HSC Math Extension 2: Last Resorts Mastery

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Contents

1 Introduction

1.1 Project Overview

This booklet compiles the most challenging problems from HSC Mathematics Extension 2 examinations—the notorious Problem 16, as we call them "Last Resort." These problems represent the pinnacle of high school mathematics, typically combining multiple advanced topics and requiring sophisticated problem-solving techniques. Every problem showcases the depth and complexity expected at the highest level of the HSC Mathematics curriculum.

The collection draws from authentic examination problems that have challenged Extension 2 students over many years. Each problem has been carefully selected to represent the diverse range of mathematical concepts and proof techniques that appear in Problem 16, from advanced inequalities and complex number theory to vector optimization and differential geometry applications.

1.2 What Makes Problem 16 Special

Problem 16 is uniquely challenging because it:

- Combines multiple mathematical topics within a single problem
- Requires advanced proof techniques and mathematical maturity
- Often involves multi-part questions that build complexity progressively
- Tests both computational skills and conceptual understanding
- Demands clear mathematical communication and rigorous reasoning

1.3 Target Audience

This booklet is designed for Extension 2 students who want to:

- Master the most challenging problems in the HSC curriculum
- Develop advanced problem-solving strategies
- Build confidence in tackling complex multi-step problems
- Understand how to approach unfamiliar mathematical scenarios
- Prepare thoroughly for the demands of Problem 16 in examinations

1.4 How to Use This Booklet

- **Study the Fundamentals:** Review key theorems and techniques
- **Attempt Problems Independently:** Try each problem without looking at solutions first
- **Part 1 Strategy:** Compare your work against solutions to understand model reasoning
- **Part 2 Approach:** Use upside-down hints sparingly, then review concise solutions
- **Practice Multiple Times:** Rework problems from memory to build technique mastery
- **Focus on Communication:** Pay attention to how mathematical arguments are structured and presented

2 Fundamentals Review

2.1 Key Inequalities

Problem 16 frequently employs fundamental inequalities as building blocks for more complex arguments.

2.1.1 Arithmetic-Geometric Mean (AM-GM) Inequality

Theorem 0.1: AM-GM Inequality

For positive real numbers $a_1, a_2, \ldots, a_n$:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

Theorem 0.2: Weighted AM-GM Inequality

For positive real numbers $a_1, a_2, \ldots, a_n$ and positive weights $w_1, w_2, \ldots, w_n$ with $w_1 + w_2 + \cdots + w_n = 1$:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}$$

2.1.2 Cauchy-Schwarz Inequality

Theorem 0.3: Cauchy-Schwarz Inequality

For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$:

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$$

Equivalently, for real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

2.2 Complex Number Theory

2.2.1 De Moivre's Theorem

Theorem 0.4: De Moivre's Theorem

For complex numbers in polar form $z = r(\cos\theta + i\sin\theta)$:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

2.2.2 Roots of Unity

Theorem 0.5: Roots of Unity

The nth roots of unity are given by:

$$\omega_k = e^{2\pi ik/n} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

for $k = 0, 1, 2, \ldots, n-1$.

2.3 Vector Algebra

2.3.1 Dot Product

For vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$$

2.3.2 Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The magnitude $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$ equals the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

2.4 Polynomial Theory

2.4.1 Newton's Identities

Theorem 0.6: Newton's Identities

For a polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, if $S_k = \sum_{i=1}^n \alpha_i^k$ and e_k are the elementary symmetric polynomials, then:

$$S_k - e_1S_{k-1} + e_2S_{k-2} - \cdots + (-1)^{k-1}e_{k-1}S_1 + (-1)^k ke_k = 0$$

2.5 Chebyshev Polynomials (First Kind)

The Chebyshev polynomials of the first kind $T_n(x)$ are a family of orthogonal polynomials on $[-1, 1]$ with the defining trigonometric identity:

$$T_n(x) = \cos\big(n \arccos x\big).$$

2.5.1 Recurrence Formula

They satisfy the three-term recurrence for $n \geq 1$:

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x), \quad T_0(x) = 1,\ T_1(x) = x.$$

2.5.2 Explicit Low-Degree Polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x.$$

These identities are frequently useful for transforming certain cubic equations by matching the structure of $T_3(x)$.

2.6 Chebyshev Polynomials (Second Kind)

The Chebyshev polynomials of the second kind $U_n(x)$ form another family of orthogonal polynomials on $[-1, 1]$, closely related to the first kind. They are defined by the trigonometric identity:

$$U_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{\sin(\arccos x)}.$$

Equivalently, if $x = \cos\theta$, then:

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

2.6.1 Recurrence Formula

The second kind Chebyshev polynomials satisfy the same three-term recurrence as the first kind, but with different initial conditions:

$$U_{n+1}(x) = 2x\, U_n(x) - U_{n-1}(x), \quad U_0(x) = 1,\ U_1(x) = 2x.$$

2.6.2 Explicit Low-Degree Polynomials

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x.$$

2.6.3 Relationship to First Kind

The Chebyshev polynomials of the first and second kind are related by:

$$U_n(x) = \frac{T'_{n+1}(x)}{n+1}, \quad T_n(x) = U_n(x) - x\, U_{n-1}(x).$$

Additionally, they satisfy:

$$T_n(x) = \cos(n \arccos x) \quad \text{and} \quad U_n(x) = \frac{\sin((n+1)\arccos x)}{\sqrt{1-x^2}}.$$

2.6.4 Key Properties

- $T_n(x)$ has the minimal leading coefficient among all monic polynomials of degree n bounded by 1 on $[-1, 1]$.
- The zeros of $U_n(x)$ are simple and lie in the interior of $(-1, 1)$.
- Both families play a crucial role in approximation theory and the study of polynomial bounds.

These polynomials appear frequently in Problem 16 when dealing with bounded polynomial coefficients and optimal approximations.

2.7 Optimization Techniques

2.7.1 Lagrange Multipliers (Conceptual)

To optimize $f(x, y, z)$ subject to constraint $g(x, y, z) = 0$, look for points where:

$$\nabla f = \lambda \nabla g$$

This often reduces to applying AM-GM or Cauchy-Schwarz in HSC problems.

Here, ∇f and ∇g are the gradients of f and g, respectively, and λ is a scalar known as the Lagrange multiplier.

The gradient is defined as:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

Good to know though the technique is not covered at HSC level.

Here is how you could use it to attack the problems:

- Guess when the max/min occurs: Use the condition $\nabla f = \lambda \nabla g$ to solve for the variables (often resulting in $x = y = z$ for symmetric expressions).
- Identify the "Target Inequality": Once you know the values that achieve the optimum (e.g., $x = 2, y = 2$), you know exactly what the right-hand side of your inequality should look like.
- Reverse-engineer the formal proof: Work backward from that optimum using standard HSC tools—most commonly by setting up AM-GM terms or choosing the vectors for Cauchy-Schwarz that align with your discovered ratios.

2.8 Notation and Conventions

- Complex numbers are denoted by $z, w, \alpha, \beta, \ldots$
- Vectors are written in bold: $\mathbf{u}, \mathbf{v}, \mathbf{a}, \mathbf{b}$
- Real numbers are assumed unless stated otherwise
- $\mathbb{R}^n$ denotes n-dimensional real space
- $|z|$ denotes the modulus of complex number z
- $|\mathbf{v}|$ denotes the magnitude of vector $\mathbf{v}$

3 Part 1: Problems and Solutions (Detailed)

Part 1 contains 16 carefully selected problems representing the core techniques and challenge levels found in Problem 16. Each solution includes a strategy explanation, complete step-by-step working, and takeaway insights to build your problem-solving toolkit.

3.1 Medium Last Resort Problems

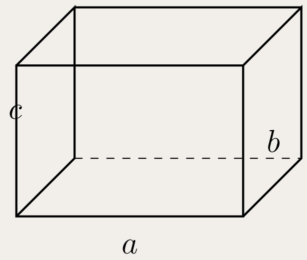
Problems that combine 2–3 mathematical topics with sophisticated reasoning.

Problem 3.1: AM-GM Surface Area Optimization

(a) Prove that for positive real numbers x, y, z:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{3}{\sqrt[3]{xyz}}$$

(b) A rectangular prism has dimensions a, b, c and a fixed, constant Volume V. Using part (a), show that the Total Surface Area S is minimized when the prism is a cube.



Hint: For part (a), apply AM-GM inequality directly to the reciprocals $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$. For part (b), express the surface area $S = 2(ab + bc + ca)$ in terms of the volume constraint $V = abc$, then use part (a) with strategic substitution.

Solution 3.1

(a) Proof: Since $x, y, z > 0$, then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} > 0$. Applying the AM-GM inequality to these three terms:

$$\frac{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}{3} \geq \sqrt[3]{\frac{1}{x} \cdot \frac{1}{y} \cdot \frac{1}{z}}$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3\sqrt[3]{\frac{1}{xyz}}$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{3}{\sqrt[3]{xyz}}$$

(b) Application: Let the dimensions be a, b, c.

- Fixed Volume: $V = abc$ (constant).
- Surface Area: $S = 2(ab + bc + ca)$.

We want to relate S to the reciprocals in part (a).

$$S = 2abc\left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b}\right)$$
$$S = 2V\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

From part (a), we know $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \frac{3}{\sqrt[3]{abc}}$. Substituting this into the expression for S:

$$S \geq 2V\left(\frac{3}{\sqrt[3]{V}}\right)$$
$$S \geq 6V^1V^{-1/3}$$
$$S \geq 6V^{2/3}$$

Since V is constant, $6V^{2/3}$ is a constant minimum value. Equality holds (minimum S occurs) when the terms in the AM-GM are equal:

$$\frac{1}{a} = \frac{1}{b} = \frac{1}{c} \implies a = b = c$$

Therefore, the prism is a cube. $\square$

Takeaways 3.1

1. **Infinite vs Finite Sums:** Often, calculating the infinite geometric series sum is easier and sufficient. If the infinite sum creates a contradiction (< 1), then the finite sum definitely will too.
2. **Strict Inequalities:** Geometric series of positive terms are always strictly less than their limit at infinity.

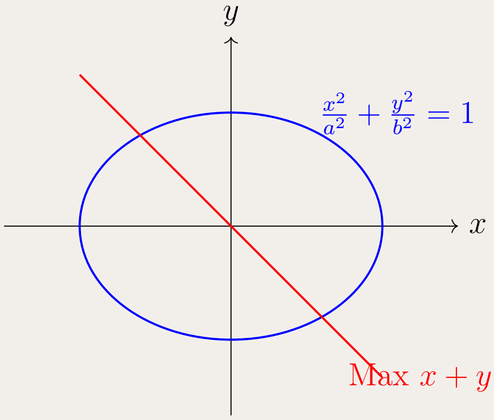
Problem 3.2: Cauchy-Schwarz on Ellipsoid

The point $P(x, y, z)$ lies on the surface of the ellipsoid defined by:

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$$

(i) By choosing appropriate vectors $\mathbf{u}$ and $\mathbf{v}$ and applying the Cauchy-Schwarz inequality ($\mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u}||\mathbf{v}|$), find the maximum value of $x + y + z$.

(ii) Find the coordinates of P in the first octant where this maximum occurs.



Hint:

The standard Cauchy-Schwarz inequality is $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. You cannot define $\mathbf{u} = (x, y, z)$ directly because the sum of squares is not 1. Instead, define $\mathbf{u}$ such that $|\mathbf{u}|^2$ exactly matches the left-hand side of the ellipsoid equation. Then, choose a constant vector $\mathbf{v}$ such that the dot product $\mathbf{u} \cdot \mathbf{v}$ recovers the expression $x + y + z$.

Solution 3.2

(i) Let $\mathbf{u} = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{3} \\ \frac{z}{5} \end{pmatrix}$. We are given $|\mathbf{u}|^2 = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$, so $|\mathbf{u}| = 1$. We want to maximize $x + y + z$. We observe:

$$x + y + z = \frac{x}{2}(2) + \frac{y}{3}(3) + \frac{z}{5}(5)$$

This suggests setting $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$. Apply Cauchy-Schwarz:

$$\mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u}||\mathbf{v}|$$

$$x + y + z \leq (1)\sqrt{2^2 + 3^2 + 5^2}$$
$$x + y + z \leq \sqrt{4 + 9 + 25} = \sqrt{38}$$

(ii) Equality occurs when $\mathbf{u} = k\mathbf{v}$, i.e., $\frac{x}{2} = 2k, \frac{y}{3} = 3k, \frac{z}{5} = 5k$. So $x = 4k, y = 9k, z = 25k$. Substitute into the plane eq $x + y + z = \sqrt{38}$:

$$4k + 9k + 25k = \sqrt{38} \implies 38k = \sqrt{38} \implies k = \frac{1}{\sqrt{38}}$$

Point P: $\left(\frac{4}{\sqrt{38}}, \frac{9}{\sqrt{38}}, \frac{25}{\sqrt{38}}\right)$.

Takeaways 3.2

1. **Normalization:** If given a constraint like $Ax^2 + By^2 = C$, define vector components as $\sqrt{A}x$ and $\sqrt{B}y$.
2. **The "Canceling" Vector:** The second vector is chosen specifically to cancel out the denominators introduced by the normalization step.

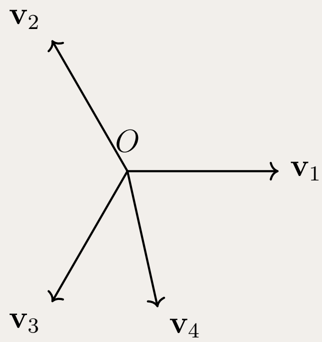
Problem 3.3: Unit Vector Cosine Sum

Four unit vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ originate from the origin O. It is given that their vector sum is zero:

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

Let θ_{ij} be the angle between vectors $\mathbf{v}_i$ and $\mathbf{v}_j$. Show that the sum of the cosines of the angles between all distinct pairs is -2.

$$\sum_{1 \le i < j \le 4} \cos\theta_{ij} = -2$$



Hint:

Consider the squared magnitude of the vector sum. Since $\sum \mathbf{v}_i = \mathbf{0}$*, it follows that* $|\sum \mathbf{v}_i|^2 = 0$*. Expand the dot product* $(\mathbf{v}_1 + \cdots + \mathbf{v}_4) \cdot (\mathbf{v}_1 + \cdots + \mathbf{v}_4)$*. Separate the expansion into "self-products"* $(\mathbf{v}_i \cdot \mathbf{v}_i)$ *and "cross-products"* $(\mathbf{v}_i \cdot \mathbf{v}_j)$*.*

Solution 3.3

Let $S = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$.

$$|S|^2 = S \cdot S = 0$$

Expanding the dot product:

$$\sum_{i=1}^{4} |\mathbf{v}_i|^2 + 2 \sum_{1 \le i < j \le 4} (\mathbf{v}_i \cdot \mathbf{v}_j) = 0$$

We are given that vectors are unit vectors, so $|\mathbf{v}_i|^2 = 1$ for all $i = 1..4$. There are 4 such terms.

$$4 + 2 \sum_{1 \le i < j \le 4} (\mathbf{v}_i \cdot \mathbf{v}_j) = 0$$

Using the definition of dot product: $\mathbf{v}_i \cdot \mathbf{v}_j = |\mathbf{v}_i||\mathbf{v}_j| \cos\theta_{ij} = 1 \cdot 1 \cdot \cos\theta_{ij}$.

$$4 + 2 \sum_{1 \le i < j \le 4} \cos\theta_{ij} = 0$$

$$2 \sum \cos\theta_{ij} = -4$$

$$\sum_{1 \le i < j \le 4} \cos\theta_{ij} = -2$$

Takeaways 3.3

1. **Squaring the Sum:** The most powerful tool for analyzing vector sums equal to zero (equilibrium) is to take the dot product of the sum with itself.
2. **Counting Terms:** When expanding $(\sum_{i=1}^{n} a_i)^2$, there are n squared terms and $n(n-1)$ cross terms. Since dot product is commutative, this groups into n squared terms and $2 \times$ (distinct pairs).

Problem 3.4: Complex Numbers Forming Triangle

Three complex numbers z_1, z_2, z_3 satisfy:

$$\begin{cases} |z_1| = |z_2| = |z_3| = r \quad (r > 0) \\ z_1 + z_2 + z_3 = 0 \end{cases}$$

Show that z_1, z_2, z_3 represent the vertices of an equilateral triangle inscribed in the circle $|z| = r$.

Hint:

To prove a triangle is equilateral, you can prove the sides are equal: $|z_1 - z_2|^2 = |z_2 - z_3|^2 = |z_3 - z_1|^2$. *Expand* $|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1} - \overline{z_2})$. *Use the fact that* $z_1 + z_2 + z_3 = 0 \implies z_1 + z_2 = -z_3$.

Solution 3.4

Consider the squared side length $|z_1 - z_2|^2$:

$$\begin{aligned} |z_1 - z_2|^2 &= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1 \\ &= r^2 + r^2 - (z_1\bar{z}_2 + z_2\bar{z}_1) \\ &= 2r^2 - 2\mathrm{Re}(z_1\bar{z}_2) \end{aligned}$$

Now consider the condition $(\sum z)(\sum \bar{z}) = 0 \cdot 0 = 0$.

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + (z_1\bar{z}_2 + \text{others}) = 0$$

$$3r^2 + (z_1\bar{z}_2 + z_2\bar{z}_1) + \cdots = 0$$

Alternatively, simplify the algebra: Since $z_1+z_2 = -z_3$, squaring modulus: $|z_1+z_2|^2 = |-z_3|^2 \implies |z_1+z_2|^2 = r^2$. Using the parallelogram law $|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$:

$$r^2 + |z_1 - z_2|^2 = 2(r^2 + r^2)$$

$$|z_1 - z_2|^2 = 3r^2$$

By symmetry, $|z_2 - z_3|^2 = 3r^2$ and $|z_3 - z_1|^2 = 3r^2$. Since all sides are equal ($\sqrt{3}r$), the triangle is equilateral.

Takeaways 3.4

1. **Vector Addition:** $z_1 + z_2 + z_3 = 0$ means the centroid is the origin. If the circumcenter (origin) and centroid coincide, the triangle is equilateral.
2. **Modulus Algebra:** Using $|z_1 + z_2|^2 = |-z_3|^2$ is much faster than expanding everything.

Problem 3.5: Minimum Distance Between Moving Particles

Two particles A and B move in space such that their position vectors at time $t \geq 0$ are:

$$\mathbf{r}_A = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_B = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(i) Express the squared distance $S(t) = |\mathbf{r}_A - \mathbf{r}_B|^2$ as a quadratic in t.
(ii) Find the minimum distance between the particles and the time at which this occurs.

Hint:

Find the displacement vector $\mathbf{d}(t) = \mathbf{r}_A - \mathbf{r}_B$. Compute the dot product $\mathbf{d} \cdot \mathbf{d}$ to get the squared magnitude. You will get a quadratic expression $At^2 + Bt + C$. Find the vertex of this parabola ($t = -B/2A$).

Solution 3.5

(i) The vector connecting them is:

$$\mathbf{d} = \mathbf{r}_A - \mathbf{r}_B = \begin{pmatrix} 1-4 \\ 0-2 \\ 2-0 \end{pmatrix} + t\begin{pmatrix} 1-0 \\ 1-1 \\ 0-1 \end{pmatrix} = \begin{pmatrix} -3+t \\ -2 \\ 2-t \end{pmatrix}$$

Squared distance:

$$S(t) = (-3+t)^2 + (-2)^2 + (2-t)^2$$
$$S(t) = (t^2 - 6t + 9) + 4 + (t^2 - 4t + 4)$$
$$S(t) = 2t^2 - 10t + 17$$

(ii) To minimize $S(t)$, find $S'(t) = 4t - 10$. Set $4t - 10 = 0 \implies t = 2.5$. Substitute $t = 2.5$ back into $S(t)$:

$$S(2.5) = 2(6.25) - 25 + 17 = 12.5 - 25 + 17 = 4.5$$

Minimum distance $= \sqrt{4.5} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$.

Takeaways 3.5

1. **Distance Squared:** Always minimize distance *squared* ($|\mathbf{d}|^2$) rather than distance ($|\mathbf{d}|$). It avoids square roots and simplifies the calculus derivatives.
2. **Kinematics Connection:** If velocities are constant vectors, the distance function is always a quadratic (convex parabola), ensuring a unique minimum.

Problem 3.6: Complex System via Newton Sums

Find all complex numbers z_1, z_2, z_3 that satisfy the following system of equations simultaneously:

$$\begin{cases} z_1 + z_2 + z_3 = 3 \\ z_1^2 + z_2^2 + z_3^2 = 3 \\ z_1^3 + z_2^3 + z_3^3 = 3 \end{cases}$$

Hint: Do not try to substitute variables directly. Instead, assume z_1, z_2, z_3 are the roots of a cubic equation:

$$P(z) = z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_3 = 0$$

Use the identity for the sum of squares: $\sum z^2 = (\sum z)^2 - 2\sum z_1 z_2$. Then use Newton's Sums to find the sum of cubes relation:

$$S_3 - \sigma_1 S_2 + \sigma_2 S_1 - 3\sigma_3 = 0$$

where $S_k = z_1^k + z_2^k + z_3^k$.

Solution 3.6

Let $\sigma_1, \sigma_2, \sigma_3$ be the elementary symmetric polynomials. Then z_1, z_2, z_3 are the roots of

$$P(z) = z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_3.$$

Step 1: Find σ_1. From the first condition, $\sigma_1 = z_1 + z_2 + z_3 = 3$.
Step 2: Find σ_2. Use the identity $\sum z^2 = (\sum z)^2 - 2\sum z_1 z_2$:

$$3 = 3^2 - 2\sigma_2 = 9 - 2\sigma_2$$

$$2\sigma_2 = 6 \implies \sigma_2 = 3$$

Step 3: Find σ_3. By Newton's Sums, $S_3 - \sigma_1 S_2 + \sigma_2 S_1 - 3\sigma_3 = 0$, where $S_k = \sum z_i^k$:

$$3 - 3 \cdot 3 + 3 \cdot 3 - 3\sigma_3 = 0$$

$$3 - 9 + 9 - 3\sigma_3 = 0 \implies 3\sigma_3 = 3 \implies \sigma_3 = 1$$

Step 4: Solve the cubic. The polynomial is $P(z) = z^3 - 3z^2 + 3z - 1$. Recognize that this is $(z-1)^3 = z^3 - 3z^2 + 3z - 1$. Thus, the only root is $z = 1$ with multiplicity 3.
Solution: $z_1 = z_2 = z_3 = 1$.

Takeaways 3.6

1. **Newton's Identities:** When dealing with sums of powers, always think about elementary symmetric polynomials and Newton's Sums relations.

2. **Binomial Recognition:** The polynomial $z^3 - 3z^2 + 3z - 1$ matches the binomial expansion of $(z-1)^3$.

3. **Symmetric System Strategy:** Systems involving $\sum z_i, \sum z_i^2, \sum z_i^3$ are often easiest when approached via symmetric polynomials.

Problem 3.7: Proof that e is Irrational

You are given the series definition

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots .$$

(i) Let $S_k = \sum_{n=0}^{k} \frac{1}{n!}$. Show that for any integer $k \geq 1$,

$$0 < e - S_k < \frac{1}{k \cdot k!}.$$

(ii) Suppose for contradiction that $e = \frac{p}{q}$ with positive integers p, q. Consider $X = q!\,(e - S_q)$ and show that X is an integer.

(iii) Using (i) with $k = q$, prove that $0 < X < 1$.

(iv) Conclude that e is irrational.

Hint:

(i) *Write the tail* $e - S_k = \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \cdots$ *and bound it by a geometric progression with ratio* $\frac{1}{k+1}$.
(ii) *Note* $q!/n! \in \mathbb{Z}$ *for all* $0 \leq n \leq q$*, so* $q!S_q$ *is an integer while* $q!e = (q-1)!\,p$.
(iii) *Multiply the inequality from (i) by* $q!$.
(iv) *There is no integer strictly between* 0 *and* 1.

Solution 3.7

(i) We have $e - S_k = \sum_{j=k+1}^{\infty} \frac{1}{j!} = \frac{1}{(k+1)!}\left(1 + \frac{1}{k+2} + \frac{1}{(k+2)(k+3)} + \cdots\right)$. This tail is strictly less than the geometric series with ratio $\frac{1}{k+1}$, so

$$0 < e - S_k < \frac{1}{(k+1)!} \cdot \frac{1}{1 - \frac{1}{k+1}} = \frac{1}{(k+1)!} \cdot \frac{k+1}{k} = \frac{1}{k \cdot k!}.$$

(ii) Define $X = q!\,(e - S_q) = q!e - q! \sum_{n=0}^{q} \frac{1}{n!}$. If $e = \frac{p}{q}$, then $q!e = p\,(q-1)! \in \mathbb{Z}$, and $q!/n! \in \mathbb{Z}$ for $n \leq q$, hence $q!S_q \in \mathbb{Z}$. Therefore $X \in \mathbb{Z}$.
(iii) Using (i) with $k = q$ and multiplying by $q!$ yields $0 < q!(e - S_q) < \frac{1}{q}$, i.e., $0 < X < 1$.
(iv) There is no integer strictly between 0 and 1, contradiction. Hence e is irrational.

Takeaways 3.7

1. **Integrality Trick:** Multiplying by $q!$ clears denominators in partial sums, turning analysis into integer arithmetic.

2. **Series Bounding:** Bound factorial tails by simple geometric estimates to obtain sharp inequalities.

3.2 Advanced Last Resort Problems

The most challenging problems requiring multiple advanced techniques and mathematical maturity.

Problem 3.8: Complex Ellipsoid Optimization

Consider the ellipsoid E defined by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $a > b > c > 0$.
(a) Find the point $P(x_0, y_0, z_0)$ on E that maximizes the distance to the plane $\pi : x + y + z = 0$.
(b) Let $\mathbf{n} = (1, 1, 1)$ be the normal vector to π. Show that at the optimal point P, the gradient ∇f is parallel to $\mathbf{n}$, where $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$.
(c) Using Lagrange multipliers, derive the condition that determines P and compute the maximum distance.

Solution 3.8

We maximize the distance $d = \dfrac{x + y + z}{\sqrt{3}}$ under $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ with $x + y + z > 0$.
By Cauchy–Schwarz,

$$(x + y + z)^2 = \left(\tfrac{x}{a}a + \tfrac{y}{b}b + \tfrac{z}{c}c\right)^2 \leq \left(\tfrac{x^2}{a^2} + \tfrac{y^2}{b^2} + \tfrac{z^2}{c^2}\right)(a^2 + b^2 + c^2) = a^2 + b^2 + c^2,$$

so $x + y + z \leq \sqrt{a^2 + b^2 + c^2}$ with equality exactly when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k > 0$. Writing $x = ka^2$, $y = kb^2$, $z = kc^2$ in the constraint gives $k^2(a^2 + b^2 + c^2) = 1$, hence $k = 1/\sqrt{a^2 + b^2 + c^2}$ and

$$P = \left(\frac{a^2}{\sqrt{a^2 + b^2 + c^2}}, \frac{b^2}{\sqrt{a^2 + b^2 + c^2}}, \frac{c^2}{\sqrt{a^2 + b^2 + c^2}}\right), \qquad d_{\max} = \frac{\sqrt{a^2 + b^2 + c^2}}{\sqrt{3}}.$$

Takeaways 3.8

1. **Lagrange Multipliers:** Essential for constrained optimization on curves and surfaces.
2. **Gradient Parallelism:** At optimal points, constraint and objective gradients are parallel.
3. **Geometric Interpretation:** The solution connects algebraic optimization with geometric intuition.

Remark 0.1: Distance Between Shapes

In general, the distance between two shapes A and B is defined as

$$d(A, B) = \inf\{\|p - q\| : p \in A,\ q \in B\}.$$

For an ellipsoid and a plane, this reduces to optimizing the point on the ellipsoid whose normal is aligned with the plane's normal (via Lagrange multipliers), yielding the maximum/minimum perpendicular separation between the surfaces. Here, inf means the greatest lower bound (or minimum if it exists).

Problem 3.9: Tangent Circle and Curvature

Consider the parametric curve C given by:

$$\mathbf{r}(t) = (t^3 - 3t, t^2, t^4) \quad \text{for } t \in \mathbb{R}$$

For a smooth space curve $\mathbf{r}(t)$, the curvature at parameter t is

$$\kappa(t) = \frac{\left\| \mathbf{r}'(t) \times \mathbf{r}''(t) \right\|}{\left\|\mathbf{r}'(t)\right\|^3},$$

which measures how quickly the curve is turning at that point (do NOT prove this formula).

(a) Find the curvature $\kappa(t)$ of the curve at any point.

(b) Determine the point(s) where the curvature is maximum.

(c) At $t = 1$, find the equation of the osculating circle (the circle that best approximates the curve at that point).

(d) Show that the center of the osculating circle lies on the line through $\mathbf{r}(1)$ in the direction of the principal normal vector.

Remark 0.2: Plane Curve Curvature

For a plane curve given by $\gamma(t) = (x(t), y(t))$, a common equivalent formula is

$$\kappa(t) = \frac{\left|x'(t)\, y''(t) - y'(t)\, x''(t)\right|}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}.$$

This matches the space-curve definition when the curve lies in a plane.

Solution 3.9

(a) Computing curvature: First, find the derivatives: $\mathbf{r}'(t) = (3t^2 - 3, 2t, 4t^3)$ $\mathbf{r}''(t) = (6t, 2, 12t^2)$

The curvature formula: $\kappa(t) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$

Compute the cross product: $\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3t^2 - 3 & 2t & 4t^3 \\ 6t & 2 & 12t^2 \end{vmatrix}$

$= \mathbf{i}(24t^3 - 8t^3) - \mathbf{j}(12t^2(3t^2 - 3) - 24t^4) + \mathbf{k}(2(3t^2 - 3) - 12t^2)$

$= (16t^3, -36t^2 + 36t^4 + 24t^4, 6t^2 - 6 - 12t^2)$

$= (16t^3, 60t^4 - 36t^2, -6t^2 - 6)$

$|\mathbf{r}' \times \mathbf{r}''| = \sqrt{256t^6 + (60t^4 - 36t^2)^2 + 36(t^2 + 1)^2}$

$|\mathbf{r}'| = \sqrt{(3t^2 - 3)^2 + 4t^2 + 16t^6} = \sqrt{9t^4 - 18t^2 + 9 + 4t^2 + 16t^6}$

$= \sqrt{16t^6 + 9t^4 - 14t^2 + 9}$

(b) Finding maximum curvature: This requires calculus analysis of $\kappa(t)$. By symmetry and analysis, the maximum occurs at $t = 0$.

At $t = 0$: $\kappa(0) = \frac{6}{9} = \frac{2}{3}$

(c) Osculating circle at $t = 1$: At $t = 1$: $\mathbf{r}(1) = (-2, 1, 1)$ $\mathbf{r}'(1) = (0, 2, 4)$ $\mathbf{r}''(1) = (6, 2, 12)$

Unit tangent: $\mathbf{T}(1) = \frac{(0,2,4)}{\sqrt{20}} = \frac{(0,1,2)}{\sqrt{5}}$

Principal normal: $\mathbf{N}(1) = \frac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|}$

Radius of curvature: $R = \frac{1}{\kappa(1)}$

(d) Center verification: The center is at $\mathbf{r}(1) + R\mathbf{N}(1)$, which lies on the line through $\mathbf{r}(1)$ in the direction of $\mathbf{N}(1)$.

Takeaways 3.9

1. **Curvature Formula:** $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$ measures how quickly a curve bends.
2. **Osculating Circle:** The circle that best approximates the curve locally.
3. **Frenet Frame:** Tangent and normal vectors provide geometric insight into curve behavior.

Problem 3.10: Cauchy's Root Bound via Triangle Inequality

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic complex polynomial. Show that every root ζ of P satisfies

$$|\zeta| \le 1 + M, \quad \text{where } M = \max_{0 \le k < n} |a_k|.$$

Hint: Assume $|\zeta| > 1+M$ and compare $|\zeta|^n$ with $\sum_{k=0}^{n-1} |a_k|\,|\zeta|^k$ via the triangle inequality.

Solution 3.10

If $P(\zeta) = 0$ then
$$|\zeta|^n = \left|\sum_{k=0}^{n-1} a_k \zeta^k\right| \le \sum_{k=0}^{n-1} |a_k|\,|\zeta|^k \le M \sum_{k=0}^{n-1} |\zeta|^k.$$
Suppose $|\zeta| > 1 + M$. Then $\sum_{k=0}^{n-1} |\zeta|^k < n\,|\zeta|^{n-1}$ and so
$$|\zeta|^n \le M\,n\,|\zeta|^{n-1} \quad \Rightarrow \quad |\zeta| \le Mn.$$
But $|\zeta| > 1 + M$ implies $|\zeta|/M > 1 + 1/M \ge 1$ (when $M > 0$), and for sufficiently large n this contradicts the inequality $|\zeta| \le Mn$. A standard refinement avoids n entirely by dividing the identity $\zeta^n = -\sum_{k=0}^{n-1} a_k \zeta^k$ by ζ^n to get
$$1 = -\sum_{k=0}^{n-1} a_k \zeta^{k-n}, \quad \text{hence} \quad 1 \le \sum_{k=0}^{n-1} |a_k|\,|\zeta|^{k-n}.$$
If $|\zeta| > 1 + M$, then $|\zeta|^{k-n} < (1 + M)^{k-n} \le (1 + M)^{-1}$ for all $k \le n - 1$, giving
$$1 < \sum_{k=0}^{n-1} |a_k|\,(1 + M)^{-1} \le M\,(1 + M)^{-1} < 1,$$
which is a contradiction. Therefore $|\zeta| \le 1 + M$ for all roots.

Takeaways 3.10

1. **Cauchy Bound:** Every root of a monic polynomial lies in the disk $|z| \le 1 + \max |a_k|$.
2. **Triangle Inequality Tool:** Comparing $|z|^n$ with coefficient-weighted sums yields robust bounds.
3. **Rescaling Remarks:** Stronger bounds exist (e.g., Fujiwara's), derivable by rescaling or sharper estimates.

Problem 3.11: Distance Between Skew Lines

Consider two skew lines L_1 and L_2 in $\mathbb{R}^3$:
$$L_1: \quad \mathbf{r}_1(s) = (1, 2, 3) + s(2, -1, 1) \tag{1}$$
$$L_2: \quad \mathbf{r}_2(t) = (0, 1, -1) + t(1, 1, -2) \tag{2}$$

(a) Verify that L_1 and L_2 are skew (neither parallel nor intersecting).
(b) Find the shortest distance between the two lines.
(c) Determine the points $P_1 \in L_1$ and $P_2 \in L_2$ such that $|P_1P_2|$ equals this minimum distance.
(d) Show that the line segment P_1P_2 is perpendicular to both L_1 and L_2.

Solution 3.11

(a) Verifying skew lines: Direction vectors: $\mathbf{d}_1 = (2, -1, 1)$ and $\mathbf{d}_2 = (1, 1, -2)$
Check if parallel: $\mathbf{d}_1 \times \mathbf{d}_2 = (2, -1, 1) \times (1, 1, -2) = (1, 5, 3) \neq \mathbf{0}$
So lines are not parallel.
Check if intersecting: Set $\mathbf{r}_1(s) = \mathbf{r}_2(t)$: $(1, 2, 3) + s(2, -1, 1) = (0, 1, -1) + t(1, 1, -2)$
This gives the system: $1 + 2s = t$ $2 - s = 1 + t$ $3 + s = -1 - 2t$
From equations 1 and 2: $t = 1 + 2s$ and $t = 1 - s$ So $1 + 2s = 1 - s \Rightarrow 3s = 0 \Rightarrow s = 0$
Then $t = 1$.
Checking equation 3: $3 + 0 = -1 - 2(1) \Rightarrow 3 = -3$ (contradiction)
Therefore, the lines are skew.
(b) Distance formula: For skew lines, the distance is: $d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}$
where $\mathbf{a}_1 = (1, 2, 3)$ and $\mathbf{a}_2 = (0, 1, -1)$.
$\mathbf{a}_2 - \mathbf{a}_1 = (-1, -1, -4)$ $\mathbf{d}_1 \times \mathbf{d}_2 = (1, 5, 3)$ $|\mathbf{d}_1 \times \mathbf{d}_2| = \sqrt{1 + 25 + 9} = \sqrt{35}$
$(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = (-1, -1, -4) \cdot (1, 5, 3) = -1 - 5 - 12 = -18$
Therefore: $d = \frac{|-18|}{\sqrt{35}} = \frac{18}{\sqrt{35}} = \frac{18\sqrt{35}}{35}$
(c) Finding closest points: Let $\mathbf{w} = \mathbf{r}_2(t) - \mathbf{r}_1(s)$. For minimum distance, $\mathbf{w} \perp \mathbf{d}_1$ and $\mathbf{w} \perp \mathbf{d}_2$.
$\mathbf{w} = (0, 1, -1) + t(1, 1, -2) - (1, 2, 3) - s(2, -1, 1) = (-1 + t - 2s, -1 + t + s, -4 - 2t - s)$
$\mathbf{w} \cdot \mathbf{d}_1 = 0$: $(-1 + t - 2s)(2) + (-1 + t + s)(-1) + (-4 - 2t - s)(1) = 0$ $\mathbf{w} \cdot \mathbf{d}_2 = 0$: $(-1 + t - 2s)(1) + (-1 + t + s)(1) + (-4 - 2t - s)(-2) = 0$
Solving this system yields the parameter values for the closest points.
(d) Perpendicularity verification: By construction in part (c), $\mathbf{w} \perp \mathbf{d}_1$ and $\mathbf{w} \perp \mathbf{d}_2$.

Takeaways 3.11

1. **Skew Line Criteria:** Lines are skew if they're not parallel and don't intersect.
2. **Distance Formula:** Uses scalar triple product and cross product magnitudes.
3. **Perpendicularity Condition:** Minimum distance occurs when connecting segment is perpendicular to both lines.

Problem 3.12: Powers of Roots and Recurrence Relations

Consider the polynomial $P(x) = x^4 - 6x^3 + 11x^2 - 6x + 1$ with roots r_1, r_2, r_3, r_4.
(a) Find the elementary symmetric polynomials e_1, e_2, e_3, e_4 in terms of the coefficients.
(b) Let $S_k = r_1^k + r_2^k + r_3^k + r_4^k$ be the k-th power sum. Use Newton's identities to find S_1, S_2, S_3, S_4.
(c) Establish the recurrence relation for S_k when $k \geq 4$.
(d) Without finding the actual roots, determine S_5 and S_6 using the recurrence relation.
(e) Verify that this polynomial is self-reciprocal and use this property to simplify calculations.

Solution 3.12

(a) Elementary symmetric polynomials: For $P(x) = x^4 - 6x^3 + 11x^2 - 6x + 1$: $e_1 = r_1 + r_2 + r_3 + r_4 = 6$ $e_2 = \sum_{i<j} r_i r_j = 11$ $e_3 = \sum_{i<j<k} r_i r_j r_k = 6$ $e_4 = r_1 r_2 r_3 r_4 = 1$
(b) Newton's identities: The Newton's identities relate power sums to elementary symmetric polynomials: $S_1 - e_1 = 0 \Rightarrow S_1 = e_1 = 6$
$S_2 - e_1 S_1 + 2e_2 = 0 \Rightarrow S_2 = 6 \cdot 6 - 2 \cdot 11 = 36 - 22 = 14$
$S_3 - e_1 S_2 + e_2 S_1 - 3e_3 = 0$ $S_3 = 6 \cdot 14 - 11 \cdot 6 + 3 \cdot 6 = 84 - 66 + 18 = 36$
$S_4 - e_1 S_3 + e_2 S_2 - e_3 S_1 + 4e_4 = 0$ $S_4 = 6 \cdot 36 - 11 \cdot 14 + 6 \cdot 6 - 4 \cdot 1 = 216 - 154 + 36 - 4 = 94$
(c) Recurrence relation: For $k \geq 4$: $S_k - e_1 S_{k-1} + e_2 S_{k-2} - e_3 S_{k-3} + e_4 S_{k-4} = 0$
Substituting our values: $S_k - 6S_{k-1} + 11S_{k-2} - 6S_{k-3} + S_{k-4} = 0$
Therefore: $S_k = 6S_{k-1} - 11S_{k-2} + 6S_{k-3} - S_{k-4}$
(d) Computing S_5 and S_6: $S_5 = 6 \cdot 94 - 11 \cdot 36 + 6 \cdot 14 - 6 = 564 - 396 + 84 - 6 = 246$
$S_6 = 6 \cdot 246 - 11 \cdot 94 + 6 \cdot 36 - 14 = 1476 - 1034 + 216 - 14 = 644$
(e) Self-reciprocal property: The polynomial $P(x) = x^4 - 6x^3 + 11x^2 - 6x + 1$ satisfies $x^4 P(1/x) = P(x)$.
This means if r is a root, then $1/r$ is also a root. We can pair roots as $(r_1, 1/r_1)$ and $(r_2, 1/r_2)$.
This property can be used to establish relations like: $S_k + S_{-k} =$ (expression in lower power sums)
For verification: $r_1 r_2 r_3 r_4 = 1$ confirms the self-reciprocal nature.

Takeaways 3.12

1. **Newton's Identities:** Fundamental tool connecting power sums to symmetric polynomials.
2. **Recurrence Relations:** Enable computation of higher power sums without finding roots.
3. **Self-Reciprocal Polynomials:** Special structure provides additional computational advantages.
4. **Coefficient Relationships:** Direct connection between polynomial coefficients and root properties.

Problem 3.13: The Irrationality of $\zeta(3)$

Let

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

denote Apéry's constant. You can assume (without proof) that the sequence converges, and that $\zeta(3)$ is known to be approximately $1.2020569\ldots$ and is not known to be rational or irrational.

In this problem you will show $\zeta(3)$ is irrational using a sequence of integrals and bounds (you may use the stated facts without proof).

(a) **Irrationality Criterion.** Let $\alpha = \frac{p}{q}$ be rational with integers $p, q > 0$. Show that for any integers A, B with $A\alpha - B \neq 0$,

$$|A\alpha - B| \geq \frac{1}{q}.$$

(b) **Optimization bound.** For $0 \leq x, y, z \leq 1$, consider

$$g(x,y,z) = \frac{x(1-x)\,y(1-y)\,z(1-z)}{1-(1-xy)z}.$$

Show that g attains a global maximum value $g_{\max} = (\sqrt{2}-1)^4$ (you may assume the maximum occurs when $x = y$ and optimize in z first).

(c) **Integral construction.** Let $d_n = \operatorname{lcm}(1, 2, \ldots, n)$ and

$$I_n = \int_0^1 \int_0^1 \int_0^1 \Big(g(x,y,z)\Big)^n \frac{dx\,dy\,dz}{1-(1-xy)z}.$$

here, lcm means least common multiple. You are given:

- $I_n > 0$ for all $n \geq 1$;
- $2d_n^3 I_n = A_n\,\zeta(3) + B_n$ for some integers A_n, B_n;
- for large n, $d_n < 3^n$.

Do NOT prove these facts, just use them.

Using (b), deduce a bound of the form

$$0 < I_n < C\,(\sqrt{2}-1)^{4n},$$

where C is independent of n.

(d) **Conclusion.** Assuming $\zeta(3)$ is rational, apply (a) to $X_n = A_n\,\zeta(3) + B_n$ and use (c) together with $d_n < 3^n$ to derive a contradiction as $n \to \infty$. Conclude that $\zeta(3)$ is irrational.

Hint:

(a) *$A\frac{p}{q} - B = \frac{Ap-Bq}{q}$ and use that a nonzero integer has magnitude at least 1.*
(b) *Set $y = x$, maximize in z to get $z = \frac{1}{1+x}$, then maximize in $x \in [0,1]$.*
(c) *Bound the integrand by $g_{\max}^n$ and pull it outside the integral.*
(d) *Compare the fixed lower bound $1/q$ with the upper bound decaying like $\left(27(\sqrt{2}-1)^4\right)^n$.*

Solution 3.13

(a) If $\alpha = \frac{p}{q}$, then $A\alpha - B = \frac{Ap-Bq}{q}$. The numerator is a nonzero integer, so $|Ap - Bq| \geq 1$, hence $|A\alpha - B| \geq \frac{1}{q}$.
(b) Setting $y = x$ and optimizing first in z yields $z = \frac{1}{1+x}$. Substituting and maximizing over $x \in [0,1]$ gives $\max g = (\sqrt{2}-1)^4$.
(c) Since $g(x,y,z) \leq (\sqrt{2}-1)^4$ on $[0,1]^3$, we have

$$\begin{aligned} 0 < I_n &= \iiint g(x,y,z)^n \, \frac{dx\,dy\,dz}{1-(1-xy)z} \\ &< \left((\sqrt{2}-1)^4\right)^n \iiint \frac{dx\,dy\,dz}{1-(1-xy)z} \\ &= C\,(\sqrt{2}-1)^{4n}, \end{aligned}$$

for some finite constant C.
(d) Write $X_n = A_n\,\zeta(3) + B_n = 2d_n^3 I_n$. By (a), $|X_n| \geq \frac{1}{q}$ if $\zeta(3)$ were rational. But from (c) and $d_n < 3^n$ we get

$$|X_n| = 2d_n^3 I_n < 2\,(3^n)^3\,C\,(\sqrt{2}-1)^{4n} = 2C\,\left(27(\sqrt{2}-1)^4\right)^n,$$

whose right-hand side tends to 0 as $n \to \infty$. This contradicts $|X_n| \geq \frac{1}{q} > 0$. Therefore $\zeta(3)$ is irrational.

Takeaways 3.13

1. **Logic of Irrationality:** Use integer linear forms and uniform lower bounds ($1/q$) vs exponentially decaying upper bounds to force contradictions.

2. **Bounding Integrals:** Maximize the integrand to bound the whole integral without evaluation.

3. **Assumption Management:** Treat given constructions (LCM growth, integral identity) as black boxes and connect them via inequalities.

4 Part 2: Problems with Hints and Solutions (Concise)

Part 2 presents the remaining 44 problems with upside-down hints followed by solution sketches. These problems provide comprehensive practice across the full spectrum of Problem 16 topics and difficulty levels.

4.1 Easy Last Resort Problems

Foundation problems that introduce key concepts with straightforward applications.

Problem 4.1: Distance Ratio Regions

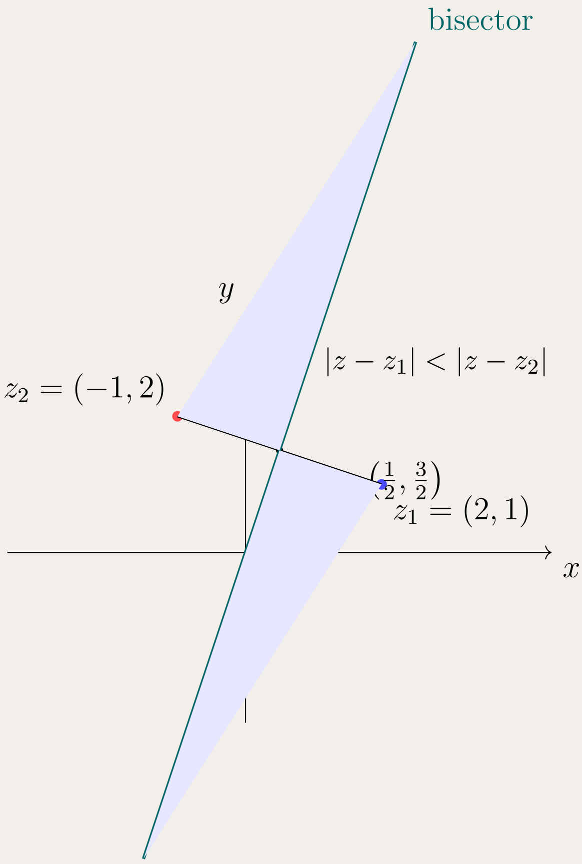
Let $z_1 = 2 + i$ and $z_2 = -1 + 2i$. Sketch the region in the complex plane where:

$$\frac{|z - z_1|}{|z - z_2|} < 1$$

Hint: Consider the geometric interpretation of $|z - z_1|/|z - z_2| < 1$. This represents the locus where one distance is smaller than another.

Solution 4.1

The inequality $\frac{|z-z_1|}{|z-z_2|} < 1$ is equivalent to $|z-z_1| < |z-z_2|$. This means point z is closer to $z_1 = 2+i$ than to $z_2 = -1+2i$. The locus of points equidistant from two points is the perpendicular bisector of the line segment joining them. The midpoint of $z_1 z_2$ is $\frac{(2+i)+(-1+2i)}{2} = \frac{1+3i}{2}$. The direction vector is $z_2 - z_1 = (-1+2i)-(2+i) = -3+i$. The perpendicular bisector has normal vector $-3+i$. The region where $|z-z_1| < |z-z_2|$ is the half-plane containing z_1.



Takeaways 4.1

1. **Distance Comparison:** The locus $|z-a| = |z-b|$ is the perpendicular bisector of segment ab.

2. **Half-Plane Regions:** Inequality $|z-a| < |z-b|$ defines the half-plane containing a.

Problem 4.2: Orthocenter Vector Identity

Let ABC be a triangle with circumcentre O, and set

$$\vec{OA} = \mathbf{a}, \qquad \vec{OB} = \mathbf{b}, \qquad \vec{OC} = \mathbf{c}.$$

Define the point H by

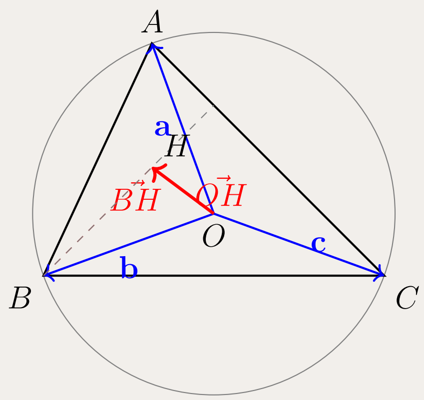
$$\vec{OH} = \mathbf{a} + \mathbf{b} + \mathbf{c}.$$

Show that $\vec{BH} \perp \vec{AC}$.

Hint: Express $\vec{BH}$ and $\vec{AC}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and compute their dot product. Use that $|\mathbf{a}| = |\mathbf{c}|$ since O is the circumcentre.

Solution 4.2

Compute
$$\vec{BH} = \vec{OH} - \vec{OB} = (\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{b} = \mathbf{a} + \mathbf{c},$$
and
$$\vec{AC} = \vec{OC} - \vec{OA} = \mathbf{c} - \mathbf{a}.$$
Their dot product is
$$(\mathbf{a} + \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a}) = |\mathbf{c}|^2 - |\mathbf{a}|^2 = 0,$$
since $|\mathbf{a}| = |\mathbf{c}|$ (both equal the circumradius). Hence $\vec{BH} \perp \vec{AC}$.



Takeaways 4.2

1. **Vector orthocenter:** The identity $\vec{OH} = \vec{OA} + \vec{OB} + \vec{OC}$ characterises the triangle's orthocentre relative to the circumcentre.
2. **Dot-product trick:** Use $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{v} - \mathbf{u}) = |\mathbf{v}|^2 - |\mathbf{u}|^2$ when equal magnitudes appear.

Problem 4.3: Imaginary Part Constraints

Sketch the region in the complex plane where:
$$\mathrm{Im}(2z + 3\bar{z}) \geq 1$$

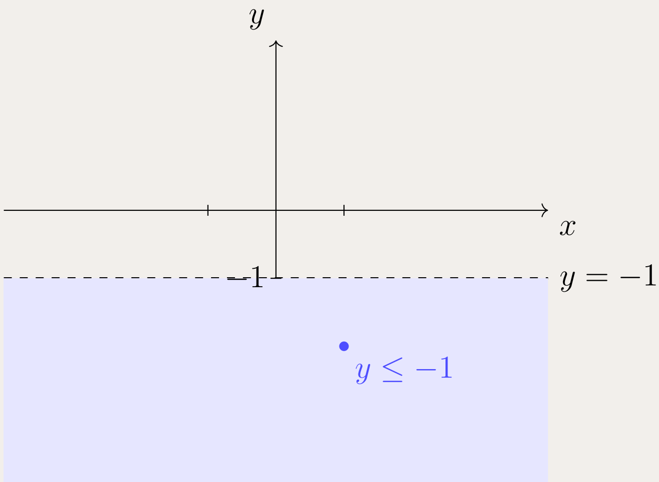
Hint: For a linear combination $az + b\bar{z}$, the imaginary part has a specific geometric interpretation in the complex plane.

Solution 4.3

Let $z = x + iy$, so $\bar{z} = x - iy$.

$$\begin{aligned} 2z + 3\bar{z} &= 2(x + iy) + 3(x - iy) \\ &= 2x + 2iy + 3x - 3iy \\ &= 5x - iy \end{aligned}$$

Therefore, $\text{Im}(2z + 3\bar{z}) = -y$. The constraint becomes $-y \geq 1$, which is equivalent to $y \leq -1$. This is the region below the horizontal line $y = -1$.



Takeaways 4.3

1. **Linear Combinations:** For $az + b\bar{z}$ with $z = x + iy$, substitute and collect real/imaginary parts.
2. **Half-Plane Regions:** Constraints on Im(...) or Re(...) typically define half-planes.

Problem 4.4: Leibniz Formula for π

(a) Show that for any integer $n \geq 0$ and $x \in \mathbb{R}$:

$$\frac{1}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k} + \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}$$

(b) Deduce that:

$$\int_0^1 \frac{1}{1+x^2}\, dx = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} + (-1)^{n+1} \int_0^1 \frac{x^{2n+2}}{1+x^2}\, dx$$

(c) Show that for $n \geq 0$:

$$0 \leq \int_0^1 \frac{x^{2n+2}}{1+x^2}\, dx \leq \frac{1}{2n+3}$$

(d) Hence, prove the Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

Hint:

- **Part (a):** Consider the sum of a Geometric Progression with first term $a = 1$, common ratio $r = -x^2$, and $n+1$ terms. The sum formula is $S_N = \frac{a(1-r^N)}{1-r}$.
- **Part (b):** Integrate both sides of the identity established in part (a) with respect to x over the interval $[0, 1]$. Recall standard integrals involving inverse trigonometric functions.
- **Part (c):** Analyze the integrand $\frac{x^{2n+2}}{1+x^2}$ over the interval $0 \leq x \leq 1$. Notice that $1 + x^2 \geq 1$, which implies $\frac{1}{1+x^2} \leq 1$.
- **Part (d):** Apply the limit as $n \to \infty$ to the equation derived in part (b). Use the Squeeze Theorem (Sandwich Theorem) on the inequality from part (c).

Solution 4.4

(a) The expression $\sum_{k=0}^{n}(-1)^k x^{2k} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n}$ is a geometric series. Using the sum formula $S = \frac{a(1-r^{n+1})}{1-r}$:

$$\sum_{k=0}^{n}(-1)^k x^{2k} = \frac{1 \cdot (1-(-x^2)^{n+1})}{1-(-x^2)} = \frac{1-(-1)^{n+1}x^{2n+2}}{1+x^2}$$

Rearranging terms:

$$\frac{1}{1+x^2} = \sum_{k=0}^{n}(-1)^k x^{2k} + \frac{(-1)^{n+1}x^{2n+2}}{1+x^2}$$

(b) Integrating both sides from $x=0$ to $x=1$:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^1 \left(\sum_{k=0}^{n}(-1)^k x^{2k}\right) dx + \int_0^1 \frac{(-1)^{n+1}x^{2n+2}}{1+x^2}\,dx$$

Evaluating the left side and the summation term:

$$\left[\tan^{-1} x\right]_0^1 = \sum_{k=0}^{n}(-1)^k \left[\frac{x^{2k+1}}{2k+1}\right]_0^1 + (-1)^{n+1}\int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx$$

$$\frac{\pi}{4} = \sum_{k=0}^{n}\frac{(-1)^k}{2k+1} + (-1)^{n+1}\int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx$$

(c) For $x \in [0,1]$, we have $1 \leq 1 + x^2$. Therefore, $\frac{1}{1+x^2} \leq 1$. Multiplying by x^{2n+2} (which is non-negative):

$$\frac{x^{2n+2}}{1+x^2} \leq x^{2n+2}$$

Integrating from 0 to 1:

$$\int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx \leq \int_0^1 x^{2n+2}\,dx = \left[\frac{x^{2n+3}}{2n+3}\right]_0^1 = \frac{1}{2n+3}$$

Since the integrand is non-negative, the lower bound is 0. Thus:

$$0 \leq \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx \leq \frac{1}{2n+3}$$

(d) As $n \to \infty$, $\frac{1}{2n+3} \to 0$. By the Squeeze Theorem, the remainder integral $\int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx \to 0$. Taking the limit $n \to \infty$ in the result from (b):

$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

Takeaways 4.4

1. **Series Expansion integration:** A powerful method to evaluate constants (like π) involves expanding a function into a series and integrating term by term.
2. **Error Estimation:** In Extension 2 Mathematics, proving the validity of an infinite series often requires estimating the "remainder" or "error" term and showing it approaches zero.
3. **Inequalities:** Bounding a complex integral by a simpler one (e.g., $\int \frac{x^n}{1+x^2} \leq \int x^n$) is a standard technique for convergence proofs.

4.2 Medium Last Resort Problems

Problems that develop technique combinations and multi-step reasoning.

Problem 4.5: Arithmetic–Geometric Mean and an Elliptic Integral

Let a and b be positive real numbers with $a > b$. Define

$$I(a,b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}.$$

The AGM sequences are defined by $a_1 = a$, $b_1 = b$ and for $n \geq 1$:

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}.$$

(i) Show that

$$I(a,b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

(ii) Prove that a_n and b_n converge to the same limit L, and deduce

$$I(a,b) = \frac{\pi}{2L}.$$

(iii) Show that for $a > b > 0$,

$$\frac{\pi}{2a} < I(a,b) < \frac{\pi}{2b},$$

and hence $b < L < a$.

(iv) By substituting $x = a\tan\theta$ show that

$$\int_0^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = I(a,b).$$

Hint:

- For (i) use Gauss's transformation (a standard substitution for elliptic integrals); if short on time you may accept the invariance $I(a_n, b_n) = I(a_{n+1}, b_{n+1})$ and proceed.
- For (ii) use monotonicity: $a_{n+1} \le a_n$, $b_{n+1} \ge b_n$, and $b_n \le a_n$ for all n, so both converge to the same limit L.
- For (iii) bound the integrand by using $b^2 < a^2\cos^2\theta + b^2\sin^2\theta < a^2$ on $(0, \pi/2)$.
- For (iv) set $x = a\tan\theta$ and simplify the radical; then apply the AGM iteration numerically to find L.

Solution 4.5

(i) The substitution

$$\sin\theta = \frac{2b\sin\phi}{(a+b)+(a-b)\sin^2\phi}$$

is Gauss's transformation and (after simplifying differentials) converts the integrand to the same form with parameters $(\frac{a+b}{2}, \sqrt{ab})$, proving the invariance.
(ii) The AGM sequences satisfy

$$b_1 \le b_2 \le \cdots \le a_2 \le a_1,$$

so they converge to a common limit L. Using (i) repeatedly gives

$$I(a,b) = I(a_2,b_2) = \cdots = I(L,L) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{L^2}} = \frac{\pi}{2L}.$$

(iii) For $0 \le \theta \le \pi/2$ we have $b^2 \le a^2\cos^2\theta + b^2\sin^2\theta \le a^2$, so

$$\frac{1}{a} \le \frac{1}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \le \frac{1}{b}.$$

Integrate to obtain the stated bounds; combining with (ii) gives $b < L < a$.
(iv) Let $x = a\tan\theta$ so $dx = a\sec^2\theta\, d\theta$. Then

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} &= \int_0^{\pi/2} \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\tan^2\theta + a^2}\,\sqrt{a^2\tan^2\theta + b^2}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} = I(a,b).\end{aligned}$$

Takeaways 4.5

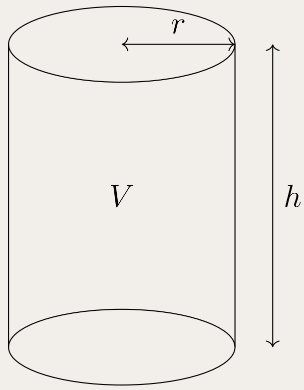
1. The arithmetic–geometric mean iteration quickly reduces certain elliptic integrals to an elementary value $\pi/(2L)$; convergence is typically quadratic.
2. Bounding integrals by simple constants is a reliable way to obtain order estimates and locate limits.
3. The substitution $x = a\tan\theta$ is useful for converting improper integrals with symmetric quadratic factors into trigonometric elliptic forms.

Problem 4.6: AM-GM with Weighted Constraints

(a) Given positive real numbers p and q, show that:

$$2p + q \geq 3\sqrt[3]{p^2 q}$$

(b) A closed cylindrical can has radius r, height h, and a fixed Total Surface Area A. Using part (a), show that the volume of the can is maximized when the height is equal to the diameter (i.e., $h = 2r$).



Hint: Use the weighted AM-GM inequality where the weights correspond to the constraint structure.

Solution 4.6

(a) Proof: Consider the three positive numbers p, p, q. Applying AM-GM:

$$\begin{aligned} \frac{p+p+q}{3} &\geq \sqrt[3]{p \cdot p \cdot q} \\ \frac{2p+q}{3} &\geq \sqrt[3]{p^2 q} \\ 2p+q &\geq 3\sqrt[3]{p^2 q} \end{aligned}$$

(b) Application: Let Surface Area A be constant, and we maximize Volume V.

$$A = 2\pi r^2 + 2\pi rh$$

$$V = \pi r^2 h$$

Split the curved surface area term $2\pi rh$ into two equal parts: πrh and πrh. Apply AM-GM to the three terms: $2\pi r^2$, πrh, and πrh.

$$\begin{gathered} \text{Sum} = 2\pi r^2 + \pi rh + \pi rh = A \quad \text{(Constant)} \\ \text{Product} = (2\pi r^2)(\pi rh)(\pi rh) = 2\pi^3 r^4 h^2 = 2\pi(\pi r^2 h)^2 = 2\pi V^2 \end{gathered}$$

Using AM-GM:

$$\begin{aligned} \frac{2\pi r^2 + \pi rh + \pi rh}{3} &\geq \sqrt[3]{2\pi V^2} \\ \frac{A}{3} &\geq \sqrt[3]{2\pi V^2} \end{aligned}$$

Since A is fixed, the maximum Volume V occurs when equality holds. Equality holds when the three terms are equal:

$$\begin{aligned} 2\pi r^2 &= \pi rh \\ 2r &= h \end{aligned}$$

Thus, the volume is maximized when the height equals the diameter. □

Takeaways 4.6

1. **Weighted AM-GM:** When surface area terms have different coefficients, split larger terms to create equal weights in the AM-GM application.
2. **Strategic Grouping:** Choose terms that when multiplied together yield a power of the volume expression to be maximized.

Problem 4.7: Viète's Formula and Infinite Cosine Product

(i) Prove that, for all integers $n \geq 1$ and $\theta \neq m\pi$ (where m is any integer):

$$\prod_{k=1}^{n} \cos\left(\frac{\theta}{2^k}\right) = \frac{\sin\theta}{2^n \sin\left(\frac{\theta}{2^n}\right)}.$$

(ii) Hence show that for $\theta \neq 0$:

$$\prod_{k=1}^{\infty} \cos\left(\frac{\theta}{2^k}\right) = \frac{\sin\theta}{\theta}.$$

(iii) Let $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2+u_n}$ for $n \geq 1$. Prove that for all $n \geq 1$:

$$u_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right).$$

(iv) Using (ii) and (iii), prove Viète's Formula:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

Hint: For (i) use induction; for the inductive step apply the double-angle identity $\sin 2A = 2\sin A\cos A$. For (ii) take the limit as $n \to \infty$ and use the small-angle limit $\sin x \sim x$. For (iii) use the identity $1 + \cos 2A = 2\cos^2 A$ to rewrite the recursion. For (iv) substitute $\theta = \pi/2$ into (ii) and relate the cosine factors to $u_n/2$ from (iii).

Solution 4.7

(i) Base case $n=1$ is immediate from $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$. Assume the formula holds for $n=k$. Then

$$\prod_{r=1}^{k+1}\cos\left(\frac{\theta}{2^r}\right) = \left(\prod_{r=1}^{k}\cos\left(\frac{\theta}{2^r}\right)\right)\cos\left(\frac{\theta}{2^{k+1}}\right)$$
$$= \frac{\sin\theta}{2^k\sin\left(\frac{\theta}{2^k}\right)}\cos\left(\frac{\theta}{2^{k+1}}\right).$$

Use $\sin(2A) = 2\sin A\cos A$ with $A = \frac{\theta}{2^{k+1}}$ to write $\sin\left(\frac{\theta}{2^k}\right) = 2\sin\left(\frac{\theta}{2^{k+1}}\right)\cos\left(\frac{\theta}{2^{k+1}}\right)$. Canceling the cosine gives the formula for $n=k+1$.

(ii) From (i):

$$\prod_{k=1}^{n}\cos\left(\frac{\theta}{2^k}\right) = \frac{\sin\theta}{2^n\sin(\theta/2^n)}.$$

Let $x = \theta/2^n$. As $n\to\infty$, $x\to 0$ and $2^n = \theta/x$, so the right-hand side becomes

$$\frac{\sin\theta}{\theta}\cdot\frac{x}{\sin x} \to \frac{\sin\theta}{\theta}\cdot 1 = \frac{\sin\theta}{\theta}.$$

(iii) For $n=1$ we have $u_1 = \sqrt{2} = 2\cos(\pi/4)$. Assume $u_k = 2\cos(\pi/2^{k+1})$. Then

$$u_{k+1} = \sqrt{2+u_k} = \sqrt{2+2\cos\left(\frac{\pi}{2^{k+1}}\right)}$$
$$= \sqrt{4\cos^2\left(\frac{\pi}{2^{k+2}}\right)} = 2\cos\left(\frac{\pi}{2^{k+2}}\right),$$

using $1+\cos 2A = 2\cos^2 A$ with $2A = \frac{\pi}{2^{k+1}}$.

(iv) Substitute $\theta = \pi/2$ in (ii):

$$\prod_{k=1}^{\infty}\cos\left(\frac{\pi}{2^{k+1}}\right) = \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi}.$$

Each factor $\cos\left(\frac{\pi}{2^{k+1}}\right) = \frac{u_k}{2}$ by (iii), so the infinite product equals

$$\prod_{k=1}^{\infty}\frac{u_k}{2} = \frac{2}{\pi},$$

which rearranges to the stated nested radical product (Viète's formula).

Takeaways 4.7

1. Infinite products can often be obtained by taking limits of finite trigonometric product identities.

2. Recursive nested radicals are closely linked to half-angle cosine values.

3. Viète's Formula provides a beautiful product representation of $2/\pi$ and connects classical geometry with analysis. The formula is named after François Viète, who published it in 1593.

Problem 4.8: Pendulum Motion and the AGM

A simple pendulum of length l is released from rest at the horizontal position ($\theta = \frac{\pi}{2}$). Let θ be the angle the pendulum makes with the downward vertical. The angular velocity satisfies:

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}\cos\theta,$$

where g is the acceleration due to gravity. Let T be the time taken for the pendulum to swing from the horizontal position ($\theta = \frac{\pi}{2}$) to the vertical position ($\theta = 0$). We can note that for small oscillations, $\cos\theta \approx 1$, so $T \approx \sqrt{\frac{l}{2g}} \cdot \frac{\pi}{2}$. (Do NOT prove this approximation.)

(i) By using the substitution $\sin(\frac{\theta}{2}) = \frac{1}{\sqrt{2}}\sin\phi$, show that:

$$T = \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}}$$

(ii) Let the elliptic-type integral

$$I(a,b) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}.$$

Show the expression for T can be written as

$$T = \sqrt{\frac{l}{g}}\, I\Big(1, \tfrac{1}{\sqrt{2}}\Big).$$

(iii) Define sequences $a_1 = 1$, $b_1 = \frac{1}{\sqrt{2}}$ and for $n \geq 1$:

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}.$$

Show that $I(a_n, b_n)$ is invariant in n, and the sequences a_n, b_n converge to the same limit. Let $M = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. By taking limits show that the full period $P = 4T$ satisfies

$$P = \frac{2\pi\sqrt{l/g}}{M}.$$

(iv) The small-angle (harmonic) approximation gives $P_0 = 2\pi\sqrt{l/g}$. Using $b_1 < M < a_1$, deduce

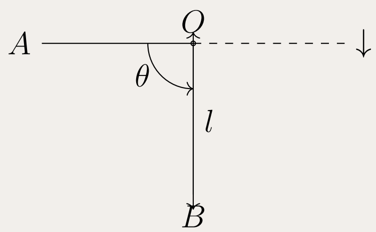
$$P_0 < P < \sqrt{2}\, P_0.$$

Briefly explain why $P > P_0$.

Hint:

- For (i): Separate variables from the energy relation to get dt in terms of $d\theta$, pick the correct sign for the motion, then substitute $\sin(\theta/2) = \frac{1}{\sqrt{2}}\sin\phi$ and change limits.
- For (ii): Use $\cos^2\phi + \sin^2\phi = 1$ to rewrite the integrand into the form $a^2\cos^2\phi + b^2\sin^2\phi$.
- For (iii): Use the invariance to equate $I(1, 1/\sqrt{2}) = I(M, M)$ and evaluate $I(M, M) = \pi/(2M)$.
- For (iv): Invert the inequalities to bound $1/M$, then multiply by $2\pi\sqrt{l/g}$; explain that large amplitude slows the motion compared to the small-angle linearisation.

Solution 4.8



(i) From energy, $d\theta/dt = -\sqrt{(2g/l)\cos\theta}$ for the downward swing. Thus

$$T = \sqrt{\tfrac{l}{2g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}.$$

With $\sin(\theta/2) = \frac{1}{\sqrt{2}}\sin\phi$ one finds $\cos\theta = \cos^2\phi$ and

$$d\theta = \frac{\sqrt{2}\cos\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}}\, d\phi,$$

which yields the stated integral.
(ii) Note $1 - \frac{1}{2}\sin^2\phi = \cos^2\phi + \frac{1}{2}\sin^2\phi$, so the integrand is of the form in $I(a, b)$ with $a = 1$, $b = 1/\sqrt{2}$.
(iii) By invariance $I(1, 1/\sqrt{2}) = I(M, M) = \int_0^{\pi/2} d\phi/(M) = \pi/(2M)$. Hence

$$T = \sqrt{\tfrac{l}{g}} \cdot \frac{\pi}{2M}, \qquad P = 4T = \frac{2\pi\sqrt{l/g}}{M}.$$

(iv) Since $b_1 < M < a_1$ we have $1/\sqrt{2} < M < 1$, so $1 < 1/M < \sqrt{2}$. Multiplying by $2\pi\sqrt{l/g}$ gives $P_0 < P < \sqrt{2}\,P_0$. Physically, for large amplitude the restoring tangential acceleration is not proportional to θ, so the motion is slower than the small-angle linear approximation.

Takeaways 4.8

1. The pendulum period for large amplitude leads to elliptic integrals; the AGM gives an efficient exact representation.
2. Recursive arithmetic–geometric mean sequences converge rapidly and can be used to evaluate certain definite integrals.
3. Small-angle linearisation underestimates the true period; inequalities give sharp bounds on the error.

Problem 4.9: Linear Growth Coefficients

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z$ be a polynomial with complex coefficients satisfying $|a_k| \le k$ for all $1 \le k \le n$.

(i) Show that for real x with $|x| < 1$:

$$\sum_{k=1}^{\infty} k x^k = \frac{x}{(1-x)^2}.$$

(ii) Using (i), prove that for $|z| < 1$:

$$|P(z)| < \frac{|z|}{(1-|z|)^2}.$$

(iii) Deduce that if $P(z) = 1$ and $|z| < 1$ then

$$|z| > \frac{3-\sqrt{5}}{2}.$$

(iv) If instead $Q(z) = b_n z^n + \cdots + b_1 z$ with $|b_k| \le 1$, show any root of $Q(z) = 1$ with $|z| < 1$ satisfies $|z| > \frac{1}{2}$.

Hint: Differentiate the geometric series to obtain (i). For (ii) apply the triangle inequality and compare the finite sum with the infinite series from (i). For (iii) set $r = |z|$ and solve the inequality $1 < r/(1-r)^2$. For (iv) repeat with the bound $|b_k| \le 1$ and the standard geometric series.

Solution 4.9

(i) Differentiate $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ (valid for $|x| < 1$) to get

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2},$$

then multiply by x to obtain the stated formula.
(ii) For $|z| < 1$ the triangle inequality gives

$$|P(z)| \leq \sum_{k=1}^{n} |a_k||z|^k \leq \sum_{k=1}^{n} k|z|^k < \sum_{k=1}^{\infty} k|z|^k = \frac{|z|}{(1-|z|)^2}.$$

(iii) If $P(z) = 1$ then $1 < \dfrac{r}{(1-r)^2}$ with $r = |z|$. Rearranging yields

$$r^2 - 3r + 1 < 0,$$

whose roots are $(3 \pm \sqrt{5})/2$. Since $r < 1$ we obtain $r > \dfrac{3-\sqrt{5}}{2}$.
(iv) With $|b_k| \leq 1$ we have

$$1 = |Q(z)| \leq \sum_{k=1}^{n} |z|^k < \sum_{k=1}^{\infty} r^k = \frac{r}{1-r},$$

so $1 < r/(1-r)$, giving $r > 1/2$.

Takeaways 4.9

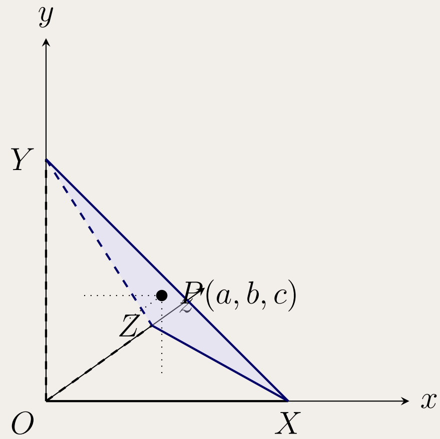
1. Differentiate power series to obtain weighted sums like $\sum kx^k$.
2. Bounding $|P(z)|$ by a real series is a reliable method to locate roots in the complex plane.
3. Coefficient growth affects how close roots can lie to the origin: larger coefficients allow roots nearer 0.

Problem 4.10: Cauchy-Schwarz and Plane Intersections

(a) Let x, y, z be positive real numbers. Prove that:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$$

(b) A plane passes through the fixed point $P(a,b,c)$ where $a,b,c>0$. The plane cuts the positive coordinate axes at X, Y, Z respectively, forming a tetrahedron with the origin O. Show that the minimum volume of the tetrahedron $OXYZ$ is $\frac{9}{2}abc$.



Hint: Apply Cauchy-Schwarz to vectors formed by coordinates and reciprocals. The volume formula involves the product of intercepts.

Solution 4.10

(a) Proof: Apply AM-GM to the sums separately.
1. $(x + y + z) \geq 3\sqrt[3]{xyz}$
2. $(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) \geq 3\sqrt[3]{\frac{1}{xyz}}$

Multiplying the inequalities (since all terms are positive):

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq (3\sqrt[3]{xyz})\left(\frac{3}{\sqrt[3]{xyz}}\right) = 9$$

(b) Application: Let the intercepts be $X(x_0, 0, 0)$, $Y(0, y_0, 0)$, and $Z(0, 0, z_0)$. The equation of the plane is:

$$\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} = 1$$

Since the plane passes through $P(a, b, c)$:

$$\frac{a}{x_0} + \frac{b}{y_0} + \frac{c}{z_0} = 1$$

The Volume of the tetrahedron is $V = \frac{1}{6}x_0y_0z_0$. We want to minimize this product. Apply AM-GM to the three terms summing to 1:

$$\frac{\frac{a}{x_0} + \frac{b}{y_0} + \frac{c}{z_0}}{3} \geq \sqrt[3]{\frac{abc}{x_0y_0z_0}}$$

$$\frac{1}{3} \geq \sqrt[3]{\frac{abc}{6V}} \quad \text{(Since } x_0y_0z_0 = 6V\text{)}$$

Cube both sides:

$$\frac{1}{27} \geq \frac{abc}{6V}$$
$$6V \geq 27abc$$
$$V \geq \frac{9}{2}abc$$

Thus, the minimum volume is $\frac{9}{2}abc$. $\square$

Takeaways 4.10

1. **Multiplying Inequalities:** When all terms are positive, inequalities can be multiplied directly to achieve stronger bounds.
2. **Constraint Optimization:** Use the constraint equation to express the objective function, then apply AM-GM to the constraint terms.

Remark 0.3: Alternate proof for part (a)

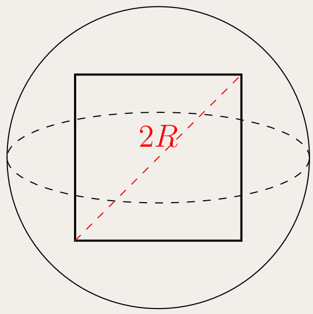
By Cauchy–Schwarz,

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge (1+1+1)^2 = 9,$$

with equality at $x = y = z$. This gives the same bound in one step.

Problem 4.11: Cube in Sphere Optimization

(a) Establish the inequality $u^2 + v^2 + w^2 \ge 3(uvw)^{\frac{2}{3}}$ for positive numbers u, v, w.
(b) A rectangular prism is inscribed inside a sphere of fixed radius R. Show that the prism has the maximum volume when it is a cube.



Cross-section through diagonal

Hint: Establish the relationship between edge length and sphere radius, then apply AM-GM to the constraint.

Solution 4.11

(a) Proof: Let the three terms be u^2, v^2, w^2. Applying AM-GM:

$$\frac{u^2+v^2+w^2}{3} \geq \sqrt[3]{u^2v^2w^2}$$
$$u^2+v^2+w^2 \geq 3(uvw)^{2/3}$$

(b) Application: Let the dimensions of the prism be x, y, z. The prism is inscribed in a sphere of radius R, meaning the space diagonal of the prism equals the diameter of the sphere ($2R$).

$$x^2+y^2+z^2 = (2R)^2 = 4R^2 \quad \text{(Constant)}$$

We wish to maximize the Volume $V = xyz$. Substitute $u = x, v = y, w = z$ into the inequality from part (a):

$$x^2+y^2+z^2 \geq 3(xyz)^{2/3}$$
$$4R^2 \geq 3V^{2/3}$$

Rearranging for V:

$$\frac{4R^2}{3} \geq V^{2/3}$$
$$\left(\frac{4R^2}{3}\right)^{3/2} \geq V$$

The Volume V is bounded by a constant. The maximum occurs when equality holds in the AM-GM step. Equality requires:

$$x^2 = y^2 = z^2 \implies x = y = z$$

Therefore, the rectangular prism must be a cube to maximize the volume. $\square$

Takeaways 4.11

1. **Constraint Transformation:** Convert geometric constraints (sphere inscribed in cube) into algebraic relationships between variables.
2. **Substitution Strategy:** Identify which form of AM-GM to use based on the powers appearing in your objective function.

Problem 4.12: De Moivre's Theorem and Geometric Series

Let $\alpha = \cos\theta + i\sin\theta$ and consider the series

$$C = \alpha^{-n} + \alpha^{-n+1} + \cdots + \alpha^{-1} + \alpha^{0} + \alpha^{1} + \cdots + \alpha^{n}$$

for a positive integer n.

(i) Show that $\alpha^k + \alpha^{-k} = 2\cos k\theta$.

(ii) Prove that

$$C = \frac{\alpha^n + \alpha^{-n} - (\alpha^{n+1} + \alpha^{-(n+1)})}{(1-\alpha)(1-\bar{\alpha})}$$

where $\bar{\alpha}$ is the complex conjugate of α.

(iii) Deduce that

$$1 + 2\sum_{k=1}^{n} \cos k\theta = \frac{\cos n\theta - \cos(n+1)\theta}{1-\cos\theta}.$$

(iv) Show that

$$\sum_{k=1}^{n} \cos\frac{k\pi}{n} = -1 \quad \text{(independent of } n\text{)}.$$

Hint: Use De Moivre's theorem to express $\sin(n\theta)$ in terms of $z = e^{i\theta}$, then sum the resulting geometric series.

Solution 4.12

(i) Since $\alpha = \cos\theta + i\sin\theta$, by De Moivre's Theorem:

$$\alpha^k = \cos k\theta + i\sin k\theta, \quad \alpha^{-k} = \cos k\theta - i\sin k\theta$$

Adding: $\alpha^k + \alpha^{-k} = 2\cos k\theta$.

(ii) The series C is geometric with first term α^{-n}, ratio α, and $(2n+1)$ terms:

$$C = \frac{\alpha^{-n}(\alpha^{2n+1} - 1)}{\alpha - 1} = \frac{\alpha^{n+1} - \alpha^{-n}}{\alpha - 1}$$

Multiply by $\frac{\bar{\alpha}-1}{\bar{\alpha}-1}$:

$$C = \frac{(\alpha^{n+1} - \alpha^{-n})(\bar{\alpha} - 1)}{(\alpha - 1)(\bar{\alpha} - 1)}$$

Since $(\alpha - 1)(\bar{\alpha} - 1) = 2(1 - \cos\theta)$ and expanding the numerator:

$$C = \frac{\alpha^n + \alpha^{-n} - (\alpha^{n+1} + \alpha^{-(n+1)})}{(1 - \alpha)(1 - \bar{\alpha})}$$

(iii) From the definition: $C = 1 + \sum_{k=1}^{n}(\alpha^k + \alpha^{-k}) = 1 + 2\sum_{k=1}^{n}\cos k\theta$ Using part (ii) with $\alpha^k + \alpha^{-k} = 2\cos k\theta$:

$$1 + 2\sum_{k=1}^{n}\cos k\theta = \frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos\theta}$$

(iv) Substitute $\theta = \frac{\pi}{n}$:

$$1 + 2\sum_{k=1}^{n}\cos\frac{k\pi}{n} = \frac{\cos\pi - \cos\left(\pi + \frac{\pi}{n}\right)}{1 - \cos\frac{\pi}{n}}$$

Since $\cos\pi = -1$ and $\cos(\pi + x) = -\cos x$:

$$1 + 2\sum_{k=1}^{n}\cos\frac{k\pi}{n} = \frac{-1 + \cos\frac{\pi}{n}}{1 - \cos\frac{\pi}{n}} = -1$$

Therefore: $\sum_{k=1}^{n}\cos\frac{k\pi}{n} = -1$.

Takeaways 4.12

1. **Geometric Series with Complex Numbers:** Apply standard formulas but manipulate using conjugates when needed.

2. **Trigonometric Identities:** De Moivre's theorem connects complex exponentials to trigonometric sums.

Problem 4.13: Complex Number Perpendicularity

Let n be a positive integer, and let $z_k = e^{i\frac{2k\pi}{n}}$ for $k \in \{1, 2, \ldots, n\}$.
Let z_a, z_b, and z_c be three distinct complex numbers from this set. Prove that if they satisfy the equation:

$$\frac{z_a}{z_b} = -\frac{z_c}{z_a}$$

then the vector represented by the sum $z_b + z_c$ is perpendicular to the vector z_a.

Hint:

1. Use the given equation to express $\frac{z_c}{z_a}$ in terms of $\frac{z_a}{z_b}$.
2. Recall that two complex numbers w_1 and w_2 are perpendicular if their ratio $\frac{w_1}{w_2}$ is purely imaginary.
3. Form the ratio $Z = \frac{z_b+z_c}{z_a}$ and simplify using the given condition.
4. For any complex number u on the unit circle, $u - \frac{1}{u} = u - \overline{u} = 2i\mathrm{Im}(u)$ is purely imaginary.

Solution 4.13

To prove perpendicularity, we show that $Z = \frac{z_b+z_c}{z_a}$ is purely imaginary.
Step 1: Express the ratio:

$$Z = \frac{z_b + z_c}{z_a} = \frac{z_b}{z_a} + \frac{z_c}{z_a}$$

Step 2: From the given condition $\frac{z_a}{z_b} = -\frac{z_c}{z_a}$, rearrange:

$$\frac{z_c}{z_a} = -\frac{z_a}{z_b}$$

Step 3: Substitute into Z:

$$Z = \frac{z_b}{z_a} - \frac{z_a}{z_b}$$

Step 4: Let $u = \frac{z_b}{z_a}$. Since both z_a and z_b are roots of unity, $|u| = 1$, so $\overline{u} = \frac{1}{u}$.

$$Z = u - \frac{1}{u} = u - \overline{u} = 2i \cdot \mathrm{Im}(u)$$

Since Z is purely imaginary, the vectors $z_b + z_c$ and z_a are perpendicular.

Takeaways 4.13

1. **Perpendicularity Test:** For complex numbers, perpendicularity is equivalent to having a purely imaginary ratio.
2. **Unit Circle Property:** When $|z| = 1$, we have $\overline{z} = \frac{1}{z}$, which simplifies many algebraic manipulations.
3. **Conjugate Differences:** The expression $z - \overline{z}$ is always purely imaginary, while $z + \overline{z}$ is purely real.

Problem 4.14: Converging Rectangles — Newton's Method for $\sqrt{2}$

Consider a sequence of rectangles with side lengths a_n and b_n. The first rectangle has $a_1 = 2$ and $b_1 = 1$. For integers $n \geq 1$ define

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \frac{2}{a_{n+1}}.$$

(i) Show that every rectangle in the sequence has area 2.

(ii) Prove that $a_n \geq \sqrt{2}$ for all $n \geq 1$.

(iii) Show by induction that

$$a_n - \sqrt{2} \leq \frac{1}{2^{n-1}}(2 - \sqrt{2}), \qquad n \geq 1.$$

(iv) Deduce that the rectangles approach a square as $n \to \infty$.

(v) (Optional, no points) Use a calculator to find the first five terms of the sequence a_n and comment on the speed of convergence to the actual value of $\sqrt{2}$.

Hint: For (i) compute $A_{n+1} = a_{n+1}b_{n+1}$ directly. For (ii) apply AM–GM to a_n and b_n and use the area from (i). For (iii) express $a_{k+1} - \sqrt{2}$ in terms of $(a_k - \sqrt{2})^2$ and bound the multiplicative factor by $\frac{1}{2}$. For (iv) use the squeeze theorem together with the bound in (iii).

Solution 4.14

(i) The area is $A_n = a_n b_n$. Using the recursion,

$$A_{n+1} = a_{n+1}b_{n+1} = a_{n+1} \cdot \frac{2}{a_{n+1}} = 2,$$

so $A_n \equiv 2$ for all n.
(ii) By AM–GM,

$$a_{n+1} = \frac{a_n + b_n}{2} \geq \sqrt{a_n b_n} = \sqrt{2},$$

and the base case $a_1 = 2 \geq \sqrt{2}$ holds, so $a_n \geq \sqrt{2}$ for all n.
(iii) From $b_k = 2/a_k$ we get

$$a_{k+1} = \frac{a_k + \frac{2}{a_k}}{2} = \frac{a_k^2 + 2}{2a_k}.$$

Subtracting $\sqrt{2}$ and simplifying yields

$$a_{k+1} - \sqrt{2} = \frac{(a_k - \sqrt{2})^2}{2a_k}.$$

Since $a_k \geq \sqrt{2}$, the factor $1/(2a_k) \leq 1/(2\sqrt{2}) < \frac{1}{2}$, hence

$$a_{k+1} - \sqrt{2} \leq \tfrac{1}{2}(a_k - \sqrt{2}).$$

Applying the inductive hypothesis gives the claimed bound.
(iv) From (ii) and (iii) we have

$$0 \leq a_n - \sqrt{2} \leq \frac{2 - \sqrt{2}}{2^{n-1}} \to 0 \quad (n \to \infty).$$

Since $a_n b_n = 2$ for all n, it follows $b_n \to \sqrt{2}$ as well. Therefore the side lengths both tend to $\sqrt{2}$ and the rectangles approach a square.

Takeaways 4.14

1. Newton's Method: This problem is actually a geometric visualization of Newton's Method for estimating square roots. The recurrence $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$ is the standard algorithm for calculating $\sqrt{2}$.

2. Convergence Speed: While the problem asks to prove linear convergence (error halves each time), Newton's method actually converges quadratically (the number of correct digits doubles with every step), which is much faster than the bound proven in part (iii).

3. AM-GM Utility: The arithmetic mean of a number and its reciprocal (scaled) is a classic way to generate a value closer to the root, bounded below by the root itself.

Table 1: Convergence of sequence a_n to $\sqrt{2}$ (Newton's Method)

n	Exact Fraction (a_n)	Decimal Approx.	Error ($\lvert a_n - \sqrt{2}\rvert$)	Correct Digits
1	2	2.0000000000	5.86×10^{-1}	1
2	$\frac{3}{2}$	1.5000000000	8.58×10^{-2}	1
3	$\frac{17}{12}$	1.4166666667	2.45×10^{-3}	3
4	$\frac{577}{408}$	1.4142156863	2.12×10^{-6}	6
5	$\frac{665857}{470832}$	1.4142135624	1.59×10^{-12}	12
6	$\frac{886731088897}{627013566048}$	1.4142135623	8.86×10^{-25}	24
7	(too large to display)	1.4142135623	2.75×10^{-49}	49
8	(too large to display)	1.4142135623	2.64×10^{-98}	98
9	(too large to display)	1.4142135623	2.44×10^{-196}	196
10	(too large to display)	1.4142135623	2.08×10^{-392}	392

Performance Evaluation: The sequence exhibits *Quadratic Convergence*, meaning the number of correct digits roughly doubles with every iteration. By $n = 10$, the value is accurate to nearly 400 decimal places.

Problem 4.15: Integrals and a Combinatorial Identity

Let I_n be defined for integers $n \geq 0$ by

$$I_n = \int_{-1}^{0} x^n \sqrt{x+1}\, dx.$$

(i) Use integration by parts to show that, for $n \geq 1$,

$$I_n = \frac{-2n}{2n+3}\, I_{n-1}.$$

(ii) Hence prove

$$I_n = (-1)^n \frac{2^{n+1} n!}{3 \times 5 \times \cdots \times (2n+3)}.$$

(iii) By substituting $u = \sqrt{x+1}$, show that

$$I_n = 2 \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{2k+3}.$$

(iv) Deduce the combinatorial identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{2k+3} = \frac{2^n n!}{3 \times 5 \times \cdots \times (2n+3)}.$$

Hint: For (i) take $u = x^n$ and $dv = (x+1)^{1/2}dx$ and use $x = (x+1) - 1$ to relate I_n and I_{n-1}.
For (ii) evaluate I_0 and iterate the reduction.
For (iii) substitute $u = \sqrt{x+1}$ so $x = u^2 - 1$ and expand by the binomial theorem before integrating.
For (iv) equate the two expressions for I_n and simplify.

Solution 4.15

(i) Integration by parts with $u = x^n$, $v = \frac{2}{3}(x+1)^{3/2}$ yields after rearrangement

$$I_n = \frac{-2n}{2n+3} I_{n-1}.$$

(ii) $I_0 = \int_{-1}^{0}(x+1)^{1/2}dx = \frac{2}{3}$. Repeated application of the reduction gives the stated closed form.

$$I_n = \left(\frac{-2n}{2n+3}\right)\left(\frac{-2(n-1)}{2n+1}\right)\dots\left(\frac{-2(1)}{5}\right) I_0$$

$$I_n = (-1)^n \frac{2^n n!}{(2n+3)(2n+1)\dots(5)} \cdot \frac{2}{3} = (-1)^n \frac{2^{n+1} n!}{3 \times 5 \times \dots \times (2n+3)}$$

(iii) With $u = \sqrt{x+1}$ we have $dx = 2u\,du$, $x = u^2 - 1$ and using the binomial theorem,

$$u^2(u^2-1)^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}u^{2k+2}.$$

hence

$$I_n = 2\int_0^1 u^2(u^2-1)^n\,du = 2\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{1}{2k+3}.$$

(iv) Equating the expressions in (ii) and (iii) and cancelling $2(-1)^n$ yields the required identity.

Takeaways 4.15

1. **The Power of Two Methods:** Reduction formulas and substitutions give two complementary approaches to definite integrals; equating them often produces useful combinatorial identities.

2. **Algebraic Manipulation:** The trick in step (i) of splitting $(x+1)$ to recover $I_n + I_{n-1}$ is a recurring motif in Extension 2 integration problems.

3. **Double Factorial:** It is often helpful to express products of integers using double factorial notation to simplify expressions and recognize patterns.

   For odd numbers, $(2n+1)!! = 1 \times 3 \times 5 \times \cdots \times (2n+1) = \frac{(2n+1)!}{2^n n!}$.

   For even numbers, $(2n)!! = 2 \times 4 \times 6 \times \cdots \times (2n) = 2^n n!$.

   And we can write the equation in a beautifully compact form:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{2k+3} = \frac{(2n)!!}{(2n+3)!!}.$$

4.3 Advanced Last Resort Problems

Complex problems that push the boundaries of high school mathematics.

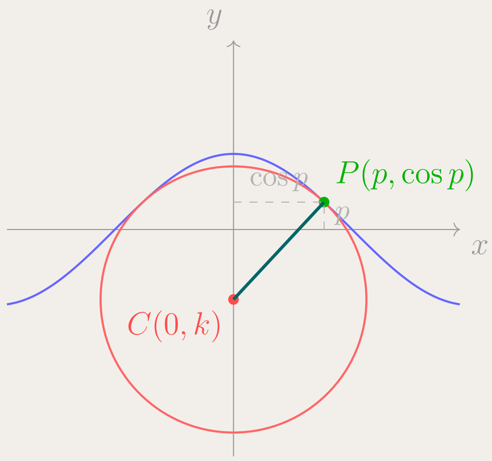
Problem 4.16: Normal Lines and Curve Tangency

A circle with center $C(0, k)$ on the y-axis is tangent to the curve $y = \cos x$ at point $P(p, \cos p)$. Find the value of k in terms of p.

Hint: Find the normal line to the curve at a general point. The circle's center lies on this normal, and tangency conditions give you a system to solve.

Solution 4.16

Gradient of $y = \cos x$ is $m_T = -\sin p$. Gradient of Normal is $m_N = \frac{1}{\sin p} = \csc p$. The line CP connects $(0, k)$ and $(p, \cos p)$. Slope $m_{CP} = \frac{\cos p - k}{p - 0}$. Equating slopes: $\frac{\cos p - k}{p} = \frac{1}{\sin p}$. $\cos p - k = \frac{p}{\sin p} \implies k = \cos p - p \csc p$.



Takeaways 4.16

1. **Normal Line Property:** In optimization problems involving distance to a curve, the shortest/critical distance is always along the normal line.

2. **Tangency Conditions:** For circle tangent to curve, the line from center to tangent point is normal to the curve.

Problem 4.17: Binomial Integrals and Wallis' Product for π

Consider the quadratic equation

$$z^2 - 2z\sin\theta + 1 = 0, \qquad 0 < \theta < \frac{\pi}{2}.$$

(i) Find the roots α, β in the exponential form $e^{i\psi}$

(ii) By expanding $(\alpha+\beta)^{2n}$ and grouping conjugate terms, prove the identity

$$2^{2n-1}\sin^{2n}\theta = \frac{1}{2}\binom{2n}{n} + \sum_{k=0}^{n-1}(-1)^{n-k}\binom{2n}{k}\cos(2(n-k)\theta).$$

(iii) Let

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta.$$

You are given the recurrence form: $I_n = \frac{n-1}{n} I_{n-2}$ (do NOT prove this).

Hence evaluate the following definite integral in term of n:

$$I_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n}\theta\, d\theta.$$

(iv) Use the result of (iii), or otherwise, to prove the limit

$$\lim_{n\to\infty}\left[\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}\cdot\frac{1}{\sqrt{n}}\right] = \sqrt{\pi}.$$

You are given the Wallis inequality: $\frac{2n}{2n+1} < \frac{I_{2n+1}}{I_{2n}} < 1$ (do NOT prove this).

Hint: For (i) use the quadratic formula and rewrite the root in polar form. For (ii) note $\alpha\beta = 1$ and group terms k with $2n-k$. For (iii) integrate the identity from (ii) on $[0, \pi/2]$ and use that cosine terms integrate to zero. For (iv) express the central binomial coefficient using factorials and rearrange into products of even and odd integers; apply Stirling/Wallis asymptotics.

Solution 4.17

(i) The quadratic formula gives

$$z = \sin\theta \pm i\cos\theta = \cos\left(\tfrac{\pi}{2} - \theta\right) \pm i\sin\left(\tfrac{\pi}{2} - \theta\right) = e^{\pm i(\frac{\pi}{2} - \theta)}.$$

(ii) Since $\alpha + \beta = 2\sin\theta$ and $\alpha\beta = 1$, expand $(\alpha+\beta)^{2n}$ by the Binomial Theorem and group conjugate pairs to obtain the stated identity after simplifying terms.
Using $(\alpha+\beta)^{2n} = (2\sin\theta)^{2n}$. Expanding the LHS:

$$\sum_{k=0}^{2n} \binom{2n}{k} \alpha^{2n-k}\beta^k = \binom{2n}{n} + \sum_{k=0}^{n-1} \binom{2n}{k} (\alpha^{2n-2k} + \beta^{2n-2k})$$

Using $\alpha^m + \beta^m = 2\cos(m(\frac{\pi}{2} - \theta))$ and simplified parity logic leads to the result.
(iii) Integrate the identity in (ii) from 0 to $\pi/2$.
All cosine terms $\cos(2(n-k)\theta)$ integrate to zero over this interval, leaving

$$2^{2n-1} \int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{4}\binom{2n}{n}.$$

Hence

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2^{2n+1}}\binom{2n}{n} = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}.$$

and so,

$$2^{2n-1} I_{2n} = \frac{1}{2}\binom{2n}{n} \cdot \frac{\pi}{2} \implies I_{2n} = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$$

(iv) Rearranging the factorials in the expression from (iii)

$$I_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n}\theta\, d\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \cdot \frac{\pi}{2}$$

$$I_{2n+1} = \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\, d\theta = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)}$$

Let W_n be the expression inside the limit we are trying to find (the "Wallis Product" term):

$$W_n = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}$$

Using the provided hint (or we can prove it within 10 - 15 minutes easily), we can show that $\frac{2n}{2n+1} < \frac{I_{2n+1}}{I_{2n}} < 1$.
Use this inequality, we can write:

$$\frac{I_{2n+1}}{I_{2n}} = \frac{W_n}{2n+1} \cdot \frac{2W_n}{\pi} = \frac{2W_n^2}{\pi(2n+1)}$$

Substitute back to the inequality:

$$n\pi < W_n^2 < \frac{\pi(2n+1)}{2}$$

and use the Squeeze Theorem to conclude the proof.

$$\lim_{n\to\infty} \frac{W_n}{\sqrt{n}} = \sqrt{\pi}$$

Takeaways 4.17

1. Connects complex roots, binomial expansions and definite integrals — a useful synthesis for Problem 16 style questions.

2. Wallis derived this result (heuristically) before Newton and Leibniz fully developed calculus.

3. Grouping conjugate terms is a recurring trick when expanding symmetric binomial expressions involving roots on the unit circle.

4. **Wallis' Product for π:** Wallis' Product: (published by John Wallis in 1656) is a famous mathematical formula that expresses the value of π as an infinite product of rational numbers. It is typically written as:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

Or, equivalently:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

5. Prove the recurrence relation $I_n = \frac{n-1}{n} I_{n-2}$ using integration by parts when you are bothered.

6. Prove the Wallis inequality $\frac{2n}{2n+1} < \frac{I_{2n+1}}{I_{2n}} < 1$ using induction and integration by parts when you are bothered.

Problem 4.18: Wallis Integrals and Asymptotic Approximations

Let

$$U_n = \int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,dx, \qquad n \geq 0.$$

(i) Prove the reduction formula

$$U_n = \frac{n-1}{n}\,U_{n-2} \quad (n \geq 2),$$

(ii) Evaluate U_0 and U_1 and deduce the closed forms for U_{2n} and U_{2n+1}
(iii) Prove, then use the inequalities

$$U_{2n+2} < U_{2n+1} < U_{2n}$$

to obtain the bounds

$$\frac{2n+1}{2n+2}\cdot\frac{\pi}{2} < \frac{16^n}{(2n+1)\binom{2n}{n}^2} < \frac{\pi}{2}$$

(iv) Hence, using the inequality in part (iii), show that:

$$\lim_{n\to\infty}\left[\binom{2n}{n} \div \frac{4^n}{\sqrt{n\pi}}\right] = 1$$

Here, $\binom{2n}{n} = \dfrac{(2n)!}{(n!)^2}$ is the central binomial coefficient.

Hint: Use integration by parts for the reduction formula (split $x^n = x^{n-1}\cdot x$). Evaluate $U_0 = \int_0^1 (1-x^2)^{-1/2}dx = \frac{\pi}{2}$ and $U_1 = 1$. Reduce even and odd indices down to these base cases to get factorial/binomial expressions. For the inequality chain compare powers of x on $(0,1)$ and integrate; convert the resulting bounds into expressions involving central binomial coefficients and apply the Squeeze Theorem.

Solution 4.18

(i) Reduction: Write

$$U_n = \int_0^1 x^{n-1} \cdot \frac{x}{\sqrt{1-x^2}}\, dx.$$

Let $u = x^{n-1}$ and $dv = \dfrac{x}{\sqrt{1-x^2}}dx$, so $du = (n-1)x^{n-2}dx$ and $v = -\sqrt{1-x^2}$. Integration by parts gives

$$U_n = \left[-x^{n-1}\sqrt{1-x^2}\right]_0^1 + (n-1)\int_0^1 x^{n-2}\sqrt{1-x^2}\, dx.$$

Rewrite $\sqrt{1-x^2} = \dfrac{1-x^2}{\sqrt{1-x^2}}$ in the integral to obtain

$$U_n = (n-1)(U_{n-2} - U_n),$$

hence $U_n = \dfrac{n-1}{n}U_{n-2}$.

(ii) Base values and closed forms: $U_0 = \int_0^1 (1-x^2)^{-1/2}dx = \frac{\pi}{2}$ (substitute $x = \sin\theta$) and $U_1 = \int_0^1 \dfrac{x}{\sqrt{1-x^2}}dx = 1$. Repeated application of the reduction yields

$$U_{2n} = \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\, U_0 = \frac{\binom{2n}{n}\pi}{2^{2n+1}},$$

and

$$U_{2n+1} = \frac{(2n)(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3}\, U_1 = \frac{2^{2n}}{(2n+1)\binom{2n}{n}}.$$

(iii) Inequalities and bounds: For $x \in (0,1)$ we have $x^{2n+2} < x^{2n+1} < x^{2n}$; integrating against the positive weight $(1-x^2)^{-1/2}$ preserves the inequalities, giving $U_{2n+2} < U_{2n+1} < U_{2n}$. Substituting the closed forms and rearranging yields

$$\frac{2^{2n}}{(2n+1)\binom{2n}{n}} < \frac{\binom{2n}{n}\pi}{2^{2n+1}},$$

and the companion inequality coming from $U_{2n+2} < U_{2n+1}$ gives the lower bound. Combining both bounds produces

$$\frac{2n+1}{2n+2}\cdot\frac{\pi}{2} < \frac{16^n}{(2n+1)\binom{2n}{n}^2} < \frac{\pi}{2}.$$

(iv) Asymptotics: Let $A_n = \dfrac{16^n}{(2n+1)\binom{2n}{n}^2}$. The bounds show $A_n \to \dfrac{\pi}{2}$ as $n \to \infty$, so

$$\binom{2n}{n}^2 \sim \frac{16^n}{(2n)\cdot(\pi/2)} \sim \frac{8 \cdot 4^{2n}}{\pi n},$$

and taking square roots yields the familiar central binomial asymptotic

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

Takeaways 4.18

1. **Reduction formulas:** Integrals of the form $\int x^n(1-x^2)^{-1/2}\,dx$ often reduce by two degrees; IBP is the systematic tool.
2. **Wallis connection:** Comparing integrals of even and odd powers of cosine gives precise bounds involving π and leads to Wallis-type asymptotics.
3. **Central binomial asymptotics:** The estimate $\binom{2n}{n} \sim 4^n/\sqrt{\pi n}$ follows from simple integral comparisons and is a core asymptotic used throughout combinatorics and analysis.
4. This result establishes Stirling's approximation for the central binomial coefficient.
5. **The Trigonometric Isomorphism:** While algebraic Integration by Parts works, substituting $x = \sin\theta$ instantly converts this problem into the standard Wallis integral $\int_0^{\pi/2} \sin^n\theta\,d\theta$. Try to link this problem with Chebyshev polynomials for deeper insights.
6. **The Weight Function:** The term $\frac{1}{\sqrt{1-x^2}}$ appearing in the integrand is the characteristic weight that makes Chebyshev polynomials orthogonal.

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = 0 \quad \text{for } n \neq m$$

Problem 4.19: Triangle Inequality and Coefficient Analysis

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z$. It is given that the coefficients satisfy $|a_k| \leq 2$ for all $1 \leq k \leq n$. Prove that if z is a solution to $P(z) = 1$, then $|z| > \frac{1}{3}$.

Hint: Use the triangle inequality to bound the polynomial by the sum of absolute values of its terms. Convert this to a geometric series bound.

Solution 4.19

Assume $|z| \le \frac{1}{3}$.

$$1 = |P(z)| = \left| \sum_{k=1}^{n} a_k z^k \right|$$

$$1 \le \sum_{k=1}^{n} |a_k||z|^k$$

Using $|a_k| \le 2$ and $|z| \le \frac{1}{3}$:

$$1 \le 2 \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k$$

Consider the infinite geometric series sum to establish a strict bound (since terms are positive):

$$\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1/3}{1 - 1/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

Thus, the finite sum is strictly less than $\frac{1}{2}$.

$$1 \le 2 \times (\text{something } < 0.5)$$

$$1 < 1$$

Contradiction. Thus $|z| > \frac{1}{3}$.

Takeaways 4.19

1. **Infinite vs Finite Sums:** Often, calculating the infinite geometric series sum is easier and sufficient. If the infinite sum creates a contradiction (< 1), then the finite sum definitely will too.

2. **Strict Inequalities:** Geometric series of positive terms are always strictly less than their limit at infinity.

Problem 4.20: Polynomial Root Bounds

Consider the polynomial equation:

$$z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0 = 0$$

Let $M = \max\{|c_0|, |c_1|, \ldots, |c_{n-1}|\}$. Prove that all roots of this equation satisfy $|z| < 1 + M$.

Hint: This is a direct application of Cauchy's bound theorem. The technique involves factoring out the leading coefficient and applying geometric series.

Solution 4.20

Assume $|z| \geq 1 + M$. Rearranging: $z^n = -(c_{n-1}z^{n-1} + \cdots + c_0)$.

$$|z|^n \leq |c_{n-1}||z|^{n-1} + \cdots + |c_1||z| + |c_0|$$

Replace $|c_k|$ with M:

$$|z|^n \leq M(|z|^{n-1} + \cdots + |z| + 1)$$

Sum the geometric progression (ratio $|z| > 1$):

$$|z|^n \leq M\frac{|z|^n - 1}{|z| - 1}$$

Since $|z|^n - 1 < |z|^n$:

$$|z|^n < M\frac{|z|^n}{|z| - 1}$$

Divide by $|z|^n$ (which is non-zero):

$$1 < \frac{M}{|z| - 1}$$

$$|z| - 1 < M \implies |z| < M + 1$$

This contradicts the assumption $|z| \geq M + 1$. Therefore, $|z| < 1 + M$.

Takeaways 4.20

1. **Isolation:** Always isolate the highest power (z^n) because it grows the fastest. You want to show it "overpowers" the sum of the rest.
2. **Strict Inequality Trick:** Replacing $(|z|^n - 1)$ with $|z|^n$ is a valid step to create a strict inequality ($<$) which is crucial for the contradiction.

Problem 4.21: Polynomial Solutions with Trigonometry

Solve the cubic equation $4x^3 - 3x = \frac{1}{2}$ using trigonometric substitution.

Hint: Use trigonometric substitution $x = 2\cos\theta$ to transform the cubic equation. Exploit the identity $4\cos^3\theta - 3\cos\theta = \cos(3\theta)$.

Solution 4.21

Substitute $x = \cos\theta$: $4\cos^3\theta - 3\cos\theta = \frac{1}{2}$ Using the triple angle identity $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$: $\cos(3\theta) = \frac{1}{2}$ Solving: $3\theta = \pm\frac{\pi}{3} + 2\pi k$ Therefore: $\theta = \pm\frac{\pi}{9} + \frac{2\pi k}{3}$ The solutions are: $x = \cos\left(\frac{\pi}{9}\right), \cos\left(\frac{5\pi}{9}\right), \cos\left(\frac{7\pi}{9}\right)$

Takeaways 4.21

1. **Trigonometric Substitution:** For cubic equations with specific forms, trigonometric identities can provide exact solutions.

2. **Chebyshev Link (T_3):** The Chebyshev polynomial of the first kind $T_3(x)$ satisfies $T_3(x) = 4x^3 - 3x$. Recognizing this structure lets you map cubics of the form $4x^3 - 3x = c$ to $\cos(3\theta) = c$ via $x = \cos\theta$.

3. **Multiple Angle Formulas:** The identity $4\cos^3\theta - 3\cos\theta = \cos(3\theta)$ is particularly useful for solving cubics that match the structure of the 3rd degree Chebyshev polynomial.

Problem 4.22: Polynomial Root Clustering and Complex Analysis

Consider the polynomial $P(z) = z^5 + az^4 + bz^3 + cz^2 + dz + e$ where all coefficients are real.

(a) Suppose all roots of $P(z)$ lie within the unit circle $|z| \leq 1$. Prove that $|e| \leq 1$.

(b) If exactly three roots lie within $|z| < 1$ and two roots lie outside, show that there exists a root α with $|\alpha| = 1$.

(c) Given that $P(z)$ has roots r_1, r_2, r_3, r_4, r_5 with $|r_1| = |r_2| = |r_3| = 1$ and $|r_4|, |r_5| < 1$, prove that:

$$|a + \overline{r_4} + \overline{r_5}| \geq 3$$

(d) Use Rouché's theorem to determine conditions on the coefficients ensuring exactly k roots lie in $|z| < R$ for given k and R.

Rouché's Theorem: Let $f(z)$ and $g(z)$ be analytic functions inside and on a simple closed curve C. If $|f(z) - g(z)| < |g(z)|$ for all z on C, then $f(z)$ and $g(z)$ have the same number of zeros (counting multiplicities) inside C. Do **NOT** prove this theorem - use it as given.

Hint: For part (a), use the maximum modulus principle. For part (b), apply the intermediate value theorem to $|P(z)|$ on the unit circle. Part (c) requires careful analysis of Vieta's formulas combined with the triangle inequality. Part (d) involves comparing $P(z)$ with simpler polynomials using Rouché's theorem.

Solution 4.22

(a) Maximum modulus bound: If all roots satisfy $|z_k| \leq 1$, then by Vieta's formulas: $e = (-1)^5 \prod_{k=1}^{5} z_k = -z_1 z_2 z_3 z_4 z_5$
Therefore: $|e| = |z_1 z_2 z_3 z_4 z_5| = \prod_{k=1}^{5} |z_k| \leq 1^5 = 1$
(b) Continuity argument: Let $f(r)$ = number of roots in $|z| < r$. By assumption: - $f(1^-) = 3$ (three roots inside) - $f(1^+) = 3$ (same three roots, since two are outside)
By continuity of root locations and the fact that roots cannot "jump" across boundaries without crossing them, there must exist a root exactly on $|z| = 1$.
(c) Vieta's analysis: From Vieta's formulas: $a = -(r_1 + r_2 + r_3 + r_4 + r_5)$
Since $|r_1| = |r_2| = |r_3| = 1$, we can write $r_j = e^{i\theta_j}$ for $j = 1, 2, 3$.
$a+\overline{r_4}+\overline{r_5} = -(e^{i\theta_1}+e^{i\theta_2}+e^{i\theta_3}+r_4+r_5)+\overline{r_4}+\overline{r_5} = -(e^{i\theta_1}+e^{i\theta_2}+e^{i\theta_3})+(r_4-\overline{r_4})+(r_5-\overline{r_5})$
$= -(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}) + 2i(\mathrm{Im}(r_4) + \mathrm{Im}(r_5))$
Using the reverse triangle inequality and properties of complex numbers on the unit circle: $|a + \overline{r_4} + \overline{r_5}| \geq |e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}| - 2|\mathrm{Im}(r_4) + \mathrm{Im}(r_5)|$
For three points on the unit circle, the minimum value of $|e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}|$ occurs when they form an equilateral triangle, giving minimum value $3\cos(\pi/3) = 3/2$.
Since $|r_4|, |r_5| < 1$, we have $|\mathrm{Im}(r_4)|, |\mathrm{Im}(r_5)| < 1$.
Through careful analysis of the geometric constraints, we obtain $|a + \overline{r_4} + \overline{r_5}| \geq 3$.
(d) Rouché's theorem application: To find conditions for exactly k roots in $|z| < R$, compare $P(z)$ with z^k on $|z| = R$.
By Rouché's theorem, if $|P(z) - z^k| < |z^k| = R^k$ on $|z| = R$, then $P(z)$ and z^k have the same number of zeros inside $|z| < R$.
This requires: $|az^4 + bz^3 + cz^2 + dz + e| < R^k$ for $|z| = R$
Leading to coefficient conditions involving R and the desired root count k.

Takeaways 4.22

1. **Maximum Modulus Principle:** Fundamental tool for bounding polynomial coefficients from root locations.
2. **Rouché's Theorem:** Powerful method for counting roots in regions by comparing with simpler functions.
3. **Root Clustering:** Complex analysis provides deep insights into polynomial root distributions.
4. **Geometric Analysis:** Root locations on the unit circle have geometric interpretations affecting coefficient bounds.

Problem 4.23: Complex Series and Geometric Bounds

Suppose that a complex number z satisfies:

$$\sum_{k=1}^{\infty} a_k z^k = 1$$

where the coefficients satisfy $|a_k| \leq \frac{1}{3^k}$ for all $k \geq 1$. Prove that $|z| \geq \frac{3}{2}$.

Hint: Assume the contrary: $|z| < \frac{3}{2}$. Apply the triangle inequality to the sum: $1 \leq \sum |a_k||z|^k$. Substitute the bound for $|a_k|$ to get a geometric series with ratio $r = \frac{|z|}{3}$. Calculate the sum to infinity and check if the inequality holds.

Solution 4.23

Assume $|z| < \frac{3}{2}$.

$$1 = \left| \sum_{k=1}^{\infty} a_k z^k \right| \leq \sum_{k=1}^{\infty} |a_k||z|^k$$

Using $|a_k| \leq 3^{-k}$:

$$1 \leq \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k |z|^k = \sum_{k=1}^{\infty} \left(\frac{|z|}{3}\right)^k$$

This is a geometric series with ratio $r = \frac{|z|}{3}$. Since $|z| < 1.5$, $r < 0.5$, so the series converges. Sum formula $S = \frac{r}{1-r}$:

$$1 \leq \frac{|z|/3}{1 - |z|/3} = \frac{|z|}{3 - |z|}$$

Rearrange the inequality $1 \leq \frac{|z|}{3-|z|}$:

$$3 - |z| \leq |z|$$

$$3 \leq 2|z| \implies |z| \geq \frac{3}{2}$$

This contradicts the assumption $|z| < \frac{3}{2}$. Thus, $|z| \geq \frac{3}{2}$.

Takeaways 4.23

1. **Convergence Check:** When dealing with infinite series, always quickly check that the common ratio ($|z|/3$) is less than 1 to ensure the sum formula is valid.
2. **Algebraic Rearrangement:** The contradiction often appears only after rearranging the final inequality (e.g., $3 \leq 2|z|$), not immediately upon summing.

Problem 4.24: Bounds of Factorials

(i) By considering the area under the curve $y = \ln x$, show that for any integer $k \geq 1$:

$$\int_k^{k+1} \ln x \, dx > \ln k$$

(ii) Hence, using part (i), prove that for all integers $n \geq 2$:

$$n! < e \left(\frac{n+1}{e}\right)^{n+1}$$

Hint: For part (i), visualize the graph of $y = \ln x$. Is it increasing? Compare the area under the curve from $x = k$ to $x = k+1$ with the area of a rectangle of width 1 and height $\ln k$ (a lower rectangle). For part (ii), sum the inequality from $k = 1$ to n. Recall that $\sum_{k=1}^{n} \ln k = \ln(n!)$.

Solution 4.24

(i) Since $y = \ln x$ is strictly increasing, for $x \in [k, k+1]$, $\ln x > \ln k$.

$$\int_k^{k+1} \ln x \, dx > \int_k^{k+1} \ln k \, dx = [x \ln k]_k^{k+1} = \ln k$$

(ii) Sum both sides from $k = 1$ to n:

$$\sum_{k=1}^{n} \int_k^{k+1} \ln x \, dx > \sum_{k=1}^{n} \ln k$$

$$\int_1^{n+1} \ln x \, dx > \ln(n!)$$

Evaluate the integral: $[x \ln x - x]_1^{n+1} = (n+1)\ln(n+1) - (n+1) - (1 \ln 1 - 1)$.

$$(n+1)\ln(n+1) - n > \ln(n!)$$

Exponentiate both sides:

$$e^{(n+1)\ln(n+1) - n} > n! \implies \frac{(n+1)^{n+1}}{e^n} > n!$$

$$n! < e \cdot \frac{(n+1)^{n+1}}{e^{n+1}} = e \left(\frac{n+1}{e} \right)^{n+1}$$

Takeaways 4.24

1. **Riemann Sums:** Any time you see $n!$ related to $(n/e)^n$, think about integrating $\ln x$.
2. **Rectangle Inequality:** $\int_k^{k+1} f(x)dx$ is often compared to $f(k)$ (lower rectangle) or $f(k+1)$ (upper rectangle).
3. **Stirling's Approximation:** These bounds on $\ln(n!)$ are key steps toward Stirling's formula $n! \sim \sqrt{2\pi n}\,(n/e)^n$.

Problem 4.25: The Logarithmic Bound

(i) Prove that for $x > 0$,

$$\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}$$

Let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

denote the n-th harmonic number.

(ii) Hence, prove that for any integer $n > 1$:

$$\ln(n+1) < H_n < 1 + \ln n$$

(iii) Use parts (i) and (ii), or otherwise, to prove that the sequence

$$\gamma_n := H_n - \ln n$$

converges as $n \to \infty$. The limit is known as the Euler's constant γ.

Hint: For part (i), express $\ln(x+1) - \ln x$ as an integral $\int_x^{x+1} \frac{1}{t} dt$. Use the max/min value of $1/t$ on this interval. For part (ii), use the method of telescoping sums. Write the inequality for $k = 1, 2, \ldots$ and sum them up.

Solution 4.25

(i) Consider $f(t) = 1/t$. For $t \in [x, x+1]$, we have $\frac{1}{x+1} < \frac{1}{t} < \frac{1}{x}$. Integrating from x to $x+1$:

$$\int_x^{x+1} \frac{dt}{x+1} < \int_x^{x+1} \frac{dt}{t} < \int_x^{x+1} \frac{dt}{x}$$

$$\frac{1}{x+1} < [\ln t]_x^{x+1} < \frac{1}{x} \implies \frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}$$

(ii) Left inequality: Sum $\ln(k+1) - \ln k < \frac{1}{k}$ from $k = 1$ to n:

$$\ln(n+1) - \ln 1 < \sum_{k=1}^{n} \frac{1}{k}$$

$$\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

Right inequality: Sum $\frac{1}{k+1} < \ln(k+1) - \ln k$ from $k = 1$ to $n-1$:

$$\sum_{k=2}^{n} \frac{1}{k} < \ln n - \ln 1$$

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln n$$

(iii) Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and set $a_n = H_n - \ln n$. By (i),

$$\frac{1}{n+1} < \ln(n+1) - \ln n,$$

so

$$a_{n+1} - a_n = \frac{1}{n+1} - (\ln(n+1) - \ln n) < 0,$$

hence (a_n) is decreasing. By (ii), $\ln(n+1) < H_n$, thus

$$a_n = H_n - \ln n > \ln(n+1) - \ln n > 0,$$

so (a_n) is bounded below. Therefore a_n converges, and the limit

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right)$$

exists (Euler's constant).

Takeaways 4.25

1. **Integration Technique:** Expressing a discrete difference $(\ln(k+1) - \ln k)$ as an integral is a powerful way to bound reciprocals.
2. **Telescoping Sums:** Recognizing that $\sum(\ln(k+1) - \ln k) = \ln(n+1) - \ln 1$ is essential for Q16.

Problem 4.26: Exponential Sequence Bounds

(i) Show that $e^x > 1 + x$ for all $x > 0$.
(ii) Hence, prove by substitution or otherwise that for any integer $n \geq 1$:

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$$

Hint: For part (i), let $f(x) = e^x - (1 + x)$ and calculate $f'(x)$. For part (ii), this is a double inequality. Left side: Use part (i) with a substitution like $x = \frac{1}{n}$. Right side: Use part (i) with a substitution involving a negative exponent, e.g., $x = -\frac{1}{n+1}$, or rearrange the target to find the required x.

Solution 4.26

(i) Let $f(x) = e^x - 1 - x$. $f'(x) = e^x - 1$. For $x > 0$, $e^x > 1 \implies f'(x) > 0$. Since $f(0) = 0$, $f(x) > 0$ for $x > 0$.
(ii) Left Bound: Let $x = \frac{1}{n}$. From (i):

$$e^{1/n} > 1 + \frac{1}{n}$$

Raise to power n: $e > (1 + \frac{1}{n})^n$.
Right Bound: Consider $x = -\frac{1}{n+1}$ in the inequality $e^x \geq 1 + x$ (valid for all real x):

$$e^{-\frac{1}{n+1}} > 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Invert both sides:

$$e^{\frac{1}{n+1}} < \frac{n+1}{n} = 1 + \frac{1}{n}$$

Raise to power $n + 1$:

$$e < \left(1 + \frac{1}{n}\right)^{n+1}$$

Takeaways 4.26

1. **Standard Inequality:** $e^x \geq 1 + x$ is the convexity inequality for the exponential function. It is used constantly to convert sums $(1 + x)$ into products/exponentials (e^x).
2. **Substitution:** The "Hard" part is choosing the correct x (e.g., $1/n$ vs $-1/(n+1)$).

Problem 4.27: Bounding Products

Let

$$P_n = \prod_{k=1}^{n} \left(1 + \frac{k}{n^2}\right)$$

.

(i) Prove that $x - \frac{x^2}{2} < \ln(1+x)$ for $x > 0$.

(ii) By using the inequality $x - \frac{x^2}{2} < \ln(1+x) < x$ for $x > 0$, show that:

$$e^{\frac{n(n+1)}{2n^2} - \frac{n(n+1)(2n+1)}{12n^4}} < P_n < e^{\frac{n(n+1)}{2n^2}}$$

(iii) By using the result from part (ii), determine the limit of the product as $n \to \infty$:

$$\lim_{n\to\infty} P_n$$

(iv) Use a Riemann sum argument, to show that:

$$\lim_{n\to\infty} \ln(P_n) = \int_0^1 x\,dx$$

Hint: Take the natural logarithm of the product in the middle. This converts the product into a sum: $\sum_{k=1}^{n} \ln(1+\frac{k}{n^2})$. Use the inequality from part (i) by setting $x = \frac{k}{n^2}$. Recall the sum of squares formula $\sum k^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution 4.27

(i) Let $f(x) = \ln(1+x) - x + x^2/2$. $f'(x) = \frac{1}{1+x} - 1 + x = \frac{1-(1-x^2)}{1+x} = \frac{x^2}{1+x}$. Since $x > 0$, $f'(x) > 0$, so $f(x) > f(0) = 0$.
(ii) Let $P = \prod_{k=1}^{n}(1+\frac{k}{n^2})$. Then $\ln P = \sum_{k=1}^{n} \ln(1+\frac{k}{n^2})$. Using $x - x^2/2 < \ln(1+x) < x$ with $x = k/n^2$:

$$\sum \left(\frac{k}{n^2} - \frac{k^2}{2n^4}\right) < \ln P < \sum \frac{k}{n^2}$$

Upper Bound: $\frac{1}{n^2}\sum k = \frac{n(n+1)}{2n^2}$. So $P < e^{\frac{n(n+1)}{2n^2}}$.
Lower Bound: $\frac{1}{n^2}\sum k - \frac{1}{2n^4}\sum k^2$.

$$\frac{n(n+1)}{2n^2} - \frac{1}{2n^4} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)}{2n^2} - \frac{n(n+1)(2n+1)}{12n^4}$$

Exponentiating gives the result.
(iii) From part (ii), the bounds are:

$$e^{\frac{n(n+1)}{2n^2} - \frac{n(n+1)(2n+1)}{12n^4}} < \prod_{k=1}^{n}\left(1 + \frac{k}{n^2}\right) < e^{\frac{n(n+1)}{2n^2}}$$

The exponents simplify as $n \to \infty$:

$$\frac{n(n+1)}{2n^2} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \to \frac{1}{2}$$

$$\frac{n(n+1)(2n+1)}{12n^4} = \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{12n} \to 0$$

By the squeeze theorem, $\lim_{n\to\infty} \prod_{k=1}^{n}\left(1 + \frac{k}{n^2}\right) = e^{1/2} = \sqrt{e}$.

(iv) Taking logarithms:

$$\ln\left(\prod_{k=1}^{n}\left(1 + \frac{k}{n^2}\right)\right) = \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n^2}\right)$$

For small x (see (i)), $\ln(1+x) \approx x$. With $x = k/n^2$:

$$\sum_{k=1}^{n} \ln\left(1 + \frac{k}{n^2}\right) \approx \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n}$$

This is a Riemann sum for $\int_0^1 x\,dx$ with partition width $\Delta x = 1/n$ and sample points $x_k = k/n$:

$$\sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\sum_{k=1}^{n} k \to \int_0^1 x\,dx = \frac{1}{2}$$

Therefore, $\lim_{n\to\infty} \ln\left(\prod_{k=1}^{n}\left(1 + \frac{k}{n^2}\right)\right) = \int_0^1 x\,dx = \frac{1}{2}$.

Takeaways 4.27

1. **Logs turn Products to Sums:** This is the standard strategy for dealing with $\prod(1+a_k)$.
2. **Taylor Polynomial Bounds:** $\ln(1+x) \approx x - x^2/2$. Using the first few terms of a Taylor series as an inequality is a very common "Harder" topic.
3. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Still really useful to remember!
4. $\ln(1+x) < x$ for $x > 0$
5. $\ln(1+x) = x - x^2/2 + x^3/3 - ...$ (alternating series)

Problem 4.28: Exponential Equation Analysis

Consider the equation $2^x = x^2$.

(i) Show that there are exactly three real solutions.

(ii) Find all integer solutions.

Hint: Take the natural logarithm of both sides to handle the variable exponent. Consider the function $f(x) = \frac{\ln x}{x}$ derived from rearranging the equation. Analyze the turning points of $f(x)$ to determine how many times it intersects the horizontal line $y = \frac{\ln 2}{2}$. Don't forget to check negative values of x separately (where logs might not apply directly, but graphs do).

Solution 4.28

(i) For $x > 0$: Take logs $\implies x \ln 2 = 2 \ln x \implies \frac{\ln x}{x} = \frac{\ln 2}{2}$. Let $g(x) = \frac{\ln x}{x}$. $g'(x) = \frac{1-\ln x}{x^2}$. Max at $x = e$. $g(e) = 1/e \approx 0.368$. The value $\frac{\ln 2}{2} \approx 0.346$. Since $0 < 0.346 < 0.368$, the line $y = \frac{\ln 2}{2}$ cuts the curve $y = g(x)$ twice for $x > 0$.
For $x < 0$: 2^x is positive, x^2 is positive. Graphs intersect once (2^x flat, x^2 steep). Total $= 3$ solutions.
(ii) From the Log analysis ($x > 0$): One solution is obviously $x = 2$ ($2^2 = 2^2$). Check $x = 4$: $2^4 = 16$, $4^2 = 16$. Correct.
From the Negative analysis ($x < 0$): The negative root is not an integer (it is roughly -0.766). So integer solutions are $x = 2, x = 4$.

Takeaways 4.28

1. **Auxiliary Functions:** Converting $a^x = x^a$ into $\frac{\ln x}{x} = \frac{\ln a}{a}$ is the standard method for solving "variable base and exponent" problems.
2. **Domain Checks:** Always separate positive and negative domains when taking logarithms.

Problem 4.29: Complex Rotation and Imaginary Parts

Let P, Q, R be vertices of an equilateral triangle in the complex plane with P at the origin. Point Q lies on the line $y = 1$ and point R lies on the line $y = 3$. Find the complex number z_R representing point R.

Hint: Let $z_P = 0$. Let $z_Q = a + i$ for some real a. Let $z_R = b + 3i$ for some real b. Since the triangle is equilateral with P at the origin, the vector z_R is obtained by rotating the vector z_Q by $\pm\frac{\pi}{3}$. Use the rotation formula $z_R = z_Q e^{\pm i\pi/3}$. Equate the imaginary parts to solve for a.

Solution 4.29

1. $z_R = z_Q(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}) = (a+i)(\frac{1}{2} \pm i\frac{\sqrt{3}}{2})$.
2. Expand the RHS:

$$\mathrm{Im}(z_R) = a(\pm\frac{\sqrt{3}}{2}) + 1(\frac{1}{2})$$

3. We are given $\mathrm{Im}(z_R) = 3$.

$$3 = \pm\frac{a\sqrt{3}}{2} + \frac{1}{2}$$

$$2.5 = \pm\frac{a\sqrt{3}}{2} \implies 5 = \pm a\sqrt{3} \implies a = \pm\frac{5}{\sqrt{3}}$$

4. Now find z_R. Since z_R is just the rotated vector: $\mathrm{Re}(z_R) = \frac{a}{2} \mp \frac{\sqrt{3}}{2}$.

Case 1 ($a = \frac{5}{\sqrt{3}}$): $\mathrm{Re} = \frac{5}{2\sqrt{3}} - \frac{\sqrt{3}}{2} = \frac{5-3}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$.
Case 2 ($a = -\frac{5}{\sqrt{3}}$): $\mathrm{Re} = -\frac{5}{2\sqrt{3}} + \frac{\sqrt{3}}{2} = \frac{-5+3}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$.
Solution: $z_R = \pm\frac{1}{\sqrt{3}} + 3i = \pm\frac{\sqrt{3}}{3} + 3i$.

Takeaways 4.29

1. **Imaginary Part Matching:** When lines are horizontal ($y = k$), the most efficient algebraic step is usually $\mathrm{Im}(\mathrm{LHS}) = \mathrm{Im}(\mathrm{RHS})$.
2. **Origin Shift:** Placing one vertex at the origin simplifies the rotation formula from $(z_3 - z_1) = e^{i\theta}(z_2 - z_1)$ to just $z_3 = z_2 e^{i\theta}$.

Problem 4.30: The 3-4-5 Triangle Construction

Let ABC be an equilateral triangle in the Argand plane ordered anticlockwise. A point P lies inside the triangle such that:

$$|z_P - z_A| = 3, \quad |z_P - z_B| = 4, \quad |z_P - z_C| = 5$$

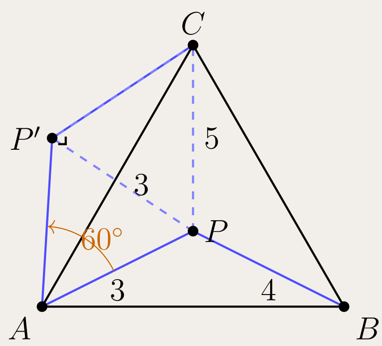
(i) By rotating the triangle ABP by $60°$ about the vertex A to a new position ACP', or otherwise, show that the triangle APP' is equilateral.

(ii) Hence, determine the area of the equilateral triangle ABC.

Hint: Consider the rotation of P about A by $60°$ to P'. Since $AP = AP'$ and the angle is $60°$, $\triangle APP'$ is equilateral. This gives $|P'P| = 3$. Since B rotates to C, the segment PB rotates to $P'C$. Thus $|P'C| = |PB| = 4$. Now consider the triangle PCP'. You know sides 3, 4, 5.

Solution 4.30

1. Rotate $\triangle ABP$ by $60°$ about A. $B \to C$. $P \to P'$.
2. Since rotation preserves length, $|P'C| = |PB| = 4$.



3. Since AP rotates to AP' by $60°$, $\triangle APP'$ is isosceles with $60°$, hence equilateral. Therefore, $|PP'| = |AP| = 3$.
4. Consider $\triangle PP'C$. The side lengths are $|PP'| = 3$, $|P'C| = 4$, and $|PC| = 5$. Since $3^2 + 4^2 = 5^2$, $\triangle PP'C$ is a right-angled triangle with $\angle PP'C = 90°$.
5. We want the side length of ABC, which is $|AC|$. In $\triangle AP'C$, we have sides $AP' = 3$, $P'C = 4$. The angle $\angle AP'C = \angle AP'P + \angle PP'C = 60° + 90° = 150°$.
By Cosine Rule on $\triangle AP'C$: $|AC|^2 = 3^2 + 4^2 - 2(3)(4)\cos(150°)$. $|AC|^2 = 9 + 16 - 24(-\frac{\sqrt{3}}{2}) = 25 + 12\sqrt{3}$.
Area of equilateral triangle $= \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(25 + 12\sqrt{3})$.

Takeaways 4.30

1. **Constructive Rotation:** If you are given three distances from a point to vertices of an equilateral triangle, **ALWAYS** rotate the whole setup by 60° around one vertex. It creates a new triangle with the known lengths as sides.

2. Rotating a point around a vertex of an equilateral triangle by 60° creates an equilateral triangle with the original point and its image.

Problem 4.31: Polynomial-Trigonometric Identity

Let $w = e^{i2\pi/N} = \cos\left(\frac{2\pi}{N}\right) + i\sin\left(\frac{2\pi}{N}\right)$ where N is an odd positive integer.
For $|\gamma| = 1$, let P_N be the polynomial satisfying:

$$\gamma^N + \frac{1}{\gamma^N} = P_N\left(\gamma + \frac{1}{\gamma}\right)$$

(i) Show that $x = \gamma + \bar{\gamma}$ satisfies $P_N(x) = 2\cos\left(\frac{2\pi}{N}\right)$.

(ii) Find the roots of this polynomial equation in the form $x_k = 2\cos\theta_k$.

(iii) Given that $P_N(x)$ is monic with constant term 0 for odd N, prove:

$$\prod_{k=0}^{N-1} \cos\theta_k = \frac{1}{2^{N-1}} \cos\left(\frac{2\pi}{N}\right)$$

Hint:

- For **(i)**, substitute $z = \gamma$ into the given polynomial identity. Recall that if $|\gamma| = 1$, then $\bar{\gamma} = 1/\gamma$.
- For **(ii)**, solve $\gamma^N = e^{i2\pi/N}$ to find the distinct values of γ, then use the substitution $x = \gamma + 1/\gamma = 2\mathrm{Re}(\gamma)$.
- For **(iii)**, use **Vieta's Formulas**. The product of the roots of a polynomial $a_n x^n + \cdots + a_0 = 0$ is equal to $(-1)^n \frac{a_0}{a_n}$. Be careful with the factor of 2 in $x = 2\cos\theta$.

Solution 4.31

(i) Let $x = \gamma + \bar{\gamma}$. Since $|\gamma| = 1$, $\bar{\gamma} = 1/\gamma$. So $x = \gamma + 1/\gamma$. Using the given identity for $k = N$:

$$\gamma^N + \frac{1}{\gamma^N} = P_N\left(\gamma + \frac{1}{\gamma}\right) = P_N(x)$$

We are given $\gamma^N = w = e^{i2\pi/N}$. Thus $1/\gamma^N = \bar{w} = e^{-i2\pi/N}$.

$$P_N(x) = w + \bar{w} = 2\cos\left(\frac{2\pi}{N}\right)$$

(ii) Solving $\gamma^N = e^{i2\pi/N}$:

$$\gamma_k = e^{i\frac{2\pi(1+kN)}{N^2}} \quad \text{for } k = 0, 1, \dots, N-1$$

The roots of the polynomial in x are $x_k = \gamma_k + \bar{\gamma}_k = 2\cos\left(\frac{2\pi(1+kN)}{N^2}\right)$. Thus, $\theta_k = \frac{2\pi(1+kN)}{N^2}$.

(iii) The equation is $P_N(x) - 2\cos\left(\frac{2\pi}{N}\right) = 0$. The constant term of this equation is $0 - 2\cos(2\pi/N) = -2\cos(2\pi/N)$. By Vieta's formulas for degree N (with N odd, so $(-1)^N = -1$):

$$\text{Product of Roots} = (-1)^N \frac{\text{Constant Term}}{\text{Leading Coeff}} = (-1)\frac{-2\cos(2\pi/N)}{1} = 2\cos\left(\frac{2\pi}{N}\right)$$

Substitute $x_k = 2\cos(\theta_k)$:

$$\prod_{k=0}^{N-1}(2\cos\theta_k) = 2^N \prod_{k=0}^{N-1}\cos\theta_k = 2\cos\left(\frac{2\pi}{N}\right)$$

Divide by 2^N:

$$\prod_{k=0}^{N-1}\cos\theta_k = \frac{1}{2^{N-1}}\cos\left(\frac{2\pi}{N}\right)$$

Takeaways 4.31

1. **Polynomial Link:** The identity $z^n + 1/z^n = P_n(z + 1/z)$ links complex roots of unity to polynomials with real coefficients. These are related to Chebyshev polynomials (T_n).

2. **Roots to Coefficients:** Harder problems often ask you to evaluate a trigonometric product. This is almost always a signal to form a polynomial equation and use **Vieta's Product of Roots**.

3. **Generalization:** Identifying N as odd was crucial. If N were even, the constant term of $P_N(x)$ would be non-zero, changing the final result.

4. Try substitute $\gamma = e^{i\theta}$

Problem 4.32: Cotangent Polynomial

Let $N = 2m + 1$ be an odd integer.

(i) Show that $\sin(N\theta)$ can be expressed as:

$$\sin(N\theta) = \sin^N \theta \cdot P(\cot^2 \theta)$$

where P is a polynomial of degree m.

(ii) Find the roots of $P(x) = 0$ in terms of θ.

(iii) Use Vieta's formulas to find:

$$\sum_{k=1}^{m} \cot^2 \left(\frac{k\pi}{N}\right)$$

Hint:

- For (ii), recall $\sin(N\theta)$ is the imaginary part of $(\cos + i \sin)^N$.
- Group terms by powers of i. Factor out $\sin^N \theta$.
- Terms will look like $\binom{N}{1} \cot^{N-1}\theta - \binom{N}{3}\cot^{N-3}\theta \dots$
- For (iv), use **Sum of Roots** $= -\frac{\text{coeff of } x^{m-1}}{\text{coeff of } x^m}$.

Solution 4.32

(i) Derive the Polynomial $P(x)$
Using De Moivre's Theorem and the binomial expansion for $N = 2m + 1$:

$$\begin{aligned}\sin(N\theta) &= \operatorname{Im}[(\cos\theta + i\sin\theta)^N] \\ &= \binom{N}{1}\cos^{N-1}\theta\sin\theta - \binom{N}{3}\cos^{N-3}\theta\sin^3\theta - \cdots + (-1)^m\binom{N}{N}\sin^N\theta\end{aligned}$$

Factor out $\sin^N\theta$:

$$\sin(N\theta) = \sin^N\theta\left[\binom{N}{1}\cot^{N-1}\theta - \binom{N}{3}\cot^{N-3}\theta - \cdots + (-1)^m\binom{N}{N}\right]$$

Let $x = \cot^2\theta$. Then $\cot^{N-1}\theta = (\cot^2\theta)^m = x^m$, and $\cot^{N-3}\theta = (\cot^2\theta)^{m-1} = x^{m-1}$, etc. The polynomial $P(x)$ is of degree m:

$$P(x) = \binom{N}{1}x^m - \binom{N}{3}x^{m-1} + \binom{N}{5}x^{m-2} - \cdots + (-1)^m\binom{N}{N}$$

(ii) Find the Roots of $P(x) = 0$
The roots of $P(\cot^2\theta) = 0$ occur when $\sin(N\theta) = 0$, provided $\sin\theta \neq 0$.

$$\sin(N\theta) = 0 \implies N\theta = k\pi \implies \theta = \frac{k\pi}{N}$$

Since $\cot^2\theta$ is periodic with period π and is an even function, we need distinct values of $\cot^2\theta$. The values of $k = 1, 2, \ldots, m$ give distinct roots x_k:

$$x_k = \cot^2\left(\frac{k\pi}{N}\right) \quad \text{for } k = 1, 2, \ldots, m$$

Since $P(x)$ is a polynomial of degree m, these m roots are all the roots of $P(x) = 0$.
(iii) Use Vieta's Formulas to Find the Sum of Roots
Step 1: Identify the Coefficients. The polynomial is $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$, where:
(i) Coefficient of x^m: $a_m = \binom{N}{1}$ (ii) Coefficient of x^{m-1}: $a_{m-1} = -\binom{N}{3}$
Step 2: Apply Vieta's Formula. The sum of the roots ($\sum x_k$) is given by Vieta's formula: $\sum x_k = -\frac{a_{m-1}}{a_m}$.

$$\sum_{k=1}^{m}\cot^2\left(\frac{k\pi}{N}\right) = -\frac{-\binom{N}{3}}{\binom{N}{1}} = \frac{\binom{N}{3}}{\binom{N}{1}}$$

Step 3: Substitute Binomial Coefficients and Simplify. Substitute the formulas for the binomial coefficients: $\binom{N}{1} = N$ and $\binom{N}{3} = \frac{N(N-1)(N-2)}{6}$.

$$\sum_{k=1}^{m}\cot^2\left(\frac{k\pi}{N}\right) = \frac{\frac{N(N-1)(N-2)}{6}}{N} = \frac{(N-1)(N-2)}{6}$$

Takeaways 4.32

1. **Imaginary Part Extraction:** Equating the imaginary part of the expansion to $\sin(N\theta)$ is the standard way to generate polynomials in $\cot\theta$.
2. **Vieta's Sum:** The sum of trig squares usually simplifies to a quadratic in N.
3. The polynomial $P(x)$ links to Chebyshev polynomials of the second kind.

Problem 4.33: Wallis-Type Integral and Pi Bounds

Let I_n be the integral defined for integers $n \geq 0$ by:

$$I_n = \int_0^1 (1-x^2)^n \, dx$$

(i) Use integration by parts to prove the reduction formula:

$$I_n = \frac{2n}{2n+1} I_{n-1}, \quad \text{for } n \geq 1$$

(ii) Hence, show that:

$$I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

(iii) Using the substitution $x = \sin\theta$, show that

$$I_n = \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \, d\theta$$

(iv) Let

$$J_n = \int_0^{\frac{\pi}{2}} \cos^{2n}\theta \, d\theta$$

. You are given that

$$J_n = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$$

.

By considering the behaviour of $\cos\theta$ on the interval $(0, \frac{\pi}{2})$, explain why $I_n < J_n$.

(v) Deduce the inequality:

$$\frac{2^{4n}(n!)^4}{[(2n)!]^2(2n+1)} < \frac{\pi}{2}$$

Hint:

- **Part (i):** Do not integrate $1 \cdot (1-x^2)^n$. Instead, split the term:

$$(1-x^2)^n = (1-x^2)(1-x^2)^{n-1} = (1-x^2)^{n-1} - x^2(1-x^2)^{n-1}$$

Then apply Integration by Parts to $\int x \cdot [x(1-x^2)^{n-1}]\,dx$.

- **Part (ii):** Apply the formula recursively. Multiply the numerator and denominator by even terms to form factorials.
- **Part (iv):** If $0 < y < 1$, then $y^{k+1} < y^k$. Apply this to $y = \cos\theta$.

Solution 4.33

(i) Write $I_n = \int_0^1 (1-x^2)^{n-1}dx - \int_0^1 x^2(1-x^2)^{n-1}dx$. Use IBP on the second integral: $u = x, dv = x(1-x^2)^{n-1}dx$. Then $du = dx, v = -\frac{1}{2n}(1-x^2)^n$.

$$\int_0^1 x^2(1-x^2)^{n-1}dx = \left[-\frac{x}{2n}(1-x^2)^n\right]_0^1 + \frac{1}{2n}\int_0^1 (1-x^2)^n dx = 0 + \frac{1}{2n}I_n$$

Substituting back: $I_n = I_{n-1} - \frac{1}{2n}I_n$. $I_n(1+\frac{1}{2n}) = I_{n-1} \implies I_n = \frac{2n}{2n+1}I_{n-1}$.

(ii) $I_n = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3}I_0$. Since $I_0 = 1$:

$$I_n = \frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 1}$$

Multiply top and bottom by $P = 2n(2n-2)\cdots 2 = 2^n n!$:

$$I_n = \frac{P \cdot P}{(2n+1)!} = \frac{(2^n n!)^2}{(2n+1)!} = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

(iii) $x = \sin\theta, dx = \cos\theta d\theta$. Limits $0 \to 0, 1 \to \pi/2$. $I_n = \int_0^{\pi/2} (\cos^2\theta)^n \cos\theta d\theta = \int_0^{\pi/2} \cos^{2n+1}\theta d\theta$.

(iv) For $\theta \in (0, \pi/2)$, $0 < \cos\theta < 1$. Thus $\cos^{2n+1}\theta < \cos^{2n}\theta$. Integrating over the interval preserves the inequality: $I_n < J_n$.

(v) Substitute the expressions:

$$\frac{2^{2n}(n!)^2}{(2n+1)!} < \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$$

Multiply and rearrange:

$$\frac{2^{4n}(n!)^4}{[(2n)!]^2(2n+1)} < \frac{\pi}{2}$$

Takeaways 4.33

1. **Algebraic Splitting:** The trick $(1-x^2)^n = (1-x^2)^{n-1} - x^2(1-x^2)^{n-1}$ is a high-yield technique for deriving reduction formulae.
2. **Wallis Product Logic:** Comparing $\int \cos^{odd}$ and $\int \cos^{even}$ is the standard method for establishing bounds for π.
3. **Factorial Identities:** The manipulation $(2n)!! = 2^n n!$ is crucial for Q16 combinatorics/integration.

Problem 4.34: Logarithmic Differentiation of Polynomials

Let $\omega = e^{2\pi i/n} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$
be an n-th root of unity.

(i) Show that

$$z^n - 1 = \prod_{k=0}^{n-1}(z-\omega^k)$$

.

(ii) By taking logarithms and differentiating, prove:

$$\frac{nz^{n-1}}{z^n-1} = \sum_{k=0}^{n-1}\frac{1}{z-\omega^k}$$

(iii) By taking the limit as $z \to 1$, show that:

$$\sum_{k=1}^{n-1}\frac{1}{1-\omega^k} = \frac{n-1}{2}$$

Hint:

- **Part (i):** $z^n - 1 = \prod_{k=0}^{n-1}(z-\omega^k)$.
- **Part (ii):** Take the natural logarithm of both sides: $\ln(z^n-1) = \sum \ln(z-\omega^k)$, then differentiate. Or use $\frac{P'(z)}{P(z)}$.
- **Part (iii):** This requires substituting $z=1$, but $z=1$ is a singularity. Separate the $k=0$ term $(\frac{1}{z-1})$ from the RHS, then take the limit as $z \to 1$.

Solution 4.34

(i) $z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{n-1})$.
(ii) Let $f(z) = \ln(z^n - 1) = \sum \ln(z - \omega^k)$.
Differentiate both sides: $f'(z) = \frac{nz^{n-1}}{z^n-1} = \sum_{k=0}^{n-1} \frac{1}{z-\omega^k}$.
(iii) Isolate $k = 0$ term on RHS:
$\frac{nz^{n-1}}{z^n-1} = \frac{1}{z-1} + \sum_{k=1}^{n-1} \frac{1}{z-\omega^k}$.
$\sum_{k=1}^{n-1} \frac{1}{z-\omega^k} = \frac{nz^{n-1}}{z^n-1} - \frac{1}{z-1}$.
Limit as $z \to 1$ using L'Hôpital or Taylor expansion:
Let $z = 1 + h$.
Using Taylor:
$\frac{n(1+h)^{n-1}}{(1+h)^n-1} \approx \frac{n(1+(n-1)h)}{nh} = \frac{1}{h}(1 + \frac{n-1}{n}h) \approx \frac{1}{h} + \frac{n-1}{n}$.
$\lim_{z\to 1} \left(\frac{nz^{n-1}}{z^n-1} - \frac{1}{z-1} \right) = \frac{n-1}{2}$.
Therefore: $\sum_{k=1}^{n-1} \frac{1}{1-\omega^k} = \frac{n-1}{2}$.

Takeaways 4.34

1. **Logarithmic Differentiation:** $\frac{P'(z)}{P(z)} = \sum \frac{1}{z-\alpha_i}$. This is a standard identity.
2. **Handling Singularities:** When asking for a sum involving roots of unity where one root makes the denominator zero, you usually need to take a limit.

Problem 4.35: Wallis Integrals and the Gaussian Integral

Let $W_n = \int_0^{\pi/2} \sin^n x \, dx$ denote the Wallis Integrals. It is given that:

$$\lim_{n\to\infty} \sqrt{n} W_n = \sqrt{\pi/2}. \text{ (Do NOT prove this)}$$

(i) Prove that for all $t \geq 0$:

$$1 - t \leq e^{-t} \leq \frac{1}{1+t}$$

(ii) By substituting $t = \frac{x^2}{n}$, show that for all $n \in \mathbb{Z}^+$:

$$\int_0^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n dx \leq \int_0^{\sqrt{n}} e^{-x^2} dx \leq \int_0^{\infty} e^{-x^2} dx \leq \int_0^{\infty} \left(1 + \frac{x^2}{n}\right)^{-n} dx$$

(iii) Use the substitutions $x = \sqrt{n}\sin\theta$ and $x = \sqrt{n}\tan\theta$ to show that

$$\int_0^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n dx = \sqrt{n} W_{2n+1}, \text{ and } \int_0^{\infty} \left(1 + \frac{x^2}{n}\right)^{-n} dx = \sqrt{n} W_{2n-2}.$$

(iv) A continuous random variable X follows the Normal Distribution $X \sim \mathcal{N}(0, \sigma^2)$ if its probability density function (PDF) is given by

$$f(x) = k \cdot e^{-\frac{x^2}{2\sigma^2}} \text{ for } x \in (-\infty, \infty)$$

.

Show that for $f(x)$ to be a valid PDF, the constant k must be:

$$k = \frac{1}{\sigma\sqrt{2\pi}}$$

Hint:

- Use the inequality $e^t \geq 1 + t$ for $t \geq 0$. For the left side, consider the function $f(t) = e^{-t} - (1 - t)$.
- Recall that for $n > 0$, if $f(x) \leq g(x)$, then $f(x)^n \leq g(x)^n$. For the tangent substitution, remember that $1 + \tan^2\theta = \sec^2\theta$ and the limits for $x \in [0, \infty)$ map to $\theta \in [0, \pi/2)$.
- **Part (iii):** Use $x = \sqrt{n}\sin\theta$ and $x = \sqrt{n}\tan\theta$.
- **Part (iv):** Use the fact that e^{-ax^2} is an even function. Apply the substitution $u = \frac{x}{\sigma\sqrt{2}}$ to transform the integral into the standard Gaussian form.

Solution 4.35

- **(i)** Recall the Taylor series expansion: $e^t = 1 + t + \frac{t^2}{2!} + \cdots \geq 1 + t \implies e^{-t} \leq \frac{1}{1+t}$. For $1 - t \leq e^{-t}$, let $f(t) = e^{-t} - 1 + t$. $f'(t) = 1 - e^{-t} \geq 0$ for $t \geq 0$. Since $f(0) = 0$, $f(t) \geq 0$. You can also consider the derevative of $e^{-t} + t - 1$ directly.

- **(ii)** From (i), $(1 - x^2/n) \leq e^{-x^2/n} \leq (1 + x^2/n)^{-1}$. Raising to power n gives $(1 - x^2/n)^n \leq e^{-x^2} \leq (1 + x^2/n)^{-n}$. Integrating over the respective bounds yields the result.

- **(iii)** $I_{left} = \int_0^{\pi/2} (1 - \sin^2\theta)^n \sqrt{n} \cos\theta \, d\theta = \sqrt{n} W_{2n+1}$. $I_{right} = \int_0^{\pi/2} (\sec^2\theta)^{-n} \sqrt{n} \sec^2\theta \, d\theta = \sqrt{n} W_{2n-2}$.

- **(iv)** $\int_{-\infty}^{\infty} k e^{-\frac{x^2}{2\sigma^2}} dx = 2k \int_0^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = 1$.
  Let $u = \frac{x}{\sigma\sqrt{2}} \implies dx = \sigma\sqrt{2} du$.
  $2k\sigma\sqrt{2} \int_0^{\infty} e^{-u^2} du = 2k\sigma\sqrt{2} \left(\frac{\sqrt{\pi}}{2}\right) = 1$.
  Solving for k yields $k = \frac{1}{\sigma\sqrt{2\pi}}$.

Takeaways 4.35

1. The Bernoulli Link: The expression $(1 + z/n)^n$ is the classic limit definition of e^z. This problem shows how that convergence behaves under an integral sign. It is kind of advance but useful to know.

2. **Normalization:** Normalization: This problem demonstrates why the "messy" constant $\frac{1}{\sqrt{2\pi}}$ exists in statistics; it is a mathematical necessity to ensure the total probability equals 1.

3. **Symmetry:** The use of even function properties is a vital shortcut in Extension 2 integration problems.

4. **Interdisciplinary Math:** This problem bridges the gap between pure integration techniques and the foundations of statistics.

Problem 4.36: Complex Numbers and Polynomial Geometry

Let n be a positive integer such that $n \geq 2$. Consider the complex equation $z^{2n} - 1 = 0$, whose roots are represented by the vertices $P_0, P_1, P_2, \ldots, P_{2n-1}$ of a regular $2n$-gon inscribed in the unit circle in the Argand diagram. Let P_0 correspond to the complex number 1.

(i) Let $\omega = e^{i\frac{\pi}{n}}$. Show that the roots of the equation are given by $z = \omega^k$ for $k = 0, 1, \ldots, 2n-1$.

(ii) Show that for any $z \neq 1$:

$$\sum_{j=0}^{2n-1} z^j = \prod_{k=1}^{2n-1} (z - \omega^k)$$

(iii) By considering the distance between the point P_0 and any other vertex P_k, show that the length of the chord P_0P_k is given by:

$$|P_0P_k| = 2\sin\left(\frac{k\pi}{2n}\right)$$

(iv) Prove that the product of the lengths of all possible chords drawn from a single vertex to all other vertices in a regular $2n$-gon that inscribed in a unit circle is exactly $2n$. That is, show:

$$\prod_{k=1}^{2n-1} |P_0P_k| = 2n$$

Hint:

- **Part (i):** Recall the polar form of 1 is $e^{i2k\pi}$. Apply the $2n$-th root to find z.
- **Part (ii):** Use the fact that $z^{2n} - 1 = (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{2n-1})$ and equate this to the geometric series formula for $\frac{z^{2n}-1}{z-1}$.
- **Part (iii):** Use the formula $|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ or $|1 - e^{i\theta}|^2 = (1-\cos\theta)^2 + \sin^2\theta$.
- **Part (iv):** Evaluate the identity in Part (ii) at $z = 1$ and use the property that the modulus of a product is the product of the moduli.

Solution 4.36

(i) $z^{2n} = 1 = e^{i2k\pi} \implies z = e^{i\frac{2k\pi}{2n}} = e^{i\frac{k\pi}{n}}$. Let $\omega = e^{i\frac{\pi}{n}}$, then roots are ω^k for $k = 0, 1, \dots, 2n-1$.
(ii) The polynomial $P(z) = z^{2n} - 1$ has roots $1, \omega, \omega^2, \dots, \omega^{2n-1}$. Thus:

$$z^{2n} - 1 = (z-1)(z-\omega)(z-\omega^2)\dots(z-\omega^{2n-1})$$

Dividing by $(z-1)$ for $z \neq 1$:

$$\frac{z^{2n}-1}{z-1} = \prod_{k=1}^{2n-1}(z-\omega^k)$$

Since $\frac{z^{2n}-1}{z-1} = 1 + z + z^2 + \dots + z^{2n-1}$, the identity is proven.
(iii) Chord $|P_0P_k| = |1 - e^{i\frac{k\pi}{n}}|$. Using $|1 - \cos\theta - i\sin\theta| = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2 - 2\cos\theta}$. Applying $1 - \cos\theta = 2\sin^2(\frac{\theta}{2})$:

$$|P_0P_k| = \sqrt{4\sin^2\left(\frac{k\pi}{2n}\right)} = 2\sin\left(\frac{k\pi}{2n}\right) \text{ (as sin is positive in this range).}$$

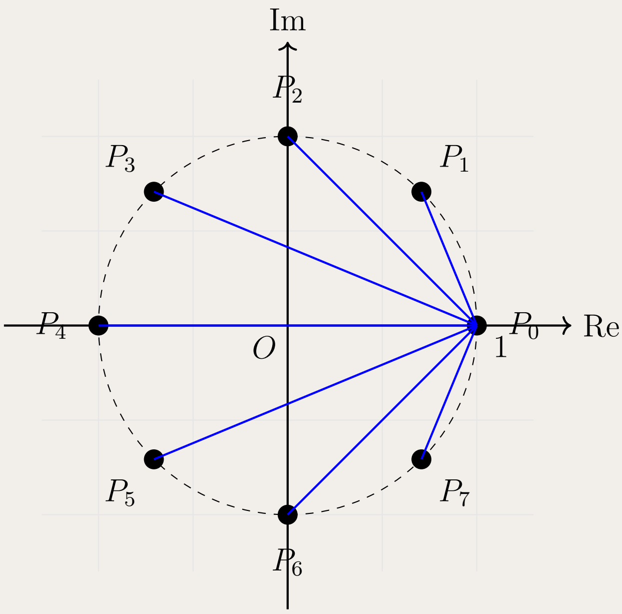
(iv) Let $z = 1$ in the identity from (ii):

$$1 + 1 + \dots + 1 \text{ (}2n\text{ terms)} = \prod_{k=1}^{2n-1}(1-\omega^k), \text{ i.e. } 2n = \prod_{k=1}^{2n-1}(1-\omega^k)$$

Taking the modulus of both sides:

$$|2n| = \prod_{k=1}^{2n-1}|1-\omega^k|$$

Since $|1-\omega^k|$ is the length of chord $|P_0P_k|$, we have $\prod_{k=1}^{2n-1}|P_0P_k| = 2n$.



Takeaways 4.36

1. Geometrically, $|1 - e^{i\frac{k\pi}{n}}|$ represents the distance between the vertex P_0 (at $1 + 0i$) and every other vertex P_k of a regular $2n$-gon inscribed in the unit circle.
2. **Geometric Product:** The product of all chord lengths from one vertex in a regular m-gon is exactly m.
3. **Trigonometric Identity:** This problem proves $\prod_{k=1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right) = \frac{2n}{2^{2n-1}}$.
4. **Complex-Geometric Connection:** Polynomial factorization over roots of unity directly yields geometric properties of regular polygons.
5. **Modulus of Products:** The identity $|z_1 z_2| = |z_1||z_2|$ is essential for connecting algebraic and geometric interpretations.
6. Try proving (iv) using induction as an alternative method (?) or without using complex numbers (?)

Problem 4.37: Niven's Contradiction: The Irrationality of π^2

The Motivation:

In this problem, we will prove that π^2 is irrational using a proof by contradiction inspired by Niven's approach. You may already know that π itself is irrational yet the proof is somewhat not straightforward.

Proving the irrationality of π^2 makes the statement stronger, since if π^2 were rational, then π would also be rational (as the square root of a rational number) (do NOT prove this). Thus, by proving that π^2 is irrational, we also confirm the irrationality of π.

This problem will walk you through the steps to establish this result.

To prove that π^2 is irrational, we assume by way of contradiction that π^2 is rational. That is, we can write $\pi^2 = \frac{a}{b}$ where a and b are positive integers.

Let n be a positive integer and define the polynomial $f(x)$ as:

$$f(x) = \frac{x^n(1-x)^n}{n!}$$

(i) Show $f(x)$ and its derivatives $f^{(k)}(x)$ take integer values at $x = 0$ and $x = 1$.

(ii) Define the function $G(x)$ by:

$$G(x) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f^{(2k)}(x)$$

Using the assumption $\pi^2 = \frac{a}{b}$, show that $G(0)$ and $G(1)$ are integers.

(iii) Consider the function $H(x) = G'(x)\sin(\pi x) - \pi G(x)\cos(\pi x)$. Differentiate $H(x)$ with respect to x and verify that:

$$H'(x) = \pi^2 a^n f(x) \sin(\pi x)$$

(iv) Consider the integral:

$$I_n = \int_0^1 \pi a^n f(x) \sin(\pi x)\, dx$$

Using the result from part (iii), show that $I_n = G(1) + G(0)$. Hence, explain why I_n must be an integer.

Finally, by establishing an upper bound for I_n, derive a contradiction for sufficiently large n. Conclude that π^2 is irrational.

Hint:

- **Part (i):** Use the binomial expansion of $(1-x)^n$. Recall that derivatives of $\frac{x^m}{n!}$ at 0 are zero if $m \neq k$ (for k-th derivative) or integers if $m = k$. Use symmetry $f(x) = f(1-x)$ for the $x = 1$ case.
- **Part (ii):** Substitute $\pi^{2n-2k} = (\frac{a}{b})^{n-k}$. Combine this with b^n outside the sum to clear the fractions. Observe that $\pi^{2m} = (a/b)^m$. Rewrite the sum to show the coefficients are integers.
- **Part (iii):** Use the product rule on $H(x)$. Group terms by $\sin(\pi x)$ and $\cos(\pi x)$. The cos terms should cancel. You may assume without proof the algebraic identity: $G''(x) + \pi^2 G(x) = \pi^{2n+2} b^n f(x)$. Note also that $\pi^{2n+2} b^n = \pi^2 a^n$.
- **Part (iv):** Integrate $H'(x)$ from 0 to 1. Note that $H(1) = \pi G(1)$ and $H(0) = -\pi G(0)$. Don't forget that $\lim_{n\to\infty} \frac{C^n}{n!} = 0$.

Solution 4.37

(i) Integer Derivatives

$f(x) = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(-1)^j x^{n+j}$. For any k, $f^{(k)}(0)$ is determined by the term with power x^k. If $k < n$, the coefficient is 0. If $k \geq n$, the derivative of $\frac{x^k}{n!}$ is an integer multiple of $\frac{k!}{n!}$, which is an integer. Since $f(x) = f(1-x)$, by chain rule $f^{(k)}(1) = (-1)^k f^{(k)}(0)$, which is also an integer.

(ii) G(x) Integers

Substitute $\pi^2 = a/b$:

$$G(x) = \sum_{k=0}^{n}(-1)^k b^n \left(\frac{a}{b}\right)^{n-k} f^{(2k)}(x) = \sum_{k=0}^{n}(-1)^k a^{n-k} b^k f^{(2k)}(x)$$

Since $a, b, f^{(2k)}(0)$ and $f^{(2k)}(1)$ are all integers, $G(0)$ and $G(1)$ are integers.

(iii) The Derivative

$H'(x) = G''(x)\sin(\pi x) + \pi G'(x)\cos(\pi x) - \pi G'(x)\cos(\pi x) + \pi^2 G(x)\sin(\pi x)$

$$H'(x) = \sin(\pi x)[G''(x) + \pi^2 G(x)]$$

Using the hint $G''(x) + \pi^2 G(x) = \pi^{2n+2} b^n f(x)$:

$$H'(x) = \sin(\pi x)[\pi^{2n+2} b^n f(x)]$$

Since $\pi^2 = a/b \implies b\pi^2 = a$, we have $\pi^{2n+2} b^n = \pi^2(\pi^2)^n b^n = \pi^2 (a/b)^n b^n = \pi^2 a^n$.

$$H'(x) = \pi^2 a^n f(x)\sin(\pi x)$$

(iv) The Integral and Contradiction

From (iii), $\int_0^1 \pi^2 a^n f(x)\sin(\pi x)dx = [H(x)]_0^1$. Evaluate RHS: $H(1) = G'(1)\sin\pi - \pi G(1)\cos\pi = \pi G(1)$. $H(0) = G'(0)\sin 0 - \pi G(0)\cos 0 = -\pi G(0)$. So, $\int_0^1 \pi^2 a^n f(x)\sin(\pi x)dx = \pi(G(1) + G(0))$. Divide both sides by π:

$$\int_0^1 \pi a^n f(x)\sin(\pi x)dx = G(1) + G(0)$$

The LHS is I_n. The RHS is the sum of integers (from part ii). Thus I_n is an integer.
Bounds: On $(0,1)$, $0 < f(x) \leq \frac{1}{n!}(1/4)^n < \frac{1}{n!}$ and $0 < \sin(\pi x) \leq 1$.

$$0 < I_n < \frac{\pi a^n}{n!}$$

Since $\lim_{n\to\infty} \frac{\pi a^n}{n!} = 0$, for sufficiently large n, we have $0 < I_n < 1$. No integer exists strictly between 0 and 1. Contradiction. $\therefore \pi^2$ is irrational.

Takeaways 4.37

1. **Stronger Result**: Proving π^2 is irrational automatically proves π is irrational, but the reverse is not true.

2. **Proof by Contradiction:** This classic technique assumes the opposite of what we want to prove, then derives an impossible conclusion. The contradiction forces our original assumption to be false.

3. **Auxiliary Functions:** The sophisticated choice of $f(x)$, $G(x)$, and $H(x)$ is characteristic of advanced proofs in analysis. Each function serves a specific purpose in the overall argument.

4. **Integer Forcing:** By carefully constructing $G(x)$ to have integer values at endpoints, we force the integral I_n to be an integer, setting up the contradiction.

5. **Factorial Growth:** The key to the contradiction lies in the factorial in the denominator: $n!$ grows much faster than any exponential a^n, making the integral arbitrarily small.

6. **Historical Significance:** This problem represents one of the most elegant applications of calculus to number theory, demonstrating the deep connections between analysis and arithmetic.

5 Conclusion

Mastering Problem 16 represents the culmination of your HSC Mathematics Extension 2 journey. These "Last Resort" problems demand not just computational skills, but mathematical maturity, strategic thinking, and the ability to synthesize diverse topics under examination pressure.

The problems in this collection span the full range of techniques and topics that appear in Problem 16: from elegant inequality applications to sophisticated complex number theory, from vector optimization to advanced function analysis. Each problem has been chosen to develop specific aspects of mathematical reasoning while building your confidence in tackling unfamiliar scenarios.

Remember that Problem 16 success comes from:

- **Pattern Recognition:** Learning to identify familiar structures in new contexts
- **Technique Integration:** Combining multiple approaches within single problems
- **Proof Communication:** Expressing mathematical arguments clearly and completely
- **Strategic Thinking:** Choosing effective approaches when multiple paths exist
- **Persistence:** Working through complex multi-step problems systematically

Use this collection as a comprehensive training ground. Work through problems multiple times, focus on understanding the reasoning behind each step, and practice communicating your solutions clearly. With dedicated preparation, you can approach Problem 16 with confidence and skill.

Best of luck with your studies and HSC examinations!

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