HSC Math Extension 2: Last Resorts Mastery

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Contents

1 Introduction

1.1 Project Overview

This booklet compiles the most challenging problems from HSC Mathematics Extension 2 examinations—the notorious Problem 16, as we call them "Last Resort." These problems represent the pinnacle of high school mathematics, typically combining multiple advanced topics and requiring sophisticated problem-solving techniques. Every problem showcases the depth and complexity expected at the highest level of the HSC Mathematics curriculum.

The collection draws from authentic examination problems that have challenged Extension 2 students over many years. Each problem has been carefully selected to represent the diverse range of mathematical concepts and proof techniques that appear in Problem 16, from advanced inequalities and complex number theory to vector optimization and differential geometry applications.

1.2 What Makes Problem 16 Special

Problem 16 is uniquely challenging because it:

- Combines multiple mathematical topics within a single problem
- Requires advanced proof techniques and mathematical maturity
- Often involves multi-part questions that build complexity progressively
- Tests both computational skills and conceptual understanding
- Demands clear mathematical communication and rigorous reasoning

1.3 Target Audience

This booklet is designed for Extension 2 students who want to:

- Master the most challenging problems in the HSC curriculum
- Develop advanced problem-solving strategies
- Build confidence in tackling complex multi-step problems
- Understand how to approach unfamiliar mathematical scenarios
- Prepare thoroughly for the demands of Problem 16 in examinations

1.4 How to Use This Booklet

- **Study the Fundamentals:** Review key theorems and techniques
- **Attempt Problems Independently:** Try each problem without looking at solutions first
- **Part 1 Strategy:** Compare your work against solutions to understand model reasoning
- **Part 2 Approach:** Use upside-down hints sparingly, then review concise solutions
- **Practice Multiple Times:** Rework problems from memory to build technique mastery
- **Focus on Communication:** Pay attention to how mathematical arguments are structured and presented

2 Fundamentals Review

2.1 Key Inequalities

Problem 16 frequently employs fundamental inequalities as building blocks for more complex arguments.

2.1.1 Arithmetic-Geometric Mean (AM-GM) Inequality

Theorem 0.1: AM-GM Inequality

For positive real numbers $a_1, a_2, \ldots, a_n$:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

Theorem 0.2: Weighted AM-GM Inequality

For positive real numbers $a_1, a_2, \ldots, a_n$ and positive weights $w_1, w_2, \ldots, w_n$ with $w_1 + w_2 + \cdots + w_n = 1$:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}$$

2.1.2 Cauchy-Schwarz Inequality

Theorem 0.3: Cauchy-Schwarz Inequality

For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$:

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$$

Equivalently, for real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

2.2 Complex Number Theory

2.2.1 De Moivre's Theorem

Theorem 0.4: De Moivre's Theorem

For complex numbers in polar form $z = r(\cos\theta + i\sin\theta)$:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

2.2.2 Roots of Unity

Theorem 0.5: Roots of Unity

The nth roots of unity are given by:

$$\omega_k = e^{2\pi ik/n} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

for $k = 0, 1, 2, \ldots, n-1$.

2.3 Vector Algebra

2.3.1 Dot Product

For vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$$

2.3.2 Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The magnitude $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$ equals the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

2.4 Polynomial Theory

2.4.1 Newton's Identities

Theorem 0.6: Newton's Identities

For a polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, if $S_k = \sum_{i=1}^{n} \alpha_i^k$ and e_k are the elementary symmetric polynomials, then:

$$S_k - e_1S_{k-1} + e_2S_{k-2} - \cdots + (-1)^{k-1}e_{k-1}S_1 + (-1)^k k e_k = 0$$

2.5 Chebyshev Polynomials (First Kind)

The Chebyshev polynomials of the first kind $T_n(x)$ are a family of orthogonal polynomials on $[-1, 1]$ with the defining trigonometric identity:

$$T_n(x) = \cos\big(n \arccos x\big).$$

2.5.1 Recurrence Formula

They satisfy the three-term recurrence for $n \geq 1$:

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x), \quad T_0(x) = 1,\ T_1(x) = x.$$

2.5.2 Explicit Low-Degree Polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x.$$

These identities are frequently useful for transforming certain cubic equations by matching the structure of $T_3(x)$.

2.6 Optimization Techniques

2.6.1 Lagrange Multipliers (Conceptual)

To optimize $f(x, y, z)$ subject to constraint $g(x, y, z) = 0$, look for points where:

$$\nabla f = \lambda \nabla g$$

This often reduces to applying AM-GM or Cauchy-Schwarz in HSC problems. Good to know though the technique is not covered at HSC level.

2.7 Notation and Conventions

- Complex numbers are denoted by $z, w, \alpha, \beta, \ldots$
- Vectors are written in bold: $\mathbf{u}, \mathbf{v}, \mathbf{a}, \mathbf{b}$
- Real numbers are assumed unless stated otherwise
- $\mathbb{R}^n$ denotes n-dimensional real space
- $|z|$ denotes the modulus of complex number z
- $|\mathbf{v}|$ denotes the magnitude of vector $\mathbf{v}$

3 Part 1: Problems and Solutions (Detailed)

Part 1 contains 15 carefully selected problems representing the core techniques and challenge levels found in Problem 16. Each solution includes a strategy explanation, complete step-by-step working, and takeaway insights to build your problem-solving toolkit.

3.1 Medium Last Resort Problems

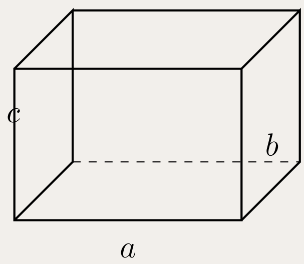
Problems that combine 2–3 mathematical topics with sophisticated reasoning.

Problem 3.1: AM-GM Surface Area Optimization

(a) Prove that for positive real numbers x, y, z:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{3}{\sqrt[3]{xyz}}$$

(b) A rectangular prism has dimensions a, b, c and a fixed, constant Volume V. Using part (a), show that the Total Surface Area S is minimized when the prism is a cube.



Hint: For part (a), apply AM-GM inequality directly to the reciprocals $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$. For part (b), express the surface area $S = 2(ab + bc + ca)$ in terms of the volume constraint $V = abc$, then use part (a) with strategic substitution.

Solution 3.1

(a) Proof: Since $x, y, z > 0$, then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} > 0$. Applying the AM-GM inequality to these three terms:

$$\frac{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}{3} \geq \sqrt[3]{\frac{1}{x} \cdot \frac{1}{y} \cdot \frac{1}{z}}$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3\sqrt[3]{\frac{1}{xyz}}$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{3}{\sqrt[3]{xyz}}$$

(b) Application: Let the dimensions be a, b, c.

- Fixed Volume: $V = abc$ (constant).
- Surface Area: $S = 2(ab + bc + ca)$.

We want to relate S to the reciprocals in part (a).

$$S = 2abc\left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b}\right)$$
$$S = 2V\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

From part (a), we know $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \frac{3}{\sqrt[3]{abc}}$. Substituting this into the expression for S:

$$S \geq 2V\left(\frac{3}{\sqrt[3]{V}}\right)$$
$$S \geq 6V^1V^{-1/3}$$
$$S \geq 6V^{2/3}$$

Since V is constant, $6V^{2/3}$ is a constant minimum value. Equality holds (minimum S occurs) when the terms in the AM-GM are equal:

$$\frac{1}{a} = \frac{1}{b} = \frac{1}{c} \implies a = b = c$$

Therefore, the prism is a cube. □

Takeaways 3.1

1. **Infinite vs Finite Sums:** Often, calculating the infinite geometric series sum is easier and sufficient. If the infinite sum creates a contradiction (< 1), then the finite sum definitely will too.

2. **Strict Inequalities:** Geometric series of positive terms are always strictly less than their limit at infinity.

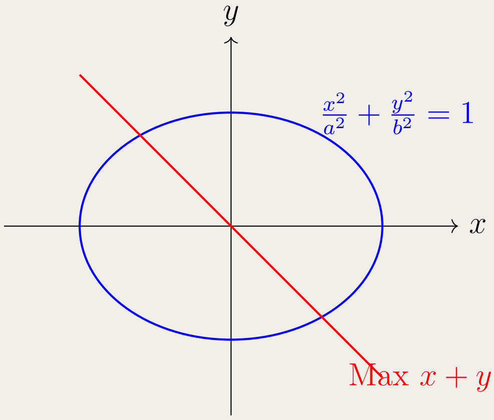
Problem 3.2: Cauchy-Schwarz on Ellipsoid

The point $P(x, y, z)$ lies on the surface of the ellipsoid defined by:

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$$

(i) By choosing appropriate vectors $\mathbf{u}$ and $\mathbf{v}$ and applying the Cauchy-Schwarz inequality ($\mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u}||\mathbf{v}|$), find the maximum value of $x + y + z$.

(ii) Find the coordinates of P in the first octant where this maximum occurs.



Hint:

The standard Cauchy-Schwarz inequality is $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. You cannot define $\mathbf{u} = (x, y, z)$ directly because the sum of squares is not 1. Instead, define $\mathbf{u}$ such that $|\mathbf{u}|^2$ exactly matches the left-hand side of the ellipsoid equation. Then, choose a constant vector $\mathbf{v}$ such that the dot product $\mathbf{u} \cdot \mathbf{v}$ recovers the expression $x + y + z$.

Solution 3.2

(i) Let $\mathbf{u} = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{3} \\ \frac{z}{5} \end{pmatrix}$. We are given $|\mathbf{u}|^2 = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$, so $|\mathbf{u}| = 1$. We want to maximize $x + y + z$. We observe:

$$x + y + z = \frac{x}{2}(2) + \frac{y}{3}(3) + \frac{z}{5}(5)$$

This suggests setting $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$. Apply Cauchy-Schwarz:

$$\mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u}||\mathbf{v}|$$

$$x + y + z \leq (1)\sqrt{2^2 + 3^2 + 5^2}$$
$$x + y + z \leq \sqrt{4 + 9 + 25} = \sqrt{38}$$

(ii) Equality occurs when $\mathbf{u} = k\mathbf{v}$, i.e., $\frac{x}{2} = 2k, \frac{y}{3} = 3k, \frac{z}{5} = 5k$. So $x = 4k, y = 9k, z = 25k$. Substitute into the plane eq $x + y + z = \sqrt{38}$:

$$4k + 9k + 25k = \sqrt{38} \implies 38k = \sqrt{38} \implies k = \frac{1}{\sqrt{38}}$$

Point P: $\left(\frac{4}{\sqrt{38}}, \frac{9}{\sqrt{38}}, \frac{25}{\sqrt{38}}\right)$.

Takeaways 3.2

1. **Normalization:** If given a constraint like $Ax^2 + By^2 = C$, define vector components as $\sqrt{A}x$ and $\sqrt{B}y$.
2. **The "Canceling" Vector:** The second vector is chosen specifically to cancel out the denominators introduced by the normalization step.

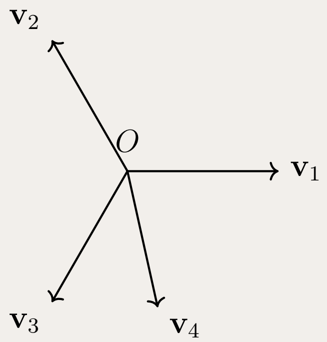
Problem 3.3: Unit Vector Cosine Sum

Four unit vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ originate from the origin O. It is given that their vector sum is zero:

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

Let θ_{ij} be the angle between vectors $\mathbf{v}_i$ and $\mathbf{v}_j$. Show that the sum of the cosines of the angles between all distinct pairs is -2.

$$\sum_{1 \le i < j \le 4} \cos\theta_{ij} = -2$$



Hint:

Consider the squared magnitude of the vector sum. Since $\sum \mathbf{v}_i = \mathbf{0}$, it follows that $|\sum \mathbf{v}_i|^2 = 0$. Expand the dot product $(\mathbf{v}_1 + \cdots + \mathbf{v}_4) \cdot (\mathbf{v}_1 + \cdots + \mathbf{v}_4)$. Separate the expansion into "self-products" $(\mathbf{v}_i \cdot \mathbf{v}_i)$ and "cross-products" $(\mathbf{v}_i \cdot \mathbf{v}_j)$.

Solution 3.3

Let $S = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$.

$$|S|^2 = S \cdot S = 0$$

Expanding the dot product:

$$\sum_{i=1}^{4} |\mathbf{v}_i|^2 + 2 \sum_{1 \le i < j \le 4} (\mathbf{v}_i \cdot \mathbf{v}_j) = 0$$

We are given that vectors are unit vectors, so $|\mathbf{v}_i|^2 = 1$ for all $i = 1..4$. There are 4 such terms.

$$4 + 2 \sum_{1 \le i < j \le 4} (\mathbf{v}_i \cdot \mathbf{v}_j) = 0$$

Using the definition of dot product: $\mathbf{v}_i \cdot \mathbf{v}_j = |\mathbf{v}_i||\mathbf{v}_j| \cos\theta_{ij} = 1 \cdot 1 \cdot \cos\theta_{ij}$.

$$4 + 2 \sum_{1 \le i < j \le 4} \cos\theta_{ij} = 0$$

$$2 \sum \cos\theta_{ij} = -4$$

$$\sum_{1 \le i < j \le 4} \cos\theta_{ij} = -2$$

Takeaways 3.3

1. **Squaring the Sum:** The most powerful tool for analyzing vector sums equal to zero (equilibrium) is to take the dot product of the sum with itself.
2. **Counting Terms:** When expanding $(\sum_{i=1}^{n} a_i)^2$, there are n squared terms and $n(n-1)$ cross terms. Since dot product is commutative, this groups into n squared terms and $2 \times$ (distinct pairs).

Problem 3.4: Complex Numbers Forming Triangle

Three complex numbers z_1, z_2, z_3 satisfy:

$$\begin{cases} |z_1| = |z_2| = |z_3| = r \quad (r > 0) \\ z_1 + z_2 + z_3 = 0 \end{cases}$$

Show that z_1, z_2, z_3 represent the vertices of an equilateral triangle inscribed in the circle $|z| = r$.

Hint:

To prove a triangle is equilateral, you can prove the sides are equal: $|z_1 - z_2|^2 = |z_2 - z_3|^2 = |z_3 - z_1|^2$. *Expand* $|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1} - \overline{z_2})$. *Use the fact that* $z_1 + z_2 + z_3 = 0 \implies z_1 + z_2 = -z_3$.

Solution 3.4

Consider the squared side length $|z_1 - z_2|^2$:

$$\begin{aligned} |z_1 - z_2|^2 &= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1 \\ &= r^2 + r^2 - (z_1\bar{z}_2 + z_2\bar{z}_1) \\ &= 2r^2 - 2\mathrm{Re}(z_1\bar{z}_2) \end{aligned}$$

Now consider the condition $(\sum z)(\sum \bar{z}) = 0 \cdot 0 = 0$.

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + (z_1\bar{z}_2 + \text{others}) = 0$$

$$3r^2 + (z_1\bar{z}_2 + z_2\bar{z}_1) + \cdots = 0$$

Alternatively, simplify the algebra: Since $z_1+z_2 = -z_3$, squaring modulus: $|z_1+z_2|^2 = |-z_3|^2 \implies |z_1+z_2|^2 = r^2$. Using the parallelogram law $|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$:

$$r^2 + |z_1 - z_2|^2 = 2(r^2 + r^2)$$

$$|z_1 - z_2|^2 = 3r^2$$

By symmetry, $|z_2 - z_3|^2 = 3r^2$ and $|z_3 - z_1|^2 = 3r^2$. Since all sides are equal ($\sqrt{3}r$), the triangle is equilateral.

Takeaways 3.4

1. **Vector Addition:** $z_1 + z_2 + z_3 = 0$ means the centroid is the origin. If the circumcenter (origin) and centroid coincide, the triangle is equilateral.
2. **Modulus Algebra:** Using $|z_1 + z_2|^2 = |-z_3|^2$ is much faster than expanding everything.

Problem 3.5: Minimum Distance Between Moving Particles

Two particles A and B move in space such that their position vectors at time $t \geq 0$ are:

$$\mathbf{r}_A = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_B = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(i) Express the squared distance $S(t) = |\mathbf{r}_A - \mathbf{r}_B|^2$ as a quadratic in t.
(ii) Find the minimum distance between the particles and the time at which this occurs.

Hint:

Find the displacement vector $\mathbf{d}(t) = \mathbf{r}_A - \mathbf{r}_B$*. Compute the dot product* $\mathbf{d} \cdot \mathbf{d}$ *to get the squared magnitude. You will get a quadratic expression* $At^2 + Bt + C$*. Find the vertex of this parabola* ($t = -B/2A$)*.*

Solution 3.5

(i) The vector connecting them is:

$$\mathbf{d} = \mathbf{r}_A - \mathbf{r}_B = \begin{pmatrix} 1-4 \\ 0-2 \\ 2-0 \end{pmatrix} + t \begin{pmatrix} 1-0 \\ 1-1 \\ 0-1 \end{pmatrix} = \begin{pmatrix} -3+t \\ -2 \\ 2-t \end{pmatrix}$$

Squared distance:

$$S(t) = (-3+t)^2 + (-2)^2 + (2-t)^2$$
$$S(t) = (t^2 - 6t + 9) + 4 + (t^2 - 4t + 4)$$
$$S(t) = 2t^2 - 10t + 17$$

(ii) To minimize $S(t)$, find $S'(t) = 4t - 10$. Set $4t - 10 = 0 \implies t = 2.5$. Substitute $t = 2.5$ back into $S(t)$:

$$S(2.5) = 2(6.25) - 25 + 17 = 12.5 - 25 + 17 = 4.5$$

Minimum distance $= \sqrt{4.5} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$.

Takeaways 3.5

1. **Distance Squared:** Always minimize distance *squared* ($|\mathbf{d}|^2$) rather than distance ($|\mathbf{d}|$). It avoids square roots and simplifies the calculus derivatives.
2. **Kinematics Connection:** If velocities are constant vectors, the distance function is always a quadratic (convex parabola), ensuring a unique minimum.

3.2 Advanced Last Resort Problems

The most challenging problems requiring multiple advanced techniques and mathematical maturity.

Problem 3.6: Complex Ellipsoid Optimization

Consider the ellipsoid E defined by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $a > b > c > 0$.
(a) Find the point $P(x_0, y_0, z_0)$ on E that maximizes the distance to the plane π : $x + y + z = 0$.
(b) Let $\mathbf{n} = (1, 1, 1)$ be the normal vector to π. Show that at the optimal point P, the gradient ∇f is parallel to $\mathbf{n}$, where $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$.
(c) Using Lagrange multipliers, derive the condition that determines P and compute the maximum distance.

Solution 3.6

We maximize the distance $d = \dfrac{x+y+z}{\sqrt{3}}$ under $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ with $x+y+z > 0$. By Cauchy–Schwarz,

$$(x+y+z)^2 = \left(\tfrac{x}{a}a + \tfrac{y}{b}b + \tfrac{z}{c}c\right)^2 \le \left(\tfrac{x^2}{a^2} + \tfrac{y^2}{b^2} + \tfrac{z^2}{c^2}\right)(a^2+b^2+c^2) = a^2+b^2+c^2,$$

so $x+y+z \le \sqrt{a^2+b^2+c^2}$ with equality exactly when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k > 0$. Writing $x = ka^2$, $y = kb^2$, $z = kc^2$ in the constraint gives $k^2(a^2+b^2+c^2) = 1$, hence $k = 1/\sqrt{a^2+b^2+c^2}$ and

$$P = \left(\frac{a^2}{\sqrt{a^2+b^2+c^2}}, \frac{b^2}{\sqrt{a^2+b^2+c^2}}, \frac{c^2}{\sqrt{a^2+b^2+c^2}}\right), \qquad d_{\max} = \frac{\sqrt{a^2+b^2+c^2}}{\sqrt{3}}.$$

Takeaways 3.6

1. **Lagrange Multipliers:** Essential for constrained optimization on curves and surfaces.
2. **Gradient Parallelism:** At optimal points, constraint and objective gradients are parallel.
3. **Geometric Interpretation:** The solution connects algebraic optimization with geometric intuition.

Remark 0.1: Distance Between Shapes

In general, the distance between two shapes A and B is defined as

$$d(A,B) = \inf\{\|p-q\| : p \in A,\ q \in B\}.$$

For an ellipsoid and a plane, this reduces to optimizing the point on the ellipsoid whose normal is aligned with the plane's normal (via Lagrange multipliers), yielding the maximum/minimum perpendicular separation between the surfaces. Here, inf means the greatest lower bound (or minimum if it exists).

Problem 3.7: Tangent Circle and Curvature

Consider the parametric curve C given by:

$$\mathbf{r}(t) = (t^3 - 3t, t^2, t^4) \quad \text{for } t \in \mathbb{R}$$

For a smooth space curve $\mathbf{r}(t)$, the curvature at parameter t is

$$\kappa(t) = \frac{\left\| \mathbf{r}'(t) \times \mathbf{r}''(t) \right\|}{\left\|\mathbf{r}'(t)\right\|^3},$$

which measures how quickly the curve is turning at that point (do NOT prove this formula).
(a) Find the curvature $\kappa(t)$ of the curve at any point.
(b) Determine the point(s) where the curvature is maximum.
(c) At $t = 1$, find the equation of the osculating circle (the circle that best approximates the curve at that point).
(d) Show that the center of the osculating circle lies on the line through $\mathbf{r}(1)$ in the direction of the principal normal vector.

Remark 0.2: Plane Curve Curvature

For a plane curve given by $\gamma(t) = (x(t), y(t))$, a common equivalent formula is

$$\kappa(t) = \frac{\left|x'(t)\, y''(t) - y'(t)\, x''(t)\right|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}.$$

This matches the space-curve definition when the curve lies in a plane.

Solution 3.7

(a) Computing curvature: First, find the derivatives: $\mathbf{r}'(t) = (3t^2 - 3, 2t, 4t^3)$ $\mathbf{r}''(t) = (6t, 2, 12t^2)$

The curvature formula: $\kappa(t) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$

Compute the cross product: $\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3t^2 - 3 & 2t & 4t^3 \\ 6t & 2 & 12t^2 \end{vmatrix}$

$= \mathbf{i}(24t^3 - 8t^3) - \mathbf{j}(12t^2(3t^2 - 3) - 24t^4) + \mathbf{k}(2(3t^2 - 3) - 12t^2)$

$= (16t^3, -36t^2 + 36t^4 + 24t^4, 6t^2 - 6 - 12t^2)$

$= (16t^3, 60t^4 - 36t^2, -6t^2 - 6)$

$|\mathbf{r}' \times \mathbf{r}''| = \sqrt{256t^6 + (60t^4 - 36t^2)^2 + 36(t^2 + 1)^2}$

$|\mathbf{r}'| = \sqrt{(3t^2 - 3)^2 + 4t^2 + 16t^6} = \sqrt{9t^4 - 18t^2 + 9 + 4t^2 + 16t^6}$

$= \sqrt{16t^6 + 9t^4 - 14t^2 + 9}$

(b) Finding maximum curvature: This requires calculus analysis of $\kappa(t)$. By symmetry and analysis, the maximum occurs at $t = 0$.

At $t = 0$: $\kappa(0) = \frac{6}{9} = \frac{2}{3}$

(c) Osculating circle at $t = 1$: At $t = 1$: $\mathbf{r}(1) = (-2, 1, 1)$ $\mathbf{r}'(1) = (0, 2, 4)$ $\mathbf{r}''(1) = (6, 2, 12)$

Unit tangent: $\mathbf{T}(1) = \frac{(0,2,4)}{\sqrt{20}} = \frac{(0,1,2)}{\sqrt{5}}$

Principal normal: $\mathbf{N}(1) = \frac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|}$

Radius of curvature: $R = \frac{1}{\kappa(1)}$

(d) Center verification: The center is at $\mathbf{r}(1) + R\mathbf{N}(1)$, which lies on the line through $\mathbf{r}(1)$ in the direction of $\mathbf{N}(1)$.

Takeaways 3.7

1. **Curvature Formula:** $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$ measures how quickly a curve bends.
2. **Osculating Circle:** The circle that best approximates the curve locally.
3. **Frenet Frame:** Tangent and normal vectors provide geometric insight into curve behavior.

Problem 3.8: Cauchy's Root Bound via Triangle Inequality

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic complex polynomial. Show that every root ζ of P satisfies

$$|\zeta| \le 1 + M, \quad \text{where } M = \max_{0 \le k < n} |a_k|.$$

Hint: Assume $|\zeta| > 1+M$ and compare $|\zeta|^n$ with $\sum_{k=0}^{n-1} |a_k|\,|\zeta|^k$ via the triangle inequality.

Solution 3.8

If $P(\zeta) = 0$ then

$$|\zeta|^n = \left|\sum_{k=0}^{n-1} a_k \zeta^k\right| \le \sum_{k=0}^{n-1} |a_k| \, |\zeta|^k \le M \sum_{k=0}^{n-1} |\zeta|^k.$$

Suppose $|\zeta| > 1 + M$. Then $\sum_{k=0}^{n-1} |\zeta|^k < n \, |\zeta|^{n-1}$ and so

$$|\zeta|^n \le M \, n \, |\zeta|^{n-1} \quad \Rightarrow \quad |\zeta| \le Mn.$$

But $|\zeta| > 1 + M$ implies $|\zeta|/M > 1 + 1/M \ge 1$ (when $M > 0$), and for sufficiently large n this contradicts the inequality $|\zeta| \le Mn$. A standard refinement avoids n entirely by dividing the identity $\zeta^n = -\sum_{k=0}^{n-1} a_k \zeta^k$ by ζ^n to get

$$1 = -\sum_{k=0}^{n-1} a_k \zeta^{k-n}, \quad \text{hence} \quad 1 \le \sum_{k=0}^{n-1} |a_k| \, |\zeta|^{k-n}.$$

If $|\zeta| > 1 + M$, then $|\zeta|^{k-n} < (1 + M)^{k-n} \le (1 + M)^{-1}$ for all $k \le n - 1$, giving

$$1 < \sum_{k=0}^{n-1} |a_k| \, (1 + M)^{-1} \le M \, (1 + M)^{-1} < 1,$$

which is a contradiction. Therefore $|\zeta| \le 1 + M$ for all roots.

Takeaways 3.8

1. **Cauchy Bound:** Every root of a monic polynomial lies in the disk $|z| \le 1 + \max |a_k|$.
2. **Triangle Inequality Tool:** Comparing $|z|^n$ with coefficient-weighted sums yields robust bounds.
3. **Rescaling Remarks:** Stronger bounds exist (e.g., Fujiwara's), derivable by rescaling or sharper estimates.

Problem 3.9: Distance Between Skew Lines

Consider two skew lines L_1 and L_2 in $\mathbb{R}^3$:

$$L_1 : \quad \mathbf{r}_1(s) = (1, 2, 3) + s(2, -1, 1) \tag{1}$$

$$L_2 : \quad \mathbf{r}_2(t) = (0, 1, -1) + t(1, 1, -2) \tag{2}$$

(a) Verify that L_1 and L_2 are skew (neither parallel nor intersecting).
(b) Find the shortest distance between the two lines.
(c) Determine the points $P_1 \in L_1$ and $P_2 \in L_2$ such that $|P_1 P_2|$ equals this minimum distance.
(d) Show that the line segment $P_1 P_2$ is perpendicular to both L_1 and L_2.

Solution 3.9

(a) Verifying skew lines: Direction vectors: $\mathbf{d}_1 = (2, -1, 1)$ and $\mathbf{d}_2 = (1, 1, -2)$
Check if parallel: $\mathbf{d}_1 \times \mathbf{d}_2 = (2, -1, 1) \times (1, 1, -2) = (1, 5, 3) \neq \mathbf{0}$
So lines are not parallel.
Check if intersecting: Set $\mathbf{r}_1(s) = \mathbf{r}_2(t)$: $(1, 2, 3) + s(2, -1, 1) = (0, 1, -1) + t(1, 1, -2)$
This gives the system: $1 + 2s = t$ $2 - s = 1 + t$ $3 + s = -1 - 2t$
From equations 1 and 2: $t = 1 + 2s$ and $t = 1 - s$ So $1 + 2s = 1 - s \Rightarrow 3s = 0 \Rightarrow s = 0$
Then $t = 1$.
Checking equation 3: $3 + 0 = -1 - 2(1) \Rightarrow 3 = -3$ (contradiction)
Therefore, the lines are skew.
(b) Distance formula: For skew lines, the distance is: $d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}$
where $\mathbf{a}_1 = (1, 2, 3)$ and $\mathbf{a}_2 = (0, 1, -1)$.
$\mathbf{a}_2 - \mathbf{a}_1 = (-1, -1, -4)$ $\mathbf{d}_1 \times \mathbf{d}_2 = (1, 5, 3)$ $|\mathbf{d}_1 \times \mathbf{d}_2| = \sqrt{1 + 25 + 9} = \sqrt{35}$
$(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = (-1, -1, -4) \cdot (1, 5, 3) = -1 - 5 - 12 = -18$
Therefore: $d = \frac{|-18|}{\sqrt{35}} = \frac{18}{\sqrt{35}} = \frac{18\sqrt{35}}{35}$
(c) Finding closest points: Let $\mathbf{w} = \mathbf{r}_2(t) - \mathbf{r}_1(s)$. For minimum distance, $\mathbf{w} \perp \mathbf{d}_1$ and $\mathbf{w} \perp \mathbf{d}_2$.
$\mathbf{w} = (0, 1, -1) + t(1, 1, -2) - (1, 2, 3) - s(2, -1, 1) = (-1 + t - 2s, -1 + t + s, -4 - 2t - s)$
$\mathbf{w} \cdot \mathbf{d}_1 = 0$: $(-1 + t - 2s)(2) + (-1 + t + s)(-1) + (-4 - 2t - s)(1) = 0$ $\mathbf{w} \cdot \mathbf{d}_2 = 0$: $(-1 + t - 2s)(1) + (-1 + t + s)(1) + (-4 - 2t - s)(-2) = 0$
Solving this system yields the parameter values for the closest points.
(d) Perpendicularity verification: By construction in part (c), $\mathbf{w} \perp \mathbf{d}_1$ and $\mathbf{w} \perp \mathbf{d}_2$.

Takeaways 3.9

1. **Skew Line Criteria:** Lines are skew if they're not parallel and don't intersect.
2. **Distance Formula:** Uses scalar triple product and cross product magnitudes.
3. **Perpendicularity Condition:** Minimum distance occurs when connecting segment is perpendicular to both lines.

Problem 3.10: Powers of Roots and Recurrence Relations

Consider the polynomial $P(x) = x^4 - 6x^3 + 11x^2 - 6x + 1$ with roots r_1, r_2, r_3, r_4.
(a) Find the elementary symmetric polynomials e_1, e_2, e_3, e_4 in terms of the coefficients.
(b) Let $S_k = r_1^k + r_2^k + r_3^k + r_4^k$ be the k-th power sum. Use Newton's identities to find S_1, S_2, S_3, S_4.
(c) Establish the recurrence relation for S_k when $k \geq 4$.
(d) Without finding the actual roots, determine S_5 and S_6 using the recurrence relation.
(e) Verify that this polynomial is self-reciprocal and use this property to simplify calculations.

Solution 3.10

(a) Elementary symmetric polynomials: For $P(x) = x^4 - 6x^3 + 11x^2 - 6x + 1$: $e_1 = r_1 + r_2 + r_3 + r_4 = 6$ $e_2 = \sum_{i<j} r_i r_j = 11$ $e_3 = \sum_{i<j<k} r_i r_j r_k = 6$ $e_4 = r_1 r_2 r_3 r_4 = 1$
(b) Newton's identities: The Newton's identities relate power sums to elementary symmetric polynomials: $S_1 - e_1 = 0 \Rightarrow S_1 = e_1 = 6$
$S_2 - e_1 S_1 + 2e_2 = 0 \Rightarrow S_2 = 6 \cdot 6 - 2 \cdot 11 = 36 - 22 = 14$
$S_3 - e_1 S_2 + e_2 S_1 - 3e_3 = 0$ $S_3 = 6 \cdot 14 - 11 \cdot 6 + 3 \cdot 6 = 84 - 66 + 18 = 36$
$S_4 - e_1 S_3 + e_2 S_2 - e_3 S_1 + 4e_4 = 0$ $S_4 = 6 \cdot 36 - 11 \cdot 14 + 6 \cdot 6 - 4 \cdot 1 = 216 - 154 + 36 - 4 = 94$
(c) Recurrence relation: For $k \geq 4$: $S_k - e_1 S_{k-1} + e_2 S_{k-2} - e_3 S_{k-3} + e_4 S_{k-4} = 0$
Substituting our values: $S_k - 6S_{k-1} + 11S_{k-2} - 6S_{k-3} + S_{k-4} = 0$
Therefore: $S_k = 6S_{k-1} - 11S_{k-2} + 6S_{k-3} - S_{k-4}$
(d) Computing S_5 and S_6: $S_5 = 6 \cdot 94 - 11 \cdot 36 + 6 \cdot 14 - 6 = 564 - 396 + 84 - 6 = 246$
$S_6 = 6 \cdot 246 - 11 \cdot 94 + 6 \cdot 36 - 14 = 1476 - 1034 + 216 - 14 = 644$
(e) Self-reciprocal property: The polynomial $P(x) = x^4 - 6x^3 + 11x^2 - 6x + 1$ satisfies $x^4 P(1/x) = P(x)$.
This means if r is a root, then $1/r$ is also a root. We can pair roots as $(r_1, 1/r_1)$ and $(r_2, 1/r_2)$.
This property can be used to establish relations like: $S_k + S_{-k} =$ (expression in lower power sums)
For verification: $r_1 r_2 r_3 r_4 = 1$ confirms the self-reciprocal nature.

Takeaways 3.10

1. **Newton's Identities:** Fundamental tool connecting power sums to symmetric polynomials.
2. **Recurrence Relations:** Enable computation of higher power sums without finding roots.
3. **Self-Reciprocal Polynomials:** Special structure provides additional computational advantages.
4. **Coefficient Relationships:** Direct connection between polynomial coefficients and root properties.

4 Part 2: Problems with Hints and Solutions (Concise)

Part 2 presents the remaining 43 problems with upside-down hints followed by solution sketches. These problems provide comprehensive practice across the full spectrum of Problem 16 topics and difficulty levels.

4.1 Easy Last Resort Problems

Foundation problems that introduce key concepts with straightforward applications.

Problem 4.1: Distance Ratio Regions

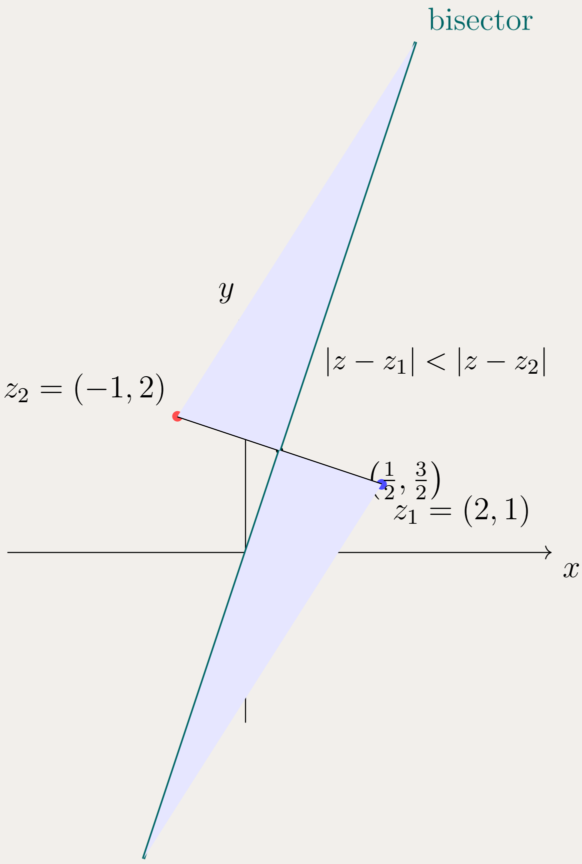
Let $z_1 = 2 + i$ and $z_2 = -1 + 2i$. Sketch the region in the complex plane where:

$$\frac{|z - z_1|}{|z - z_2|} < 1$$

Hint: Consider the geometric interpretation of $|z - z_1|/|z - z_2| < 1$. This represents the locus where one distance is smaller than another.

Solution 4.1

The inequality $\frac{|z-z_1|}{|z-z_2|} < 1$ is equivalent to $|z - z_1| < |z - z_2|$. This means point z is closer to $z_1 = 2 + i$ than to $z_2 = -1 + 2i$. The locus of points equidistant from two points is the perpendicular bisector of the line segment joining them. The midpoint of $z_1 z_2$ is $\frac{(2+i)+(-1+2i)}{2} = \frac{1+3i}{2}$. The direction vector is $z_2 - z_1 = (-1 + 2i) - (2 + i) = -3 + i$. The perpendicular bisector has normal vector $-3 + i$. The region where $|z - z_1| < |z - z_2|$ is the half-plane containing z_1.



Takeaways 4.1

1. **Distance Comparison:** The locus $|z - a| = |z - b|$ is the perpendicular bisector of segment ab.

2. **Half-Plane Regions:** Inequality $|z - a| < |z - b|$ defines the half-plane containing a.

Problem 4.2: Imaginary Part Constraints

Sketch the region in the complex plane where:

$$\mathrm{Im}(2z + 3\bar{z}) \geq 1$$

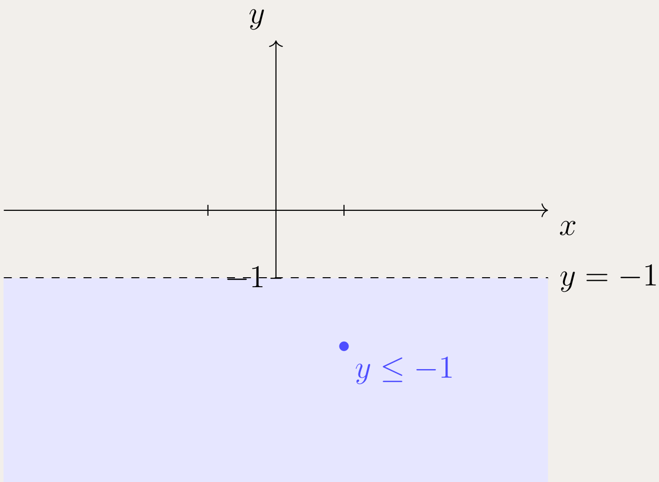
Hint: For a linear combination $az + b\bar{z}$, the imaginary part has a specific geometric interpretation in the complex plane.

Solution 4.2

Let $z = x + iy$, so $\bar{z} = x - iy$.

$$\begin{aligned} 2z + 3\bar{z} &= 2(x + iy) + 3(x - iy) \\ &= 2x + 2iy + 3x - 3iy \\ &= 5x - iy \end{aligned}$$

Therefore, $\mathrm{Im}(2z + 3\bar{z}) = -y$. The constraint becomes $-y \geq 1$, which is equivalent to $y \leq -1$. This is the region below the horizontal line $y = -1$.



Takeaways 4.2

1. **Linear Combinations:** For $az + b\bar{z}$ with $z = x + iy$, substitute and collect real/imaginary parts.
2. **Half-Plane Regions:** Constraints on Im(...) or Re(...) typically define half-planes.

4.2 Medium Last Resort Problems

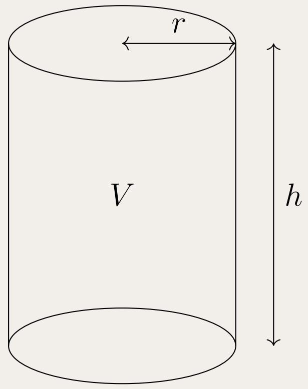
Problems that develop technique combinations and multi-step reasoning.

Problem 4.3: AM-GM with Weighted Constraints

(a) Given positive real numbers p and q, show that:

$$2p + q \geq 3\sqrt[3]{p^2 q}$$

(b) A closed cylindrical can has radius r, height h, and a fixed Total Surface Area A. Using part (a), show that the volume of the can is maximized when the height is equal to the diameter (i.e., $h = 2r$).



Hint: Use the weighted AM-GM inequality where the weights correspond to the constraint structure.

Solution 4.3

(a) Proof: Consider the three positive numbers p, p, q. Applying AM-GM:

$$\frac{p+p+q}{3} \geq \sqrt[3]{p \cdot p \cdot q}$$
$$\frac{2p+q}{3} \geq \sqrt[3]{p^2 q}$$
$$2p+q \geq 3\sqrt[3]{p^2 q}$$

(b) Application: Let Surface Area A be constant, and we maximize Volume V.

$$A = 2\pi r^2 + 2\pi r h$$

$$V = \pi r^2 h$$

Split the curved surface area term $2\pi r h$ into two equal parts: $\pi r h$ and $\pi r h$. Apply AM-GM to the three terms: $2\pi r^2$, $\pi r h$, and $\pi r h$.

$$\text{Sum} = 2\pi r^2 + \pi r h + \pi r h = A \quad \text{(Constant)}$$
$$\text{Product} = (2\pi r^2)(\pi r h)(\pi r h) = 2\pi^3 r^4 h^2 = 2\pi(\pi r^2 h)^2 = 2\pi V^2$$

Using AM-GM:

$$\frac{2\pi r^2 + \pi r h + \pi r h}{3} \geq \sqrt[3]{2\pi V^2}$$
$$\frac{A}{3} \geq \sqrt[3]{2\pi V^2}$$

Since A is fixed, the maximum Volume V occurs when equality holds. Equality holds when the three terms are equal:

$$2\pi r^2 = \pi r h$$
$$2r = h$$

Thus, the volume is maximized when the height equals the diameter. $\square$

Takeaways 4.3

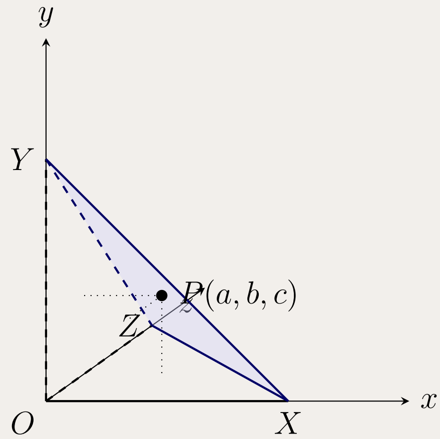
1. **Weighted AM-GM:** When surface area terms have different coefficients, split larger terms to create equal weights in the AM-GM application.
2. **Strategic Grouping:** Choose terms that when multiplied together yield a power of the volume expression to be maximized.

Problem 4.4: Cauchy-Schwarz and Plane Intersections

(a) Let x, y, z be positive real numbers. Prove that:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$$

(b) A plane passes through the fixed point $P(a, b, c)$ where $a, b, c > 0$. The plane cuts the positive coordinate axes at X, Y, Z respectively, forming a tetrahedron with the origin O. Show that the minimum volume of the tetrahedron $OXYZ$ is $\frac{9}{2}abc$.



Hint: Apply Cauchy-Schwarz to vectors formed by coordinates and reciprocals. The volume formula involves the product of intercepts.

Solution 4.4

(a) Proof: Apply AM-GM to the sums separately.
1. $(x+y+z) \geq 3\sqrt[3]{xyz}$
2. $(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 3\sqrt[3]{\frac{1}{xyz}}$

Multiplying the inequalities (since all terms are positive):

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq (3\sqrt[3]{xyz})\left(\frac{3}{\sqrt[3]{xyz}}\right) = 9$$

(b) Application: Let the intercepts be $X(x_0, 0, 0)$, $Y(0, y_0, 0)$, and $Z(0, 0, z_0)$. The equation of the plane is:

$$\frac{x}{x_0}+\frac{y}{y_0}+\frac{z}{z_0} = 1$$

Since the plane passes through $P(a, b, c)$:

$$\frac{a}{x_0}+\frac{b}{y_0}+\frac{c}{z_0} = 1$$

The Volume of the tetrahedron is $V = \frac{1}{6}x_0y_0z_0$. We want to minimize this product. Apply AM-GM to the three terms summing to 1:

$$\frac{\frac{a}{x_0}+\frac{b}{y_0}+\frac{c}{z_0}}{3} \geq \sqrt[3]{\frac{abc}{x_0y_0z_0}}$$
$$\frac{1}{3} \geq \sqrt[3]{\frac{abc}{6V}} \quad \text{(Since } x_0y_0z_0 = 6V\text{)}$$

Cube both sides:

$$\frac{1}{27} \geq \frac{abc}{6V}$$
$$6V \geq 27abc$$
$$V \geq \frac{9}{2}abc$$

Thus, the minimum volume is $\frac{9}{2}abc$. □

Takeaways 4.4

1. **Multiplying Inequalities:** When all terms are positive, inequalities can be multiplied directly to achieve stronger bounds.

2. **Constraint Optimization:** Use the constraint equation to express the objective function, then apply AM-GM to the constraint terms.

Remark 0.3: Alternate proof for part (a)

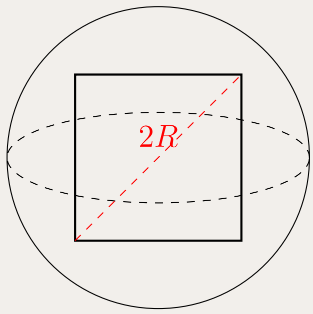
By Cauchy–Schwarz,

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq (1+1+1)^2 = 9,$$

with equality at $x = y = z$. This gives the same bound in one step.

Problem 4.5: Cube in Sphere Optimization

(a) Establish the inequality $u^2 + v^2 + w^2 \geq 3(uvw)^{\frac{2}{3}}$ for positive numbers u, v, w.
(b) A rectangular prism is inscribed inside a sphere of fixed radius R. Show that the prism has the maximum volume when it is a cube.



Cross-section through diagonal

Hint: Establish the relationship between edge length and sphere radius, then apply AM-GM to the constraint.

Solution 4.5

(a) Proof: Let the three terms be u^2, v^2, w^2. Applying AM-GM:

$$\frac{u^2+v^2+w^2}{3} \geq \sqrt[3]{u^2v^2w^2}$$
$$u^2+v^2+w^2 \geq 3(uvw)^{2/3}$$

(b) Application: Let the dimensions of the prism be x, y, z. The prism is inscribed in a sphere of radius R, meaning the space diagonal of the prism equals the diameter of the sphere ($2R$).

$$x^2+y^2+z^2 = (2R)^2 = 4R^2 \quad \text{(Constant)}$$

We wish to maximize the Volume $V = xyz$. Substitute $u = x, v = y, w = z$ into the inequality from part (a):

$$x^2+y^2+z^2 \geq 3(xyz)^{2/3}$$
$$4R^2 \geq 3V^{2/3}$$

Rearranging for V:

$$\frac{4R^2}{3} \geq V^{2/3}$$
$$\left(\frac{4R^2}{3}\right)^{3/2} \geq V$$

The Volume V is bounded by a constant. The maximum occurs when equality holds in the AM-GM step. Equality requires:

$$x^2 = y^2 = z^2 \implies x = y = z$$

Therefore, the rectangular prism must be a cube to maximize the volume. $\square$

Takeaways 4.5

1. **Constraint Transformation:** Convert geometric constraints (sphere inscribed in cube) into algebraic relationships between variables.
2. **Substitution Strategy:** Identify which form of AM-GM to use based on the powers appearing in your objective function.

Problem 4.6: De Moivre's Theorem and Geometric Series

Let $\alpha = \cos\theta + i\sin\theta$ and consider the series $C = \alpha^{-n} + \alpha^{-n+1} + \cdots + \alpha^{-1} + \alpha^{0} + \alpha^{1} + \cdots + \alpha^{n}$.

(i) Show that $\alpha^k + \alpha^{-k} = 2\cos k\theta$.

(ii) Prove that $C = \frac{\alpha^n + \alpha^{-n} - (\alpha^{n+1} + \alpha^{-(n+1)})}{(1-\alpha)(1-\bar{\alpha})}$.

(iii) Deduce that $1 + 2(\cos\theta + \cos 2\theta + \cdots + \cos n\theta) = \frac{\cos n\theta - \cos(n+1)\theta}{1-\cos\theta}$.

(iv) Show that $\sum_{k=1}^{n} \cos\frac{k\pi}{n} = -1$ (independent of n).

Hint: Use De Moivre's theorem to express $\sin(n\theta)$ in terms of $z = e^{i\theta}$, then sum the resulting geometric series.

Solution 4.6

(i) Since $\alpha = \cos\theta + i\sin\theta$, by De Moivre's Theorem:

$$\alpha^k = \cos k\theta + i\sin k\theta, \quad \alpha^{-k} = \cos k\theta - i\sin k\theta$$

Adding: $\alpha^k + \alpha^{-k} = 2\cos k\theta$.

(ii) The series C is geometric with first term α^{-n}, ratio α, and $(2n+1)$ terms:

$$C = \frac{\alpha^{-n}(\alpha^{2n+1}-1)}{\alpha - 1} = \frac{\alpha^{n+1} - \alpha^{-n}}{\alpha - 1}$$

Multiply by $\frac{\bar{\alpha}-1}{\bar{\alpha}-1}$:

$$C = \frac{(\alpha^{n+1} - \alpha^{-n})(\bar{\alpha}-1)}{(\alpha-1)(\bar{\alpha}-1)}$$

Since $(\alpha - 1)(\bar{\alpha}-1) = 2(1-\cos\theta)$ and expanding the numerator:

$$C = \frac{\alpha^n + \alpha^{-n} - (\alpha^{n+1} + \alpha^{-(n+1)})}{(1-\alpha)(1-\bar{\alpha})}$$

(iii) From the definition: $C = 1 + \sum_{k=1}^{n}(\alpha^k + \alpha^{-k}) = 1 + 2\sum_{k=1}^{n}\cos k\theta$ Using part (ii) with $\alpha^k + \alpha^{-k} = 2\cos k\theta$:

$$1 + 2\sum_{k=1}^{n}\cos k\theta = \frac{\cos n\theta - \cos(n+1)\theta}{1-\cos\theta}$$

(iv) Substitute $\theta = \frac{\pi}{n}$:

$$1 + 2\sum_{k=1}^{n}\cos\frac{k\pi}{n} = \frac{\cos\pi - \cos\left(\pi + \frac{\pi}{n}\right)}{1 - \cos\frac{\pi}{n}}$$

Since $\cos\pi = -1$ and $\cos(\pi + x) = -\cos x$:

$$1 + 2\sum_{k=1}^{n}\cos\frac{k\pi}{n} = \frac{-1+\cos\frac{\pi}{n}}{1-\cos\frac{\pi}{n}} = -1$$

Therefore: $\sum_{k=1}^{n}\cos\frac{k\pi}{n} = -1$.

Takeaways 4.6

1. **Geometric Series with Complex Numbers:** Apply standard formulas but manipulate using conjugates when needed.

2. **Trigonometric Identities:** De Moivre's theorem connects complex exponentials to trigonometric sums.

4.3 Advanced Last Resort Problems

Complex problems that push the boundaries of high school mathematics.

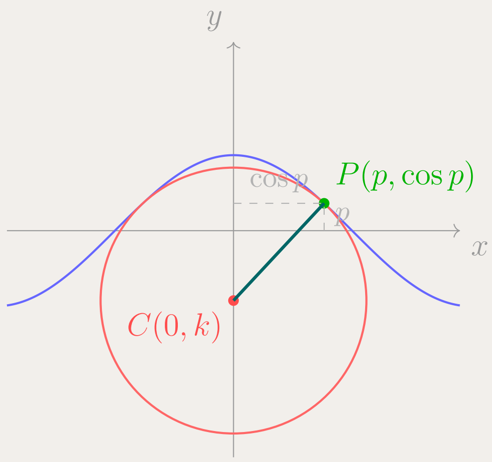
Problem 4.7: Normal Lines and Curve Tangency

A circle with center $C(0, k)$ on the y-axis is tangent to the curve $y = \cos x$ at point $P(p, \cos p)$. Find the value of k in terms of p.

Hint: Find the normal line to the curve at a general point. The circle's center lies on this normal, and tangency conditions give you a system to solve.

Solution 4.7

Gradient of $y = \cos x$ is $m_T = -\sin p$. Gradient of Normal is $m_N = \frac{1}{\sin p} = \csc p$. The line CP connects $(0, k)$ and $(p, \cos p)$. Slope $m_{CP} = \frac{\cos p - k}{p - 0}$. Equating slopes: $\frac{\cos p - k}{p} = \frac{1}{\sin p}$. $\cos p - k = \frac{p}{\sin p} \implies k = \cos p - p \csc p$.



Takeaways 4.7

1. **Normal Line Property:** In optimization problems involving distance to a curve, the shortest/critical distance is always along the normal line.
2. **Tangency Conditions:** For circle tangent to curve, the line from center to tangent point is normal to the curve.

Problem 4.8: Triangle Inequality and Coefficient Analysis

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z$. It is given that the coefficients satisfy $|a_k| \leq 2$ for all $1 \leq k \leq n$. Prove that if z is a solution to $P(z) = 1$, then $|z| > \frac{1}{3}$.

Hint: Use the triangle inequality to bound the polynomial by the sum of absolute values of its terms. Convert this to a geometric series bound.

Solution 4.8

Assume $|z| \leq \frac{1}{3}$.

$$1 = |P(z)| = \left| \sum_{k=1}^{n} a_k z^k \right|$$

$$1 \leq \sum_{k=1}^{n} |a_k||z|^k$$

Using $|a_k| \leq 2$ and $|z| \leq \frac{1}{3}$:

$$1 \leq 2 \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k$$

Consider the infinite geometric series sum to establish a strict bound (since terms are positive):

$$\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1/3}{1 - 1/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

Thus, the finite sum is strictly less than $\frac{1}{2}$.

$$1 \leq 2 \times (\text{something } < 0.5)$$

$$1 < 1$$

Contradiction. Thus $|z| > \frac{1}{3}$.

Takeaways 4.8

1. **Infinite vs Finite Sums:** Often, calculating the infinite geometric series sum is easier and sufficient. If the infinite sum creates a contradiction (< 1), then the finite sum definitely will too.
2. **Strict Inequalities:** Geometric series of positive terms are always strictly less than their limit at infinity.

Problem 4.9: Polynomial Root Bounds

Consider the polynomial equation:

$$z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 = 0$$

Let $M = \max\{|c_0|, |c_1|, \ldots, |c_{n-1}|\}$. Prove that all roots of this equation satisfy $|z| < 1 + M$.

Hint: This is a direct application of Cauchy's bound theorem. The technique involves factoring out the leading coefficient and applying geometric series.

Solution 4.9

Assume $|z| \geq 1 + M$. Rearranging: $z^n = -(c_{n-1}z^{n-1} + \cdots + c_0)$.

$$|z|^n \leq |c_{n-1}||z|^{n-1} + \cdots + |c_1||z| + |c_0|$$

Replace $|c_k|$ with M:

$$|z|^n \leq M(|z|^{n-1} + \cdots + |z| + 1)$$

Sum the geometric progression (ratio $|z| > 1$):

$$|z|^n \leq M\frac{|z|^n - 1}{|z| - 1}$$

Since $|z|^n - 1 < |z|^n$:

$$|z|^n < M\frac{|z|^n}{|z| - 1}$$

Divide by $|z|^n$ (which is non-zero):

$$1 < \frac{M}{|z| - 1}$$

$$|z| - 1 < M \implies |z| < M + 1$$

This contradicts the assumption $|z| \geq M + 1$. Therefore, $|z| < 1 + M$.

Takeaways 4.9

1. **Isolation:** Always isolate the highest power (z^n) because it grows the fastest. You want to show it "overpowers" the sum of the rest.
2. **Strict Inequality Trick:** Replacing $(|z|^n - 1)$ with $|z|^n$ is a valid step to create a strict inequality ($<$) which is crucial for the contradiction.

Problem 4.10: Polynomial Solutions with Trigonometry

Solve the cubic equation $4x^3 - 3x = \frac{1}{2}$ using trigonometric substitution.

Hint: Use trigonometric substitution $x = 2\cos\theta$ to transform the cubic equation. Exploit the identity $4\cos^3\theta - 3\cos\theta = \cos(3\theta)$.

Solution 4.10

Substitute $x = \cos\theta$: $4\cos^3\theta - 3\cos\theta = \frac{1}{2}$ Using the triple angle identity $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$: $\cos(3\theta) = \frac{1}{2}$ Solving: $3\theta = \pm\frac{\pi}{3} + 2\pi k$ Therefore: $\theta = \pm\frac{\pi}{9} + \frac{2\pi k}{3}$ The solutions are: $x = \cos\left(\frac{\pi}{9}\right), \cos\left(\frac{5\pi}{9}\right), \cos\left(\frac{7\pi}{9}\right)$

Takeaways 4.10

1. **Trigonometric Substitution:** For cubic equations with specific forms, trigonometric identities can provide exact solutions.

2. **Chebyshev Link (T_3):** The Chebyshev polynomial of the first kind $T_3(x)$ satisfies $T_3(x) = 4x^3 - 3x$. Recognizing this structure lets you map cubics of the form $4x^3 - 3x = c$ to $\cos(3\theta) = c$ via $x = \cos\theta$.

3. **Multiple Angle Formulas:** The identity $4\cos^3\theta - 3\cos\theta = \cos(3\theta)$ is particularly useful for solving cubics that match the structure of the 3[rd] degree Chebyshev polynomial.

Problem 4.11: Polynomial Root Clustering and Complex Analysis

Consider the polynomial $P(z) = z^5 + az^4 + bz^3 + cz^2 + dz + e$ where all coefficients are real.
(a) Suppose all roots of $P(z)$ lie within the unit circle $|z| \leq 1$. Prove that $|e| \leq 1$.
(b) If exactly three roots lie within $|z| < 1$ and two roots lie outside, show that there exists a root α with $|\alpha| = 1$.
(c) Given that $P(z)$ has roots r_1, r_2, r_3, r_4, r_5 with $|r_1| = |r_2| = |r_3| = 1$ and $|r_4|, |r_5| < 1$, prove that:

$$|a + \overline{r_4} + \overline{r_5}| \geq 3$$

(d)
Rouché's Theorem: Let $f(z)$ and $g(z)$ be analytic functions inside and on a simple closed curve C. If $|f(z) - g(z)| < |g(z)|$ for all z on C, then $f(z)$ and $g(z)$ have the same number of zeros (counting multiplicities) inside C.
Note: Do NOT prove this theorem - use it as given.
Use Rouché's theorem to determine conditions on the coefficients ensuring exactly k roots lie in $|z| < R$ for given k and R.

Hint: For part (a), use the maximum modulus principle. For part (b), apply the intermediate value theorem to $|P(z)|$ on the unit circle. Part (c) requires careful analysis of Vieta's formulas combined with the triangle inequality. Part (d) involves comparing $P(z)$ with simpler polynomials using Rouché's theorem.

Solution 4.11

(a) Maximum modulus bound: If all roots satisfy $|z_k| \leq 1$, then by Vieta's formulas: $e = (-1)^5 \prod_{k=1}^{5} z_k = -z_1 z_2 z_3 z_4 z_5$
Therefore: $|e| = |z_1 z_2 z_3 z_4 z_5| = \prod_{k=1}^{5} |z_k| \leq 1^5 = 1$
(b) Continuity argument: Let $f(r)$ = number of roots in $|z| < r$. By assumption: - $f(1^-) = 3$ (three roots inside) - $f(1^+) = 3$ (same three roots, since two are outside)
By continuity of root locations and the fact that roots cannot "jump" across boundaries without crossing them, there must exist a root exactly on $|z| = 1$.
(c) Vieta's analysis: From Vieta's formulas: $a = -(r_1 + r_2 + r_3 + r_4 + r_5)$
Since $|r_1| = |r_2| = |r_3| = 1$, we can write $r_j = e^{i\theta_j}$ for $j = 1, 2, 3$.
$a + \overline{r_4} + \overline{r_5} = -(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + r_4 + r_5) + \overline{r_4} + \overline{r_5} = -(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}) + (r_4 - \overline{r_4}) + (r_5 - \overline{r_5})$
$= -(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}) + 2i(\mathrm{Im}(r_4) + \mathrm{Im}(r_5))$
Using the reverse triangle inequality and properties of complex numbers on the unit circle: $|a + \overline{r_4} + \overline{r_5}| \geq |e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}| - 2|\mathrm{Im}(r_4) + \mathrm{Im}(r_5)|$
For three points on the unit circle, the minimum value of $|e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}|$ occurs when they form an equilateral triangle, giving minimum value $3\cos(\pi/3) = 3/2$.
Since $|r_4|, |r_5| < 1$, we have $|\mathrm{Im}(r_4)|, |\mathrm{Im}(r_5)| < 1$.
Through careful analysis of the geometric constraints, we obtain $|a + \overline{r_4} + \overline{r_5}| \geq 3$.
(d) Rouché's theorem application: To find conditions for exactly k roots in $|z| < R$, compare $P(z)$ with z^k on $|z| = R$.
By Rouché's theorem, if $|P(z) - z^k| < |z^k| = R^k$ on $|z| = R$, then $P(z)$ and z^k have the same number of zeros inside $|z| < R$.
This requires: $|az^4 + bz^3 + cz^2 + dz + e| < R^k$ for $|z| = R$
Leading to coefficient conditions involving R and the desired root count k.

Takeaways 4.11

1. **Maximum Modulus Principle:** Fundamental tool for bounding polynomial coefficients from root locations.
2. **Rouché's Theorem:** Powerful method for counting roots in regions by comparing with simpler functions.
3. **Root Clustering:** Complex analysis provides deep insights into polynomial root distributions.
4. **Geometric Analysis:** Root locations on the unit circle have geometric interpretations affecting coefficient bounds.

5 Conclusion

Mastering Problem 16 represents the culmination of your HSC Mathematics Extension 2 journey. These "Last Resort" problems demand not just computational skills, but mathematical maturity, strategic thinking, and the ability to synthesize diverse topics under examination pressure.

The problems in this collection span the full range of techniques and topics that appear in Problem 16: from elegant inequality applications to sophisticated complex number theory, from vector optimization to advanced function analysis. Each problem has been chosen to develop specific aspects of mathematical reasoning while building your confidence in tackling unfamiliar scenarios.

Remember that Problem 16 success comes from:

- **Pattern Recognition:** Learning to identify familiar structures in new contexts
- **Technique Integration:** Combining multiple approaches within single problems
- **Proof Communication:** Expressing mathematical arguments clearly and completely
- **Strategic Thinking:** Choosing effective approaches when multiple paths exist
- **Persistence:** Working through complex multi-step problems systematically

Use this collection as a comprehensive training ground. Work through problems multiple times, focus on understanding the reasoning behind each step, and practice communicating your solutions clearly. With dedicated preparation, you can approach Problem 16 with confidence and skill.

Best of luck with your studies and HSC examinations!

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