Limits of Generalized Mean Expectations

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1 Overview

This document evaluates the limits of expectations of generalized means H_p over the unit hypercube $[0,1]^n$ as $n \to \infty$, where $H_p = \left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{1/p}$ for various values of p, including special cases such as minimum, maximum, harmonic, geometric, arithmetic, root mean square, and cubic means.

2 Problem Statements

2.1 Generalized Mean

For a real parameter $p \neq 0$, the generalized mean (or power mean) is defined as:

$$H_p(\mathbf{x}) = \left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{1/p}.$$

We seek to evaluate:

Problem 1. *For $p \in \mathbb{R} \setminus \{0\}$, find*

$$\lim_{n\to\infty} \int_{[0,1]^n} \left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{1/p} dx_1 \cdots dx_n.$$

2.2 Special Cases

We also consider the following special cases:

Problem 2 (Minimum Mean)**.**

$$\lim_{n\to\infty} \int_{[0,1]^n} \min(x_1, x_2, \ldots, x_n)\, dx_1 \cdots dx_n.$$

Problem 3 (Maximum Mean)**.**

$$\lim_{n\to\infty} \int_{[0,1]^n} \max(x_1, x_2, \ldots, x_n)\, dx_1 \cdots dx_n.$$

Problem 4 (Harmonic Mean)**.**

$$\lim_{n\to\infty} \int_{[0,1]^n} \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}\, dx_1 \cdots dx_n.$$

Problem 5 (Geometric Mean)**.**

$$\lim_{n\to\infty} \int_{[0,1]^n} \left(\prod_{i=1}^{n} x_i \right)^{1/n} dx_1 \cdots dx_n.$$

Problem 6 (Arithmetic Mean)**.**

$$\lim_{n\to\infty} \int_{[0,1]^n} \frac{1}{n} \sum_{i=1}^{n} x_i \, dx_1 \cdots dx_n.$$

Problem 7 (Root Mean Square)**.**

$$\lim_{n\to\infty} \int_{[0,1]^n} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^2 \right)^{1/2} dx_1 \cdots dx_n.$$

Problem 8 (Cubic Mean)**.**

$$\lim_{n\to\infty} \int_{[0,1]^n} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^3 \right)^{1/3} dx_1 \cdots dx_n.$$

3 Key Ideas

The solutions rely on two fundamental tools:

1. **The Fubini-Tonelli Theorem**: This allows us to exchange the order of integration and factorize integrals over independent variables. For independent random variables $X_1, \ldots, X_n$ uniformly distributed on $[0,1]$, we can write:

$$E[f(X_1, \ldots, X_n)] = \int_{[0,1]^n} f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n = \int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n.$$

2. **Strong Law of Large Numbers (SLLN)**: For i.i.d. random variables $X_1, X_2, \ldots$ with finite expectation $E[X]$, we have:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E[X] \quad \text{almost surely.}$$

For the case when $E[X] = \infty$ (as in the harmonic mean), a generalized version applies, stating that the sample mean converges to infinity almost surely.

4 Solutions

4.1 Solution for Generalized Mean H_p

For $p \neq 0$, we interpret the integral as an expectation:

$$I_n^{(p)} = \int_{[0,1]^n} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{1/p} dx_1 \cdots dx_n = E\left[\left(\frac{1}{n} \sum_{i=1}^{n} X_i^p \right)^{1/p} \right],$$

where $X_1, \ldots, X_n$ are i.i.d. Unif$(0,1)$.

By the Strong Law of Large Numbers, $\frac{1}{n}\sum_{i=1}^n X_i^p \to E[X_1^p]$ almost surely. Since $E[X_1^p] = \int_0^1 x^p\,dx = \frac{1}{p+1}$ for $p > -1$, we have:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^p = \frac{1}{p+1} \quad \text{a.s.}$$

Taking the $1/p$-th power (which is continuous for $p \neq 0$):

$$\lim_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^n X_i^p\right)^{1/p} = \left(\frac{1}{p+1}\right)^{1/p} \quad \text{a.s.}$$

Since the generalized mean is bounded (for $x_i \in [0,1]$ and $p > 0$, we have $H_p \in [0,1]$; for $p < 0$, appropriate bounds exist), the Dominated Convergence Theorem applies:

$$\lim_{n\to\infty} I_n^{(p)} = E\left[\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^n X_i^p\right)^{1/p}\right] = \left(\frac{1}{p+1}\right)^{1/p}.$$

Solution.

$$\boxed{\lim_{n\to\infty}\int_{[0,1]^n}\left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{1/p} dx_1\cdots dx_n = \left(\frac{1}{p+1}\right)^{1/p}}$$

4.2 Solution for Minimum Mean

The minimum corresponds to the limit of the generalized mean as $p \to -\infty$. Let $M_n = \min(X_1, \ldots, X_n)$ for i.i.d. $X_i \sim$ Unif$(0,1)$. The probability that $M_n > t$ is:

$$P(M_n > t) = P(X_1 > t, \ldots, X_n > t) = (1-t)^n, \quad 0 < t < 1.$$

Using the identity $E[M_n] = \int_0^1 P(M_n > t)\,dt$:

$$E[M_n] = \int_0^1 (1-t)^n\,dt = \frac{1}{n+1}.$$

Taking the limit:

$$\lim_{n\to\infty} E[M_n] = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Solution.

$$\boxed{\lim_{n\to\infty}\int_{[0,1]^n}\min(x_1, x_2, \ldots, x_n)\,dx_1\cdots dx_n = 0}$$

4.3 Solution for Maximum Mean

The maximum corresponds to the limit of the generalized mean as $p \to +\infty$. Let $M_n' = \max(X_1, \ldots, X_n)$ for i.i.d. $X_i \sim$ Unif$(0,1)$. The cumulative distribution function is:

$$F_{M_n'}(t) = P(M_n' \le t) = P(X_1 \le t, \ldots, X_n \le t) = t^n, \quad 0 \le t \le 1.$$

The probability density function is $f_{M_n'}(t) = nt^{n-1}$, so:

$$E[M_n'] = \int_0^1 t\cdot nt^{n-1}\,dt = n\int_0^1 t^n\,dt = \frac{n}{n+1}.$$

Taking the limit:

$$\lim_{n\to\infty} E[M_n'] = \lim_{n\to\infty}\frac{n}{n+1} = 1.$$

Solution.

$$\boxed{\lim_{n\to\infty}\int_{[0,1]^n}\max(x_1,x_2,\ldots,x_n)\,dx_1\cdots dx_n = 1}$$

4.4 Solution for Harmonic Mean

The harmonic mean corresponds to $p=-1$ in the generalized mean. This case demonstrates the application of the generalized SLLN when the expectation is infinite. The harmonic mean is $H_n = \frac{n}{\sum_{i=1}^n \frac{1}{X_i}}$. Let $Y_i = \frac{1}{X_i}$. Then:

$$E[Y_i] = \int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}[\ln x]_t^1 = \infty.$$

Since $E[Y_i]=\infty$, the generalized SLLN implies:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n Y_i = \infty \quad \text{a.s.}$$

Therefore:

$$\lim_{n\to\infty} H_n = \lim_{n\to\infty}\frac{1}{\frac{1}{n}\sum_{i=1}^n Y_i} = 0 \quad \text{a.s.}$$

Since $0 < H_n \le 1$ (harmonic mean of numbers in $(0,1]$ is bounded), the Dominated Convergence Theorem applies:

$$\lim_{n\to\infty} E[H_n] = E\left[\lim_{n\to\infty} H_n\right] = 0.$$

Solution.

$$\boxed{\lim_{n\to\infty}\int_{[0,1]^n}\frac{n}{\sum_{i=1}^n \frac{1}{x_i}}\,dx_1\cdots dx_n = 0}$$

4.5 Solution for Geometric Mean

The geometric mean corresponds to the limit $p\to 0$ in the generalized mean, signifying the exponential of the arithmetic mean of logarithms. The geometric mean is $G_n = \left(\prod_{i=1}^n X_i\right)^{1/n}$. By the Fubini-Tonelli Theorem and independence:

$$E[G_n] = \int_{[0,1]^n}\left(\prod_{i=1}^n x_i\right)^{1/n} dx_1\cdots dx_n = \prod_{i=1}^n \int_0^1 x_i^{1/n}\,dx_i = \left(\int_0^1 x^{1/n}\,dx\right)^n.$$

Since $\int_0^1 x^{1/n}\,dx = \frac{n}{n+1}$:

$$E[G_n] = \left(\frac{n}{n+1}\right)^n.$$

Taking the limit:

$$\lim_{n\to\infty} E[G_n] = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e}.$$

Solution.

$$\boxed{\lim_{n\to\infty}\int_{[0,1]^n}\left(\prod_{i=1}^n x_i\right)^{1/n} dx_1\cdots dx_n = \frac{1}{e}}$$

4.6 Solution for Arithmetic Mean

The arithmetic mean corresponds to $p = 1$, which is the most straightforward case. The arithmetic mean is $A_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. By the SLLN:

$$\lim_{n\to\infty} A_n = E[X_1] = \int_0^1 x\,dx = \frac{1}{2} \quad \text{a.s.}$$

Since $A_n \in [0,1]$, the Dominated Convergence Theorem gives:

$$\lim_{n\to\infty} E[A_n] = E\left[\lim_{n\to\infty} A_n\right] = \frac{1}{2}.$$

Solution.

$$\boxed{\lim_{n\to\infty} \int_{[0,1]^n} \frac{1}{n}\sum_{i=1}^{n} x_i\,dx_1\cdots dx_n = \frac{1}{2}}$$

4.7 Solution for Root Mean Square

The root mean square is $R_n = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{1/2}$. By the SLLN:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i^2 = E[X_1^2] = \int_0^1 x^2\,dx = \frac{1}{3} \quad \text{a.s.}$$

Taking the square root:

$$\lim_{n\to\infty} R_n = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} \quad \text{a.s.}$$

Since $R_n \in [0,1]$, the Dominated Convergence Theorem applies:

$$\lim_{n\to\infty} E[R_n] = \frac{1}{\sqrt{3}}.$$

Solution.

$$\boxed{\lim_{n\to\infty} \int_{[0,1]^n} \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{1/2} dx_1\cdots dx_n = \frac{1}{\sqrt{3}}}$$

4.8 Solution for Cubic Mean

The cubic mean is $C_n = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^3\right)^{1/3}$. By the SLLN:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i^3 = E[X_1^3] = \int_0^1 x^3\,dx = \frac{1}{4} \quad \text{a.s.}$$

Taking the cube root:

$$\lim_{n\to\infty} C_n = \left(\frac{1}{4}\right)^{1/3} = \frac{1}{\sqrt[3]{4}} \quad \text{a.s.}$$

Since $C_n \in [0,1]$, the Dominated Convergence Theorem applies:

$$\lim_{n\to\infty} E[C_n] = \frac{1}{\sqrt[3]{4}}.$$

Solution.

$$\boxed{\lim_{n\to\infty} \int_{[0,1]^n} \left(\frac{1}{n}\sum_{i=1}^{n} x_i^3\right)^{1/3} dx_1\cdots dx_n = \frac{1}{\sqrt[3]{4}}}$$

5 Further Works and Open Problems

Several directions for future research emerge:

1. **Alternative proof for Harmonic Mean**: For the harmonic mean case ($p = -1$), an alternative proof can be constructed using bounds and the squeeze theorem, without relying on the Strong Law of Large Numbers. The key idea is to establish upper and lower bounds for $E[H_n]$ that both converge to zero.

   Specifically, for any $\epsilon > 0$, we can partition the integration domain based on the minimum value $m = \min(x_1, \ldots, x_n)$. On the region where $m > \epsilon$, we have $\sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{n}{\epsilon}$, which gives $H_n \leq \epsilon$. On the complement where $m \leq \epsilon$, we use the fact that $H_n \leq 1$ and that the measure of this region is $1 - (1-\epsilon)^n \to 0$ as $n \to \infty$. By carefully bounding $E[H_n]$ from above using these two regions and noting that $E[H_n] \geq 0$ trivially, the squeeze theorem yields $\lim_{n\to\infty} E[H_n] = 0$.

2. **Generalized means for** $p < -1$: The case $p < -1$ requires careful analysis since $E[X_1^p]$ may not exist. The behavior of the limit as $p \to -1^-$ (approaching the harmonic mean from below) is of particular interest.

3. **Non-uniform distributions**: Extending these results to other probability distributions on $[0, 1]$ or to unbounded domains would provide a more general theory.

4. **Convergence rates**: While we have established the limits, the rate of convergence (e.g., $O(1/n)$ for arithmetic mean) could be studied more systematically.

5. **Multivariate extensions**: Generalizing to integrals over $[0, 1]^{n\times d}$ for $d > 1$ or to other geometric domains.

6. **Connection to order statistics**: The minimum and maximum cases are directly related to order statistics. A unified treatment connecting all means through order statistics might yield deeper insights.

6 Conclusions

We have evaluated the limits of expectations of various generalized means over the unit hypercube $[0, 1]^n$ as $n \to \infty$. The key results are summarized in Table **??**.

Mean Type	**Limit**
Generalized Mean H_p ($p \neq 0$)	$\left(\frac{1}{p+1}\right)^{1/p}$
Minimum	0
Maximum	1
Harmonic Mean ($p = -1$)	0
Geometric Mean ($p \to 0$)	$\frac{1}{e}$
Arithmetic Mean ($p = 1$)	$\frac{1}{2}$
Root Mean Square ($p = 2$)	$\frac{1}{\sqrt{3}}$
Cubic Mean ($p = 3$)	$\frac{1}{\sqrt[3]{4}}$

Table 1: Summary of limit results for various means.

The solutions demonstrate the power of probabilistic methods (interpreting integrals as expectations) combined with fundamental convergence theorems (SLLN, Dominated Convergence

Theorem) to solve high-dimensional integration problems. The results show interesting patterns: the minimum and harmonic means converge to 0, the maximum converges to 1, while other means converge to intermediate values that depend on the parameter p.

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