HSC Math Extension 2: Integration Mastery

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Contents

1 Introduction

1.1 Project Overview

This booklet compiles high-quality integration problems curated specifically for the HSC Mathematics Extension 2 syllabus. Every problem covers essential integration techniques including substitution, integration by parts, partial fractions, reduction formulae, volumes of solids, and definite integral properties. Detailed reasoning showcases advanced problem-solving strategies that build from fundamental techniques to complex multi-step applications.

1.2 Target Audience

The explanations are crafted for Extension 2 students aiming to master integration and develop advanced problem-solving skills. Each solution in Part 1 explicitly states the strategy, justifies technique choices, and provides complete step-by-step working so that high-school learners can follow every transition. Part 2 offers hints and concise solutions to encourage independent problem-solving.

1.3 How to Use This Booklet

- Review the fundamentals section before attempting problems to refresh key techniques.
- Attempt problems in Part 1 without looking at solutions; compare your work against detailed solutions to understand model reasoning.
- For Part 2, try each problem first, then check the upside-down hint if needed, and finally review the solution sketch.
- Practice problems multiple times, working from memory to reinforce technique mastery.
- Use the appendices as quick references for formulas, techniques, and decision-making flowcharts.

1.4 Integration Techniques Overview

The problems in this collection cover:

- **Basic Techniques:** Reverse chain rule, standard integrals, u-substitution
- **Advanced Substitution:** Trigonometric substitution, t-formula, rationalizing substitutions
- **Integration by Parts:** Single and multiple applications, LIATE rule
- **Partial Fractions:** Linear, quadratic, and repeated factors
- **Reduction Formulae:** Deriving and applying recurrence relations
- **Volumes of Solids:** Disk method, washer method, cylindrical shells, general slicing
- **Definite Integral Properties:** Symmetry, King's property, inequalities

2 Fundamentals Review

This section provides a comprehensive review of integration techniques essential for HSC Extension 2. Use this as a reference while working through problems.

3 Standard Integrals & The "Reverse Chain Rule"

Before applying complex techniques, always check if the integral fits a standard form or the reverse chain rule.

- **Logarithmic Form:**
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$
- **Power Rule (General):**
$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C \quad (\text{where } n \neq -1)$$
- **Inverse Trigonometric Forms:**
$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

4 Integration by Parts

Used for integrating products of functions (e.g., xe^x, $x\ln x$, $e^x\cos x$).

The Formula:
$$\int u\,dv = uv - \int v\,du$$

Strategy (LIATE): Choose u based on this priority list (top to bottom):

1. **L** – Logarithmic ($\ln x$)
2. **I** – Inverse Trigonometric ($\tan^{-1} x$)
3. **A** – Algebraic ($x^2, 3x$)
4. **T** – Trigonometric ($\sin x, \cos x$)
5. **E** – Exponential (e^x)

5 Integration by Substitution

5.1 General Substitution (u-sub)

Used to simplify composite functions. Let $u = g(x)$, then find $du = g'(x)dx$.

5.2 Trigonometric Substitution

Used when the integrand contains quadratic roots:

- $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$ (uses $1 - \sin^2\theta = \cos^2\theta$)
- $\sqrt{a^2 + x^2}$: Let $x = a\tan\theta$ (uses $1 + \tan^2\theta = \sec^2\theta$)
- $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$ (uses $\sec^2\theta - 1 = \tan^2\theta$)

5.3 The t-Formula Substitution

Used for rational functions involving $\sin x$ and $\cos x$. Let $t = \tan\left(\frac{x}{2}\right)$.

$$\begin{aligned} dx &= \frac{2}{1+t^2}\,dt \\ \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \end{aligned}$$

6 Partial Fractions

Used to integrate rational functions $\frac{P(x)}{Q(x)}$ where $\deg(P) < \deg(Q)$. If $\deg(P) \geq \deg(Q)$, perform **polynomial long division** first.

- **Distinct Linear Factors:**

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

- **Repeated Linear Factors:**

$$\frac{1}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

- **Irreducible Quadratic Factors:**

$$\frac{1}{(x-a)(x^2+b)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+b}$$

7 Trigonometric Integrals

7.1 Integrals of $\sin^m x \cos^n x$

- **One power is odd:** Save one factor of the odd power for du. Convert the rest using $\sin^2 x + \cos^2 x = 1$.
- **Both powers even:** Use double angle formulae:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

7.2 Integrals of $\tan^m x \sec^n x$

- **If** $\sec$ **power is even:** Save $\sec^2 x$ for du. Convert remaining sec to tan.
- **If** $\tan$ **power is odd:** Save $\sec x \tan x$ for du. Convert remaining tan to sec.

8 Reduction Formulas (I_n)

Involves finding a recurrence relation using **Integration by Parts**.

Typical form:

$$I_n = \int x^n e^x \, dx \quad \text{or} \quad I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

Steps: Apply parts, manipulate the integral to find I_{n-1} or I_{n-2}, then rearrange for I_n.

9 Definite Integral Properties

- **Odd Function:** If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\,dx = 0$.
- **Even Function:** If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
- **Reflection (King's) Property:**

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

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10 Part 1: Problems and Solutions (Detailed)

Part 1 contains three sets of problems—basic, medium, and advanced. Each set provides five problems with comprehensive solutions. Every solution includes a strategy paragraph explaining technique selection, complete step-by-step working with annotations, and a takeaways box highlighting key insights.

10.1 Basic Integration Problems

Problem 10.1: Partial Fractions

Find the indefinite integral:

$$\int \frac{x^2 - 2x + 9}{(4 - x)(x^2 + 1)}\, dx$$

Strategy: This integral requires partial fraction decomposition. The denominator contains a linear factor $(4 - x)$ and an irreducible quadratic factor $(x^2 + 1)$, so we set up the decomposition with constants A for the linear part and $Bx + C$ for the quadratic part.

Solution 10.1

Step 1: Set up the decomposition
Since the denominator has $(4-x)$ (linear) and (x^2+1) (irreducible quadratic):

$$\frac{x^2-2x+9}{(4-x)(x^2+1)} = \frac{A}{4-x} + \frac{Bx+C}{x^2+1}$$

Step 2: Find coefficients
Multiply both sides by $(4-x)(x^2+1)$:

$$x^2-2x+9 = A(x^2+1) + (Bx+C)(4-x)$$

Finding A: Let $x=4$:

$$\begin{aligned} 16-8+9 &= A(17)+0 \\ 17 &= 17A \\ A &= 1 \end{aligned}$$

Finding B and C: Substitute $A=1$ and expand:

$$x^2-2x+9 = x^2+1+4Bx-Bx^2+4C-Cx$$

Group by powers of x:

$$x^2-2x+9 = (1-B)x^2+(4B-C)x+(1+4C)$$

Equating coefficients:

- x^2: $1 = 1-B \implies B=0$
- x^0: $9 = 1+4C \implies C=2$
- Check x^1: $-2 = 4(0)-2 = -2$ ✓

Step 3: Integrate
With $A=1$, $B=0$, $C=2$:

$$\int \left(\frac{1}{4-x} + \frac{2}{x^2+1}\right) dx$$

For $\int \frac{1}{4-x} dx$, let $u = 4-x$, then $du = -dx$:

$$\int \frac{1}{4-x} dx = -\ln|4-x| + C_1$$

For $\int \frac{2}{x^2+1} dx$:

$$2\int \frac{1}{x^2+1} dx = 2\arctan(x) + C_2$$

Final Answer:

$$\boxed{\int \frac{x^2-2x+9}{(4-x)(x^2+1)} dx = -\ln|4-x| + 2\arctan(x) + C}$$

Takeaways 10.1

- **Partial Fractions Setup:** Linear factors use constant numerators (A), irreducible quadratics use linear numerators ($Bx + C$).
- **Strategic Value Selection:** Choose x values that eliminate terms (e.g., $x = 4$ eliminates the $(4 - x)$ factor).
- **Coefficient Matching:** After substitution, equate coefficients of like powers to find remaining constants.
- **Standard Integral Recognition:** $\int \frac{1}{a-x} dx = -\ln|a - x|$ and $\int \frac{1}{x^2+1} dx = \arctan(x)$ are key formulas.

Problem 10.2: Integration by Parts

Use integration by parts to evaluate:

$$\int_1^e x \ln x \, dx$$

Strategy: This is a classic integration by parts problem. Apply the LIATE rule to choose which function to differentiate: Logarithmic functions come before Algebraic functions, so let $u = \ln x$.

Solution 10.2

Using the integration by parts formula $\int u\,dv = uv - \int v\,du$:
Step 1: Choose u and dv using LIATE:

$$u = \ln x \qquad\qquad dv = x\,dx$$

Step 2: Differentiate u and integrate dv:

$$du = \frac{1}{x}\,dx \qquad\qquad v = \frac{x^2}{2}$$

Step 3: Apply the formula:

$$\begin{aligned}
\int_1^e x \ln x\,dx &= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}\,dx \\
&= \left[\frac{x^2}{2}\ln x\right]_1^e - \frac{1}{2}\int_1^e x\,dx \\
&= \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^e
\end{aligned}$$

Step 4: Evaluate at the limits (using $\ln(e) = 1$ and $\ln(1) = 0$):

$$= \left(\frac{e^2}{2} - \frac{e^2}{4}\right) - \left(0 - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4} = \boxed{\frac{e^2+1}{4}}$$

Takeaways 10.2

- **LIATE Rule:** Prioritize L(og) > I(nverse trig) > A(lgebraic) > T(rig) > E(xponential) for u
- **Boundary Terms:** Apply limits after integration: $[uv]_a^b - \int_a^b v\,du$
- **Special Values:** Remember $\ln(e) = 1$ and $\ln(1) = 0$

Problem 10.3: Reverse Chain Rule

Find the indefinite integral:

$$\int \frac{2x+3}{x^2+2x+2}\,dx$$

Strategy: Recognize that the numerator can be split to match the derivative of the denominator plus a constant. This allows us to use both logarithmic and inverse trigonometric standard forms.

Solution 10.3

Step 1: Check the derivative of the denominator:

$$\frac{d}{dx}(x^2 + 2x + 2) = 2x + 2$$

Step 2: Split the numerator:

$$2x + 3 = (2x + 2) + 1$$

Step 3: Separate the integral:

$$I = \int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

Step 4: First integral uses logarithm form $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|$:

$$\int \frac{2x+2}{x^2+2x+2}\,dx = \ln(x^2 + 2x + 2)$$

(Note: $x^2 + 2x + 2 = (x+1)^2 + 1 > 0$ always)
Step 5: Second integral requires completing the square:

$$x^2 + 2x + 2 = (x+1)^2 + 1$$

Apply arctangent form:

$$\int \frac{1}{(x+1)^2+1}\,dx = \arctan(x+1)$$

Final Answer:

$$\boxed{\int \frac{2x+3}{x^2+2x+2}\,dx = \ln(x^2 + 2x + 2) + \arctan(x+1) + C}$$

Takeaways 10.3

- **Numerator Splitting:** Match part of numerator to $f'(x)$ when denominator is $f(x)$
- **Standard Forms:** $\frac{f'}{f} \to \ln|f|$ and $\frac{1}{a^2+u^2} \to \frac{1}{a}\arctan(\frac{u}{a})$
- **Completing the Square:** Essential for identifying inverse trig forms

Problem 10.4: Algebraic Substitution

Use an appropriate substitution to evaluate:

$$\int_{\sqrt{10}}^{\sqrt{13}} x^3\sqrt{x^2-9}\,dx$$

Strategy: The radical $\sqrt{x^2-9}$ suggests $u = x^2 - 9$. This also helps manage the x^3 term since $x^3dx = x^2 \cdot xdx$.

Solution 10.4

Step 1: Let $u = x^2 - 9$, then:

$$\frac{du}{dx} = 2x \implies x\,dx = \frac{1}{2}du$$

Since $x^2 = u + 9$:

$$x^3\,dx = x^2 \cdot x\,dx = (u+9)\cdot\frac{1}{2}du$$

Step 2: Transform limits:

- $x = \sqrt{10} \implies u = 10 - 9 = 1$
- $x = \sqrt{13} \implies u = 13 - 9 = 4$

Step 3: Substitute:

$$\int_{\sqrt{10}}^{\sqrt{13}} x^3\sqrt{x^2-9}\,dx = \int_1^4 (u+9)\sqrt{u}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_1^4 (u^{3/2} + 9u^{1/2})\,du$$

Step 4: Integrate:

$$= \frac{1}{2}\left[\frac{2u^{5/2}}{5} + 6u^{3/2}\right]_1^4 = \left[\frac{u^{5/2}}{5} + 3u^{3/2}\right]_1^4$$

Step 5: Evaluate:

$$\begin{aligned} \text{At } u = 4: &\quad \frac{32}{5} + 24 = \frac{152}{5} \\ \text{At } u = 1: &\quad \frac{1}{5} + 3 = \frac{16}{5} \end{aligned}$$

Final Answer:

$$\boxed{\frac{152}{5} - \frac{16}{5} = \frac{136}{5} = 27.2}$$

Takeaways 10.4

- **Radical Substitution:** For $\sqrt{x^2 \pm a^2}$, try $u = x^2 \pm a^2$
- **Limit Transformation:** Always convert limits for definite integrals
- **Fractional Powers:** $\int u^n du = \frac{u^{n+1}}{n+1}$ works for all $n \neq -1$

Problem 10.5: Definite Integral Property

Which of the following is equal to $\displaystyle\int_0^{2a} f(x)\,dx$?

(A) $\displaystyle\int_0^a (f(x) - f(2a - x))\,dx$

(B) $\displaystyle\int_0^a (f(x) + f(2a - x))\,dx$

(C) $\displaystyle 2\int_0^a f(x - a)\,dx$

(D) $\displaystyle\int_0^a \frac{1}{2} f(2x)\,dx$

Strategy: Split the integral at $x = a$, then use substitution on the second part to transform its limits to match $[0, a]$.

Solution 10.5

Step 1: Split the interval:

$$I = \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$$

Step 2: For the second integral, let $u = 2a - x$:

- Then $du = -dx$
- When $x = a$: $u = a$
- When $x = 2a$: $u = 0$

Step 3: Substitute:

$$\begin{aligned}\int_a^{2a} f(x)\,dx &= \int_a^{0} f(2a-u)(-du)\\ &= \int_0^{a} f(2a-u)\,du\\ &= \int_0^{a} f(2a-x)\,dx \quad \text{(dummy variable)}\end{aligned}$$

Step 4: Combine:

$$\begin{aligned}I &= \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx\\ &= \int_0^{a} \left(f(x) + f(2a-x)\right)\,dx\end{aligned}$$

Answer: $\boxed{\text{B}}$

Takeaways 10.5

- **Interval Splitting:** $\int_a^b = \int_a^c + \int_c^b$ for any $c \in [a,b]$
- **Reflection Substitution:** $u = 2a - x$ reflects the interval about the midpoint
- **Dummy Variables:** In definite integrals, the variable name doesn't matter
- **King's Property:** This is a special case: $\int_0^{2a} f(x)dx = \int_0^a [f(x)+f(2a-x)]dx$

10.2 Medium Integration Problems

Problem 10.6: Reduction Formula for Powers of Cotangent

Let $I_n = \displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n}\theta \, d\theta$ for integers $n \geq 0$.

(i) Show that $I_n = \dfrac{1}{2n-1} - I_{n-1}$ for $n > 0$, given that $\dfrac{d}{d\theta}\cot\theta = -\csc^2\theta$.

(ii) Hence, or otherwise, calculate I_2.

Strategy: This is a classic reduction formula problem. For part (i), we'll combine $I_n + I_{n-1}$ and use the identity $\cot^2\theta + 1 = \csc^2\theta$ to simplify, then apply substitution. For part (ii), we'll use the recurrence relation iteratively, starting from I_0.

Solution 10.6

Part (i):
Consider $I_n + I_{n-1}$:

$$
\begin{aligned}
I_n + I_{n-1} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n}\theta \, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2(n-1)}\theta \, d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\cot^{2n}\theta + \cot^{2n-2}\theta\right) d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n-2}\theta \left(\cot^2\theta + 1\right) d\theta
\end{aligned}
$$

Using the Pythagorean identity $\cot^2\theta + 1 = \csc^2\theta$:

$$
I_n + I_{n-1} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n-2}\theta \cdot \csc^2\theta \, d\theta
$$

Substitution: Let $u = \cot\theta$, then $du = -\csc^2\theta \, d\theta$.
Change limits:

- When $\theta = \frac{\pi}{4}$: $u = \cot\left(\frac{\pi}{4}\right) = 1$
- When $\theta = \frac{\pi}{2}$: $u = \cot\left(\frac{\pi}{2}\right) = 0$

Substituting:

$$
\begin{aligned}
I_n + I_{n-1} &= \int_1^0 u^{2n-2}(-du) \\
&= \int_0^1 u^{2n-2} \, du \\
&= \left[\frac{u^{2n-1}}{2n-1}\right]_0^1 \\
&= \frac{1}{2n-1}
\end{aligned}
$$

Rearranging: $\boxed{I_n = \dfrac{1}{2n-1} - I_{n-1}}$

Part (ii):
We need I_2. First, calculate I_0 and I_1.
Calculate I_0:

$$
I_0 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 \, d\theta = [\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
$$

Calculate I_1: Using the formula with $n = 1$:

$$
I_1 = \frac{1}{2(1) - 1} - I_0 = 1 - \frac{\pi}{4}
$$

Calculate I_2: Using the formula with $n = 2$:

$$
\begin{aligned}
I_2 &= \frac{1}{2(2) - 1} - I_1 \\
&= \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) \\
&= \frac{1}{3} - 1 + \frac{\pi}{4}
\end{aligned}
$$

Takeaways 10.6

- **Reduction Formula Strategy:** Adding consecutive terms $(I_n + I_{n-1})$ can reveal simplifying identities.
- **Trigonometric Identities:** $\cot^2\theta + 1 = \csc^2\theta$ is key for cotangent integrals.
- **Substitution Choice:** $u = \cot\theta$ works well because $du = -\csc^2\theta\, d\theta$ matches our integral.
- **Iterative Application:** Build from base case (I_0) through recurrence to find any I_n.

Problem 10.7: King's Rule with t-Formula

Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{u}{1 + \sin u + \cos u}\, du$$

by first using the substitution $u = \frac{\pi}{2} - x$.

Strategy: Use King's property (reflection about midpoint) to create a self-referencing equation that simplifies the numerator. Then apply the t-formula (Weierstrass substitution) to handle the trigonometric denominator.

Solution 10.7

Let $I = \int_0^{\frac{\pi}{2}} \frac{u}{1+\sin u+\cos u}\, du \quad \ldots(1)$
Step 1: Apply substitution $u = \frac{\pi}{2} - x$, so $du = -dx$.
Change limits: $u = 0 \implies x = \frac{\pi}{2}$, $u = \frac{\pi}{2} \implies x = 0$

$$\begin{aligned} I &= \int_{\frac{\pi}{2}}^{0} \frac{\frac{\pi}{2} - x}{1 + \sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)}(-dx) \\ &= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{1 + \cos x + \sin x}\, dx \end{aligned}$$

Relabel: $I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-u}{1+\sin u+\cos u}\, du \quad \ldots(2)$
Step 2: Add equations (1) and (2):

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \frac{u + (\frac{\pi}{2} - u)}{1 + \sin u + \cos u}\, du \\ 2I &= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin u + \cos u}\, du \end{aligned}$$

Step 3: Apply t-formula. Let $t = \tan(\frac{u}{2})$:

$$du = \frac{2}{1+t^2}dt, \quad \sin u = \frac{2t}{1+t^2}, \quad \cos u = \frac{1-t^2}{1+t^2}$$

Limits: $u = 0 \implies t = 0$, $u = \frac{\pi}{2} \implies t = 1$

$$\begin{aligned} 2I &= \frac{\pi}{2} \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt \\ &= \frac{\pi}{2} \int_0^1 \frac{2}{(1+t^2) + 2t + (1-t^2)}\, dt \\ &= \frac{\pi}{2} \int_0^1 \frac{2}{2+2t}\, dt \\ &= \frac{\pi}{2} \int_0^1 \frac{1}{1+t}\, dt \end{aligned}$$

Step 4: Integrate:

$$2I = \frac{\pi}{2}[\ln|1+t|]_0^1 = \frac{\pi}{2}(\ln 2 - \ln 1) = \frac{\pi}{2}\ln 2$$

Final Answer:

$$\boxed{I = \frac{\pi}{4}\ln 2}$$

Takeaways 10.7

- **King's Property:** For $\int_a^b f(x)dx$, use $\int_a^b f(a+b-x)dx$ to create symmetry
- **Self-Reference:** Adding two forms of same integral can simplify complex expressions
- **t-Formula:** $t = \tan(\frac{x}{2})$ converts rational trig functions to rational algebraic functions
- **Workflow:** Simplify limits first (King's), then tackle trig terms (t-formula)

Problem 10.8: Reduction Formula - Logarithmic Powers

The integral I_n is defined by:

$$I_n = \int_1^e (\ln x)^n \, dx \quad \text{for integers } n \geq 0$$

Show that $I_n = e - nI_{n-1}$ for $n \geq 1$.

Strategy: Use integration by parts with $u = (\ln x)^n$ and $dv = dx$. The derivative of u will reduce the power of the logarithm.

Solution 10.8

Step 1: Choose u and dv:

$$\begin{aligned} u &= (\ln x)^n & dv &= dx \\ du &= n(\ln x)^{n-1} \cdot \frac{1}{x} \, dx & v &= x \end{aligned}$$

Step 2: Apply parts formula:

$$\begin{aligned} I_n &= [x(\ln x)^n]_1^e - \int_1^e x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} \, dx \\ &= [x(\ln x)^n]_1^e - n \int_1^e (\ln x)^{n-1} \, dx \end{aligned}$$

Step 3: Evaluate boundary term:

- At $x = e$: $e(\ln e)^n = e(1)^n = e$
- At $x = 1$: $1(\ln 1)^n = 1(0)^n = 0$ (for $n \geq 1$)

Step 4: Recognize I_{n-1}:

$$I_n = e - nI_{n-1} \quad \blacksquare$$

Takeaways 10.8

- **Parts for Reduction:** Choose u as the term you want to reduce in power
- **Logarithm Priority:** $\ln x$ is top choice for u in LIATE
- **Boundary Evaluation:** Special values like $\ln(e) = 1$ and $\ln(1) = 0$ simplify calculations
- **Recurrence Relations:** Reduction formulae connect I_n to simpler I_{n-1} or I_{n-2}

Problem 10.9: Applications - Particle Dynamics

A particle of mass m kg moves along a horizontal line with initial velocity V_0 m/s. The motion is resisted by a constant force of mk newtons and a variable force of mv^2 newtons, where k is a positive constant and v m/s is the velocity at time t seconds.
Show that the distance travelled when the particle comes to rest is $\frac{1}{2}\ln\left(\frac{k+V_0^2}{k}\right)$ metres.

Strategy: Apply Newton's Second Law, use the kinematic identity $a = v\frac{dv}{dx}$ to change variables, then separate and integrate.

Solution 10.9

Step 1: Establish equation of motion using $F = ma$:
Total resistive force: $F = -(mk + mv^2)$

$$ma = -m(k + v^2) \implies a = -(k + v^2)$$

Step 2: Change variable to displacement using $a = v\frac{dv}{dx}$:

$$v\frac{dv}{dx} = -(k + v^2)$$

Step 3: Separate variables:

$$dx = -\frac{v}{k + v^2}\, dv$$

Step 4: Integrate with limits:

- Initial: $x = 0, v = V_0$
- Final: $x = D, v = 0$ (at rest)

$$\int_0^D dx = \int_{V_0}^0 -\frac{v}{k + v^2}\, dv$$

LHS: D
RHS: Note that $\frac{d}{dv}(k + v^2) = 2v$, so:

$$\int \frac{v}{k + v^2}\, dv = \frac{1}{2}\ln(k + v^2)$$

Step 5: Evaluate:

$$\begin{aligned} D &= -\left[\frac{1}{2}\ln(k + v^2)\right]_{V_0}^0 \\ &= -\frac{1}{2}(\ln k - \ln(k + V_0^2)) \\ &= \frac{1}{2}(\ln(k + V_0^2) - \ln k) \\ &= \boxed{\frac{1}{2}\ln\left(\frac{k + V_0^2}{k}\right) \text{ metres}} \quad \blacksquare \end{aligned}$$

Takeaways 10.9

- **Variable Change:** For distance problems, use $a = v\frac{dv}{dx}$ instead of $a = \frac{dv}{dt}$
- **Separation:** Rearrange to get all x terms on one side, all v terms on other
- **Reverse Chain Rule:** $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|$ is crucial for rational integrands
- **Log Laws:** $\ln a - \ln b = \ln(\frac{a}{b})$ simplifies final answers

Problem 10.10: Complex Numbers Method

(i) Show that for any integer n, $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$.

(ii) By expanding $(e^{i\theta} + e^{-i\theta})^4$, show that

$$\cos^4\theta = \frac{1}{8}\left(\cos(4\theta) + 4\cos(2\theta) + 3\right)$$

Strategy: Use Euler's formula to convert complex exponentials to trigonometric form, then use binomial theorem and group terms to derive the power reduction formula.

Solution 10.10

Part (i):
Using Euler's formula $e^{ix} = \cos x + i\sin x$:

$$\begin{aligned} e^{in\theta} &= \cos(n\theta) + i\sin(n\theta) \\ e^{-in\theta} &= \cos(n\theta) - i\sin(n\theta) \quad (\text{using } \cos(-x) = \cos x,\ \sin(-x) = -\sin x) \end{aligned}$$

Adding:

$$e^{in\theta} + e^{-in\theta} = 2\cos(n\theta) \quad \blacksquare$$

Part (ii):
From part (i) with $n = 1$: $2\cos\theta = e^{i\theta} + e^{-i\theta}$
Raise to power 4:

$$16\cos^4\theta = (e^{i\theta} + e^{-i\theta})^4$$

Let $z = e^{i\theta}$. By Binomial Theorem:

$$\begin{aligned} (z + z^{-1})^4 &= z^4 + 4z^3 \cdot z^{-1} + 6z^2 \cdot z^{-2} + 4z \cdot z^{-3} + z^{-4} \\ &= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4} \\ &= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6 \end{aligned}$$

Using part (i) identity:

$$\begin{aligned} 16\cos^4\theta &= 2\cos(4\theta) + 4(2\cos(2\theta)) + 6 \\ 16\cos^4\theta &= 2\cos(4\theta) + 8\cos(2\theta) + 6 \end{aligned}$$

Divide by 16:

$$\cos^4\theta = \frac{1}{8}(\cos(4\theta) + 4\cos(2\theta) + 3) \quad \blacksquare$$

Takeaways 10.10

- **Euler's Bridge:** $e^{ix} = \cos x + i \sin x$ connects exponentials and trig functions
- **Power Reduction:** Complex methods convert high powers of trig to sums of single-angle terms
- **Binomial Expansion:** $(a+b)^n$ with $b = a^{-1}$ creates symmetric terms
- **Integration Advantage:** Reduced forms integrate easily (useful for $\int \cos^4 \theta d\theta$)

10.3 Advanced Integration Problems

Problem 10.11: Advanced Reduction Formula with Induction

(i) Let $J_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$ where $n \geq 0$ is an integer.
Show that $J_n = \frac{n-1}{n} J_{n-2}$ for all integers $n \geq 2$.

(ii) Let $I_n = \int_0^1 x^n (1-x)^n \, dx$ where n is a positive integer.
By using the substitution $x = \sin^2 \theta$, or otherwise,
show that $I_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1} \theta \, d\theta$.

(iii) Hence, or otherwise, show that $I_n = \frac{n}{4n+2} I_{n-1}$, for all integers $n \geq 1$.

Strategy: This is a sophisticated multi-part problem connecting two different integral forms through substitution and reduction formulae. Part (i) uses integration by parts to derive a recurrence for J_n. Part (ii) employs trigonometric substitution to relate I_n to J_{2n+1}. Part (iii) combines the previous results to establish the recurrence for I_n.

Solution 10.11

Part (i):
We use integration by parts on $J_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$.
Let:

$$u = \sin^{n-1}\theta \implies du = (n-1)\sin^{n-2}\theta\cos\theta\, d\theta$$
$$dv = \sin\theta\, d\theta \implies v = -\cos\theta$$

Applying $\int u\, dv = [uv] - \int v\, du$:

$$J_n = \left[-\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos\theta(n-1)\sin^{n-2}\theta\cos\theta\, d\theta$$

The boundary term vanishes: $[-\sin^{n-1}(\pi/2)\cos(\pi/2)] - [-\sin^{n-1}(0)\cos(0)] = 0 - 0 = 0$.

$$\begin{aligned}
J_n &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\cos^2\theta\, d\theta \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta(1-\sin^2\theta)\, d\theta \quad [\text{using } \cos^2\theta = 1-\sin^2\theta] \\
&= (n-1)\left(\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\, d\theta - \int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta\right) \\
&= (n-1)J_{n-2} - (n-1)J_n
\end{aligned}$$

Rearranging:

$$\begin{aligned}
J_n + (n-1)J_n &= (n-1)J_{n-2} \\
nJ_n &= (n-1)J_{n-2}
\end{aligned}$$

$$\boxed{J_n = \frac{n-1}{n}J_{n-2}}$$

Part (ii):
Substitute $x = \sin^2\theta$, so $dx = 2\sin\theta\cos\theta\, d\theta$.
Limits: $x = 0 \implies \theta = 0$; $x = 1 \implies \theta = \frac{\pi}{2}$.

$$\begin{aligned}
I_n &= \int_0^1 x^n(1-x)^n\, dx \\
&= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^n(1-\sin^2\theta)^n \cdot 2\sin\theta\cos\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^{2n}\theta\cos^{2n}\theta \cdot 2\sin\theta\cos\theta\, d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\cos^{2n+1}\theta\, d\theta
\end{aligned}$$

Using $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$:

$$\begin{aligned}
I_n &= 2\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2\theta\right)^{2n+1} d\theta \\
&= \frac{2}{2^{2n+1}}\int_0^{\frac{\pi}{2}} \sin^{2n+1} 2\theta\, d\theta
\end{aligned}$$

Takeaways 10.11

- **Integration by Parts for Reduction:** Choosing $u = \sin^{n-1}\theta$ and $dv = \sin\theta\, d\theta$ systematically reduces the power.
- **Trigonometric Substitution Power:** $x = \sin^2\theta$ transforms algebraic integrals into trigonometric ones.
- **Connecting Different Integrals:** Parts (i) and (ii) establish relationships that combine in part (iii).
- **Symmetry Properties:** $\int_0^\pi \sin^k\phi\, d\phi = 2\int_0^{\pi/2} \sin^k\phi\, d\phi$ simplifies definite integrals.
- **Multi-Part Problems:** Each part builds on previous results—read all parts before starting!

Problem 10.12: Reduction Formula with Factorial Series

Let $J_n = \int_0^1 x^n e^{-x}\, dx$, where n is a non-negative integer.

(i) Show that $J_0 = 1 - \frac{1}{e}$.

(ii) Show that $J_n \leq \frac{1}{n+1}$.

(iii) Show that $J_n = nJ_{n-1} - \frac{1}{e}$, for $n \geq 1$.

(iv) Using parts (i) and (iii), show by mathematical induction that for all $n \geq 0$,

$$J_n = n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}$$

(v) Using parts (ii) and (iv) prove that $e = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}$.

Strategy: This sophisticated problem connects integration, reduction formulae, induction, and limits. Part (i) establishes base case, (ii) bounds, (iii) recurrence, (iv) explicit formula via induction, and (v) uses squeeze theorem.

Solution 10.12

Part (i):

$$J_0 = \int_0^1 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = -e^{-1} - (-1) = \boxed{1 - \frac{1}{e}}$$

Part (ii):
For $x \in [0,1]$: $e^{-x} \le e^0 = 1$, so $x^n e^{-x} \le x^n$.

$$J_n = \int_0^1 x^n e^{-x}\,dx \le \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \boxed{\frac{1}{n+1}}$$

Part (iii):
Use integration by parts: $u = x^n$, $dv = e^{-x}dx$

$$du = nx^{n-1}dx, \quad v = -e^{-x}$$

$$\begin{aligned} J_n &= \left[-x^n e^{-x}\right]_0^1 + \int_0^1 nx^{n-1}e^{-x}\,dx \\ &= \left(-\frac{1}{e} - 0\right) + nJ_{n-1} = \boxed{nJ_{n-1} - \frac{1}{e}} \end{aligned}$$

Part (iv):
Base case $(n=0)$: LHS: $J_0 = 1 - \frac{1}{e}$. RHS: $0! - \frac{0!}{e}(1) = 1 - \frac{1}{e}$. ✓
Inductive step: Assume true for $n = k$: $J_k = k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}$
From part (iii): $J_{k+1} = (k+1)J_k - \frac{1}{e}$

$$\begin{aligned} J_{k+1} &= (k+1)\left[k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right] - \frac{1}{e} \\ &= (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k}\frac{1}{r!} - \frac{1}{e} \\ &= (k+1)! - \frac{(k+1)!}{e}\left[\sum_{r=0}^{k}\frac{1}{r!} + \frac{1}{(k+1)!}\right] \\ &= \boxed{(k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}} \end{aligned}$$

By induction, the formula holds for all $n \ge 0$. ■
Part (v):
From part (iv): $\frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!} = n! - J_n$
Rearranging: $\sum_{r=0}^{n}\frac{1}{r!} = e - \frac{eJ_n}{n!}$
From part (ii): $0 \le J_n \le \frac{1}{n+1}$, so: $0 \le \frac{eJ_n}{n!} \le \frac{e}{(n+1)!}$
As $n \to \infty$: $(n+1)! \to \infty$, thus $\frac{e}{(n+1)!} \to 0$.
By Squeeze Theorem: $\lim_{n\to\infty}\frac{eJ_n}{n!} = 0$

Therefore: $\boxed{e = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}}$ ■

Takeaways 10.12

- **Proof Architecture:** Each part builds toward the final limit result
- **Induction with Series:** Factorial notation and series manipulation are key
- **Squeeze Theorem:** Upper bound from (ii) + explicit formula from (iv) $\implies$ limit
- **e as Series:** This proves the famous expansion $e = \sum_{r=0}^{\infty} \frac{1}{r!}$

Problem 10.13: Substitution Proof

It is given that:

$$A = \int_2^4 \frac{e^x}{x-1}\,dx$$

Show that:

$$\int_{m-4}^{m-2} \frac{e^{-x}}{x-m+1}\,dx = kA$$

where k and m are constants, and determine the value of k.

Strategy: Transform the second integral using substitution $u = m - x$ to reverse limits and match the form of A. Factor out constants carefully.

Solution 10.13

Let $I = \int_{m-4}^{m-2} \frac{e^{-x}}{x-m+1}\, dx$

Step 1: Substitute $u = m - x$, so $x = m - u$ and $dx = -du$.
Change limits:

- $x = m - 4 \implies u = 4$
- $x = m - 2 \implies u = 2$

Step 2: Transform the integral:

$$\begin{aligned} I &= \int_4^2 \frac{e^{-(m-u)}}{(m-u)-m+1}(-du) \\ &= \int_4^2 \frac{e^{u-m}}{-u+1}(-du) \\ &= \int_4^2 \frac{e^{-m} \cdot e^u}{-(u-1)}(-du) \\ &= \int_4^2 \frac{e^{-m}e^u}{u-1} du \end{aligned}$$

Step 3: Reverse limits (introduces negative sign):

$$\begin{aligned} I &= -\int_2^4 \frac{e^{-m}e^u}{u-1} du \\ &= -e^{-m}\int_2^4 \frac{e^u}{u-1} du \end{aligned}$$

Step 4: Recognize A:
Since $\int_2^4 \frac{e^u}{u-1} du = A$ (dummy variable):

$$I = -e^{-m}A$$

Final Answer:

$$\boxed{k = -e^{-m}}$$

Takeaways 10.13

- **Reversing Substitution:** $u = m - x$ reverses limit order and transforms exponentials
- **Limit Transformation:** Always check how limits change under substitution
- **Dummy Variables:** $\int_a^b f(x)dx = \int_a^b f(u)du$ for definite integrals
- **Constant Extraction:** e^{-m} can be factored out as it's independent of u

Problem 10.14: Applications - Inclined Plane Dynamics

An object of mass 5 kg is on a slope inclined at 60 to the horizontal. The acceleration due to gravity is g m/s^2 and velocity down the slope is v m/s.
The object experiences two resistive forces (acting up the slope): one of magnitude $2v$ N and one of $2v^2$ N.

(i) Show that the resultant force down the slope is $\frac{5\sqrt{3}}{2}g - 2v - 2v^2$ newtons.

(ii) There is one value of v such that the object slides at constant speed. Find this value (in m/s, to 1 d.p.) given $g = 10$.

Strategy: Resolve forces parallel to slope using $F = mg\sin\theta$, then apply Newton's First Law for constant velocity (equilibrium).

Solution 10.14

Part (i):
Forces parallel to slope:

- **Down slope:** Component of weight: $F_g = mg\sin(60) = 5g \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}g$
- **Up slope:** Resistance: $R = 2v + 2v^2$

Resultant force (taking down-slope as positive):

$$\boxed{F_{net} = \frac{5\sqrt{3}}{2}g - 2v - 2v^2 \text{ N}} \quad \blacksquare$$

Part (ii):
Constant speed $\implies F_{net} = 0$ (Newton's First Law):

$$\frac{5\sqrt{3}}{2}(10) - 2v - 2v^2 = 0$$

$$25\sqrt{3} - 2v - 2v^2 = 0$$

$$2v^2 + 2v - 25\sqrt{3} = 0$$

Using quadratic formula with $a = 2$, $b = 2$, $c = -25\sqrt{3}$:

$$v = \frac{-2 \pm \sqrt{4 + 200\sqrt{3}}}{4}$$

Taking positive root (velocity down slope):

$$v = \frac{-2 + \sqrt{4 + 346.41}}{4} = \frac{-2 + \sqrt{350.41}}{4} \approx \frac{-2 + 18.72}{4} \approx 4.18$$

Answer: $\boxed{v = 4.2 \text{ m/s}}$ (to 1 d.p.)

Takeaways 10.14

- **Force Resolution:** Always resolve perpendicular and parallel to the plane
- **Constant Velocity:** Zero acceleration $\implies$ balanced forces
- **Sign Convention:** Define positive direction (here: down-slope)
- **Quadratic Reality Check:** Reject negative velocity (unphysical)

Problem 10.15: Volumes of Revolution with Ratio

Region A is bounded by $y = 1$ and $x^2 + y^2 = 1$ between $x = 0$ and $x = 1$.
Region B is bounded by $y = 1$ and $y = \ln x$ between $x = 1$ and $x = e$.
The volume of solid formed when region A is rotated about the x-axis is V_A. The volume of solid formed when region B is rotated about the x-axis is V_B.
Show that the ratio $V_A : V_B$ is $1 : 3$.

Strategy: Use washer method: $V = \pi \int_a^b [(y_{outer})^2 - (y_{inner})^2] dx$. For region A, outer is line $y = 1$, inner is circle. For region B, outer is $y = 1$, inner is $\ln x$.

Solution 10.15

Calculate V_A:
Region A: outer $y = 1$, inner $y = \sqrt{1-x^2}$ (upper semicircle), from $x = 0$ to $x = 1$:

$$\begin{aligned} V_A &= \pi \int_0^1 [1^2 - (\sqrt{1-x^2})^2]dx = \pi \int_0^1 [1 - (1-x^2)]dx \\ &= \pi \int_0^1 x^2 dx = \pi \left[\frac{x^3}{3}\right]_0^1 = \frac{\pi}{3} \end{aligned}$$

Calculate V_B:
Region B: outer $y = 1$, inner $y = \ln x$, from $x = 1$ to $x = e$:

$$V_B = \pi \int_1^e [1 - (\ln x)^2]dx = \pi \left[x - \int (\ln x)^2 dx\right]_1^e$$

For $\int (\ln x)^2 dx$, use parts twice:
Let $u = (\ln x)^2$, $dv = dx \implies du = \frac{2\ln x}{x}dx$, $v = x$:

$$\begin{aligned} \int (\ln x)^2 dx &= x(\ln x)^2 - 2\int \ln x dx \\ &= x(\ln x)^2 - 2(x\ln x - x) \\ &= x(\ln x)^2 - 2x\ln x + 2x \end{aligned}$$

Therefore:

$$\begin{aligned} V_B &= \pi[x - (x(\ln x)^2 - 2x\ln x + 2x)]_1^e \\ &= \pi[-x(\ln x)^2 + 2x\ln x - x]_1^e \end{aligned}$$

At $x = e$: $-e(1)^2 + 2e(1) - e = 0$
At $x = 1$: $-1(0)^2 + 2(1)(0) - 1 = -1$

$$V_B = \pi[0 - (-1)] = \pi$$

Calculate Ratio:

$$V_A : V_B = \frac{\pi}{3} : \pi = \frac{1}{3} : 1 = \boxed{1:3} \quad \blacksquare$$

Takeaways 10.15

- **Washer Method:** Subtract inner curve squared from outer curve squared
- **Repeated Parts:** $\int (\ln x)^2 dx$ requires applying by parts twice
- **Boundary Simplification:** $\ln e = 1$ and $\ln 1 = 0$ simplify evaluation
- **Geometric Insight:** Volume comparisons often yield simple ratios

11 Part 2: Problems with Hints Only

Part 2 presents additional problems with upside-down hints. Try each problem first, then rotate the page to read the hint if needed. These 45 problems provide additional practice across all difficulty levels.

11.1 Basic Integration Problems

Problem 11.1

Evaluate $\int (3x+1)^5 \, dx$

Hint: Let $u = 3x + 1$. Then $du = 3dx$, so $dx = \frac{1}{3}du$. The integral becomes $\frac{1}{3}\int u^5 du = \frac{u^6}{18} + C$.

Problem 11.2

Find $\int \frac{x}{\sqrt{x^2+4}} \, dx$

Hint: Let $u = x^2 + 4$. Then $du = 2x\,dx$. Rewrite as $\frac{1}{2}\int u^{-1/2} du = \sqrt{u} + C = \sqrt{x^2+4} + C$.

Problem 11.3

Evaluate $\int 2xe^{x^2} \, dx$

Hint: The derivative of x^2 is $2x$. Pattern: $\int f'(x)e^{f(x)} dx = e^{f(x)} + C$. Answer: $e^{x^2} + C$.

Problem 11.4

Find $\int \frac{3}{9+x^2} \, dx$

Hint: Factor: $\frac{3}{9+x^2} = \frac{1}{3} \cdot \frac{1}{1+(x/3)^2}$. Let $u = x/3$. Answer: $\arctan(x/3) + C$.

Problem 11.5

Evaluate $\displaystyle\int \sin^3 2x \cos 2x\, dx$

Hint: Let $u = \sin 2x$. Then $du = 2\cos 2x\, dx$. Integral: $\frac{1}{2}\int u^3 du = \frac{1}{8}\sin^4 2x + C$.

Problem 11.6

Find $\displaystyle\int xe^x\, dx$

Hint: LIATE: $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$. Answer: $xe^x - e^x + C = e^x(x-1) + C$.

Problem 11.7

Evaluate $\displaystyle\int \frac{2x+3}{x^2+3x+1}\, dx$

Hint: Numerator = derivative of denominator. Pattern: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C = \ln|x^2+3x+1|+C$.

Problem 11.8

Find $\displaystyle\int \sin^2 x\, dx$

Hint: Use $\sin^2 x = \frac{1-\cos 2x}{2}$. Integrate: $\frac{x}{2} - \frac{\sin 2x}{4} + C$.

Problem 11.9

Evaluate $\displaystyle\int_0^2 x\sqrt{1+x^2}\, dx$

Hint: Let $u = 1 + x^2$, $du = 2x\,dx$. Limits: $u = 1$ to $u = 5$. Answer: $\frac{1}{3}[(5)^{3/2} - (1)^{3/2}] = \frac{5\sqrt{5}-1}{3}$.

Problem 11.10

Find $\displaystyle\int \frac{1}{\sqrt{25 - x^2}}\,dx$

Hint: Factor: $\sqrt{25 - x^2} = 5\sqrt{1 - (x/5)^2}$. Let $u = x/5$. Answer: $\arcsin(x/5) + C$.

Problem 11.11

Evaluate $\displaystyle\int \frac{1}{(x-1)(x+2)}\,dx$

Hint: $\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$. Solving: $A = \frac{1}{3}$, $B = -\frac{1}{3}$. Answer: $\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$.

Problem 11.12

Find $\displaystyle\int x\cos x\,dx$

Hint: Let $u = x$, $dv = \cos x\,dx$. Then $v = \sin x$. Answer: $x\sin x + \cos x + C$.

Problem 11.13

Evaluate $\displaystyle\int \frac{6x^2}{(x^3+1)^4}\,dx$

Hint: Let $u = x^3 + 1$, $du = 3x^2dx$. Then $6x^2dx = 2du$. Answer: $2 \cdot \frac{u^{-3}}{-3} = -\frac{2}{3(x^3+1)^3} + C$.

Problem 11.14

Given $f(x)$ is even, evaluate $\int_{-2}^{2} f(x)\,dx$ if $\int_{0}^{2} f(x)\,dx = 5$

Hint: For even functions: $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$. Answer: $2 \times 5 = 10$.

Problem 11.15

Find $\int \frac{1}{x^2+4x+13}\,dx$

Hint: Complete square: $x^2+4x+13=(x+2)^2+9$. Let $u=\frac{x+2}{3}$. Answer: $\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)+C$.

11.2 Medium Integration Problems

Problem 11.16

Evaluate $\int x^2 e^x\,dx$

Hint: Apply by parts twice. First: $u_1 = x^2$, $dv_1 = e^x dx$. Second: $u_2 = 2x$, $dv_2 = e^x dx$. Answer: $e^x(x^2-2x+2)+C$.

Problem 11.17

Find $\int \frac{2x+3}{(x-1)^2}\,dx$

Hint: $\frac{2x+3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$. Solving: $A=2$, $B=5$. Answer: $2\ln|x-1| - \frac{5}{x-1} + C$.

Problem 11.18

Evaluate $\displaystyle\int \frac{1}{\sqrt{6x - x^2}}\,dx$

Hint: Complete square: $6x - x^2 = 9 - (x-3)^2$. Let $u = x - 3$. Answer: $\arcsin\left(\frac{x-3}{3}\right) + C$.

Problem 11.19

Evaluate $\displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$

Hint: Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$. King's: $I = \int_0^{\pi/2} \frac{\pi/2 - x}{\cos x + \sin x} dx$. Add: $2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$. Use $t = \tan(x/2)$.

Problem 11.20

Find $\displaystyle\int \sqrt{16 - x^2}\,dx$

Hint: Let $x = 4\sin\theta$, $dx = 4\cos\theta\,d\theta$. Then $\sqrt{16 - x^2} = 4\cos\theta$. Answer: $8\arcsin(x/4) + \frac{x\sqrt{16-x^2}}{2} + C$.

Problem 11.21

Derive and use: If $I_n = \displaystyle\int_0^{\pi/2} \sin^n x\,dx$, find I_4

Hint: By parts: $I_n = \frac{n-1}{n} I_{n-2}$. Calculate: $I_0 = \frac{\pi}{2}$, $I_2 = \frac{\pi}{4}$, $I_4 = \frac{3}{4} \cdot \frac{\pi}{4} = \frac{3\pi}{16}$.

Problem 11.22

Evaluate $\displaystyle\int \frac{x^2+1}{(x-1)(x^2+1)}\,dx$

Hint: Simplify first: $\frac{x^2+1}{(x-1)(x^2+1)} = \frac{1}{x-1}$. Answer: $\ln|x-1| + C$.

Problem 11.23

Find $\displaystyle\int x^2 \ln x\,dx$

Hint: LIATE: $u = \ln x$, $dv = x^2 dx$. Then $v = \frac{x^3}{3}$. Answer: $\frac{x^3 \ln x}{3} - \frac{x^3}{9} + C = \frac{x^3}{9}(3\ln x - 1) + C$.

Problem 11.24

Evaluate $\displaystyle\int_{-1}^{1} \frac{x^2}{1+e^x}\,dx$

Hint: Let $I = \int_{-1}^{1} \frac{x^2}{1+e^x} dx$. King's property: $I = \int_{-1}^{1} \frac{x^2 e^x}{1+e^x} dx$. Add: $2I = \int_{-1}^{1} x^2 dx = \frac{2}{3}$, so $I = \frac{1}{3}$.

Problem 11.25

Evaluate $\displaystyle\int \sin^4 x\,dx$

Hint: Use $\sin^2 x = \frac{1-\cos 2x}{2}$. Then $\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x+\cos^2 2x}{4}$. Use $\cos^2 2x = \frac{1+\cos 4x}{2}$.

Problem 11.26

Find the volume when the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 4$ is rotated about the x-axis.

Hint: $V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x \, dx = \pi \cdot \frac{16}{2} = 8\pi$ cubic units.

Problem 11.27

Evaluate $\displaystyle\int_0^{\pi/2} \frac{1}{3 + 5\cos x} \, dx$

Hint: Let $t = \tan(x/2)$. Then $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$. Limits: $t = 0$ to $t = 1$. Answer: $\frac{\ln 3}{4}$.

Problem 11.28

A particle moves with velocity $v = 2t - 3$ m/s. If $s(0) = 5$m, find $s(4)$.

Hint: $s(t) = \int v \, dt = t^2 - 3t + C$. Use $s(0) = 5$: $C = 5$. Then $s(4) = 16 - 12 + 5 = 9$m.

Problem 11.29

Find $\displaystyle\int \frac{3x + 5}{x^2 + 4} \, dx$

Hint: Split: $\frac{3x}{x^2+4} + \frac{5}{x^2+4}$. First gives $\frac{3}{2}\ln(x^2+4)$, second gives $\frac{5}{2}\arctan(x/2)$. Answer: $\frac{3}{2}\ln(x^2+4) + \frac{5}{2}\arctan(x/2) + C$.

Problem 11.30

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to find $\displaystyle\int e^x \cos x \, dx$

Hint: Consider $\int e^x e^{ix} dx = \int e^{(1+i)x} dx = \frac{e^{(1+i)x}}{1+i}$. Multiply by $\frac{1-i}{1-i}$, take real part. Answer: $\frac{e^x(\cos x + \sin x)}{2} + C$.

11.3 Advanced Integration Problems

Problem 11.31

Evaluate $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$

Hint: Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$. Then $\sqrt{x^2+9} = 3\sec\theta$. Use $\tan^2\theta\sec\theta = \sec^3\theta - \sec\theta$. Answer involves $\frac{x\sqrt{x^2+9}}{2} - \frac{9}{2}\ln|x+\sqrt{x^2+9}|$.

Problem 11.32

Let $I_n = \displaystyle\int_0^{\pi/2} \cos^n x\,dx.$

(a) Show $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$

(b) Hence find I_6

Hint: (a) $I_n = \int \cos^{n-1} x \cdot \cos x\,dx$. By parts: $u = \cos^{n-1} x$, $dv = \cos x\,dx$. (b) $I_0 = \frac{\pi}{2}$, $I_2 = \frac{\pi}{4}$, $I_4 = \frac{3\pi}{16}$, $I_6 = \frac{5}{6}\cdot\frac{3\pi}{16} = \frac{5\pi}{32}$.

Problem 11.33

Evaluate $\displaystyle\int \frac{x^3+2x+1}{x^2(x^2+1)}\,dx$

Hint: $\frac{x^3+2x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$. Solve: $A = 0$, $B = 1$, $C = 1$, $D = 1$. Answer: $-\frac{1}{x} + \frac{\ln(x^2+1)}{2} + \arctan x + C$.

Problem 11.34

Find the volume when the region between $y = x^2$ and $y = \sqrt{x}$ for $0 \leq x \leq 1$ is rotated about the x-axis.

Hint: Washer: $V = \pi\int_0^1[(\sqrt{x})^2 - (x^2)^2]dx = \pi\int_0^1(x-x^4)dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}$.

Problem 11.35

Show that $\displaystyle\int_0^1 \ln(1+x)\,dx = \int_0^1 \frac{x}{1+x}\,dx$

Hint: Let $u = 1+x$ in LHS. Then $x = u-1$, $dx = du$. Limits: $u=1$ to $u=2$. LHS $= \int_1^2 \ln u\,du$. By parts: $\int \ln u\,du = u\ln u - u$. Evaluate and simplify to show both equal $2\ln 2 - 1$.

Problem 11.36

Evaluate $\displaystyle\int e^x \sin x\,dx$

Hint: Let $I = \int e^x \sin x\,dx$. By parts twice creates: $I = e^x \sin x - e^x \cos x - I$. Solve: $2I = e^x(\sin x - \cos x)$, so $I = \frac{e^x(\sin x - \cos x)}{2} + C$.

Problem 11.37

Find $\displaystyle\int \frac{\sqrt{9-x^2}}{x}\,dx$

Hint: Let $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$. Then $\sqrt{9-x^2} = 3\cos\theta$. Integral: $\int \frac{3\cos\theta}{3\sin\theta}\cdot 3\cos\theta\,d\theta = 3\int \cot\theta\cos\theta\,d\theta$. Use $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

Problem 11.38

Evaluate $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$

Hint: Let $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$. King's: $I = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$. Add: $2I = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx$. Let $u = \cos x$. Answer: $\frac{\pi^2}{4}$.

Problem 11.39

Let $I_n = \int_0^1 x^n e^x \, dx$ for $n \geq 0$.

(a) Show $I_n = e - nI_{n-1}$ for $n \geq 1$

(b) Find I_4 and use it to find $\sum_{k=0}^{4} \frac{4!}{k!}$

Hint: (a) By parts: $u = x^n$, $dv = e^x dx$. (b) Calculate: $I_0 = e - 1$, $I_1 = 1$, $I_2 = e - 2$, $I_3 = 6 - 2e$, $I_4 = 9e - 24$. Pattern: $I_n = e\sum_{k=0}^{n}(-1)^k \frac{n!}{k!} + (-1)^{n+1} n!$.

Problem 11.40

Evaluate $\int \frac{x^2 - 3x + 5}{(x-1)(x^2+4)} \, dx$

Hint: $\frac{x^2-3x+5}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$. Solve: $A = \frac{3}{5}$, $B = \frac{2}{5}$, $C = -\frac{6}{5}$. Integrate separately: ln, ln, arctan terms.

Problem 11.41

Find $\int (\ln x)^2 \, dx$

Hint: Let $u = (\ln x)^2$, $dv = dx$. Then $du = \frac{2\ln x}{x} dx$, $v = x$. Apply by parts again to $\int \ln x \, dx$. Answer: $x(\ln x)^2 - 2x \ln x + 2x + C = x[(\ln x)^2 - 2\ln x + 2] + C$.

Problem 11.42

A particle undergoes SHM with $\ddot{x} = -4(x-3)$. At $t = 0$, $x = 8$ and $\dot{x} = 0$.

(a) Find the amplitude and period

(b) Find the time taken to first reach $x = 1$

Hint: (a) Center $c = 3$, $\omega^2 = 4$ so $\omega = 2$. From $x(0) = 8$: amplitude $A = 5$. Period $T = \frac{2\pi}{\omega} = \pi$. (b) $x = 3 + 5\cos(2t)$. Solve $1 = 3 + 5\cos(2t)$: $t = \frac{1}{2}\arccos(-\frac{2}{5}) \approx 0.58$s.

Problem 11.43

Let $I_n = \displaystyle\int \frac{1}{(x^2+1)^n}\,dx$ for $n \geq 1$.
Show that $(2n-1)I_n = \frac{x}{(x^2+1)^{n-1}} + (2n-2)I_{n-1}$

Hint: Write $I_n = \int \frac{x^2+1-x^2}{(x^2+1)^n}dx = I_{n-1} - \int \frac{x^2}{(x^2+1)^n}dx$. By parts on second term: $u = x$, $dv = \frac{x}{(x^2+1)^n}dx$. Rearrange to get reduction formula.

Problem 11.44

Find the volume when the region bounded by $y = x^2$, $y = 0$, $x = 2$ is rotated about the y-axis.

Hint: Shell method: $V = 2\pi \int_0^2 x \cdot x^2\,dx = 2\pi \int_0^2 x^3 dx = 2\pi \cdot \frac{16}{4} = 8\pi$ cubic units.

Problem 11.45

Prove that $\displaystyle\int_0^{\pi/2} \ln(\sin x)\,dx = -\frac{\pi}{2}\ln 2$

Hint: Let $I = \int_0^{\pi/2} \ln(\sin x)dx$. King's: $I = \int_0^{\pi/2} \ln(\cos x)dx$. Add: $2I = \int_0^{\pi/2} [\ln(\sin x) + \ln(\cos x)]dx = \int_0^{\pi/2} \ln(\sin x \cos x)dx$. Use $\sin x \cos x = \frac{\sin 2x}{2}$. Substitute $u = 2x$.

12 Appendices

12.1 Appendix A: Formula Sheet

Standard Integrals

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$
$$\int \frac{1}{x} \, dx = \ln|x| + C$$
$$\int e^x \, dx = e^x + C$$
$$\int a^x \, dx = \frac{a^x}{\ln a} + C \qquad (a > 0, a \neq 1)$$
$$\int \sin x \, dx = -\cos x + C$$
$$\int \cos x \, dx = \sin x + C$$
$$\int \sec^2 x \, dx = \tan x + C$$
$$\int \csc^2 x \, dx = -\cot x + C$$
$$\int \sec x \tan x \, dx = \sec x + C$$
$$\int \csc x \cot x \, dx = -\csc x + C$$

Inverse Trigonometric Forms

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad (|x| < a)$$
$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{x\sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \sec^{-1}\left(\frac{|x|}{a}\right) + C \quad (|x| > a)$$

Integration by Parts

$$\int u \, dv = uv - \int v \, du$$

LIATE Rule for choosing u:

- **L** Logarithmic ($\ln x$, $\log x$)
- **I** Inverse Trigonometric ($\sin^{-1} x$, $\tan^{-1} x$, etc.)
- **A** Algebraic (x^2, $3x$, etc.)
- **T** Trigonometric ($\sin x$, $\cos x$, etc.)
- **E** Exponential (e^x, a^x)

Reverse Chain Rule

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \qquad \text{(Logarithmic form)}$$
$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C \qquad (n \neq -1)$$

Trigonometric Identities

$\sin^2 x + \cos^2 x = 1$	$1 + \tan^2 x = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$	$\sin 2x = 2\sin x \cos x$
$\cos 2x = \cos^2 x - \sin^2 x$	$\sin^2 x = \frac{1-\cos 2x}{2}$
$\cos^2 x = \frac{1+\cos 2x}{2}$	

Definite Integral Properties

- **Odd Function:** If $f(-x) = -f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$
- **Even Function:** If $f(-x) = f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
- **King's Property:** $\displaystyle\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$

12.2 Appendix B: Index of Problems by Technique

Substitution Methods

Basic u-Substitution:

- Part 1, Easy #4: $\int \frac{2x+1}{\sqrt{x^2+x+3}}dx$ (algebraic substitution)
- Part 2, Easy #1: $\int (3x+1)^5 dx$ (linear substitution)
- Part 2, Easy #2: $\int \frac{x}{\sqrt{x^2+4}}dx$ (radical with u-substitution)
- Part 2, Easy #5: $\int \sin^3 2x \cos 2x\,dx$ (trig substitution)
- Part 2, Easy #9: $\int_0^2 x\sqrt{1+x^2}dx$ (definite with substitution)
- Part 2, Easy #13: $\int \frac{6x^2}{(x^3+1)^4}dx$ (reverse chain rule power)

Trigonometric Substitution ($\sqrt{a^2 \pm x^2}$ forms):

- Part 1, Hard #1: Three-part trig substitution with reduction
- Part 2, Medium #3: $\int \frac{1}{\sqrt{6x-x^2}}dx$ (completing square + arcsin)
- Part 2, Medium #5: $\int \sqrt{16-x^2}dx$ (standard $\sqrt{a^2-x^2}$ form)
- Part 2, Hard #1: $\int \frac{x^2}{\sqrt{x^2+9}}dx$ (standard $\sqrt{x^2+a^2}$ form)
- Part 2, Hard #7: $\int \frac{\sqrt{9-x^2}}{x}dx$ (mixed trig substitution)

t-Formula ($t = \tan(x/2)$):

- Part 1, Medium #2: King's property combined with t-formula

- Part 2, Medium #12: $\int_0^{\pi/2} \frac{1}{3+5\cos x} dx$ (standard t-formula)

Substitution Transformation Proofs:

- Part 1, Hard #3: Proving two integrals equal via substitution
- Part 2, Hard #5: $\int_0^1 \ln(1+x)dx = \int_0^1 \frac{x}{1+x} dx$ proof

Integration by Parts

Single Application:

- Part 1, Easy #2: $\int x \ln x \, dx$ (LIATE: logarithm)
- Part 2, Easy #6: $\int xe^x dx$ (algebraic × exponential)
- Part 2, Easy #12: $\int x \cos x \, dx$ (algebraic × trig)
- Part 2, Medium #8: $\int x^2 \ln x \, dx$ (higher power × ln)

Multiple Applications:

- Part 2, Medium #1: $\int x^2 e^x dx$ (by parts twice)
- Part 2, Hard #11: $\int (\ln x)^2 dx$ (nested logarithms)

Cyclic Method:

- Part 1, Medium #5: $\int e^x \sin x \, dx$ using complex numbers
- Part 2, Hard #6: $\int e^x \sin x \, dx$ (traditional cyclic method)
- Part 2, Medium #15: $\int e^x \cos x \, dx$ using Euler's formula

Partial Fractions

Linear Factors:

- Part 1, Easy #1: Linear + irreducible quadratic
- Part 2, Easy #11: $\int \frac{1}{(x-1)(x+2)} dx$ (two linear factors)

Repeated Factors:

- Part 2, Medium #2: $\int \frac{2x+3}{(x-1)^2} dx$ (repeated linear)

Irreducible Quadratics:

- Part 2, Hard #3: $\int \frac{x^3+2x+1}{x^2(x^2+1)} dx$ (quadratic in denominator)
- Part 2, Hard #10: $\int \frac{x^2-3x+5}{(x-1)(x^2+4)} dx$ (linear + quadratic)

Simplification First:

- Part 2, Medium #7: $\int \frac{x^2+1}{(x-1)(x^2+1)} dx$ (cancel common factor)

Reduction Formulae

Derivation and Application:

- Part 1, Medium #1: $I_n = \int \cot^n x \, dx$ (cotangent reduction, 2-part)
- Part 1, Medium #3: $I_n = \int (\ln x)^n dx$ (logarithm reduction)
- Part 2, Medium #6: $I_n = \int_0^{\pi/2} \sin^n x \, dx$ (sine reduction)
- Part 2, Hard #2: $I_n = \int_0^{\pi/2} \cos^n x \, dx$ (cosine reduction with induction)
- Part 2, Hard #13: $I_n = \int \frac{1}{(x^2+1)^n} dx$ (rational reduction)

With Mathematical Induction:

- Part 1, Hard #2: 5-part problem with factorial series and limit proof for e

Reverse Chain Rule

Recognition Patterns:

- Part 1, Easy #3: $\int \left(\frac{1}{x+1} + \frac{2x}{x^2+1}\right) dx$ (ln + arctan)
- Part 2, Easy #3: $\int 2xe^{x^2} dx$ (exponential pattern)
- Part 2, Easy #7: $\int \frac{2x+3}{x^2+3x+1} dx$ (logarithm pattern)
- Part 2, Medium #14: $\int \frac{3x+5}{x^2+4} dx$ (split into ln + arctan)

Trigonometric Techniques

Power-Reduction Identities:

- Part 2, Easy #8: $\int \sin^2 x \, dx$ ($\sin^2 x = \frac{1-\cos 2x}{2}$)
- Part 2, Medium #10: $\int \sin^4 x \, dx$ (double application)

Combined Methods:

- Part 1, Medium #2: King's property + t-formula

Definite Integral Properties

Even/Odd Functions:

- Part 2, Easy #14: $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ for even functions

King's Property ($\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$):

- Part 1, Easy #5: MCQ using King's property
- Part 2, Medium #4: $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$ (King's + t-formula)
- Part 2, Medium #9: $\int_{-1}^{1} \frac{x^2}{1+e^x} dx$ (symmetry with exponential)
- Part 2, Hard #8: $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ (complex denominator)
- Part 2, Hard #15: $\int_0^{\pi/2} \ln(\sin x) dx$ (logarithm with King's)

Volumes of Revolution

Disk Method:

- Part 2, Medium #11: $y = \sqrt{x}$ rotated about x-axis

Washer Method:

- Part 1, Hard #5: Two regions (circle and logarithm) with ratio proof
- Part 2, Hard #4: Between $y = x^2$ and $y = \sqrt{x}$

Shell Method:

- Part 2, Hard #14: $y = x^2$ rotated about y-axis

Mechanics Applications

Particle Motion:

- Part 1, Medium #4: Velocity integration with $F = ma$
- Part 2, Medium #13: Position from velocity with initial conditions

Simple Harmonic Motion (SHM):

- Part 1, Hard #4: Inclined plane with quadratic solution
- Part 2, Hard #12: Amplitude, period, and timing calculations

Special Techniques

Completing the Square:

- Part 2, Easy #15: $\int \frac{1}{x^2+4x+13}dx$ (arctan form)
- Part 2, Medium #3: Combined with substitution for arcsin

Complex Numbers Method:

- Part 1, Medium #5: Using Euler's formula $e^{i\theta}$
- Part 2, Medium #15: $\int e^x \cos x \, dx$ via complex exponentials

Series and Limits:

- Part 1, Hard #2: Factorial series limit proof for e
- Part 2, Hard #9: $I_n = \int_0^1 x^n e^x dx$ with series sum

Standard Forms:

- Part 2, Easy #4: $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan(x/a) + C$
- Part 2, Easy #10: $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin(x/a) + C$

Note: This index helps identify problems by technique. Many problems combine multiple methods—refer to solution strategies for complete technique breakdowns.

12.3 Appendix C: Common Substitutions Guide

Basic U-Substitution

When to use: The integrand contains a function and its derivative (or a constant multiple).

How: Let $u = g(x)$ where $g(x)$ is the inner function. Then $du = g'(x)\,dx$.

Examples:

- $\int x\sqrt{x^2+1}\,dx$: Let $u = x^2 + 1$
- $\int \sin x \cos x\,dx$: Let $u = \sin x$ (or $u = \cos x$)
- $\int \frac{x}{x^2+5}\,dx$: Let $u = x^2 + 5$

Trigonometric Substitutions

For $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$ or $x = a\cos\theta$

Uses: $1 - \sin^2\theta = \cos^2\theta$

Example: $\int \frac{1}{\sqrt{9-x^2}}\,dx$ with $x = 3\sin\theta$

For $\sqrt{a^2 + x^2}$: Let $x = a\tan\theta$

Uses: $1 + \tan^2\theta = \sec^2\theta$

Example: $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ with $x = 2\tan\theta$

For $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$ or $x = a\cosh t$

Uses: $\sec^2\theta - 1 = \tan^2\theta$

Example: $\int x^3\sqrt{x^2-9}\,dx$ with $x = 3\sec\theta$

t-Formula Substitution

When to use: Rational functions of $\sin x$ and $\cos x$

Substitution: $t = \tan\left(\frac{x}{2}\right)$

Formulas:

$$\begin{aligned} dx &= \frac{2}{1+t^2}\,dt \\ \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \end{aligned}$$

Example: $\int \frac{1}{2+\sin x}\,dx$

Exponential and Logarithmic Substitutions

For integrands with e^x: Often let $u = e^x$, then $du = e^x\,dx$

For integrands with $\ln x$: Often use integration by parts with $u = \ln x$

Examples:

- $\int \frac{e^x}{1+e^x}\,dx$: Let $u = 1 + e^x$
- $\int \frac{\ln x}{x}\,dx$: Let $u = \ln x$

Completing the Square

When to use: Quadratic expressions in denominators or under square roots

How: Rewrite $ax^2 + bx + c$ as $a[(x+h)^2 + k]$

Example: $\int \frac{1}{x^2+4x+13}\,dx$

Complete the square: $x^2 + 4x + 13 = (x+2)^2 + 9$

Then use $u = x + 2$ and apply $\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

Quick Reference Table

Integrand Contains	Try Substitution	Uses Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$
$f(x)$ and $f'(x)$	$u = f(x)$	Reverse chain rule
Rational trig	$t = \tan(x/2)$	t-formula identities
e^x and algebra	$u = e^x$	$du = e^x dx$

12.4 Appendix D: Integration by Parts Decision Tree

Integration by Parts Formula

$$\boxed{\int u\,dv = uv - \int v\,du}$$

The LIATE Rule

Choose u according to this priority (top to bottom):

1. **L**ogarithmic functions: $\ln x$, $\log_a x$
2. **I**nverse trigonometric functions: $\sin^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$
3. **A**lgebraic functions: x^n, polynomials
4. **T**rigonometric functions: $\sin x$, $\cos x$, $\tan x$
5. **E**xponential functions: e^x, a^x

The remaining factor becomes dv.

Decision Flowchart

1. **Identify the product:** Is the integrand a product of two different types of functions?
   - If YES → Proceed to step 2
   - If NO → Consider other methods (substitution, partial fractions)
2. **Apply LIATE:** Choose u as the function highest in the LIATE priority list
3. **Determine dv:** The remaining factor (and dx) becomes dv
4. **Can you find v?** Integrate dv to get v
   - If YES → Proceed to step 5
   - If NO → Reconsider choice of u and dv
5. **Is $\int v\,du$ simpler?** Compare $\int v\,du$ with the original integral
   - If SIMPLER → Good choice! Proceed with integration
   - If SAME COMPLEXITY → May need to apply parts again
   - If MORE COMPLEX → Try different u and dv

Special Cases

Cyclic Integrals: When $\int v\,du$ returns to the original form

Example: $\int e^x \sin x\,dx$ or $\int e^x \cos x\,dx$

Strategy: Apply integration by parts twice, then solve algebraically for the original integral.

Reduction Formulae: When seeking a recurrence relation I_n in terms of I_{n-1}

Strategy: Choose u and dv to reduce the power of the function.

Definite Integrals: Don't forget to apply limits to uv term!

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

Common Examples

Integral	**Choose u**	**Choose dv**	**Why**
$\int xe^x\,dx$	$u = x$	$dv = e^x dx$	A before E
$\int x\sin x\,dx$	$u = x$	$dv = \sin x\,dx$	A before T
$\int \ln x\,dx$	$u = \ln x$	$dv = dx$	L is highest
$\int x^2e^x\,dx$	$u = x^2$	$dv = e^x dx$	A before E
$\int \tan^{-1} x\,dx$	$u = \tan^{-1} x$	$dv = dx$	I is high

Tips and Warnings

- **Tip 1:** If $u = \ln x$ or $u = \tan^{-1} x$, set $dv = dx$
- **Tip 2:** For $\int x^n e^{ax} dx$ or $\int x^n \sin(ax) dx$, apply parts n times
- **Tip 3:** Watch your signs! Especially with $v = -\cos x$ or $v = -e^{-x}$
- **Warning:** Don't forget the constant of integration $+C$ for indefinite integrals
- **Warning:** For definite integrals, apply limits to the $[uv]$ term before integrating $\int v\,du$

13 Conclusion

Integration is a cornerstone technique in the HSC Mathematics Extension 2 course. Mastery requires understanding when and how to apply different techniques, recognizing patterns, and practicing extensively. Use these problems to build confidence, develop systematic approaches, and strengthen your ability to communicate complete mathematical solutions. Best of luck with your studies and HSC examinations!

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