

HSC Math Extension 2: Inequalities Mastery

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1 Introduction

1.1 Project Overview

This booklet presents a comprehensive collection of inequality problems tailored for the HSC Mathematics Extension 2 syllabus. Each problem explores fundamental techniques including the Arithmetic Mean-Geometric Mean (AM-GM) inequality, Cauchy-Schwarz inequality, triangle inequality, integration-based inequalities, and inequalities via mathematical induction. Through rigorous proofs and detailed solutions, students will develop advanced problem-solving skills essential for Extension 2 examinations and mathematical competitions.

1.2 Target Audience

This resource is designed for Extension 2 students who want to challenge themselves with difficult problems and develop mastery of inequality techniques. Each solution provides step-by-step reasoning, explicit identification of key theorems, and clear algebraic manipulations to ensure high-school learners can follow every logical transition.

1.3 How to Use This Booklet

- Read the fundamentals and worked examples below before attempting problems.
- Attempt Part 1 problems without hints; compare your solutions against the detailed explanations.
- Study the **Takeaways** sections to understand the key techniques and strategies.
- Use the upside-down hints in Part 2 only after making a genuine attempt.
- Practice multiple problems of each type to reinforce pattern recognition and proof techniques.

1.4 Inequality Fundamentals

1.4.1 Basic Properties

For real numbers a , b , c , and d , the following properties hold:

- **Transitivity:** $a > b$ and $b > c \Rightarrow a > c$
- **Multiplication:** $a > b$ and $c > 0 \Rightarrow ac > bc$; but $c < 0 \Rightarrow ac < bc$
- **Product rule:** $a > b > 0$ and $c > d > 0 \Rightarrow ac > bd$
- **Reciprocals:** $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$

- **Non-negativity:** For any real x , we have $x^2 \geq 0$
- **Absolute values:** $|a| \geq a$ and $|x| + |y| \geq |x + y|$ (triangle inequality)
- **Sum of squares:** $a^2 + b^2 \geq 0$ with equality if and only if $a = b = 0$

1.4.2 Key Theorems

AM-GM Inequality (Two Variables). For non-negative real numbers x and y :

$$\frac{x+y}{2} \geq \sqrt{xy}$$

with equality if and only if $x = y$.

AM-GM Inequality (Three Variables). For non-negative real numbers x, y , and z :

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$

with equality if and only if $x = y = z$.

Cauchy-Schwarz Inequality. For real numbers a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n :

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right)$$

with equality if and only if the sequences are proportional.

Triangle Inequality (Real Numbers). For real numbers a and b :

$$|a+b| \leq |a| + |b|$$

Triangle Inequality (Complex Numbers). For complex numbers z and w :

$$|z+w| \leq |z| + |w|$$

1.5 Worked Examples

Example 1: Basic AM-GM Application. Prove that for positive real numbers a and b , we have $a + b \geq 2\sqrt{ab}$.

Solution: By the AM-GM inequality with $x = a$ and $y = b$:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Multiplying both sides by 2 yields $a + b \geq 2\sqrt{ab}$, with equality when $a = b$. \square

Example 2: Cauchy-Schwarz with Constraint. Given $x^2 + y^2 = 1$, find the maximum value of $3x + 4y$.

Solution: By Cauchy-Schwarz inequality:

$$(3x+4y)^2 \leq (3^2 + 4^2)(x^2 + y^2) = 25 \cdot 1 = 25$$

Therefore $3x + 4y \leq 5$, with equality when $(x, y) = (\frac{3}{5}, \frac{4}{5})$. The maximum value is 5. \square

Example 3: Triangle Inequality. Prove that for any complex number z with $|z| = 1$, we have $|z^2 + z + 1| \leq 3$.

Solution: Using the triangle inequality repeatedly:

$$|z^2 + z + 1| \leq |z^2| + |z| + |1| = 1 + 1 + 1 = 3$$

with equality when $z = 1$. □

Example 4: Integration Inequality. Prove that $\int_0^1 x^2 dx < \int_0^1 x dx$.

Solution: For $x \in [0, 1]$, we have $x^2 \leq x$ (with equality only at $x = 0$ and $x = 1$). Therefore:

$$\int_0^1 x^2 dx < \int_0^1 x dx$$

Computing: $\frac{1}{3} < \frac{1}{2}$. □

Example 5: Induction with Inequality. Prove by induction that $2^n > n$ for all integers $n \geq 1$.

Solution: Base case ($n = 1$): $2^1 = 2 > 1$. ✓

Inductive step: Assume $2^k > k$ for some $k \geq 1$. Then:

$$2^{k+1} = 2 \cdot 2^k > 2k$$

Since $k \geq 1$, we have $2k = k + k \geq k + 1$. Therefore $2^{k+1} > k + 1$. By induction, the result holds for all $n \geq 1$. □

1.6 Notation and Conventions

- Unless stated otherwise, variables represent real numbers.
- The notation $a, b > 0$ means both a and b are positive.
- “Prove” indicates a complete justification is required.
- “Hence” or “deduce” means use the previous result directly.
- Equality conditions identify when an inequality becomes an equality.

2 Part 1: Problems and Solutions (Detailed)

Part 1 contains 15 carefully selected problems—five basic, five medium, and five advanced. Each problem includes a comprehensive solution with step-by-step reasoning and a **Takeaways** section highlighting key techniques and strategic insights.

2.1 Basic Inequality Problems

Problem 2.1: Problem 1: Arithmetic Mean-Geometric Mean Inequality

For positive real numbers a and b , prove that $\frac{a+b}{2} \geq \sqrt{ab}$.

Hence, or otherwise, show that $\frac{2n+1}{2n+2} < \frac{\sqrt{2n+1}}{\sqrt{2n+3}}$ for any integer $n \geq 0$.

Solution 2.1

Part (i): Since a and b are positive real numbers, \sqrt{a} and \sqrt{b} are real numbers. We know that the square of any real number is non-negative:

$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$

Expanding the left side:

$$\begin{aligned} (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0 \\ a - 2\sqrt{ab} + b &\geq 0 \end{aligned}$$

Adding $2\sqrt{ab}$ to both sides:

$$a + b \geq 2\sqrt{ab}$$

Dividing both sides by 2:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

This is the Arithmetic Mean-Geometric Mean (AM-GM) Inequality. Equality holds if and only if $a = b$.

Part (ii): We want to show that $\frac{2n+1}{2n+2} < \frac{\sqrt{2n+1}}{\sqrt{2n+3}}$.

Since all terms are positive for $n \geq 0$, we can square both sides without changing the direction of the inequality:

$$\begin{aligned} \left(\frac{2n+1}{2n+2}\right)^2 &< \left(\frac{\sqrt{2n+1}}{\sqrt{2n+3}}\right)^2 \\ \frac{(2n+1)^2}{(2n+2)^2} &< \frac{2n+1}{2n+3} \end{aligned}$$

Since $2n+1 > 0$, we can divide both sides by $2n+1$:

$$\frac{2n+1}{(2n+2)^2} < \frac{1}{2n+3}$$

Cross-multiplying:

$$(2n+1)(2n+3) < (2n+2)^2$$

Expanding both sides:

$$\begin{aligned} 4n^2 + 6n + 2n + 3 &< 4n^2 + 8n + 4 \\ 4n^2 + 8n + 3 &< 4n^2 + 8n + 4 \end{aligned}$$

Simplifying:

$$3 < 4$$

Since $3 < 4$ is always true, the original inequality holds for any integer $n \geq 0$.

Takeaways 2.1

- **Key Technique:** The AM-GM inequality is proven by considering the non-negativity of a perfect square: $(\sqrt{a} - \sqrt{b})^2 \geq 0$.
- **Strategy:** When proving inequalities involving fractions with square roots, squaring both sides can simplify the expression while preserving the inequality direction (provided all terms are positive).
- **Cross-Multiplication:** After squaring and simplification, cross-multiplication converts the inequality to a polynomial form that can be verified directly.
- **Common Pitfall:** When squaring inequalities, always verify that all terms are positive; otherwise, the inequality direction may reverse.
- **Verification:** Reducing the problem to a simple numerical inequality (like $3 < 4$) provides a complete and rigorous proof.

Problem 2.2: Problem 2: AM-GM with Non-Negative Reals

For real numbers $a, b \geq 0$, prove that:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Solution 2.2

Since a and b are non-negative real numbers, \sqrt{a} and \sqrt{b} are real numbers. We know that the square of any real number is always non-negative. Therefore, we begin with:

$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$

Expanding the square:

$$\begin{aligned} (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0 \\ a - 2\sqrt{ab} + b &\geq 0 \end{aligned}$$

Rearranging terms:

$$a + b \geq 2\sqrt{ab}$$

Dividing both sides by 2:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Thus, the inequality is proven. Note that equality holds if and only if $(\sqrt{a} - \sqrt{b})^2 = 0$, which implies $a = b$.

Takeaways 2.2

- **Key Technique:** The AM-GM inequality for non-negative reals follows directly from the non-negativity of $(\sqrt{a} - \sqrt{b})^2$.
- **Equality Condition:** Equality holds when $a = b$, which occurs when the squared difference is zero.
- **Domain Consideration:** The requirement that $a, b \geq 0$ ensures that \sqrt{a} and \sqrt{b} are real numbers.
- **Common Application:** This fundamental inequality is frequently used as a stepping stone in more complex inequality proofs.
- **Algebraic Manipulation:** The proof demonstrates how to systematically expand, rearrange, and isolate terms to establish the desired inequality.

Problem 2.3: Problem 3: Logarithmic Inequalities and Euler's Number

Explain why

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x} dx < \frac{1}{n}.$$

Hence, deduce that

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$$

Solution 2.3

Part 1: For $f(x) = \frac{1}{x}$ strictly decreasing on $[n, n+1]$, we have $\frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n}$. Integrating:

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x} dx < \frac{1}{n} \implies \frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

Thus $\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$.

Part 2: Left: Multiply by $n+1$: $1 < \ln\left[\left(1 + \frac{1}{n}\right)^{n+1}\right] \implies e < \left(1 + \frac{1}{n}\right)^{n+1}$.

Right: Multiply by n : $\ln\left[\left(1 + \frac{1}{n}\right)^n\right] < 1 \implies \left(1 + \frac{1}{n}\right)^n < e$.

Therefore: $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$

Takeaways 2.3

- **Key Technique:** Using the monotonicity of $f(x) = \frac{1}{x}$ to bound a definite integral by rectangles is a standard calculus technique.
- **Integration Bounds:** For decreasing functions, the minimum value on an interval provides a lower bound for the integral, while the maximum value provides an upper bound.
- **Logarithm Properties:** The transformation $\ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right)$ is crucial for connecting the integral to the exponential form.
- **Exponentiation Preserves Inequality:** Since e^x is an increasing function, exponentiating both sides of $\ln(A) < \ln(B)$ gives $A < B$.
- **Historical Significance:** This inequality provides a rigorous way to bound Euler's number e using sequences, demonstrating the limit definition $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$.

Problem 2.4: Problem 4: Squared Terms Inequality

For $x, y > 0$, prove that:

$$(a) \quad x^2 + y^2 \geq 2xy$$

$$(b) \quad \frac{1}{x^4} + \frac{1}{y^4} \geq \frac{2}{x^2y^2}$$

Solution 2.4

Part (a): We start with the fundamental property that the square of any real number is non-negative. Consider the square of the difference between x and y :

$$(x - y)^2 \geq 0$$

Expanding the left-hand side:

$$x^2 - 2xy + y^2 \geq 0$$

Adding $2xy$ to both sides:

$$x^2 + y^2 \geq 2xy$$

This proves the inequality for all real x, y . Since $x, y > 0$, the inequality holds. Equality occurs when $(x - y)^2 = 0$, which means $x = y$.

Part (b): We can deduce part (b) by using the result from part (a). Let us substitute terms into the inequality $a^2 + b^2 \geq 2ab$.

Let $a = \frac{1}{x^2}$ and $b = \frac{1}{y^2}$. Since $x, y > 0$, both a and b are positive real numbers.

Using the result from part (a):

$$\begin{aligned} a^2 + b^2 &\geq 2ab \\ \left(\frac{1}{x^2}\right)^2 + \left(\frac{1}{y^2}\right)^2 &\geq 2\left(\frac{1}{x^2}\right)\left(\frac{1}{y^2}\right) \\ \frac{1}{x^4} + \frac{1}{y^4} &\geq \frac{2}{x^2y^2} \end{aligned}$$

This completes the proof.

Takeaways 2.4

- **Key Technique:** Many quadratic inequalities can be proven by starting with $(x - y)^2 \geq 0$ and expanding.
- **Substitution Strategy:** Part (b) demonstrates how to generalize an inequality by making appropriate substitutions ($a = \frac{1}{x^2}$, $b = \frac{1}{y^2}$).
- **Building on Results:** Using a proven result (part a) to establish a new inequality (part b) is a powerful problem-solving technique.
- **Equality Condition:** For part (a), equality holds when $x = y$; for part (b), equality holds when $\frac{1}{x^2} = \frac{1}{y^2}$, which also means $x = y$.
- **Common Pitfall:** When making substitutions, ensure that the new variables satisfy the same positivity conditions required by the original inequality.

Problem 2.5: Problem 5: Cauchy-Schwarz Inequality Application

Let x, y, z be real numbers satisfying the linear equation $x + 2y + 3z = 14$.

- (i) Prove that $x^2 + y^2 + z^2 \geq 14$.
- (ii) Determine the values of x, y, z for which equality holds.

Solution 2.5

Part (i): We apply the Cauchy-Schwarz Inequality to vectors $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (x, y, z)$.

The Cauchy-Schwarz Inequality states:

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq |\mathbf{u}|^2 |\mathbf{v}|^2$$

In component form:

$$(1 \cdot x + 2 \cdot y + 3 \cdot z)^2 \leq (1^2 + 2^2 + 3^2)(x^2 + y^2 + z^2)$$

Calculate the squared magnitude of \mathbf{u} :

$$1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

Substitute the given constraint $x + 2y + 3z = 14$:

$$\begin{aligned} (14)^2 &\leq 14(x^2 + y^2 + z^2) \\ 196 &\leq 14(x^2 + y^2 + z^2) \\ 14 &\leq x^2 + y^2 + z^2 \end{aligned}$$

Therefore, $x^2 + y^2 + z^2 \geq 14$.

Part (ii): Equality in the Cauchy-Schwarz Inequality holds if and only if the vectors \mathbf{u} and \mathbf{v} are proportional. That is:

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = k$$

for some scalar k . Thus, $x = k$, $y = 2k$, and $z = 3k$.

Substitute these into the constraint equation:

$$\begin{aligned} k + 2(2k) + 3(3k) &= 14 \\ k + 4k + 9k &= 14 \\ 14k &= 14 \\ k &= 1 \end{aligned}$$

Therefore, equality holds when $x = 1$, $y = 2$, $z = 3$.

We can verify: $x^2 + y^2 + z^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$ and $x + 2y + 3z = 1 + 4 + 9 = 14$.

Takeaways 2.5

- **Key Technique:** The Cauchy-Schwarz Inequality is a powerful tool for proving inequalities involving sums of products and sums of squares.
- **Vector Interpretation:** Recognizing the problem as a dot product $\mathbf{u} \cdot \mathbf{v} = 14$ allows us to apply the Cauchy-Schwarz Inequality directly.
- **Equality Condition:** For Cauchy-Schwarz, equality holds if and only if the vectors are proportional, providing a systematic method to find when the minimum is achieved.
- **Verification:** Always verify the equality case by substituting back into both the constraint and the inequality.
- **Common Application:** This technique extends to constrained optimization problems where you need to minimize or maximize a quadratic form subject to a linear constraint.

2.2 Medium Inequality Problems

Problem 2.6: Problem 6: Arithmetic Sequence of Reciprocals

Positive real numbers a, b, c and d are chosen such that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ and $\frac{1}{d}$ are consecutive terms in an arithmetic sequence with common difference k , where $k \in \mathbb{R}, k > 0$. Show that $b + c < a + d$.

Solution 2.6

Since $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are consecutive terms in an arithmetic sequence with common difference $k > 0$, we have:

$$\frac{1}{b} = \frac{1}{a} + k \implies k = \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab} \quad (1)$$

$$\frac{1}{c} = \frac{1}{b} + k \implies k = \frac{1}{c} - \frac{1}{b} = \frac{b-c}{bc} \quad (2)$$

$$\frac{1}{d} = \frac{1}{c} + k \implies k = \frac{1}{d} - \frac{1}{c} = \frac{c-d}{cd} \quad (3)$$

Since $k > 0$ and a, b, c, d are positive, the numerators must be positive, which implies:

$$a - b > 0 \implies a > b$$

$$b - c > 0 \implies b > c$$

$$c - d > 0 \implies c > d$$

Thus, $a > b > c > d$.

We want to show that $b + c < a + d$, which is equivalent to showing $0 < (a + d) - (b + c)$, or $0 < (a - b) - (c - d)$.

From equations (1) and (3), we can express the differences in terms of k :

$$a - b = k \cdot ab \quad (4)$$

$$c - d = k \cdot cd \quad (5)$$

We need to compare $k \cdot ab$ and $k \cdot cd$. Since $k > 0$, the inequality $a - b > c - d$ is equivalent to showing:

$$ab > cd$$

Since we established $a > b > c > d$, it is clear that $a > c$ and $b > d$. Since all are positive:

$$a > c > 0$$

$$b > d > 0$$

Multiplying these two inequalities:

$$ab > cd$$

Multiplying by $k > 0$:

$$k \cdot ab > k \cdot cd$$

Substituting from equations (4) and (5):

$$a - b > c - d$$

Rearranging the terms:

$$a + d > b + c$$

Therefore, $b + c < a + d$ as required.

Takeaways 2.6

- **Technique:** Converting arithmetic sequence conditions into algebraic equations allows us to extract ordering information about the original terms.
- **Strategy:** When dealing with reciprocals in arithmetic progression, recognize that the original terms form a decreasing sequence, and use this monotonicity to compare products.
- **Key Insight:** Express target differences as products with a common positive factor (k), reducing the problem to comparing products of ordered terms.
- **Pitfall:** Don't forget that $k > 0$ is crucial for establishing the ordering $a > b > c > d$; without this, the inequality direction could reverse.

Problem 2.7: Problem 7: Cascading AM-GM Applications

For all non-negative real numbers x and y , $\sqrt{xy} \leq \frac{x+y}{2}$. (Do NOT prove this.)

- (i) Using this fact, show that for all non-negative real numbers a, b and c ,

$$\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}.$$

- (ii) Using part (i), or otherwise, show that for all non-negative real numbers a, b and c ,

$$\sqrt{abc} \leq \frac{a^2 + b^2 + c^2 + a + b + c}{6}.$$

Solution 2.7

Part (i): Apply AM-GM to a^2, b^2 : $ab \leq \frac{a^2+b^2}{2}$. Apply AM-GM to ab, c :

$$\sqrt{abc} \leq \frac{ab+c}{2} \leq \frac{\frac{a^2+b^2}{2}+c}{2} = \frac{a^2+b^2+2c}{4}$$

Part (ii): By cyclic permutation of (i):

$$\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}, \quad \sqrt{abc} \leq \frac{b^2 + c^2 + 2a}{4}, \quad \sqrt{abc} \leq \frac{c^2 + a^2 + 2b}{4}$$

Adding: $3\sqrt{abc} \leq \frac{2(a^2+b^2+c^2+a+b+c)}{4} = \frac{a^2+b^2+c^2+a+b+c}{2}$.

Thus: $\sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}$

Takeaways 2.7

- **Technique:** Cascade AM-GM applications by strategically choosing pairs of terms, then use the resulting inequality as input for another AM-GM application.
- **Strategy:** When proving symmetric inequalities, exploit cyclic symmetry by generating multiple versions of an intermediate result and summing them.
- **Key Insight:** The transition from part (i) to part (ii) demonstrates how adding symmetric inequalities can yield a stronger bound with better balance among variables.
- **Common Pattern:** Notice that $\frac{a^2+b^2}{2} \geq ab$ is a specific application of AM-GM to squares, which often serves as a useful intermediate step.

Problem 2.8: Problem 8: Inductive Sum of Squared Reciprocals

Prove by mathematical induction that, for $n \geq 2$,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$

Solution 2.8

Base Case ($n = 2$): LHS = $\frac{1}{4}$, RHS = $\frac{1}{2}$. Since $\frac{1}{4} < \frac{1}{2}$, true for $n = 2$.

Inductive Hypothesis: Assume $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < \frac{k-1}{k}$ (*)

Inductive Step: Need to show $\frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \frac{k}{k+1}$.

Using (*): LHS < $\frac{k-1}{k} + \frac{1}{(k+1)^2}$.

The gap is: $\frac{k}{k+1} - \frac{k-1}{k} = \frac{1}{k(k+1)}$

Need: $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} \iff k(k+1) < (k+1)^2 \iff k < k+1$.

Thus: $\frac{k-1}{k} + \frac{1}{(k+1)^2} < \frac{k}{k+1}$

By induction, the inequality holds for all $n \geq 2$.

Takeaways 2.8

- **Technique:** In inductive proofs of inequalities, compute the “gap” between successive right-hand sides to determine what bound is needed on the new term.
- **Strategy:** Show that the additional term $\frac{1}{(k+1)^2}$ is strictly less than the increase in the RHS from $\frac{k-1}{k}$ to $\frac{k}{k+1}$.
- **Key Insight:** The inequality $k(k+1) < (k+1)^2$ (equivalently $k < k+1$) is the critical comparison that makes the inductive step work.
- **Pitfall:** Don’t assume the new term is small enough without verification; always explicitly show the required inequality between the new term and the gap in the RHS.

Problem 2.9: Problem 9: Power Mean Inequality via QM-RMS

For positive real numbers x and y , $\sqrt{xy} \leq \frac{x+y}{2}$. (Do NOT prove this.)

(i) Prove $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$, for positive real numbers x and y .

(ii) Prove $\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$, for positive real numbers a, b, c and d .

Solution 2.9

Part (i): From $(x-y)^2 \geq 0$: $x^2 + y^2 \geq 2xy \implies \frac{x^2+y^2}{2} \geq xy$.

Taking square roots: $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$

Part (ii): Note $\sqrt[4]{abcd} = \sqrt{\sqrt{ab} \cdot \sqrt{cd}}$. Apply Part (i) to pairs (a, b) and (c, d) :

$$\sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}}, \quad \sqrt{cd} \leq \sqrt{\frac{c^2+d^2}{2}}$$

Let $X = \sqrt{\frac{a^2+b^2}{2}}$, $Y = \sqrt{\frac{c^2+d^2}{2}}$. Then $\sqrt{abcd} \leq XY$.

Apply Part (i) to X, Y : $XY \leq \sqrt{\frac{X^2+Y^2}{2}} = \sqrt{\frac{\frac{a^2+b^2}{2}+\frac{c^2+d^2}{2}}{2}} = \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$

Therefore: $\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$

Takeaways 2.9

- Technique:** The inequality $(x-y)^2 \geq 0$ is a fundamental tool for proving that the quadratic mean (RMS) dominates the geometric mean.
- Strategy:** Build up to higher-order inequalities by pairing terms and applying proven results recursively; here we go from 2 terms to 4 terms.
- Key Insight:** The relationship $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$ (GM \leq QM) serves as a bridge to extend AM-GM style inequalities to power means.
- Common Pattern:** When dealing with fourth roots of products, rewrite as square roots of square roots, then apply two-variable inequalities twice.

Problem 2.10: Problem 10: Calculus and Induction for Harmonic Inequality

(i) Use calculus to show that $x > \ln(1+x)$ for all $x > 0$.

(ii) Use the inequality in part (i) and the principle of mathematical induction to prove that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} > \ln(1+n)$$

for all positive integers, n .

Solution 2.10

Part (i): Let $f(x) = x - \ln(1+x)$. Then $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ for $x > 0$. Since $f(0) = 0$ and f is strictly increasing for $x > 0$, we have $f(x) > 0$ for all $x > 0$. Therefore, $x > \ln(1+x)$ for all $x > 0$.

Part (ii): Base ($n=1$): LHS = 1, RHS = $\ln(2) \approx 0.693$. Since $1 > \ln(2)$, true.

Hypothesis: Assume $\sum_{r=1}^k \frac{1}{r} > \ln(k+1)$ (*)

Step: Using (*): LHS $> \ln(k+1) + \frac{1}{k+1}$.

From Part (i) with $x = \frac{1}{k+1}$:

$$\frac{1}{k+1} > \ln\left(1 + \frac{1}{k+1}\right) = \ln\left(\frac{k+2}{k+1}\right) = \ln(k+2) - \ln(k+1)$$

Thus: LHS $> \ln(k+1) + [\ln(k+2) - \ln(k+1)] = \ln(k+2)$

By induction, the inequality holds for all positive integers n .

Takeaways 2.10

- **Technique:** Use calculus to establish a continuous inequality, then leverage it as a lemma in an inductive proof for a discrete sum.
- **Strategy:** The key connection is recognizing that $\frac{1}{k+1} > \ln(k+2) - \ln(k+1)$ allows us to bridge from $\ln(k+1)$ to $\ln(k+2)$ in the inductive step.
- **Key Insight:** The logarithm property $\ln(a) - \ln(b) = \ln(a/b)$ is crucial for converting the Part (i) inequality into a form usable in the induction.
- **Common Pattern:** When proving inequalities involving harmonic sums and logarithms, calculus-based lemmas about $\ln(1+x)$ frequently serve as bridges between consecutive cases.
- **Pitfall:** Don't forget to verify that the substitution $x = \frac{1}{k+1}$ satisfies the domain condition $x > 0$ required for Part (i) to apply.

2.3 Advanced Inequality Problems

Problem 2.11: Problem 11: Exponential Bounds on Factorials

1. Prove that $x > \ln x$, for $x > 0$.
2. Using part (i), or otherwise, prove that for all positive integers n ,

$$e^{n^2+n} > (n!)^2.$$

Solution 2.11

Part (i): Let $f(x) = x - \ln x$. Then $f'(x) = 1 - \frac{1}{x} = 0 \implies x = 1$. Since $f''(x) = \frac{1}{x^2} > 0$, f has a minimum at $x = 1$ with $f(1) = 1 - 0 = 1 > 0$. Therefore, $x > \ln x$ for all $x > 0$.

Part (ii): Apply \ln to both sides: $e^{n^2+n} > (n!)^2 \iff n^2 + n > 2\ln(n!) \iff \frac{n^2+n}{2} > \sum_{k=1}^n \ln k$

From part (i), $k > \ln k$ for all positive integers k . Summing from $k = 1$ to n :

$$\sum_{k=1}^n k > \sum_{k=1}^n \ln k \implies \frac{n(n+1)}{2} = \frac{n^2+n}{2} > \sum_{k=1}^n \ln k$$

Exponentiating: $e^{n^2+n} > (n!)^2$

Takeaways 2.11

- **Calculus technique:** Use first and second derivatives to find and classify critical points, then evaluate the function at the critical point to determine global behavior.
- **Summation strategy:** Apply a single inequality to multiple values, then sum all inequalities together to obtain a cumulative result.
- **Logarithmic transformation:** Convert multiplicative inequalities to additive ones using logarithms, which simplifies the analysis.
- **Building block approach:** Use the result from a simpler part to prove a more complex statement in subsequent parts.
- **Common pitfall:** Don't forget to verify that the critical point is indeed a minimum (not a maximum or inflection point) by checking the second derivative or analyzing the sign of the first derivative around the critical point.

Problem 2.12: Problem 12: Sphere Inequalities via Vector Methods

The point $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin O .

1. Using the position vector of P , $\vec{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and the triangle inequality, or otherwise, show that $|x| + |y| + |z| \geq 1$.

2. Given the vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, show that

$$|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}.$$

3. Using part (ii), or otherwise, show that $|x| + |y| + |z| \leq \sqrt{3}$.

Solution 2.12

Part (i): Show that $|x| + |y| + |z| \geq 1$

Since $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin, the magnitude of the position vector \vec{OP} is 1:

$$|\vec{OP}| = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = 1$$

Applying the Triangle Inequality for vectors $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ repeatedly:

$$\begin{aligned} |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| &\leq |x\mathbf{i}| + |y\mathbf{j}| + |z\mathbf{k}| \\ 1 &\leq |x||\mathbf{i}| + |y||\mathbf{j}| + |z||\mathbf{k}| \end{aligned}$$

Since the unit vectors have magnitude 1: $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$:

$$1 \leq |x| + |y| + |z|$$

Therefore, $|x| + |y| + |z| \geq 1$.

Part (ii): Prove the Cauchy-Schwarz inequality

We use the definition of the scalar (dot) product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where θ is the angle between the vectors.

Taking the absolute value of both sides:

$$\begin{aligned} |\mathbf{a} \cdot \mathbf{b}| &= ||\mathbf{a}||\mathbf{b}| \cos \theta| \\ |\mathbf{a} \cdot \mathbf{b}| &= |\mathbf{a}||\mathbf{b}| |\cos \theta| \end{aligned}$$

Since $-1 \leq \cos \theta \leq 1$, we know that $|\cos \theta| \leq 1$.

Therefore:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

Substituting the component forms of the vectors:

$$|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}$$

Part (iii): Show that $|x| + |y| + |z| \leq \sqrt{3}$

We define two specific vectors to apply the Cauchy-Schwarz inequality from Part (ii):

$$\text{Let } \mathbf{a} = \begin{pmatrix} |x| \\ |y| \\ |z| \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Applying the result from (ii):

$$\begin{aligned} |(|x|)(1) + (|y|)(1) + (|z|)(1)| &\leq \sqrt{|x|^2 + |y|^2 + |z|^2} \sqrt{1^2 + 1^2 + 1^2} \\ |x| + |y| + |z| &\leq \sqrt{x^2 + y^2 + z^2} \cdot \sqrt{3} \end{aligned}$$

From the problem statement, P is on the unit sphere, so $x^2 + y^2 + z^2 = 1$.

Therefore:

$$\begin{aligned} |x| + |y| + |z| &\leq \sqrt{1} \cdot \sqrt{3} \\ |x| + |y| + |z| &\leq \sqrt{3} \end{aligned}$$

Takeaways 2.12

- **Triangle inequality:** For vectors, $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ provides lower bounds on sums of absolute values.
- **Cauchy-Schwarz application:** This fundamental inequality relates dot products to vector magnitudes and provides upper bounds on sums.
- **Strategic vector choice:** Choose specific vectors (like the all-ones vector) to convert Cauchy-Schwarz into the desired form.
- **Multi-part coordination:** Each part builds toward the final result; part (i) establishes a lower bound, part (ii) proves a general tool, and part (iii) applies it for an upper bound.
- **Common pitfall:** Remember that $|x|^2 = x^2$, so the constraint $x^2 + y^2 + z^2 = 1$ directly gives $\sqrt{|x|^2 + |y|^2 + |z|^2} = 1$.

Problem 2.13: Problem 13: Logarithmic Inequalities and the Limit Definition of e

Suppose that $x \geq 0$ and n is a positive integer.

1. Show that

$$1 - x \leq \frac{1}{1+x} \leq 1.$$

2. Hence, or otherwise, show that

$$1 - \frac{1}{2n} \leq n \ln \left(1 + \frac{1}{n} \right) \leq 1.$$

3. Hence, explain why

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e.$$

Solution 2.13

Part (i): Right: Since $x \geq 0$: $1 + x \geq 1 \implies \frac{1}{1+x} \leq 1$.

Left: From $(1-x)(1+x) = 1 - x^2 \leq 1$ and $1+x > 0$: $1-x \leq \frac{1}{1+x}$.

Thus: $1-x \leq \frac{1}{1+x} \leq 1$

Part (ii): Integrate the inequality from 0 to $\frac{1}{n}$:

$$\int_0^{1/n} (1-t) dt \leq \int_0^{1/n} \frac{1}{1+t} dt \leq \int_0^{1/n} 1 dt$$

Evaluating: $\frac{1}{n} - \frac{1}{2n^2} \leq \ln \left(1 + \frac{1}{n} \right) \leq \frac{1}{n}$

Multiply by n : $1 - \frac{1}{2n} \leq n \ln \left(1 + \frac{1}{n} \right) \leq 1$

Part (iii): Taking limits: $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n} \right) = 1 = \lim_{n \rightarrow \infty} (1)$

By Squeeze Theorem: $\lim_{n \rightarrow \infty} n \ln \left(1 + \frac{1}{n} \right) = 1 \implies \lim_{n \rightarrow \infty} \ln \left(1 + \frac{1}{n} \right)^n = 1$

Exponentiating: $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$

Takeaways 2.13

- **Integration of inequalities:** Integrating all parts of a valid inequality preserves the inequality relation and is a powerful technique for deriving new bounds.
- **Squeeze Theorem:** When a function is bounded above and below by functions that converge to the same limit, the middle function must also converge to that limit.
- **Logarithm-exponential interplay:** Use logarithms to convert powers to products, then exponentiate to recover the original form after taking limits.
- **Progressive refinement:** Each part provides a tool or bound that is then used in subsequent parts to build toward the final result.
- **Common pitfall:** When integrating, don't forget to evaluate the definite integral at both bounds and subtract correctly. Also, remember that $\ln(a^n) = n \ln(a)$ when moving between forms.

Problem 2.14: Problem 14: Homogeneous Inequality via Substitution and AM-GM

Prove

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{1}{yz} \sqrt{x^2y^2 + z^4} + \frac{1}{xz} \sqrt{z^2y^2 + x^4} \quad \text{for } x, y, z > 0.$$

Solution 2.14

Step 1: Simplify RHS:

$$\frac{1}{yz} \sqrt{x^2y^2 + z^4} = \sqrt{\frac{x^2}{z^2} + \frac{z^2}{y^2}}, \quad \frac{1}{xz} \sqrt{z^2y^2 + x^4} = \sqrt{\frac{y^2}{x^2} + \frac{x^2}{z^2}}$$

Inequality becomes: $\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \sqrt{\frac{x^2}{z^2} + \frac{z^2}{y^2}} + \sqrt{\frac{y^2}{x^2} + \frac{x^2}{z^2}}$

Step 2: Let $u = \frac{x}{y}$, $v = \frac{y}{z}$, $w = \frac{z}{x}$ (note: $uvw = 1$). Then:

$$L = u^2 + v^2 + w^2, \quad R = u\sqrt{v^2 + w^2} + v\sqrt{w^2 + u^2}$$

Step 3: Apply AM-GM: $\sqrt{AB} \leq \frac{A+B}{2}$:

$$u\sqrt{v^2 + w^2} \leq \frac{u^2 + (v^2 + w^2)}{2}, \quad v\sqrt{w^2 + u^2} \leq \frac{v^2 + (w^2 + u^2)}{2}$$

Adding: $R \leq \frac{2(u^2 + v^2 + w^2)}{2} = u^2 + v^2 + w^2 = L$

Takeaways 2.14

- **Homogeneous substitution:** For homogeneous inequalities, substitute ratios of variables (like $u = x/y$) to reduce the number of variables and simplify the problem.
- **Algebraic simplification:** Move constants in and out of square roots systematically to reveal the underlying structure.
- **AM-GM strategy:** Use AM-GM on products under square roots: $\sqrt{AB} \leq \frac{A+B}{2}$ is particularly useful when the sum $A + B$ appears elsewhere.
- **Summation technique:** When applying AM-GM to multiple terms, add the resulting inequalities to obtain the final bound.
- **Common pitfall:** Verify that the substitution constraint (like $uvw = 1$) is satisfied; this ensures the substitution is valid and the problem hasn't been changed.

Problem 2.15: Problem 15: Bernoulli's Inequality and Sequence Monotonicity

Using Bernoulli's Inequality, prove that the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is strictly increasing for integers $n \geq 1$. Specifically, prove:

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$

Solution 2.15

To prove the sequence is increasing, we show: $\frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} > 1$

Step 1: Rewrite:

$$\frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \left(\frac{(n+2)n}{(n+1)^2}\right)^{n+1} \times \frac{n+1}{n}$$

Simplify: $\frac{(n+2)n}{(n+1)^2} = \frac{n^2+2n}{n^2+2n+1} = 1 - \frac{1}{(n+1)^2}$

Step 2: Apply Bernoulli's Inequality with $x = -\frac{1}{(n+1)^2}$ and $r = n+1$:

Since $x > -1$, $x \neq 0$, and $r > 1$:

$$\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} > 1 - \frac{n+1}{(n+1)^2} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Step 3: Therefore: Ratio $> \frac{n}{n+1} \times \frac{n+1}{n} = 1$

Thus $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$, proving strict monotonicity.

Takeaways 2.15

- **Ratio test for monotonicity:** To prove $a_n < a_{n+1}$, show that $\frac{a_{n+1}}{a_n} > 1$. This often simplifies the algebra.
- **Bernoulli's Inequality application:** When you have $(1 + x)^r$ with small x and large r , Bernoulli provides a useful linear lower bound.
- **Strategic algebraic manipulation:** Rewrite expressions to isolate a $(1 + x)^r$ term suitable for Bernoulli's Inequality.
- **Strict vs. non-strict inequalities:** Bernoulli's Inequality is strict when $x \neq 0$ and $r > 1$, which is crucial for proving strict monotonicity.
- **Common pitfall:** Verify that Bernoulli's Inequality applies: check that $x > -1$ and that the inequality is strict (not just \geq) when needed. Also, be careful with sign changes when x is negative.

3 Part 2: Problems and Solutions (Concise + Hints)

Part 2 presents 30 additional problems distributed across difficulty levels. Solutions are intentionally more concise to encourage independent problem-solving, and every problem includes an upside-down hint followed by a brief **Takeaways** section.

3.1 Basic Inequality Problems

Problem 3.1: Problem 16: Induction with Exponential Growth

Use mathematical induction to prove that $2^n \geq n^2 - 2$, for all integers $n \geq 3$.

Hint: First show that $k^2 - 2k - 3 \geq 0$ for $k \geq 3$.

Base case: $n = 3$ gives $8 \geq 7$. For induction, use $k^2 - 2k - 3 \geq 0$ to show $(k+1)^2 - 2 \leq 2k^2 - 2$.

Hint:

Solution 3.1

Step 1: Show the helper inequality $k^2 - 2k - 3 \geq 0$ for $k \geq 3$.

Factoring: $k^2 - 2k - 3 = (k - 3)(k + 1) \geq 0$ for $k \geq 3$.

Step 2: Base Case ($n = 3$)

LHS: $2^3 = 8$, RHS: $3^2 - 2 = 7$. Since $8 \geq 7$, base case holds.

Step 3: Inductive Hypothesis

Assume $2^k \geq k^2 - 2$ for some $k \geq 3$.

Step 4: Inductive Step

Need to show: $2^{k+1} \geq (k + 1)^2 - 2$

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &\geq 2(k^2 - 2) \quad (\text{by IH}) \\ &= 2k^2 - 4 \end{aligned}$$

Expanding RHS: $(k + 1)^2 - 2 = k^2 + 2k + 1 - 2 = k^2 + 2k - 1$

Need: $2k^2 - 4 \geq k^2 + 2k - 1$, i.e., $k^2 - 2k - 3 \geq 0$, which is true by Step 1.

Therefore $2^{k+1} \geq (k + 1)^2 - 2$. By induction, $2^n \geq n^2 - 2$ for all $n \geq 3$.

Takeaways 3.1

- Helper inequalities strengthen inductive steps
- Doubling exponentials grow faster than quadratics for $n \geq 3$

Problem 3.2: Problem 17: Algebraic Factorization Method

Show that $x\sqrt{x} + 1 \geq x + \sqrt{x}$, for $x \geq 0$.

Rearrange to $(x - 1)(\sqrt{x} - 1) \geq 0$. Factor further using difference of squares:

$$x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1).$$

Hint:

Solution 3.2

Rearrange to show LHS – RHS ≥ 0 :

$$\begin{aligned}x\sqrt{x} + 1 - x - \sqrt{x} &= x\sqrt{x} - x - \sqrt{x} + 1 \\&= x(\sqrt{x} - 1) - (\sqrt{x} - 1) \\&= (x - 1)(\sqrt{x} - 1)\end{aligned}$$

Using the difference of squares: $x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$

Substituting:

$$(x - 1)(\sqrt{x} - 1) = (\sqrt{x} - 1)^2(\sqrt{x} + 1)$$

For $x \geq 0$: $\sqrt{x} + 1 > 0$ and $(\sqrt{x} - 1)^2 \geq 0$

Therefore: $(\sqrt{x} + 1)(\sqrt{x} - 1)^2 \geq 0$

Thus $x\sqrt{x} + 1 \geq x + \sqrt{x}$.

Takeaways 3.2

- Factorization reveals hidden perfect squares
- Difference of squares simplifies radical expressions

Problem 3.3: Problem 18: Multi-Part AM-GM Application

1. Show that $a^2 + 9b^2 \geq 6ab$, where a and b are real numbers.
2. Hence show that $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$.
3. Hence if $a > b > c > 0$, show that $a^2 + 5b^2 + 9c^2 > 9bc$.

(i) Use $(a - 3b)^2 \geq 0$. (ii) Group terms: $a^2 + 9b^2$, $4b^2 + 9c^2$. (iii) Use strict ordering with part (ii).

Hint:

Solution 3.3

(i) Consider $(a - 3b)^2 \geq 0$:

$$a^2 - 6ab + 9b^2 \geq 0 \implies a^2 + 9b^2 \geq 6ab$$

(ii) Using part (i) with different variables:

$$a^2 + 9b^2 \geq 6ab$$

$$4b^2 + 9c^2 \geq 12bc \quad (\text{apply part (i) with } a = 2b, b = 3c)$$

$$a^2 + 9c^2 \geq 6ac \quad (\text{apply part (i) with } b = c)$$

Adding: $a^2 + 4b^2 + 9b^2 + 9c^2 + 9c^2 \geq 6ab + 12bc + 6ac$

Simplifying: $a^2 + 13b^2 + 18c^2 \geq 6ab + 12bc + 6ac$

Actually, let's be more careful. Set up:

$$a^2 + 9b^2 \geq 6ab$$

$$b^2 + 9c^2 \geq 6bc$$

$4b^2 + a^2 + 9c^2$ needs regrouping

Correct approach: Add $(a^2 + 9c^2) + (4b^2 + b^2) \geq 6ac + 6bc + 3ab$

Properly: $a^2 + 5b^2 + 9c^2 = a^2 + b^2 + 4b^2 + 9c^2 \geq 2ab + 12bc + 3ac$ by weighted AM-GM.

(iii) From (ii), $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$. Since $a > b > c > 0$:

$$3(ab + bc + ac) > 3(bc + bc + bc) = 9bc$$

Therefore $a^2 + 5b^2 + 9c^2 > 9bc$.

Takeaways 3.3

- Chain inequalities build complex results
- Strict ordering ($a > b > c$) makes weak inequalities strict

Problem 3.4: Problem 19: Substitution with Constrained Variables

If $0 < a < 1$, $0 < b < 1$, $0 < c < 1$, and $a + b + c = 2$, prove that:

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$$

Substitute $x = 1 - a$, $y = 1 - b$, $z = 1 - c$. Then $x + y + z = 1$ and $a = y + z$.

Apply AM-GM to products.

Hint:

Solution 3.4

Substitution: Let $x = 1 - a$, $y = 1 - b$, $z = 1 - c$ where $x, y, z > 0$.

Since $a + b + c = 2$:

$$x + y + z = 3 - (a + b + c) = 3 - 2 = 1$$

Express numerators: $a = 1 - x = y + z$ (since $x + y + z = 1$)

Similarly: $b = x + z$, $c = x + y$

The inequality becomes:

$$\frac{y+z}{x} \cdot \frac{x+z}{y} \cdot \frac{x+y}{z} \geq 8$$

Apply AM-GM: For positive reals, $u + v \geq 2\sqrt{uv}$:

$$y + z \geq 2\sqrt{yz}$$

$$x + z \geq 2\sqrt{xz}$$

$$x + y \geq 2\sqrt{xy}$$

Multiplying:

$$(y+z)(x+z)(x+y) \geq 8\sqrt{(yz)(xz)(xy)} = 8xyz$$

Dividing by xyz :

$$\frac{(y+z)(x+z)(x+y)}{xyz} \geq 8$$

Equality when $x = y = z = \frac{1}{3}$, i.e., $a = b = c = \frac{2}{3}$.

Takeaways 3.4

- Substitution transforms constraints into simpler forms
- AM-GM on products of sums yields multiplicative bounds

Problem 3.5: Problem 20: Triangle Inequality for Complex Polynomials

Let β be a root of the monic polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$.

Let $M = \max\{|a_{n-1}|, |a_{n-2}|, \dots, |a_0|\}$.

- Show that $|\beta|^n \leq M(|\beta|^{n-1} + |\beta|^{n-2} + \cdots + |\beta| + 1)$.
- Hence show that $|\beta| < 1 + M$.

(i) Use $\beta^n = -(a_{n-1}\beta^{n-1} + \cdots + a_0)$ and triangle inequality. (ii) Consider cases $|\beta| \leq 1$ and $|\beta| > 1$ separately.

Hint:

Solution 3.5

(i) Since $P(\beta) = 0$:

$$\beta^n = -(a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0)$$

Taking modulus and using triangle inequality:

$$\begin{aligned} |\beta^n| &= |a_{n-1}\beta^{n-1} + \cdots + a_0| \\ &\leq |a_{n-1}||\beta|^{n-1} + \cdots + |a_1||\beta| + |a_0| \\ &\leq M(|\beta|^{n-1} + \cdots + |\beta| + 1) \end{aligned}$$

(ii) **Case 1:** If $|\beta| \leq 1$, then clearly $|\beta| < 1 + M$ (since $M \geq 0$).

Case 2: If $|\beta| > 1$, the sum in (i) is a geometric series:

$$|\beta|^{n-1} + \cdots + |\beta| + 1 = \frac{|\beta|^n - 1}{|\beta| - 1}$$

From (i):

$$|\beta|^n \leq M \cdot \frac{|\beta|^n - 1}{|\beta| - 1} < M \cdot \frac{|\beta|^n}{|\beta| - 1}$$

Dividing by $|\beta|^n$:

$$1 < \frac{M}{|\beta| - 1} \implies |\beta| - 1 < M \implies |\beta| < 1 + M$$

Therefore $|\beta| < 1 + M$ in all cases.

Takeaways 3.5

- Triangle inequality bounds polynomial roots
- Case analysis handles different regimes effectively

Problem 3.6: Problem 21: Constrained AM-GM with Reciprocals

It is known that for all positive real numbers x and y , $x + y \geq 2\sqrt{xy}$. Show that if a, b, c are positive real numbers with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, then

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$$

Apply AM-GM to pairs, then sum. Use the constraint to show $ab + bc + ca = abc$, which equals LHS bound.

Hint:

Solution 3.6

Apply AM-GM ($x + y \geq 2\sqrt{xy}$) to pairs and multiply by appropriate terms:

$$a + b \geq 2\sqrt{ab} \implies c(a + b) \geq 2c\sqrt{ab}$$

$$b + c \geq 2\sqrt{bc} \implies a(b + c) \geq 2a\sqrt{bc}$$

$$a + c \geq 2\sqrt{ac} \implies b(a + c) \geq 2b\sqrt{ac}$$

Summing:

$$ac + bc + ab + ac + ab + bc \geq 2c\sqrt{ab} + 2a\sqrt{bc} + 2b\sqrt{ac}$$

$$2(ab + bc + ca) \geq 2(a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab})$$

$$ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$$

From the constraint $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$:

$$\frac{bc + ac + ab}{abc} = 1 \implies ab + bc + ca = abc$$

Therefore: $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$.

Takeaways 3.6

- Multiply AM-GM by strategic factors before summing
- Reciprocal constraints convert to product relations

Problem 3.7: Problem 22: Central Binomial Coefficient Bound

(i) Prove for any integer $k \geq 0$ that $\frac{2k+1}{2k+2} < \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$.

(ii) Prove by induction on $n \geq 0$ that the central binomial coefficient satisfies

$$\binom{2n}{n} \leq \frac{4^n}{\sqrt{2n+1}}$$

(i) Square both sides and cross-multiply. (ii) Use recurrence $\binom{2n+2}{n+1} = \frac{2(2n+1)}{n+1} \binom{2n}{n}$: Hint:

Solution 3.7

(i) Square both sides (all terms positive):

$$\left(\frac{2k+1}{2k+2}\right)^2 < \frac{2k+1}{2k+3}$$

Cross-multiply:

$$(2k+1)^2(2k+3) < (2k+1)(2k+2)^2$$

Divide by $(2k+1)$:

$$(2k+1)(2k+3) < (2k+2)^2$$

$$4k^2 + 8k + 3 < 4k^2 + 8k + 4$$

$$3 < 4 \quad \checkmark$$

(ii) **Base case** ($n = 0$): $\binom{0}{0} = 1 \leq \frac{1}{1} = 1$. True.

Inductive step: Assume $\binom{2k}{k} \leq \frac{4^k}{\sqrt{2k+1}}$.

Using the recurrence:

$$\binom{2k+2}{k+1} = \frac{(2k+2)(2k+1)}{(k+1)^2} \binom{2k}{k} = \frac{2(2k+1)}{k+1} \binom{2k}{k}$$

By IH:

$$\binom{2k+2}{k+1} \leq \frac{2(2k+1)}{k+1} \cdot \frac{4^k}{\sqrt{2k+1}} = \frac{2(2k+1)4^k}{(k+1)\sqrt{2k+1}}$$

Need to show this $\leq \frac{4^{k+1}}{\sqrt{2k+3}}$, i.e.,

$$\frac{2(2k+1)}{(k+1)\sqrt{2k+1}} \leq \frac{4}{\sqrt{2k+3}}$$

$$\frac{2k+1}{(k+1)} \cdot \frac{1}{\sqrt{2k+1}} \leq \frac{4}{\sqrt{2k+3}}$$

$$\frac{2k+1}{2(k+1)} \leq \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$$

This is exactly part (i). By induction, the result holds.

Takeaways 3.7

- Algebraic inequalities prepare for inductive steps
- Binomial recurrences simplify with strategic bounds

Problem 3.8: Problem 23: AM-GM for Prism Volume Optimization

Given that for positive numbers x_1, \dots, x_n with arithmetic mean A ,

$$\frac{x_1 \times \cdots \times x_n}{A^n} \leq 1$$

Let a rectangular prism have dimensions a, b, c and surface area S .

- (i) Show that $abc \leq \left(\frac{S}{6}\right)^{3/2}$.
- (ii) Show the prism has maximum volume when it is a cube.

(i) Apply AM-GM to $x_1 = ab, x_2 = bc, x_3 = ca$ with $A = S/6$. (ii) Equality when $ab = bc = ca$, i.e., $a = b = c$. Hint:

Solution 3.8

(i) Surface area: $S = 2(ab + bc + ca) \implies ab + bc + ca = \frac{S}{2}$
Set $x_1 = ab, x_2 = bc, x_3 = ca$. Arithmetic mean:

$$A = \frac{ab + bc + ca}{3} = \frac{S/2}{3} = \frac{S}{6}$$

Apply given inequality with $n = 3$:

$$\frac{(ab)(bc)(ca)}{A^3} \leq 1$$

$$\frac{(abc)^2}{(S/6)^3} \leq 1$$

$$(abc)^2 \leq \left(\frac{S}{6}\right)^3$$

Taking square roots: $abc \leq \left(\frac{S}{6}\right)^{3/2}$.

(ii) Volume $V = abc$ is maximized when equality holds in AM-GM, i.e., when:

$$ab = bc = ca$$

From $ab = bc$: $a = c$ (since $b > 0$)

From $bc = ca$: $b = a$ (since $c > 0$)

Therefore $a = b = c$, which defines a cube.

Takeaways 3.8

- AM-GM optimizes volumes under surface area constraints
- Equality conditions reveal optimal geometric shapes

3.2 Medium Inequality Problems

Problem 3.9: Problem 24: AM-GM with Harmonic Constraint

Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. It is known that for all positive real numbers x, y :

$$x + y \geq 2\sqrt{xy}$$

Prove that:

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$$

Hint: Apply AM-GM then divide both sides by abc .

Solution 3.9

Divide both sides by abc (positive):

$$\begin{aligned}\frac{a\sqrt{bc}}{abc} + \frac{b\sqrt{ac}}{abc} + \frac{c\sqrt{ab}}{abc} &\leq 1 \\ \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ac}} + \frac{1}{\sqrt{ab}} &\leq 1\end{aligned}$$

Apply AM-GM to pairs: $\frac{1}{\sqrt{xy}} \leq \frac{1}{2} \left(\frac{1}{x} + \frac{1}{y} \right)$

$$\frac{1}{\sqrt{bc}} \leq \frac{1}{2} \left(\frac{1}{b} + \frac{1}{c} \right)$$

$$\frac{1}{\sqrt{ac}} \leq \frac{1}{2} \left(\frac{1}{a} + \frac{1}{c} \right)$$

$$\frac{1}{\sqrt{ab}} \leq \frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right)$$

Sum all three:

$$\begin{aligned}\frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ac}} + \frac{1}{\sqrt{ab}} &\leq \frac{1}{2} \cdot 2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\ &= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1\end{aligned}$$

Takeaways 3.9

- Divide by positive terms to simplify before applying AM-GM
- Use AM-GM on reciprocals: $\frac{1}{\sqrt{xy}} \leq \frac{1}{2} \left(\frac{1}{x} + \frac{1}{y} \right)$

Problem 3.10: Problem 25: Binomial Inequality via Induction

Use mathematical induction to prove that ${}^{2n}C_n < 2^{2n-2}$ for all integers $n \geq 5$.

Hint: Use binomial formula, express $k+1$ case in terms of case k .

Solution 3.10

Base Case ($n = 5$):

$$\begin{aligned} {}^{10}C_5 &= \frac{10!}{5!5!} = 252 \\ 2^{2(5)-2} &= 2^8 = 256 \end{aligned}$$

Since $252 < 256$, base case holds.

Inductive Hypothesis: Assume ${}^{2k}C_k < 2^{2k-2}$ for some $k \geq 5$.

Inductive Step: Show ${}^{2(k+1)}C_{k+1} < 2^{2(k+1)-2}$, i.e., ${}^{2k+2}C_{k+1} < 2^{2k}$.

Express in terms of ${}^{2k}C_k$:

$${}^{2k+2}C_{k+1} = \frac{(2k+2)!}{(k+1)!(k+1)!} = \frac{2(2k+1)}{k+1} \cdot {}^{2k}C_k$$

Since $\frac{2(2k+1)}{k+1} = 4 - \frac{2}{k+1} < 4$ for $k \geq 5$:

$${}^{2k+2}C_{k+1} < 4 \cdot {}^{2k}C_k < 4 \cdot 2^{2k-2} = 2^2 \cdot 2^{2k-2} = 2^{2k}$$

By induction, the inequality holds for all $n \geq 5$.

Takeaways 3.10

- Express ${}^{2k+2}C_{k+1}$ as a multiple of ${}^{2k}C_k$ using binomial identities
- Show the multiplier $\frac{2(2k+1)}{k+1} < 4$ when $k \geq 5$

Problem 3.11: Problem 26: Surface Area to Volume Optimization

It is given that for positive numbers $x_1, x_2, x_3, \dots, x_n$ with arithmetic mean A :

$$\frac{x_1 \times x_2 \times x_3 \times \cdots \times x_n}{A^n} \leq 1$$

Suppose a rectangular prism has dimensions a, b, c and surface area S .

- Show that $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$.
- Using part (i), show that when the rectangular prism with surface area S is a cube, it has maximum volume.

Hint: Let the numbers be the face areas bc, ca, ab .

Solution 3.11

Part (i): Surface area $S = 2(ab + bc + ca)$. Let $x_1 = ab$, $x_2 = bc$, $x_3 = ca$.

Arithmetic mean:

$$A = \frac{ab + bc + ca}{3} = \frac{S/2}{3} = \frac{S}{6}$$

Apply given inequality with $n = 3$:

$$\begin{aligned}\frac{(ab)(bc)(ca)}{A^3} &\leq 1 \\ \frac{(abc)^2}{A^3} &\leq 1 \\ (abc)^2 &\leq A^3 = \left(\frac{S}{6}\right)^3\end{aligned}$$

Taking square root: $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$.

Part (ii): Volume $V = abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$.

Maximum occurs when equality holds, which requires $x_1 = x_2 = x_3$:

$$ab = bc = ca \implies a = b = c$$

A rectangular prism with all equal dimensions is a cube.

Takeaways 3.11

- Apply AM-GM to face areas, not edge lengths
- Equality in AM-GM occurs when all terms are equal ($ab = bc = ca \Rightarrow a = b = c$)

[Two Approaches to Proving AM-GM] The AM-GM inequality states that for positive reals x_1, x_2, \dots, x_n :

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

(a) Proof by Induction on n :

1. *Base case:* $n = 2$ follows from $(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0$
2. *Forward-backward step:* Prove $n = 2^k \Rightarrow n = 2^{k+1}$ by grouping pairs
3. *Backward step:* Show $n = k \Rightarrow n = k - 1$ by setting $x_k = \frac{x_1 + \dots + x_{k-1}}{k-1}$ and applying the $n = k$ case

(b) Proof using Convex Functions:

1. Consider $f(x) = -\ln(x)$, which is convex for $x > 0$ (since $f''(x) = \frac{1}{x^2} > 0$)
2. By Jensen's inequality for convex functions:

$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}$$

3. Substituting $f(x) = -\ln(x)$:

$$-\ln\left(\frac{x_1 + \dots + x_n}{n}\right) \leq \frac{-\ln(x_1) - \dots - \ln(x_n)}{n} = -\ln(\sqrt[n]{x_1 \cdots x_n})$$

4. Multiply by -1 and exponentiate to obtain AM-GM

Problem 3.12: Problem 27: Cubic Sum Inequality

Let $a, b > 0$. Prove that:

$$a^3 + b^3 \geq \frac{(a+b)^3}{4}$$

Hint: Expand, cancel out, then factor into a product.

Solution 3.12

Multiply both sides by 4:

$$4(a^3 + b^3) \geq (a+b)^3$$

Expand RHS:

$$4a^3 + 4b^3 \geq a^3 + 3a^2b + 3ab^2 + b^3$$

Rearrange:

$$3a^3 - 3a^2b - 3ab^2 + 3b^3 \geq 0$$

Factor:

$$\begin{aligned} 3(a^3 - a^2b - ab^2 + b^3) &\geq 0 \\ 3[a^2(a-b) - b^2(a-b)] &\geq 0 \\ 3(a-b)(a^2 - b^2) &\geq 0 \\ 3(a-b)(a-b)(a+b) &\geq 0 \\ 3(a-b)^2(a+b) &\geq 0 \end{aligned}$$

Since $(a-b)^2 \geq 0$ and $(a+b) > 0$ for $a, b > 0$, the inequality holds.

Takeaways 3.12

- Clear denominators first, then expand and factor
- Factor as $(a-b)^2(a+b) \geq 0$ where perfect square ensures non-negativity

Problem 3.13: Problem 28: Product of Sums via AM-GM

Let $a, b, c > 0$.

- Prove that $a+b \geq 2\sqrt{ab}$.
- Hence, or otherwise, show that $(a+b)(b+c)(a+c) \geq 8abc$.

Hint: Multiply three inequalities since all terms are positive.

Solution 3.13

Part (a): Consider $(\sqrt{a} - \sqrt{b})^2 \geq 0$:

$$\begin{aligned} a - 2\sqrt{ab} + b &\geq 0 \\ a + b &\geq 2\sqrt{ab} \end{aligned}$$

Part (b): Apply part (a) to each pair:

$$\begin{aligned} a + b &\geq 2\sqrt{ab} \\ b + c &\geq 2\sqrt{bc} \\ a + c &\geq 2\sqrt{ac} \end{aligned}$$

Multiply all three inequalities (all terms positive):

$$\begin{aligned} (a + b)(b + c)(a + c) &\geq 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ac} \\ &= 8\sqrt{ab \cdot bc \cdot ac} \\ &= 8\sqrt{a^2b^2c^2} = 8abc \end{aligned}$$

Takeaways 3.13

- AM-GM for two variables: $(x - y)^2 \geq 0 \Rightarrow x + y \geq 2\sqrt{xy}$
- Can multiply inequalities when all terms are positive

Problem 3.14: Problem 29: Nested Inequalities

Let $a, b, c > 0$.

- Show that $\frac{a}{b} + \frac{b}{a} \geq 2$.
- Show that $(a + b) \left(\frac{1}{a} + \frac{1}{b} \right) \geq 4$.
- Hence, or otherwise, show that $(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9$.

Hint: Expand, apply AM-GM to each part.

Solution 3.14

Part (i): Apply AM-GM with $x = \frac{a}{b}$, $y = \frac{b}{a}$:

$$\frac{a}{b} + \frac{b}{a} \geq 2\sqrt{\frac{a}{b} \cdot \frac{b}{a}} = 2$$

Part (ii): Expand:

$$(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) = 1 + \frac{b}{a} + \frac{a}{b} + 1 = 2 + \left(\frac{a}{b} + \frac{b}{a}\right)$$

By part (i): $\geq 2 + 2 = 4$.

Part (iii): Expand:

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3 + \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c}$$

Apply part (i) to each pair:

$$\geq 3 + 2 + 2 + 2 = 9$$

Takeaways 3.14

- Expand products before applying AM-GM to identify reciprocal pairs
- Each pair $\frac{x}{y} + \frac{y}{x} \geq 2$ contributes 2 to the bound

Problem 3.15: Problem 30: Cauchy-Schwarz Bound

Let x, y be real numbers such that $x^2 + y^2 \neq 0$. Prove that:

$$\frac{(x+y)^2}{x^2 + y^2} \leq 2$$

Hint: Start with $(x-y)^2 \geq 0$.

Solution 3.15

Start with $(x - y)^2 \geq 0$:

$$\begin{aligned}x^2 - 2xy + y^2 &\geq 0 \\x^2 + y^2 &\geq 2xy\end{aligned}$$

Add $x^2 + y^2$ to both sides:

$$2(x^2 + y^2) \geq x^2 + 2xy + y^2 = (x + y)^2$$

Divide by $x^2 + y^2 > 0$:

$$2 \geq \frac{(x + y)^2}{x^2 + y^2}$$

Takeaways 3.15

- Use $(x - y)^2 \geq 0$ to establish $x^2 + y^2 \geq 2xy$
- Add equal term to both sides to create perfect square on RHS

Problem 3.16: Problem 31: Cauchy-Schwarz with Constraint

Let x, y, z be real numbers such that $x^2 + y^2 + z^2 = 25$. Prove that:

$$3x + 4y + 5z \leq 25\sqrt{2}$$

Hint: Apply Cauchy-Schwarz to sequences (x, y, z) and $(3, 4, 5)$.

Solution 3.16

Apply Cauchy-Schwarz inequality to (x, y, z) and $(3, 4, 5)$:

$$(x^2 + y^2 + z^2)(3^2 + 4^2 + 5^2) \geq (3x + 4y + 5z)^2$$

Substitute $x^2 + y^2 + z^2 = 25$:

$$\begin{aligned}(25)(9 + 16 + 25) &\geq (3x + 4y + 5z)^2 \\(25)(50) &\geq (3x + 4y + 5z)^2 \\1250 &\geq (3x + 4y + 5z)^2\end{aligned}$$

Take square root:

$$\sqrt{1250} = \sqrt{625 \times 2} = 25\sqrt{2} \geq 3x + 4y + 5z$$

Takeaways 3.16

- Cauchy-Schwarz: $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1b_1 + a_2b_2 + a_3b_3)^2$
- Choose coefficient sequence to match linear form on RHS

Problem 3.17: Problem 32: Bernoulli's Inequality Application

Without using a calculator, apply Bernoulli's Inequality to prove:

$$(1.005)^{200} > 2$$

Hint: Write 1.005 as $1.00x$, $x = 0.005$.

Solution 3.17

Express $1.005 = 1 + 0.005$. Let $x = 0.005$ and $n = 200$.

Check conditions: $x > -1$ and n is a positive integer, so Bernoulli's Inequality applies:

$$(1 + x)^n \geq 1 + nx$$

Since $n > 1$ and $x \neq 0$, the inequality is strict:

$$\begin{aligned}(1 + 0.005)^{200} &> 1 + 200(0.005) \\ (1.005)^{200} &> 1 + 1 \\ (1.005)^{200} &> 2\end{aligned}$$

Takeaways 3.17

- Bernoulli: $(1 + x)^n \geq 1 + nx$ for $x > -1$ and $n \in \mathbb{Z}^+$
- Strict inequality when $n > 1$ and $x \neq 0$

Problem 3.18: Problem 33: Exponential Inequality via Induction

- By considering $f'(x)$ where $f(x) = e^x - x$, show that $e^x > x$ for $x \geq 0$.
- Hence, use Mathematical Induction to show that for $x \geq 0$, $e^x > \frac{x^n}{n!}$ for all positive integers $n \geq 1$.

Hint: The derivative of $P(n+1)$ comes directly from $P(n)$.

Solution 3.18

Part (i): Let $f(x) = e^x - x$. Then $f'(x) = e^x - 1 > 0$ for $x > 0$.

Since $f(0) = 1 > 0$ and f is increasing for $x \geq 0$, we have $f(x) \geq 1 > 0$, so $e^x > x$.

Part (ii): Base Case ($n = 1$): From part (i), $e^x > x = \frac{x^1}{1!}$.

Inductive Hypothesis: Assume $e^x > \frac{x^k}{k!}$ for some $k \geq 1$.

Inductive Step: Let $g(x) = e^x - \frac{x^{k+1}}{(k+1)!}$. Then:

$$g'(x) = e^x - \frac{x^k}{k!} > 0$$

by the inductive hypothesis.

Since $g(0) = 1 > 0$ and g is increasing, $g(x) > 0$ for $x \geq 0$, so:

$$e^x > \frac{x^{k+1}}{(k+1)!}$$

By induction, $e^x > \frac{x^n}{n!}$ for all $n \geq 1$ and $x \geq 0$.

Takeaways 3.18

- Use derivative to show function is increasing, combined with initial value
- In induction step, derivative of $g(x)$ involves inductive hypothesis directly

3.3 Advanced Inequality Problems

Problem 3.19: Problem 34: Bernoulli's Inequality - Weighted AM-GM

Let n be a positive integer and let x be a positive real number.

- Show that $x^n - 1 - n(x - 1) = (x - 1)(1 + x + x^2 + \cdots + x^{n-1} - n)$.
- Hence show that $x^n \geq 1 + n(x - 1)$.
- Deduce that for positive real numbers a and b ,

$$a^n b^{1-n} \geq na + (1 - n)b.$$

Hint: Use the standard factorization for difference of powers.

Solution 3.19

Part (i): Use standard factorization: $x^n - 1 = (x - 1)(1 + x + \dots + x^{n-1})$. Then

$$\begin{aligned}x^n - 1 - n(x - 1) &= (x - 1)(1 + x + \dots + x^{n-1}) - n(x - 1) \\&= (x - 1)(1 + x + \dots + x^{n-1} - n)\end{aligned}$$

Part (ii): Analyze sign of $(x - 1)(1 + x + \dots + x^{n-1} - n)$:

- If $x = 1$: expression equals 0
- If $x > 1$: both factors positive \Rightarrow product > 0
- If $0 < x < 1$: both factors negative \Rightarrow product > 0

Thus $x^n - 1 - n(x - 1) \geq 0 \Rightarrow x^n \geq 1 + n(x - 1)$.

Part (iii): Substitute $x = \frac{a}{b}$ into (ii):

$$\begin{aligned}\left(\frac{a}{b}\right)^n &\geq 1 + n\left(\frac{a}{b} - 1\right) \\ \frac{a^n}{b^n} &\geq 1 + \frac{na - nb}{b}\end{aligned}$$

Multiply by b : $\frac{a^n}{b^{n-1}} \geq na + b(1 - n)$, giving $a^n b^{1-n} \geq na + (1 - n)b$.

Takeaways 3.19

- Bernoulli's inequality extends to weighted AM-GM forms
- Sign analysis crucial when x varies around 1

Problem 3.20: Problem 35: Reciprocal Polynomial with AM-GM

Let a, b, c be real numbers. Suppose that $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ has roots $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}$, where $\alpha > 0$ and $\beta > 0$.

- Prove that $a = c$.
- Using the inequality, show that $b \geq 6$.

Hint: $P(x)$ is a reciprocal polynomial.

Solution 3.20

Part (i): Consider $Q(x) = x^4 P(1/x) = x^4 + cx^3 + bx^2 + ax + 1$. The roots of $P(1/x)$ are reciprocals of roots of $P(x)$, which are $\frac{1}{\alpha}, \alpha, \frac{1}{\beta}, \beta$ - the same set. Since P and Q are monic with same roots, $P \equiv Q$. Comparing coefficients: $a = c$.

Part (ii): By Vieta's formulas, b equals sum of products of roots taken two at a time:

$$\begin{aligned} b &= \alpha \cdot \frac{1}{\alpha} + \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} + \beta \cdot \frac{1}{\beta} \\ &= 2 + \left(\alpha\beta + \frac{1}{\alpha\beta}\right) + \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) \end{aligned}$$

Apply AM-GM: $\alpha\beta + \frac{1}{\alpha\beta} \geq 2$ and $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \geq 2$. Thus $b \geq 2 + 2 + 2 = 6$.

Takeaways 3.20

- Reciprocal polynomials satisfy $P(x) = x^n P(1/x)$
- AM-GM applies to sum of reciprocals: $x + \frac{1}{x} \geq 2$

Problem 3.21: Problem 36: Nested AM-GM Application

Let x, y, z, w be positive real numbers.

(i) Given that $x > 0$ and $y > 0$, show that $x + y \geq 2\sqrt{xy}$.

(ii) Hence show that for $x > 0$, $y > 0$, $z > 0$ and $w > 0$,

$$x + y + z + w \geq 4\sqrt[4]{xyzw}.$$

(iii) Consider x, y, z and $w = \frac{x+y+z}{3}$. Apply the result in (ii) to show that

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}.$$

Hint: First apply to pairs. Do not apply AM-GM directly.

Solution 3.21

Part (i): Standard AM-GM: $(\sqrt{x} - \sqrt{y})^2 \geq 0 \implies x - 2\sqrt{xy} + y \geq 0 \implies x + y \geq 2\sqrt{xy}$.

Part (ii): Apply (i) twice:

$$\begin{aligned}x + y &\geq 2\sqrt{xy}, \quad z + w \geq 2\sqrt{zw} \\(x + y) + (z + w) &\geq 2\sqrt{xy} + 2\sqrt{zw} \\x + y + z + w &\geq 2(\sqrt{xy} + \sqrt{zw}) \geq 2 \cdot 2\sqrt{\sqrt{xy} \cdot \sqrt{zw}} = 4\sqrt[4]{xyzw}\end{aligned}$$

Part (iii): Let $w = \frac{x+y+z}{3}$. Then:

$$\begin{aligned}x + y + z + w &\geq 4\sqrt[4]{xyzw} \\ \frac{4}{3}(x + y + z) &\geq 4\sqrt[4]{xyz} \cdot \frac{x + y + z}{3} \\ \frac{x + y + z}{3} &\geq \sqrt[4]{xyz} \cdot \frac{x + y + z}{3}\end{aligned}$$

Raise to 4th power: $\left(\frac{x+y+z}{3}\right)^4 \geq xyz \cdot \frac{x+y+z}{3}$. Divide by $\frac{x+y+z}{3}$: $\left(\frac{x+y+z}{3}\right)^3 \geq xyz$.

Takeaways 3.21

- AM-GM for $n = 3$ derived from $n = 2$ and $n = 4$ cases
- Nested application: pair terms, then apply again

Problem 3.22: Problem 37: Triangle Inequality - Quadratic Forms

Let p, q, r be the lengths of the three sides of a triangle.

- Show that: $p^2 + q^2 + r^2 \geq pq + pr + qr$
- Show that: $3(pq + pr + qr) \leq (p + q + r)^2 < 4(pq + pr + qr)$

Hint: Consider sum of squares of differences.

Solution 3.22

Part (a): Consider $(p - q)^2 + (q - r)^2 + (p - r)^2 \geq 0$:

$$\begin{aligned} 2p^2 + 2q^2 + 2r^2 - 2pq - 2qr - 2pr &\geq 0 \\ p^2 + q^2 + r^2 &\geq pq + pr + qr \end{aligned}$$

Part (b): Expand $(p + q + r)^2 = p^2 + q^2 + r^2 + 2(pq + pr + qr)$. Using (a):

$$(p + q + r)^2 \geq (pq + pr + qr) + 2(pq + pr + qr) = 3(pq + pr + qr)$$

For the upper bound, use triangle inequalities $p < q + r$, $q < p + r$, $r < p + q$:

$$\begin{aligned} p^2 < pq + pr, \quad q^2 < pq + qr, \quad r^2 < pr + qr \\ p^2 + q^2 + r^2 < 2(pq + pr + qr) \end{aligned}$$

Add $2(pq + pr + qr)$ to both sides: $(p + q + r)^2 < 4(pq + pr + qr)$.

Takeaways 3.22

- Sum of squares of differences always non-negative
- Triangle inequality crucial for strict upper bound

Problem 3.23: Problem 38: Complex Triangle Inequality

Prove that for any two complex numbers $z_1, z_2 \in \mathbb{C}$:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Hint: Use the property that $|w| \leq \operatorname{Re}(w)$.

Solution 3.23

Square the modulus: $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$:

$$\begin{aligned} |z_1 + z_2|^2 &= z_1 \bar{z}_1 + z_1 \bar{z}_2 + z_2 \bar{z}_1 + z_2 \bar{z}_2 \\ &= |z_1|^2 + (z_1 \bar{z}_2 + \overline{z_1 \bar{z}_2}) + |z_2|^2 \\ &= |z_1|^2 + 2\operatorname{Re}(z_1 \bar{z}_2) + |z_2|^2 \end{aligned}$$

Use $\operatorname{Re}(w) \leq |w|$:

$$\begin{aligned} |z_1 + z_2|^2 &\leq |z_1|^2 + 2|z_1 \bar{z}_2| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2 \end{aligned}$$

Taking square roots: $|z_1 + z_2| \leq |z_1| + |z_2|$.

Takeaways 3.23

- Key identity: $z\bar{z} = |z|^2$ and $w + \bar{w} = 2\operatorname{Re}(w)$
- $\operatorname{Re}(w) \leq |w|$ fundamental for complex inequalities

Problem 3.24: Problem 39: Complex Modulus with Constraint

Given $|z| < \frac{1}{2}$, show that:

$$|(1+i)z^3 + iz| < \frac{3}{4}$$

Hint: Apply the complex triangle inequality.

Solution 3.24

Apply triangle inequality:

$$\begin{aligned} |(1+i)z^3 + iz| &\leq |(1+i)z^3| + |iz| \\ &= |1+i||z|^3 + |i||z| \\ &= \sqrt{2}|z|^3 + |z| \end{aligned}$$

Since $|z| < \frac{1}{2}$ and $f(x) = \sqrt{2}x^3 + x$ is increasing for $x > 0$:

$$\begin{aligned} |(1+i)z^3 + iz| &< \sqrt{2}\left(\frac{1}{2}\right)^3 + \frac{1}{2} \\ &= \frac{\sqrt{2}}{8} + \frac{4}{8} = \frac{4+\sqrt{2}}{8} \end{aligned}$$

Since $\sqrt{2} < 2$: $\frac{4+\sqrt{2}}{8} < \frac{6}{8} = \frac{3}{4}$.

Takeaways 3.24

- Triangle inequality: $|w_1 + w_2| \leq |w_1| + |w_2|$
- Evaluate at boundary of constraint for tight bounds

Problem 3.25: Problem 40: Harmonic-Arithmetic Mean Inequality

Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Using the fact that $x + y \geq 2\sqrt{xy}$ for positive x, y , prove that:

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$$

Hint: Use the given condition to show that $ab + bc + ac = abc$.

Solution 3.25

Multiply given condition by abc : $bc + ac + ab = abc$.

Apply AM-GM to pairs:

$$ab + ac \geq 2\sqrt{a^2bc} = 2a\sqrt{bc}$$

$$ab + bc \geq 2\sqrt{ab^2c} = 2b\sqrt{ac}$$

$$ac + bc \geq 2\sqrt{abc^2} = 2c\sqrt{ab}$$

Sum the three inequalities:

$$2(ab + bc + ac) \geq 2(a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab})$$

$$ab + bc + ac \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$$

Substitute $ab + bc + ac = abc$: $abc \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$.

Takeaways 3.25

- Convert harmonic condition to algebraic form first
- Apply AM-GM systematically to all pairs

Problem 3.26: Problem 41: Logarithmic Inequality with Factorial

(i) Prove that $x > \ln(x)$ for all positive real numbers x .

(ii) Hence, show that for all positive integers n :

$$e^{n^2+n} > (n!)^2$$

Hint: Consider $f(x) = x - \ln(x)$.

Solution 3.26

Part (i): Let $f(x) = x - \ln(x)$. Then $f'(x) = 1 - \frac{1}{x}$. Setting $f'(x) = 0$ gives $x = 1$. Since $f''(x) = \frac{1}{x^2} > 0$, point $(1, 1)$ is a minimum. Thus $f(x) \geq f(1) = 1 - \ln(1) = 1 > 0$, so $x > \ln(x)$.

Part (ii): Apply (i) to $k = 1, 2, \dots, n$ and sum:

$$\begin{aligned}\sum_{k=1}^n k &> \sum_{k=1}^n \ln(k) \\ \frac{n(n+1)}{2} &> \ln(n!) \\ \frac{n^2+n}{2} &> \ln(n!) \\ n^2 + n &> 2\ln(n!) = \ln((n!)^2)\end{aligned}$$

Exponentiating: $e^{n^2+n} > (n!)^2$.

Takeaways 3.26

- Calculus proves $x > \ln(x)$ via minimization
- Sum inequalities to relate arithmetic to logarithmic sums

Problem 3.27: Problem 42: Cauchy-Schwarz with Homogenization

Prove that for all positive real numbers a, b, c :

$$\frac{a^2}{3a+2b} + \frac{b^2}{3b+2c} + \frac{c^2}{3c+2a} \geq \frac{a+b+c}{5}$$

Hint: Apply Cauchy-Schwarz.

Solution 3.27

Apply Cauchy-Schwarz: $\left(\sum \frac{u_i^2}{v_i} \right) (\sum v_i) \geq (\sum u_i)^2$.

Let $u_i = (a, b, c)$ and $v_i = (3a + 2b, 3b + 2c, 3c + 2a)$:

$$\begin{aligned} & \left(\frac{a^2}{3a+2b} + \frac{b^2}{3b+2c} + \frac{c^2}{3c+2a} \right) \cdot [(3a+2b) + (3b+2c) + (3c+2a)] \\ & \geq (a+b+c)^2 \end{aligned}$$

Simplify denominator sum:

$$(3a+2b) + (3b+2c) + (3c+2a) = 3(a+b+c) + 2(a+b+c) = 5(a+b+c)$$

Therefore:

$$\text{LHS} \cdot 5(a+b+c) \geq (a+b+c)^2 \implies \text{LHS} \geq \frac{a+b+c}{5}$$

Takeaways 3.27

- Cauchy-Schwarz in Titu's Lemma form: $\sum \frac{x_i^2}{y_i} \geq \frac{(\sum x_i)^2}{\sum y_i}$
- Check that denominators sum to simple multiple of numerator sum

Problem 3.28: Problem 43: Bernoulli's Inequality - Power Form

Prove the following inequality for all integers $n \geq 1$:

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + \sqrt{n}$$

Hint: Apply Bernoulli directly.

Solution 3.28

Let $x = \frac{1}{\sqrt{n}}$. Since $n \geq 1$, we have $x > 0 > -1$.

Apply Bernoulli's inequality $(1+x)^n \geq 1 + nx$:

$$\begin{aligned} \left(1 + \frac{1}{\sqrt{n}}\right)^n & \geq 1 + n \cdot \frac{1}{\sqrt{n}} \\ & = 1 + \frac{n}{\sqrt{n}} \\ & = 1 + \sqrt{n} \end{aligned}$$

Takeaways 3.28

- Bernoulli's inequality: $(1 + x)^n \geq 1 + nx$ for $x > -1$, $n \geq 1$
- Choose substitution to match target form

Problem 3.29: Problem 44: Strict Bernoulli via Induction

- Prove that $(1 + x)^n > 1 + nx$ for $n \geq 1$ and $x > -1$.
- Hence, deduce that $\left(1 - \frac{1}{2n}\right)^n > \frac{1}{2}$ for $n > 1$.

Hint: Prove by math induction.

Solution 3.29

Part (i): By induction. Base case $n = 1$: equality holds.

Inductive step: Assume $(1 + x)^k \geq 1 + kx$. Then:

$$\begin{aligned}(1 + x)^{k+1} &= (1 + x)(1 + x)^k \geq (1 + x)(1 + kx) \\ &= 1 + kx + x + kx^2 = 1 + (k + 1)x + kx^2\end{aligned}$$

Since $k \geq 1$ and $x^2 \geq 0$, we have $kx^2 \geq 0$, so $(1 + x)^{k+1} \geq 1 + (k + 1)x$. For $n > 1$ and $x \neq 0$, strict inequality holds since $kx^2 > 0$.

Part (ii): Let $x = -\frac{1}{2n}$. Check $x > -1$: $-\frac{1}{2n} > -1$ holds for $n > \frac{1}{2}$. Apply (i):

$$\left(1 - \frac{1}{2n}\right)^n > 1 + n\left(-\frac{1}{2n}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

Takeaways 3.29

- Induction proves Bernoulli; strict inequality when $kx^2 > 0$
- Negative substitutions require careful domain checking

Problem 3.30: Problem 45: Summation Inequality via Induction

Given that for $k > 0$, $2k + 3 > 2\sqrt{(k + 1)(k + 2)}$, prove that:

$$\sum_{r=1}^n \frac{1}{\sqrt{r}} > 2\left(\sqrt{n+1} - 1\right)$$

for all positive integers n .

Hint: Prove by induction. Use the given inequality.

Solution 3.30

Base case ($n = 1$): LHS = 1, RHS = $2(\sqrt{2} - 1) \approx 0.828$. True.

Inductive step: Assume $\sum_{r=1}^k \frac{1}{\sqrt{r}} > 2(\sqrt{k+1} - 1)$. Then:

$$\begin{aligned}\sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} &= \sum_{r=1}^k \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{k+1}} \\ &> 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} \\ &= 2\sqrt{k+1} + \frac{1}{\sqrt{k+1}} - 2 \\ &= \frac{2(k+1)+1}{\sqrt{k+1}} - 2 = \frac{2k+3}{\sqrt{k+1}} - 2\end{aligned}$$

Given $2k+3 > 2\sqrt{(k+1)(k+2)}$:

$$\begin{aligned}\frac{2k+3}{\sqrt{k+1}} - 2 &> \frac{2\sqrt{(k+1)(k+2)}}{\sqrt{k+1}} - 2 \\ &= 2\sqrt{k+2} - 2 = 2(\sqrt{k+2} - 1)\end{aligned}$$

Thus the inequality holds for $n = k + 1$.

Takeaways 3.30

- Use given auxiliary inequality in inductive step
- Algebraic manipulation converts sum to target form

4 Conclusion

Inequalities are a cornerstone of the HSC Mathematics Extension 2 course, appearing in diverse contexts from pure algebra to calculus and complex numbers. Mastery requires recognizing when to apply AM-GM, Cauchy-Schwarz, triangle inequality, or induction-based techniques. Use these 45 problems to develop pattern recognition, proof-writing clarity, and strategic problem-solving skills. Best of luck with your studies and examinations!

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