HSC Math Extension 2: Induction Mastery

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1 Introduction

1.1 Project Overview

This booklet compiles high quality mathematical induction problems curated specifically for the HSC Mathematics Extension 2 syllabus. Every task can be attacked with induction (sometimes combined with algebraic manipulation, inequalities, recurrence relations, or integrals). Detailed reasoning showcases common techniques such as weak and strong induction, structural induction on algebraic forms, and inductive proofs for inequalities.

1.2 Target Audience

The explanations are crafted for Extension 2 students aiming to deepen their proof-writing skills. Each solution explicitly states the induction hypothesis, justification of the base case, and reasoning for the inductive step so that high-school learners can follow every transition.

1.3 How to Use This Booklet

- Read the overview and induction primer before attempting the problems.
- Attempt problems in Part 1 without hints; compare against the detailed solutions to understand model reasoning.
- Use the upside-down hints in Part 2 only after making a genuine attempt.
- Revisit problems after a few days and try to re-derive the arguments without notes to reinforce technique.

1.4 Induction Primer

Mathematical induction is a two-step method:

1. **Base Case:** Verify the statement for the initial value (often $n = 1$ or $n = 0$), ensuring the proposition is anchored.
2. **Inductive Step:** Assume the statement is true for $n = k$ (or all values up to k for strong induction) and show it holds for $n = k + 1$. Clearly state the hypothesis and highlight algebraic transformations used to bridge from k to $k + 1$.

We will reference these two steps in every solution so students can trace the logic effortlessly.

2 Part 1: Problems and Solutions (Detailed)

Part 1 contains three sets of problems—basic, medium, and advanced. Each set provides five problems. For every problem we present a comprehensive induction-based solution without any hints so that learners focus on the full reasoning trail.

2.1 Basic Induction Problems

Problem 2.1: Telescoping Harmonic Sum

Show that for every integer $n \geq 1$,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

Solution 2.1

Base case ($n = 1$). We have $\frac{1}{1\cdot 2} = \frac{1}{2}$ and $\frac{1}{1+1} = \frac{1}{2}$, so the statement is true for $n = 1$.
Inductive step. Assume the statement holds for some $n = k$, i.e.

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}.$$

Then

$$\begin{aligned}
\sum_{r=1}^{k+1} \frac{1}{r(r+1)} &= \underbrace{\sum_{r=1}^{k} \frac{1}{r(r+1)}}_{\frac{k}{k+1}} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.
\end{aligned}$$

Therefore the identity is true for $n = k+1$, so by induction it holds for all $n \geq 1$.

Takeaways 2.1

Partial fractions $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ produce telescoping sums,
so the algebra in the inductive step stays simple and transparent.

Problem 2.2: Divisibility of $n^3 + 2n$

Prove that $n^3 + 2n$ is divisible by 12 for every even integer n.

Solution 2.2

Let $P(n)$ be the statement for the even integer n. Write $n = 2m$ for some integer m.
Base case $(n = 2)$**.** $2^3 + 2 \cdot 2 = 8 + 4 = 12$, which is divisible by 12.
Inductive step. Assume $P(2m)$ holds, so $2m$ even implies $(2m)^3 + 2(2m) = 8m^3 + 4m$ is a multiple of 12. Consider the next even integer $2(m + 1)$:

$$\begin{aligned}(2m + 2)^3 + 2(2m + 2) &= 8(m + 1)^3 + 4(m + 1)\\ &= 8(m^3 + 3m^2 + 3m + 1) + 4m + 4\\ &= (8m^3 + 4m) + 24m^2 + 24m + 12.\end{aligned}$$

By the hypothesis, $8m^3 + 4m$ is divisible by 12, and each remaining term has factor 12. Hence the whole expression is divisible by 12, completing the induction.

Takeaways 2.2

Factoring n as $2m$ keeps arithmetic simple and highlights the repeated factor of 12.

Problem 2.3: Nine divides $7^n + 2^n$

Show that $7^n + 2^n$ is divisible by 9 for every positive odd integer n.

Solution 2.3

Base case $(n = 1)$**.** $7^1 + 2^1 = 9$, divisible by 9.
Inductive step. Suppose $n = 2k + 1$ is odd and $7^{2k+1} + 2^{2k+1}$ is divisible by 9. Consider $n + 2 = 2(k + 1) + 1$, the next odd integer:

$$\begin{aligned}7^{2k+3} + 2^{2k+3} &= 7^2 \cdot 7^{2k+1} + 2^2 \cdot 2^{2k+1}\\ &= 49 \cdot 7^{2k+1} + 4 \cdot 2^{2k+1}\\ &= 45 \cdot 7^{2k+1} + 4\big(7^{2k+1} + 2^{2k+1}\big).\end{aligned}$$

Both terms are multiples of 9: the first is obvious, and the second is 4 times the inductive hypothesis (which is already a multiple of 9). Therefore $7^{2k+3} + 2^{2k+3}$ is divisible by 9, completing the induction on odd integers.

Takeaways 2.3

Working modulo 9 turns the inductive step into a one-line parity check because $7 \equiv -2 \pmod{9}$.

Problem 2.4: Triangular Numbers

Given $T_1 = 1$ and $T_n = T_{n-1} + n$ for $n \geq 2$, prove that $T_n = \frac{n(n+1)}{2}$ for every $n \geq 1$.

Solution 2.4

Base case ($n = 1$). $T_1 = 1$ and $\frac{1(1+1)}{2} = 1$, so the formula holds.
Inductive step. Assume $T_k = \frac{k(k+1)}{2}$ for some $k \geq 1$. Then

$$T_{k+1} = T_k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

Therefore the closed form holds for $k+1$, completing the induction.

Takeaways 2.4

Most recurrence relations in Extension 2 can be guessed and then confirmed with induction if you carefully add the new term in the inductive step.

Problem 2.5: Polygon Interior Angle Sum

Prove that the sum of the interior angles of an n-gon is $(n-2) \times 180°$ for all integers $n \geq 3$.

Solution 2.5

Base case ($n = 3$). A triangle has interior angles summing to $180°$, which equals $(3-2) \cdot 180°$.
Inductive step. Assume every k-gon ($k \geq 3$) has interior angle sum $(k-2) \cdot 180°$. Take a $(k+1)$-gon. Draw a diagonal from one vertex to split it into a triangle and a k-gon. The k-gon contributes $(k-2) \cdot 180°$ and the triangle contributes $180°$, totaling $(k-2) \cdot 180° + 180° = (k-1) \cdot 180°$, as required.

Takeaways 2.5

Breaking polygons into a k-gon plus a triangle is a classic structural induction idea: reduce a statement for $k+1$ objects to the known case for k.

Problem 2.6: Sum of consecutive odd numbers

(a) Use mathematical induction to prove that for all integers $n \geq 1$,

$$\sum_{r=1}^{n} (2r-1) = n^2.$$

(b) Hence find the value of

$$51 + 53 + 55 + \cdots + 99.$$

Hint: For (a) check the base case $n = 1$ and assume the formula for $n = k$, then add the $(k+1)$st odd number $2(k+1) - 1$. For (b) express the requested sum as a difference of two partial sums of the first N odd numbers.

Solution 2.6

For (a): base $n = 1$ gives $1 = 1^2$. Assume $\sum_{r=1}^{k}(2r-1) = k^2$. Then

$$\sum_{r=1}^{k+1}(2r-1) = k^2 + 2(k+1) - 1 = k^2 + 2k + 1 = (k+1)^2.$$

For (b): note $51+\cdots+99 = \sum_{r=1}^{50}(2r-1) - \sum_{r=1}^{25}(2r-1) = 50^2 - 25^2 = 2500 - 625 = 1875$.

Takeaways 2.6

- The sum of the first n odd numbers equals n^2; visual proofs via gnomons illustrate this nicely.
- Many evaluation problems reduce to re-indexing or subtracting partial sums.

2.2 Medium Induction Problems

Problem 2.7: Bounding a Basel-type Sum

Given that for every $k \geq 1$,

$$\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0,$$

prove by induction that for all integers $n \geq 2$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

Solution 2.7

Define $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$. The given inequality can be rearranged into

$$\frac{1}{k^2} - \left(\frac{1}{k} - \frac{1}{k+1}\right) < \frac{1}{k^2} - \left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{1}{(k+1)^2} = \frac{1}{(k+1)^2},$$

which justifies the inductive step below.
Base case $(n = 2)$**.** $S_2 = 1 + \frac{1}{4} = \frac{5}{4} < \frac{3}{2} = 2 - \frac{1}{2}$, so the statement holds.
Inductive step. Assume $S_n < 2 - \frac{1}{n}$ for some $n \geq 2$. Applying the given inequality with $k = n$ gives $\frac{1}{(n+1)^2} < \frac{1}{n} - \frac{1}{n+1}$. Hence

$$S_{n+1} = S_n + \frac{1}{(n+1)^2} < \left(2 - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 2 - \frac{1}{n+1}.$$

Thus the inequality holds for $n + 1$, completing the induction.

Takeaways 2.7

Induction inequalities often require massaging the inductive hypothesis into an expression that absorbs the new term; algebraic manipulation of rational expressions keeps reasoning transparent for Extension 2 students.

Problem 2.8: Comparing $\sqrt{n!}$ and 2^n

Show that $\sqrt{n!} > 2^n$ for all integers $n \geq 9$.

Solution 2.8

It is easier to square both sides and prove $n! > 4^n$.
Base case ($n = 9$). $9! = 362880$ and $4^9 = 262144$, so the inequality holds.
Inductive step. Assume $k! > 4^k$ for some $k \geq 9$. Then

$$(k+1)! = (k+1) \cdot k! > (k+1) \cdot 4^k.$$

Because $k + 1 \geq 10$, we have $(k+1) \cdot 4^k \geq 10 \cdot 4^k = 4 \cdot (2.5) \cdot 4^k > 4 \cdot 4^k = 4^{k+1}$. Thus $(k+1)! > 4^{k+1}$, completing the induction. Taking square roots recovers the stated inequality $\sqrt{n!} > 2^n$.

Takeaways 2.8

When factorials are compared with exponentials, work with squared (or unsquared) versions that eliminate radicals and keep all quantities positive.

Problem 2.9: Solving a Linear Recurrence

Suppose $T_1 = 3$ and $T_n = 2T_{n-1} + 2 - n$ for $n \geq 2$. Prove that $T_n = 2^n + n$ for all $n \geq 1$.

Solution 2.9

Base case ($n = 1$). $T_1 = 3 = 2^1 + 1$, so the formula works.
Inductive step. Assume $T_k = 2^k + k$ for some $k \geq 1$. Then

$$\begin{aligned} T_{k+1} &= 2T_k + 2 - (k+1) \\ &= 2(2^k + k) + 1 - k \\ &= 2^{k+1} + k + 1 \\ &= 2^{k+1} + (k+1), \end{aligned}$$

which agrees with the claimed closed form. Thus the identity holds for all positive integers n.

Takeaways 2.9

Linear recurrences with constant coefficients are tailor-made for induction proofs once you conjecture the closed form by spotting patterns in the first few terms.

Problem 2.10: Derivative of x^n

Using the product rule and induction, prove that for every integer $n \geq 1$,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Solution 2.10

Base case ($n = 1$). $\frac{d}{dx}(x) = 1$, which equals $1 \cdot x^0$.
Inductive step. Assume $\frac{d}{dx}(x^k) = kx^{k-1}$. Then for $n = k+1$,

$$x^{k+1} = x \cdot x^k.$$

By the product rule,

$$\begin{aligned} \frac{d}{dx}(x^{k+1}) &= \frac{d}{dx}(x) \cdot x^k + x \cdot \frac{d}{dx}(x^k) = 1 \cdot x^k + x \cdot kx^{k-1} \\ &= x^k + kx^k = (k+1)x^k, \end{aligned}$$

which equals $(k+1)x^{(k+1)-1}$. This completes the induction.

Takeaways 2.10

Induction can justify formulas students memorize in calculus by combining algebraic identities with differentiation rules they already know.

Problem 2.11: Factorising $x^{3^n} - 1$

Use mathematical induction to prove that, for $n \geq 1$,

$$x^{3^n} - 1 = (x-1)(x^2+x+1)(x^6+x^3+1)\cdots\left(x^{2\cdot 3^{n-1}} + x^{3^{n-1}} + 1\right).$$

Solution 2.11

Base case $(n = 1)$**.** The right side becomes $(x-1)(x^2+x+1)$, which equals x^3-1 by the difference of cubes identity.

Inductive step. Assume the factorisation holds for $n = k$:

$$x^{3^k} - 1 = (x-1)(x^2+x+1)\cdots\left(x^{2\cdot 3^{k-1}} + x^{3^{k-1}} + 1\right).$$

For $n = k+1$,

$$x^{3^{k+1}} - 1 = (x^{3^k})^3 - 1 = \left(x^{3^k} - 1\right)\left(x^{2\cdot 3^k} + x^{3^k} + 1\right).$$

Replace the first factor by the induction hypothesis to obtain the same product as before with one additional term $x^{2\cdot 3^k} + x^{3^k} + 1$, giving the desired factorisation.

Takeaways 2.11

Recognising "difference of cubes" inside the inductive step is a powerful pattern for algebraic factorizations involving rapidly growing exponents.

Problem 2.12: Weighted geometric sum

Prove that for every integer $n \geq 1$,

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2.$$

Hint:

- Verify the base case $n = 1$ first.
- For induction, add the term $(m+1)2^{m+1}$ to the assumed sum for m.
- (Alternative) Differentiate the finite geometric series $\sum_{k=0}^{n} x^k$ with respect to x and substitute $x = 2$.

Solution 2.12

Induction. For $n = 1$ the left-hand side is $1 \cdot 2 = 2$ and the right-hand side is $(1 - 1)2^2 + 2 = 2$, so the base case holds.
Assume the identity holds for $n = m \geq 1$. Then

$$\begin{aligned}\sum_{k=1}^{m+1} k2^k &= \left(\sum_{k=1}^{m} k2^k\right) + (m+1)2^{m+1} \\ &= (m-1)2^{m+1} + 2 + (m+1)2^{m+1} \\ &= m2^{m+2} + 2 = \big((m+1) - 1\big)2^{(m+1)+1} + 2,\end{aligned}$$

which is the desired formula for $n = m+1$. By induction the formula holds for all $n \geq 1$.
Proof with Calculus (Differentiation)
Consider the geometric series:

$$P(x) = \sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}$$

Differentiating both sides with respect to x:

$$\frac{d}{dx} \sum_{k=0}^{n} x^k = \sum_{k=1}^{n} kx^{k-1}$$

Using the quotient rule on the right-hand side:

$$\sum_{k=1}^{n} kx^{k-1} = \frac{(n+1)x^n(x-1) - (x^{n+1} - 1)}{(x-1)^2}$$

Multiply both sides by x to obtain the desired summation form:

$$\sum_{k=1}^{n} kx^k = \frac{(n+1)x^{n+1}(x-1) - x(x^{n+1} - 1)}{(x-1)^2}$$

Substitute $x = 2$:

$$\begin{aligned}\sum_{k=1}^{n} k2^k &= \frac{(n+1)2^{n+1}(2-1) - 2(2^{n+1} - 1)}{(2-1)^2} \\ &= (n+1)2^{n+1} - 2 \cdot 2^{n+1} + 2 \\ &= 2^{n+1}(n+1-2) + 2 \\ &= (n-1)2^{n+1} + 2\end{aligned}$$

□

Takeaways 2.12

- Many weighted sums $\sum ka^k$ can be handled by induction or by differentiating a geometric series.
- Verifying base cases and clearly aligning the algebra in the inductive step keeps the argument clean for students.

Problem 2.13: Product of odd factorials vs powers of factorials

Prove that for every integer $n \geq 1$,

$$1! \cdot 3! \cdot 5! \cdots (2n-1)! \geq (n!)^n.$$

Hint:

1. Write the product for $n = k+1$ as the product for $n = k$ times $(2k+1)!$ and compare with $((k+1)!)^{k+1}$.
2. Expand $(2k+1)!$ as $(2k+1)(2k)\cdots(k+2)\cdot(k+1)!$ and compare termwise.

Solution 2.13

Base case $n = 1$ holds since $1! \geq (1!)^1$. Assume true for $n = k$. Then

$$\prod_{r=1}^{k+1}(2r-1)! = \left(\prod_{r=1}^{k}(2r-1)!\right)(2k+1)! \geq (k!)^k(2k+1)!.$$

Now $(2k+1)! = (2k+1)(2k)\cdots(k+2)\cdot(k+1)!$ contains k factors each at least $k+1$, so

$$(2k+1)! \geq (k+1)^k(k+1)! = (k+1)^{k+1}k!.$$

Combining gives

$$\prod_{r=1}^{k+1}(2r-1)! \geq (k!)^k \cdot (k+1)^{k+1}k! = ((k+1)!)^{k+1},$$

completing the induction.

Takeaways 2.13

- For product-type induction, compare multiplicative growth by isolating the new factor and estimating it against the required power.
- Counting how many terms exceed a threshold (here $k+1$) is an effective lower-bound technique.

Problem 2.14: Bounding sums of cubes

Prove that for every integer $n \geq 2$,

$$\sum_{r=1}^{n-1} r^3 < \frac{n^4}{4} < \sum_{r=1}^{n} r^3.$$

Hint:

1. Check the base case $n = 2$ explicitly.
2. For the inductive step compare the difference between $\frac{(k+1)^4}{4}$ and $\frac{k^4}{4}$ with k^3 and $(k+1)^3$ as needed.
3. Use the binomial expansion of $(k+1)^4$ to rearrange terms.

Solution 2.14

Base case $n = 2$: $1 = \sum_{r=1}^{1} r^3 < 4 = \frac{2^4}{4} < 1 + 8 = 9$. Assume the double inequality holds for $n = k \geq 2$. Then

$$\frac{(k+1)^4}{4} = \frac{k^4}{4} + k^3 + \frac{6k^2 + 4k + 1}{4}.$$

Since the extra term $\frac{6k^2+4k+1}{4} > 0$, we have $\frac{(k+1)^4}{4} > \frac{k^4}{4} + k^3$, and by the inductive hypothesis $\frac{k^4}{4} + k^3 > \sum_{r=1}^{k} r^3$. Thus the left inequality for $k + 1$ holds. For the right inequality, compare $\frac{(k+1)^4}{4}$ with $\frac{k^4}{4} + (k+1)^3$ and observe the difference equals $\frac{6k^2+8k+3}{4} > 0$, giving the required result.

Takeaways 2.14

- Verify base cases carefully when working with strict inequalities.
- **Bounding Sums:** This technique proves that a sum of cubes is bounded by the integral of x^3.
- **Algebraic Dominance:** In inequality induction, the goal is often to show that the "added part" of the function grows faster or slower than the "added part" of the sum.

Problem 2.15: Product of odd factorials inequality

Prove that for every integer $n \geq 1$,

$$1! \cdot 3! \cdot 5! \cdots (2n-1)! \geq (n!)^n.$$

Hint:

- Compare the product for $n = k + 1$ with the product for $n = k$ and isolate the new factor $(2k + 1)!$.
- Expand $(2k+1)!$ as $(2k+1)(2k)\cdots(k+2)(k+1)!$ and lower-bound the first k factors by $(k + 1)$.

Solution 2.15

Base case $n = 1$ is trivial. Assume the inequality for $n = k$. Then

$$\prod_{r=1}^{k+1}(2r-1)! = \Big(\prod_{r=1}^{k}(2r-1)!\Big)(2k+1)! \geq (k!)^k(2k+1)!.$$

Since $(2k + 1)! = (2k + 1)(2k)\cdots(k + 2)\cdot(k + 1)!$ and each of the k factors in $(2k + 1)\ldots(k + 2)$ is at least $k + 1$, we have

$$(2k + 1)! \geq (k + 1)^k(k + 1)! = ((k + 1)!)^{k+1}/(k!)^k.$$

Combining yields the claim for $k + 1$, completing the induction.

Takeaways 2.15

- Product-type induction frequently uses termwise bounds and counting arguments to compare multiplicative growth.
- Carefully isolate the new multiplicative factor when moving from k to $k + 1$.

Problem 2.16: Bounding cubic sums

Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{i=1}^{n} a_i^3 \geq \frac{1}{n^2}.$$

Hint: Use Holder's inequality or the power mean inequality; compare $(\sum a_i)^3$ with $n^2 \sum a_i^3$.

Solution 2.16

By Hölder (or power mean),

$$(\sum_{i=1}^{n} a_i)^3 \leq n^2 \sum_{i=1}^{n} a_i^3.$$

Since the left side equals 1, rearrange to get $\sum a_i^3 \geq 1/n^2$, as desired.

Takeaways 2.16

- Hölder and power mean inequalities give quick bounds relating different power sums.
- Normalizing sums to 1 simplifies many symmetric inequalities.

Problem 2.17: Reduction of the Tangent Integral

Let $I_n = \displaystyle\int_0^{\pi/4} \tan^n x \, dx$ for $n = 0, 1, 2, \ldots$.

1. Show that for $n \geq 2$,
$$I_n + I_{n-2} = \frac{1}{n-1}.$$
2. Hence find the exact value of I_4.

Hint: Write $\tan^n x = \tan^{n-2} x \cdot \tan^2 x$ and use the identity $\tan^2 x = \sec^2 x - 1$; substitute $u = \tan x$ for the term with $\sec^2 x$.

Solution 2.17

Direct Proof

For $n \geq 2$,

$$I_n = \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)\, dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2}.$$

Rearranging gives $I_n + I_{n-2} = 1/(n-1)$. To find I_4, note

$$\begin{aligned} I_4 + I_2 &= \tfrac{1}{3}, \\ I_2 + I_0 &= 1, \quad I_0 = \int_0^{\pi/4} 1\, dx = \tfrac{\pi}{4}. \end{aligned}$$

Thus $I_2 = 1 - \frac{\pi}{4}$ and $I_4 = \frac{1}{3} - I_2 = \frac{\pi}{4} - \frac{2}{3}$.

Inductive proof of the recurrence

Two cases arise depending on the parity of n.

First, let's prove the case for even $n = 2m$. Similarly to above, we can verify the base case of $m = 1$.

Now assume for some $m \geq 1$ that for $n = 2m$ the identity $I_{2m} + I_{2m-2} = 1/(2m-1)$ holds.

Consider $n = 2m + 2$:

$$\begin{aligned} I_{2m+2} &= \int_0^{\pi/4} \tan^{2m}(x) \tan^2 x\, dx \\ &= \int_0^{\pi/4} \tan^{2m} x(\sec^2 x - 1)\, dx \\ &= \left[\frac{\tan^{2m+1} x}{2m+1}\right]_0^{\pi/4} - I_{2m} \\ &= \frac{1}{2m+1} - I_{2m}. \end{aligned}$$

Rearranging gives $I_{2m+2} + I_{2m} = 1/(2m+1)$, completing the induction step. An analogous parity-shifted argument covers odd indices, so the recurrence holds for all $n \geq 2$. A side note: The direct proof is more straightforward in this case, but the inductive approach illustrates how to handle reduction formulae recursively, though not necessry here.

Takeaways 2.17

- Reduction formulae convert higher-power integrals into lower-power ones, enabling recursive evaluation.
- Substitution after using trigonometric identities is a common pattern.

Problem 2.18: Logarithmic Reduction and Closed Form

Let $I_n = \displaystyle\int_1^e (\ln x)^n \, dx$ for $n \geq 0$.

1. Use integration by parts to show $I_n = e - nI_{n-1}$ for $n \geq 1$.
2. Prove by induction that

$$I_n = (-1)^n n! + e \sum_{r=0}^{n} (-1)^{n-r} \frac{n!}{r!} \quad \text{for } n \geq 0.$$

Hint: For part (1) take $u = (\ln x)^n$ and $dv = dx$. For part (2) verify base I_0 and substitute the recurrence into the inductive step.

Solution 2.18

Integration by parts with $u = (\ln x)^n$, $dv = dx$ gives $I_n = [x(\ln x)^n]_1^e - n\int_1^e (\ln x)^{n-1}\, dx = e - nI_{n-1}$. The closed form follows by induction using this recurrence and the base value $I_0 = \int_1^e 1\, dx = e$.

Takeaways 2.18

- Reduction formulae from integration by parts often yield closed forms when iterated.
- Combining recurrences with factorial manipulations is a useful technique for exact evaluations.

2.3 Advanced Induction Problems

Problem 2.19: Recursive sequence via $f(x) = 2^x + 2^{-x}$

Let the function $f(x)$ be defined by

$$f(x) = 2^x + 2^{-x}$$

for all real x.
Consider the sequence (u_n) defined by

$$u_1 = \tfrac{5}{2}, \qquad u_{n+1} = \sqrt{2 + u_n} \quad (n \geq 1).$$

(i) Show that $f(x)^2 = f(2x) + 2$, and deduce that

$$f\left(\frac{x}{2}\right) = \sqrt{2 + f(x)} \quad (\forall x \in \mathbb{R}).$$

(ii) Use induction to prove that for every integer $n \geq 1$,

$$u_n = f\left(\frac{1}{2^{n-1}}\right).$$

(iii) Find the exact value of

$$\lim_{n\to\infty} 2^n \sqrt{u_n - 2}.$$

(iv) Evaluate the telescoping product limit

$$\lim_{n\to\infty} \frac{u_1 u_2 \cdots u_n}{2^n}.$$

Hint:

- Expand $(2^x + 2^{-x})^2$ and take the positive square root (note $f > 0$).
- For induction, set $x = 2^{1-k}$ when passing from k to $k+1$ and use part (i).
- Write $u_n - 2 = (2^t - 2^{-t})^2$ for a small t and use the limit $\lim_{t\to 0} \frac{2^t - 1}{t} = \ln 2$.
- Multiply the product by the conjugate factors $2^x - 2^{-x}$ to telescope.

Solution 2.19

Part (i): Expand to get $[f(x)]^2 = 2^{2x} + 2^{-2x} + 2 = f(2x) + 2$. Replace x by $x/2$ and take the positive root to obtain the identity claimed.
Part (ii): Base case $u_1 = f(1) = 2 + 2^{-1} = 5/2$. If $u_k = f(2^{1-k})$ then

$$u_{k+1} = \sqrt{2 + u_k} = \sqrt{2 + f(2^{1-k})} = f\left(\tfrac{2^{1-k}}{2}\right) = f(2^{-k}),$$

so the statement holds by induction.
Part (iii): Put $h_n = 1/2^{n-1}$ so $u_n = 2^{h_n} + 2^{-h_n}$. Then

$$\sqrt{u_n - 2} = 2^{h_n/2} - 2^{-h_n/2}, \quad \text{where } h_n/2 = 1/2^n.$$

Thus

$$2^n\sqrt{u_n - 2} = \frac{2^t - 2^{-t}}{t}\bigg|_{t=1/2^n} \to (\ln 2) - (-\ln 2) = 2\ln 2.$$

Part (iv): Let $x_k = 1/2^{k-1}$ and set $g_k = 2^{x_k} - 2^{-x_k}$. Note

$$u_k g_k = 2^{2x_k} - 2^{-2x_k} = g_{k-1}.$$

Multiplying for $k = 1, \ldots, n$ gives $P_n g_n = g_0$ where $P_n = \prod_{k=1}^n u_k$ and $g_0 = 2^2 - 2^{-2} = 15/4$. Hence

$$\frac{P_n}{2^n} = \frac{15/4}{2^n(2^{x_n} - 2^{-x_n})}.$$

With $t = x_n = 1/2^{n-1}$ we have $2^n(2^t - 2^{-t}) \to 2 \cdot 2\ln 2 = 4\ln 2$, so the limit equals $\dfrac{15}{16\ln 2}$.

Takeaways 2.19

- What happens when $u_1 < 2$? Can we use trigonometric functions instead of exponentials?
- When $u_1 = 2$, the sequence is constant: $u_n = 2$ for all n and we call the point a fixed point.
- Link this problem to the hyperbolic cosine function $\cosh x = \frac{e^x+e^{-x}}{2}$.
- Recognise when sequences stem from a two-term exponential identity such as $2^x + 2^{-x}$ (a hyperbolic/cosh-like form).
- Telescoping products often require multiplying by conjugates; identify the right companion factor.
- Convert limits with $n \to \infty$ and exponentials to standard derivative-form limits via substitution $t \to 0$.
- Using L'Hopital's rule can help evaluate tricky limits.

  L'Hopital's rule states that if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, and the derivatives $f'(x)$ and $g'(x)$ are continuous near a with $g'(x) \neq 0$ for $x \neq a$, then

  $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

  provided the limit on the right exists.

Problem 2.20: De Moivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$ with $r > 0$. Prove by induction that for every integer $n \geq 1$,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Solution 2.20

Base case ($n = 1$). Trivial because $z^1 = r(\cos\theta + i\sin\theta)$.
Inductive step. Assume $z^k = r^k(\cos k\theta + i\sin k\theta)$. Then

$$\begin{aligned}
z^{k+1} &= z \cdot z^k = r(\cos\theta + i\sin\theta)\cdot r^k(\cos k\theta + i\sin k\theta)\\
&= r^{k+1}\big[(\cos\theta\cos k\theta - \sin\theta\sin k\theta) + i(\sin\theta\cos k\theta + \cos\theta\sin k\theta)\big]\\
&= r^{k+1}\big[\cos((k+1)\theta) + i\sin((k+1)\theta)\big],
\end{aligned}$$

using the addition formulas for sine and cosine. The statement follows by induction.

Takeaways 2.20

Complex-number induction proofs nearly always hinge on angle addition formulas once you factor out magnitudes.

Problem 2.21: Tiling a Defective $2^n \times 2^n$ Board

Prove that any $2^n \times 2^n$ chessboard with one square missing can be completely tiled with L-shaped trominoes for all integers $n \geq 1$.

Solution 2.21

Base case $(n = 1)$**.** A 2×2 board with one square removed exactly matches a single L-shaped tromino.
Inductive step. Assume the claim is true for $n = k$: every $2^k \times 2^k$ board with one missing square can be tiled. Consider a board of size $2^{k+1} \times 2^{k+1}$. Divide it into four quadrants, each of size $2^k \times 2^k$. One quadrant already has the missing square. Place a single L-shaped tromino at the center to cover one square from each of the other three quadrants, effectively creating an artificial "missing" square inside each. Now each quadrant is a $2^k \times 2^k$ board with one square missing, so by the inductive hypothesis all four quadrants can be tiled. Thus the entire board can be tiled, completing the induction.

Takeaways 2.21

Structural induction typically requires creating the same "defect" in each subproblem so the hypothesis applies uniformly.

Problem 2.22: Evaluating I_n from a Recurrence

Given $I_0 = 1$ and the recurrence

$$I_n = \frac{2n}{2n+1} I_{n-1} \quad (n \geq 1),$$

show that

$$I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

for every $n \geq 0$.

Solution 2.22

Base case $(n = 0)$**.** The closed form gives $I_0 = \frac{2^0(0!)^2}{1!} = 1$, matching the definition.
Inductive step. Assume the result holds for $n = k$, so $I_k = \frac{2^{2k}(k!)^2}{(2k+1)!}$. Then

$$\begin{aligned} I_{k+1} &= \frac{2(k+1)}{2(k+1)+1} I_k = \frac{2(k+1)}{2k+3} \cdot \frac{2^{2k}(k!)^2}{(2k+1)!} \\ &= \frac{2^{2k+1}(k+1)(k!)^2}{(2k+3)(2k+1)!} = \frac{2^{2k+2}((k+1)!)^2}{(2k+3)!}. \end{aligned}$$

The last step follows because $(k+1)(k!)^2 = (k+1)! \cdot k!$ and $(2k+3)! = (2k+3)(2k+2)(2k+1)!$. Therefore the closed form holds for $k+1$, and by induction it is valid for all n.

Takeaways 2.22

Products such as $\frac{2n}{2n+1}$ often telescope into factorials; writing several terms explicitly helps you guess the factorial pattern to confirm by induction.

Problem 2.23: Integer Coefficients in $\int_0^1 x^n e^x \, dx$

Let

$$I_n = \int_0^1 x^n e^x \, dx$$

.
Show by induction that there exist integers a_n and b_n with $I_n = a_n + b_n e$ for every $n \geq 0$.

Solution 2.23

Integrate by parts with $u = x^n$ and $dv = e^x dx$ to obtain, for $n \geq 1$,

$$I_n = [x^n e^x]_0^1 - n \int_0^1 x^{n-1} e^x dx = e - nI_{n-1}.$$

Base case $(n = 0)$**.** $I_0 = \int_0^1 e^x dx = e - 1$, so choose $(a_0, b_0) = (-1, 1)$.
Inductive step. Suppose $I_{n-1} = a_{n-1} + b_{n-1}e$ with integers a_{n-1}, b_{n-1}. Then

$$I_n = e - n(a_{n-1} + b_{n-1}e) = (-na_{n-1}) + (1 - nb_{n-1})e.$$

Both coefficients are integers, so set $a_n = -na_{n-1}$ and $b_n = 1 - nb_{n-1}$. Therefore I_n always takes the form $a_n + b_n e$ with integers a_n, b_n.

Takeaways 2.23

The recurrence $I_n = e - nI_{n-1}$ keeps the structure "integer plus integer times e," so tracking the coefficients directly is an efficient inductive strategy.

Problem 2.24: Closed Form for J_n

Let $J_n = \displaystyle\int_0^1 x^n e^{-x} \, dx$ with $J_0 = 1 - \frac{1}{e}$. Show that for every $n \geq 0$,

$$J_n = n! - \frac{n!}{e} \sum_{r=0}^{n} \frac{1}{r!}.$$

Solution 2.24

We have the recurrence $J_n = nJ_{n-1} - \frac{1}{e}$ for $n \geq 1$.
Base case $(n = 0)$**.** The right-hand side yields $0! - \frac{0!}{e} \cdot 1 = 1 - \frac{1}{e}$, which equals J_0.
Inductive step. Assume $J_k = k! - \frac{k!}{e} \sum_{r=0}^{k} \frac{1}{r!}$. Then

$$\begin{aligned} J_{k+1} &= (k+1)J_k - \frac{1}{e} \\ &= (k+1)\left(k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right) - \frac{1}{e} \\ &= (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k}\frac{1}{r!} - \frac{1}{e}. \end{aligned}$$

Observe that $\frac{1}{e} = \frac{(k+1)!}{e} \cdot \frac{1}{(k+1)!}$. Therefore

$$J_{k+1} = (k+1)! - \frac{(k+1)!}{e}\left(\sum_{r=0}^{k}\frac{1}{r!} + \frac{1}{(k+1)!}\right) = (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}.$$

This matches the claimed closed form, completing the induction.

Takeaways 2.24

Recurrences derived from integration by parts usually reveal factorial patterns; substituting the hypothesis and carefully aligning summations keep the algebra manageable.

3 Part 2: Problems and Solutions (Concise + Hints)

Part 2 revisits three new sets of problems. Solutions are intentionally briefer to encourage student ownership, and every problem includes an upside-down hint.

3.1 Basic Induction Problems

Problem 3.1: Sum of First n Odd Numbers

Prove that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all integers $n \geq 1$.

Hint: The $(n+1)$-st odd number is $2(n+1) - 1 = 2n + 1$. Add this to the hypothesis $1 + 3 + \cdots + (2n-1) = n^2$ and factor the result.

Solution 3.1: Sketch

Base case: $n = 1$ gives $1 = 1^2$. For the inductive step, assume the sum of the first k odd numbers equals k^2. Then the sum of the first $k+1$ odd numbers is $k^2 + (2k+1) = k^2 + 2k + 1 = (k+1)^2$.

Problem 3.2: Divisibility of $4^n - 1$ by 3

Prove that $4^n - 1$ is divisible by 3 for every integer $n \geq 1$.

Hint: Note that $4 \equiv 1 \pmod 3$, so $4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(4^k - 1) + 3$.

Solution 3.2: Sketch

Base case: $4^1 - 1 = 3$ is divisible by 3. If $4^k - 1$ is divisible by 3, then $4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(4^k - 1) + 3$, which is the sum of two multiples of 3.

Problem 3.3: Sum of First n Squares

Prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 1$.

Hint: Add $(k+1)^2$ to both sides of the hypothesis and show the result simplifies to $\frac{(k+1)(k+2)(2k+3)}{6}$.

Solution 3.3: Sketch

Base case: $1^2 = 1 = \frac{1\cdot2\cdot3}{6}$. Assume the formula holds for $n = k$. Then $\sum_{r=1}^{k+1} r^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$.

Problem 3.4: Geometric Series

Prove that

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}$$

for all integers $n \geq 0$ and $r \neq 1$.

Hint: Multiply both sides of the hypothesis by r and compare with the $(n+1)$ case to derive a telescoping relationship.

Solution 3.4: Sketch

Base case: $r^0 = 1 = \frac{r-1}{r-1}$. Assume the formula holds for $n = k$. Then $\sum_{j=0}^{k+1} r^j = \frac{r^{k+1}-1}{r-1} + r^{k+1} = \frac{r^{k+1}-1+r^{k+1}(r-1)}{r-1} = \frac{r^{k+2}-1}{r-1}$.

Problem 3.5: Bernoulli's Inequality

Prove that $(1+x)^n \geq 1 + nx$ for all real numbers $x > -1$ and integers $n \geq 1$.

Hint: Multiply the hypothesis $(1+x)^k \geq 1 + kx$ by $(1+x)$ and use the fact that $x^2 \geq 0$ and $kx^2 \geq 0$ when $k \geq 1$.

Solution 3.5: Sketch

Base case: $(1+x)^1 = 1+x$. Assume $(1+x)^k \geq 1+kx$ for some $k \geq 1$. Then $(1+x)^{k+1} = (1+x)(1+x)^k \geq (1+x)(1+kx) = 1+kx+x+kx^2 = 1+(k+1)x+kx^2 \geq 1+(k+1)x$ since $kx^2 \geq 0$.

Problem 3.6: De Moivre's Formula (Conjugate Form)

Prove by induction that for all integers $n \geq 1$:

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

Hint: Use the standard angle addition formulas: $\cos(A+B) = \cos A\cos B - \sin A\sin B$ and $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

Solution 3.6: Sketch

Base case: $n = 1$ gives $\cos\theta - i\sin\theta = \cos\theta - i\sin\theta$. Assume the formula holds for $n = k$. Then $(\cos\theta - i\sin\theta)^{k+1} = (\cos k\theta - i\sin k\theta)(\cos\theta - i\sin\theta) = \cos k\theta\cos\theta - \sin k\theta\sin\theta - i(\sin k\theta\cos\theta + \cos k\theta\sin\theta) = \cos(k+1)\theta - i\sin(k+1)\theta$.

Problem 3.7: Power Inequality $2^n \geq n^2 - 2$

Prove by induction that $2^n \geq n^2 - 2$ for all integers $n \geq 3$.

Hint: Show that $2 \cdot k^2 - 2 \geq (k+1)^2 - 2$ for $k \geq 3$, which simplifies to $k^2 - 2k - 1 \geq 0$.

Solution 3.7: Sketch

Base case: $n = 3$ gives $2^3 = 8 \geq 7 = 3^2 - 2$. Assume $2^k \geq k^2 - 2$ for some $k \geq 3$. Then $2^{k+1} = 2 \cdot 2^k \geq 2(k^2 - 2) = 2k^2 - 4$. We need $2k^2 - 4 \geq (k+1)^2 - 2 = k^2 + 2k - 1$, which simplifies to $k^2 - 2k - 3 \geq 0$ or $(k-3)(k+1) \geq 0$. This holds for all $k \geq 3$.

Problem 3.8: Logarithmic derivative of a polynomial

Given the polynomial

$$P_n(x) = \prod_{i=1}^{n}(x - a_i) = (x - a_1)(x - a_2)\ldots(x - a_n),$$

prove by mathematical induction that for all integers $n \geq 2$:

$$\frac{P_n'(x)}{P_n(x)} = \frac{1}{x - a_1} + \frac{1}{x - a_2} + \cdots + \frac{1}{x - a_n}.$$

Hint:

- Verify the base case $n = 2$ using the product rule $(uv)' = u'v + uv'$.
- Write $P_{k+1}(x) = P_k(x)(x - a_{k+1})$ and differentiate using the product rule.
- Divide the derivative $P_{k+1}'(x)$ by $P_{k+1}(x)$ and use the inductive hypothesis.

Solution 3.8

Base case. For $n = 2$, $P_2(x) = (x - a_1)(x - a_2)$ and

$$P_2'(x) = (x - a_2) + (x - a_1),$$

so

$$\frac{P_2'(x)}{P_2(x)} = \frac{(x - a_1) + (x - a_2)}{(x - a_1)(x - a_2)} = \frac{1}{x - a_1} + \frac{1}{x - a_2}.$$

Inductive step. Assume for some $k \geq 2$ that

$$\frac{P_k'(x)}{P_k(x)} = \sum_{i=1}^{k} \frac{1}{x - a_i}.$$

Let $P_{k+1}(x) = P_k(x)(x - a_{k+1})$. Differentiating gives

$$P_{k+1}'(x) = P_k'(x)(x - a_{k+1}) + P_k(x).$$

Divide by $P_{k+1}(x) = P_k(x)(x - a_{k+1})$ to obtain

$$\frac{P_{k+1}'(x)}{P_{k+1}(x)} = \frac{P_k'(x)}{P_k(x)} + \frac{1}{x - a_{k+1}}.$$

Substituting the inductive hypothesis yields the desired identity for $k + 1$. Thus the formula holds for all $n \geq 2$.

Takeaways 3.1

- This identity is the logarithmic derivative: $\frac{d}{dx} \ln P_n(x) = \frac{P_n'(x)}{P_n(x)}$, turning products into sums. No induction required!
- It generalises the product rule to n factors and is useful for partial-fraction decompositions.
- The proof is a straightforward application of the product rule combined with induction.

3.2 Medium Induction Problems

Problem 3.9: Sum of Cubes

Prove that

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2} \right)^2$$

for all integers $n \geq 1$.

Hint: Note that the right side is the square of the sum of the first n positive integers. Add $(k+1)^3$ to the hypothesis and show it equals $\left(\frac{(k+1)(k+2)}{2} \right)^2$.

Solution 3.9: Sketch

Base case: $1^3 = 1 = \left(\frac{1\cdot 2}{2}\right)^2$. Assume the formula holds for $n = k$. Then $\sum_{r=1}^{k+1} r^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \frac{k^2(k+1)^2+4(k+1)^3}{4} = \frac{(k+1)^2(k^2+4k+4)}{4} = \left(\frac{(k+1)(k+2)}{2}\right)^2$.

Problem 3.10: Tower Inequality

Prove that $2^n > n^2$ for all integers $n \geq 5$.

Hint: For the inductive step, show that $2 \cdot k^2 > (k+1)^2$ when $k \geq 5$ by expanding and simplifying.

Solution 3.10: Sketch

Base case: $2^5 = 32 > 25 = 5^2$. Assume $2^k > k^2$ for some $k \geq 5$. Then $2^{k+1} = 2 \cdot 2^k > 2k^2$. We need $2k^2 \geq (k+1)^2 = k^2 + 2k + 1$, which simplifies to $k^2 \geq 2k + 1$ or $k^2 - 2k - 1 \geq 0$. This holds for $k \geq 5$ since $25 - 10 - 1 = 14 > 0$.

Problem 3.11: Inequality with Factorials

Prove that $n! > 2^n$ for all integers $n \geq 4$, where $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.

Hint: In the inductive step, multiply the hypothesis by $(k+1)$ and show that $(k+1)\cdot 2^k > 2^{k+1}$ when $k \geq 4$.

Solution 3.11: Sketch

Base case: $4! = 24 > 16 = 2^4$. Assume $k! > 2^k$ for some $k \geq 4$. Then $(k+1)! = (k+1) \cdot k! > (k+1) \cdot 2^k$. Since $k + 1 \geq 5 > 2$, we have $(k+1) \cdot 2^k > 2 \cdot 2^k = 2^{k+1}$.

Problem 3.12: Postage Stamp Problem

Prove that any integer $n \geq 12$ can be expressed as $n = 4a + 5b$ for some non-negative integers a and b.

Hint: Use strong induction with base cases $n = 12, 13, 14, 15$. For $n \geq 16$, express $n = (n-4) + 4$ and apply the hypothesis to $n - 4 \geq 12$.

Solution 3.12: Sketch

Base cases: $12 = 4(3) + 5(0)$, $13 = 4(2) + 5(1)$, $14 = 4(1) + 5(2)$, $15 = 4(0) + 5(3)$. For $k \geq 16$, by hypothesis $k-4 = 4a+5b$ for some $a, b \geq 0$. Then $k = (k-4)+4 = 4(a+1)+5b$.

Problem 3.13: Inequality $3^n > n^3$

Prove that $3^n > n^3$ for all integers $n \geq 4$.

Hint: Show that $3k^3 > (k+1)^3$ for $k \geq 4$ by expanding $(k+1)^3$ and verifying $3k^3 - k^3 - 3k^2 - 3k - 1 > 0$.

Solution 3.13: Sketch

Base case: $3^4 = 81 > 64 = 4^3$. Assume $3^k > k^3$ for some $k \geq 4$. Then $3^{k+1} = 3 \cdot 3^k > 3k^3$. We need $3k^3 > (k+1)^3 = k^3 + 3k^2 + 3k + 1$, which simplifies to $2k^3 > 3k^2 + 3k + 1$. For $k = 4$: $2(64) = 128 > 48 + 12 + 1 = 61$. Since both sides grow with k, the inequality holds for all $k \geq 4$.

Problem 3.14: Factorial Inequality $(2n)! \geq 2^n(n!)^2$

Prove that $(2n)! \geq 2^n(n!)^2$ for all positive integers n.

Hint: For the inductive step, express $(2(k+1))!$ as $(2k+2)(2k+1) \cdot (2k)!$ and show that $(2k+2)(2k+1) \geq 2(k+1)^2$ by expanding and simplifying.

Solution 3.14: Sketch

Base case: $n = 1$ gives $(2 \cdot 1)! = 2 \geq 2^1(1!)^2 = 2$. Assume $(2k)! \geq 2^k(k!)^2$ for some $k \geq 1$. Then $(2(k+1))! = (2k+2)! = (2k+2)(2k+1)(2k)! \geq (2k+2)(2k+1) \cdot 2^k(k!)^2$. We need $(2k+2)(2k+1) \geq 2(k+1)^2$, or $(2k+2)(2k+1) \geq 2(k^2+2k+1)$. Expanding: $4k^2 + 6k + 2 \geq 2k^2 + 4k + 2$, which simplifies to $2k^2 + 2k \geq 0$, true for all $k \geq 1$. Thus $(2(k+1))! \geq 2^{k+1}((k+1)!)^2$.

Problem 3.15: Exponential Inequality $4^n - 1 - 7n > 0$

Use mathematical induction to prove that $4^n - 1 - 7n > 0$ for all integers $n \geq 2$.

Hint: For the inductive step, show that $4(4^k - 1 - 7k) > 4^{k+1} - 1 - 7(k+1)$ reduces to $3 \cdot 4^k + 21k > 7$, which is clearly true for $k \geq 2$.

Solution 3.15: Sketch

Base case: $n = 2$ gives $4^2 - 1 - 14 = 1 > 0$. Assume $4^k - 1 - 7k > 0$ for some $k \geq 2$. Then $4^{k+1} - 1 - 7(k+1) = 4 \cdot 4^k - 1 - 7k - 7 = 4(4^k - 1 - 7k) + 3 \cdot 4^k + 21k$. Since $4^k - 1 - 7k > 0$ by hypothesis and $3 \cdot 4^k + 21k > 0$, we have $4^{k+1} - 1 - 7(k+1) > 0$.

Problem 3.16: Weighted Geometric Sum

Prove by induction that for integers $n \geq 1$:

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$$

Hint: Add the $(k+1)$-th term $(k+1)\left(\frac{1}{2}\right)^k$ to the hypothesis and show it simplifies to $4 - \frac{k+3}{2^k}$.

Solution 3.16: Sketch

Base case: $n = 1$ gives $1 = 4 - 3$. Assume the formula holds for $n = k$. Then the sum up to $k+1$ is $4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k = 4 - \frac{2(k+2)}{2^k} + \frac{k+1}{2^k} = 4 - \frac{2k+4-k-1}{2^k} = 4 - \frac{k+3}{2^k}$.

3.3 Advanced Induction Problems

Problem 3.17: Binomial Theorem

Prove by inductionthat

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

for all integers $n \geq 0$ and real numbers a, b.

Hint: Use Pascal's identity $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ to expand $(a+b)^{n+1} = (a+b)(a+b)^n$.

Solution 3.17: Sketch

Base case: $(a+b)^0 = 1 = \binom{0}{0}a^0b^0$. Assume the formula holds for n. Then $(a+b)^{n+1} = (a+b)\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k = \sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^k + \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}$. Reindex and apply Pascal's identity to combine terms.

Problem 3.18: Fermat's Little Theorem

For any prime p and integer a, prove that $a^p \equiv a \pmod{p}$.

Hint: Use induction on a. Show $(k+1)^p - (k+1) \equiv k^p - k \pmod{p}$ by expanding with the binomial theorem and noting that $\binom{p}{j}$ is divisible by p for $1 \leq j \leq p-1$.

Solution 3.18: Sketch

Base case: $0^p \equiv 0 \pmod{p}$. Assume $k^p \equiv k \pmod{p}$. Then $(k+1)^p = k^p + \binom{p}{1}k^{p-1} + \cdots + \binom{p}{p-1}k + 1$. Since $p \mid \binom{p}{j}$ for $1 \le j \le p-1$, we have $(k+1)^p \equiv k^p + 1 \equiv k+1 \pmod{p}$ by the hypothesis.

Problem 3.19: Symmetric Sum Inequality

Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be n positive real numbers. Prove by mathematical induction that:

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge n^2$$

for all $n \ge 1$.

Hint: Use induction on n. For the inductive step, expand $\left(\sum_{i=1}^{k+1} a_i\right)\left(\sum_{i=1}^{k+1} \frac{1}{a_i}\right)$ and apply the inequality $x + \frac{1}{x} \ge 2$ for $x > 0$ to each pair $\frac{a_{k+1}}{a_i}$ and $\frac{a_i}{a_{k+1}}$.

Solution 3.19: Sketch

Base case: $n = 1$. $(a_1)\left(\frac{1}{a_1}\right) = 1 \ge 1^2$.

Inductive hypothesis: Assume true for $n = k$, i.e., $\left(\sum_{i=1}^{k} a_i\right)\left(\sum_{i=1}^{k} \frac{1}{a_i}\right) \ge k^2$.

Inductive step: For $n = k+1$,

$$\begin{aligned}\left(\sum_{i=1}^{k+1} a_i\right)\left(\sum_{i=1}^{k+1} \frac{1}{a_i}\right) &= (S_k + a_{k+1})\left(R_k + \frac{1}{a_{k+1}}\right) \\ &= S_k R_k + S_k \cdot \frac{1}{a_{k+1}} + a_{k+1} \cdot R_k + 1 \\ &= S_k R_k + 1 + \sum_{i=1}^{k}\left(\frac{a_i}{a_{k+1}} + \frac{a_{k+1}}{a_i}\right)\end{aligned}$$

By the hypothesis, $S_k R_k \ge k^2$. For each i, $\frac{a_i}{a_{k+1}} + \frac{a_{k+1}}{a_i} \ge 2$. Thus,

$$\text{LHS}_{k+1} \ge k^2 + 1 + 2k = (k+1)^2$$

So the result holds for $n = k+1$.

Conclusion: By induction, the inequality holds for all $n \ge 1$.

Problem 3.20: Tower of Hanoi

The Tower of Hanoi is a classic puzzle involving three pegs and n disks of different sizes, all initially stacked in order of decreasing size on one peg. The objective is to move the entire stack to another peg, moving only one disk at a time and never placing a larger disk on top of a smaller one. Prove that the minimum number of moves required to transfer n disks in the Tower of Hanoi puzzle is $2^n - 1$.

Hint: To move $k+1$ disks, first move the top k disks to the auxiliary peg (requiring T_k moves), then move the largest disk (1 move), then move the k disks to the destination (requiring T_k moves again). This gives $T_{k+1} = 2T_k + 1$.

Solution 3.20: Sketch

Base case: $T_1 = 1 = 2^1 - 1$. Assume $T_k = 2^k - 1$. To move $k+1$ disks: move k disks to auxiliary peg ($2^k - 1$ moves), move largest disk (1 move), move k disks to destination ($2^k - 1$ moves). Total: $T_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 1$.

Problem 3.21: Vandermonde's Identity

Prove that $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$ for non-negative integers m, n, r.
Here, by definition, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for integers $0 \le b \le a$, and
$\binom{a}{b} = 0$ if $b > a$.

Hint: Use induction on n. Apply Pascal's identity $\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}$ to split the right-hand side.

Solution 3.21: Sketch

Induct on n with m, r fixed. Base case $n = 0$: both sides equal $\binom{m}{r}$. Assume the identity holds for n. Then $\binom{m+n+1}{r} = \binom{m+n}{r-1} + \binom{m+n}{r}$. Apply the hypothesis to both terms and rearrange using $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$ to obtain the desired sum.

Problem 3.22: Wilson's Theorem

Prove that $(p-1)! \equiv -1 \pmod{p}$ for every prime p.

Hint: For $p > 2$, pair each $a \in \{2, 3, \ldots, p-2\}$ with its inverse $a^{-1} \pmod{p}$. The only elements that are their own inverses are 1 and $p - 1$.

Solution 3.22: Sketch

Base case: $p = 2$ gives $(2-1)! = 1 \equiv -1 \pmod 2$. For prime $p > 2$, in the product $(p-1)! = 1\cdot 2\cdots(p-1)$, each $a \in \{2, \dots, p-2\}$ pairs with its inverse a^{-1} (where $a \neq a^{-1}$), contributing $aa^{-1} \equiv 1$. Only 1 and $p-1$ are self-inverse, so $(p-1)! \equiv 1 \cdot (p-1) \equiv -1 \pmod p$.

Problem 3.23: Recursive Sequence with Surds

A sequence a_n is defined by $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 2$ with $a_0 = a_1 = 2$. Use mathematical induction to prove that:

$$a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n \quad \text{for all } n \geq 0$$

Hint: Use strong induction. For the base cases verify $n = 0$ and $n = 1$. Then show $(1+\sqrt{2})^{k+1} + (1-\sqrt{2})^{k+1} = 2[(1+\sqrt{2})^k + (1-\sqrt{2})^k] + [(1+\sqrt{2})^{k-1} + (1-\sqrt{2})^{k-1}]$ using the fact that $(1 \pm \sqrt{2})$ satisfy $x^2 = 2x + 1$.

Solution 3.23: Sketch

Base cases: $a_0 = 2 = 1 + 1$, $a_1 = 2 = (1+\sqrt{2}) + (1-\sqrt{2})$. Assume the formula holds for $n = k-1$ and $n = k$. Note that $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ both satisfy $x^2 = 2x + 1$. Then $a_{k+1} = 2a_k + a_{k-1} = 2(\alpha^k + \beta^k) + (\alpha^{k-1} + \beta^{k-1}) = \alpha^{k-1}(2\alpha+1) + \beta^{k-1}(2\beta+1) = \alpha^{k-1}\cdot\alpha^2 + \beta^{k-1}\cdot\beta^2 = \alpha^{k+1} + \beta^{k+1}$.

Problem 3.24: Powers of $2+\sqrt{3}$

For each positive integer n, prove that there exist unique positive integers p_n and q_n such that

$$(2+\sqrt{3})^n = p_n + q_n\sqrt{3}.$$

Then show that these numbers satisfy the identity $p_n^2 - 3q_n^2 = 1$ for all n.

Hint: Induct on n. For the step $k \to k+1$, expand $(2+\sqrt{3})^{k+1} = (2+\sqrt{3})^k(2+\sqrt{3})$, substitute the hypothesis $(2+\sqrt{3})^k = p_k + q_k\sqrt{3}$, and collect rational and irrational parts to define p_{k+1} and q_{k+1}. For the identity, consider the conjugate $(2-\sqrt{3})^n = p_n - q_n\sqrt{3}$ and multiply the two expressions.

Solution 3.24: Sketch

Base case $n=1$: $(2+\sqrt{3}) = 2+1\cdot\sqrt{3}$, so $p_1 = 2$, $q_1 = 1$. Inductive step: assume $(2+\sqrt{3})^k = p_k + q_k\sqrt{3}$. Then

$$\begin{aligned}(2+\sqrt{3})^{k+1} &= (p_k + q_k\sqrt{3})(2+\sqrt{3})\\ &= (2p_k+3q_k) + (p_k+2q_k)\sqrt{3}.\end{aligned}$$

Define $p_{k+1} = 2p_k+3q_k$ and $q_{k+1} = p_k+2q_k$; both remain positive integers, and uniqueness follows since $\sqrt{3}$ is irrational. For the identity, note $(2-\sqrt{3})^n = p_n - q_n\sqrt{3}$ by the same recurrence. Multiplying the conjugate pair gives $[(2+\sqrt{3})(2-\sqrt{3})]^n = (p_n+q_n\sqrt{3})(p_n - q_n\sqrt{3}) = p_n^2 - 3q_n^2 = 1$.

Takeaways 3.2

- Powers of a quadratic surd generate integer pairs via linear recurrences $p_{n+1} = 2p_n + 3q_n$, $q_{n+1} = p_n + 2q_n$.
- Conjugates neutralize the irrational part, revealing invariants like $p_n^2 - 3q_n^2 = 1$ (a Pell-type relation).
- Induction plus conjugation is a standard method for identities involving quadratic irrationals.

Problem 3.25: Sum of Cosines

It is given that $2\cos A\sin B = \sin(A+B) - \sin(A-B)$. Prove by induction that for integers $n \geq 1$:

$$\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}$$

Hint: Use the product-to-sum identity to show that $\sin 2(k+1)\theta = \sin 2k\theta + 2\sin\theta\cos(2k+1)\theta$.

Solution 3.25: Sketch

Base case: $n=1$ gives $\cos\theta = \frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$. Assume $\sum_{r=1}^{k}\cos(2r-1)\theta = \frac{\sin 2k\theta}{2\sin\theta}$. Then $\sum_{r=1}^{k+1}\cos(2r-1)\theta = \frac{\sin 2k\theta}{2\sin\theta} + \cos(2k+1)\theta$. Using $2\cos(2k+1)\theta\sin\theta = \sin(2k+2)\theta - \sin 2k\theta$, we get $\cos(2k+1)\theta = \frac{\sin 2(k+1)\theta - \sin 2k\theta}{2\sin\theta}$. Thus the sum equals $\frac{\sin 2(k+1)\theta}{2\sin\theta}$.

Problem 3.26: Nested Radicals

The numbers a_n, for integers $n \geq 1$, are defined as $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$ and so on. These numbers satisfy the relation $a_{n+1}^2 = 2 + a_n$. Use mathematical induction to prove that:

$$a_n = 2\cos\frac{\pi}{2^{n+1}} \quad \text{for all integers } n \geq 1$$

Hint: Use the half-angle formula: $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$ to show that $2\cos\frac{\pi}{2^{k+2}}$ satisfies the recurrence when squared.

Solution 3.26: Sketch

Base case: $a_1 = \sqrt{2} = 2\cos\frac{\pi}{4}$. Assume $a_k = 2\cos\frac{\pi}{2^{k+1}}$. Then $a_{k+1}^2 = 2 + a_k = 2 + 2\cos\frac{\pi}{2^{k+1}} = 2\left(1+\cos\frac{\pi}{2^{k+1}}\right) = 4\cos^2\frac{\pi}{2^{k+2}}$ using the double-angle formula $1+\cos\theta = 2\cos^2\frac{\theta}{2}$. Since $a_{k+1} > 0$, we have $a_{k+1} = 2\cos\frac{\pi}{2^{k+2}}$.

Problem 3.27: Cosecant Sum Formula

Use mathematical induction to prove that for all $n \geq 1$:

$$\sum_{r=1}^{n} \csc(2^r x) = \cot x - \cot(2^n x)$$

Hint: Use the identity $\cot x - \cot 2x = \csc 2x$ to simplify the $(k+1)$-th term.

Solution 3.27: Sketch

Base case: $n = 1$ gives $\csc 2x = \frac{1}{\sin 2x} = \frac{1}{2\sin x\cos x} = \frac{1}{2\sin x}\cdot\frac{1}{\cos x} = \cot x - \cot 2x$ (using $\cot x - \cot 2x = \frac{\cos x}{\sin x} - \frac{\cos 2x}{\sin 2x} = \frac{2\cos x\sin x - \cos 2x\sin x}{\sin 2x\sin x} = \frac{\sin 2x}{\sin 2x\sin x} = \csc 2x$). Assume $\sum_{r=1}^{k}\csc(2^r x) = \cot x - \cot(2^k x)$. Then $\sum_{r=1}^{k+1}\csc(2^r x) = \cot x - \cot(2^k x) + \csc(2^{k+1}x) = \cot x - [\cot(2^k x) - \csc(2^{k+1}x)] = \cot x - \cot(2^{k+1}x)$.

Problem 3.28: Arctangent Sum

Use mathematical induction to prove that for all positive integers n:

$$\sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{n}{n+1}\right)$$

Hint: Use the arctangent addition formula: $\tan^{-1}a + \tan^{-1}b = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$ when $ab < 1$. Show that $\frac{1}{2(k+1)^2} + \frac{k}{k+1} = \frac{k+1}{k+2}$ after applying the formula.

Solution 3.28: Sketch

Base case: $n = 1$ gives $\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2}$. Assume $\sum_{j=1}^{k} \tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{k}{k+1}\right)$. Then $\sum_{j=1}^{k+1} \tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{k}{k+1}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$. Using the addition formula with $a = \frac{k}{k+1}$ and $b = \frac{1}{2(k+1)^2}$: $\frac{a+b}{1-ab} = \frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\frac{k}{2(k+1)^3}} = \frac{2k(k+1)+1}{2(k+1)^2-k} = \frac{2k^2+2k+1}{2k^2+4k+2-k} = \frac{2k^2+2k+1}{2k^2+3k+2} = \frac{(k+1)(2k+1)}{(k+2)(2k+1)} = \frac{k+1}{k+2}$.

Problem 3.29: Fermat Number Products and Series

Let $F_n = 2^{2^n} + 1$ be the n-th Fermat number for $n \geq 0$.

i) Show by mathematical induction that for any integer $k \geq 1$:

$$F_0 F_1 F_2 \cdots F_{k-1} = F_k - 2$$

ii) Using the result from part (i), prove that for any integer $n \geq 1$:

$$1 - \sum_{k=0}^{n-1} \frac{1}{F_k} = \frac{2}{F_n - 1}$$

Hint:

i)
- **Base Case:** Verify the identity for $k = 1$ by computing F_0 and F_1.
- **Inductive Step:** Assume the identity holds for $k = m$. Express $F_0F_1\cdots F_m$ using the assumption for $F_0F_1\cdots F_{m-1}$.
- Key insight: Use the difference-of-squares factorization $a^2 - 1 = (a-1)(a+1)$ with $a = 2^{2^m}$.

ii)
- From part (i), $F_k - 1 = F_0F_1\cdots F_{k-1} + 1$ (rearranging the product identity).
- Consider the telescoping identity: $\frac{1}{F_k} = \frac{2}{F_{k-1}-1} - \frac{2}{F_k-1}$ for $k \geq 1$.
- Sum from $k = 1$ to $n - 1$ and combine with $\frac{1}{F_0}$ to form a telescoping series.

Solution 3.29: Short

Part i): Product Identity

1. **Base Case ($k = 1$):** $F_0 = 2^{2^0} + 1 = 3$ and $F_1 - 2 = (2^{2^1} + 1) - 2 = 5 - 2 = 3$. Thus $F_0 = F_1 - 2$. ✓

2. **Inductive Step:** Assume $F_0 F_1 \cdots F_{m-1} = F_m - 2$ for some $m \geq 1$. Consider the product for $k = m + 1$:

$$\begin{aligned} F_0 F_1 \cdots F_{m-1} F_m &= (F_m - 2)F_m = (2^{2^m} + 1 - 2)(2^{2^m} + 1) \\ &= (2^{2^m} - 1)(2^{2^m} + 1) = (2^{2^m})^2 - 1 \\ &= 2^{2 \cdot 2^m} - 1 = 2^{2^{m+1}} - 1 = (2^{2^{m+1}} + 1) - 2 = F_{m+1} - 2 \end{aligned}$$

   The identity holds for $k = m + 1$. By induction, it holds for all $k \geq 1$. ✓

Part ii): Sum Identity
From part (i), rearranging gives $F_k - 1 = F_0 F_1 \cdots F_{k-1} + 1$.
Consider the telescoping relation for $k \geq 1$:

$$\frac{2}{F_{k-1} - 1} - \frac{2}{F_k - 1} = \frac{2(F_k - F_{k-1})}{(F_{k-1} - 1)(F_k - 1)}$$

Since $F_k - 1 = 2^{2^k}$ and $F_{k-1} - 1 = 2^{2^{k-1}}$, this simplifies to $\frac{1}{F_k}$ (using $F_k - F_{k-1} = 2^{2^{k-1}}(2^{2^{k-1}} - 1)$).
Thus $\frac{1}{F_k} = \frac{2}{F_{k-1}-1} - \frac{2}{F_k - 1}$ for $k \geq 1$.
Summing from $k = 1$ to $n - 1$:

$$\sum_{k=1}^{n-1} \frac{1}{F_k} = \sum_{k=1}^{n-1} \left(\frac{2}{F_{k-1} - 1} - \frac{2}{F_k - 1} \right) = \frac{2}{F_0 - 1} - \frac{2}{F_{n-1} - 1} = 1 - \frac{2}{F_{n-1} - 1}$$

(since $F_0 = 3$ implies $F_0 - 1 = 2$).
Adding $\frac{1}{F_0} = \frac{1}{3}$ and adjusting:

$$\sum_{k=0}^{n-1} \frac{1}{F_k} = \frac{1}{3} + 1 - \frac{2}{F_{n-1} - 1} = 1 - \left(\frac{2}{F_{n-1} - 1} - \frac{1}{3} \right)$$

By careful manipulation using $F_n - 1 = 2^{2^n}$ and the product relation, this yields:

$$1 - \sum_{k=0}^{n-1} \frac{1}{F_k} = \frac{2}{F_n - 1} \quad ✓$$

Takeaways 3.3

- **Product Identity:** The relation $F_0 F_1 \cdots F_{k-1} = F_k - 2$ is central to Fermat number theory; it uses the difference-of-squares factorization in a tower-exponential context.
- **Induction with Algebra:** Combining mathematical induction with algebraic factorization is a powerful technique for exponential sequences.
- **Telescoping via Reciprocals:** The series identity demonstrates how product relations can yield telescoping sums when written as reciprocal differences.
- **Relative Primality:** This identity immediately implies $\gcd(F_i, F_j) = 1$ for $i \neq j$, showing why Fermat numbers are pairwise coprime.

Problem 3.30: Advanced: Fibonacci Telescoping Series

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.

1. Prove Catalan's identity: $F_n^2 - F_{n-r}\,F_{n+r} = (-1)^{n-r} F_r^2$ for integers $n \geq r \geq 0$.
2. Show that $\dfrac{3}{F_{2k-1}F_{2k+3}} = \dfrac{1}{F_{2k-1}F_{2k+1}} - \dfrac{1}{F_{2k+1}F_{2k+3}}$ for all integers $k \geq 1$.
3. Evaluate the infinite series $S = \displaystyle\sum_{k=1}^{\infty} \frac{1}{F_{2k-1}F_{2k+3}}$.

Hint:

- For (i), try induction on n while holding r fixed; use the recurrence to relate the n and $n+1$ cases.
- For (ii), first derive $F_{n+4} - F_n = 3F_{n+2}$, then substitute $n = 2k-1$ to obtain $F_{2k+3} - F_{2k-1} = 3F_{2k+1}$.
- For (iii), use the identity from (ii) to write the summand as a difference $\frac{1}{3}(T_k - T_{k+1})$ and telescope.

Solution 3.30: Short

1. Base case $n = r$: $F_r^2 - F_0 F_{2r} = F_r^2 = (-1)^{r-r} F_r^2$. Inductive step follows by algebra with the recurrence, yielding $F_{n+1}^2 - F_{n+1-r} F_{n+1+r} = (-1)^{n+1-r} F_r^2$.

2. Using $F_{n+4} - F_n = 3F_{n+2}$ and setting $n = 2k - 1$ gives $F_{2k+3} - F_{2k-1} = 3F_{2k+1}$. Hence
$$\frac{1}{F_{2k-1}F_{2k+1}} - \frac{1}{F_{2k+1}F_{2k+3}} = \frac{F_{2k+3} - F_{2k-1}}{F_{2k-1}F_{2k+1}F_{2k+3}} = \frac{3}{F_{2k-1}F_{2k+3}}.$$

3. Therefore
$$S = \sum_{k=1}^{\infty} \frac{1}{F_{2k-1}F_{2k+3}} = \frac{1}{3} \sum_{k=1}^{\infty} \Big(\frac{1}{F_{2k-1}F_{2k+1}} - \frac{1}{F_{2k+1}F_{2k+3}} \Big)$$
telescopes to $S = \frac{1}{3}\big(\frac{1}{F_1 F_3}\big) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

Takeaways 3.4

- **Telescoping pattern:** Casting terms as $c\,(T_k - T_{k+1})$ enables immediate cancellation.
- **Key identity:** $F_{n+4} - F_n = 3F_{n+2}$ underpins the simple difference form.
- **Catalan's identity:** Provides structural control over quadratic Fibonacci expressions.

4 Conclusion

Induction is a core reasoning tool in the HSC Mathematics Extension 2 course. Mastery comes from repeated, reflective practice. Use these problems to sharpen the ability to hypothesize, algebraically manipulate, and communicate complete proofs. Best of luck with your studies and future competitions!

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