HSC Math Extension 2: Vectors Mastery

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1 Introduction

1.1 Project Overview

This booklet compiles high quality vector problems curated specifically for the HSC Mathematics Extension 2 syllabus. The collection covers all essential topics in 3D vectors including geometric proofs, line equations, dot and cross products, spheres, planes, and distance calculations. Each problem is selected to challenge students while building toward exam readiness.

1.2 Target Audience

These problems are designed for Extension 2 students who want to master vectors through systematic practice. Part 1 provides detailed solutions showing complete algebraic steps, geometric intuition, and key takeaways. Part 2 offers hints and concise solutions to encourage independent problem-solving.

1.3 How to Use This Booklet

- Read the vectors primer below to review fundamental concepts and notation.
- Attempt problems in Part 1 independently; study the detailed solutions to understand model approaches.
- Use the upside-down hints in Part 2 only after making a genuine attempt.
- Focus on understanding geometric meaning alongside algebraic manipulation.
- Revisit challenging problems after a few days to reinforce techniques.

1.3.1 Notations

Throughout this booklet, we use the following standard mathematical notations:

Vectors:

- $\mathbf{v}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$ — vectors in bold font
- $\overrightarrow{AB}$ — directed line segment from point A to point B
- $\vec{AB}$ — alternative notation for directed line segment
- $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ — standard unit vectors along x, y, z axes
- $|\mathbf{v}|$ — magnitude (length) of vector $\mathbf{v}$
- $\hat{\mathbf{u}}$ — unit vector in the direction of $\mathbf{u}$

- $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ — column vector notation
- (x, y, z) — coordinate/component notation

Vector Operations:

- $\mathbf{a} \cdot \mathbf{b}$ — dot product (scalar product) of vectors **a** and **b**
- $\mathbf{a} \times \mathbf{b}$ — cross product (vector product) of vectors **a** and **b**
- $\mathbf{a} \perp \mathbf{b}$ — vectors **a** and **b** are perpendicular
- $\mathbf{a} \parallel \mathbf{b}$ — vectors **a** and **b** are parallel
- $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$ — projection of vector **a** onto vector **b**

Lines, Planes, and Spheres:

- $\mathbf{r}$ — position vector of a general point
- $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ — parametric equation of a line through **a** with direction **b**
- λ, μ, t — scalar parameters
- $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ — vector equation of a plane through **a** with normal **n**
- $|\mathbf{r} - \mathbf{c}| = R$ — vector equation of a sphere with center **c** and radius R
- $ax + by + cz = d$ — Cartesian equation of a plane
- $(x - h)^2 + (y - k)^2 + (z - \ell)^2 = R^2$ — Cartesian equation of a sphere

Angles and Geometry:

- θ — angle between vectors or geometric objects
- $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ — formula for angle between vectors
- $\angle ABC$ — angle at vertex B formed by rays BA and BC
- $\triangle ABC$ — triangle with vertices A, B, C
- $ABCD$ — quadrilateral with vertices listed in order

Other Symbols:

- $\mathbb{R}$ — the set of real numbers
- $\in$ — belongs to (element of)
- $\Leftrightarrow$ — if and only if (equivalence)
- $\Rightarrow$ — implies
- $\equiv$ — identically equal to
- $\approx$ — approximately equal to

1.4 Vectors Primer

1.4.1 Coordinates in 3D

In three dimensions, every point is described by an ordered triple (x, y, z), representing its position along the x, y, and z axes. This extension from 2D to 3D allows us to model space more realistically and solve geometric problems involving depth.

1.4.2 Vectors in Three Dimensions

A vector in 3D is an object with both magnitude and direction, written as $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ point along the x, y, and z axes, respectively, and form a basis for all vectors in space.

1.4.3 The Dot Product

The dot product (or scalar product) combines two vectors to produce a real number. It measures how much one vector extends in the direction of another and is fundamental for finding angles, projections, and testing perpendicularity in 3D geometry.

1.4.4 Applications of the Dot Product

The dot product is used to calculate angles between vectors, determine orthogonality, find projections, and solve geometric problems such as distances from points to planes or lines. It is also essential in physics for work and energy calculations.

1.4.5 Vector Proofs in Geometry

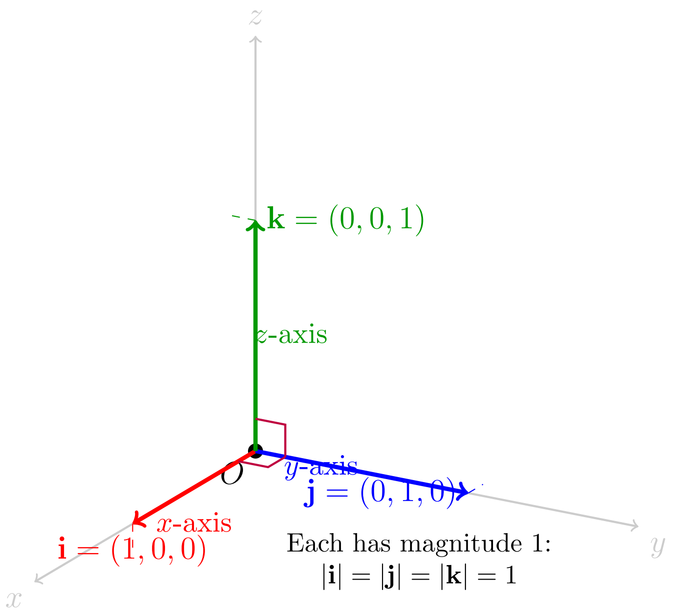
Vector methods provide elegant, coordinate-free proofs for geometric properties of shapes like triangles, parallelograms, and tetrahedrons. By expressing points and relationships with vectors, many results can be derived more simply than with traditional coordinate geometry.

1.4.6 Vector Equation of a Line

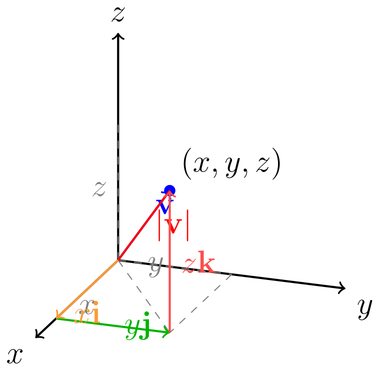
In 3D, the equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ describes a line through point $\mathbf{a}$ in the direction of vector $\mathbf{b}$. This form is more flexible than $y = mx + b$ and is well-suited for problems involving intersections and distances in space.

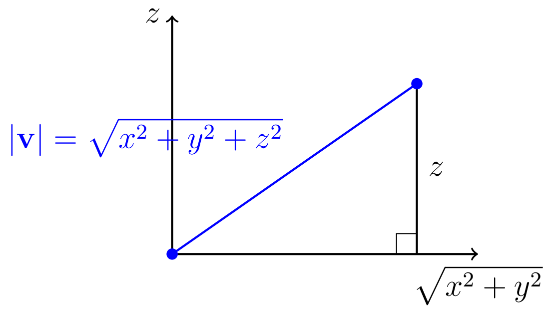
1.4.7 Vector Equations of Circles, Spheres and Planes

Circles and spheres can be described using vector equations such as $|\mathbf{r} - \mathbf{c}| = R$, where $\mathbf{c}$ is the center and R the radius. Planes are described by $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is a normal vector. These forms unify many geometric objects under a common vector framework.



Standard unit vectors: $\mathbf{i}$ (red), $\mathbf{j}$ (blue), and $\mathbf{k}$ (green) are mutually perpendicular unit vectors that form the basis of 3D coordinate space. Any vector can be written as a combination of these three directions.





Left: A vector $\mathbf{v}$ in 3D space decomposed into components $x\mathbf{i}$, $y\mathbf{j}$, and $z\mathbf{k}$. *Right:* The magnitude formula derived using Pythagoras' theorem twice.

1.4.8 The Dot Product (Scalar Product)

The dot product is the central algebraic tool for vectors in Extension 2.

- **Definition:** $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ and algebraically: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- **Perpendicular Vectors:** $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$
- **Angle Between Vectors:** $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
- **Scalar Projection:** $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$
- **Vector Projection:** $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\mathbf{b}$

Remark (Useful Identity for Vector Proofs):

A powerful formula relating the dot product and cross product is:

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

Derivation: Using $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$:

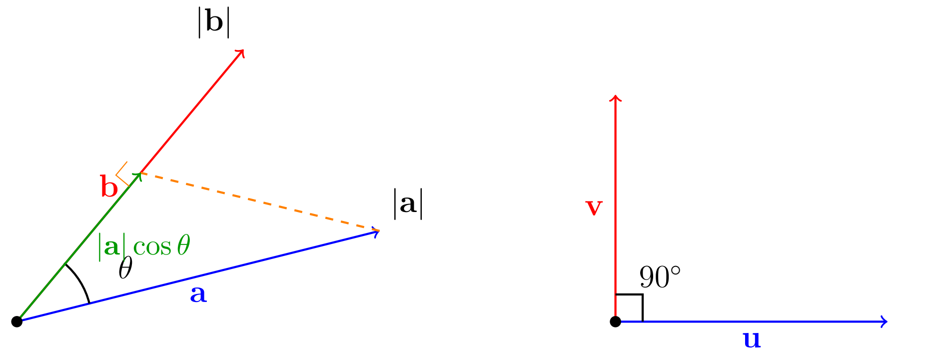
$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \quad (\text{using } \sin^2\theta + \cos^2\theta = 1) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2\theta \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \end{aligned}$$

Connection to Cauchy-Schwarz: This identity is equivalent to the Cauchy-Schwarz inequality in 3D:

$$(\mathbf{a} \cdot \mathbf{b})^2 \leq |\mathbf{a}|^2|\mathbf{b}|^2$$

Since $|\mathbf{a} \times \mathbf{b}|^2 \geq 0$, the identity immediately gives the Cauchy-Schwarz inequality. Equality holds when vectors are parallel (cross product is zero).

Applications: This formula is particularly useful in geometric proofs involving perpendicular distances, areas, and relationships between dot and cross products without explicitly computing angles.



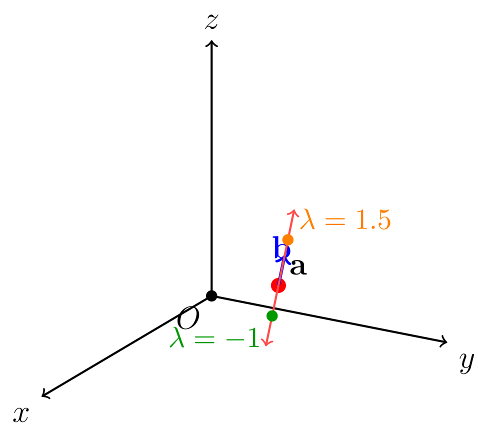
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ $\qquad$ $\mathbf{u} \cdot \mathbf{v} = 0$ (perpendicular)

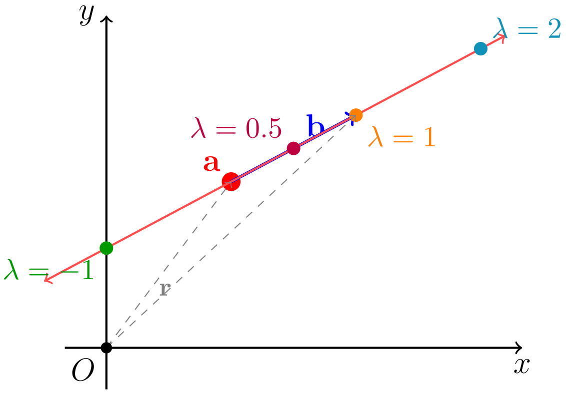
Left: The dot product $\mathbf{a} \cdot \mathbf{b}$ equals $|\mathbf{a}|$ times the projection of $\mathbf{b}$ onto $\mathbf{b}$ (or vice versa). *Right:* When vectors are perpendicular ($\theta = 90°$), the dot product is zero.

1.4.9 Vector Equation of a Line

Extension 2 uses vector form rather than $y = mx + b$.

- **Parametric Form:** $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where $\mathbf{a}$ is a point on the line, $\mathbf{b}$ is the direction vector, and λ is a parameter
- **Parallel Lines:** Direction vectors are scalar multiples
- **Skew Lines:** In 3D, lines can be non-parallel and non-intersecting

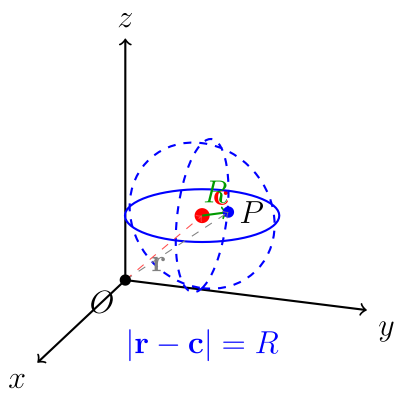


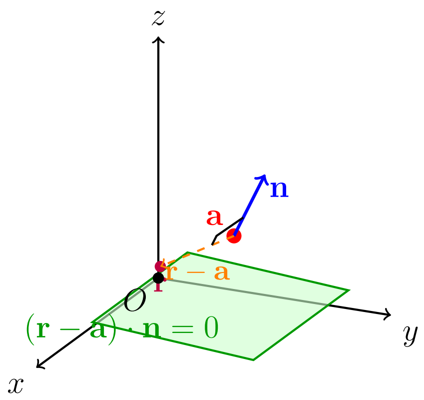


Left: A line in 3D space defined by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ is a fixed point and $\mathbf{b}$ is the direction vector. *Right:* Different values of λ generate different points along the line; $\lambda = 0$ gives point $\mathbf{a}$, positive λ extends forward, negative λ extends backward.

1.4.10 Spheres & Planes

- **Sphere:** Vector form $|\mathbf{r} - \mathbf{c}| = R$ or Cartesian $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$
- **Plane (Point-Normal):** $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ where $\mathbf{n}$ is normal to the plane
- **Plane (Cartesian):** $ax + by + cz = d$ where (a, b, c) is the normal vector





Left: A sphere centered at $\mathbf{c}$ with radius R. Any point $\mathbf{r}$ on the sphere satisfies $|\mathbf{r} - \mathbf{c}| = R$. *Right:* A plane containing point $\mathbf{a}$ with normal vector $\mathbf{n}$. Any point $\mathbf{r}$ on the plane has $(\mathbf{r} - \mathbf{a}) \perp \mathbf{n}$, so their dot product is zero.

1.4.11 Geometric Proofs

Use pure vector logic (without coordinates) to prove theorems about parallelograms, triangles, centroids, medians, and 3D shapes like tetrahedrons.

1.4.12 The Cross Product (Optional Advanced Technique)

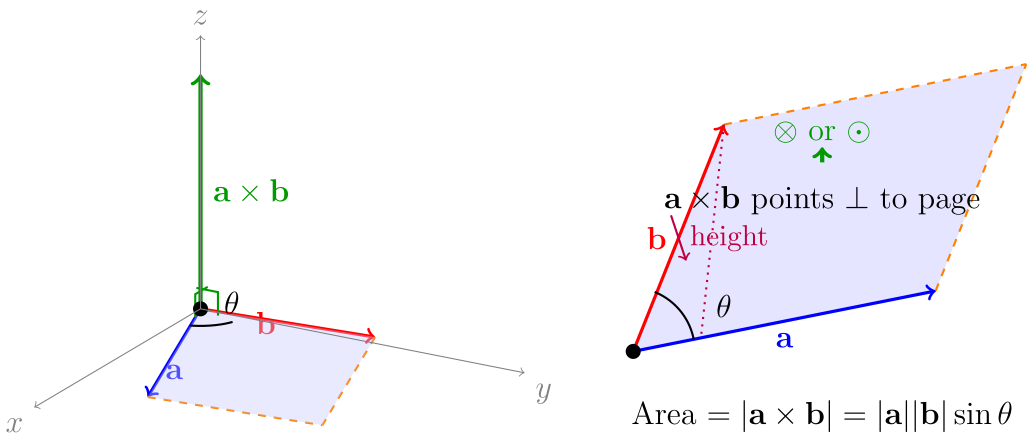
While not explicitly in the NESA syllabus, the cross product is a powerful tool for certain problems.

Definition: For $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Properties:

- $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ (area of parallelogram)
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (anticommutative)



Left: The cross product $\mathbf{a} \times \mathbf{b}$ (green) is perpendicular to both $\mathbf{a}$ (blue) and $\mathbf{b}$ (red), using the right-hand rule for direction. *Right:* The magnitude $|\mathbf{a} \times \mathbf{b}|$ equals the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.

Worked Example: Distance from Point to Line

Problem: Find the distance from point $P(1, 2, 0)$ to the line $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Solution using Cross Product: Let $A(1, 0, 1)$ be a point on the line and $\mathbf{d} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ be the direction vector.

The vector from A to P is $\vec{AP} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.

Using the cross product:

$$\vec{AP} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

Distance: $d = \frac{|\vec{AP} \times \mathbf{d}|}{|\mathbf{d}|} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$ units.

Comparison with Dot Product Method: The dot product method uses projection. The component of $\vec{AP}$ perpendicular to the line is found by:

$$\vec{AP}_\perp = \vec{AP} - \text{proj}_{\mathbf{d}}\vec{AP} = \vec{AP} - \frac{\vec{AP} \cdot \mathbf{d}}{\mathbf{d} \cdot \mathbf{d}}\mathbf{d}$$

Then $d = |\vec{AP}_\perp|$. Both methods yield the same result, but cross product is often more direct for distance calculations.

2 Part 1: Problems and Solutions (Detailed)

Part 1 contains 15 carefully selected problems covering all 10 topic areas. Each problem includes complete solutions with all algebraic steps, geometric diagrams where helpful, and key takeaways. Solutions use compact notation to fit on one A4 page while maintaining clarity.

2.1 Part 1 Basic Problems (Easy)

Problem 2.1: Vector Projection Formula

The vector $\mathbf{a}$ is $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and the vector $\mathbf{b}$ is $\begin{pmatrix}2\\0\\-4\end{pmatrix}$.

(i) Find $\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$.

(ii) Show that $\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$ is perpendicular to $\mathbf{b}$.

Solution 2.1

Given: $\mathbf{a} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}2\\0\\-4\end{pmatrix}$

(i) Calculate the dot products:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (1)(2) + (2)(0) + (3)(-4) = 2 - 12 = -10\\ \mathbf{b}\cdot\mathbf{b} &= 4 + 0 + 16 = 20\end{aligned}$$

Thus:

$$\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \frac{-10}{20}\begin{pmatrix}2\\0\\-4\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}2\\0\\-4\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

(ii) Let $\mathbf{v} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$. Then:

$$\mathbf{v} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\2\\1\end{pmatrix}$$

Check perpendicularity: $\mathbf{v}\cdot\mathbf{b} = (2)(2)+(2)(0)+(1)(-4) = 4-4 = 0 \quad \therefore$ perpendicular.

Takeaways 2.1

The expression $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$ is the **vector projection** of $\mathbf{a}$ onto $\mathbf{b}$. The remainder $\mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}$ is the component perpendicular to $\mathbf{b}$. This orthogonal decomposition is fundamental: any vector can be split into parallel and perpendicular components relative to another vector. Always verify perpendicularity by checking that the dot product equals zero.

Problem 2.2: Shortest Distance from Point to Line

$\mathbf{r}_1$ and $\mathbf{r}_2$ are two lines with vector equations:

$\mathbf{r}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix}2\\0\\0\end{pmatrix} + \mu\begin{pmatrix}1\\3\\2\end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$

(i) Show that these two lines intersect.

(ii) Find the angle between the lines.

(iii) Find the shortest distance from point $P(1, 2, 0)$ to line $\mathbf{r}_1$.

Solution 2.2

(i) Equate components: $1 = 2 + \mu$ gives $\mu = -1$. Then $\lambda = 3\mu = -3$. Check z: $1 + \lambda = 1 - 3 = -2$ and $2\mu = -2$. Consistent. Lines intersect at $(1, -3, -2)$.

(ii) Direction vectors: $\mathbf{d}_1 = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix}1\\3\\2\end{pmatrix}$

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = 0 + 3 + 2 = 5$$

$$|\mathbf{d}_1| = \sqrt{2}, \quad |\mathbf{d}_2| = \sqrt{1 + 9 + 4} = \sqrt{14}$$

$$\cos\theta = \tfrac{5}{\sqrt{2}\sqrt{14}} = \tfrac{5}{2\sqrt{7}} \implies \theta \approx 19.1^\circ$$

(iii) Let $A(1, 0, 1)$ be on line $\mathbf{r}_1$, direction $\mathbf{d} = \begin{pmatrix}0\\1\\1\end{pmatrix}$.

$$\vec{AP} = \begin{pmatrix}0\\2\\-1\end{pmatrix}$$

Cross product: $\vec{AP} \times \mathbf{d} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 2 & -1\\ 0 & 1 & 1\end{vmatrix} = \mathbf{i}(2 + 1) = 3\mathbf{i}$

$$D = \tfrac{|\vec{AP} \times \mathbf{d}|}{|\mathbf{d}|} = \tfrac{3}{\sqrt{2}} = \tfrac{3\sqrt{2}}{2} \text{ units}$$

Takeaways 2.2

To find line intersection in 3D: equate components and solve the system (3 equations, 2 unknowns). Consistent solution means intersection; inconsistent means skew lines. The **cross product method** for point-to-line distance is efficient: $d = \frac{|\vec{AP} \times \mathbf{d}|}{|\mathbf{d}|}$ where A is any point on the line. This uses the geometric interpretation that $|\vec{AP} \times \mathbf{d}|$ equals the area of a parallelogram with base $|\mathbf{d}|$ and height d.

Problem 2.3: Parallelogram Area via Cross Product

The adjacent sides of a parallelogram are represented by vectors $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Show that the area of the parallelogram is $6\sqrt{10}$ square units.

Solution 2.3

Area of parallelogram $= |\mathbf{a} \times \mathbf{b}|$. Calculate the cross product:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} \\ &= \mathbf{i}(3 \cdot 3 - (-1)(-1)) - \mathbf{j}(4 \cdot 3 - (-1)(2)) + \mathbf{k}(4(-1) - 3 \cdot 2) \\ &= \mathbf{i}(9 - 1) - \mathbf{j}(12 + 2) + \mathbf{k}(-4 - 6) \\ &= 8\mathbf{i} - 14\mathbf{j} - 10\mathbf{k}\end{aligned}$$

Magnitude:

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{64 + 196 + 100} = \sqrt{360} = \sqrt{36 \cdot 10} = 6\sqrt{10}$$

Takeaways 2.3

The **cross product** $\mathbf{a} \times \mathbf{b}$ produces a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, with magnitude equal to the area of the parallelogram they span. Key properties: (1) anti-commutative: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$; (2) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$. For area calculations, only the magnitude matters. Remember the determinant pattern for 3×3 cross products.

Problem 2.4: Cosine Difference Formula Proof

Let $\mathbf{a}$ and $\mathbf{b}$ be 2-dimensional unit vectors, inclined to the x-axis at angles α and β respectively. You may assume $\mathbf{a} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$ and $\mathbf{b} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$. Prove that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Solution 2.4

Method 1: Geometric dot product
The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \alpha - \beta$. Since both are unit vectors:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = (1)(1)\cos(\alpha - \beta) = \cos(\alpha - \beta)$$

Method 2: Algebraic dot product

$$\mathbf{a} \cdot \mathbf{b} = (\cos\alpha)(\cos\beta) + (\sin\alpha)(\sin\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Equating the two expressions: $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad \square$

Takeaways 2.4

This elegant proof demonstrates the power of vectors in deriving trigonometric identities. The key insight: the dot product has both a **geometric definition** ($|\mathbf{a}||\mathbf{b}|\cos\theta$) and an **algebraic definition** (sum of component products). Equating these yields the cosine difference formula. Similar approaches can derive $\cos(\alpha + \beta)$, $\sin(\alpha \pm \beta)$, and other compound angle formulas. This vector method is often cleaner than traditional geometric proofs.

Problem 2.5: Perpendicular Vectors Condition

Consider two vectors $\mathbf{u} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = p\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. For what values of p are $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ perpendicular?

Solution 2.5

First compute $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$:

$$\mathbf{u} - \mathbf{v} = (-2 - p)\mathbf{i} + (-1 - 1)\mathbf{j} + (3 - 2)\mathbf{k} = (-2 - p)\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{u} + \mathbf{v} = (-2 + p)\mathbf{i} + (-1 + 1)\mathbf{j} + (3 + 2)\mathbf{k} = (-2 + p)\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}$$

For perpendicularity, their dot product must equal zero:

$$\begin{aligned}
(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= 0 \\
(-2 - p)(-2 + p) + (-2)(0) + (1)(5) &= 0 \\
4 - p^2 + 5 &= 0 \\
9 - p^2 &= 0 \\
p^2 &= 9 \\
p &= \pm 3
\end{aligned}$$

Takeaways 2.5

Perpendicularity problems always reduce to setting the dot product equal to zero. This problem illustrates the algebraic identity: $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2$ (vector form of difference of squares). The general principle: two vectors are perpendicular if and only if the sum of products of corresponding components equals zero. When the problem involves a parameter, solve the resulting equation carefully—don't forget both positive and negative solutions for squared terms.

2.2 Part 1 Medium Problems

Problem 2.6: Line Tangent to Sphere

A sphere has centre at $(3, -3, 4)$ and radius 6 units.

A line has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$.

(i) Write down the vector equation of the sphere.

(ii) Determine whether the line is a tangent to the sphere, clearly justifying your conclusion.

Solution 2.6

(i) Let centre $\mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}$ and radius $R = 6$. Vector equation:

$$\left| \mathbf{r} - \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} \right| = 6$$

(ii) Substitute line into sphere. Cartesian form: $(x-3)^2 + (y+3)^2 + (z-4)^2 = 36$. Parametric line: $x = 1 + 2\lambda$, $y = 5 - \lambda$, $z = 4 - \lambda$.

$$\begin{aligned} (1 + 2\lambda - 3)^2 + (5 - \lambda + 3)^2 + (4 - \lambda - 4)^2 &= 36 \\ (2\lambda - 2)^2 + (8 - \lambda)^2 + (-\lambda)^2 &= 36 \\ 4\lambda^2 - 8\lambda + 4 + 64 - 16\lambda + \lambda^2 + \lambda^2 &= 36 \\ 6\lambda^2 - 24\lambda + 68 &= 36 \\ 6\lambda^2 - 24\lambda + 32 &= 0 \\ 3\lambda^2 - 12\lambda + 16 &= 0 \end{aligned}$$

Discriminant: $\Delta = (-12)^2 - 4(3)(16) = 144 - 192 = -48 < 0$
No real solutions $\implies$ **line does NOT intersect sphere** (not a tangent).

Takeaways 2.6

For line-sphere intersection: substitute parametric line equations into sphere equation to get a quadratic in λ. The discriminant determines the nature: $\Delta > 0$ (2 intersections, secant), $\Delta = 0$ (1 intersection, tangent), $\Delta < 0$ (no intersection). A tangent touches the sphere at exactly one point, requiring distance from centre to line equals radius. Always check discriminant rather than just counting solutions.

Problem 2.7: Perpendicular Lines and Plane Equation

Consider lines $L_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(i) Show that L_1 and L_2 intersect and are perpendicular, stating the point of intersection.

(ii) Deduce that the plane containing both lines has equation $y + z = 1$.

(iii) Find the perpendicular distance from the origin to this plane.

Solution 2.7

(i) Directions: $\mathbf{d}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Perpendicular: $\mathbf{d}_1 \cdot \mathbf{d}_2 = 2 - 1 - 1 = 0$ ✓

Intersection: Equate $(3 + 2\lambda, 2 - \lambda, -1 + \lambda) = (-1 + \mu, 1 + \mu, -\mu)$

$$\begin{aligned} x : 3 + 2\lambda = -1 + \mu &\implies \mu - 2\lambda = 4 \quad (1) \\ y : 2 - \lambda = 1 + \mu &\implies \mu + \lambda = 1 \quad (2) \\ z : -1 + \lambda = -\mu &\implies \mu + \lambda = 1 \quad (3) \end{aligned}$$

(1) − (2): $-3\lambda = 3 \implies \lambda = -1$, so $\mu = 2$. Point: $(3 - 2, 2 + 1, -1 - 1) = (1, 3, -2)$ ✓

(ii) Plane through $(1, 3, -2)$ spanned by $\mathbf{d}_1, \mathbf{d}_2$. Normal vector:

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 0\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

Simplified: $\mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Plane: $0x + y + z = k$. Using $(1, 3, -2)$: $3 + (-2) = 1 \implies k = 1$.

Thus $y + z = 1$.

(iii) Distance from origin $(0, 0, 0)$ to plane $y + z - 1 = 0$:

$$D = \frac{|0+0-1|}{\sqrt{0^2+1^2+1^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \text{ units}$$

Takeaways 2.7

When two lines intersect, they determine a unique plane. To find the plane's Cartesian equation: (1) verify intersection, (2) compute normal via cross product of direction vectors, (3) use point-normal form. The cross product automatically gives a perpendicular vector. For distance from point (x_0, y_0, z_0) to plane $Ax + By + Cz + D = 0$: $d = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$. Memorize this formula.

Problem 2.8: Three Conditions on Vectors

Which of the following is a true statement about the lines
$\ell_1 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ and $\ell_2 = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$?

A. ℓ_1 and ℓ_2 are the same line.

B. ℓ_1 and ℓ_2 are not parallel and they intersect.

C. ℓ_1 and ℓ_2 are parallel and they do not intersect.

D. ℓ_1 and ℓ_2 are not parallel and they do not intersect.

Solution 2.8

Direction vectors: $\mathbf{d}_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$

Step 1: Check parallelism

$$\mathbf{d}_2 = -1 \cdot \mathbf{d}_1$$

Lines are **parallel** (eliminates B and D).
Step 2: Check if coincident Test if point $P_1 = (-1, 2, 5)$ from ℓ_1 lies on ℓ_2:

$$\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

Component equations:

$$\begin{aligned} x : -1 = 3 + \mu &\implies \mu = -4 \\ y : 2 = -10 - 3\mu &\implies \mu = -4 \\ z : 5 = 1 - \mu &\implies \mu = -4 \end{aligned}$$

Consistent value $\mu = -4$ for all components $\implies$ point lies on ℓ_2.
Answer: A (same line).

Takeaways 2.8

For two lines to be identical, two conditions must hold: (1) direction vectors are scalar multiples (parallel), and (2) they share at least one common point. If parallel but don't share a point, they're distinct parallel lines. In 3D, non-parallel lines can be skew (no intersection). The systematic approach: first check parallelism via direction vectors, then test a point from one line on the other. This problem reinforces the distinction between "parallel" and "coincident."

Problem 2.9: Projectile Motion Vector Proof

A particle is projected from the origin with initial velocity u m/s at angle θ to the horizontal. The acceleration vector is $\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$.

(i) Show that the position vector is $\mathbf{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$.

(ii) Show that the Cartesian equation of the path is $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$.

(iii) Given $u^2 > gR$, prove there are two distinct values of θ for which the particle lands at $x = R$.

Solution 2.9

(i) Initial velocity: $\mathbf{v}_0 = u\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. Integrate acceleration:

$$\mathbf{v}(t) = \mathbf{v}_0 + \int \mathbf{a}\,dt = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$$

Integrate velocity (starting from origin):

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$

(ii) From $x = ut\cos\theta$, we get $t = \frac{x}{u\cos\theta}$. Substitute into y:

$$\begin{aligned} y &= u\sin\theta \cdot \tfrac{x}{u\cos\theta} - \tfrac{1}{2}g\left(\tfrac{x}{u\cos\theta}\right)^2 \\ &= x\tan\theta - \tfrac{gx^2}{2u^2\cos^2\theta} \end{aligned}$$

(iii) At landing, $y = 0$ and $x = R$:

$$0 = R\tan\theta - \frac{gR^2}{2u^2\cos^2\theta}$$

Multiply by $\cos^2\theta$: $0 = R\sin\theta\cos\theta - \frac{gR^2}{2u^2}$
Using $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$\sin(2\theta) = \frac{gR}{u^2}$$

Since $u^2 > gR$, we have $\frac{gR}{u^2} < 1$. Thus 2θ has two solutions in $[0, 180^\circ]$: one acute, one obtuse. This gives two distinct values of θ in $[0, 90^\circ]$.

Takeaways 2.9

Projectile motion problems integrate naturally with vectors: acceleration $\to$ velocity $\to$ position. The key insight for part (iii): for a given range R, there are two launch angles (complementary angles to 45°) that achieve the same horizontal distance, provided the initial speed is sufficient. The condition $u^2 > gR$ ensures the target is within reach. The identity $\sin(2\theta) = 2\sin\theta\cos\theta$ is crucial for converting between different forms.

Problem 2.10: Distance from Point to Line and Sphere

Consider point B with position vector $\mathbf{b}$ and line $\ell : \mathbf{a} + \lambda\mathbf{d}$, where $|\mathbf{d}| = 1$ and λ is a parameter. Let $f(\lambda)$ be the distance between a point on ℓ and point B.

(i) Find λ_0, the value of λ that minimises f, in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{d}$.

(ii) Let P be the point with position vector $\mathbf{a} + \lambda_0\mathbf{d}$. Show that PB is perpendicular to the direction of ℓ.

(iii) Hence find the shortest distance between line ℓ and the sphere of radius 1 centred at origin O, in terms of $\mathbf{d}$ and $\mathbf{a}$.

Solution 2.10

(i) Minimize $S(\lambda) = f(\lambda)^2 = |(\mathbf{a} + \lambda\mathbf{d}) - \mathbf{b}|^2$:

$$\begin{aligned} S(\lambda) &= |\mathbf{a} - \mathbf{b}|^2 + 2\lambda(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} + \lambda^2|\mathbf{d}|^2 \\ \tfrac{dS}{d\lambda} &= 2(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} + 2\lambda = 0 \\ \lambda_0 &= -(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{d} \end{aligned}$$

(ii) $\vec{PB} = \mathbf{b} - (\mathbf{a} + \lambda_0\mathbf{d}) = (\mathbf{b} - \mathbf{a}) - \lambda_0\mathbf{d}$ Check: $\vec{PB} \cdot \mathbf{d} = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{d} - \lambda_0(\mathbf{d} \cdot \mathbf{d}) = \lambda_0 - \lambda_0(1) = 0$ ✓

(iii) For origin (set $\mathbf{b} = \mathbf{0}$): $\lambda_{min} = -\mathbf{a} \cdot \mathbf{d}$. Point on line closest to O: $\mathbf{p} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{d})\mathbf{d}$

$$|\mathbf{p}|^2 = |\mathbf{a}|^2 - 2(\mathbf{a} \cdot \mathbf{d})^2 + (\mathbf{a} \cdot \mathbf{d})^2 = |\mathbf{a}|^2 - (\mathbf{a} \cdot \mathbf{d})^2$$

Shortest distance to sphere: $\sqrt{|\mathbf{a}|^2 - (\mathbf{a} \cdot \mathbf{d})^2} - 1$ or equivalently $|\mathbf{a} \times \mathbf{d}| - 1$.

Takeaways 2.10

Point-to-line distance problems use calculus (minimize squared distance) or projection (subtract parallel component). The perpendicularity in part (ii) confirms the minimum—the shortest path is always perpendicular to the line. For sphere-line distance, first find line-to-centre distance, then subtract radius. The identity $|\mathbf{a} - (\mathbf{a} \cdot \mathbf{d})\mathbf{d}|^2 = |\mathbf{a}|^2 - (\mathbf{a} \cdot \mathbf{d})^2$ (when $|\mathbf{d}| = 1$) represents the perpendicular component of $\mathbf{a}$ to the line, which can also be expressed as $|\mathbf{a} \times \mathbf{d}|^2$ using the cross product.

2.3 Part 1 Advanced Problems (Hard)

Problem 2.11: Position Vector Ratios and Parallelogram Division

Point C divides interval AB so that $\frac{CB}{AC} = \frac{m}{n}$. Position vectors of A and B are $\mathbf{a}, \mathbf{b}$.

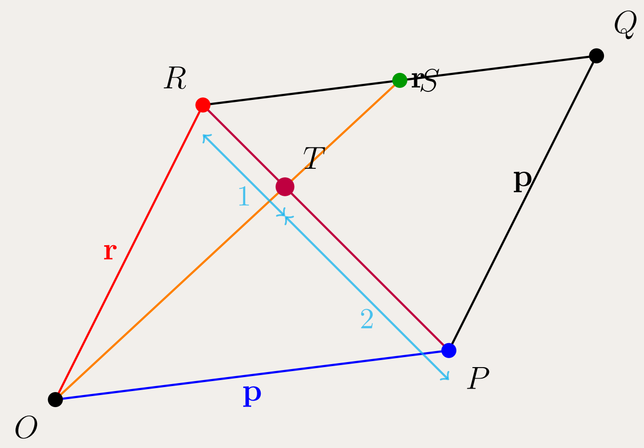
(i) Show that $\vec{AC} = \frac{n}{m+n}(\mathbf{b} - \mathbf{a})$.

(ii) Prove that $\vec{OC} = \frac{m}{m+n}\mathbf{a} + \frac{n}{m+n}\mathbf{b}$.

Let $OPQR$ be a parallelogram with $\vec{OP} = \mathbf{p}$, $\vec{OR} = \mathbf{r}$. S is the midpoint of QR, T is the intersection of PR and OS.

(iii) Show that $\vec{OT} = \frac{2}{3}\mathbf{r} + \frac{1}{3}\mathbf{p}$.

(iv) Prove that T divides PR in the ratio $2:1$.



Parallelogram $OPQR$ with $\vec{OP} = \mathbf{p}$, $\vec{OR} = \mathbf{r}$. Point S is the midpoint of QR, and T is where diagonal PR intersects OS, dividing PR in ratio $2:1$.

Solution 2.11

(i) Given $\frac{CB}{AC} = \frac{m}{n}$, let $\vec{AC} = k(\mathbf{b} - \mathbf{a})$ for some k. Then $C = A + k(\mathbf{b} - \mathbf{a})$. Since $\vec{CB} = \mathbf{b} - \mathbf{c} = (1-k)(\mathbf{b} - \mathbf{a})$ and $\vec{AC} = k(\mathbf{b} - \mathbf{a})$:

$$\frac{CB}{AC} = \frac{1-k}{k} = \frac{m}{n} \implies n(1-k) = mk \implies n = k(m+n) \implies k = \frac{n}{m+n}$$

Thus $\vec{AC} = \frac{n}{m+n}(\mathbf{b} - \mathbf{a})$.

(ii) $\vec{OC} = \vec{OA} + \vec{AC} = \mathbf{a} + \frac{n}{m+n}(\mathbf{b} - \mathbf{a}) = \mathbf{a}\left(1 - \frac{n}{m+n}\right) + \frac{n}{m+n}\mathbf{b} = \frac{m}{m+n}\mathbf{a} + \frac{n}{m+n}\mathbf{b}$

(iii) In parallelogram $OPQR$: $\vec{OQ} = \mathbf{p} + \mathbf{r}$. S is midpoint of QR:

$$\vec{OS} = \frac{1}{2}(\vec{OQ} + \vec{OR}) = \frac{1}{2}(\mathbf{p} + \mathbf{r} + \mathbf{r}) = \frac{1}{2}\mathbf{p} + \mathbf{r}$$

T lies on OS: $\vec{OT} = s(\frac{1}{2}\mathbf{p} + \mathbf{r})$ for some s. T also lies on PR. Since $\vec{OP} = \mathbf{p}$ and $\vec{OR} = \mathbf{r}$:

$$\vec{OT} = (1-t)\mathbf{p} + t\mathbf{r}$$

for some $t \in [0,1]$. Equating: $s \cdot \frac{1}{2}\mathbf{p} + s\mathbf{r} = (1-t)\mathbf{p} + t\mathbf{r}$
Since $\mathbf{p}, \mathbf{r}$ are independent: $\frac{s}{2} = 1 - t$ and $s = t$. From second: $s = t$. Substitute into first: $\frac{s}{2} = 1 - s \implies \frac{3s}{2} = 1 \implies s = \frac{2}{3}$. Thus: $\vec{OT} = \frac{2}{3}(\frac{1}{2}\mathbf{p} + \mathbf{r}) = \frac{1}{3}\mathbf{p} + \frac{2}{3}\mathbf{r}$ (or equivalently $\frac{2}{3}\mathbf{r} + \frac{1}{3}\mathbf{p}$).

(iv) From part (iii), $t = \frac{2}{3}$ means T divides PR such that $\vec{PT} = \frac{2}{3}\vec{PR}$, so $PT : TR = 2 : 1$.

Takeaways 2.11

The **section formula** $\vec{OC} = \frac{m\mathbf{a}+n\mathbf{b}}{m+n}$ (when C divides AB in ratio $n : m$ from A) is fundamental for position vectors. For geometric proofs: (1) express target point in terms of base vectors using two different paths, (2) equate and solve using linear independence. Parts (iii)-(iv) show how medians and diagonals in parallelograms create consistent ratios—here T is the centroid-like point dividing both segments in ratio $2 : 1$.

Problem 2.12: Triangle Inequality and Cauchy-Schwarz on Sphere

(i) Point $P(x, y, z)$ lies on the unit sphere centred at origin O. Using the triangle inequality, show that $|x| + |y| + |z| \geq 1$.

(ii) Given vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, show that $|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$.

(iii) As in part (i), point $P(x, y, z)$ lies on the unit sphere. Using part (ii), show that $|x| + |y| + |z| \leq \sqrt{3}$.

Solution 2.12

(i) Position vector: $\vec{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with $|\vec{OP}| = 1$.
Triangle inequality: $|\mathbf{u} + \mathbf{v} + \mathbf{w}| \leq |\mathbf{u}| + |\mathbf{v}| + |\mathbf{w}|$

$$1 = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| \leq |x\mathbf{i}| + |y\mathbf{j}| + |z\mathbf{k}| = |x| + |y| + |z|$$

(ii) Cauchy-Schwarz from dot product: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ Taking absolute value: $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta| \leq |\mathbf{a}||\mathbf{b}|$ (since $|\cos\theta| \leq 1$). In components: $|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$

(iii) Let $\mathbf{a} = \begin{pmatrix} |x| \\ |y| \\ |z| \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Apply Cauchy-Schwarz:

$$|x| + |y| + |z| = |x|(1) + |y|(1) + |z|(1) \leq \sqrt{x^2 + y^2 + z^2}\sqrt{1^2 + 1^2 + 1^2}$$

Since P is on unit sphere: $x^2 + y^2 + z^2 = 1$. Thus:

$$|x| + |y| + |z| \leq \sqrt{1} \cdot \sqrt{3} = \sqrt{3}$$

Takeaways 2.12

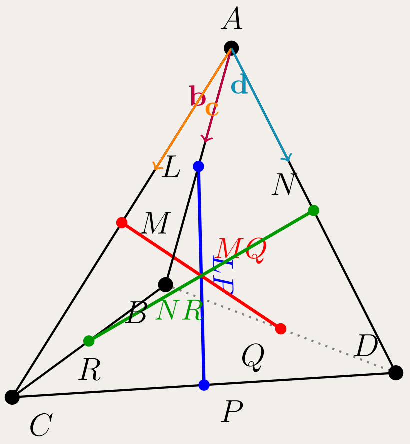
This problem demonstrates two fundamental inequalities. The **triangle inequality** $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ gives lower bounds. The **Cauchy-Schwarz inequality** $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ gives upper bounds. Both arise from the dot product definition. Part (iii) uses a clever choice of vectors to convert a sum constraint into a dot product. Combined: $1 \leq |x| + |y| + |z| \leq \sqrt{3}$ for points on the unit sphere—tight bounds achieved at corners of inscribed cube.

Problem 2.13: Tetrahedron Bimedians Equality

On triangular pyramid $ABCD$, L, M, N, P, Q, R are midpoints of edges AB, AC, AD, CD, BD, BC respectively. Let $\mathbf{b} = \vec{AB}$, $\mathbf{c} = \vec{AC}$, $\mathbf{d} = \vec{AD}$.

(i) Show that $\vec{LP} = \frac{1}{2}(-\mathbf{b} + \mathbf{c} + \mathbf{d})$.

(ii) Given that $\vec{MQ} = \frac{1}{2}(\mathbf{b} - \mathbf{c} + \mathbf{d})$ and $\vec{NR} = \frac{1}{2}(\mathbf{b} + \mathbf{c} - \mathbf{d})$, prove that $|AB|^2 + |AC|^2 + |AD|^2 + |BC|^2 + |BD|^2 + |CD|^2 = 4(|LP|^2 + |MQ|^2 + |NR|^2)$.



Tetrahedron $ABCD$ with midpoints: L (of AB), M (of AC), N (of AD), P (of CD), Q (of BD), R (of BC). The three bimedians LP, MQ, NR connect midpoints of opposite edges.

Solution 2.13

(i) L is midpoint of AB: $\vec{AL} = \frac{1}{2}\mathbf{b}$. P is midpoint of CD: $\vec{AP} = \frac{1}{2}(\vec{AC}+\vec{AD}) = \frac{1}{2}(\mathbf{c}+\mathbf{d})$.

$$\vec{LP} = \vec{AP} - \vec{AL} = \tfrac{1}{2}(\mathbf{c}+\mathbf{d}) - \tfrac{1}{2}\mathbf{b} = \tfrac{1}{2}(-\mathbf{b}+\mathbf{c}+\mathbf{d})$$

(ii) LHS: Sum of squared edge lengths.

$$\begin{aligned}\text{LHS} &= |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 + |\mathbf{c}-\mathbf{b}|^2 + |\mathbf{d}-\mathbf{b}|^2 + |\mathbf{d}-\mathbf{c}|^2 \\ &= b^2 + c^2 + d^2 + (c^2 + b^2 - 2\mathbf{b}\cdot\mathbf{c}) + (d^2 + b^2 - 2\mathbf{b}\cdot\mathbf{d}) \\ &\quad + (d^2 + c^2 - 2\mathbf{c}\cdot\mathbf{d}) \\ &= 3(b^2 + c^2 + d^2) - 2(\mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d})\end{aligned}$$

RHS: Calculate $4|LP|^2$:

$$\begin{aligned}|LP|^2 &= \tfrac{1}{4}(-\mathbf{b}+\mathbf{c}+\mathbf{d})\cdot(-\mathbf{b}+\mathbf{c}+\mathbf{d}) \\ &= \tfrac{1}{4}(b^2 + c^2 + d^2 - 2\mathbf{b}\cdot\mathbf{c} - 2\mathbf{b}\cdot\mathbf{d} + 2\mathbf{c}\cdot\mathbf{d}) \\ 4|LP|^2 &= b^2 + c^2 + d^2 - 2\mathbf{b}\cdot\mathbf{c} - 2\mathbf{b}\cdot\mathbf{d} + 2\mathbf{c}\cdot\mathbf{d}\end{aligned}$$

Similarly:

$$\begin{aligned}4|MQ|^2 &= b^2 + c^2 + d^2 - 2\mathbf{b}\cdot\mathbf{c} + 2\mathbf{b}\cdot\mathbf{d} - 2\mathbf{c}\cdot\mathbf{d} \\ 4|NR|^2 &= b^2 + c^2 + d^2 + 2\mathbf{b}\cdot\mathbf{c} - 2\mathbf{b}\cdot\mathbf{d} - 2\mathbf{c}\cdot\mathbf{d}\end{aligned}$$

Sum:

$$\begin{aligned}\text{RHS} &= 4(|LP|^2 + |MQ|^2 + |NR|^2) \\ &= 3(b^2 + c^2 + d^2) + (-2-2+2)\mathbf{b}\cdot\mathbf{c} + (-2+2-2)\mathbf{b}\cdot\mathbf{d} \\ &\quad + (2-2-2)\mathbf{c}\cdot\mathbf{d} \\ &= 3(b^2 + c^2 + d^2) - 2(\mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d}) = \text{LHS} \quad \square\end{aligned}$$

Takeaways 2.13

Bimedians (segments joining midpoints of opposite edges) in a tetrahedron have remarkable properties. This identity relates all six edge lengths to the three bimedian lengths—a 3D analogue of the parallelogram law. The result is known as **Commandino's theorem** (16th century): the sum of squares of all edges equals four times the sum of squares of the three bimedians.

The proof strategy: express everything in terms of base vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}$, expand dot products carefully, and verify algebraic cancellation. Note the symmetry in coefficients $(+2, -2, -2)$ cycling through the three cross terms.

Special case: When $D = A$ (degenerate tetrahedron), $ABCD$ collapses to triangle ABC. Then $\mathbf{d} = \mathbf{0}$, and the identity reduces to: $|AB|^2 + |AC|^2 + |BC|^2 = 4(|LP|^2 + |MQ|^2 + |NR|^2)$, which becomes the triangle median formula: the sum of squares of the sides equals $\frac{4}{3}$ times the sum of squares of the medians. Such identities are useful in crystallography and structural analysis.

Problem 2.14: Point-to-Line Distance and Sphere Intersection

Consider the line ℓ with equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(i) Find the perpendicular distance from point $P(1, 2, 0)$ to the line ℓ.

(ii) Find the shortest distance from the origin to the line ℓ.

(iii) Determine the points where line ℓ intersects the sphere $x^2 + y^2 + z^2 = 4$.

Solution 2.14

(i) Let $A(1, 0, 1)$ be on ℓ, direction $\mathbf{d} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\vec{AP} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

Cross product:

$$\vec{AP} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}(2+1) = 3\mathbf{i}$$

Distance: $d = \frac{|\vec{AP} \times \mathbf{d}|}{|\mathbf{d}|} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$ units.

(ii) Use projection method. Let $\vec{OA} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Projection of $\vec{OA}$ onto $\mathbf{d}$: $\text{proj}_{\mathbf{d}} \vec{OA} = \frac{\vec{OA} \cdot \mathbf{d}}{|\mathbf{d}|^2} \mathbf{d} = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Perpendicular component: $\vec{OA}_{\perp} = \vec{OA} - \text{proj}_{\mathbf{d}} \vec{OA} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

Distance: $|\vec{OA}_{\perp}| = \sqrt{1 + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$ units.

(iii) Substitute line into sphere: $x = 1$, $y = \lambda$, $z = 1 + \lambda$.

$$1 + \lambda^2 + (1 + \lambda)^2 = 4$$

$$1 + \lambda^2 + 1 + 2\lambda + \lambda^2 = 4$$

$$2\lambda^2 + 2\lambda - 2 = 0 \implies \lambda^2 + \lambda - 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

Points: For $\lambda_1 = \frac{-1+\sqrt{5}}{2}$: $(1, \frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$

For $\lambda_2 = \frac{-1-\sqrt{5}}{2}$: $(1, \frac{-1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$

Takeaways 2.14

Three key distance techniques demonstrated: (1) **cross product** for point-to-line (geometric), (2) **projection** for point-to-line (algebraic), (3) **substitution** for line-sphere intersection. Both methods in (i)-(ii) should give same result (verify as practice). For intersections, substitute parametric equations into the surface equation to get a quadratic—the number of real solutions indicates the geometric relationship. The golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ appearing in λ is a nice coincidence!

3 Part 2: Problems and Solutions (Concise + Hints)

Part 2 contains 45 additional problems distributed across three difficulty levels. Each problem includes an upside-down hint and a concise solution sketch to encourage independent thinking.

3.1 Part 2 Basic Problems

Problem 3.1: Vector to Cartesian Line Equation

What is the Cartesian equation of the line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \end{pmatrix}$?

Hint: Eliminate the parameter λ by expressing it from one equation and substituting into the other.

Solution 3.1: Sketch

Write parametric equations: $x = 1 + 2\lambda$, $y = 3 + 4\lambda$. From the first equation, $\lambda = \frac{x-1}{2}$. Substitute into second: $y = 3 + 4 \cdot \frac{x-1}{2} = 3 + 2(x-1) = 2x + 1$. Therefore $y = 2x + 1$ or $2x - y + 1 = 0$.

Problem 3.2: Sketch 3D Helix

Sketch the curve described by $\mathbf{r}(t) = -5\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + t\,\mathbf{k}$ for $0 \le t \le 4\pi$.

Hint: The x and y components trace a circle while z increases linearly with t.

Solution 3.2: Sketch

In the xy-plane, $x^2 + y^2 = 25\cos^2 t + 25\sin^2 t = 25$, which is a circle of radius 5. As t increases from 0 to 4π, the point moves around the circle twice while rising from $z = 0$ to $z = 4\pi$. This creates a helix spiraling upward around the z-axis.

Problem 3.3: Angle Between 3D Vectors

Find the angle between vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ to the nearest degree.

Hint: Use $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

Solution 3.3: Sketch

$\mathbf{a}\cdot\mathbf{b} = (2)(-3) + (0)(1) + (4)(2) = -6 + 8 = 2$. $|\mathbf{a}| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$, $|\mathbf{b}| = \sqrt{9+1+4} = \sqrt{14}$. Therefore $\cos\theta = \frac{2}{2\sqrt{5}\sqrt{14}} = \frac{1}{\sqrt{70}}$, so $\theta = \cos^{-1}(1/\sqrt{70}) \approx 83°$.

Problem 3.4: Unit Vector in Direction

Given $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$, find a vector parallel to $\mathbf{u}$ with length 5 units.

Hint: Find the unit vector in the direction of $\mathbf{u}$, then scale by 5.

Solution 3.4: Sketch

$|\mathbf{u}| = \sqrt{4+4+25} = \sqrt{33}$. Unit vector: $\hat{\mathbf{u}} = \frac{1}{\sqrt{33}}(2\mathbf{i} - 2\mathbf{j} - 5\mathbf{k})$. Required vector: $5\hat{\mathbf{u}} = \frac{5}{\sqrt{33}}(2\mathbf{i} - 2\mathbf{j} - 5\mathbf{k})$.

Problem 3.5: Distance to Plane and Axis

Find the distance from point $P(1, 4, 3)$ to: (a) the xz-plane, (b) the y-axis.

Hint: (a) Distance to xz-plane is the y-coordinate. (b) Distance to y-axis uses x and z coordinates.

Solution 3.5: Sketch

Part (a): Distance from $P(1,4,3)$ to the xz-plane
The xz-plane consists of all points where $y = 0$. The perpendicular distance from point $P(1,4,3)$ to this plane is simply the absolute value of the y-coordinate:

$$\text{Distance} = |y| = |4| = 4 \text{ units}$$

The closest point on the xz-plane to P is $(1,0,3)$, which we can verify: $\sqrt{(1-1)^2 + (4-0)^2 + (3-3)^2} = 4$.

Part (b): Distance from $P(1,4,3)$ to the y-axis
The y-axis consists of all points of the form $(0,y,0)$ where $y \in \mathbb{R}$. To find the closest point on the y-axis to $P(1,4,3)$, we note that this point has the same y-coordinate as P. Therefore, the closest point is $(0,4,0)$.
The distance from $P(1,4,3)$ to $(0,4,0)$ is:

$$\text{Distance} = \sqrt{(1-0)^2 + (4-4)^2 + (3-0)^2} = \sqrt{1+0+9} = \sqrt{10} \text{ units}$$

Geometric interpretation: The distance to the y-axis is the radius of the circle in a plane perpendicular to the y-axis passing through P. This radius involves only the x and z coordinates: $\sqrt{x^2+z^2} = \sqrt{1^2+3^2} = \sqrt{10}$.

Problem 3.6: Unit Vector Perpendicular to Two Vectors

Find a unit vector perpendicular to both $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the standard unit vectors along the x-, y-, and z-axes respectively.

Hint: Use the cross product $\mathbf{u} \times \mathbf{v}$, then normalize.

Solution 3.6: Sketch

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{i}(1) - \mathbf{j}(1) + \mathbf{k}(-1) = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Magnitude: $|\mathbf{w}| = \sqrt{1+1+1} = \sqrt{3}$.
Unit vector: $\frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$.

Problem 3.7: Point on Line

Consider the line passing through points $A(0,2,1)$ and $B(2,7,4)$. Find the values of a and b such that the point $P(a,1,b)$ also lies on this line.

Hint: Find parametric equations using direction vector $(2,5,3)$, then solve for the parameter when $y = 1$.

Solution 3.7: Sketch

Step 1: Find the direction vector
The direction vector of the line is:

$$\vec{AB} = B - A = (2, 7, 4) - (0, 2, 1) = (2, 5, 3)$$

Step 2: Write parametric equations
Using point $A(0, 2, 1)$ and direction vector $(2, 5, 3)$, the parametric equations of the line are:

$$\begin{aligned} x &= 0 + 2t = 2t \\ y &= 2 + 5t \\ z &= 1 + 3t \end{aligned}$$

where t is the parameter.
Step 3: Find the parameter when $y = 1$
Since point P has y-coordinate equal to 1, we substitute into the y equation:

$$1 = 2 + 5t \Rightarrow 5t = -1 \Rightarrow t = -\frac{1}{5}$$

Step 4: Find a and b
Substituting $t = -\frac{1}{5}$ into the x and z equations:

$$a = x = 2\left(-\frac{1}{5}\right) = -\frac{2}{5}$$

$$b = z = 1 + 3\left(-\frac{1}{5}\right) = 1 - \frac{3}{5} = \frac{2}{5}$$

Therefore, $a = -\frac{2}{5}$ and $b = \frac{2}{5}$, giving the point $P\left(-\frac{2}{5}, 1, \frac{2}{5}\right)$.
Verification: We can verify by checking that this point satisfies the line equation with $t = -\frac{1}{5}$.

Problem 3.8: Equal Magnitude Perpendicular Vectors

Prove that if $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors of equal magnitude, then $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$.

Hint: Show that $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = 0$.

Solution 3.8: Sketch

To prove perpendicularity, we need to show that the dot product equals zero.
Expand the dot product using distributivity:

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}$$

Since the dot product is commutative ($\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$), the middle terms cancel:

$$\begin{aligned} &= \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} \\ &= |\mathbf{u}|^2 - |\mathbf{v}|^2 \end{aligned}$$

Given that $|\mathbf{u}| = |\mathbf{v}|$, we have:

$$|\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$$

Therefore, $(\mathbf{u}-\mathbf{v}) \perp (\mathbf{u}+\mathbf{v})$.

Takeaways 3.1

- This result has a beautiful geometric interpretation: if vectors $\mathbf{u}$ and $\mathbf{v}$ have equal magnitude, then the vectors $\mathbf{u}+\mathbf{v}$ (the diagonal) and $\mathbf{u}-\mathbf{v}$ (the other diagonal) form perpendicular diagonals.
- **Rectangle interpretation:** The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{u}-\mathbf{v}$ form a *rhombus* (not a rectangle in general), since $|\mathbf{u}| = |\mathbf{v}|$ makes all four sides equal length. The diagonals of this rhombus are perpendicular, which is what we just proved.
- Special case: If $\mathbf{u} \perp \mathbf{v}$ *and* $|\mathbf{u}| = |\mathbf{v}|$, then the rhombus becomes a *square*, and the diagonals are both perpendicular and equal in length.
- This identity is useful in many geometric proofs and demonstrates the power of algebraic vector manipulation to prove geometric relationships.

Problem 3.9: Direction Cosines

Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ for a position vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ making angles α, β, γ with the positive x-, y-, and z-axes respectively. (The quantities $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of the vector $\mathbf{r}$.)

Hint: Use dot products with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to find the direction cosines.

Solution 3.9: Sketch

$\cos\alpha = \frac{a}{|\mathbf{r}|}$, $\cos\beta = \frac{b}{|\mathbf{r}|}$, $\cos\gamma = \frac{c}{|\mathbf{r}|}$. Therefore $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2+b^2+c^2}{|\mathbf{r}|^2} = \frac{|\mathbf{r}|^2}{|\mathbf{r}|^2} = 1$.

Problem 3.10: Line Intersection by Components

Find the intersection point of lines $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$.

Hint: Equate components and solve the system for λ_1 and λ_2.

Solution 3.10: Sketch

Equating: $3 + \lambda_1 = 3 - 2\lambda_2$, $-1 + 2\lambda_1 = -6 + \lambda_2$, $7 + \lambda_1 = 2 + 3\lambda_2$. From first equation: $\lambda_1 = -2\lambda_2$. Substitute into second: $-1 - 4\lambda_2 = -6 + \lambda_2 \Rightarrow \lambda_2 = 1$, so $\lambda_1 = -2$. Check third equation: $7 - 2 = 5 = 2 + 3$ ✓. Intersection point: $(1, -5, 5)$.

Problem 3.11: Multiple Choice: Cartesian Equation

What is the Cartesian equation of $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \end{pmatrix}$?

Options:
A. $2y + x = 7$,
B. $y - 2x = -5$,
C. $y + 2x = 5$,
D. $2y - x = -1$.

Hint: Eliminate λ from $x = 1 - 2\lambda$ and $y = 3 + 4\lambda$.

Solution 3.11: Sketch

From $x = 1 - 2\lambda$, get $\lambda = \frac{1-x}{2}$. Substitute: $y = 3 + 4 \cdot \frac{1-x}{2} = 3 + 2(1-x) = 5 - 2x$, giving $y + 2x = 5$. Answer: C.

Problem 3.12: Closest Point on Line to Origin

Find the point on line $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ closest to the origin. That is, find the value of λ such that the distance from the point $\mathbf{r}(\lambda)$ to the origin $O(0, 0, 0)$ is minimized.

Hint: The closest point occurs when $\mathbf{r}$ is perpendicular to the direction vector.

Solution 3.12: Sketch

Let $\mathbf{r} = \begin{pmatrix} 1+2\lambda \\ 4+\lambda \\ 6+\lambda \end{pmatrix}$. For closest point: $\mathbf{r}\cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0$. This gives $2(1+2\lambda)+(4+\lambda)+(6+\lambda) = 0 \Rightarrow 6\lambda + 12 = 0 \Rightarrow \lambda = -2$. Point: $(-3, 2, 4)$.

Problem 3.13: Unit Vector Perpendicular to Two

Find a unit vector perpendicular to $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

Hint: Compute the cross product and normalize the result.

Solution 3.13: Sketch

Cross product: $\mathbf{a} \times \mathbf{b} = \begin{bmatrix} (-2)(-1) - (1)(4) \\ (1)(1) - (3)(-1) \\ (3)(4) - (-2)(1) \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 14 \end{bmatrix}$. Magnitude: $\sqrt{4+16+196} = 6\sqrt{6}$. Unit vector: $\frac{1}{6\sqrt{6}} \begin{bmatrix} -2 \\ 4 \\ 14 \end{bmatrix} = \frac{1}{3\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix}$.

Problem 3.14: Perpendicular Dot Product Proof

Prove that for non-zero vectors $\mathbf{a}, \mathbf{b}$, the equation $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2$ holds only if $\mathbf{a} \perp \mathbf{b}$.

Hint: Expand the left side and compare with the right side.

Solution 3.14: Sketch

Expanding: $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2$. Comparing with RHS: $|\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \Rightarrow 2\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow \mathbf{a}\cdot\mathbf{b} = 0$, which means $\mathbf{a} \perp \mathbf{b}$.

3.2 Part 2 Medium Problems

Problem 3.15: Line Intersection in 3D

Find the intersection point of lines $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$.

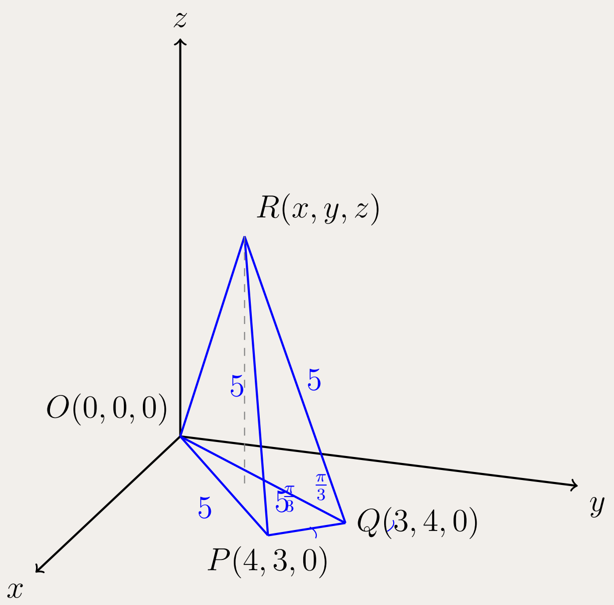
Hint: Set up three equations by equating components, then solve for parameters.

Solution 3.15: Sketch

Equating components: $3+\lambda = 1-2\mu$, $-1+2\lambda = -6+\mu$, $7+\lambda = 2+3\mu$. From equations 1 and 3: $\lambda = -2\mu$ and $\lambda = -5+3\mu$. Solving: $\mu = 1$, $\lambda = -2$. Verify in equation 2. Intersection point: $(1, -5, 5)$.

Problem 3.16: Triangular Pyramid — Find Vertex Coordinates

The diagram below shows a triangular pyramid with vertices $O(0,0,0)$, $P(4,3,0)$, $Q(3,4,0)$ and $R(x,y,z)$, where x, y, z are positive real numbers. Given that $\angle RPO = \angle RQO = \frac{\pi}{3}$ and $|\vec{PR}| = |\vec{QR}| = 5$, find the coordinates of R.



Hint:

- Calculate $|OP|$ and $|OQ|$ to recognise any special triangle.
- Use the fact that equal sides with included $60°$ imply equilateral properties.
- Set up three sphere-distance equations for $R(x, y, z)$ from O, P, Q and subtract to get linear relations for x, y.

Solution 3.16: Sketch

Compute $|OP| = \sqrt{4^2+3^2} = 5$ and $|OQ| = 5$. Since $|PR| = 5$ and $\angle RPO = 60°$, triangle RPO is equilateral so $|OR| = 5$. Thus

$$\begin{aligned} x^2+y^2+z^2 &= 25, \\ (x-4)^2+(y-3)^2+z^2 &= 25, \\ (x-3)^2+(y-4)^2+z^2 &= 25. \end{aligned}$$

Subtracting the first from the second and third gives the linear system

$$\begin{aligned} 8x+6y &= 25, \\ 6x+8y &= 25, \end{aligned}$$

so by symmetry $x = y = \frac{25}{14}$. Substituting into $x^2+y^2+z^2 = 25$ yields $z = \dfrac{5\sqrt{47}}{7}$.

Takeaways

- Intersection of spheres often reduces 3D location problems to solving linear systems by subtraction.
- Symmetry in the base points can force $x = y$ and simplify algebra.

Problem 3.17: Perpendicular Intersecting Lines

Consider the line $L_1 : \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and the line $L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ q \\ -1 \end{pmatrix}$.
Given that these two lines intersect and are perpendicular to each other, find the values of p and q.
(Note: Two conditions must be satisfied: the lines must intersect at some point, and their direction vectors must be perpendicular.)

Hint: Use perpendicularity condition (dot product $= 0$) and intersection condition (solve system).

Solution 3.17: Sketch

Step 1: Find q from perpendicularity. Direction vectors: $\mathbf{d}_1 = (1, 2, 1)$ and $\mathbf{d}_2 = (2, q, -1)$. Since $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$: $(1)(2) + (2)(q) + (1)(-1) = 0 \Rightarrow 2q = -1 \Rightarrow q = -\frac{1}{2}$.

Step 2: Set up intersection condition. The lines intersect when $\begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1/2 \\ -1 \end{pmatrix}$. This gives: $3 + \mu = 1 + 2\lambda$, $-1 + 2\mu = -\frac{1}{2}\lambda$, and $7 + \mu = p - \lambda$.

Step 3: Solve for μ and λ. From first equation: $\mu = 2\lambda - 2$. Substituting into second: $-1 + 2(2\lambda - 2) = -\frac{1}{2}\lambda \Rightarrow \frac{9}{2}\lambda = 5 \Rightarrow \lambda = \frac{10}{9}$. Thus $\mu = \frac{2}{9}$.

Step 4: Find p. From third equation: $p = 7 + \frac{2}{9} + \frac{10}{9} = 7 + \frac{4}{3} = \frac{25}{3}$.

Answer: $p = \dfrac{25}{3}$ and $q = -\dfrac{1}{2}$.

Problem 3.18: Tetrahedron Collinearity

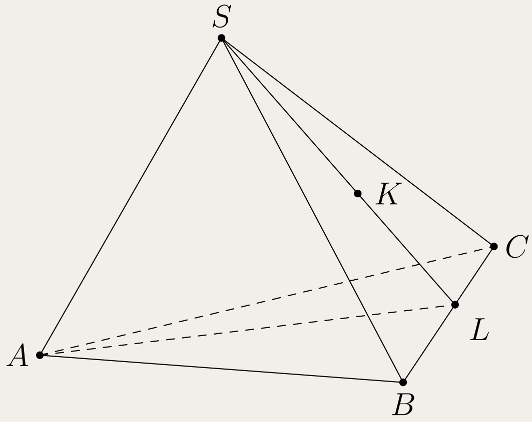
(i) The two non-parallel vectors $\vec{u}$ and $\vec{v}$ satisfy $\lambda\vec{u} + \mu\vec{v} = \vec{0}$ for some real numbers λ and μ.

Show that $\lambda = \mu = 0$.

(ii) The two non-parallel vectors $\vec{u}$ and $\vec{v}$ satisfy $\lambda_1\vec{u} + \mu_1\vec{v} = \lambda_2\vec{u} + \mu_2\vec{v}$ for some real numbers $\lambda_1, \lambda_2, \mu_1$ and μ_2.

Using part (i), or otherwise, show that $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$.

The diagram below shows a tetrahedron with vertices A, B, C and S.
The point K is defined by $\vec{SK} = \frac{1}{4}\vec{SB} + \frac{1}{3}\vec{SC}$, as shown in the diagram.
The point L is the point of intersection of the straight lines SK and BC.



(iii) Using part (ii), or otherwise, determine the position of L by showing that $\vec{BL} = \frac{4}{7}\vec{BC}$.

(iv) The point P is defined by $\vec{AP} = -6\vec{AB} - 8\vec{AC}$.

Does P lie on the line AL? Justify your answer.

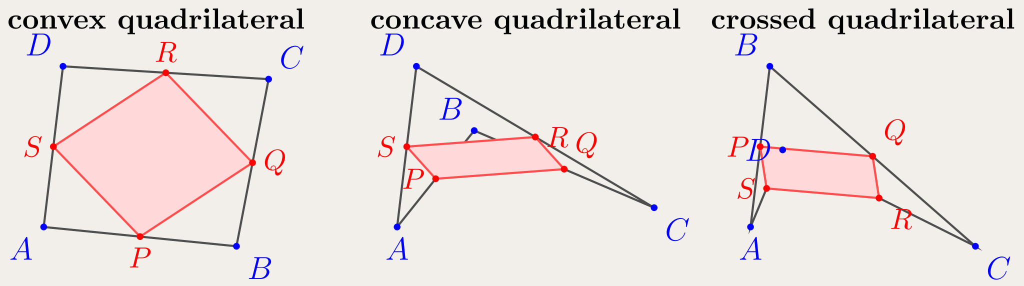
Hint:(i) Use proof by contradiction: if $\lambda \neq 0$, then $\vec{u} = -\frac{\mu}{\lambda}\vec{v}$, contradicting non-parallel assumption. (ii) Rearrange to get $(\lambda_1 - \lambda_2)\vec{u} + (\mu_1 - \mu_2)\vec{v} = \vec{0}$ and apply part (i). (iii) Express $\vec{SL}$ in two ways: as $m\vec{SK}$ and as $(1-k)\vec{SB} + k\vec{SC}$, then equate coefficients. (iv) Express $\vec{AL}$ in terms of $\vec{AB}$ and $\vec{AC}$, then check if $\vec{AP} = c\vec{AL}$ for some scalar c.

Solution 3.18: Sketch

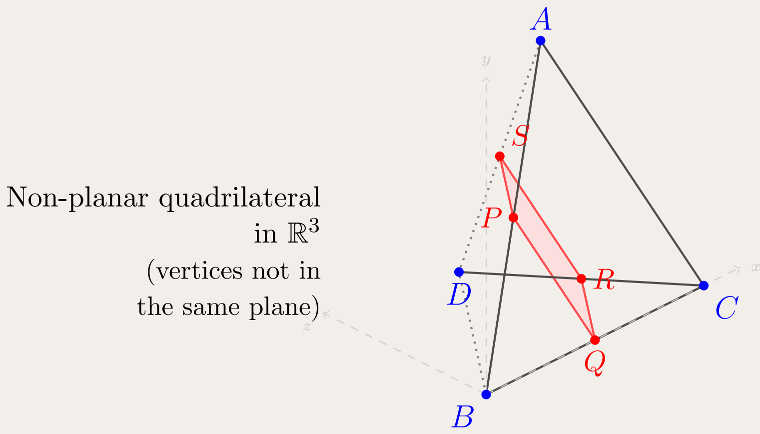
(i) Assume $\lambda \neq 0$. Then $\vec{u} = -\frac{\mu}{\lambda}\vec{v}$, implying $\vec{u} \parallel \vec{v}$, contradiction. Thus $\lambda = 0$, which gives $\mu = 0$.
(ii) Rearranging: $(\lambda_1 - \lambda_2)\vec{u} + (\mu_1 - \mu_2)\vec{v} = \vec{0}$. By part (i), both coefficients equal zero.
(iii) Since S, K, L are collinear: $\vec{SL} = m\vec{SK} = \frac{m}{4}\vec{SB} + \frac{m}{3}\vec{SC}$. Since L is on BC: $\vec{SL} = (1-k)\vec{SB} + k\vec{SC}$ where $\vec{BL} = k\vec{BC}$. Equating coefficients: $\frac{m}{4} = 1 - k$ and $\frac{m}{3} = k$. Solving: $m = \frac{12}{7}$ and $k = \frac{4}{7}$.
(iv) $\vec{AL} = \vec{AB} + \frac{4}{7}\vec{BC} = \vec{AB} + \frac{4}{7}(\vec{AC} - \vec{AB}) = \frac{3}{7}\vec{AB} + \frac{4}{7}\vec{AC}$. Check: $\vec{AP} = -6\vec{AB} - 8\vec{AC} = -14(\frac{3}{7}\vec{AB} + \frac{4}{7}\vec{AC}) = -14\vec{AL}$. Yes, P lies on line AL (extended backwards through A).

Problem 3.19: Varignon's Theorem - Midpoint Parallelogram

Let $ABCD$ be any quadrilateral in three-dimensional space (not necessarily planar). Let P, Q, R, and S be the midpoints of sides AB, BC, CD, and DA respectively.



3D Non-Planar Quadrilateral:



(i) Express the position vectors of P, Q, R, and S in terms of the position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ of points A, B, C, and D respectively.

(ii) Show that $\overrightarrow{PQ} = \overrightarrow{SR}$.

(iii) Hence, prove that $PQRS$ is a parallelogram.

(iv) If $ABCD$ is a parallelogram, what special type of quadrilateral is $PQRS$? Justify your answer.

Note: This result is known as Varignon's Theorem (1731). It states that the midpoints of any quadrilateral always form a parallelogram, regardless of the shape of the original quadrilateral.

Hint:(i) Use the midpoint formula: the position vector of the midpoint of two points is the average of their position vectors. (ii) Express both $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$, then show they are equal. (iii) Use the result from (ii) - if one pair of opposite sides is equal and parallel, then $PQRS$ is a parallelogram. (iv) When $ABCD$ is a parallelogram, use $\overrightarrow{DC} = \overrightarrow{AB}$ and investigate the relationship between adjacent sides of $PQRS$.

Solution 3.19: Sketch

(i) Using the midpoint formula:

$$\begin{aligned}\mathbf{p} &= \frac{\mathbf{a}+\mathbf{b}}{2}\\ \mathbf{q} &= \frac{\mathbf{b}+\mathbf{c}}{2}\\ \mathbf{r} &= \frac{\mathbf{c}+\mathbf{d}}{2}\\ \mathbf{s} &= \frac{\mathbf{d}+\mathbf{a}}{2}\end{aligned}$$

(ii) Calculate $\overrightarrow{PQ}$:

$$\begin{aligned}\overrightarrow{PQ} = \mathbf{q}-\mathbf{p} &= \frac{\mathbf{b}+\mathbf{c}}{2} - \frac{\mathbf{a}+\mathbf{b}}{2}\\ &= \frac{\mathbf{b}+\mathbf{c}-\mathbf{a}-\mathbf{b}}{2} = \frac{\mathbf{c}-\mathbf{a}}{2}\end{aligned}$$

Calculate $\overrightarrow{SR}$:

$$\begin{aligned}\overrightarrow{SR} = \mathbf{r}-\mathbf{s} &= \frac{\mathbf{c}+\mathbf{d}}{2} - \frac{\mathbf{d}+\mathbf{a}}{2}\\ &= \frac{\mathbf{c}+\mathbf{d}-\mathbf{d}-\mathbf{a}}{2} = \frac{\mathbf{c}-\mathbf{a}}{2}\end{aligned}$$

Therefore $\overrightarrow{PQ} = \overrightarrow{SR}$.
(iii) Since $\overrightarrow{PQ} = \overrightarrow{SR}$, the sides PQ and SR are equal in length and parallel. By definition, a quadrilateral with one pair of opposite sides equal and parallel is a parallelogram. Therefore $PQRS$ is a parallelogram.
(iv) If $ABCD$ is a parallelogram, then $\overrightarrow{AB} = \overrightarrow{DC}$, which means $\mathbf{b}-\mathbf{a} = \mathbf{c}-\mathbf{d}$.
Calculate $\overrightarrow{PS}$:

$$\overrightarrow{PS} = \mathbf{s}-\mathbf{p} = \frac{\mathbf{d}+\mathbf{a}}{2} - \frac{\mathbf{a}+\mathbf{b}}{2} = \frac{\mathbf{d}-\mathbf{b}}{2}$$

Calculate $\overrightarrow{QR}$:

$$\overrightarrow{QR} = \mathbf{r}-\mathbf{q} = \frac{\mathbf{c}+\mathbf{d}}{2} - \frac{\mathbf{b}+\mathbf{c}}{2} = \frac{\mathbf{d}-\mathbf{b}}{2}$$

So $\overrightarrow{PS} = \overrightarrow{QR}$, meaning both pairs of opposite sides are equal and parallel.
Additionally, from (ii): $|\overrightarrow{PQ}| = \frac{1}{2}|\mathbf{c}-\mathbf{a}| = \frac{1}{2}|\overrightarrow{AC}|$ (half the diagonal AC).
Similarly: $|\overrightarrow{PS}| = \frac{1}{2}|\mathbf{d}-\mathbf{b}| = \frac{1}{2}|\overrightarrow{BD}|$ (half the diagonal BD).
If $ABCD$ is a parallelogram with equal diagonals, then $PQRS$ is a rhombus (all four sides equal). If the diagonals of $ABCD$ are perpendicular, then $PQRS$ is a rectangle.

Problem 3.20: Force Vector Analysis

A particle is subject to two forces and undergoes a displacement in three-dimensional space.

(i) Force $\mathbf{F}_1$ has magnitude 12 newtons in the direction of vector $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Show that $\mathbf{F}_1 = 8\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}$ newtons.

(ii) Force $\mathbf{F}_1$ from part (i) and a second force, $\mathbf{F}_2 = -6\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$ newtons, both act upon the particle.

Show that the resultant force acting on the particle is given by:

$$\mathbf{F}_3 = 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k} \text{ newtons}$$

(iii) The particle moves through a displacement $\mathbf{d} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ metres under the action of the resultant force $\mathbf{F}_3$ from part (ii).

Calculate the work done by the resultant force on the particle.

Note: Here $\mathbf{i}$, $\mathbf{j}$, *and* $\mathbf{k}$ *are the standard unit vectors in the* x, y, *and* z *directions respectively. Work done is calculated using* $W = \mathbf{F} \cdot \mathbf{d}$ *and is measured in joules (J) when force is in newtons (N) and displacement is in metres (m).*

Hint:(i) Find the unit vector in the given direction by dividing by its magnitude, then multiply by 12 to get the force vector. (ii) Add the two force vectors component-wise to find the resultant. (iii) Use the dot product formula $W = \mathbf{F} \cdot \mathbf{d}$ to compute the work done.

Solution 3.20: Sketch

(i) Direction vector: $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ has magnitude $|\mathbf{v}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$.
Unit vector in this direction: $\hat{\mathbf{u}} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.
Force with magnitude 12 N:

$$\mathbf{F}_1 = 12\hat{\mathbf{u}} = 12 \times \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 4(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 8\mathbf{i} - 8\mathbf{j} + 4\mathbf{k} \text{ N}$$

(ii) Resultant force (vector addition):

$$\begin{aligned}
\mathbf{F}_3 &= \mathbf{F}_1 + \mathbf{F}_2 \\
&= (8\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}) + (-6\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}) \\
&= (8-6)\mathbf{i} + (-8+12)\mathbf{j} + (4+4)\mathbf{k} \\
&= 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k} \text{ N}
\end{aligned}$$

(iii) Work done is $W = \mathbf{F}_3 \cdot \mathbf{d}$:

$$\begin{aligned}
W &= (2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \\
&= (2)(1) + (4)(1) + (8)(2) \\
&= 2 + 4 + 16 \\
&= 22 \text{ J}
\end{aligned}$$

The work done by the resultant force is 22 joules.

Problem 3.21: Double Angle with Vectors

Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$.
Let θ be the acute angle between these two vectors. Find the exact value of $\sin 2\theta$.
Note: Here $\mathbf{i}$, $\mathbf{j}$, *and* $\mathbf{k}$ *are the standard unit vectors in the x, y, and z directions respectively.*

Hint: Find $\cos\theta$ from dot product, then $\sin\theta$ from Pythagorean identity, and use $\sin 2\theta = 2\sin\theta\cos\theta$.

Solution 3.21: Sketch

$|\mathbf{a}| = 3$, $|\mathbf{b}| = 6$, $\mathbf{a}\cdot\mathbf{b} = 2$. Thus $\cos\theta = \frac{2}{18} = \frac{1}{9}$. Then $\sin\theta = \sqrt{1 - \frac{1}{81}} = \frac{4\sqrt{5}}{9}$. Therefore $\sin 2\theta = 2 \cdot \frac{4\sqrt{5}}{9} \cdot \frac{1}{9} = \frac{8\sqrt{5}}{81}$.

Problem 3.22: Vector Projection

Find the integer value of m such that the projection of $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ onto $\mathbf{b} = \mathbf{i} + m\mathbf{j} - \mathbf{k}$ equals $-\frac{11}{18}\mathbf{b}$.
Note: Here $\mathbf{i}$, $\mathbf{j}$, *and* $\mathbf{k}$ *are the standard unit vectors in the* x, y, *and* z *directions respectively.*

Hint: Use projection formula: $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$.

Solution 3.22: Sketch

$\mathbf{a}\cdot\mathbf{b} = 2 - 3m - 1 = 1 - 3m$, $|\mathbf{b}|^2 = 1 + m^2 + 1 = m^2 + 2$. Set $\frac{1-3m}{m^2+2} = -\frac{11}{18}$. Cross-multiply: $18(1-3m) = -11(m^2+2)$, giving $11m^2 - 54m + 40 = 0$. Factor: $(11m - 10)(m - 4) = 0$. Since m is an integer, $m = 4$.

Problem 3.23: Perpendicular Vectors Condition

For $\mathbf{u} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = p\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, find values of p such that $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ are perpendicular.
Note: Here $\mathbf{i}$, $\mathbf{j}$, *and* $\mathbf{k}$ *are the standard unit vectors in the* x, y, *and* z *directions respectively.*

Hint: Use $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$.

Solution 3.23: Sketch

$|\mathbf{u}|^2 = 4 + 1 + 9 = 14$. $|\mathbf{v}|^2 = p^2 + 1 + 4 = p^2 + 5$. Setting equal: $14 = p^2 + 5 \Rightarrow p^2 = 9 \Rightarrow p = \pm 3$.

Problem 3.24: Parallel and Perpendicular Lines

For lines with direction vectors $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} a-2 \\ -7 \\ 7 \end{pmatrix}$, find a if they are: (a) parallel, (b) perpendicular.

Hint: Parallel: direction vectors are scalar multiples. Perpendicular: dot product equals zero.

Solution 3.24: Sketch

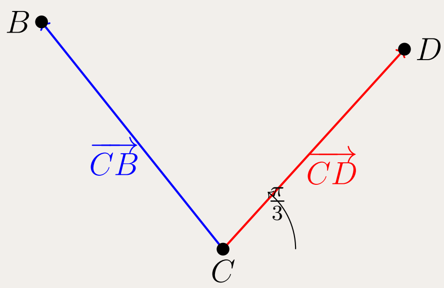
(a) For parallel: $\frac{3}{a-2} = \frac{-3}{-7} = \frac{3}{7}$. Solving: $3 \cdot 7 = 3(a-2) \Rightarrow a = 9$. (b) For perpendicular: $3(a-2) + (-3)(-7) + 3(7) = 0 \Rightarrow 3a + 36 = 0 \Rightarrow a = -12$.

Problem 3.25: Angle BCD Using Dot Product

Relative to a fixed origin, the points B, C, and D are defined respectively by the position vectors:

$$\begin{aligned} \mathbf{b} &= \mathbf{i} - \mathbf{j} + 2\mathbf{k} \\ \mathbf{c} &= 2\mathbf{i} - \mathbf{j} + \mathbf{k} \\ \mathbf{d} &= a\mathbf{i} - 2\mathbf{j} \end{aligned}$$

where a is a real constant.
Given that the magnitude of angle BCD is $\frac{\pi}{3}$ (i.e., $\angle BCD = \frac{\pi}{3}$), find the value of a.



(Note: The angle BCD is the angle at vertex C between rays CB and CD.)

Hint: Find vectors $\overrightarrow{CB}$ and $\overrightarrow{CD}$ emanating from vertex C, then use the dot product formula $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$ with $\cos(\pi/3) = 1/2$. Be careful to check that your solution gives a positive value for $1 - a$.

Solution 3.25: Sketch

Step 1: Find vectors from C: $\overrightarrow{CB} = \mathbf{b} - \mathbf{c} = -\mathbf{i} + \mathbf{k}$ and $\overrightarrow{CD} = \mathbf{d} - \mathbf{c} = (a-2)\mathbf{i} - \mathbf{j} - \mathbf{k}$.
Step 2: Calculate magnitudes: $|\overrightarrow{CB}| = \sqrt{2}$ and $|\overrightarrow{CD}| = \sqrt{(a-2)^2 + 2}$.
Step 3: Dot product: $\overrightarrow{CB} \cdot \overrightarrow{CD} = -(a-2) + 0 - 1 = 1 - a$.
Step 4: Use formula with $\cos(\pi/3) = 1/2$: $1 - a = \sqrt{2} \cdot \sqrt{(a-2)^2 + 2} \cdot \frac{1}{2}$
Step 5: Square both sides: $4(1-a)^2 = 2[(a-2)^2 + 2]$, which simplifies to $a^2 = 4$, giving $a = \pm 2$.
Step 6: Check constraint: Since the RHS is positive, we need $1 - a > 0$, so $a < 1$. Therefore $a = -2$.

Problem 3.26: Direction Cosines Sum

Let $\mathbf{r}$ be a non-zero position vector in three-dimensional space with magnitude $r = |\mathbf{r}|$. Suppose $\mathbf{r}$ makes angles α, β, and γ with the positive x-axis, y-axis, and z-axis respectively.
The **direction cosines** of $\mathbf{r}$ are defined as:

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$$

Prove that:

$$l^2 + m^2 + n^2 = 1$$

That is, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Hint: Express direction cosines as ratios of components to magnitude.

Solution 3.26: Sketch

$\cos\alpha = \frac{a}{|\mathbf{r}|}$, $\cos\beta = \frac{b}{|\mathbf{r}|}$, $\cos\gamma = \frac{c}{|\mathbf{r}|}$. Sum: $\frac{a^2+b^2+c^2}{|\mathbf{r}|^2} = \frac{|\mathbf{r}|^2}{|\mathbf{r}|^2} = 1$.

Problem 3.27: Section Formula Proof

Prove that if C divides AB in ratio $m:n$, then $\overrightarrow{OC} = \frac{m\mathbf{a}+n\mathbf{b}}{m+n}$ where $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$.

Hint: Use $\overrightarrow{AC} = \frac{n}{m+n}\overrightarrow{AB}$ and vector addition.

Solution 3.27: Sketch

Since $\frac{CB}{AC} = \frac{m}{n}$, we have $\overrightarrow{AC} = \frac{n}{m+n}(\mathbf{b}-\mathbf{a})$. Then $\overrightarrow{OC} = \mathbf{a}+\overrightarrow{AC} = \mathbf{a}+\frac{n}{m+n}(\mathbf{b}-\mathbf{a}) = \frac{m\mathbf{a}+n\mathbf{b}}{m+n}$.

Problem 3.28: Skew or Intersecting Lines

Consider the two lines:

$$L_1: \quad \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

$$L_2: \quad \mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -11 \\ 3 \end{pmatrix}$$

Determine whether lines L_1 and L_2 are skew or intersecting.

Hint: Check if direction vectors are parallel. If not, solve system to see if it's consistent.

Solution 3.28: Sketch

Direction vectors not parallel (check ratios). Solve system: from equations 1 and 3, find $\lambda = 6$, $\mu = 3$. Check equation 2: $4(6) + 11(3) = 57 \neq 9$. System inconsistent, so lines are skew.

Problem 3.29: Line Through Points, Intersection Check

A line passes through the points $A(1, 3, -2)$ and $B(2, -1, 5)$.

(i) Show that the vector equation of the line AB is given by:

$$\mathbf{r} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + \lambda_1(\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}), \quad \lambda_1 \in \mathbb{R}$$

(ii) Determine if the point $C(3, 4, 9)$ lies on the line.

(iii) Consider a second line with parametric equations $x = 1 - \lambda_2$, $y = 2 + 3\lambda_2$, $z = -1 + \lambda_2$.

Assuming this line is neither parallel nor perpendicular to AB, determine whether the two lines intersect or are skew.

Hint: (i) Find the direction vector $\overrightarrow{AB}$ and use the point A to write the vector equation. (ii) Substitute the coordinates of C into the line equation and check if a consistent value of λ_1 exists for all three components. (iii) Set the two lines equal component-wise, solve for both parameters, then verify if the solution is consistent across all three equations.

Solution 3.29: Sketch

(i) Direction vector: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2-1 \\ -1-3 \\ 5-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix} = \mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.
Using point A as the position vector: $\mathbf{r} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + \lambda_1(\mathbf{i} - 4\mathbf{j} + 7\mathbf{k})$.
(ii) If C lies on the line: $\begin{pmatrix} 3 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$.
From x-component: $3 = 1 + \lambda_1 \Rightarrow \lambda_1 = 2$.
Check y-component: $y = 3 - 4(2) = -5 \neq 4$. Since the y-coordinate doesn't match, point C does not lie on the line.
(iii) Line 1: $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$. Line 2: $\mathbf{r}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$.
Equate components: From x: $1 + \lambda_1 = 1 - \lambda_2 \Rightarrow \lambda_1 = -\lambda_2$. From y: $3 - 4\lambda_1 = 2 + 3\lambda_2$.
Substitute $\lambda_1 = -\lambda_2$: $3 + 4\lambda_2 = 2 + 3\lambda_2 \Rightarrow \lambda_2 = -1$, so $\lambda_1 = 1$.
Check z-component: LHS: $-2 + 7(1) = 5$. RHS: $-1 + (-1) = -2$. Since $5 \neq -2$, the system is inconsistent. The lines do not intersect and are not parallel, therefore they are skew.

Problem 3.30: Line-Sphere Intersection Angle

Let S be the sphere with center at the origin and equation $x^2 + y^2 + z^2 = 10$.
Let ℓ be the line with equation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(i) Find A and B, the points of intersection of S and ℓ.

(ii) Find $\angle AOB$ to the nearest degree, where O is the origin.

Hint:(i) Write the line in parametric form ($x = 1$, $y = 2 + \lambda$, $z = 1 - \lambda$), substitute into the sphere equation, and solve the resulting quadratic equation for λ. (ii) Use the dot product formula $\cos\theta = \frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}$ to find the angle between the two position vectors.

Solution 3.30: Sketch

(i) Parametric form: $x = 1$, $y = 2 + \lambda$, $z = 1 - \lambda$.
Substitute into $x^2+y^2+z^2 = 10$: $1+(2+\lambda)^2+(1-\lambda)^2 = 10$ $1+4+4\lambda+\lambda^2+1-2\lambda+\lambda^2 = 10$ $2\lambda^2 + 2\lambda + 6 = 10$ $\lambda^2 + \lambda - 2 = 0$ $(\lambda + 2)(\lambda - 1) = 0$
So $\lambda = 1$ or $\lambda = -2$.
When $\lambda = 1$: $A(1, 3, 0)$. When $\lambda = -2$: $B(1, 0, 3)$.
(ii) $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$.
$\overrightarrow{OA} \cdot \overrightarrow{OB} = (1)(1) + (3)(0) + (0)(3) = 1$.
$|\overrightarrow{OA}| = \sqrt{1^2 + 3^2 + 0^2} = \sqrt{10}$, $|\overrightarrow{OB}| = \sqrt{1^2 + 0^2 + 3^2} = \sqrt{10}$.
$\cos\theta = \frac{1}{\sqrt{10}\cdot\sqrt{10}} = \frac{1}{10} = 0.1$.
Therefore $\theta = \cos^{-1}(0.1) \approx 84.26^\circ \approx 84^\circ$.

Problem 3.31: Linear Combination of Vectors

Consider the following four vectors in three-dimensional space:

$$\begin{aligned} \mathbf{a} &= 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \\ \mathbf{b} &= \mathbf{i} - \mathbf{j} + 2\mathbf{k} \\ \mathbf{c} &= -\mathbf{i} + 2\mathbf{j} - \mathbf{k} \\ \mathbf{u} &= 5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k} \end{aligned}$$

Find the values of the scalars λ, μ, and ν that satisfy the equation:

$$\mathbf{u} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$$

Note: This means expressing vector $\mathbf{u}$ *as a linear combination of vectors* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$. *Any vector in 3D space can be written as a combination of three non-coplanar (non-parallel) vectors.*

Hint: Substitute the given vectors into the equation $\mathbf{u} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$, then equate the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to form a system of three linear equations. Solve the system by elimination.

Solution 3.31: Sketch

Substituting: $5\mathbf{i}+5\mathbf{j}+5\mathbf{k} = \lambda(2\mathbf{i}+3\mathbf{j}+\mathbf{k}) + \mu(\mathbf{i}-\mathbf{j}+2\mathbf{k}) + \nu(-\mathbf{i}+2\mathbf{j}-\mathbf{k})$.
Equating coefficients gives the system:

$$\begin{aligned} 2\lambda + \mu - \nu &= 5 \quad (1) \\ 3\lambda - \mu + 2\nu &= 5 \quad (2) \\ \lambda + 2\mu - \nu &= 5 \quad (3) \end{aligned}$$

Subtract (1) from (3): $-\lambda + \mu = 0 \Rightarrow \lambda = \mu$.
Substitute $\mu = \lambda$ into (2): $3\lambda - \lambda + 2\nu = 5 \Rightarrow 2\lambda + 2\nu = 5 \Rightarrow \lambda + \nu = \frac{5}{2}$.
Substitute $\mu = \lambda$ into (1): $2\lambda + \lambda - \nu = 5 \Rightarrow 3\lambda - \nu = 5$.
Add these two results: $4\lambda = \frac{5}{2} + 5 = \frac{15}{2} \Rightarrow \lambda = \frac{15}{8}$.
Since $\lambda = \mu$: $\mu = \frac{15}{8}$.
From $3\lambda - \nu = 5$: $\nu = 3 \cdot \frac{15}{8} - 5 = \frac{45}{8} - \frac{40}{8} = \frac{5}{8}$.
Therefore: $\lambda = \frac{15}{8}$, $\mu = \frac{15}{8}$, $\nu = \frac{5}{8}$.

Takeaways 3.2

Key Concepts:

- **Linear Combination of Vectors:** Any vector can potentially be expressed as a linear combination of other vectors: $\mathbf{v} = \lambda\mathbf{u}_1 + \mu\mathbf{u}_2 + \nu\mathbf{u}_3$. The scalars λ, μ, ν are the coefficients.

- **Equating Coefficients Method:** When vectors are expressed in component form (using $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$), equate the coefficients of each unit vector separately to create a system of linear equations.

- **Solving Linear Systems:** The systematic approach involves:

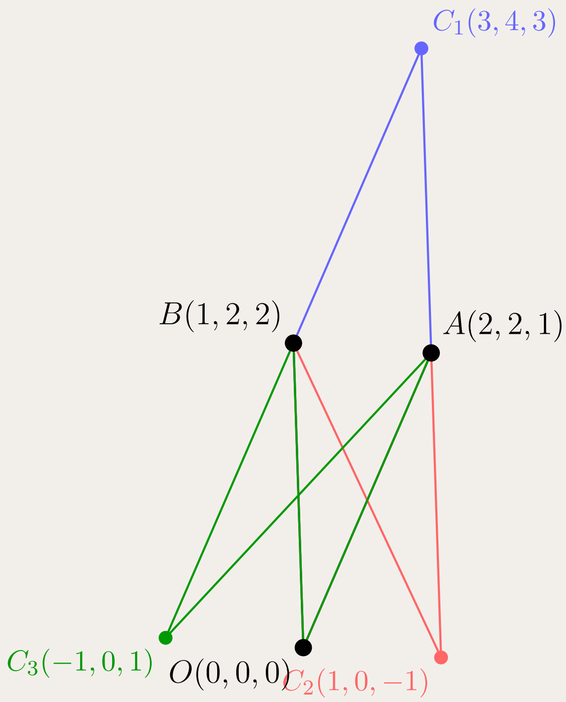
  1. Look for relationships by adding/subtracting equations
  2. Find simple relationships (e.g., $\lambda = \mu$) to reduce variables
  3. Substitute to eliminate variables progressively
  4. Solve for remaining variables and back-substitute

- **Verification Strategy:** Always substitute final values back into the original equations to verify correctness. Here: $2(\frac{15}{8}) + \frac{15}{8} - \frac{5}{8} = \frac{30+15-5}{8} = \frac{40}{8} = 5$ ✓

- **Linear Independence:** The fact that a unique solution exists ($\lambda = \mu = \frac{15}{8}$, $\nu = \frac{5}{8}$) indicates that the three given vectors are linearly independent and span $\mathbb{R}^3$. Any vector in 3D space can be expressed as a linear combination of these three vectors.

Problem 3.32: Parallelogram Fourth Vertex

Three vertices of a parallelogram are $O(0,0,0)$, $A(2,2,1)$, $B(1,2,2)$. Find all possible positions of the fourth vertex.



Three possible parallelograms with vertices O, A, B
(shown in blue, red, and green)

Hint: Consider three cases: which point is opposite to which, giving three parallelograms.

Solution 3.32: Sketch

Given three vertices $O(0,0,0)$, $A(2,2,1)$, and $B(1,2,2)$, we need to find all possible fourth vertices C such that $OABC$ forms a parallelogram (in some order).
In a parallelogram, opposite sides are equal and parallel. This gives us three distinct cases depending on which vertex is opposite to which.
Case 1: C_1 opposite to O (parallelogram $OABC_1$ with $OA \parallel BC_1$ and $OB \parallel AC_1$)
Using the parallelogram property: $\overrightarrow{OC_1} = \overrightarrow{OA} + \overrightarrow{OB}$.
Calculate:

$$\begin{aligned}\overrightarrow{OA} &= (2,2,1)\\ \overrightarrow{OB} &= (1,2,2)\\ \overrightarrow{OC_1} &= (2,2,1) + (1,2,2) = (3,4,3)\end{aligned}$$

Therefore, $C_1 = (3,4,3)$.
Verification: $\overrightarrow{OA} = (2,2,1)$ and $\overrightarrow{BC_1} = (3,4,3) - (1,2,2) = (2,2,1)$ ✓
Case 2: C_2 opposite to B (parallelogram OAC_2B with $OA \parallel C_2B$ and $OC_2 \parallel AB$)
Using the parallelogram property: $\overrightarrow{OC_2} = \overrightarrow{OA} - \overrightarrow{OB}$.
Calculate:

$$\overrightarrow{OC_2} = (2,2,1) - (1,2,2) = (1,0,-1)$$

Therefore, $C_2 = (1,0,-1)$.
Verification: $\overrightarrow{OA} = (2,2,1)$ and $\overrightarrow{C_2B} = (1,2,2) - (1,0,-1) = (0,2,3)$... Actually, $\overrightarrow{AC_2} = (1,0,-1) - (2,2,1) = (-1,-2,-2)$ and $\overrightarrow{OB} = (1,2,2)$. Let me verify: $\overrightarrow{OC_2} + \overrightarrow{OB} = (1,0,-1) + (1,2,2) = (2,2,1) = \overrightarrow{OA}$ ✓
Case 3: C_3 opposite to A (parallelogram OBC_3A with $OB \parallel C_3A$ and $OC_3 \parallel BA$)
Using the parallelogram property: $\overrightarrow{OC_3} = \overrightarrow{OB} - \overrightarrow{OA}$.
Calculate:

$$\overrightarrow{OC_3} = (1,2,2) - (2,2,1) = (-1,0,1)$$

Therefore, $C_3 = (-1,0,1)$.
Verification: $\overrightarrow{OB} = (1,2,2)$ and $\overrightarrow{C_3A} = (2,2,1) - (-1,0,1) = (3,2,0)$... Let me verify: $\overrightarrow{OC_3} + \overrightarrow{OA} = (-1,0,1) + (2,2,1) = (1,2,2) = \overrightarrow{OB}$ ✓
Final Answer: The three possible positions for the fourth vertex are:

$$C_1 = (3,4,3), \quad C_2 = (1,0,-1), \quad C_3 = (-1,0,1)$$

3.3 Part 2 Advanced Problems

Problem 3.33: Triangle Inequality and Cauchy-Schwarz

Let $P(x, y, z)$ be a point on the unit sphere centered at the origin, so that $x^2+y^2+z^2 = 1$. Prove:

(i) $|x| + |y| + |z| \geq 1$

(ii) For any two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, the Cauchy-Schwarz inequality states:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| \cdot |\mathbf{b}|$$

Equivalently: $|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}$

Derive this inequality from the dot product formula.

(iii) $|x| + |y| + |z| \leq \sqrt{3}$

Hint: Use triangle inequality for part (i) and Cauchy-Schwarz with vector $(1, 1, 1)$ for part (iii).

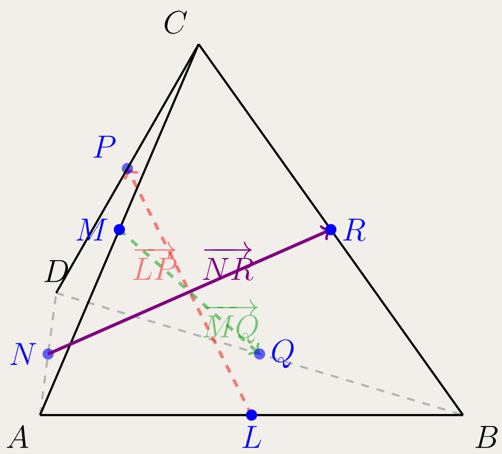
Solution 3.33: Sketch

(i) By triangle inequality: $1 = |\mathbf{r}| \leq |x| + |y| + |z|$. (ii) From dot product: $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. (iii) Apply Cauchy-Schwarz with $\mathbf{a} = (|x|, |y|, |z|)$, $\mathbf{b} = (1, 1, 1)$: $|x| + |y| + |z| \leq \sqrt{x^2 + y^2 + z^2}\sqrt{3} = \sqrt{3}$.

Problem 3.34: Bimedians of Tetrahedron

On the triangular pyramid (tetrahedron) $ABCD$, let:

- L is the midpoint of AB
- M is the midpoint of AC
- N is the midpoint of AD
- P is the midpoint of CD
- Q is the midpoint of BD
- R is the midpoint of BC



Note: The three colored arrows show the bimedians $\overrightarrow{LP}$ (red), $\overrightarrow{MQ}$ (green), and $\overrightarrow{NR}$ (violet). Each bimedian connects the midpoint of one edge to the midpoint of the opposite edge.

Let $\mathbf{b} = \overrightarrow{AB}$, $\mathbf{c} = \overrightarrow{AC}$ and $\mathbf{d} = \overrightarrow{AD}$.

(i) Show that $\overrightarrow{LP} = \frac{1}{2}(-\mathbf{b} + \mathbf{c} + \mathbf{d})$.

(ii) It can be shown that:

$$\overrightarrow{MQ} = \frac{1}{2}(\mathbf{b} - \mathbf{c} + \mathbf{d}) \quad \text{and} \quad \overrightarrow{NR} = \frac{1}{2}(\mathbf{b} + \mathbf{c} - \mathbf{d})$$

(Do NOT prove these.)

Prove that:

$$|AB|^2 + |AC|^2 + |AD|^2 + |BC|^2 + |BD|^2 + |CD|^2 = 4\left(|LP|^2 + |MQ|^2 + |NR|^2\right)$$

Note: The segments LP, MQ, and NR are called the bimedians of the tetrahedron. This problem proves that the sum of the squares of all six edges equals four times the sum of the squares of the three bimedians.

Hint: (i) Express $\overrightarrow{AL}$ and $\overrightarrow{AP}$ in terms of $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$, then find $\overrightarrow{LP} = \overrightarrow{AP} - \overrightarrow{AL}$. (ii) Expand the left side as the sum of squared magnitudes of all six edges. Expand the right side by computing $4|LP|^2$, $4|MQ|^2$, and $4|NR|^2$ using dot products. Show both sides equal $3(b^2 + c^2 + d^2) - 2(\mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d})$.

Solution 3.34: Sketch

(i) Since L is midpoint of AB: $\overrightarrow{AL} = \frac{1}{2}\mathbf{b}$. Since P is midpoint of CD: $\overrightarrow{AP} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$. Therefore: $\overrightarrow{LP} = \overrightarrow{AP} - \overrightarrow{AL} = \frac{1}{2}(\mathbf{c} + \mathbf{d}) - \frac{1}{2}\mathbf{b} = \frac{1}{2}(-\mathbf{b} + \mathbf{c} + \mathbf{d})$.
(ii) LHS: The six edges are AB, AC, AD, $BC = \mathbf{c} - \mathbf{b}$, $BD = \mathbf{d} - \mathbf{b}$, $CD = \mathbf{d} - \mathbf{c}$.

$$\begin{aligned} LHS &= |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 + |\mathbf{c} - \mathbf{b}|^2 + |\mathbf{d} - \mathbf{b}|^2 + |\mathbf{d} - \mathbf{c}|^2 \\ &= 3(b^2 + c^2 + d^2) - 2(\mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d}) \end{aligned}$$

RHS: Using part (i) and the given expressions:

$$\begin{aligned} 4|LP|^2 &= b^2 + c^2 + d^2 - 2\mathbf{b}\cdot\mathbf{c} - 2\mathbf{b}\cdot\mathbf{d} + 2\mathbf{c}\cdot\mathbf{d} \\ 4|MQ|^2 &= b^2 + c^2 + d^2 - 2\mathbf{b}\cdot\mathbf{c} + 2\mathbf{b}\cdot\mathbf{d} - 2\mathbf{c}\cdot\mathbf{d} \\ 4|NR|^2 &= b^2 + c^2 + d^2 + 2\mathbf{b}\cdot\mathbf{c} - 2\mathbf{b}\cdot\mathbf{d} - 2\mathbf{c}\cdot\mathbf{d} \end{aligned}$$

Summing: $RHS = 3(b^2 + c^2 + d^2) - 2(\mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d}) = LHS$.

Takeaways 3.3

Key Concepts and Connections:

- **Bimedians of a Tetrahedron:** The three bimedians are segments connecting midpoints of opposite edges in a tetrahedron. This problem proves a beautiful relationship: the sum of squares of all six edge lengths equals four times the sum of squares of the three bimedian lengths.

- **Connection to Apollonius's Theorem:** This result is a 3D generalization of Apollonius's Theorem. In 2D, Apollonius's Theorem states that for a triangle with sides a, b, c and median m_c to side c:

  $$a^2 + b^2 = 2m_c^2 + \frac{c^2}{2}$$

  Equivalently: $2(a^2 + b^2 + c^2) = 4(m_a^2 + m_b^2 + m_c^2)$ where m_a, m_b, m_c are the three medians.

  The tetrahedron bimedian theorem extends this median-edge relationship from triangles (2D) to tetrahedra (3D).

- **Algebraic Technique:** The solution demonstrates a powerful technique: expanding dot products systematically and observing how cross-terms cancel when summing. Notice how the coefficients ± 2 in the dot product terms sum to -2 for each pair $(\mathbf{b} \cdot \mathbf{c})$, $(\mathbf{b} \cdot \mathbf{d})$, and $(\mathbf{c} \cdot \mathbf{d})$.

- **Symmetric Structure:** The three bimedian expressions have a beautiful symmetry:

  $$\overrightarrow{LP} = \frac{1}{2}(-\mathbf{b} + \mathbf{c} + \mathbf{d})$$
  $$\overrightarrow{MQ} = \frac{1}{2}(\mathbf{b} - \mathbf{c} + \mathbf{d})$$
  $$\overrightarrow{NR} = \frac{1}{2}(\mathbf{b} + \mathbf{c} - \mathbf{d})$$

  Each expression has exactly one negative sign, cycling through the three vectors. This symmetry guarantees that when we compute $|\overrightarrow{LP}|^2 + |\overrightarrow{MQ}|^2 + |\overrightarrow{NR}|^2$, the cross-terms will combine correctly.

- **Geometric Insight:** This theorem provides a metric relationship in tetrahedra, analogous to how Pythagoras relates sides in right triangles or how Apollonius relates medians in triangles. It's a fundamental identity in 3D geometry that can be used to prove other properties or solve optimization problems involving tetrahedra.

Problem 3.35: Triangle Intersection Ratios

The diagram shows triangle OQA.
The point P lies on OA so that $OP : OA = 3 : 5$.
The point B lies on OQ so that $OB : OQ = 1 : 3$.
The point R is the intersection of AB and PQ.
The point T is chosen on AQ so that O, R and T are collinear.
Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\overrightarrow{PR} = k\overrightarrow{PQ}$ where k is a real number.

(i) Show that $\overrightarrow{OR} = \frac{3}{5}(1-k)\mathbf{a} + 3k\mathbf{b}$.

(ii) Writing $\overrightarrow{AR} = h\overrightarrow{AB}$, where h is a real number, it can be shown that $\overrightarrow{OR} = (1-h)\mathbf{a} + h\mathbf{b}$. (Do NOT prove this.)

Show that $k = \frac{1}{6}$.

(iii) Find $\overrightarrow{OT}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Hint: (i) Express $\overrightarrow{OR}$ as $\overrightarrow{OP} + \overrightarrow{PR}$. Use $\overrightarrow{OP} = \frac{3}{5}\mathbf{a}$ and $\overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ}$ where $\overrightarrow{OQ} = 3\mathbf{b}$. (ii) Equate coefficients of $\mathbf{a}$ and $\mathbf{b}$ from the two expressions for $\overrightarrow{OR}$. Use the relationship $3k = h$ and solve the system. (iii) Since O, R, T are collinear: $\overrightarrow{OT} = \lambda\overrightarrow{OR}$. Since T is on AQ: $\overrightarrow{OT} = (1-\mu)\mathbf{a} + 3\mu\mathbf{b}$. Equate coefficients.

Solution 3.35: Sketch

(i) $\overrightarrow{OR} = \overrightarrow{OP} + k\overrightarrow{PQ}$. Since $\overrightarrow{OP} = \frac{3}{5}\mathbf{a}$ and $\overrightarrow{OQ} = 3\mathbf{b}$, we have: $\overrightarrow{PQ} = -\frac{3}{5}\mathbf{a} + 3\mathbf{b}$.
Therefore: $\overrightarrow{OR} = \frac{3}{5}\mathbf{a} + k(-\frac{3}{5}\mathbf{a} + 3\mathbf{b}) = \frac{3}{5}(1-k)\mathbf{a} + 3k\mathbf{b}$.
(ii) Equating coefficients from both expressions: For $\mathbf{b}$: $3k = h$. For $\mathbf{a}$: $\frac{3}{5}(1-k) = 1-h$.
Substitute $h = 3k$ into second equation: $\frac{3}{5}(1-k) = 1 - 3k \Rightarrow 3(1-k) = 5(1-3k) \Rightarrow 3 - 3k = 5 - 15k \Rightarrow 12k = 2 \Rightarrow k = \frac{1}{6}$.
(iii) From part (ii): $\overrightarrow{OR} = \frac{3}{5}(1-\frac{1}{6})\mathbf{a} + 3(\frac{1}{6})\mathbf{b} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.
Since O, R, T collinear: $\overrightarrow{OT} = \lambda\overrightarrow{OR} = \frac{\lambda}{2}(\mathbf{a} + \mathbf{b})$.
Since T on AQ: $\overrightarrow{OT} = \mathbf{a} + \mu(\overrightarrow{OQ} - \mathbf{a}) = (1-\mu)\mathbf{a} + 3\mu\mathbf{b}$.
Equating coefficients: $\frac{\lambda}{2} = 1 - \mu$ and $\frac{\lambda}{2} = 3\mu$. From these: $1 - \mu = 3\mu \Rightarrow \mu = \frac{1}{4}$.
Therefore: $\overrightarrow{OT} = (1 - \frac{1}{4})\mathbf{a} + 3(\frac{1}{4})\mathbf{b} = \frac{3}{4}\mathbf{a} + \frac{3}{4}\mathbf{b} = \frac{3}{4}(\mathbf{a} + \mathbf{b})$.

Problem 3.36: Circle Intersection of Sets

Let A and B be two distinct points in three-dimensional space. Let M be the midpoint of AB.
Let S_1 be the set of all points P such that $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$.
Let S_2 be the set of all points N such that $\left|\overrightarrow{AN}\right| = \left|\overrightarrow{MN}\right|$.
The intersection of S_1 and S_2 is the circle S.
What is the radius of the circle S?

A. $\dfrac{\left|\overrightarrow{AB}\right|}{2}$

B. $\dfrac{\left|\overrightarrow{AB}\right|}{4}$

C. $\dfrac{\sqrt{3}\left|\overrightarrow{AB}\right|}{2}$

D. $\dfrac{\sqrt{3}\left|\overrightarrow{AB}\right|}{4}$

Hint:(Step 1) Recognize that $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$ defines a sphere with diameter AB centered at M. (Step 2) The condition $|\overrightarrow{AN}| = |\overrightarrow{MN}|$ describes the perpendicular bisecting plane of segment AM. (Step 3) Use the Pythagorean relationship $r^2 + d^2 = R^2_{sphere}$ where d is the distance from the sphere center to the plane.

Solution 3.36: Sketch

Step 1: Analyze set S_1. The condition $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$ means vectors $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are perpendicular. The locus of points P that subtend a right angle to segment AB is a sphere with diameter AB. Center: M (midpoint of AB). Radius: $R_{sphere} = \frac{1}{2}|\overrightarrow{AB}|$.
Step 2: Analyze set S_2. The condition $|\overrightarrow{AN}| = |\overrightarrow{MN}|$ describes points equidistant from A and M, which is the perpendicular bisecting plane of segment AM.
Step 3: Find intersection. The intersection of a sphere (centered at M) and a plane is a circle. Using the Pythagorean relationship: $r^2 + d^2 = R^2_{sphere}$ where d is the perpendicular distance from center M to the plane.
Step 4: Calculate distances. Since the plane passes through the midpoint of AM, and $|\overrightarrow{AM}| = \frac{1}{2}|\overrightarrow{AB}|$, we have:

$$d = \frac{1}{2}|\overrightarrow{AM}| = \frac{1}{2}\left(\frac{1}{2}|\overrightarrow{AB}|\right) = \frac{1}{4}|\overrightarrow{AB}|$$

Step 5: Solve for radius r.

$$\begin{aligned} r^2 &= R^2_{sphere} - d^2 = \left(\frac{|\overrightarrow{AB}|}{2}\right)^2 - \left(\frac{|\overrightarrow{AB}|}{4}\right)^2 \\ &= \frac{|\overrightarrow{AB}|^2}{4} - \frac{|\overrightarrow{AB}|^2}{16} = \frac{4|\overrightarrow{AB}|^2 - |\overrightarrow{AB}|^2}{16} = \frac{3|\overrightarrow{AB}|^2}{16} \\ r &= \frac{\sqrt{3}|\overrightarrow{AB}|}{4} \end{aligned}$$

Therefore, the answer is **D**.

Problem 3.37: Complex Numbers and Centroid

The complex numbers x, w, and z are all different and all have modulus 1 (i.e., they lie on the unit circle in the complex plane).
The **centroid** (also called the center of mass) of three points is the average of their positions, given by $G = \frac{1}{3}(x + w + z)$.
Prove that $\frac{1}{3}(x + w + z)$ is never a cube root of xwz.
Note: A cube root of xwz is any complex number K satisfying $K^3 = xwz$.

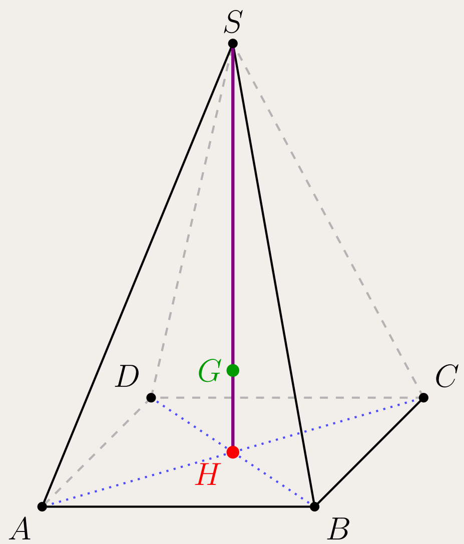
Hint: Show that the centroid has modulus strictly less than 1 (using the triangle inequality with strict inequality since the points are distinct), while any cube root of xwz has modulus exactly 1.

Solution 3.37: Sketch

Given: $|x| = |w| = |z| = 1$ and x, w, z are all different.
Step 1: Find modulus of any cube root. Let K be any cube root of xwz, so $K^3 = xwz$. Taking modulus: $|K|^3 = |xwz| = |x||w||z| = 1 \cdot 1 \cdot 1 = 1$. Therefore $|K| = 1$. Thus, any cube root of xwz lies on the unit circle.
Step 2: Find modulus of centroid. The centroid is $G = \frac{1}{3}(x + w + z)$. By triangle inequality: $|x + w + z| \leq |x| + |w| + |z| = 3$. Equality holds only if x, w, z have the same argument (same direction), which means $x = w = z$. But they are given to be different, so strict inequality holds: $|x + w + z| < 3$. Therefore: $|G| = \left|\frac{x+w+z}{3}\right| < 1$.
Conclusion: Since $|G| < 1$ but $|K| = 1$ for any cube root K, we have $G \neq K$. Therefore, the centroid is never a cube root of the product.

Problem 3.38: Pyramid Centroid

Consider a pyramid $SABCD$ with square base $ABCD$ and apex S. Let H be the center of the square base (the intersection point of the diagonals AC and BD).



Note: H (red) is the center of the square base, and G (green) is the centroid of the pyramid. The blue dotted lines show vectors from H to all five vertices.
The centroid G of the pyramid is the point such that:

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} + \overrightarrow{GS} = \vec{0}$$

Show that G lies on the line HS with $\overrightarrow{HG} = \frac{1}{5}\overrightarrow{HS}$.

Hint: Express each $\overrightarrow{GA}$, in terms of $\overrightarrow{GH}$ and $\overrightarrow{HA}$. Use the fact that H is the center of the square base, so $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD} = \vec{0}$ by symmetry.

Solution 3.38

Finding the centroid G
The centroid G of the five vertices satisfies:

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} + \overrightarrow{GS} = \vec{0}$$

Express each vector using the triangle rule: $\overrightarrow{GA} = \overrightarrow{GH} + \overrightarrow{HA}$:

$$\begin{aligned}\vec{0} &= \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} + \overrightarrow{GS} \\ &= (\overrightarrow{GH} + \overrightarrow{HA}) + (\overrightarrow{GH} + \overrightarrow{HB}) + (\overrightarrow{GH} + \overrightarrow{HC}) + (\overrightarrow{GH} + \overrightarrow{HD}) + (\overrightarrow{GH} + \overrightarrow{HS}) \\ &= 5\overrightarrow{GH} + (\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD}) + \overrightarrow{HS}\end{aligned}$$

Since H is the center of the square base, by symmetry the vectors from H to the four base vertices cancel:

$$\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD} = \vec{0}$$

(This follows because H is the midpoint of both diagonals: $\overrightarrow{HA} = -\overrightarrow{HC}$ and $\overrightarrow{HB} = -\overrightarrow{HD}$.)
Therefore:

$$\vec{0} = 5\overrightarrow{GH} + \overrightarrow{HS}$$

Rearranging:

$$5\overrightarrow{GH} = -\overrightarrow{HS} = \overrightarrow{SH}$$

Therefore:

$$\overrightarrow{GH} = \frac{1}{5}\overrightarrow{SH}$$

Or equivalently:

$$\overrightarrow{HG} = \frac{1}{5}\overrightarrow{HS}$$

This shows that G lies on the line segment HS, located $\frac{1}{5}$ of the way from H to S.
Answer: The centroid G divides the segment from the center of the base to the apex in the ratio $HG : GS = 1 : 4$.

Problem 3.39: Line-Sphere Intersection Points

Find intersection points of line $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and sphere $(x-1)^2 + (y-3)^2 + (z+4)^2 = 81$.

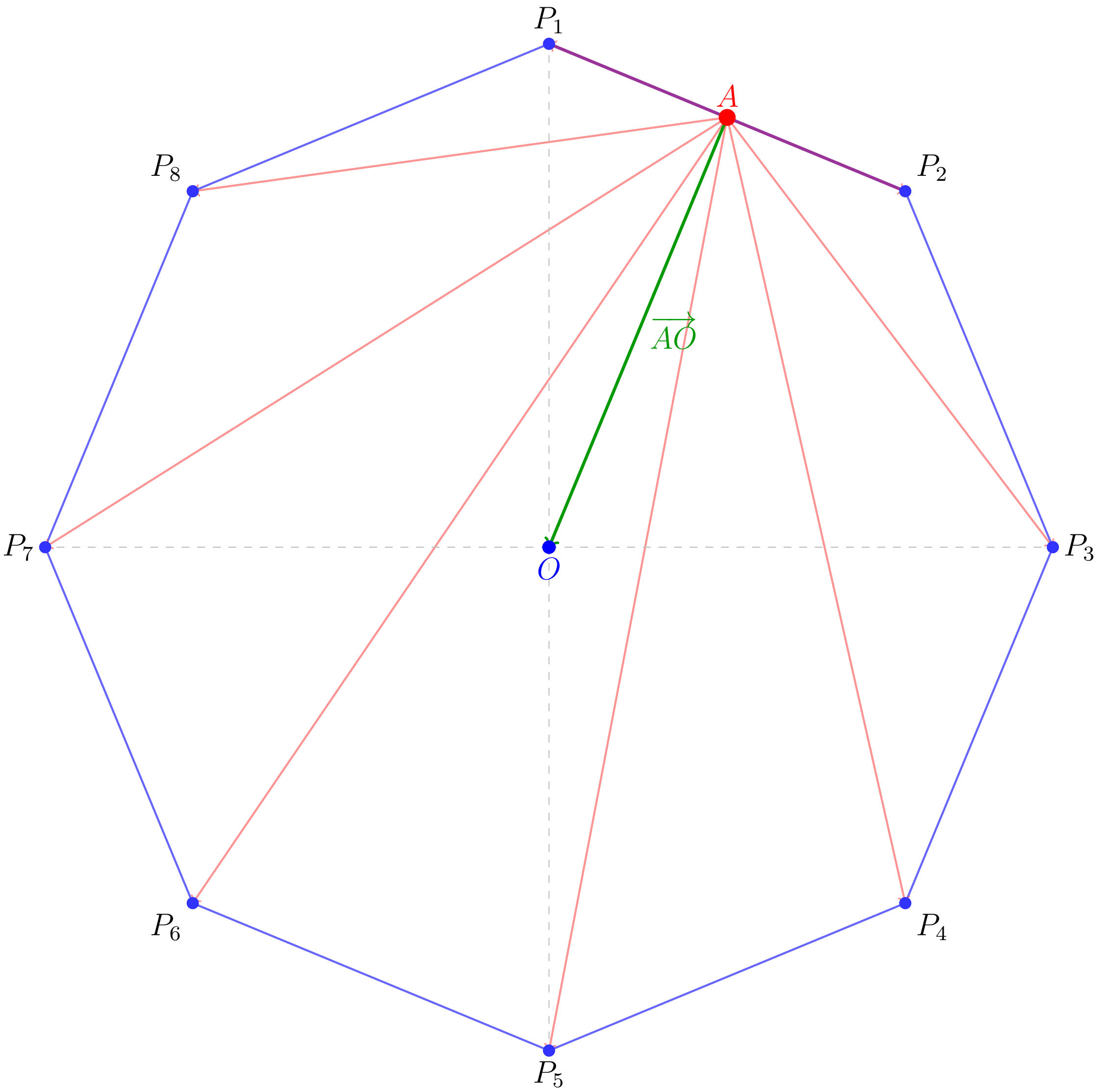
Hint: Substitute parametric equations into sphere equation and solve resulting quadratic.

Solution 3.39: Sketch

Parametric: $x = 1 + t$, $y = 3 + 2t$, $z = -4 + 2t$. Substitute: $(t)^2 + (2t)^2 + (2t)^2 = 81 \Rightarrow 9t^2 = 81 \Rightarrow t = \pm 3$. Points: $(4, 9, 2)$ when $t = 3$ and $(-2, -3, -10)$ when $t = -3$.

Problem 3.40: Regular Octagon Vector Sum

In regular octagon with side length 4, find magnitude of sum of vectors from midpoint of one side to all vertices.



Note: The regular octagon has vertices $P_1, P_2, \ldots, P_8$, center O (blue), and A (red) is the midpoint of side P_1P_2 (shown in violet). Red arrows show vectors from A to all eight vertices. The green arrow highlights $\overrightarrow{AO}$.

Hint: Use symmetry: sum of vectors from center to vertices is zero. Express via center.

Solution 3.40

Step 1: Use vector decomposition via the center.
For any point A and the center O, we can write each vector from A to a vertex P_i as:

$$\overrightarrow{AP_i} = \overrightarrow{AO} + \overrightarrow{OP_i}$$

Summing over all 8 vertices:

$$\begin{aligned}\sum_{i=1}^{8} \overrightarrow{AP_i} &= \sum_{i=1}^{8} (\overrightarrow{AO} + \overrightarrow{OP_i}) \\ &= 8\overrightarrow{AO} + \sum_{i=1}^{8} \overrightarrow{OP_i}\end{aligned}$$

Step 2: Apply symmetry to eliminate the sum from center.
By symmetry of a regular octagon, the 8 vertices are evenly distributed around the center O. The vectors $\overrightarrow{OP_1}, \overrightarrow{OP_2}, \dots, \overrightarrow{OP_8}$ point in directions separated by $45°$ and all have equal magnitude. Their vector sum is:

$$\sum_{i=1}^{8} \overrightarrow{OP_i} = \vec{0}$$

Therefore:

$$\sum_{i=1}^{8} \overrightarrow{AP_i} = 8\overrightarrow{AO}$$

Step 3: Calculate the apothem $|AO|$.
The apothem is the perpendicular distance from center to midpoint of a side. Consider the right triangle formed by O, midpoint A, and vertex P_1. The central angle between adjacent vertices is $45°$, so half this angle at O is $22.5°$. With opposite side $|AP_1| = 2$ (half the side length) and adjacent side $|AO|$:

$$\tan(22.5°) = \frac{2}{|AO|} \quad \Rightarrow \quad |AO| = 2\cot(22.5°)$$

Step 4: Evaluate $\cot(22.5°)$.
Using the half-angle formula: $\tan(22.5°) = \sqrt{2} - 1$. Therefore:

$$\cot(22.5°) = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \sqrt{2}+1$$

Thus: $|AO| = 2(\sqrt{2}+1)$.
Step 5: Find the final magnitude.
From Step 2, we have $\sum_{i=1}^{8} \overrightarrow{AP_i} = 8\overrightarrow{AO}$.
Taking magnitudes:

$$\left|\sum_{i=1}^{8} \overrightarrow{AP_i}\right| = |8\overrightarrow{AO}| = 8|AO| = 8 \cdot 2(\sqrt{2}+1) = 16(\sqrt{2}+1)$$

Answer: The magnitude of the sum of vectors is $\boxed{16(\sqrt{2}+1)}$ or approximately $\boxed{16(2.414) \approx 38.63}$.

4 Conclusion

Vectors are a powerful tool in HSC Mathematics Extension 2, unifying geometry, algebra, and analytic methods. Mastery comes from understanding both the algebraic manipulation and the geometric intuition behind each technique. Use these 60 problems to develop fluency across all vector topics, from basic computations to sophisticated proofs.

Best of luck with your studies and the HSC examination!

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Repository: https://github.com/vuhung16au/math-olympiad-ml/tree/main/HSC-Vectors