HSC Math Extension 2: Complex Numbers Mastery

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1 Introduction

1.1 Project Overview

This booklet compiles high quality complex numbers problems curated specifically for the HSC Mathematics Extension 2 syllabus. Every problem explores fundamental and advanced techniques involving complex numbers—from basic arithmetic and form conversions to De Moivre's theorem, Argand diagram geometry, polynomial roots, and geometric transformations. Detailed reasoning showcases common techniques such as modulus-argument manipulation, Euler's formula applications, and geometric interpretations.

1.2 Target Audience

The explanations are crafted for Extension 2 students aiming to deepen their complex number skills. Each solution explicitly states the conversion steps, theorem applications, and geometric reasoning so that high-school learners can follow every transition.

1.3 How to Use This Booklet

- Read the overview and complex numbers primer before attempting the problems.
- Attempt problems in Part 1 without hints; compare against the detailed solutions to understand model reasoning.
- Use the upside-down hints in Part 2 only after making a genuine attempt.
- Revisit problems after a few days and try to re-derive the arguments without notes to reinforce technique.

1.4 Key Topics Covered

This booklet comprehensively covers the following essential topics from the HSC Mathematics Extension 2 Complex Numbers syllabus:

- **Arithmetic and Quadratic Equations:** Operations with complex numbers, solving quadratic equations with complex roots, and understanding the relationship between coefficients and roots.
- **Argand Diagram:** Visual representation of complex numbers as points or vectors in the complex plane, interpreting geometric relationships and transformations.
- **Euler's Theorem:** The fundamental relationship $e^{i\theta} = \cos\theta + i\sin\theta$ connecting exponential, trigonometric, and complex number representations.

- **De Moivre's Theorem:** Computing powers and roots of complex numbers using $(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$, with applications to trigonometric identities.
- **Complex Numbers and Polynomials:** Understanding how complex numbers arise as polynomial roots, factorization over complex numbers, and the Fundamental Theorem of Algebra.
- **Zeros of a Polynomial:** Finding and interpreting zeros, multiplicity of roots, and relationships between roots and coefficients via Vieta's formulas.
- **Complex Coefficients of a Polynomial:** Working with polynomials that have complex coefficients and understanding when conjugate pairs appear.
- **Vector Problems:** Representing complex numbers as vectors, operations such as addition and scalar multiplication, and applications to geometry including rotations and translations.
- **Roots of Complex Numbers:** Finding n-th roots using De Moivre's theorem, understanding the geometric distribution of roots on circles, and unity roots.
- **Curves and Regions:** Describing and sketching loci such as circles, lines, rays, and regions in the complex plane defined by equations and inequalities involving modulus and argument.

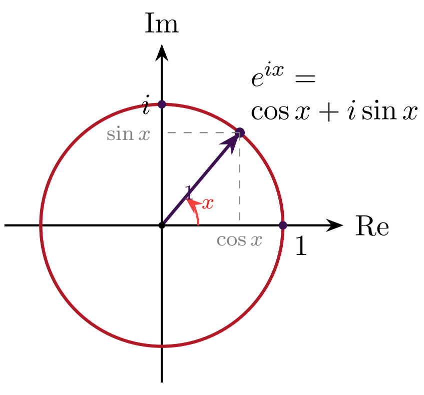
1.5 Complex Numbers Primer

1.5.1 Key Theorems and Formulas

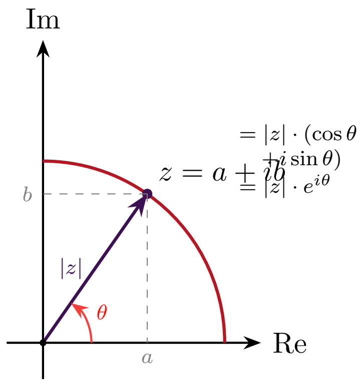
Euler's Theorem:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This connects exponential form with polar form and is fundamental to many proofs.



Unit Circle: e^{ix}



General Form:
$z = |z|e^{i\theta}$

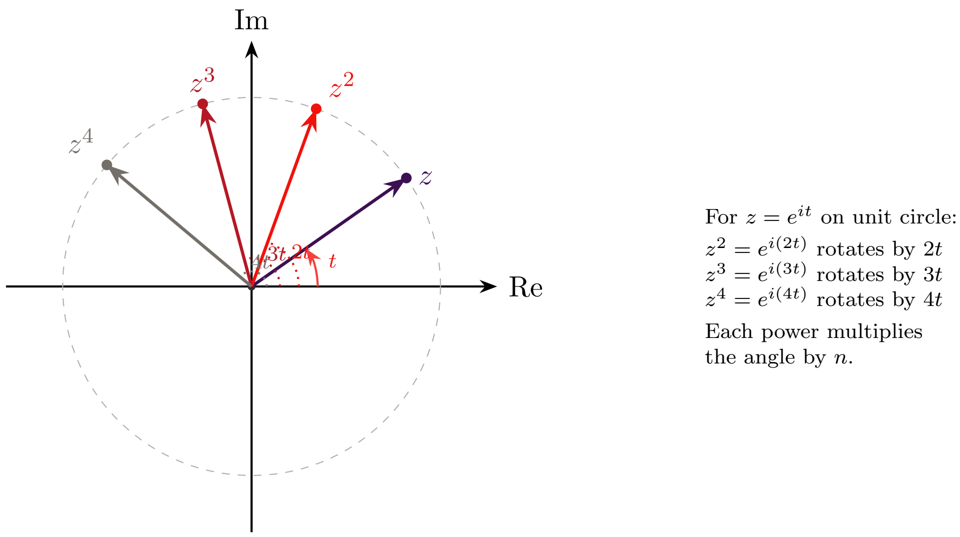
De Moivre's Theorem: For any integer n,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

or equivalently, $(re^{i\theta})^n = r^n e^{in\theta}$.

This theorem is invaluable for:

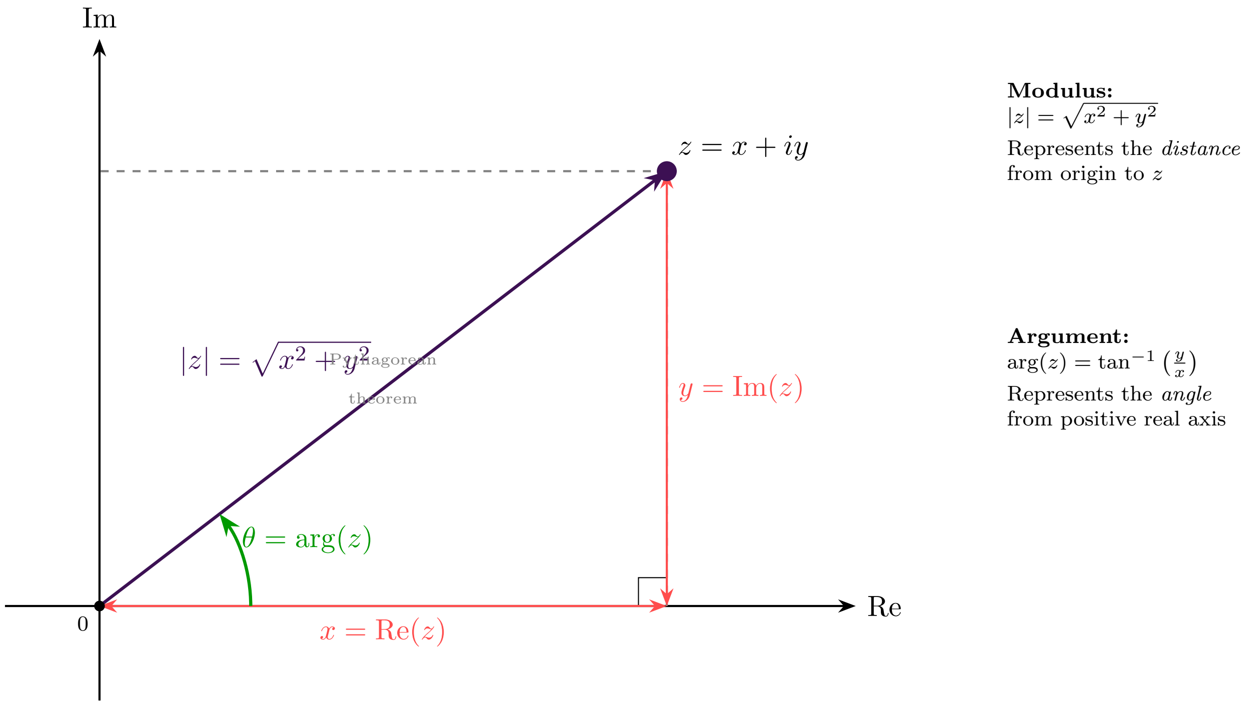
- Finding powers of complex numbers
- Finding n-th roots of complex numbers
- Proving trigonometric identities



Powers of z with $|z| = 1$ and $\arg(z) = t$

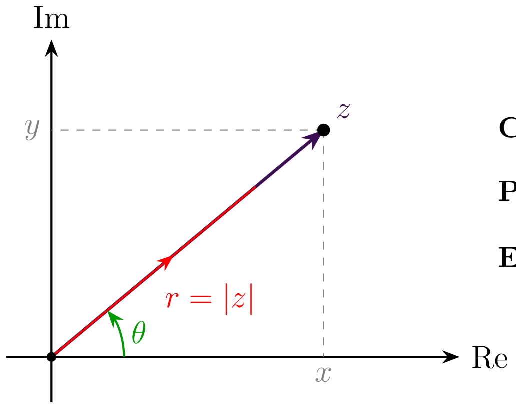
Modulus and Argument: For $z = x + iy$:

- Modulus: $|z| = \sqrt{x^2 + y^2}$
- Argument: $\arg(z) = \tan^{-1}(y/x)$ (with appropriate quadrant adjustments)
- Properties: $|z_1 z_2| = |z_1||z_2|$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ (modulo 2π)



1.5.2 Forms of Complex Numbers

Cartesian Form: $z = x + iy$
Polar Form: $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg(z)$
Exponential Form: $z = re^{i\theta}$
Conversions between these forms are essential skills tested frequently in Extension 2.



Cartesian: $z = x + iy$

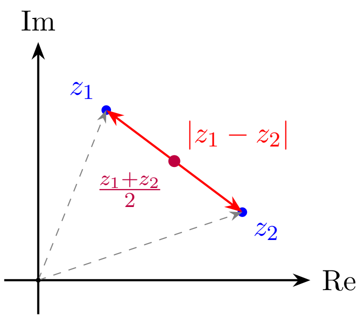
Polar: $z = r(\cos\theta + i\sin\theta)$

Exponential: $z = re^{i\theta}$

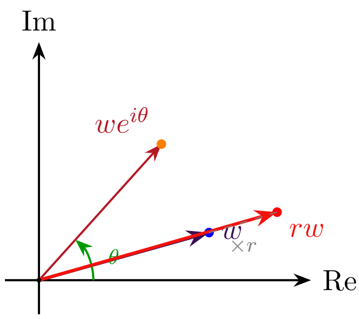
1.5.3 Argand Diagram

The Argand diagram visualizes complex numbers as points in the plane, with the real part on the horizontal axis and the imaginary part on the vertical axis. Geometric interpretations include:

- $|z_1 - z_2|$ represents the distance between z_1 and z_2
- $(z_1 + z_2)/2$ represents the midpoint
- Multiplication by $e^{i\theta}$ rotates by angle θ
- Multiplication by r scales by factor r



Distance & Midpoint



Rotation & Scaling

1.5.4 2D Rotation with Euler's Equation

The multiplication of two complex numbers implies a rotation in 2D space. When we multiply a complex number z by $e^{i\phi}$, it rotates z by angle ϕ while preserving its modulus.

Key Insight: For complex numbers $e^{i\theta}$ and $e^{i\phi}$ on the unit circle:

$$e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)}$$

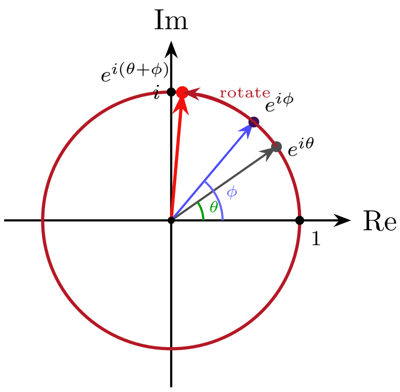
This shows that multiplication adds the angles, which is equivalent to rotation.

General Rotation: To rotate any complex number $z = x + iy = |z|e^{i\theta}$ by angle ϕ:

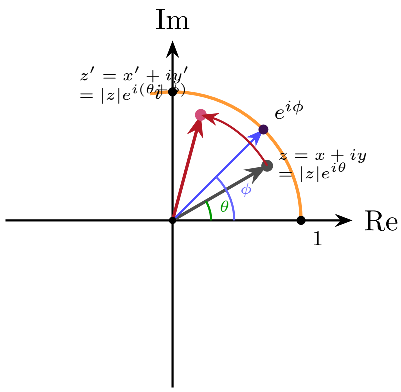
$$\begin{aligned} z' &= z \cdot e^{i\phi} = |z|e^{i\theta} \cdot e^{i\phi} \\ &= |z|e^{i(\theta+\phi)} \\ &= (x+iy)e^{i\phi} \\ &= (x+iy)(\cos\phi + i\sin\phi) \\ &= \cos\phi \cdot x - \sin\phi \cdot y + i(\sin\phi \cdot x + \cos\phi \cdot y) \end{aligned}$$

Matrix Form: This rotation can be expressed as a 2D rotation matrix:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Multiplication implies rotation



2D rotation of complex number

Note: This concept extends to 3D rotations using *quaternions*, which use a rotation axis vector $\mathbf{u}$ instead of a single complex number. The quaternion rotation formula is $\cos\theta + \mathbf{u}\sin\theta$ where $\mathbf{u}$ is the unit vector along the rotation axis.

1.5.5 Notation and Conventions

Throughout this booklet:

- $\bar{z}$ denotes the complex conjugate of z
- $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts
- $\arg(z)$ denotes the principal argument in $(-\pi, \pi]$
- Angles are in radians unless otherwise specified

2 Part 1: Problems and Solutions (Detailed)

Part 1 contains three sets of problems—basic, medium, and advanced. Each set provides five problems. For every problem we present a comprehensive solution without any hints so that learners focus on the full reasoning trail.

2.1 Basic Complex Numbers Problems

Problem 2.1: Basic Complex Arithmetic

Let $z = 5 - i$ and $w = 2 + 3i$. What is the value of $2z + \overline{w}$?

Solution 2.1

First, compute $2z$:

$$2z = 2(5 - i) = 10 - 2i.$$

Next, find the conjugate of w:

$$\overline{w} = \overline{2 + 3i} = 2 - 3i.$$

Now add these results:

$$2z + \overline{w} = (10 - 2i) + (2 - 3i) = 12 - 5i.$$

Therefore, $2z + \overline{w} = 12 - 5i$.

Takeaways 2.1

The conjugate of $a + bi$ is $a - bi$. When adding complex numbers, combine real parts with real parts and imaginary parts with imaginary parts.

Problem 2.2: Finding Square Roots

What value of z satisfies $z^2 = 7 - 24i$?

Solution 2.2

Let $z = a + bi$ where a, b are real. Then:

$$z^2 = (a+bi)^2 = a^2 - b^2 + 2abi = 7 - 24i.$$

Equating real and imaginary parts:

$$\begin{aligned} a^2 - b^2 &= 7 \\ 2ab &= -24. \end{aligned}$$

From the second equation: $ab = -12$, so $b = -\frac{12}{a}$.
Substitute into the first equation:

$$a^2 - \left(-\frac{12}{a}\right)^2 = 7 \implies a^2 - \frac{144}{a^2} = 7.$$

Multiply by a^2:

$$a^4 - 7a^2 - 144 = 0.$$

Let $u = a^2$:

$$u^2 - 7u - 144 = 0 \implies (u-16)(u+9) = 0.$$

Since $u = a^2 \geq 0$, we have $u = 16$, so $a^2 = 16$ giving $a = \pm 4$.
If $a = 4$: $b = -\frac{12}{4} = -3$, so $z = 4 - 3i$.
If $a = -4$: $b = -\frac{12}{-4} = 3$, so $z = -4 + 3i$.
Verify: $(4-3i)^2 = 16 - 24i + 9i^2 = 16 - 24i - 9 = 7 - 24i$. ✓
Therefore, $z = 4 - 3i$ or $z = -4 + 3i$.

Takeaways 2.2

To find square roots of complex numbers in Cartesian form, let $z = a + bi$ and equate real and imaginary parts after expanding z^2. This gives a system of two equations in two unknowns.

Problem 2.3: Complex Roots of Quadratics

Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where p and q are real, what are the values of p and q?

Solution 2.3

Since the polynomial has real coefficients and $z = 3 + i$ is a root, its complex conjugate $\overline{z} = 3 - i$ must also be a root.
By Vieta's formulas:

$$p = -(z_1 + z_2) = -\big((3+i) + (3-i)\big) = -6,$$

$$q = z_1 \cdot z_2 = (3+i)(3-i) = 9 - i^2 = 9 - (-1) = 10.$$

Therefore, $p = -6$ and $q = 10$.

Takeaways 2.3

For polynomials with real coefficients, complex roots occur in conjugate pairs. Use Vieta's formulas: sum of roots $= -p$ and product of roots $= q$ for $z^2 + pz + q = 0$.

Problem 2.4: Powers of i

Write i^9 in the form $a + ib$ where a and b are real.

Solution 2.4

Note the pattern of powers of i:

$$\begin{aligned}
i^1 &= i \\
i^2 &= -1 \\
i^3 &= i^2 \cdot i = -i \\
i^4 &= (i^2)^2 = 1 \\
i^5 &= i^4 \cdot i = i
\end{aligned}$$

The pattern repeats every 4 powers. To find i^9:

$$9 = 4 \cdot 2 + 1,$$

so $i^9 = i^{4 \cdot 2+1} = (i^4)^2 \cdot i = 1^2 \cdot i = i$.
Therefore, $i^9 = 0 + 1i$, giving $a = 0$ and $b = 1$.

Takeaways 2.4

Powers of i repeat with period 4: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. Use division by 4 to find $i^n = i^{n \bmod 4}$.

Problem 2.5: Polar Form Conversion

Write $1 + i$ in the form $r(\cos\theta + i\sin\theta)$.

Solution 2.5

For $z = 1 + i$, first find the modulus:

$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Next, find the argument. Since z is in the first quadrant:

$$\theta = \arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

Therefore:

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

Takeaways 2.5

To convert $z = x+iy$ to polar form: (1) Find modulus $r = \sqrt{x^2 + y^2}$. (2) Find argument $\theta = \tan^{-1}(y/x)$ (adjust for quadrant). (3) Write $z = r(\cos\theta + i\sin\theta)$.

2.2 Medium Complex Numbers Problems

Problem 2.6: Rhombus on Argand Diagram

On the Argand diagram, the complex numbers 0, $1+i\sqrt{3}$, $\sqrt{3}+i$ and z form a rhombus. Find z in the form $a + ib$.

Solution 2.6

In a rhombus, opposite sides are parallel and equal in length. Since we have vertices at 0, $1 + i\sqrt{3}$, $\sqrt{3} + i$, and z, and they form a rhombus with one vertex at the origin, we need z to be the fourth vertex.
The rhombus has sides:

- From 0 to $1 + i\sqrt{3}$
- From 0 to $\sqrt{3} + i$
- From $1 + i\sqrt{3}$ to z
- From $\sqrt{3} + i$ to z

For a rhombus with one vertex at the origin, if two adjacent vertices are at w_1 and w_2, then the fourth vertex is at $w_1 + w_2$ (by the parallelogram law).
Therefore:

$$z = (1 + i\sqrt{3}) + (\sqrt{3} + i) = (1 + \sqrt{3}) + i(\sqrt{3} + 1) = (1 + \sqrt{3})(1 + i).$$

Simplifying:

$$z = 1 + \sqrt{3} + i(1 + \sqrt{3}) = (1 + \sqrt{3})(1 + i).$$

Therefore, $z = (1 + \sqrt{3}) + i(1 + \sqrt{3})$.

Takeaways 2.6

In the Argand diagram, a parallelogram (including rhombus) with vertices at 0, w_1, w_2, and z has its fourth vertex at $z = w_1 + w_2$ by the parallelogram law of vector addition.

Problem 2.7: De Moivre's Theorem for Real Results

Given $-\sqrt{3} - i = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$, show that $(-\sqrt{3} - i)^6$ is a real number.

Solution 2.7

Using De Moivre's theorem:

$$\begin{aligned}(-\sqrt{3} - i)^6 &= \left[2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)\right]^6 \\ &= 2^6\left(\cos\left(6\cdot\left(-\frac{5\pi}{6}\right)\right) + i\sin\left(6\cdot\left(-\frac{5\pi}{6}\right)\right)\right) \\ &= 64\left(\cos(-5\pi) + i\sin(-5\pi)\right).\end{aligned}$$

Now evaluate the trigonometric values:

$$\cos(-5\pi) = \cos(5\pi) = \cos(\pi) = -1,$$

$$\sin(-5\pi) = -\sin(5\pi) = -\sin(\pi) = 0.$$

Therefore:

$$(-\sqrt{3} - i)^6 = 64(-1 + 0i) = -64.$$

Since the imaginary part is zero, $(-\sqrt{3} - i)^6 = -64$ is indeed a real number.

Takeaways 2.7

A complex number in polar form $r(\cos\theta + i\sin\theta)$ raised to power n has argument $n\theta$. If $n\theta$ is a multiple of π, then $\sin(n\theta) = 0$ and the result is real.

Problem 2.8: Division in Polar Form

Let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 + i$. Find the modulus-argument form of $\frac{\alpha}{\beta}$.

Solution 2.8

First, convert α and β to polar form.
For $\alpha = 1 + i\sqrt{3}$:

$$|\alpha| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2,$$

$$\arg(\alpha) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

So $\alpha = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.
For $\beta = 1 + i$:

$$|\beta| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

$$\arg(\beta) = \tan^{-1}(1) = \frac{\pi}{4}.$$

So $\beta = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.
Now compute the quotient:

$$\frac{\alpha}{\beta} = \frac{2}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right).$$

Takeaways 2.8

For division in polar form: $\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$.

Problem 2.9: Powers and Cartesian Form

Express $(\sqrt{3} - i)^7$ in the form $x + iy$.

Solution 2.9

First, convert $\sqrt{3} - i$ to polar form:

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2,$$

$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6} \text{ (fourth quadrant).}$$

So $\sqrt{3} - i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$.
Apply De Moivre's theorem:

$$(\sqrt{3} - i)^7 = 2^7\left(\cos\left(7 \cdot \left(-\frac{\pi}{6}\right)\right) + i\sin\left(7 \cdot \left(-\frac{\pi}{6}\right)\right)\right)$$

$$= 128\left(\cos\left(-\frac{7\pi}{6}\right) + i\sin\left(-\frac{7\pi}{6}\right)\right).$$

Now evaluate:

$$\cos\left(-\frac{7\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) = \cos\left(\pi + \frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2},$$

$$\sin\left(-\frac{7\pi}{6}\right) = -\sin\left(\frac{7\pi}{6}\right) = -\sin\left(\pi + \frac{\pi}{6}\right) = -\left(-\sin\frac{\pi}{6}\right) = \frac{1}{2}.$$

Therefore:

$$(\sqrt{3} - i)^7 = 128\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) = -64\sqrt{3} + 64i.$$

Takeaways 2.9

To compute high powers of complex numbers: (1) Convert to polar form. (2) Apply De Moivre's theorem. (3) Simplify the angle using periodicity and reference angles. (4) Convert back to Cartesian form.

Problem 2.10: Locus as a Curve

The point P on the Argand diagram represents $z = x + iy$ satisfying $z^2 + \bar{z}^2 = 8$. Find the equation of the curve in terms of x and y and state what type of curve it is.

Solution 2.10

Let $z = x + iy$, so $\bar{z} = x - iy$.
Then:

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy,$$
$$\bar{z}^2 = (x-iy)^2 = x^2 - y^2 - 2ixy.$$

Therefore:

$$z^2 + \bar{z}^2 = (x^2 - y^2 + 2ixy) + (x^2 - y^2 - 2ixy) = 2(x^2 - y^2) = 8.$$

Simplifying:

$$x^2 - y^2 = 4.$$

This can be rewritten as:

$$\frac{x^2}{4} - \frac{y^2}{4} = 1,$$

which is a rectangular hyperbola with center at the origin, with branches along the x-axis at $x = \pm 2$.

Takeaways 2.10

To find the locus of points satisfying a complex equation: (1) Substitute $z = x + iy$ and $\bar{z} = x - iy$. (2) Expand and simplify. (3) Equate real and imaginary parts. (4) Identify the curve type.

2.3 Advanced Complex Numbers Problems

Problem 2.11: Geometric Proof with Complex Numbers

Points A and B represent complex numbers z and w on a circle centered at O. Point C represents $z + w$ and also lies on the circle. Show geometrically that $\angle AOB = \frac{2\pi}{3}$.

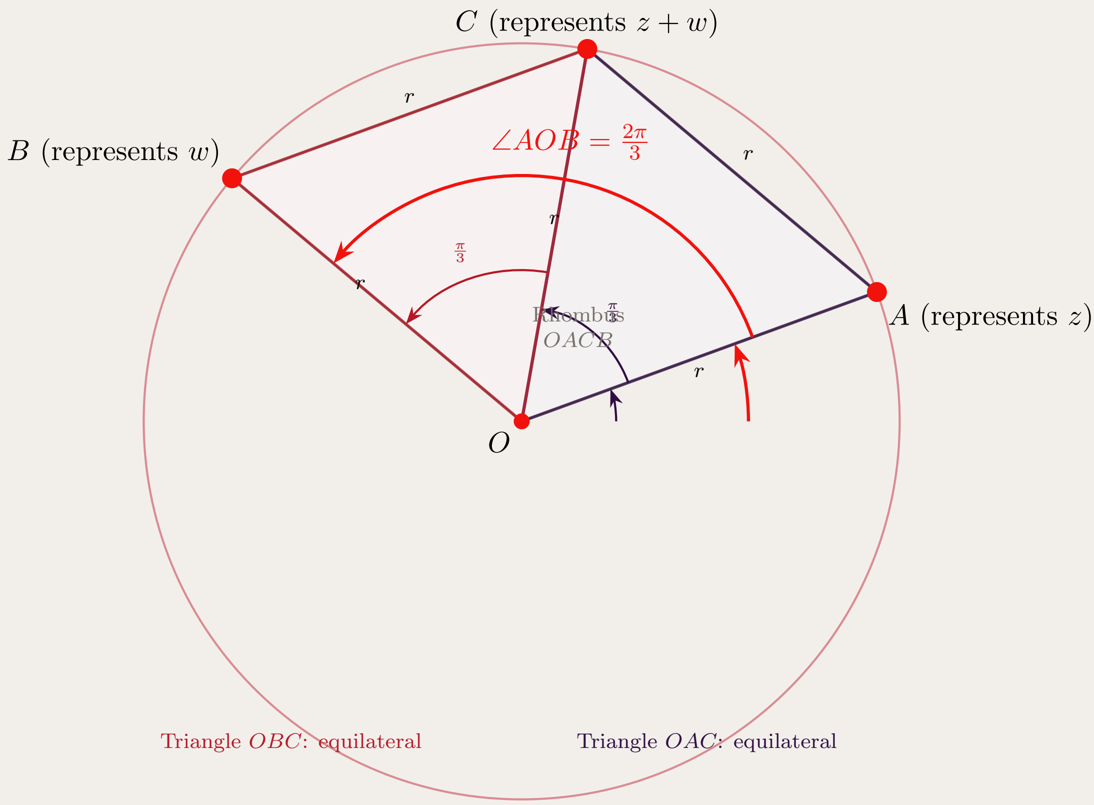
Solution 2.11

Given that $|z| = |w| = |z+w| = r$ (the radius of the circle), we need to find $\angle AOB$.
Since $OACB$ forms a parallelogram (by vector addition), and we know:

- $OA = |z| = r$
- $OB = |w| = r$
- $OC = |z+w| = r$
- $AC = |w| = r$ (opposite side of parallelogram)
- $BC = |z| = r$ (opposite side of parallelogram)

All sides of the rhombus $OACB$ have length r. Moreover, diagonal OC also has length r.
Consider triangle OAC: all three sides have length r (since $OA = AC = OC = r$), so it is equilateral. Therefore, $\angle AOC = \frac{\pi}{3}$.
Similarly, triangle OBC has $OB = BC = OC = r$, so it is also equilateral. Therefore, $\angle BOC = \frac{\pi}{3}$.
Thus:

$$\angle AOB = \angle AOC + \angle BOC = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}.$$



Takeaways 2.11

When complex numbers are represented geometrically, vector addition corresponds to the parallelogram law. A rhombus with all sides equal to its diagonal consists of two equilateral triangles.

Problem 2.12: Complex Division and Real Parts

Let $z = 2(\cos\theta + i\sin\theta)$. Show that the real part of $\frac{1}{1-z}$ is $\frac{1-2\cos\theta}{5-4\cos\theta}$.

Solution 2.12

First, compute $1 - z$:

$$1 - z = 1 - 2(\cos\theta + i\sin\theta) = (1 - 2\cos\theta) - 2i\sin\theta.$$

To find $\frac{1}{1-z}$, multiply numerator and denominator by the conjugate of $1 - z$:

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{(1-2\cos\theta) - 2i\sin\theta} \cdot \frac{(1-2\cos\theta) + 2i\sin\theta}{(1-2\cos\theta) + 2i\sin\theta} \\ &= \frac{(1-2\cos\theta) + 2i\sin\theta}{(1-2\cos\theta)^2 + (2\sin\theta)^2}.\end{aligned}$$

Compute the denominator:

$$\begin{aligned}(1-2\cos\theta)^2 + 4\sin^2\theta &= 1 - 4\cos\theta + 4\cos^2\theta + 4\sin^2\theta \\ &= 1 - 4\cos\theta + 4(\cos^2\theta + \sin^2\theta) \\ &= 1 - 4\cos\theta + 4 \\ &= 5 - 4\cos\theta.\end{aligned}$$

Therefore:

$$\frac{1}{1-z} = \frac{(1-2\cos\theta) + 2i\sin\theta}{5 - 4\cos\theta}.$$

The real part is:

$$\operatorname{Re}\left(\frac{1}{1-z}\right) = \frac{1-2\cos\theta}{5-4\cos\theta}.$$

Takeaways 2.12

To find the real part of a complex fraction, rationalize by multiplying by the conjugate of the denominator. Use the identity $\cos^2\theta + \sin^2\theta = 1$ to simplify.

Problem 2.13: Finding Complex Roots of Polynomials

Two zeros of $P(x) = x^4 - 12x^3 + 59x^2 - 138x + 130$ are $a + ib$ and $a + 2ib$ where a, b are real and $b > 0$. Find a and b.

Solution 2.13

Since $P(x)$ has real coefficients, complex roots occur in conjugate pairs. If $a + ib$ is a root, then $a - ib$ is also a root. Similarly, if $a + 2ib$ is a root, then $a - 2ib$ is also a root. Therefore, the four roots are: $a + ib$, $a - ib$, $a + 2ib$, $a - 2ib$.
By Vieta's formulas, the sum of roots equals the negative of the coefficient of x^3 divided by the leading coefficient:

$$(a + ib) + (a - ib) + (a + 2ib) + (a - 2ib) = 4a = 12,$$

so $a = 3$.
The product of roots equals the constant term:

$$[(a + ib)(a - ib)][(a + 2ib)(a - 2ib)] = (a^2 + b^2)(a^2 + 4b^2) = 130.$$

Substituting $a = 3$:

$$(9 + b^2)(9 + 4b^2) = 130.$$

Expand:

$$81 + 36b^2 + 9b^2 + 4b^4 = 130,$$
$$4b^4 + 45b^2 + 81 = 130,$$
$$4b^4 + 45b^2 - 49 = 0.$$

Let $u = b^2$:

$$4u^2 + 45u - 49 = 0.$$

Using the quadratic formula:

$$u = \frac{-45 \pm \sqrt{45^2 + 4 \cdot 4 \cdot 49}}{8} = \frac{-45 \pm \sqrt{2025 + 784}}{8} = \frac{-45 \pm \sqrt{2809}}{8} = \frac{-45 \pm 53}{8}.$$

Since $u = b^2 \geq 0$: $u = \frac{-45+53}{8} = \frac{8}{8} = 1$.
Therefore, $b^2 = 1$, so $b = 1$ (since $b > 0$).
Thus, $a = 3$ and $b = 1$.

Takeaways 2.13

For polynomials with real coefficients, complex roots come in conjugate pairs. Use Vieta's formulas (sum and product of roots) along with conjugate pairing to set up equations for unknown parameters.

Problem 2.14: Trigonometric Identity via Complex Numbers

Show that $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

Solution 2.14

By De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta.$$

Expand the left side using the binomial theorem:

$$\begin{aligned}(\cos\theta + i\sin\theta)^4 &= \sum_{k=0}^{4}\binom{4}{k}\cos^{4-k}\theta(i\sin\theta)^k \\ &= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta \\ &\quad + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta \\ &= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta \\ &\quad - 4i\cos\theta\sin^3\theta + \sin^4\theta \\ &= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) \\ &\quad + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta).\end{aligned}$$

Equating real parts:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Takeaways 2.14

De Moivre's theorem combined with binomial expansion provides a systematic way to derive multiple-angle formulas. Equate real and imaginary parts to obtain separate identities for $\cos(n\theta)$ and $\sin(n\theta)$.

Problem 2.15: Conjugate Pairs and De Moivre

Show that $(1+i)^n + (1-i)^n = 2(\sqrt{2})^n\cos\frac{n\pi}{4}$.

Solution 2.15

First, convert $1+i$ and $1-i$ to polar form.
For $1+i$:

$$|1+i| = \sqrt{2}, \quad \arg(1+i) = \frac{\pi}{4},$$

so $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\pi/4}$.
For $1-i$:

$$|1-i| = \sqrt{2}, \quad \arg(1-i) = -\frac{\pi}{4},$$

so $1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}e^{-i\pi/4}$.
By De Moivre's theorem:

$$(1+i)^n = (\sqrt{2})^n e^{in\pi/4} = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right),$$

$$(1-i)^n = (\sqrt{2})^n e^{-in\pi/4} = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right).$$

Adding:

$$\begin{aligned}(1+i)^n + (1-i)^n &= (\sqrt{2})^n\left[\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + \left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)\right] \\ &= (\sqrt{2})^n \cdot 2\cos\frac{n\pi}{4} \\ &= 2(\sqrt{2})^n\cos\frac{n\pi}{4}.\end{aligned}$$

Takeaways 2.15

For conjugate pairs z and $\bar{z}$ in polar form, $z^n + \bar{z}^n = 2|z|^n\cos(n\arg(z))$ because the imaginary parts cancel. This is useful for simplifying sums involving conjugate powers.

3 Part 2: Problems and Solutions (Concise + Hints)

Part 2 revisits three new sets of problems. Solutions are intentionally briefer to encourage student ownership, and every problem includes an upside-down hint.

3.1 Basic Complex Numbers Problems

Problem 3.1: Multiplication by i as Rotation

Consider the complex number $z = 3+2i$ on the Argand diagram. Find the complex number iz and describe its geometric relationship to z.

Hint: Multiplication by i rotates a complex number by 90 counterclockwise about the origin while preserving its modulus. If $z = a+bi$, calculate iz algebraically.

Solution 3.1: Sketch

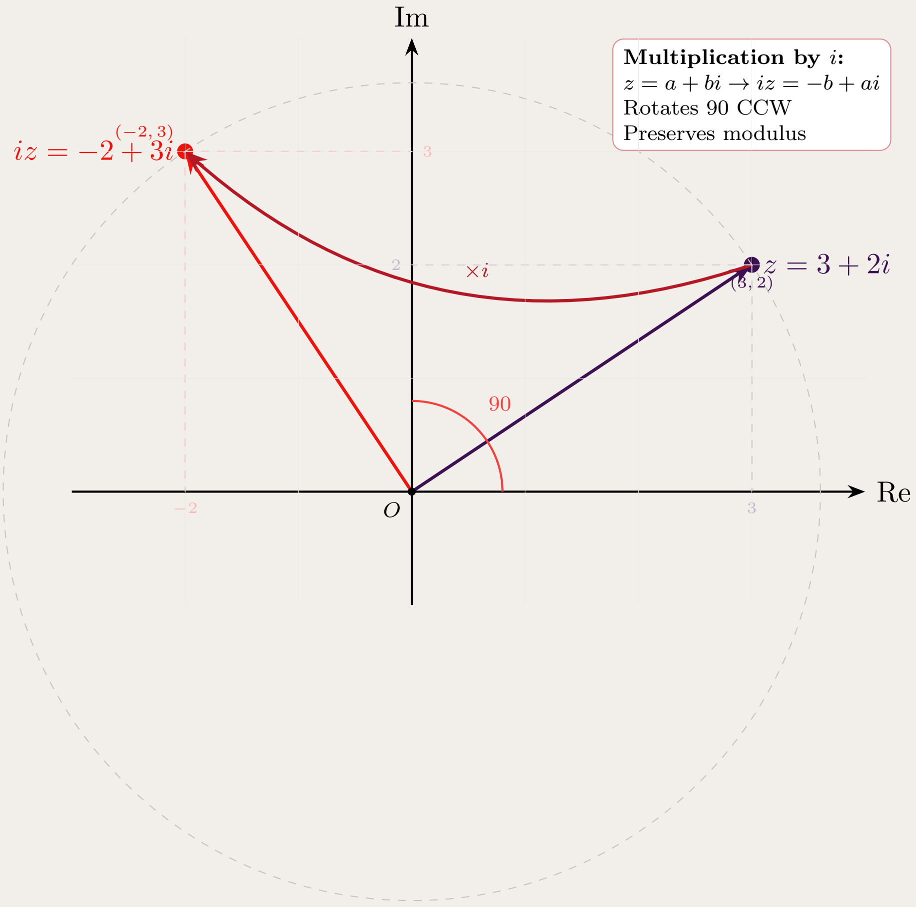
Algebraic calculation:
Given $z = 3 + 2i$, we compute:

$$\begin{aligned} iz &= i(3+2i) \\ &= 3i + 2i^2 \\ &= 3i + 2(-1) \\ &= -2 + 3i \end{aligned}$$

Therefore: $iz = -2 + 3i$
General rule: If $z = a + bi$, then:

$$iz = i(a+bi) = ai + bi^2 = -b + ai$$

This shows that $(a, b) \to (-b, a)$, which is a 90 counterclockwise rotation.
Verification for our example: - Original point: $z = 3 + 2i$ at position $(3, 2)$ - Rotated point: $iz = -2 + 3i$ at position $(-2, 3)$ - Check: $(3, 2) \to (-2, 3)$ is correct



Key observations:

- The modulus is preserved: $|z| = |iz| = \sqrt{13}$
- The transformation $(a, b) \to (-b, a)$ is a 90 counterclockwise rotation
- Both points lie on the same circle centered at the origin
- This geometric property makes i rotation in the complex plane

Problem 3.2: Quadrant Determination

The Argand diagram shows z in the first quadrant and w in the second quadrant. Which complex number could lie in the 3rd quadrant: $-w$, $2iz$, $\bar{z}$, or $w - z$?

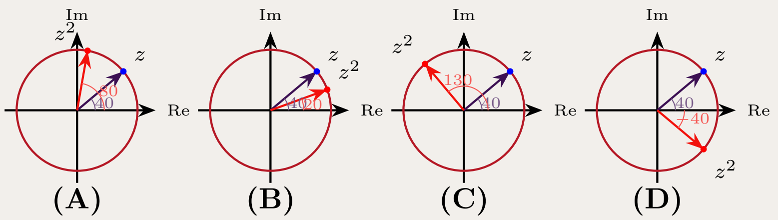
Hint: Check the signs of the real and imaginary parts of each option. Third quadrant means both parts are negative.

Solution 3.2: Sketch

If z is in Q1: $\mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0$. If w is in Q2: $\mathrm{Re}(w) < 0, \mathrm{Im}(w) > 0$. Then $w - z$ has $\mathrm{Re}(w - z) = \mathrm{Re}(w) - \mathrm{Re}(z) < 0$ and $\mathrm{Im}(w - z) = \mathrm{Im}(w) - \mathrm{Im}(z)$ which could be negative if $\mathrm{Im}(z) > \mathrm{Im}(w)$. Answer: (D) $w - z$.

Problem 3.3: Doubling the Argument

A complex number z is on the unit circle at angle $\theta = 40$. Which diagram best shows the location of $\frac{z^2}{|z|}$?



Hint: If $z = e^{i\theta}$, then $z^2 = e^{i2\theta}$ and $|z| = 1$.

Solution 3.3: Sketch

Since $|z| = 1$, we have $\frac{z^2}{|z|} = z^2$. By De Moivre's theorem, z^2 has the same modulus but double the argument. If z is at angle $\theta = 40$, then z^2 is at angle $2\theta = 80$.
Analysis of options:

(A) **CORRECT**: Shows z^2 at angle $80 = 2 \times 40$. This correctly applies De Moivre's theorem.

(B) Incorrect: Shows z^2 at angle $20 = 40/2$. This halves the angle instead of doubling it.

(C) Incorrect: Shows z^2 at angle $130 = 40 + 90$. This adds a fixed angle, which would represent multiplication by i.

(D) Incorrect: Shows z^2 at angle -40. This reflects across the real axis, which would give the conjugate $\bar{z}$.

Answer: (A)

Problem 3.4: Polynomial with Complex Zero

Which polynomial could have $2+i$ as a zero, given that k is real: (A) $x^3 - 4x^2 + kx$, (B) $x^3 - 4x^2 + kx + 5$, (C) $x^3 - 5x^2 + kx$, (D) $x^3 - 5x^2 + kx + 5$?

Hint: If the polynomial has real coefficients and $2+i$ is a root, then $2-i$ must also be a root. The sum of these roots is 4.

Solution 3.4: Sketch

Roots $2+i$ and $2-i$ sum to 4. Let the third root be r. Then sum of all roots $= 4+r$. From the coefficient of x^2: $-(4+r) = -4$ or -5. If -4, then $r = 0$, giving option (A) or (B). Check: $(x)(x^2 - 4x + 5) = x^3 - 4x^2 + 5x$, but we need $+5$ constant. Answer: (B).

Problem 3.5: Squaring Complex Numbers

Let $z = 3+i$ and $w = 2-5i$. Find z^2 in the form $x+iy$.

Hint: Use $(a+bi)^2 = a^2 - b^2 + 2abi$.

Solution 3.5: Sketch

$z^2 = (3+i)^2 = 9 + 6i + i^2 = 9 + 6i - 1 = 8 + 6i$.

Problem 3.6: Conjugate Subtraction

Let $z = 4 + i$ and $w = \bar{z}$. Find $w - z$ in the form $x + iy$.

Hint: If $z = 4 + i$, then $\bar{z} = 4 - i$.

Solution 3.6: Sketch

$w = \bar{z} = 4 - i$. Then $w - z = (4 - i) - (4 + i) = -2i$.

Problem 3.7: Division by Complex Number

Let $z = 2 + i$. Find $\frac{4}{z}$ in the form $x + iy$.

Hint: Multiply numerator and denominator by $\bar{z} = 2 - i$.

Solution 3.7: Sketch

$\frac{4}{2+i} = \frac{4(2-i)}{(2+i)(2-i)} = \frac{8-4i}{4-i^2} = \frac{8-4i}{5} = \frac{8}{5} - \frac{4}{5}i.$

Problem 3.8: Complex Multiplication

Let $z = 3 + i$ and $w = 1 - i$. Find zw in the form $x + iy$.

Hint: Use FOIL: $(a + bi)(c + di) = ac - bd + (ad + bc)i$.

Solution 3.8: Sketch

$zw = (3 + i)(1 - i) = 3 - 3i + i - i^2 = 3 - 2i + 1 = 4 - 2i.$

Problem 3.9: Another Division

Let $w = 1 - i$. Find $\frac{6}{w}$ in the form $x + iy$.

Hint: Multiply by conjugate: $\bar{w} = 1 + i$.

Solution 3.9: Sketch

$\frac{6}{1-i} = \frac{6(1+i)}{(1-i)(1+i)} = \frac{6+6i}{1-i^2} = \frac{6+6i}{2} = 3 + 3i.$

Problem 3.10: Dividing by Conjugate

Let $z = 4 + i$ and $w = \bar{z}$. Find $\frac{z}{w}$ in the form $x + iy$.

Hint: $w = 4 - i$, so $\frac{z}{w} = \frac{4+i}{4-i}$.

Solution 3.10: Sketch

$\frac{4+i}{4-i} = \frac{(4+i)(4+i)}{(4-i)(4+i)} = \frac{16+8i+i^2}{16-i^2} = \frac{15+8i}{17} = \frac{15}{17} + \frac{8}{17}i.$

Problem 3.11: Modulus from Polar Form

Let $z = \frac{1}{2}(\cos\theta + i\sin\theta)$. Find $|z|$.

Hint: For $z = r(\cos\theta + i\sin\theta)$, we have $|z| = r$.

Solution 3.11: Sketch

$|z| = \frac{1}{2}$.

Problem 3.12: Division in Cartesian Form

Express $\frac{3-i}{2+i}$ in the form $x + iy$.

Hint: Multiply by $\frac{2-i}{2-i}$.

Solution 3.12: Sketch

$\frac{3-i}{2+i} = \frac{(3-i)(2-i)}{(2+i)(2-i)} = \frac{6-3i-2i+i^2}{4-i^2} = \frac{5-5i}{5} = 1 - i.$

Problem 3.13: Addition with Conjugate

Let $z = 1 + 3i$ and $w = 2 - i$. Find $z + \bar{w}$.

Hint: $\bar{w} = 2 + i$.

Solution 3.13: Sketch

$z + \bar{w} = (1 + 3i) + (2 + i) = 3 + 4i.$

Problem 3.14: Product with Conjugate

Evaluate $w\bar{z}$ given $w = -1 + 4i$ and $z = 2 - i$.

Hint: $\bar{z} = 2 + i$.

Solution 3.14: Sketch

$w\bar{z} = (-1+4i)(2+i) = -2 - i + 8i + 4i^2 = -2 + 7i - 4 = -6 + 7i.$

Problem 3.15: Polar Form of Given Number

Let $w = 1 + i\sqrt{3}$. Express w in modulus-argument form.

Hint: $|w| = \sqrt{1+3} = 2$, and $\arg(w) = \tan^{-1}(\sqrt{3}/1) = \pi/3$.

Solution 3.15: Sketch

$w = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$

Problem 3.16: Showing a Power is Real

Let $z = \sqrt{3} - i$. Show that z^6 is real.

Hint: Convert to polar form: $z = 2(\cos(-\pi/6) + i\sin(-\pi/6))$.

Solution 3.16: Sketch

$z^6 = 2^6(\cos(-\pi) + i\sin(-\pi)) = 64(-1 + 0i) = -64$, which is real.

Problem 3.17: Principal Argument

Evaluate $\mathrm{Arg}(z)$ for $z = -\sqrt{3} + i$.

Hint: z is in the second quadrant. Use $\tan\theta = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}$.

Solution 3.17: Sketch

Reference angle is $\pi/6$. Since z is in Q2, $\mathrm{Arg}(z) = \pi - \pi/6 = \frac{5\pi}{6}$.

Problem 3.18: Polynomial with Complex Root

It is given that $z = 2 + i$ is a root of $z^3 + az^2 - 7z + 15 = 0$, where a is a real number. What is the value of a?

(A) -1

(B) 1

(C) 7

(D) -7

Hint: Since the polynomial has real coefficients and $2 + i$ is a root, then $2 - i$ must also be a root. Use Vieta's formulas or substitute the root directly.

Solution 3.18: Sketch

Method 1: Direct substitution
Since $z = 2 + i$ is a root, substitute into the equation:

$$(2+i)^3 + a(2+i)^2 - 7(2+i) + 15 = 0$$

Calculate $(2+i)^2$:

$$(2+i)^2 = 4 + 4i + i^2 = 4 + 4i - 1 = 3 + 4i$$

Calculate $(2+i)^3 = (2+i)(3+4i)$:

$$(2+i)(3+4i) = 6 + 8i + 3i + 4i^2 = 6 + 11i - 4 = 2 + 11i$$

Substitute into the equation:

$$\begin{aligned}(2+11i) + a(3+4i) - 7(2+i) + 15 &= 0\\(2+11i) + (3a+4ai) + (-14-7i) + 15 &= 0\\(2+3a-14+15) + (11+4a-7)i &= 0\\(3+3a) + (4+4a)i &= 0\end{aligned}$$

For this to equal zero, both real and imaginary parts must be zero:

$$\begin{aligned}\text{Real part: } 3+3a = 0 &\implies a = -1\\\text{Imaginary part: } 4+4a = 0 &\implies a = -1\end{aligned}$$

Method 2: Using conjugate roots
Since the polynomial has real coefficients, if $2+i$ is a root, then $2-i$ is also a root. Let the third root be r. By Vieta's formulas, the sum of roots equals $-a$:

$$(2+i) + (2-i) + r = -a$$

$$4 + r = -a$$

The product of roots equals -15:

$$(2+i)(2-i) \cdot r = -15$$

$$(4-i^2) \cdot r = -15$$

$$5r = -15$$

$$r = -3$$

Therefore: $4 + (-3) = -a \implies a = -1$
Answer: (A) $a = -1$

Problem 3.19: Power Using De Moivre's Theorem

(i) Write the number $\sqrt{3} + i$ in modulus-argument form.

(ii) Hence, or otherwise, write $(\sqrt{3}+i)^7$ in exact Cartesian form.

Hint: For part (i): Find $|z| = \sqrt{3+1} = 2$ and $\arg(z) = \tan^{-1}(1/\sqrt{3}) = \pi/6$. For part (ii): Use De Moivre's theorem to find $(re^{i\theta})^7 = r^7e^{i7\theta}$.

Solution 3.19: Sketch

Part (i): Convert to modulus-argument form
Let $z = \sqrt{3} + i$.
Calculate the modulus:

$$|z| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

Calculate the argument. Since z is in the first quadrant:

$$\arg(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Therefore, in modulus-argument form:

$$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2e^{i\pi/6}$$

Part (ii): Calculate z^7 using De Moivre's theorem
By De Moivre's theorem:

$$\begin{aligned} z^7 &= \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^7 \\ &= 2^7\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) \\ &= 128\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) \end{aligned}$$

Calculate the trigonometric values. The angle $\frac{7\pi}{6}$ is in the third quadrant (reference angle $\frac{\pi}{6}$):

$$\cos\frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$
$$\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

Therefore:

$$\begin{aligned} z^7 &= 128\left(-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) \\ &= 128\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) \\ &= -64\sqrt{3} - 64i \end{aligned}$$

Answer: $(\sqrt{3}+i)^7 = -64\sqrt{3} - 64i$
Verification (optional): We can verify that $|z^7| = |z|^7 = 2^7 = 128$ and $\arg(z^7) = 7\arg(z) = 7\cdot\frac{\pi}{6} = \frac{7\pi}{6}$ (in the range $(-\pi, \pi]$).

3.2 Medium Complex Numbers Problems

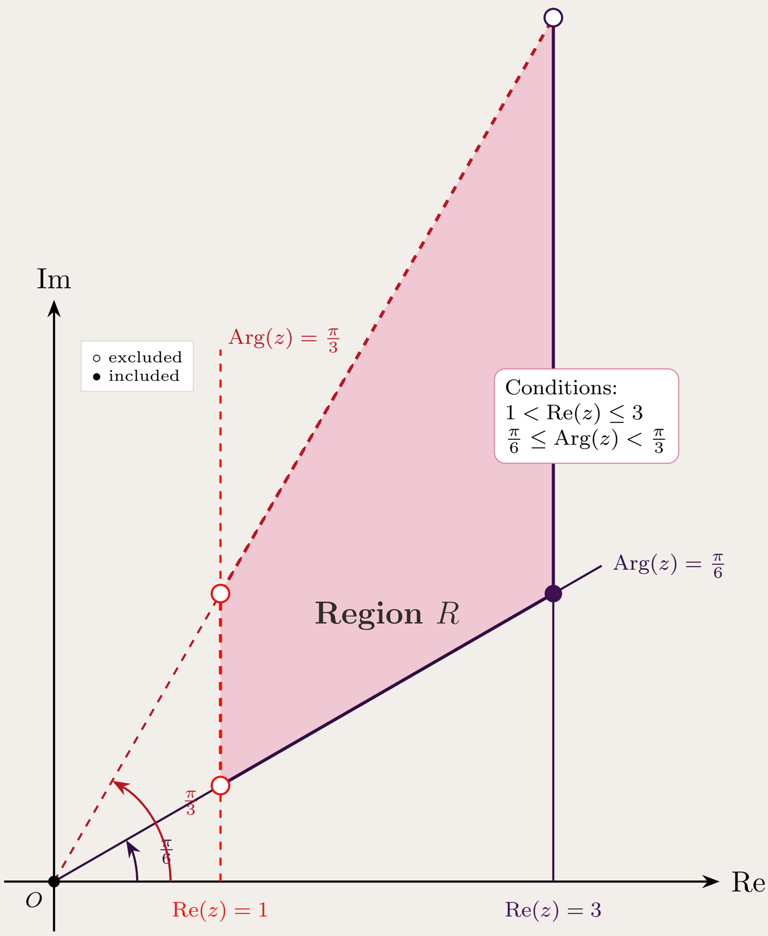
Problem 3.20: Region with Real and Argument Constraints

Let R be the region in the complex plane defined by $1 < \mathrm{Re}(z) \leq 3$ and $\frac{\pi}{6} \leq \mathrm{Arg}(z) < \frac{\pi}{3}$. Sketch this region.

Hint: The region is bounded by two vertical lines ($x = 1$ and $x = 3$) and two rays from the origin at angles $\pi/6$ and $\pi/3$.

Solution 3.20: Sketch

Draw vertical lines at $x = 1$ (dashed, excluded) and $x = 3$ (solid, included). Draw rays from origin at angles 30° (solid) and 60° (dashed). The region is the intersection in the first quadrant.



Problem 3.21: Statement Analysis

Which statement about complex numbers is true?

(A) For $z = x + iy$, the argument is given by $\arg(z) = \arctan(y/x)$ for all $z \neq 0$.

(B) For any two complex numbers z_1 and z_2, we have $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$.

(C) If two complex numbers have the same modulus and their arguments satisfy $\theta_1 = \theta_2 + 2\pi k$ for some integer k, then they are equal (modulo 2π).

(D) For $z = x + iy$ with $x > 0$, the argument is $\arg(z) = \arctan(y/x)$.

Hint: Check each statement. Remember that $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ only modulo 2π.

Solution 3.21: Sketch

- **(A) False:** The formula $\arg(z) = \arctan(y/x)$ doesn't account for the quadrant. For example, if $z = -1 + i$, then $\arctan(-1) = -\pi/4$, but the actual argument is $3\pi/4$ (in the second quadrant).
- **(B) False:** Principal argument addition requires modulo 2π adjustment. For instance, if $\text{Arg}(z_1) = 3\pi/4$ and $\text{Arg}(z_2) = 3\pi/4$, then $\text{Arg}(z_1) + \text{Arg}(z_2) = 3\pi/2$, but we need to adjust to the principal range $(-\pi, \pi]$, giving $\text{Arg}(z_1 z_2) = -\pi/2$.
- **(C) True (with proper interpretation):** If $|z_1| = |z_2|$ and $\arg(z_1) = \arg(z_2) + 2\pi k$ for integer k, then $z_1 = z_2$ since arguments that differ by $2\pi k$ represent the same direction.
- **(D) True (but limited):** This is correct when $x > 0$ (first and fourth quadrants), as the arctan function gives the correct argument in these quadrants without adjustment.

Answer: The best answer is **(D)** for its specific domain, though **(C)** is also correct if "equality" means modulo 2π. Statement (D) is the most straightforward true statement.

Problem 3.22: Roots of Unity Application

Suppose that $x + \frac{1}{x} = -1$. What is the value of $x^{2016} + \frac{1}{x^{2016}}$?

Hint: Solve $x^2 + x + 1 = 0$ to find $x = e^{i2\pi/3}$ or $x = e^{-i2\pi/3}$. These are cube roots of unity.

Solution 3.22: Sketch

$x^2 + x + 1 = 0$ gives $x = e^{\pm i2\pi/3}$. Since $x^3 = 1$, we have $x^{2016} = x^{3 \cdot 672} = 1$. Thus $x^{2016} + \frac{1}{x^{2016}} = 1 + 1 = 2$.

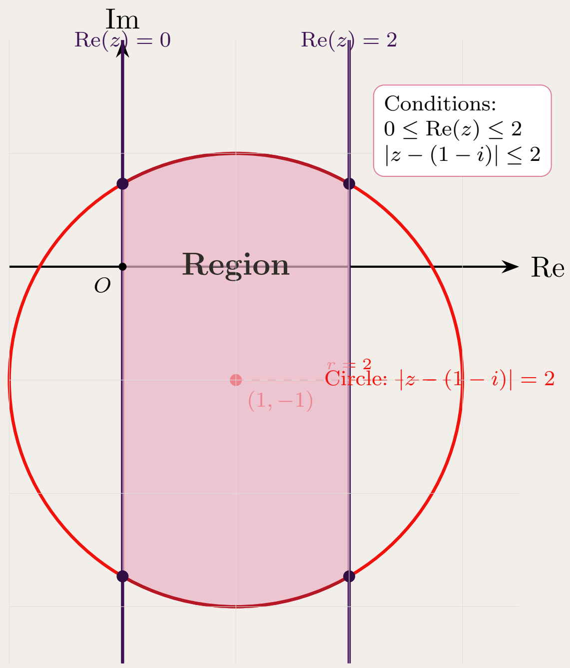
Problem 3.23: Shading a Region

On an Argand diagram, shade the region where $0 \le \mathrm{Re}(z) \le 2$ and $|z-(1-i)| \le 2$ both hold.

Hint: The first inequality is a vertical strip. The second is a disk centered at $(1,-1)$ with radius 2.

Solution 3.23: Sketch

Draw vertical lines at $x = 0$ and $x = 2$. Draw circle centered at $(1,-1)$ with radius 2. Shade the intersection.



Problem 3.24: Modulus-Argument Division

Given $\frac{1+\sqrt{3}i}{1+i} = \frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}i$, express $\frac{1+i\sqrt{3}}{1+i}$ in modulus-argument form by converting both numerator and denominator.

Hint: $|1+i\sqrt{3}| = 2$, $\arg(1+i\sqrt{3}) = \pi/3$; $|1+i| = \sqrt{2}$, $\arg(1+i) = \pi/4$.

Solution 3.24: Sketch

$\frac{1+i\sqrt{3}}{1+i} = \frac{2}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right) = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right).$

Problem 3.25: Power in Polar Form

Let $\beta = 1 - i\sqrt{3}$. Express β^5 in modulus-argument form.

Hint: $|\beta| = 2$, $\arg(\beta) = -\pi/3$.

Solution 3.25: Sketch

$\beta = 2(\cos(-\pi/3) + i\sin(-\pi/3))$. Then $\beta^5 = 32(\cos(-5\pi/3) + i\sin(-5\pi/3)) = 32(\cos(\pi/3) + i\sin(\pi/3))$.

Problem 3.26: Deriving Trigonometric Value

Find the exact value of $\sin\frac{\pi}{12}$ using the angle subtraction formula. Note that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4} = 60 - 45 = 15$.

Hint: Use the angle subtraction formula: $\sin(A - B) = \sin A\cos B - \cos A\sin B$.
Recall the exact values:

- $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$
- $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Solution 3.26: Sketch

We use the fact that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.
Applying the angle subtraction formula:

$$\begin{aligned}
\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

Alternative approach using complex numbers:
Consider the division $\frac{1+i\sqrt{3}}{1+i}$. In modulus-argument form:

- Numerator: $|1+i\sqrt{3}| = 2$, $\arg(1+i\sqrt{3}) = \frac{\pi}{3}$
- Denominator: $|1+i| = \sqrt{2}$, $\arg(1+i) = \frac{\pi}{4}$

Therefore:

$$\frac{1+i\sqrt{3}}{1+i} = \frac{2}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right) = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

Computing the division algebraically:

$$\frac{1+i\sqrt{3}}{1+i} = \frac{(1+i\sqrt{3})(1-i)}{(1+i)(1-i)} = \frac{1-i+i\sqrt{3}+\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} + i\frac{\sqrt{3}-1}{2}$$

Comparing imaginary parts: $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$.
Answer: $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

Problem 3.27: Marking Rotated Point

A point P represents the complex number $z = 3 + 2i$ on the Argand diagram. Mark the point R representing iz.

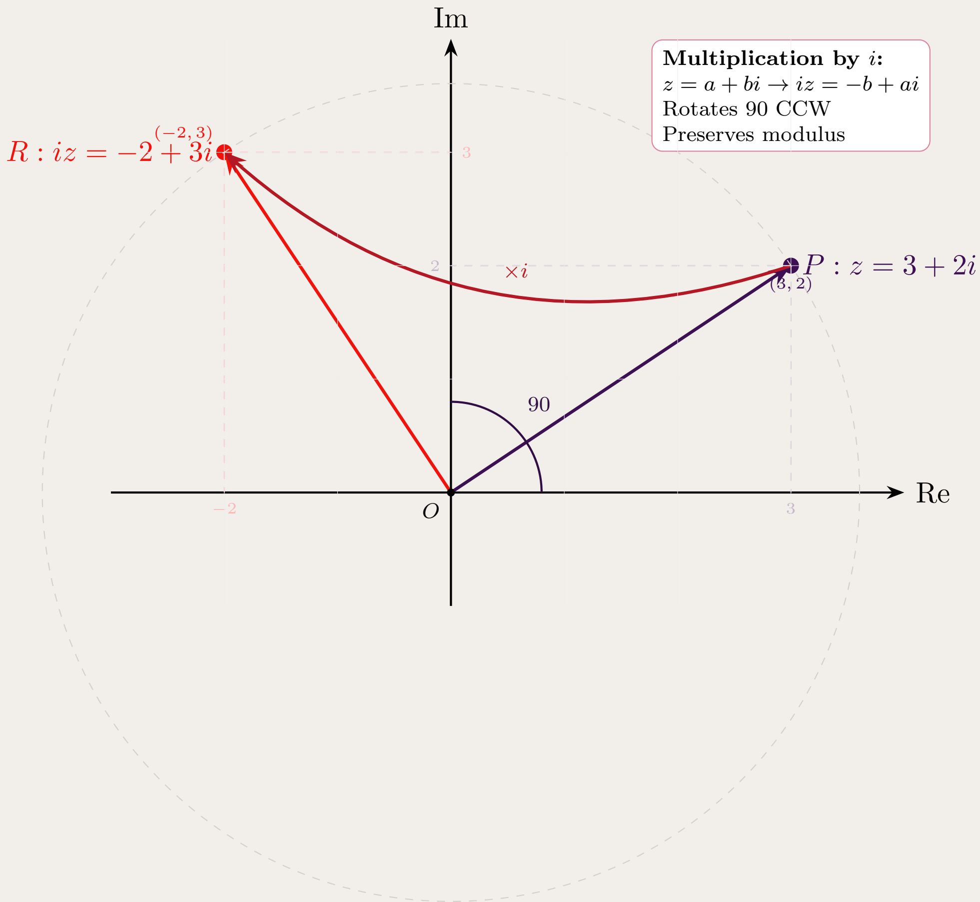
Hint: Multiply by i rotates 90° counterclockwise. If $z = a + bi$, then $iz = i(a + bi) = ai + bi^2 = -b + ai$.

Solution 3.27: Sketch

If z is at (a, b), then iz is at $(-b, a)$. Draw accordingly.
For $z = 3 + 2i$ at point $(3, 2)$:

$$iz = i(3 + 2i) = 3i + 2i^2 = 3i - 2 = -2 + 3i$$

So R is at $(-2, 3)$.



Key observation: Multiplying by i rotates any complex number by 90 counterclockwise while keeping the same distance from the origin.

Problem 3.28: Conjugate Root Property

$2 + i$ is a root of $P(z) = z^3 + rz^2 + sz + 20$ where r, s are real. State why $2 - i$ is also a root.

Hint: Polynomials with real coefficients have complex roots in conjugate pairs.

Solution 3.28: Sketch

Since $P(z)$ has real coefficients, if $2 + i$ is a root, then $\overline{2 + i} = 2 - i$ must also be a root.

Problem 3.29: Factorizing Over Reals

Factorise $P(z) = z^3 + rz^2 + sz + 20$ over the real numbers, given roots $2+i$ and $2-i$.

Hint: $(z-(2+i))(z-(2-i)) = (z-2)^2 - (i)^2 = z^2 - 4z + 5$.

Solution 3.29: Sketch

$(z-2-i)(z-2+i) = z^2 - 4z + 5$. Divide $P(z)$ by $z^2 - 4z + 5$ to find the third factor. $P(z) = (z^2 - 4z + 5)(z+4)$ after polynomial division.

Problem 3.30: Vector Addition Point

Points P and Q represent the complex numbers $z = 2 + i$ and $w = 1 + 3i$ respectively on the Argand diagram. Mark point T representing $z + w$.

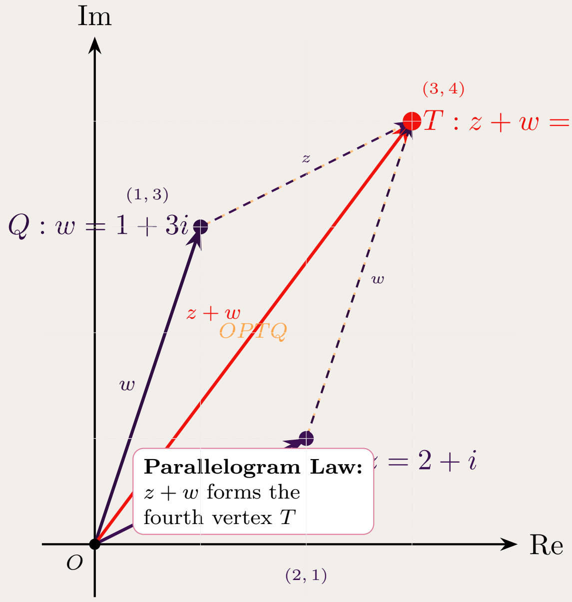
Hint: Use parallelogram law: T is the fourth vertex of parallelogram $OPTQ$ where O is the origin.

Solution 3.30: Sketch

From O to P is vector z, from O to Q is vector w. The sum $z+w$ is at the opposite corner of the parallelogram.
For $z = 2+i$ and $w = 1+3i$:

$$z + w = (2+i) + (1+3i) = 3+4i$$

So T is at $(3, 4)$.



Key observations:

- The parallelogram $OPTQ$ has OP parallel to QT (both represent vector z)
- Similarly, OQ is parallel to PT (both represent vector w)
- Vector addition: starting from O, go to P (add z), then to T (add w)
- Alternatively: starting from O, go to Q (add w), then to T (add z)
- The diagonal OT represents the sum $z+w$

Problem 3.31: Sum in Polar Form

Let $z = -2-2i$ and $w = 3+i$. Express $z+w$ in modulus-argument form.

Hint: First find $z+w = 1-i$, then convert to polar.

Solution 3.31: Sketch

$z+w = 1-i$. $|z+w| = \sqrt{2}$, $\arg(z+w) = -\pi/4$. So $z+w = \sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$.

Problem 3.32: Division in Mixed Form

Express $\frac{z}{w}$ in form $x+iy$ where $z=-2-2i, w=3+i$.

Hint: $\frac{-2-2i}{3+i} = \frac{(-2-2i)(3-i)}{(3+i)(3-i)}$.

Solution 3.32: Sketch

$\frac{(-2-2i)(3-i)}{10} = \frac{-6+2i-6i+2i^2}{10} = \frac{-8-4i}{10} = -\frac{4}{5} - \frac{2}{5}i.$

Problem 3.33: Sketching with Argument and Circle

Sketch the region: $-\frac{\pi}{4} \leq \arg z \leq 0$ and $|z-(1-i)| \leq 1$.

Hint: The first is a wedge in the fourth quadrant. The second is a disk centered at $(1,-1)$.

Solution 3.33: Sketch

Draw rays at angles $-45°$ and $0°$ from origin. Draw circle centered at $(1,-1)$ radius 1. Shade intersection.

Problem 3.34: High Power via De Moivre

Express z^9 in form $x+iy$ for $z=\sqrt{3}-i$.

Hint: $z = 2(\cos(-\pi/6) + i\sin(-\pi/6))$.

Solution 3.34: Sketch

$z^9 = 2^9(\cos(-3\pi/2) + i\sin(-3\pi/2)) = 512(0+i) = 512i.$

Problem 3.35: Polar Form of Specific Number

Write $z=-1+i\sqrt{3}$ in modulus-argument form.

Hint: $|z|=2$, z is in the second quadrant with reference angle $\pi/3$.

Solution 3.35: Sketch

$z = 2(\cos(2\pi/3) + i\sin(2\pi/3)).$

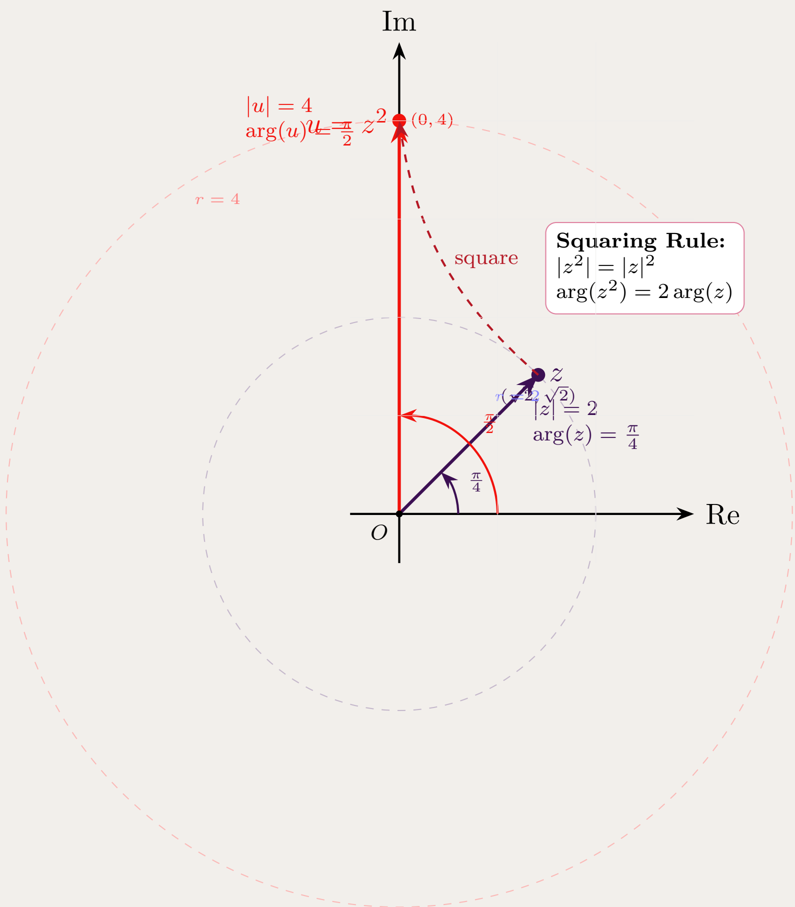
Problem 3.36: Plotting a Squared Number

Given $|z| = 2, \arg(z) = \frac{\pi}{4}$, plot $u = z^2$.

Hint: $|u| = |z|^2 = 4$, $\arg(u) = 2\arg(z) = \pi/2$.

Solution 3.36: Sketch

u has modulus 4 and argument $\pi/2$, so $u = 4i$. Plot at $(0, 4)$.



Problem 3.37: Fifth Roots of Unity and Cosine Sum

Let ω be a non-real root of $z^5 = 1$

a) Show that: $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

b) Hence, show that: $\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$

Hint: For part (a): The fifth roots of unity are roots of $z^5 - 1 = 0$. Factor as $(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$. For part (b): Use $\omega = e^{i2\pi/5}$ and $\omega^4 = e^{i8\pi/5} = e^{-i2\pi/5} = \bar{\omega}$. Add $\omega + \omega^4$ using Euler's formula.

Solution 3.37: Sketch

(a) Since $\omega^5 = 1$, factor $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1) = 0$. As $\omega \neq 1$ (non-real), we have $z^4 + z^3 + z^2 + z + 1 = 0$, so $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. $\square$
(b) Let $\omega = e^{i2\pi/5}$. Then $\omega^4 = e^{-i2\pi/5} = \bar{\omega}$ and $\omega^3 = \overline{\omega^2}$. From (a):

$$1 + (\omega + \omega^4) + (\omega^2 + \omega^3) = 0 \implies 1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$$

Therefore $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. $\square$

Problem 3.38: Cube Roots of Unity Properties

If ω is a complex cube root of unity, show that:

a) $1 + \omega + \omega^2 = 0$

b) $1 + \omega + \bar{\omega} = 0$

c) $(6\omega + 1)(6\omega^2 + 1) = 31$

d) $(1 + \omega^2)^3(2 + 3\omega + 3\omega^2) = 1$

Hint: A complex cube root of unity satisfies $\omega^3 = 1$ and $\omega \neq 1$. Use $\omega = e^{i2\pi/3}$ and the fundamental identity $1 + \omega + \omega^2 = 0$ to prove all parts. For (b), note that $\omega^2 = \bar{\omega}$. For (c) and (d), expand and repeatedly substitute $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

Solution 3.38: Sketch

Key facts: A complex cube root of unity ω satisfies:

- $\omega^3 = 1$
- $\omega = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
- $\omega^2 = e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
- $\omega^2 = \bar{\omega}$ (conjugates)

Part (a): Show $1 + \omega + \omega^2 = 0$
Since ω is a cube root of unity: $\omega^3 = 1$, so $\omega^3 - 1 = 0$.
Factor: $(\omega - 1)(\omega^2 + \omega + 1) = 0$
Since $\omega \neq 1$ (it's a complex root), we must have:

$$\omega^2 + \omega + 1 = 0$$

Therefore: $1 + \omega + \omega^2 = 0 \quad \square$
Part (b): Show $1 + \omega + \bar{\omega} = 0$
Since $\omega = e^{i2\pi/3}$ and $\omega^2 = e^{i4\pi/3} = e^{-i2\pi/3}$, we have:

$$\omega^2 = \overline{\omega}$$

From part (a): $1 + \omega + \omega^2 = 0$
Substituting $\omega^2 = \bar{\omega}$:

$$1 + \omega + \bar{\omega} = 0 \quad \square$$

Part (c): Show $(6\omega + 1)(6\omega^2 + 1) = 31$
Expand the product:

$$\begin{aligned}(6\omega + 1)(6\omega^2 + 1) &= 36\omega \cdot \omega^2 + 6\omega + 6\omega^2 + 1 \\ &= 36\omega^3 + 6\omega + 6\omega^2 + 1\end{aligned}$$

Use $\omega^3 = 1$:

$$= 36(1) + 6\omega + 6\omega^2 + 1 = 36 + 1 + 6(\omega + \omega^2)$$

From part (a): $\omega + \omega^2 = -1$
Therefore:

$$= 37 + 6(-1) = 37 - 6 = 31 \quad \square$$

Part (d): From (a), $1+\omega^2 = -\omega$, so $(1+\omega^2)^3 = (-\omega)^3 = -\omega^3 = -1$. Also, $2+3\omega+3\omega^2 = 2 + 3(-1) = -1$. Therefore $(1 + \omega^2)^3(2 + 3\omega + 3\omega^2) = (-1)(-1) = 1$. $\quad \square$

Problem 3.39: Circle Locus in Complex Plane

Show that the following relationship geometrically describes a circle in the complex plane:

$$\frac{1}{z} - \frac{1}{\bar{z}} = i$$

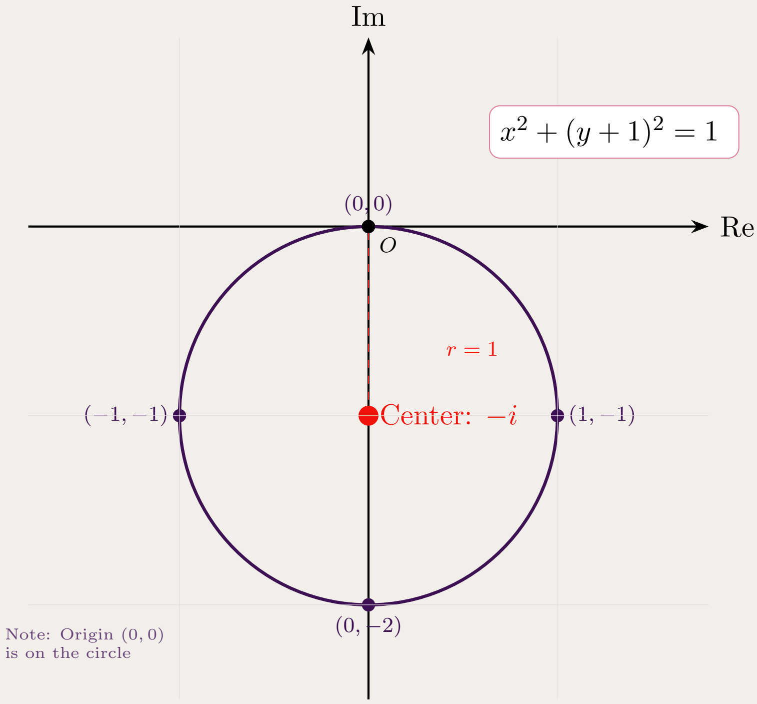
Hint: Let $z = x + iy$ where $x, y \in \mathbb{R}$. Express $\frac{1}{z}$ and $\frac{1}{\bar{z}}$ in terms of x and y, then separate real and imaginary parts. The result should be an equation of the form $(x-a)^2 + (y-b)^2 = r^2$.

Solution 3.39: Sketch

Let $z = x + iy$. Then:

$$\frac{1}{z} - \frac{1}{\bar{z}} = \frac{x - iy}{x^2 + y^2} - \frac{x + iy}{x^2 + y^2} = \frac{-2iy}{x^2 + y^2}$$

Given this equals i: $\frac{-2y}{x^2+y^2} = 1 \implies x^2 + y^2 + 2y = 0 \implies x^2 + (y+1)^2 = 1$. This is a circle centered at $-i$ with radius 1.



Verification: The origin $(0,0)$ satisfies the equation: $0^2 + (0+1)^2 = 1$ ✓
Geometric interpretation: For any point z on this circle, the difference $\frac{1}{z} - \frac{1}{\bar{z}}$ equals i. This relates the reciprocals and their conjugates in a specific way that constrains z to lie on a circle.

Problem 3.40: Square OABC with Complex Numbers

$OABC$ is a square with point A represented by the complex number $z = 2 + i$.

a) Let w represent the complex number at point C. Prove that: $z^2 + w^2 = 0$

b) Find the complex numbers represented by B and C.

c) Find the complex number represented by the vector $\overrightarrow{AC}$

Hint: For a square $OABC$ with O at the origin, C is obtained by rotating A by 90 counterclockwise, so $w = iz$. For part (a), substitute and use $i^2 = -1$. For part (b), B is the fourth vertex, obtained by $B = A + C$ (vector addition).

Solution 3.40: Sketch

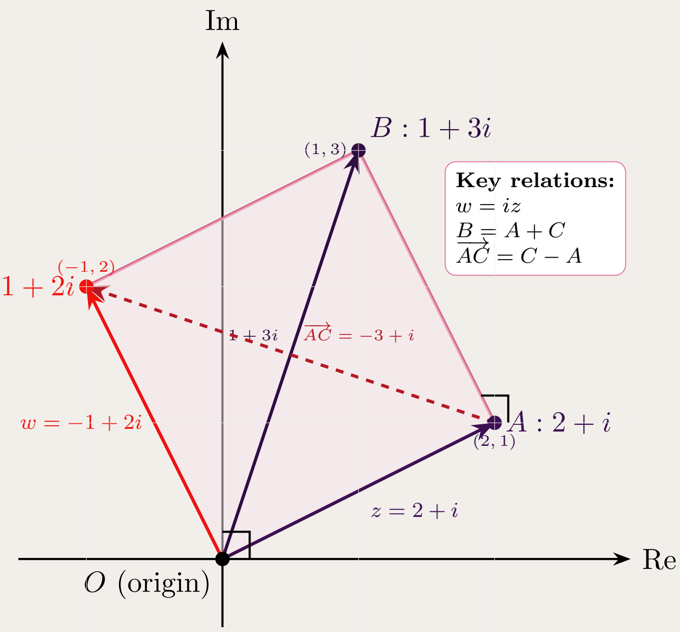
Given: Square $OABC$ with O at origin and A at $z = 2 + i$.
Part (a): Prove $z^2 + w^2 = 0$
For a square with one vertex at the origin O and adjacent vertex at A (represented by z), the next vertex C (going counterclockwise) is obtained by rotating OA by 90 counterclockwise.
Rotation by 90 is achieved by multiplying by i:

$$w = iz$$

Now calculate $z^2 + w^2$:

$$\begin{aligned} z^2 + w^2 &= z^2 + (iz)^2 \\ &= z^2 + i^2 z^2 \\ &= z^2 + (-1)z^2 \\ &= z^2 - z^2 \\ &= 0 \quad \square \end{aligned}$$

(b) C: $w = iz = i(2+i) = 2i-1 = -1+2i$. B: $B = A+C = (2+i)+(-1+2i) = 1+3i$.
(c) $\overrightarrow{AC} = C - A = (-1+2i) - (2+i) = -3+i$.



Problem 3.41: Square ABCD Vertices with Complex Numbers

$ABCD$ is a square in the complex plane. The vertices A and B represent the complex numbers $9+i$ and $4+13i$ respectively. Find the complex numbers corresponding to vertices C and D.

Hint: The vector from A to B is $\overrightarrow{AB} = B - A = (4+13i) - (9+i) = -5+12i$. To find C, rotate $\overrightarrow{AB}$ by 90 counterclockwise (multiply by i) and add to B. Similarly for D.

Solution 3.41: Sketch

$\overrightarrow{AB} = B - A = (4+13i) - (9+i) = -5+12i$. Rotate by 90: $\overrightarrow{BC} = i(-5+12i) = -12-5i$. Thus $C = B + \overrightarrow{BC} = (4+13i) + (-12-5i) = -8+8i$. Similarly, $D = A + \overrightarrow{BC} = (9+i) + (-12-5i) = -3-4i$.
Answers: $C = -8+8i$, $D = -3-4i$.

Problem 3.42: Square ABCD Vector Relationship

The points A, B, C and D on the Argand diagram represent the complex numbers a, b, c and d respectively. The points form a square $ABCD$.
By using vectors, or otherwise, show that $c = (1+i)d - ia$.

Hint: Use the fact that in a square, consecutive sides are equal in length and perpendicular. The key relationship is $\overrightarrow{DC} = i \cdot \overrightarrow{AD}$ (rotation by 90).

Solution 3.42: Proof

In square $ABCD$, $\overrightarrow{DC} = i \cdot \overrightarrow{AD}$ (rotation by 90). Thus:

$$c - d = i(d-a) = id - ia \implies c = d + id - ia = (1+i)d - ia \quad \square$$

3.3 Advanced Complex Numbers Problems

Problem 3.43: Complex Region with Exclusion

Sketch the region on the Argand diagram where $|z - \bar{z}| < 2$ and $|z - 1| \geq 1$ hold simultaneously.

Hint: $|z - \bar{z}| = |2iy| = 2|y|$, so the first inequality gives $|y| < 1$. The second is the exterior of a circle.

Solution 3.43: Sketch

The region $|y| < 1$ is a horizontal strip $-1 < y < 1$. The region $|z-1| \geq 1$ is outside the circle centered at $(1, 0)$ with radius 1. Shade the intersection: horizontal strip with circular hole.

Problem 3.44: Isosceles Right Triangle

Triangle ABC is represented by z_1, z_2, z_3. It is isosceles and right-angled at B. Explain why $(z_1 - z_2)^2 = -(z_3 - z_2)^2$.

Hint: The vectors BA and BC are perpendicular and equal in length. Multiplication by i rotates by 90°.

Solution 3.44: Sketch

$BA = z_1 - z_2$, $BC = z_3 - z_2$. Since $BA \perp BC$ and $|BA| = |BC|$, we have $BA = \pm i \cdot BC$. Squaring: $(z_1 - z_2)^2 = (\pm i)^2 (z_3 - z_2)^2 = -1 \cdot (z_3 - z_2)^2$.

Problem 3.45: Square from Triangle

Three vertices of a square $ABCD$ in the complex plane are represented by complex numbers z_1 (point A), z_2 (point B), and z_3 (point C). Given that consecutive sides of a square are perpendicular and equal in length, find the complex number z_4 representing vertex D in terms of z_1, z_2, z_3.

Hint: Since $ABCD$ is a square, side AB is perpendicular to side BC, and $|AB| = |BC|$. This means $BA = i \cdot BC$ (or $-i \cdot BC$). Use the same relationship for sides CD and DA.

Solution 3.45: Sketch

For a square $ABCD$, consecutive sides are equal in length and perpendicular.
The vector from B to A is $z_1 - z_2$, and from B to C is $z_3 - z_2$.
Since $BA \perp BC$ and $|BA| = |BC|$, we have:

$$z_1 - z_2 = i(z_3 - z_2) \quad \text{(rotating } BC \text{ by 90 counterclockwise)}$$

Similarly, for the square to close, the vector from C to D must equal the vector from B to C rotated 90:

$$z_4 - z_3 = i(z_2 - z_3)$$

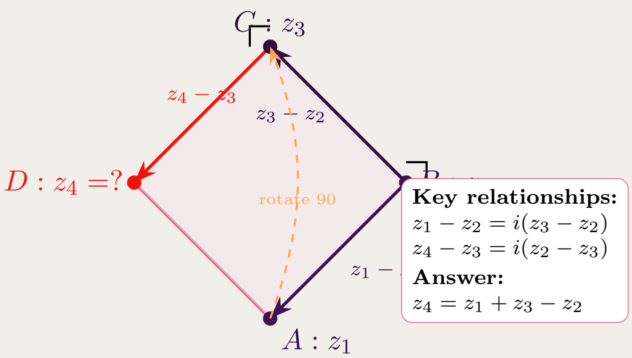
Solving for z_4:

$$\begin{aligned} z_4 &= z_3 + i(z_2 - z_3) \\ &= z_3 + iz_2 - iz_3 \\ &= z_2 \cdot i + z_3(1 - i) \\ &= iz_2 + z_3(1 - i) \end{aligned}$$

Alternative formula: Using the parallelogram property, we can also write:

$$z_4 = z_1 + z_3 - z_2$$

This works because $\vec{AD} = \vec{BC}$, so $z_4 - z_1 = z_3 - z_2$.



Verification: Both formulas are equivalent:

$$\begin{aligned} iz_2 + z_3(1 - i) &= iz_2 + z_3 - iz_3 \\ &= z_3 + i(z_2 - z_3) \end{aligned}$$

And using $z_1 - z_2 = i(z_3 - z_2)$, we get $z_1 = z_2 + i(z_3 - z_2) = iz_3 + z_2(1 - i)$.
Therefore: $z_4 = z_1 + z_3 - z_2$

Problem 3.46: Factoring Polynomial

Express $P(x) = x^4 - 12x^3 + 59x^2 - 138x + 130$ as product of quadratic factors, given roots $3 + i$ and $3 + 2i$.

Hint: The roots are $3 \pm i$ and $3 \pm 2i$. Group conjugate pairs.

Solution 3.46: Sketch

$(x-(3+i))(x-(3-i)) = (x-3)^2+1 = x^2-6x+10$. $(x-(3+2i))(x-(3-2i)) = (x-3)^2+4 = x^2-6x+13$. Thus $P(x) = (x^2-6x+10)(x^2-6x+13)$.

Problem 3.47: Real Root Count

Deduce that $x^4 - 3x^3 + 5x^2 + 7x - 8 = 0$ has exactly two real roots.

Hint: A degree-4 polynomial with real coefficients has either 0, 2, or 4 real roots. Check behavior or use Descartes' rule.

Solution 3.47: Sketch

Complex roots come in conjugate pairs. If there are 4 complex roots, they form 2 pairs, leaving 0 real roots. If 1 pair of complex roots, then 2 real roots. Since the polynomial has sign changes, it must have at least one positive real root. Testing values or using intermediate value theorem confirms exactly 2 real roots.

Problem 3.48: Vector Addition and Angles

Consider a sequence of unit complex numbers defined by:

$$z_n = \cos(\alpha + n\beta) + i\sin(\alpha + n\beta) = e^{i(\alpha+n\beta)}$$

for $n = 0, 1, 2, 3, \ldots$
On the Argand diagram, we construct points by cumulative vector addition:

$$\begin{aligned} P_0 &= z_0 \\ P_1 &= z_0 + z_1 \\ P_2 &= z_0 + z_1 + z_2 \\ P_3 &= z_0 + z_1 + z_2 + z_3 \\ &\vdots \end{aligned}$$

These points form a polygonal path. Using vector addition, prove that all external angles of this polygon are equal to β. That is, show that $\theta_0 = \theta_1 = \theta_2 = \cdots = \beta$, where θ_n is the external angle at vertex P_n.

Hint: Each z_n is a unit vector at angle $\alpha + n\beta$. The angle between consecutive vectors is constant. The external angle is the amount the direction turns from one segment to the next.

Solution 3.48: Sketch

Understanding the geometry:
The polygonal path is formed by placing unit vectors $z_0, z_1, z_2, \ldots$ tip-to-tail. Each vector z_n is a unit complex number (lies on the unit circle) with argument $\alpha + n\beta$.
Step 1: Arguments of consecutive vectors
For any consecutive pair of vectors:

$$\arg(z_n) = \alpha + n\beta$$
$$\arg(z_{n+1}) = \alpha + (n+1)\beta = \alpha + n\beta + \beta$$

The difference in arguments is:

$$\arg(z_{n+1}) - \arg(z_n) = \beta$$

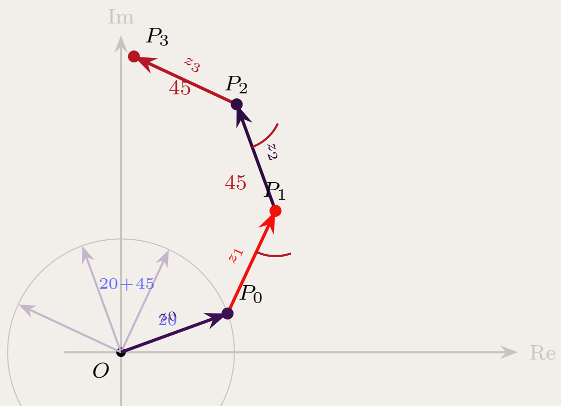
Step 2: Geometric interpretation
The segment from P_n to P_{n+1} is represented by the vector z_{n+1} (pointing in direction $\arg(z_{n+1})$).
The segment from P_{n-1} to P_n is represented by the vector z_n (pointing in direction $\arg(z_n)$).
The external angle θ_n at vertex P_n is the angle by which the path turns. This is precisely the difference in the directions of consecutive vectors:

$$\theta_n = \arg(z_{n+1}) - \arg(z_n) = \beta$$

Step 3: Conclusion
Since this difference is constant for all n, we have:

$$\theta_0 = \theta_1 = \theta_2 = \cdots = \beta$$

All external angles equal β, which means the polygon turns by the same angle at each vertex.
Special case: If $\beta = \frac{2\pi}{n}$, the path closes after n steps, forming a regular n-sided polygon.



Vector addition spiral with constant external angle 45

The diagram shows unit vectors z_0, z_1, z_2, z_3 (shown faintly from origin) being added tip-to-tail to form the polygonal path $P_0 \to P_1 \to P_2 \to P_3$. Each turn angle (external angle) is $\beta = 45$.

Problem 3.49: Modulus via Trigonometry

Show $|z| = 2\sin\theta$ for $z = 1 - \cos 2\theta + i\sin 2\theta$.

Hint: Use $1 - \cos 2\theta = 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

Solution 3.49: Sketch

$|z|^2 = (1-\cos 2\theta)^2 + \sin^2 2\theta = (2\sin^2\theta)^2 + (2\sin\theta\cos\theta)^2 = 4\sin^4\theta + 4\sin^2\theta\cos^2\theta = 4\sin^2\theta(\sin^2\theta + \cos^2\theta) = 4\sin^2\theta$. Thus $|z| = 2|\sin\theta|$.

Problem 3.50: Euler's Formula Identity

Show $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$.

Hint: Use $e^{i\phi} = \cos\phi + i\sin\phi$.

Solution 3.50: Sketch

$e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$ and $e^{-in\theta} = \cos(n\theta) - i\sin(n\theta)$. Adding: $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$.

Problem 3.51: Modulus and Angle Calculation with Complex Exponentials

Let z be the complex number $z = e^{\frac{i\pi}{6}}$ and w be the complex number $w = e^{\frac{3i\pi}{4}}$.

(i) By first writing z and w in Cartesian form, or otherwise, show that

$$|z+w|^2 = \frac{4 - \sqrt{6} + \sqrt{2}}{2}.$$

(ii) The complex numbers z, w and $z+w$ are represented in the complex plane by the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively, where O is the origin.

Show that $\angle AOC = \frac{7\pi}{24}$.

(iii) Deduce that $\cos\frac{7\pi}{24} = \frac{\sqrt{8-2\sqrt{6}+2\sqrt{2}}}{4}$.

Hint: For (i): Convert to Cartesian form and compute $|z+w|^2 = x^2 + y^2$. For (ii): Since $|z| = |w| = 1$, the parallelogram $OACB$ is a rhombus, and the diagonal bisects the angle. For (iii): Use the relationship $|z+w| = 2\cos(\angle AOC)$ in a rhombus.

Solution 3.51: Full Solution

Part (i)

First, we convert z and w to Cartesian form:

$$z = e^{\frac{i\pi}{6}} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w = e^{\frac{3i\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

Now, find $z + w$:

$$z + w = \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right) + i\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{3} - \sqrt{2}}{2} + i\frac{1+\sqrt{2}}{2}$$

Calculate the square of the modulus $|z+w|^2 = x^2 + y^2$:

$$\begin{aligned}
|z+w|^2 &= \left(\frac{\sqrt{3}-\sqrt{2}}{2}\right)^2 + \left(\frac{1+\sqrt{2}}{2}\right)^2 \\
&= \frac{3 - 2\sqrt{6} + 2}{4} + \frac{1 + 2\sqrt{2} + 2}{4} \\
&= \frac{5 - 2\sqrt{6} + 3 + 2\sqrt{2}}{4} \\
&= \frac{8 - 2\sqrt{6} + 2\sqrt{2}}{4} \\
&= \frac{4 - \sqrt{6} + \sqrt{2}}{2} \quad \checkmark
\end{aligned}$$

Part (ii)

Note that $|z| = |e^{\frac{i\pi}{6}}| = 1$ and $|w| = |e^{\frac{3i\pi}{4}}| = 1$. Since the moduli are equal ($OA = OB = 1$), the parallelogram $OACB$ formed by vector addition is a **rhombus**.

In a rhombus, the main diagonal $\overrightarrow{OC}$ bisects the angle between the adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Find $\angle AOB$:

$$\arg(z) = \frac{\pi}{6} = \frac{4\pi}{24}, \quad \arg(w) = \frac{3\pi}{4} = \frac{18\pi}{24}$$

$$\angle AOB = \arg(w) - \arg(z) = \frac{18\pi}{24} - \frac{4\pi}{24} = \frac{14\pi}{24}$$

Since $\overrightarrow{OC}$ bisects $\angle AOB$, the angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is half of $\angle AOB$:

$$\angle AOC = \frac{1}{2} \times \frac{14\pi}{24} = \frac{7\pi}{24} \quad \checkmark$$

Part (iii)

We use the geometry of the rhombus. The length of the diagonal OC can be calculated using the property that in a rhombus with side length 1 and angle θ between the diagonal and the side:

$$OC = 2 \times OA \times \cos(\angle AOC)$$

Substituting the known values:

$$|z+w| = 2(1)\cos\left(\frac{7\pi}{24}\right) = 2\cos\left(\frac{7\pi}{24}\right)$$

Squaring both sides:

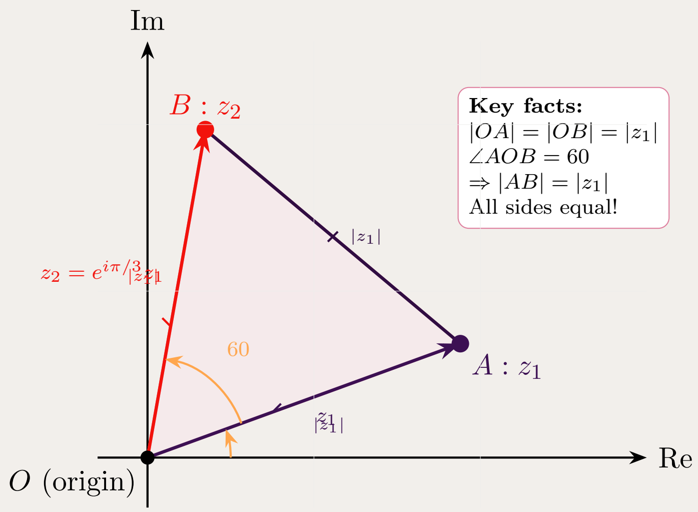
Problem 3.52: Equilateral Triangle

Let z_1 be a non-zero complex number represented by point A on the Argand diagram, and let $z_2 = e^{i\pi/3} z_1$ be represented by point B. The origin is represented by point O. Prove that triangle $\triangle OAB$ is equilateral.

Hint: Multiplication by $e^{i\pi/3}$ rotates z_1 by angle $\pi/3 = 60$ while preserving its modulus. Therefore $|z_1| = |z_2|$ and the angle $\angle AOB = \pi/3$.

Solution 3.52: Sketch

Given $z_2 = e^{i\pi/3} z_1$, we show all three sides of triangle OAB are equal.
Step 1: Since $|e^{i\pi/3}| = 1$, we have $|OA| = |z_1|$ and $|OB| = |z_2| = |e^{i\pi/3} z_1| = |z_1|$.
Step 2: Since $z_2 = e^{i\pi/3} z_1$, the angle $\angle AOB = \arg(z_2) - \arg(z_1) = \frac{\pi}{3} = 60$.
Step 3: By law of cosines: $|AB|^2 = |z_1|^2 + |z_1|^2 - 2|z_1|^2 \cos(\pi/3) = 2|z_1|^2 - |z_1|^2 = |z_1|^2$, so $|AB| = |z_1|$.
Conclusion: $|OA| = |OB| = |AB| = |z_1|$, so triangle OAB is equilateral.



Alternative: $|AB| = |z_2 - z_1| = |z_1||e^{i\pi/3} - 1| = |z_1| \cdot |(1/2 + i\sqrt{3}/2) - 1| = |z_1|\sqrt{1/4 + 3/4} = |z_1|$. $\square$

Problem 3.53: Algebraic Identity

Consider an equilateral triangle $\triangle OAB$ where O is the origin, A represents complex number z_1, and B represents $z_2 = e^{i\pi/3} z_1$ (i.e., z_2 is obtained by rotating z_1 by 60). Prove the algebraic identity:

$$z_1^2 + z_2^2 = z_1 z_2$$

Hint: Let $\omega = e^{i\pi/3}$, so $z_2 = \omega z_1$. Show that ω satisfies $\omega^2 - \omega + 1 = 0$ by using the fact that $\omega^3 = e^{i\pi} = -1$.

Solution 3.53: Sketch

Let $\omega = e^{i\pi/3}$ (a primitive 6th root of unity). Then $z_2 = \omega z_1$.
Left-hand side:

$$\begin{aligned} z_1^2 + z_2^2 &= z_1^2 + (\omega z_1)^2 \\ &= z_1^2 + \omega^2 z_1^2 \\ &= z_1^2(1+\omega^2) \end{aligned}$$

Right-hand side:

$$\begin{aligned} z_1 z_2 &= z_1 \cdot \omega z_1 \\ &= \omega z_1^2 \end{aligned}$$

To prove: We need to show that $1 + \omega^2 = \omega$, or equivalently:

$$\omega^2 - \omega + 1 = 0$$

Key observation: Since $\omega = e^{i\pi/3}$, we have:

$$\omega^3 = e^{i\pi/3 \cdot 3} = e^{i\pi} = -1$$

Therefore:

$$\omega^3 + 1 = 0$$

This factors as:

$$(\omega+1)(\omega^2 - \omega + 1) = 0$$

Since $\omega = e^{i\pi/3} = \cos 60 + i \sin 60 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \neq -1$, we must have:

$$\omega^2 - \omega + 1 = 0$$

Therefore $1 + \omega^2 = \omega$, which gives us:

$$z_1^2(1+\omega^2) = z_1^2 \cdot \omega = \omega z_1^2 = z_1 z_2$$

Thus: $z_1^2 + z_2^2 = z_1 z_2 \quad \square$
Alternative direct verification: We can also verify by direct calculation:

$$\begin{aligned} \omega^2 = e^{i2\pi/3} = \cos 120 + i \sin 120 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 + \omega^2 = 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega \quad \checkmark \end{aligned}$$

Problem 3.54: Finding Non-Real Roots

Find the two complex roots of $P(x) = x^5 - 10x^2 + 15x - 6$.

Hint: Test for rational roots first (like $x = 1$). Factor out linear factors, then solve the remaining polynomial.

Solution 3.54: Sketch

$P(1) = 1 - 10 + 15 - 6 = 0$, so $(x-1)$ is a factor. Polynomial division gives $P(x) = (x-1)(x^4 + x^3 + x^2 - 9x + 6)$. Continue factoring or use numerical/algebraic methods to find complex roots.

Problem 3.55: Expansion Result

Using De Moivre's theorem and the binomial expansion, derive the formula for $\cos 5\theta$ in terms of $\cos\theta$. Hence show that:

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

Hint: Use Euler's formula: $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$. Expand the left side using the binomial theorem, then equate real parts. Express $\sin^2\theta$ in terms of $\cos\theta$ using $\sin^2\theta = 1 - \cos^2\theta$.

Solution 3.55: Sketch

By De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

Expand the left side using binomial theorem:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^5 &= \sum_{k=0}^{5}\binom{5}{k}\cos^{5-k}\theta(i\sin\theta)^k \\
&= \binom{5}{0}\cos^5\theta + \binom{5}{1}\cos^4\theta(i\sin\theta) + \binom{5}{2}\cos^3\theta(i\sin\theta)^2 \\
&\quad + \binom{5}{3}\cos^2\theta(i\sin\theta)^3 + \binom{5}{4}\cos\theta(i\sin\theta)^4 + \binom{5}{5}(i\sin\theta)^5 \\
&= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
&\quad - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta
\end{aligned}$$

Separate real and imaginary parts:
Real part:

$$\mathrm{Re} = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

This equals $\cos 5\theta$.
Express in terms of $\cos\theta$ only:
Use $\sin^2\theta = 1 - \cos^2\theta$:

$$\begin{aligned}
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\
&= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta) \\
&= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\
&= (1+10+5)\cos^5\theta + (-10-10)\cos^3\theta + 5\cos\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}$$

Therefore: $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ $\square$

Problem 3.56: Symmetry Identity

Show $\alpha^k + \alpha^{-k} = 2\cos k\theta$ where $\alpha = \cos\theta + i\sin\theta$.

Hint: $\alpha = e^{i\theta}$ and $\alpha^{-1} = e^{-i\theta}$.

Solution 3.56: Sketch

$\alpha^k = e^{ik\theta} = \cos k\theta + i\sin k\theta$, $\alpha^{-k} = e^{-ik\theta} = \cos k\theta - i\sin k\theta$. Adding: $\alpha^k + \alpha^{-k} = 2\cos k\theta$.

Problem 3.57: Equilateral Triangle Centroid

Let $\triangle ABC$ be an equilateral triangle with vertices A, B, C represented by complex numbers a, b, c respectively, oriented anticlockwise. Let $\omega = e^{i2\pi/3} = \cos 120 + i \sin 120$ be a primitive cube root of unity. Prove that the centroid of the triangle is at the origin if and only if:

$$a + b\omega + c\omega^2 = 0$$

Hint: For an equilateral triangle centered at the origin, the vertices are related by 120 rotations. Use the fact that $\omega = e^{i2\pi/3}$ satisfies $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

Solution 3.57: Sketch

Key facts about $\omega = e^{i2\pi/3}$:

- $\omega^3 = e^{i2\pi} = 1$ (cube root of unity)
- $1 + \omega + \omega^2 = 0$ (sum of cube roots of unity)
- $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

If the triangle is centered at origin:
For an equilateral triangle centered at the origin with one vertex at a, the other two vertices are obtained by rotating by 120 and 240:

$$b = \omega a \quad \text{and} \quad c = \omega^2 a$$

Now compute:

$$\begin{aligned} a + b\omega + c\omega^2 &= a + (\omega a)\cdot\omega + (\omega^2 a)\cdot\omega^2 \\ &= a + \omega^2 a + \omega^4 a \\ &= a(1 + \omega^2 + \omega^4) \end{aligned}$$

Since $\omega^3 = 1$, we have $\omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega$.
Therefore:

$$a + b\omega + c\omega^2 = a(1 + \omega^2 + \omega) = a(1 + \omega + \omega^2)$$

Using the fundamental identity $1 + \omega + \omega^2 = 0$:

$$a + b\omega + c\omega^2 = a \cdot 0 = 0$$

Conversely: If $a + b\omega + c\omega^2 = 0$ and the triangle is equilateral, then its centroid is at the origin.
The centroid is at $\frac{a+b+c}{3}$. For an equilateral triangle with the given property, we can show:

$$a + b + c = 0$$

This follows because if $a + b\omega + c\omega^2 = 0$, and we also have the relations for an equilateral triangle, then the centroid must be at the origin.
Therefore: $a + b\omega + c\omega^2 = 0 \quad \square$
Geometric interpretation: This identity expresses the fact that in an equilateral triangle centered at the origin, the weighted sum of the vertices (with weights $1, \omega, \omega^2$) is zero, reflecting the rotational symmetry of the configuration.

Problem 3.58: Real Sum and Product

Given that $z + w$ and zw are both real, prove that either $z = \overline{w}$ or $\mathrm{Im}(z) = \mathrm{Im}(w) = 0$.

Hint: Let $z = a + bi$ and $w = c + di$. Use the conditions that $z + w$ and zw are real to derive equations for b and d in terms of a and c.

Solution 3.58: Proof

Let $z = a + bi$, $w = c + di$ where $a, b, c, d \in \mathbb{R}$.
Step 1: If $z + w$ is real, then $\mathrm{Im}(z + w) = b + d = 0$, so $d = -b$. (Eq 1)
Step 2: If zw is real, then $\mathrm{Im}(zw) = \mathrm{Im}[(a + bi)(c + di)] = ad + bc = 0$. (Eq 2)
Step 3: Substitute Eq 1 into Eq 2: $a(-b) + bc = 0 \implies b(c - a) = 0$.
Case 1: If $b = 0$, then $d = -b = 0$, so $\mathrm{Im}(z) = \mathrm{Im}(w) = 0$ (both real).
Case 2: If $b \neq 0$, then $c = a$. With $d = -b$: $w = a - bi = \bar{z}$, so $z = \bar{w}$.
Conclusion: Either $\mathrm{Im}(z) = \mathrm{Im}(w) = 0$ (both real) or $z = \bar{w}$ (conjugates). $\square$

Problem 3.59: Complex Argument Region

The complex numbers w and z both have modulus 1, and $\frac{\pi}{2} < \mathrm{Arg}\left(\frac{z}{w}\right) < \pi$, where Arg denotes the principal argument.
For real numbers x and y, consider the complex number $\frac{xz+yw}{z}$.
On an xy-plane, clearly sketch the region that contains all points (x, y) for which

$$\frac{\pi}{2} < \mathrm{Arg}\left(\frac{xz + yw}{z}\right) < \pi.$$

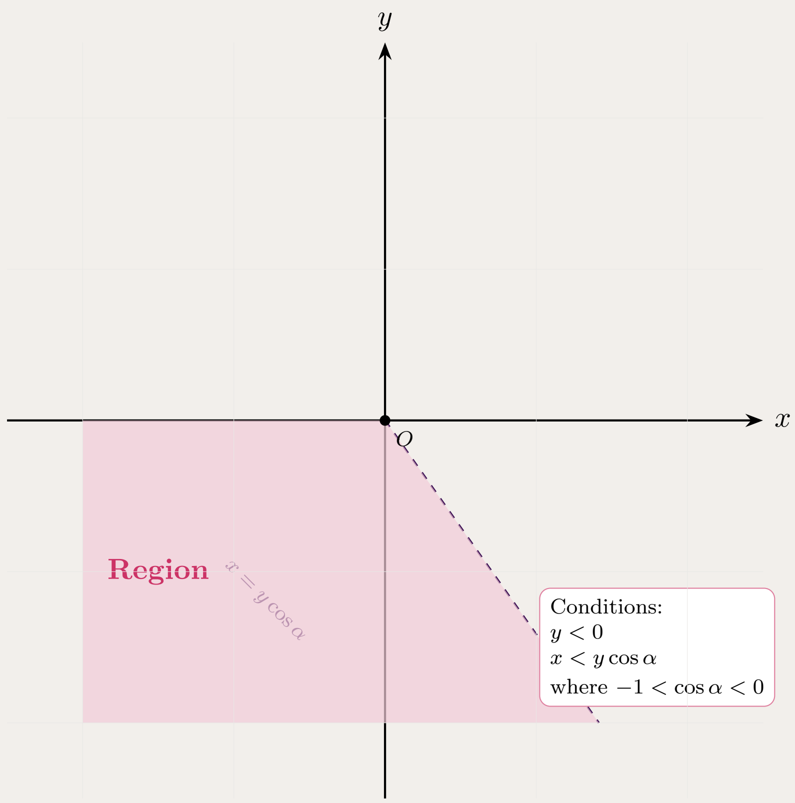
Hint: Simplify $\frac{xz+yw}{z} = x + y\frac{w}{z}$. Let $\alpha = \mathrm{Arg}(w/z)$ where $\frac{\pi}{2} < \alpha < \pi$. Then $w/z = e^{i\alpha}$. The expression becomes $x + ye^{i\alpha}$. For the argument to be in the given range, the resulting complex number must lie in the second quadrant.

Solution 3.59: Sketch

Simplify: $\frac{xz+yw}{z} = x + y\frac{w}{z}$. Since $\frac{\pi}{2} < \operatorname{Arg}(z/w) < \pi$, we have $-\pi < \operatorname{Arg}(w/z) < -\frac{\pi}{2}$, so $w/z = e^{i\alpha}$ where $-\pi < \alpha < -\frac{\pi}{2}$. Thus:

$$x + y(\cos\alpha + i\sin\alpha) = (x + y\cos\alpha) + i(y\sin\alpha)$$

For this in Q2 (second quadrant): $x + y\cos\alpha < 0$ and $y\sin\alpha > 0$. Since $\sin\alpha < 0$ and $\cos\alpha < 0$ in Q3, we need $y < 0$ and $x < y\cos\alpha$. The region is a wedge below the x-axis bounded by the line $x = y\cos\alpha$.



Problem 3.60: Geometric Series with n-th Roots

If $z = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$, and n is a positive integer, prove:

(i) $1 + z + z^2 + \cdots + z^{2n-1} = 0$

(ii) $1 + z + z^2 + \cdots + z^{n-1} = 1 + i\cot\left(\frac{\pi}{2n}\right)$

Hint: For part (i): Note that $z = e^{i\pi/n}$, so $z^{2n} = e^{i2\pi} = 1$. Use the geometric series formula. For part (ii): Split the sum at z^{n-1} and use the fact that $z^n = e^{i\pi} = -1$.

Solution 3.60: Proof

Setup: Given $z = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n} = e^{i\pi/n}$
Part (i): Prove $1 + z + z^2 + \cdots + z^{2n-1} = 0$
Calculate z^{2n}:

$$z^{2n} = \left(e^{i\pi/n}\right)^{2n} = e^{i\cdot 2\pi} = 1$$

Therefore z is a $2n$-th root of unity.
The sum $S = 1 + z + z^2 + \cdots + z^{2n-1}$ is a geometric series with first term 1, common ratio z, and $2n$ terms.
Using the geometric series formula:

$$S = \frac{z^{2n} - 1}{z - 1} = \frac{1 - 1}{z - 1} = \frac{0}{z - 1} = 0$$

(Note: Since $z = e^{i\pi/n} \neq 1$ for $n \geq 2$, the denominator is non-zero.)
Therefore: $1 + z + z^2 + \cdots + z^{2n-1} = 0 \quad \square$
Part (ii): Prove $1 + z + z^2 + \cdots + z^{n-1} = 1 + i\cot\left(\frac{\pi}{2n}\right)$
Let $S_n = 1 + z + z^2 + \cdots + z^{n-1}$
From part (i), we know:

$$1 + z + z^2 + \cdots + z^{2n-1} = 0$$

We can split this sum:

$$(1 + z + \cdots + z^{n-1}) + (z^n + z^{n+1} + \cdots + z^{2n-1}) = 0$$

The second sum can be factored:

$$z^n + z^{n+1} + \cdots + z^{2n-1} = z^n(1 + z + \cdots + z^{n-1}) = z^n \cdot S_n$$

Since $z^n = e^{i\pi} = -1$:

$$S_n + (-1)\cdot S_n = 0$$

$$S_n - S_n = 0$$

Using geometric series: $S_n = \frac{z^n - 1}{z-1} = \frac{-2}{z-1}$ (since $z^n = -1$). With $z - 1 = (\cos\frac{\pi}{n} - 1) + i\sin\frac{\pi}{n}$, multiply by conjugate. Denominator: $2(1 - \cos\frac{\pi}{n}) = 4\sin^2\frac{\pi}{2n}$. Using $\cos\frac{\pi}{n} - 1 = -2\sin^2\frac{\pi}{2n}$ and $\sin\frac{\pi}{n} = 2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}$, we get:

$$S_n = \frac{4\sin^2\frac{\pi}{2n} + i\cdot 4\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}{4\sin^2\frac{\pi}{2n}} = 1 + i\cot\frac{\pi}{2n} \quad \square$$

Therefore: $1 + z + z^2 + \cdots + z^{n-1} = 1 + i\cot\left(\frac{\pi}{2n}\right) \quad \square$

Problem 3.61: Fifth Roots of -1 and Cosine Values

By finding the fifth roots of -1, find the exact values of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$.

Hint: The fifth roots of -1 are solutions to $z^5 = -1$, or equivalently $z^5 + 1 = 0$. Use $-1 = e^{i\pi}$, so $z = e^{i\pi(2k+1)/5}$ for $k = 0, 1, 2, 3, 4$. The polynomial $z^5 + 1$ factors as $(z+1)(z^4 - z^3 + z^2 - z + 1) = 0$. Use the quartic to find relationships involving $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$.

Solution 3.61: Sketch

Step 1: Solve $z^5 = -1 = e^{i\pi}$ using De Moivre: $z = e^{i\pi(2k+1)/5}$ for $k = 0, 1, 2, 3, 4$.
Five roots: $z_0 = e^{i\pi/5}$, $z_1 = e^{i3\pi/5}$, $z_2 = -1$, $z_3 = e^{-i3\pi/5} = \bar{z}_1$, $z_4 = e^{-i\pi/5} = \bar{z}_0$.
Step 2: Factor: $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1) = 0$. The four non-real roots $z_0, z_1, \bar{z}_0, \bar{z}_1$ satisfy $z^4 - z^3 + z^2 - z + 1 = 0$.
Step 3: Use Vieta's formulas
Sum of roots: $(z_0 + \bar{z}_0) + (z_1 + \bar{z}_1) = 1 \implies 2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} = 1 \implies \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$.
Sum of products (coeff. of z^2): Using $z_0 z_4 = |z_0|^2 = 1$, $z_1 z_3 = |z_1|^2 = 1$, and simplifying other products gives: $\alpha\beta = -\frac{1}{4}$ where $\alpha = \cos\frac{\pi}{5}$ and $\beta = \cos\frac{3\pi}{5}$.
From $\alpha + \beta = \frac{1}{2}$ and $\alpha\beta = -\frac{1}{4}$, solve $4t^2 - 2t - 1 = 0$ to get $t = \frac{1 \pm \sqrt{5}}{4}$. Since $\cos\frac{\pi}{5} > 0$ and $\cos\frac{3\pi}{5} < 0$:

$$\boxed{\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}} \quad \text{and} \quad \boxed{\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}}$$

Problem 3.62: Euler's Formula and Integration

(i) Show that for any integer n, $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$.

(ii) By expanding $\left(e^{i\theta} + e^{-i\theta}\right)^4$, show that

$$\cos^4\theta = \frac{1}{8}\big(\cos(4\theta) + 4\cos(2\theta) + 3\big).$$

(iii) Hence, or otherwise, find $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$.

Hint: For part (i): Use Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$. For part (ii): Use part (i) to write $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, then raise to the 4th power using the binomial theorem. For part (iii): Integrate the expression from part (ii) term by term.

Solution 3.62: Sketch

Part (i): Show $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$
Using Euler's formula:

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$
$$e^{-in\theta} = \cos(-n\theta) + i\sin(-n\theta) = \cos(n\theta) - i\sin(n\theta)$$

Adding:

$$e^{in\theta} + e^{-in\theta} = 2\cos(n\theta) \quad \square$$

(ii) From (i), $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, so $\cos^4\theta = \frac{(e^{i\theta}+e^{-i\theta})^4}{16}$. Expand using binomial: $(e^{i\theta} + e^{-i\theta})^4 = e^{i4\theta} + 4e^{i2\theta} + 6 + 4e^{-i2\theta} + e^{-i4\theta} = 2\cos 4\theta + 8\cos 2\theta + 6$. Thus $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. $\square$

(iii) $\displaystyle\int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{1}{8}\left[\frac{\sin 4\theta}{4} + 2\sin 2\theta + 3\theta\right]_0^{\pi/2} = \frac{1}{8}\cdot\frac{3\pi}{2} = \boxed{\frac{3\pi}{16}}$

4 Conclusion

Complex numbers are a core component of the HSC Mathematics Extension 2 course. Mastery comes from repeated, reflective practice across all forms—Cartesian, polar, and exponential—and understanding their geometric interpretations. Use these problems to sharpen the ability to convert between forms, apply De Moivre's theorem, interpret loci geometrically, and communicate complete mathematical reasoning. Best of luck with your studies and HSC examinations!

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Repository: https://github.com/vuhung16au/math-olympiad-ml/tree/main/HSC-ComplexNumbers