solution 3.22: Sketch

$\frac{1+i\sqrt{3}}{1+i} = \frac{2}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right) = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$.

Problem 3.23: Power in Polar Form

Let $\beta = 1 - i\sqrt{3}$. Express β^5 in modulus-argument form.

Hint:$|\beta| = 2$, $\arg(\beta) = -\pi/3$.

Solution 3.23: Sketch

$\beta = 2(\cos(-\pi/3) + i\sin(-\pi/3))$. Then $\beta^5 = 32(\cos(-5\pi/3) + i\sin(-5\pi/3)) = 32(\cos(\pi/3) + i\sin(\pi/3))$.

Problem 3.24: Deriving Trigonometric Value

Using previous results, find the exact value of $\sin\frac{\pi}{12}$.

Hint: From $\frac{\alpha}{\beta} = \sqrt{2}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$, compute explicitly or use half-angle formulas.

Solution 3.24: Sketch

$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$. Using angle subtraction: $\sin\frac{\pi}{12} = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$.

Problem 3.25: Marking Rotated Point

Points $P(z)$ and $Q(w)$ are on the Argand diagram. Mark the point R representing iz.

Hint: Multiply by i rotates $90°$ counterclockwise.

Solution 3.25: Sketch

If z is at (a, b), then iz is at $(-b, a)$. Draw accordingly.

Problem 3.26: Conjugate Root Property

$2 + i$ is a root of $P(z) = z^3 + rz^2 + sz + 20$ where r, s are real. State why $2 - i$ is also a root.

Hint: Polynomials with real coefficients have complex roots in conjugate pairs.

Solution 3.26: Sketch

Since $P(z)$ has real coefficients, if $2 + i$ is a root, then $\overline{2 + i} = 2 - i$ must also be a root.

Problem 3.27: Factorizing Over Reals

Factorise $P(z) = z^3 + rz^2 + sz + 20$ over the real numbers, given roots $2 + i$ and $2 - i$.

Hint: $(z-(2+i))(z-(2-i)) = (z-2)^2 - (i)^2 = z^2 - 4z + 5$.

Solution 3.27: Sketch

$(z-2-i)(z-2+i) = z^2 - 4z + 5$. Divide $P(z)$ by $z^2 - 4z + 5$ to find the third factor. $P(z) = (z^2 - 4z + 5)(z+4)$ after polynomial division.

Problem 3.28: Vector Addition Point

Points $P(z)$ and $Q(w)$ are given. Mark point T representing $z + w$.

Hint: Use parallelogram law: T is the fourth vertex of parallelogram $OPTQ$.

Solution 3.28: Sketch

From O to P is vector z, from O to Q is vector w. The sum $z + w$ is at the opposite corner of the parallelogram.

Problem 3.29: Sum in Polar Form

Let $z = -2 - 2i$ and $w = 3 + i$. Express $z + w$ in modulus-argument form.

Hint: First find $z + w = 1 - i$, then convert to polar.

Solution 3.29: Sketch

$z+w = 1-i$. $|z+w| = \sqrt{2}$, $\arg(z+w) = -\pi/4$. So $z+w = \sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$.

Problem 3.30: Division in Mixed Form

Express $\frac{z}{w}$ in form $x + iy$ where $z = -2 - 2i, w = 3 + i$.

Hint: $\frac{-2-2i}{3+i} = \frac{(-2-2i)(3-i)}{(3+i)(3-i)}$.

Solution 3.30: Sketch

$\frac{(-2-2i)(3-i)}{10} = \frac{-6+2i-6i+2i^2}{10} = \frac{-8-4i}{10} = -\frac{4}{5} - \frac{2}{5}i$.

Problem 3.31: Sketching with Argument and Circle

Sketch the region: $-\frac{\pi}{4} \leq \arg z \leq 0$ and $|z-(1-i)| \leq 1$.

Hint: The first is a wedge in the fourth quadrant. The second is a disk centered at $(1,-1)$.

Solution 3.31: Sketch

Draw rays at angles $-45°$ and $0°$ from origin. Draw circle centered at $(1,-1)$ radius 1. Shade intersection.

Problem 3.32: High Power via De Moivre

Express z^9 in form $x+iy$ for $z=\sqrt{3}-i$.

Hint:$z = 2(\cos(-\pi/6) + i\sin(-\pi/6))$.

Solution 3.32: Sketch

$z^9 = 2^9(\cos(-3\pi/2) + i\sin(-3\pi/2)) = 512(0+i) = 512i$.

Problem 3.33: Polar Form of Specific Number

Write $z = -1 + i\sqrt{3}$ in modulus-argument form.

Hint:$|z| = 2$, z is in the second quadrant with reference angle $\pi/3$.

Solution 3.33: Sketch

$z = 2(\cos(2\pi/3) + i\sin(2\pi/3))$.

Problem 3.34: Plotting a Squared Number

Given $|z| = 2, \arg(z) = \frac{\pi}{4}$, plot $u = z^2$.

Hint:$|u| = |z|^2 = 4$, $\arg(u) = 2\arg(z) = \pi/2$.

Solution 3.34: Sketch

u has modulus 4 and argument $\pi/2$, so $u = 4i$. Plot at $(0,4)$.

3.3 Advanced Complex Numbers Problems

Problem 3.35: Complex Region with Exclusion

Sketch the region on the Argand diagram where $|z - \bar{z}| < 2$ and $|z - 1| \geq 1$ hold simultaneously.

Hint: $|z - \bar{z}| = |2iy| = 2|y|$, so the first inequality gives $|y| < 1$. The second is the exterior of a circle.

Solution 3.35: Sketch

The region $|y| < 1$ is a horizontal strip $-1 < y < 1$. The region $|z - 1| \geq 1$ is outside the circle centered at $(1, 0)$ with radius 1. Shade the intersection: horizontal strip with circular hole.

Problem 3.36: Isosceles Right Triangle

Triangle ABC is represented by z_1, z_2, z_3. It is isosceles and right-angled at B. Explain why $(z_1 - z_2)^2 = -(z_3 - z_2)^2$.

Hint: The vectors BA and BC are perpendicular and equal in length. Multiplication by i rotates by $90°$.

Solution 3.36: Sketch

$BA = z_1 - z_2$, $BC = z_3 - z_2$. Since $BA \perp BC$ and $|BA| = |BC|$, we have $BA = \pm i \cdot BC$. Squaring: $(z_1 - z_2)^2 = (\pm i)^2 (z_3 - z_2)^2 = -1 \cdot (z_3 - z_2)^2$.

Problem 3.37: Square from Triangle

Use the result from the previous problem. If $ABCD$ is a square, find the complex number for D in terms of z_1, z_2, z_3.

Hint: D is obtained from C by the same transformation that takes A to C via B.

Solution 3.37: Sketch

From $(z_1 - z_2)^2 = -(z_3 - z_2)^2$, we have $z_1 - z_2 = i(z_3 - z_2)$ (choosing sign). For square, $z_4 - z_3 = i(z_2 - z_3)$, giving $z_4 = z_3 + i(z_2 - z_3) = z_2 + z_3 - iz_2 + iz_3 = z_2(1 - i) + z_3(1 + i)$. Alternatively: $z_4 = z_1 + z_3 - z_2$.

Problem 3.38: Factoring Polynomial

Express $P(x) = x^4 - 12x^3 + 59x^2 - 138x + 130$ as product of quadratic factors, given roots $3+i$ and $3+2i$.

Hint: The roots are $3 \pm i$ and $3 \pm 2i$. Group conjugate pairs.

Solution 3.38: Sketch

$(x-(3+i))(x-(3-i)) = (x-3)^2+1 = x^2-6x+10$. $(x-(3+2i))(x-(3-2i)) = (x-3)^2+4 = x^2-6x+13$. Thus $P(x) = (x^2-6x+10)(x^2-6x+13)$.

Problem 3.39: Real Root Count

Deduce that $x^4 - 3x^3 + 5x^2 + 7x - 8 = 0$ has exactly two real roots.

Hint: A degree-4 polynomial with real coefficients has either 0, 2, or 4 real roots. Check behavior or use Descartes' rule.

Solution 3.39: Sketch

Complex roots come in conjugate pairs. If there are 4 complex roots, they form 2 pairs, leaving 0 real roots. If 1 pair of complex roots, then 2 real roots. Since the polynomial has sign changes, it must have at least one positive real root. Testing values or using intermediate value theorem confirms exactly 2 real roots.

Problem 3.40: Vector Addition and Angles

Points P_0, P_1, P_2, P_3 correspond to $z_0, z_0+z_1, z_0+z_1+z_2, \ldots$ where $z_n = \cos(\alpha+n\beta) + i\sin(\alpha+n\beta)$. Using vector addition, explain why external angles $\theta_0 = \theta_1 = \theta_2 = \beta$.

Hint: Each z_n is a unit vector at angle $\alpha + n\beta$. The angle between consecutive vectors is constant.

Solution 3.40: Sketch

Vector z_n has argument $\alpha + n\beta$, and z_{n+1} has argument $\alpha + (n+1)\beta$. The difference is β, which is the external angle between consecutive segments. Thus all external angles equal β.

Problem 3.41: Modulus via Trigonometry

Show $|z| = 2\sin\theta$ for $z = 1 - \cos 2\theta + i\sin 2\theta$.

Hint: Use $1 - \cos 2\theta = 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

Solution 3.41: Sketch

$|z|^2 = (1-\cos 2\theta)^2 + \sin^2 2\theta = (2\sin^2\theta)^2 + (2\sin\theta\cos\theta)^2 = 4\sin^4\theta + 4\sin^2\theta\cos^2\theta = 4\sin^2\theta(\sin^2\theta + \cos^2\theta) = 4\sin^2\theta$. Thus $|z| = 2|\sin\theta|$.

Problem 3.42: Euler's Formula Identity

Show $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$.

Hint: Use $e^{i\phi} = \cos\phi + i\sin\phi$.

Solution 3.42: Sketch

$e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$ and $e^{-in\theta} = \cos(n\theta) - i\sin(n\theta)$. Adding: $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$.

Problem 3.43: Modulus of Sum

Let $z = e^{i\pi/6}, w = e^{i3\pi/4}$. Show $|z+w|^2 = \frac{4-\sqrt{6}+\sqrt{2}}{2}$.

Hint: $|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w}) = |z|^2 + |w|^2 + z\bar{w} + \bar{z}w$.

Solution 3.43: Sketch

$|z| = |w| = 1$. $z\bar{w} = e^{i(\pi/6-3\pi/4)} = e^{-i7\pi/12}$, $\bar{z}w = e^{i7\pi/12}$. Thus $|z+w|^2 = 2 + 2\cos(7\pi/12) = 2(1+\cos(7\pi/12))$. Evaluate $\cos(7\pi/12) = \cos(105°) = -\sin(15°) = -\frac{\sqrt{6}-\sqrt{2}}{4}$. Result follows.

Problem 3.44: Equilateral Triangle

$z_2 = e^{i\pi/3}z_1$. Explain why $\triangle OAB$ (points $0, z_1, z_2$) is equilateral.

Hint: $|z_1| = |z_2|$ since multiplication by $e^{i\pi/3}$ preserves modulus. The angle $\angle z_1 O z_2 = \pi/3 = 60°$.

Solution 3.44: Sketch

$|OA| = |z_1|$, $|OB| = |z_2| = |z_1|$. The angle between them is $\pi/3$. By the law of cosines, $|AB|^2 = 2|z_1|^2(1-\cos\pi/3) = 2|z_1|^2(1-1/2) = |z_1|^2$. Thus $|AB| = |OA| = |OB|$, making it equilateral.

Problem 3.45: Algebraic Identity

Prove $z_1^2 + z_2^2 = z_1 z_2$ for the equilateral triangle in the previous problem.

Hint: $z_2 = e^{i\pi/3} z_1 = \omega z_1$ where $\omega = e^{i\pi/3}$ satisfies $\omega^2 - \omega + 1 = 0$ (approximately).

Solution 3.45: Sketch

Let $\omega = e^{i\pi/3}$. Then $z_2 = \omega z_1$. We have $z_1^2 + z_2^2 = z_1^2 + \omega^2 z_1^2 = z_1^2(1+\omega^2)$. Also, $z_1 z_2 = z_1 \cdot \omega z_1 = \omega z_1^2$. Need to show $1 + \omega^2 = \omega$, i.e., $\omega^2 - \omega + 1 = 0$. Since $\omega = e^{i\pi/3}$, we have $\omega^3 = e^{i\pi} = -1$, so ω satisfies $\omega^3 + 1 = 0$, which factors as $(\omega+1)(\omega^2-\omega+1) = 0$. Since $\omega \neq -1$, we have $\omega^2 - \omega + 1 = 0$.

Problem 3.46: Finding Non-Real Roots

Find the two complex roots of $P(x) = x^5 - 10x^2 + 15x - 6$.

Hint: Test for rational roots first (like $x = 1$). Factor out linear factors, then solve the remaining polynomial.

Solution 3.46: Sketch

$P(1) = 1 - 10 + 15 - 6 = 0$, so $(x-1)$ is a factor. Polynomial division gives $P(x) = (x-1)(x^4 + x^3 + x^2 - 9x + 6)$. Continue factoring or use numerical/algebraic methods to find complex roots.

Problem 3.47: Expansion Result

Using expansion of $\cos 5\theta$, show a specific result (problem statement incomplete in original).

Hint: Use $(e^{i\theta})^5 = e^{i5\theta}$ and expand with binomial theorem.

Solution 3.47: Sketch

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$. Expand left side and equate real parts to derive $\cos 5\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Problem 3.48: Symmetry Identity

Show $\alpha^k + \alpha^{-k} = 2\cos k\theta$ where $\alpha = \cos\theta + i\sin\theta$.

Hint: $\alpha = e^{i\theta}$ and $\alpha^{-1} = e^{-i\theta}$.

Solution 3.48: Sketch

$\alpha^k = e^{ik\theta} = \cos k\theta + i\sin k\theta$, $\alpha^{-k} = e^{-ik\theta} = \cos k\theta - i\sin k\theta$. Adding: $\alpha^k + \alpha^{-k} = 2\cos k\theta$.

Problem 3.49: Equilateral Triangle Centroid

Let $w = e^{i2\pi/3}$. Show that if $\triangle ABC$ is anticlockwise and equilateral, then $a + bw + cw^2 = 0$.

Hint: For an equilateral triangle centered at origin, vertices satisfy $b = wa$, $c = w^2a$ where w is a cube root of unity.

Solution 3.49: Sketch

If the triangle is equilateral with center at origin, then the vertices are related by 120° rotations: $b = wa$, $c = w^2a$. Then $a + bw + cw^2 = a + wa \cdot w + w^2a \cdot w^2 = a(1 + w^2 + w^4) = a(1 + w^2 + w)$ since $w^3 = 1$. Since w is a primitive cube root of unity, $1 + w + w^2 = 0$. Thus $a + bw + cw^2 = 0$.

4 Conclusion

Complex numbers are a core component of the HSC Mathematics Extension 2 course. Mastery comes from repeated, reflective practice across all forms—Cartesian, polar, and exponential—and understanding their geometric interpretations. Use these problems to sharpen the ability to convert between forms, apply De Moivre's theorem, interpret loci geometrically, and communicate complete mathematical reasoning. Best of luck with your studies and HSC examinations!

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