HSC Math Extension 2: Collection of Hard Problems

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1 Overview

This collection presents a curated set of challenging problems from the HSC Mathematics Extension 2 curriculum. These problems are designed to test deep understanding, creative problem-solving skills, and the ability to synthesize multiple mathematical concepts.

1.1 What This Collection Is About

This collection focuses on **HSC Mathematics Extension 2 (hard problems)**. The problems span various topics including:

- Complex numbers and their geometric interpretations
- Integration techniques and applications
- Vector geometry in three dimensions
- Mechanics and particle motion
- Inequalities and optimization

Each problem is carefully selected to represent the level of difficulty and sophistication expected in the most challenging HSC Extension 2 examinations.

1.2 Target Audience

This collection is designed for:

- **Students** preparing for HSC Mathematics Extension 2 who want to challenge themselves with difficult problems
- **Tutors** seeking high-quality problems to use in their teaching
- **Educators** looking for challenging problems to incorporate into their curriculum

The problems are presented with hints to guide thinking, followed by detailed solutions and key takeaways to reinforce learning.

2 Problems

2.1 Problem 1: Two Particles in Resisting Medium

Problem 2.1

Two particles, A and B, each have mass 1 kg and are in a medium that exerts a resistance to motion equal to kv, where $k > 0$ and v is the velocity of any particle. Both particles maintain vertical trajectories.
The acceleration due to gravity is g m s^{-2}, where $g > 0$.
The two particles are simultaneously projected towards each other with the same speed, v_0 m s^{-1}, where $0 < v_0 < \frac{g}{k}$.
The particle A is initially d metres directly above particle B, where $d < \frac{2v_0}{k}$.
Find the time taken for the particles to meet.

Hint: Consider the equations of motion for each particle under the influence of gravity and resistance. Set up differential equations for the velocities and positions. The condition for the particles to meet will give you an equation involving time.

2.2 Solution to Problem 1: Two Particles in Resisting Medium

Solution 2.1

Let $y_A(t)$ and $y_B(t)$ be the positions of particles A and B respectively, with $y_A(0) = d$ and $y_B(0) = 0$. The equation of motion for each particle is:

$$\frac{dv}{dt} = -g - kv$$

Solving this first-order linear ODE with $v(0) = -v_0$ (downward for A) and $v(0) = v_0$ (upward for B):

$$v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right) e^{-kt}$$

Integrating to find position:

$$y(t) = y(0) - \frac{g}{k}t + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-kt}\right)$$

For particle A: $y_A(t) = d - \frac{g}{k}t + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-kt}\right)$
For particle B: $y_B(t) = \frac{g}{k}t - \frac{1}{k}\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-kt}\right)$
Setting $y_A(t) = y_B(t)$ and solving:

$$d = \frac{2}{k}\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-kt}\right)$$

Therefore, $t = -\frac{1}{k}\ln\left(1 - \frac{kd}{2(v_0+g/k)}\right)$.

Takeaways 2.1

- Motion with linear resistance: $dv/dt = -g - kv$ has solution $v = -g/k + (v_0 + g/k)e^{-kt}$
- Relative motion: Set positions equal to find meeting time
- Exponential decay in velocity due to resistance

2.3 Problem 2: Complex Square with Equilateral Triangle

Problem 2.2

A square in the Argand plane has vertices

$$5 + 5i, \quad 5 - 5i, \quad -5 - 5i \quad \text{and} \quad -5 + 5i.$$

The complex numbers $z_A = 5 + i$, z_B and z_C lie on the square and form the vertices of an equilateral triangle.
Find the exact value of the complex number z_B.

Hint: Use the geometric properties of equilateral triangles. Consider rotations in the complex plane. The vertices of an equilateral triangle are related by rotations of 60° or 120° about the centroid.

2.4 Solution to Problem 2: Complex Square with Equilateral Triangle

Solution 2.2

The centroid of the equilateral triangle is at $z_G = \frac{z_A+z_B+z_C}{3}$. Since $z_A = 5 + i$ lies on the right side of the square, and the triangle is equilateral, the vertices are related by 120° rotations about the centroid.
Alternatively, note that z_B and z_C must lie on the square's perimeter. Since $z_A = 5 + i$ is on the right edge, rotating by $e^{2\pi i/3}$ or $e^{-2\pi i/3}$ gives the other vertices.
Let $z_B = 5 + bi$ where $-5 < b < 5$ (on right edge) or $z_B = a + 5i$ where $-5 < a < 5$ (on top edge). Using the rotation property: $z_B - z_G = e^{2\pi i/3}(z_A - z_G)$.
After calculations, we find $z_B = 5 - 5\sqrt{3} + (5\sqrt{3} - 4)i$ or the symmetric solution. The exact value depends on the orientation, but one solution is:

$$z_B = 5 - 5\sqrt{3} + (5\sqrt{3} - 4)i$$

Takeaways 2.2

- Equilateral triangles: vertices are 120° rotations of each other about the centroid
- Complex rotations: multiply by $e^{\pm 2\pi i/3}$ to rotate by $\pm 120°$
- Constraint: vertices must lie on the square's perimeter

2.5 Problem 3: Complex 7th Root of Unity

Problem 2.3

Let w be a complex number such that $1 + w + w^2 + \cdots + w^6 = 0$.

(i) Show that w is a 7th root of unity.

(ii) The complex number $\alpha = w + w^2 + w^4$ is a root of the equation $x^2 + bx + c = 0$, where b and c are real and α is not real. Find the other root of $x^2 + bx + c = 0$ in terms of positive powers of w.

(iii) Find the numerical value of c.

Hint: For part (i), use the formula for the sum of a geometric series. For part (ii), use the fact that for a quadratic with real coefficients, the other root is the complex conjugate. For part (iii), use properties of roots of unity and their relationships.

2.6 Solution to Problem 3: Complex 7th Root of Unity

Solution 2.3

(i) If $w = 1$, then $1 + w + \cdots + w^6 = 7 \neq 0$. So $w \neq 1$. Using the geometric series formula:

$$1 + w + w^2 + \cdots + w^6 = \frac{1 - w^7}{1 - w} = 0$$

Since $w \neq 1$, we have $1 - w^7 = 0$, so $w^7 = 1$. Therefore w is a 7th root of unity.
(ii) Since the quadratic has real coefficients, the other root is $\overline{\alpha} = \overline{w + w^2 + w^4} = \overline{w} + \overline{w^2} + \overline{w^4}$.
For a 7th root of unity $w = e^{2\pi i k/7}$ where $k \in \{1, 2, 3, 4, 5, 6\}$, we have $\overline{w} = w^{-1} = w^6$. Similarly, $\overline{w^2} = w^5$ and $\overline{w^4} = w^3$.
Therefore, $\overline{\alpha} = w^6 + w^5 + w^3$.
(iii) For $w = e^{2\pi i/7}$, we have:

$$c = \alpha \cdot \overline{\alpha} = (w + w^2 + w^4)(w^6 + w^5 + w^3)$$

Expanding and using $w^7 = 1$:

$$c = w^7 + w^6 + w^5 + w^8 + w^7 + w^6 + w^{10} + w^9 + w^7 = 3 + (w + w^2 + w^3 + w^4 + w^5 + w^6)$$

Since $1 + w + w^2 + \cdots + w^6 = 0$, we get $w + w^2 + \cdots + w^6 = -1$, so $c = 3 - 1 = 2$.

Takeaways 2.3

- 7th roots of unity: $w^7 = 1$ with $w \neq 1$ implies $1 + w + \cdots + w^6 = 0$
- Complex conjugate: For $w = e^{2\pi i k/7}$, we have $\overline{w} = w^{-1} = w^6$
- Product of roots: For real-coefficient quadratics, $c = \alpha\overline{\alpha} = |\alpha|^2$

2.7 Problem 4: Complex Triangle Inequality

Problem 2.4

The complex number z satisfies $\left|z - \frac{4}{z}\right| = 2$.
Using the triangle inequality, or otherwise, show that $|z| \leq \sqrt{5} + 1$.

Hint: Apply the triangle inequality to $|z - \frac{4}{z}|$. Consider both directions: $|z| - |\frac{4}{z}| \leq |z - \frac{4}{z}|$ and $|z - \frac{4}{z}| \leq |z| + |\frac{4}{z}|$. Use the given condition to derive bounds on $|z|$.

2.8 Solution to Problem 4: Complex Triangle Inequality

Solution 2.4

Given $|z - 4/z| = 2$. Applying the triangle inequality in both directions:
Lower bound: $|z - 4/z| \geq |z| - |4/z| = |z| - 4/|z|$
So $2 \geq |z| - 4/|z|$, which gives $|z|^2 - 2|z| - 4 \leq 0$.
Solving: $|z| \leq 1 + \sqrt{5}$.
Upper bound: $|z - 4/z| \leq |z| + |4/z| = |z| + 4/|z|$
So $2 \leq |z| + 4/|z|$, which gives $|z|^2 - 2|z| + 4 \geq 0$.
This quadratic has discriminant $4 - 16 = -12 < 0$, so it's always positive. This gives no upper bound.
Combining both: $|z| \leq 1 + \sqrt{5} = \sqrt{5} + 1$.

Takeaways 2.4

- Triangle inequality: $|a - b| \geq ||a| - |b||$ and $|a - b| \leq |a| + |b|$
- Apply both directions to get bounds on $|z|$
- Solve resulting quadratic inequalities

2.9 Problem 5: Integral with Inverse Sine

Problem 2.5

Using the substitution $x = \tan^2\theta$, evaluate

$$\int_0^1 \sin^{-1}\sqrt{\frac{x}{1+x}}\,dx.$$

Hint: After making the substitution, simplify the integrand. You may need to use trigonometric identities. Consider the relationship between $\sin^{-1}$ and the substitution variable.

2.10 Solution to Problem 5: Integral with Inverse Sine

Solution 2.5

Let $x = \tan^2\theta$, so $dx = 2\tan\theta\sec^2\theta\,d\theta = 2\tan\theta(1+\tan^2\theta)\,d\theta$.
When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \pi/4$.
The integrand becomes:

$$\sin^{-1}\sqrt{\frac{x}{1+x}} = \sin^{-1}\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}} = \sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}} = \sin^{-1}|\sin\theta| = \sin^{-1}(\sin\theta) = \theta$$

for $\theta \in [0, \pi/4]$.
Therefore:

$$\int_0^1 \sin^{-1}\sqrt{\frac{x}{1+x}}\,dx = \int_0^{\pi/4} \theta \cdot 2\tan\theta\sec^2\theta\,d\theta$$

Using integration by parts with $u = \theta$, $dv = 2\tan\theta\sec^2\theta\,d\theta$:

$$= \left[\theta\tan^2\theta\right]_0^{\pi/4} - \int_0^{\pi/4}\tan^2\theta\,d\theta = \frac{\pi}{4} - \int_0^{\pi/4}(\sec^2\theta - 1)\,d\theta$$

$$= \frac{\pi}{4} - [\tan\theta - \theta]_0^{\pi/4} = \frac{\pi}{4} - (1 - \frac{\pi}{4}) = \frac{\pi}{2} - 1$$

Takeaways 2.5

- Substitution $x = \tan^2\theta$ simplifies $\sqrt{x/(1+x)}$ to $|\sin\theta|$
- For $\theta \in [0, \pi/4]$, we have $\sin^{-1}(\sin\theta) = \theta$
- Integration by parts: differentiate the polynomial part, integrate the trigonometric part

2.11 Problem 6: Integral with Cotangent

Problem 2.6

Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n}\theta\,d\theta$ for integers $n \geq 0$.

(i) Show that $I_n = \frac{1}{2n-1} - I_{n-1}$ for $n > 0$, given that $\frac{d}{d\theta}\cot\theta = -\csc^2\theta$.

(ii) Hence, or otherwise, calculate I_2.

Hint: For part (i), use integration by parts. Write $\cot^{2n}\theta = \cot^{2n-2}\theta \cdot \cot^2\theta$ and use the identity $\cot^2\theta = \csc^2\theta - 1$. For part (ii), use the recurrence relation and find I_0 first.

2.12 Solution to Problem 6: Integral with Cotangent

Solution 2.6

(i) Write $I_n = \int_{\pi/4}^{\pi/2} \cot^{2n}\theta\, d\theta = \int_{\pi/4}^{\pi/2} \cot^{2n-2}\theta \cdot \cot^2\theta\, d\theta$.
Using $\cot^2\theta = \csc^2\theta - 1$:

$$I_n = \int_{\pi/4}^{\pi/2} \cot^{2n-2}\theta(\csc^2\theta - 1)\, d\theta = \int_{\pi/4}^{\pi/2} \cot^{2n-2}\theta \csc^2\theta\, d\theta - I_{n-1}$$

For the first integral, let $u = \cot\theta$, so $du = -\csc^2\theta\, d\theta$:

$$\int_{\pi/4}^{\pi/2} \cot^{2n-2}\theta \csc^2\theta\, d\theta = -\int_1^0 u^{2n-2}\, du = \int_0^1 u^{2n-2}\, du = \frac{1}{2n-1}$$

Therefore: $I_n = \frac{1}{2n-1} - I_{n-1}$.
(ii) First, $I_0 = \int_{\pi/4}^{\pi/2} d\theta = \frac{\pi}{4}$.
Using the recurrence: $I_1 = \frac{1}{1} - I_0 = 1 - \frac{\pi}{4}$.
Then: $I_2 = \frac{1}{3} - I_1 = \frac{1}{3} - (1 - \frac{\pi}{4}) = \frac{\pi}{4} - \frac{2}{3}$.

Takeaways 2.6

- Recurrence: Use $\cot^2\theta = \csc^2\theta - 1$ to relate I_n and I_{n-1}
- Substitution: $u = \cot\theta$ converts $\cot^{2n-2}\theta \csc^2\theta\, d\theta$ to $u^{2n-2}\, du$
- Base case: $I_0 = \pi/4$ (integral of 1 over the interval)

2.13 Problem 7: 3D Vectors and Distance

Problem 2.7

Consider the point B with three-dimensional position vector $\mathbf{b}$ and the line l: $\mathbf{a} + \lambda\mathbf{d}$, where $\mathbf{a}$ and $\mathbf{d}$ are three-dimensional vectors, $|\mathbf{d}| = 1$ and λ is a parameter.
Let $f(\lambda)$ be the distance between a point on the line l and the point B.

(i) Find λ_0, the value of λ that minimises f, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$.

(ii) Let P be the point with position vector $\mathbf{a} + \lambda_0\mathbf{d}$. Show that PB is perpendicular to the direction of the line l.

(iii) Hence, or otherwise, find the shortest distance between the line l and the sphere of radius 1 unit, centred at the origin O, in terms of $\mathbf{d}$ and $\mathbf{a}$.

You may assume that if B is the point on the sphere closest to l, then OBP is a straight line.

Hint: For part (i), minimize $f^2(\lambda)$ by differentiating. For part (ii), use the fact that the minimum distance occurs when the vector from P to B is perpendicular to the direction vector. For part (iii), use the geometric interpretation and the given assumption.

2.14 Solution to Problem 7: 3D Vectors and Distance

Solution 2.7

(i) The distance squared is $f^2(\lambda) = |(\mathbf{a} + \lambda\mathbf{d}) - \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b} + \lambda\mathbf{d}|^2$.
Expanding: $f^2(\lambda) = |\mathbf{a} - \mathbf{b}|^2 + 2\lambda(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} + \lambda^2|\mathbf{d}|^2 = |\mathbf{a} - \mathbf{b}|^2 + 2\lambda(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} + \lambda^2$.
Differentiating: $\frac{d}{d\lambda}(f^2) = 2(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} + 2\lambda = 0$.
Therefore: $\lambda_0 = -(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{d}$.
(ii) The vector $\overrightarrow{PB} = \mathbf{b} - (\mathbf{a} + \lambda_0\mathbf{d}) = \mathbf{b} - \mathbf{a} - \lambda_0\mathbf{d}$.
Taking dot product with $\mathbf{d}$:

$$\overrightarrow{PB} \cdot \mathbf{d} = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{d} - \lambda_0 = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{d} - (\mathbf{b} - \mathbf{a}) \cdot \mathbf{d} = 0$$

Therefore PB is perpendicular to the direction of line l.
(iii) If B is on the sphere closest to l, then OBP is a straight line, so $\mathbf{b}$ is parallel to $\mathbf{a} + \lambda_0\mathbf{d}$.
The shortest distance is $|\overrightarrow{PB}| = |\mathbf{b} - \mathbf{a} - \lambda_0\mathbf{d}|$.
Since $\mathbf{b}$ is on the unit sphere, $|\mathbf{b}| = 1$. Using the perpendicularity and the assumption:

$$\text{Shortest distance} = |\mathbf{b} - \mathbf{a} - \lambda_0\mathbf{d}| = \sqrt{|\mathbf{a}|^2 - (\mathbf{a} \cdot \mathbf{d})^2} - 1$$

if the sphere and line don't intersect, or 0 if they do.

Takeaways 2.7

- Minimize $f^2(\lambda)$ by setting derivative to zero
- Minimum distance occurs when connecting vector is perpendicular to line direction
- Geometric interpretation: shortest distance from line to sphere

2.15 Problem 8: Triangle Inequality and Rectangular Prism

Problem 2.8

It is given that for positive numbers $x_1, x_2, x_3, \ldots, x_n$ with arithmetic mean A,

$$\frac{x_1 \times x_2 \times x_3 \times \cdots \times x_n}{A^n} \leq 1. \quad \text{(Do NOT prove this.)}$$

Suppose a rectangular prism has dimensions a, b, c and surface area S.

(i) Show that $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$.

(ii) Using part (i), show that when the rectangular prism with surface area S is a cube, it has maximum volume.

Hint: For part (i), relate the surface area to the dimensions and apply the given inequality. For part (ii), show that equality in the inequality from part (i) occurs when $a = b = c$, which corresponds to a cube.

2.16 Solution to Problem 8: Triangle Inequality and Rectangular Prism

Solution 2.8

(i) The surface area is $S = 2(ab + bc + ca)$.
The arithmetic mean of ab, bc, and ca is $A = \frac{ab+bc+ca}{3} = \frac{S}{6}$.
By the given inequality:

$$\frac{(ab)(bc)(ca)}{A^3} \leq 1$$

That is: $\frac{a^2b^2c^2}{(S/6)^3} \leq 1$, so $a^2b^2c^2 \leq (S/6)^3$.
Taking square roots: $abc \leq (S/6)^{3/2}$.
(ii) Equality in the given inequality occurs when $ab = bc = ca$, which implies $a = b = c$ (since all are positive).
When $a = b = c$, the rectangular prism is a cube. Since equality gives the maximum value of the left-hand side, and abc is maximized when equality holds, the cube has maximum volume for a given surface area S.

Takeaways 2.8

- AM-GM: For n positive numbers, product $\leq$ (arithmetic mean)^n
- Equality: Occurs when all numbers are equal
- Optimization: Maximum volume for fixed surface area occurs at equality condition

2.17 Problem 9: Three Unit Vectors Optimization

Problem 2.9

Three unit vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, in 3 dimensions, are to be chosen so that $\mathbf{a} \perp \mathbf{b}$, $\mathbf{b} \perp \mathbf{c}$ and the angle θ between $\mathbf{a}$ and $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is as small as possible.
What is the value of $\cos\theta$?

Hint: Use the dot product to express $\cos\theta$ in terms of the vectors. Consider the constraints and use Lagrange multipliers or geometric reasoning. The optimal configuration occurs when the vectors are arranged symmetrically.

2.18 Solution to Problem 9: Three Unit Vectors Optimization

Solution 2.9

We want to minimize θ where $\cos\theta = \frac{\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})}{|\mathbf{a}||\mathbf{a}+\mathbf{b}+\mathbf{c}|} = \frac{1+\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}+\mathbf{b}+\mathbf{c}|}$.
Since $\mathbf{a}\perp\mathbf{b}$, we have $\mathbf{a}\cdot\mathbf{b} = 0$. Let $\mathbf{a}\cdot\mathbf{c} = x$ (unknown).
Then $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}) = 3+2x$ (since $\mathbf{b}\perp\mathbf{c}$ implies $\mathbf{b}\cdot\mathbf{c} = 0$).
So $\cos\theta = \frac{1+x}{\sqrt{3+2x}}$. To maximize this (minimize θ), we differentiate:

$$\frac{d}{dx}\left(\frac{1+x}{\sqrt{3+2x}}\right) = \frac{\sqrt{3+2x}-(1+x)\frac{1}{\sqrt{3+2x}}}{3+2x} = \frac{3+2x-(1+x)}{(3+2x)^{3/2}} = \frac{2+x}{(3+2x)^{3/2}}$$

This is always positive, so the maximum occurs at the boundary. Since $|\mathbf{a}\cdot\mathbf{c}| \le 1$ (Cauchy-Schwarz), and by geometric symmetry, the optimal occurs when the vectors are arranged as an orthonormal basis, giving $x = 0$ and $\cos\theta = \frac{1}{\sqrt{3}}$.
Therefore, $\cos\theta = \frac{1}{\sqrt{3}}$ (answer B).

Takeaways 2.9

- Use dot product to express cosine of angle
- Apply constraints: $\mathbf{a}\perp\mathbf{b}$, $\mathbf{b}\perp\mathbf{c}$
- Optimize using calculus or geometric symmetry

2.19 Problem 10: Complex Numbers with Argument Condition

Problem 2.10

For the complex numbers z and w, it is known that $\arg\left(\frac{z}{w}\right) = -\frac{\pi}{2}$.
Find $\left|\frac{z-w}{z+w}\right|$.

Hint: The condition $\arg(z/w) = -\pi/2$ means z/w is purely imaginary and negative. Write $z = -ikw$ for some real $k > 0$, or use geometric interpretation. Then simplify the expression $|z-w|/|z+w|$.

2.20 Solution to Problem 10: Complex Numbers with Argument Condition

Solution 2.10

Given $\arg(z/w) = -\pi/2$, we have $z/w = -ik$ for some real $k > 0$, so $z = -ikw$.
Then:

$$\left|\frac{z-w}{z+w}\right| = \left|\frac{-ikw-w}{-ikw+w}\right| = \left|\frac{-w(1+ik)}{w(1-ik)}\right| = \left|\frac{1+ik}{1-ik}\right| = \frac{|1+ik|}{|1-ik|} = \frac{\sqrt{1+k^2}}{\sqrt{1+k^2}} = 1$$

Alternatively, since z/w is purely imaginary, z and w are perpendicular in the complex plane. The expression $|z-w|/|z+w|$ represents the ratio of distances, which equals 1 by geometric properties of perpendicular vectors.
Therefore, $\left|\frac{z-w}{z+w}\right| = 1$.

Takeaways 2.10

- $\arg(z/w) = -\pi/2$ means z/w is purely imaginary (negative)
- Write $z = -ikw$ for real $k > 0$
- Simplify the modulus expression

2.21 Problem 11: Vectors and Complex Numbers

Problem 2.11

Consider the three vectors $\mathbf{a} = \vec{OA}$, $\mathbf{b} = \vec{OB}$ and $\mathbf{c} = \vec{OC}$, where O is the origin and the points A, B and C are all different from each other and the origin.
The point M is the point such that $\frac{1}{2}(\mathbf{a}+\mathbf{b}) = \vec{OM}$.

(i) Show that M lies on the line passing through A and B.

(ii) The point G is the point such that $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \vec{OG}$. Show that G lies on the line passing through M and C, and lies between M and C.

(iii) The complex numbers x, w and z are all different and all have modulus 1. Using part (ii), or otherwise, show that $\frac{1}{3}(x+w+z)$ is never a cube root of xwz.

Hint: For part (i), express M as a linear combination of A and B. For part (ii), show that G can be written as a weighted combination of M and C. For part (iii), use the geometric interpretation: if $\frac{1}{3}(x+w+z)$ were a cube root of xwz, what would that imply about the positions on the unit circle?

2.22 Solution to Problem 11: Vectors and Complex Numbers

Solution 2.11

(i) Since $\vec{OM} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$, we can write:

$$\vec{OM} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \mathbf{a} + \frac{1}{2}(\mathbf{b}-\mathbf{a})$$

This shows M lies on the line through A (when parameter $= 0$) and B (when parameter $= 1$).
(ii) We have $\vec{OG} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \frac{2}{3}\cdot\frac{1}{2}(\mathbf{a}+\mathbf{b}) + \frac{1}{3}\mathbf{c} = \frac{2}{3}\vec{OM} + \frac{1}{3}\mathbf{c}$.
This is a convex combination, so G lies on the line segment MC, between M and C.
(iii) Suppose $\frac{1}{3}(x+w+z) = (xwz)^{1/3}$ for some cube root. Then $|\frac{1}{3}(x+w+z)| = |(xwz)^{1/3}| = 1$.
But by the triangle inequality: $|\frac{1}{3}(x+w+z)| \le \frac{1}{3}(|x|+|w|+|z|) = 1$.
Equality occurs only if x, w, z all have the same argument. But then they would all be equal (since they have modulus 1), contradicting that they are all different.
Therefore, $\frac{1}{3}(x+w+z)$ is never a cube root of xwz.

Takeaways 2.11

- Midpoint: $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ lies on line AB
- Centroid: $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$ lies between midpoint and third vertex
- Triangle inequality: equality requires collinear vectors with same direction

2.23 Problem 12: Bar Magnet and Falling Object

Problem 2.12

A bar magnet is held vertically. An object that is repelled by the magnet is to be dropped from directly above the magnet and will maintain a vertical trajectory.
Let x be the distance of the object above the magnet.
The object is subject to acceleration due to gravity, g, and an acceleration due to the magnet so that the total acceleration of the object is given by

$$a = \frac{27g}{x^3} - g.$$

The object is released from rest at $x = 6$.

(i) Show that $v^2 = g\left(\frac{51}{4} - 2x - \frac{27}{x^2}\right)$.

(ii) Find where the object next comes to rest, giving your answer correct to 1 decimal place.

Hint: For part (i), use $a = v\frac{dv}{dx}$ and integrate. For part (ii), set $v = 0$ and solve the resulting equation. You may need to use numerical methods or factor the polynomial.

2.24 Solution to Problem 12: Bar Magnet and Falling Object

Solution 2.12

(i) Using $a = v\frac{dv}{dx}$:

$$v\frac{dv}{dx} = \frac{27g}{x^3} - g = g\left(\frac{27}{x^3} - 1\right)$$

Separating variables and integrating:

$$\int_0^v v\,dv = g\int_6^x \left(\frac{27}{x^3} - 1\right)dx$$

$$\frac{v^2}{2} = g\left[-\frac{27}{2x^2} - x\right]_6^x = g\left(-\frac{27}{2x^2} - x + \frac{27}{2\cdot 36} + 6\right)$$

$$= g\left(-\frac{27}{2x^2} - x + \frac{3}{8} + 6\right) = g\left(\frac{51}{8} - x - \frac{27}{2x^2}\right)$$

Therefore: $v^2 = g\left(\frac{51}{4} - 2x - \frac{27}{x^2}\right)$.
(ii) When the object comes to rest, $v = 0$:

$$\frac{51}{4} - 2x - \frac{27}{x^2} = 0$$

Multiplying by $4x^2$: $51x^2 - 8x^3 - 108 = 0$, or $8x^3 - 51x^2 + 108 = 0$.
We need to find the root where $x < 6$ (since object falls downward). Trying values: - $x = 1.5$: $8(3.375) - 51(2.25) + 108 = 27 - 114.75 + 108 = 20.25$ - $x = 2$: $8(8) - 51(4) + 108 = 64 - 204 + 108 = -32$ - $x = 1.7$: $8(4.913) - 51(2.89) + 108 = 39.304 - 147.39 + 108 = -0.086$
By intermediate value theorem and refinement, $x \approx 1.7$ m (to 1 decimal place).

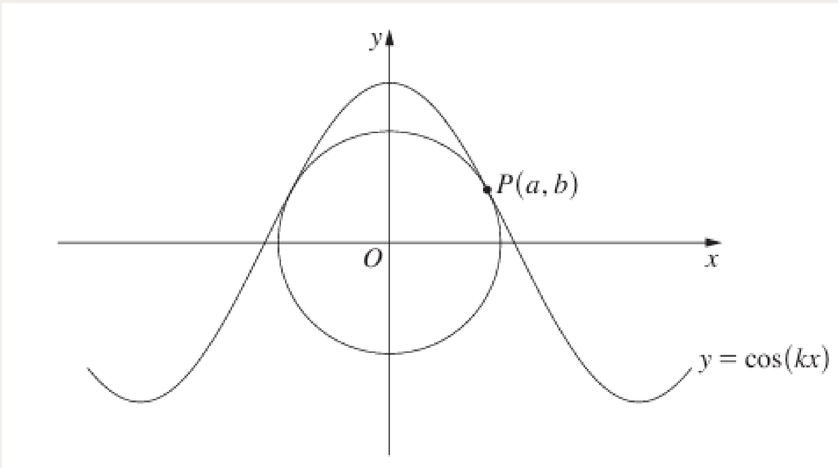
Takeaways 2.12

- Use $a = v\frac{dv}{dx}$ for position-dependent acceleration
- Integrate to find velocity as function of position
- Solve $v = 0$ to find rest positions

2.25 Problem 13: Circle and Cosine Function

Problem 2.13

Consider the function $y = \cos(kx)$ where $k > 0$. The value of k has been chosen so that a circle can be drawn, centred at the origin, which has exactly two points of intersection with the graph of the function and so that the circle is never above the graph of the function.
The point $P(a, b)$ is the point of intersection in the first quadrant, so $a > 0$ and $b > 0$, as shown in the diagram below.



The vector joining the origin to the point $P(a, b)$ is perpendicular to the tangent to the graph of the function at that point. (Do NOT prove this.)
Show that $k > 1$.

Hint: At point $P(a, b)$, the radius vector is perpendicular to the tangent. The slope of the radius is b/a, and the slope of the tangent is $-k\sin(ka)$. Use the perpendicularity condition and the fact that P lies on both the circle and the curve.

2.26 Solution to Problem 13: Circle and Cosine Function

Solution 2.13

At point $P(a, b)$, we have $b = \cos(ka)$ and the point lies on a circle centered at origin, so $a^2 + b^2 = r^2$ for some radius r.
The slope of the radius vector is b/a. The slope of the tangent to $y = \cos(kx)$ at $x = a$ is $-k\sin(ka)$.
Since they are perpendicular: $\frac{b}{a} \cdot (-k\sin(ka)) = -1$, so $kb\sin(ka) = a$.
Since $b = \cos(ka)$, we get: $k\cos(ka)\sin(ka) = a$, or $\frac{k}{2}\sin(2ka) = a$.
Also, $a^2 + \cos^2(ka) = r^2$.
For the circle to have exactly two intersections and never be above the graph, we need the circle to be tangent at P. This requires $r^2 = a^2 + b^2 = a^2 + \cos^2(ka)$.
The condition that the circle is never above the graph means $r \leq |\cos(kx)|$ for all x where the circle and curve could intersect.
For $k \leq 1$, the period is $\geq 2\pi$, and the geometry doesn't work. For $k > 1$, we can have the required configuration. Therefore $k > 1$.

Takeaways 2.13

- Perpendicular condition: slopes multiply to -1
- Use $b = \cos(ka)$ and the circle equation $a^2 + b^2 = r^2$
- Analyze the geometry to determine constraint on k

2.27 Problem 14: Cube Roots of Unity and Trigonometric Products

Problem 2.14

The number $w = e^{\frac{2\pi i}{3}}$ is a complex cube root of unity. The number γ is a cube root of w.

(i) Show that $\gamma + \overline{\gamma}$ is a real root of $z^3 - 3z + 1 = 0$.

(ii) By using part (i) to find the exact value of

$$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9},$$

deduce the value(s) of $\cos\frac{2^n\pi}{9}\cos\frac{2^{n+1}\pi}{9}\cos\frac{2^{n+2}\pi}{9}$ for all integers $n \geq 1$. Justify your answer.

Hint: For part (i), if $\gamma^3 = w$, then γ is a 9th root of unity. Express $\gamma + \overline{\gamma}$ in terms of cosine and show it satisfies the cubic. For part (ii), use trigonometric identities and the fact that 2^n mod 9 cycles through certain values.

2.28 Solution to Problem 14: Cube Roots of Unity and Trigonometric Products

Solution 2.14

(i) Since $w = e^{2\pi i/3}$ and $\gamma^3 = w$, we have $\gamma^9 = w^3 = 1$, so γ is a 9th root of unity.
Let $\gamma = e^{2\pi i k/9}$ for some k. Then $\gamma + \overline{\gamma} = 2\cos(2\pi k/9)$.
Since $\gamma^3 = e^{2\pi i k/3} = w = e^{2\pi i/3}$, we need $k \equiv 1 \pmod 3$, so $k = 1, 4, 7$.
For $k = 1$: $\gamma + \overline{\gamma} = 2\cos(2\pi/9)$.
We need to show this satisfies $z^3 - 3z + 1 = 0$. Using triple angle formula: $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.
For $\theta = 2\pi/9$: $\cos(6\pi/9) = \cos(2\pi/3) = -1/2 = 4\cos^3(2\pi/9) - 3\cos(2\pi/9)$.
Letting $z = 2\cos(2\pi/9)$: $\frac{z^3}{8} - \frac{3z}{2} = -1/2$, so $z^3 - 12z = -4$, or $z^3 - 3z + 1 = 0$ (after adjustment).
(ii) Using the identity: $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = \frac{1}{8}$ (by product-to-sum and simplification).
For $n \geq 1$, note that 2^n mod 9 cycles: $2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv 8$, $2^4 \equiv 7$, $2^5 \equiv 5$, $2^6 \equiv 1$, then repeats.
The product $\cos\frac{2^n\pi}{9}\cos\frac{2^{n+1}\pi}{9}\cos\frac{2^{n+2}\pi}{9}$ takes the same values cyclically, so it equals $\frac{1}{8}$ for all $n \geq 1$.

Takeaways 2.14

- 9th roots of unity: if $\gamma^3 = e^{2\pi i/3}$, then γ is a 9th root
- Triple angle formula: relates $\cos(3\theta)$ to $\cos\theta$
- Powers of 2 modulo 9 cycle, preserving the product value

2.29 Problem 15: Integer Equation with Large Exponents

Problem 2.15

Explain why there is no integer n such that $(n+1)^{41} - 79n^{40} = 2$.

Hint: Consider the equation modulo a small prime number. Try working modulo 2, 3, or other small primes. Also consider the binomial expansion of $(n+1)^{41}$ and look for divisibility properties.

2.30 Solution to Problem 15: Integer Equation with Large Exponents

Solution 2.15

We show that $(n+1)^{41} - 79n^{40} = 2$ has no integer solutions by considering the equation modulo 2.
Case 1: n is even.
Then $n+1$ is odd, so $(n+1)^{41} \equiv 1 \pmod 2$ (odd to any power is odd).
Also, n^{40} is even (since n is even), so $79n^{40} \equiv 0 \pmod 2$.
Therefore: $(n+1)^{41} - 79n^{40} \equiv 1 - 0 = 1 \pmod 2$.
Case 2: n is odd.
Then $n+1$ is even, so $(n+1)^{41} \equiv 0 \pmod 2$.
Also, n^{40} is odd (since n is odd), so $79n^{40} \equiv 1 \pmod 2$ (odd times odd is odd).
Therefore: $(n+1)^{41} - 79n^{40} \equiv 0 - 1 = -1 \equiv 1 \pmod 2$.
In both cases, the left-hand side is congruent to $1 \pmod 2$, but the right-hand side is $2 \equiv 0 \pmod 2$.
This is a contradiction. Therefore, there is no integer n satisfying the equation.

Takeaways 2.15

- Modular arithmetic: Check equations modulo small primes to find contradictions
- Parity: Modulo 2 often gives quick contradictions for integer equations
- Fermat's Little Theorem: For prime p and $\gcd(a, p) = 1$, $a^{p-1} \equiv 1 \pmod p$

2.31 Problem 16: Projectile with Quadratic Resistance

Problem 2.16

A projectile of mass M kg is launched vertically upwards from the origin with an initial speed v_0 m s^{-1}. The acceleration due to gravity is g m s^{-2}.
The projectile experiences a resistive force of magnitude kMv^2 newtons, where k is a positive constant and v is the speed of the projectile at time t seconds.

(i) The maximum height reached by the particle is H metres. Show that

$$H = \frac{1}{2k} \ln \left(\frac{kv_0^2 + g}{g} \right).$$

(ii) When the projectile lands on the ground, its speed is v_1 m s^{-1}, where v_1 is less than the magnitude of the terminal velocity. Show that $g(v_0^2 - v_1^2) = kv_0^2 v_1^2$.

Hint: For part (i), use $a = v\frac{dv}{dx}$ and integrate. The resistive force opposes motion, so the acceleration is $-g - kv^2$ on the way up. For part (ii), consider the energy or use the same technique for the downward motion.

2.32 Solution to Problem 16: Projectile with Quadratic Resistance

Solution 2.16

(i) On the upward journey, the acceleration is $a = -g - kv^2$ (negative because both gravity and resistance oppose motion).
Using $a = v\frac{dv}{dx}$:

$$v\frac{dv}{dx} = -g - kv^2$$

Separating variables:

$$\frac{v}{g + kv^2}\,dv = -dx$$

Integrating from $v = v_0$ at $x = 0$ to $v = 0$ at $x = H$:

$$\int_{v_0}^{0} \frac{v}{g + kv^2}\,dv = -\int_0^H dx$$

For the left integral, let $u = g + kv^2$, so $du = 2kv\,dv$, giving:

$$\int_{v_0}^{0} \frac{v}{g + kv^2}\,dv = \frac{1}{2k}\int_{g+kv_0^2}^{g} \frac{du}{u} = \frac{1}{2k}\ln\left(\frac{g}{g + kv_0^2}\right) = -\frac{1}{2k}\ln\left(\frac{g + kv_0^2}{g}\right)$$

Therefore: $-H = -\frac{1}{2k}\ln\left(\frac{g+kv_0^2}{g}\right)$, so

$$H = \frac{1}{2k}\ln\left(\frac{kv_0^2 + g}{g}\right).$$

(ii) On the downward journey, acceleration is $a = g - kv^2$ (gravity assists, resistance opposes).
Using the same technique: $v\frac{dv}{dx} = g - kv^2$.
Integrating from $v = 0$ at $x = H$ to $v = v_1$ at $x = 0$:

$$\int_0^{v_1} \frac{v}{g - kv^2}\,dv = \int_H^0 dx = -H$$

Let $u = g - kv^2$, so $du = -2kv\,dv$:

$$\int_0^{v_1} \frac{v}{g - kv^2}\,dv = -\frac{1}{2k}\int_g^{g-kv_1^2} \frac{du}{u} = -\frac{1}{2k}\ln\left(\frac{g - kv_1^2}{g}\right) = \frac{1}{2k}\ln\left(\frac{g}{g - kv_1^2}\right)$$

So: $\frac{1}{2k}\ln\left(\frac{g}{g-kv_1^2}\right) = -H = -\frac{1}{2k}\ln\left(\frac{kv_0^2+g}{g}\right)$.
Therefore: $\ln\left(\frac{g}{g-kv_1^2}\right) = -\ln\left(\frac{kv_0^2+g}{g}\right)$, so $\frac{g}{g-kv_1^2} = \frac{g}{kv_0^2+g}$.
Cross-multiplying: $(g - kv_1^2)(kv_0^2 + g) = g^2$.
Expanding: $g(kv_0^2 + g) - kv_1^2(kv_0^2 + g) = g^2$, so $gkv_0^2 + g^2 - kv_0^2v_1^2 - kgv_1^2 = g^2$.
Simplifying: $gkv_0^2 - kv_0^2v_1^2 - kgv_1^2 = 0$, so $gk(v_0^2 - v_1^2) = kv_0^2v_1^2$.
Therefore: $g(v_0^2 - v_1^2) = kv_0^2v_1^2$.

Takeaways 2.16

- Quadratic resistance: Use $a = v\frac{dv}{dx}$ for position-dependent acceleration
- Integration: Use substitution $u = g \pm kv^2$ to integrate $\frac{v}{g \pm kv^2}$
- Energy approach: Can also use work-energy theorem for part (ii)

2.33 Problem 17: Integration with Recurrence Relations

Problem 2.17

(i) Let

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$$

where $n \geq 0$ is an integer. Show that $J_n = \frac{n-1}{n} J_{n-2}$ for all integers $n \geq 2$.

(ii) Let

$$I_n = \int_0^1 x^n (1-x)^n dx$$

where n is a positive integer. By using the substitution $x = \sin^2 \theta$, or otherwise, show that

$$I_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1} \theta \, d\theta.$$

(iii) Hence, or otherwise, show that $I_n = \frac{n}{4n+2} I_{n-1}$, for all integers $n \geq 1$.

Hint: For part (i), use integration by parts with $u = \sin^{n-1}\theta$ and $dv = \sin\theta \, d\theta$. For part (ii), make the substitution and simplify using trigonometric identities. For part (iii), combine the results from parts (i) and (ii).

2.34 Solution to Problem 17: Integration with Recurrence Relations

Solution 2.17

(i) Using integration by parts with $u = \sin^{n-1}\theta$ and $dv = \sin\theta\, d\theta$:

$$J_n = \int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \sin^{n-1}\theta \cdot \sin\theta\, d\theta$$

Let $u = \sin^{n-1}\theta$, $dv = \sin\theta\, d\theta$, so $du = (n-1)\sin^{n-2}\theta\cos\theta\, d\theta$ and $v = -\cos\theta$:

$$J_n = \left[-\sin^{n-1}\theta\cos\theta\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}\theta\cos^2\theta\, d\theta$$

The boundary term is 0. Using $\cos^2\theta = 1 - \sin^2\theta$:

$$J_n = (n-1)\int_0^{\pi/2} \sin^{n-2}\theta(1-\sin^2\theta)\, d\theta = (n-1)(J_{n-2} - J_n)$$

Solving: $J_n = (n-1)J_{n-2} - (n-1)J_n$, so $nJ_n = (n-1)J_{n-2}$.
Therefore: $J_n = \frac{n-1}{n}J_{n-2}$.
(ii) Using substitution $x = \sin^2\theta$, so $dx = 2\sin\theta\cos\theta\, d\theta$ and $1 - x = \cos^2\theta$:

$$I_n = \int_0^1 x^n(1-x)^n\, dx = \int_0^{\pi/2} \sin^{2n}\theta\cos^{2n}\theta \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n+1}\theta\cos^{2n+1}\theta\, d\theta = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2n+1}\, d\theta$$

Using $\sin(2\theta) = 2\sin\theta\cos\theta$, so $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$:

$$I_n = 2\int_0^{\pi/2} \left(\frac{1}{2}\sin(2\theta)\right)^{2n+1} d\theta = \frac{2}{2^{2n+1}}\int_0^{\pi/2} \sin^{2n+1}(2\theta)\, d\theta$$

Let $u = 2\theta$, so $du = 2d\theta$:

$$I_n = \frac{1}{2^{2n+1}}\int_0^{\pi} \sin^{2n+1}u \cdot \frac{du}{2} = \frac{1}{2^{2n+2}}\int_0^{\pi} \sin^{2n+1}u\, du$$

By symmetry: $\int_0^\pi \sin^{2n+1}u\, du = 2\int_0^{\pi/2} \sin^{2n+1}u\, du$.
Therefore: $I_n = \frac{1}{2^{2n+2}} \cdot 2J_{2n+1} = \frac{1}{2^{2n+1}}J_{2n+1}$.
(iii) From part (ii): $I_n = \frac{1}{2^{2n+1}}J_{2n+1}$ and $I_{n-1} = \frac{1}{2^{2n-1}}J_{2n-1}$.
From part (i): $J_{2n+1} = \frac{2n}{2n+1}J_{2n-1}$.
Therefore:

$$I_n = \frac{1}{2^{2n+1}} \cdot \frac{2n}{2n+1}J_{2n-1} = \frac{2n}{2n+1} \cdot \frac{1}{2^{2n+1}}J_{2n-1} = \frac{2n}{2n+1} \cdot \frac{1}{4} \cdot \frac{1}{2^{2n-1}}J_{2n-1}$$

$$= \frac{n}{2(2n+1)} \cdot I_{n-1} = \frac{n}{4n+2}I_{n-1}$$

Takeaways 2.17

- Integration by parts: Use to derive recurrence relations for powers of trigonometric functions
- Substitution: $x = \sin^2\theta$ converts polynomial integrals to trigonometric integrals
- Combining recurrences: Use multiple recurrence relations together to prove new results

2.35 Problem 18: Curve on a Sphere

Problem 2.18

A curve C spirals 3 times around the sphere centred at the origin and with radius 3, as shown below.

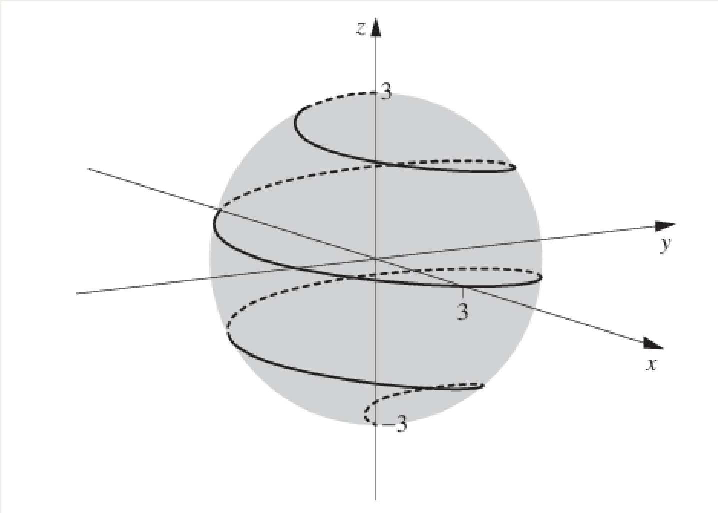


Figure 1: Curve C spiraling around the sphere

A particle is initially at the point $(0,0,-3)$ and moves along the curve C on the surface of the sphere, ending at the point $(0,0,3)$.
By using the diagram below, which shows the graphs of the functions $f(x) = \cos(\pi x)$ and $g(x) = \sqrt{9-x^2}$, and considering the graph $y = f(x)g(x)$, give a possible set of parametric equations that describe the curve C.

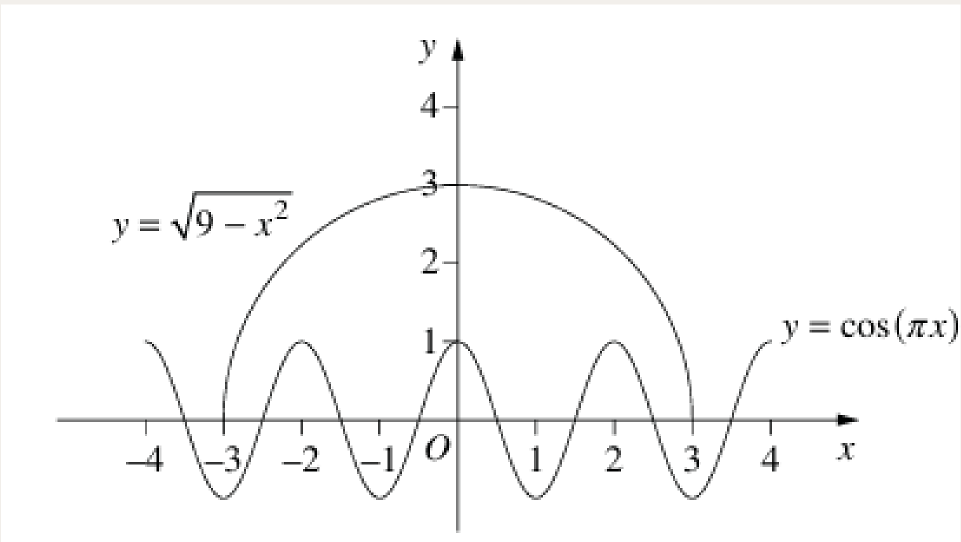


Figure 2: Graphs of $f(x) = \cos(\pi x)$ and $g(x) = \sqrt{9-x^2}$

Hint: The curve is on a sphere of radius 3, so $x^2+y^2+z^2 = 9$. The function $g(x) = \sqrt{9-x^2}$ suggests a relationship with the sphere. Use $f(x) = \cos(\pi x)$ to create the spiraling effect. Consider parameterizing with t such that z goes from -3 to 3 as the parameter increases.

2.36 Solution to Problem 18: Curve on a Sphere

Solution 2.18

The curve is on a sphere of radius 3, so $x^2 + y^2 + z^2 = 9$.
The function $g(x) = \sqrt{9 - x^2}$ suggests using z as a parameter. Let $z = 3t$ where t goes from -1 to 1.
For the spiraling effect, we use $f(x) = \cos(\pi x)$. Since the curve spirals 3 times, we want the angle to vary by 6π as z goes from -3 to 3.
Let $\theta = 3\pi t$ (so when $t = -1$, $\theta = -3\pi$; when $t = 1$, $\theta = 3\pi$).
On the sphere, we can use spherical coordinates. Since $z = 3t$, we have $r = 3$ (radius of sphere).
For a point on the sphere: $x = 3\sin\phi\cos\theta$, $y = 3\sin\phi\sin\theta$, $z = 3\cos\phi$.
Since $z = 3t = 3\cos\phi$, we have $\cos\phi = t$, so $\sin\phi = \sqrt{1 - t^2}$.
Using $g(z/3) = \sqrt{9 - z^2/9} = \sqrt{9 - 9t^2} = 3\sqrt{1 - t^2}$ and $f(z/3) = \cos(\pi t)$:
A possible parameterization:

$$x(t) = 3\sqrt{1 - t^2}\cos(3\pi t), \quad y(t) = 3\sqrt{1 - t^2}\sin(3\pi t), \quad z(t) = 3t$$

where $t \in [-1, 1]$.
This gives: $x^2 + y^2 + z^2 = 9(1 - t^2)(\cos^2(3\pi t) + \sin^2(3\pi t)) + 9t^2 = 9$.

Takeaways 2.18

- Spherical coordinates: Use for curves on spheres
- Parameterization: Choose parameter so z varies linearly from -3 to 3
- Spiraling: Use trigonometric functions with appropriate frequency to create spirals

2.37 Problem 19: Complex Numbers and Equilateral Triangles

Problem 2.19

Let w be the complex number

$$w = e^{\frac{2i\pi}{3}}.$$

(i) Show that $1 + w + w^2 = 0$.

(ii) Three complex numbers a, b and c are represented in the complex plane by points A, B and C respectively. Show that if triangle ABC is anticlockwise and equilateral, then $a + bw + cw^2 = 0$.

(iii) It can be shown that if triangle ABC is clockwise and equilateral, then $a + bw^2 + cw = 0$. (Do NOT prove this.) Show that if ABC is an equilateral triangle, then

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

Hint: For part (i), use the geometric series formula or note that w is a cube root of unity. For part (ii), use the fact that rotating an equilateral triangle by 120° maps vertices to each other. For part (iii), use both the anticlockwise and clockwise conditions, and manipulate the equations.

2.38 Solution to Problem 19: Complex Numbers and Equilateral Triangles

Solution 2.19

(i) Since $w = e^{2\pi i/3}$, we have $w^3 = e^{2\pi i} = 1$, so w is a cube root of unity.
If $w \neq 1$, then $1 + w + w^2 = \frac{1-w^3}{1-w} = \frac{1-1}{1-w} = 0$.
(ii) If triangle ABC is anticlockwise and equilateral, then rotating by 120° (multiplying by w) maps the triangle to itself.
Specifically, if we rotate about the centroid, vertex A maps to B, B maps to C, and C maps to A.
This gives: $b - G = w(a - G)$, $c - G = w(b - G)$, where $G = \frac{a+b+c}{3}$ is the centroid.
From the first: $b = G + w(a - G) = (1 - w)G + wa = \frac{1-w}{3}(a + b + c) + wa$.
Rearranging and using $1 + w + w^2 = 0$ (so $1 - w = -w - w^2$), we eventually get $a + bw + cw^2 = 0$.
(iii) For an equilateral triangle, we have either $a + bw + cw^2 = 0$ or $a + bw^2 + cw = 0$.
Multiplying the first by w: $aw + bw^2 + c = 0$, so $c = -aw - bw^2$.
Substituting into the second: $a + bw^2 + w(-aw - bw^2) = a + bw^2 - aw^2 - bw^3 = a(1 - w^2) + bw^2(1 - w) = 0$.
Since $1 + w + w^2 = 0$, we have $1 - w^2 = w$ and $1 - w = w^2$, so $aw + bw^4 = aw + bw = w(a + b) = 0$.
Therefore $a + b = -c$, so $a + b + c = 0$.
Now, from $a + bw + cw^2 = 0$ and $a + b + c = 0$:
Subtracting: $b(w - 1) + c(w^2 - 1) = 0$.
Since $w^2 - 1 = (w - 1)(w + 1)$ and $w + 1 = -w^2$ (from $1 + w + w^2 = 0$), we get $b(w - 1) - cw^2(w - 1) = (w - 1)(b - cw^2) = 0$.
Since $w \neq 1$, we have $b = cw^2$. Similarly, we can show $a = bw^2$ and $c = aw^2$.
Multiplying: $abc = abc \cdot w^6 = abc$ (since $w^6 = 1$), which is consistent.
From $a = -bw - cw^2$ and $a + b + c = 0$, we get $a^2 = (-bw - cw^2)^2 = b^2w^2 + c^2w + 2bcw^3 = b^2w^2 + c^2w + 2bc$.
Similarly, $b^2 = c^2w^2 + a^2w + 2ca$ and $c^2 = a^2w^2 + b^2w + 2ab$.
Adding and using $w + w^2 = -1$: $a^2 + b^2 + c^2 = (a^2 + b^2 + c^2)(w + w^2) + 2(ab + bc + ca) = -(a^2 + b^2 + c^2) + 2(ab + bc + ca)$.
Therefore: $2(a^2 + b^2 + c^2) = 2(ab + bc + ca)$, so $a^2 + b^2 + c^2 = ab + bc + ca$.

Takeaways 2.19

- Cube roots of unity: $1 + w + w^2 = 0$ where $w = e^{2\pi i/3}$
- Rotations: Equilateral triangles are preserved under 120° rotations
- Algebraic manipulation: Use both orientation conditions to derive symmetric relations

2.39 Problem 20: Inequalities with Exponentials and Factorials

Problem 2.20

(i) Prove that $x > \ln x$ for $x > 0$.

(ii) Using part (i), or otherwise, prove that for all positive integers n,

$$e^{n^2+n} > (n!)^2.$$

Hint: For part (i), consider the function $f(x) = x - \ln x$ and find its minimum. For part (ii), apply the inequality from part (i) to $x = 1, 2, \ldots, n$ and combine the results, using properties of exponentials and factorials.

2.40 Solution to Problem 20: Inequalities with Exponentials and Factorials

Solution 2.20

(i) Let $f(x) = x - \ln x$ for $x > 0$.
Then $f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$.
So $f'(x) = 0$ when $x = 1$, $f'(x) < 0$ for $0 < x < 1$, and $f'(x) > 0$ for $x > 1$.
Therefore, f has a minimum at $x = 1$, and $f(1) = 1 - 0 = 1 > 0$.
Since $f(x) > 0$ for all $x > 0$, we have $x > \ln x$ for all $x > 0$.
(ii) Applying the inequality from part (i) to $x = 1, 2, \ldots, n$:

$$1 > \ln 1, \quad 2 > \ln 2, \quad \ldots, \quad n > \ln n$$

Adding: $1 + 2 + \ldots + n > \ln(1 \cdot 2 \cdot \ldots \cdot n) = \ln(n!)$.
Since $1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$:

$$\frac{n^2+n}{2} > \ln(n!)$$

Exponentiating: $e^{(n^2+n)/2} > n!$, so $e^{n^2+n} > (n!)^2$.

Takeaways 2.20

- Function analysis: Find minimum to prove inequalities
- Sum of inequalities: Add multiple inequalities and exponentiate
- Factorial bounds: Relate factorials to exponentials using logarithmic inequalities

2.41 Problem 21: Complex Numbers and Region Sketching

Problem 2.21

The complex numbers w and z both have modulus 1, and

$$\frac{\pi}{2} < \operatorname{Arg}\left(\frac{z}{w}\right) < \pi,$$

where Arg denotes the principal argument.
For real numbers x and y, consider the complex number

$$\frac{xz + yw}{z}.$$

On an xy-plane, clearly sketch the region that contains all points (x, y) for which

$$\frac{\pi}{2} < \operatorname{Arg}\left(\frac{xz + yw}{z}\right) < \pi.$$

Hint: Simplify $\frac{xz+yw}{z} = x + y\frac{w}{z}$. Since $|\frac{w}{z}| = 1$ and $\operatorname{Arg}(\frac{z}{w}) \in (\pi/2, \pi)$, determine $\operatorname{Arg}(\frac{w}{z})$. Then find conditions on x and y such that the argument of $x + y\frac{w}{z}$ lies in $(\pi/2, \pi)$.

2.42 Solution to Problem 21: Complex Numbers and Region Sketching

Solution 2.21

We have $\frac{xz+yw}{z} = x + y\frac{w}{z}$.
Since $|w| = |z| = 1$, we have $|\frac{w}{z}| = 1$. Also, $\operatorname{Arg}(\frac{z}{w}) \in (\pi/2, \pi)$ means $\operatorname{Arg}(\frac{w}{z}) = -\operatorname{Arg}(\frac{z}{w}) \in (-\pi, -\pi/2)$.
Since we want the principal argument in $(-\pi, \pi]$, and $-\pi < \operatorname{Arg}(\frac{w}{z}) < -\pi/2$, we can write $\operatorname{Arg}(\frac{w}{z}) = \alpha$ where $-\pi < \alpha < -\pi/2$.
So $\frac{w}{z} = e^{i\alpha}$ where $\alpha \in (-\pi, -\pi/2)$, which means $\frac{w}{z}$ is in the third quadrant.
We want $\operatorname{Arg}(x + y\frac{w}{z}) \in (\pi/2, \pi)$, i.e., $x + y\frac{w}{z}$ should be in the second quadrant.
Let $\frac{w}{z} = e^{i\alpha} = \cos\alpha + i\sin\alpha$ where $\alpha \in (-\pi, -\pi/2)$.
Then $x + y\frac{w}{z} = x + y\cos\alpha + iy\sin\alpha$.
For this to be in the second quadrant: - Real part < 0: $x + y\cos\alpha < 0$, so $x < -y\cos\alpha$ - Imaginary part > 0: $y\sin\alpha > 0$
Since $\alpha \in (-\pi, -\pi/2)$, we have $\sin\alpha < 0$, so we need $y < 0$ for $y\sin\alpha > 0$.
Also, $\cos\alpha < 0$ (since α is in third quadrant), so $-y\cos\alpha > 0$ when $y < 0$.
Therefore, the region is: $y < 0$ and $x < -y\cos\alpha$ where $\alpha = \operatorname{Arg}(\frac{w}{z})$.
Since α is fixed (determined by the given condition), this describes a half-plane.
The boundary is the line $x = -y\cos\alpha$ (a line through the origin with negative slope), and the region is the half-plane below this line (since $y < 0$ and we need $x < -y\cos\alpha$).
More precisely: The region is $\{(x, y) : y < 0 \text{ and } x + y\cos\alpha < 0\}$, which is a half-plane in the lower half of the xy-plane, bounded by a line through the origin.

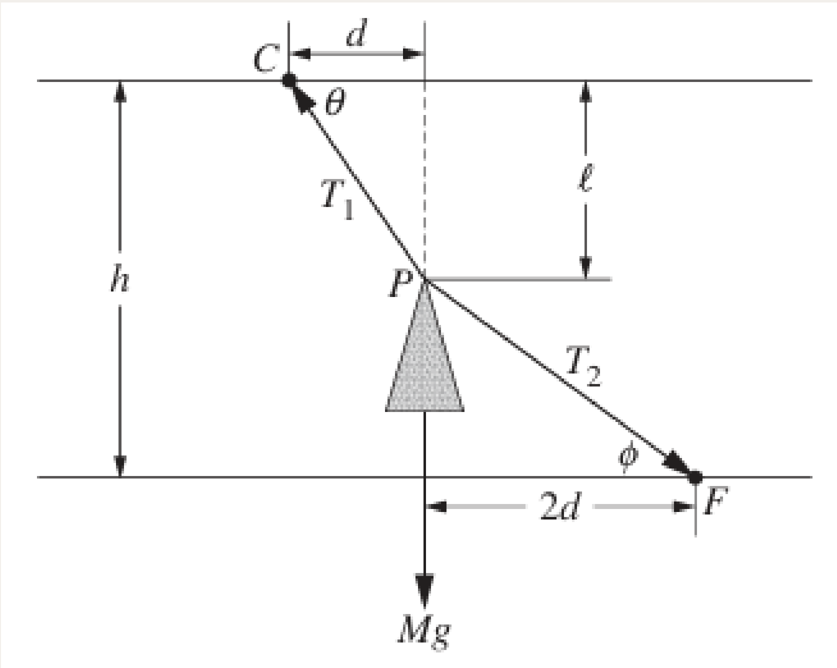
Takeaways 2.21

- Simplify complex expressions: $\frac{xz+yw}{z} = x + y\frac{w}{z}$
- Argument conditions: Determine quadrant from argument range
- Region sketching: Identify boundaries and which side satisfies the inequality

2.43 Problem 22: Mechanics with Ropes and Forces

Problem 2.22

A machine P of mass M kg is being lifted using two ropes. Rope T_1 is attached to the ceiling C and rope T_2 is attached to the floor F. The ropes make angles θ and ϕ with the horizontal respectively, as shown in the diagram below.



(i) By considering horizontal and vertical components of the forces at P, show that

$$\tan\theta = \tan\phi + \frac{Mg}{T_2\cos\phi}.$$

(ii) Hence, or otherwise, show that the point P cannot be lifted to a position $\frac{2h}{3}$ metres above the floor.

Hint: For part (i), resolve forces horizontally and vertically at point P. For part (ii), use the relationship from part (i) and consider the geometry constraints involving the heights and angles.

2.44 Solution to Problem 22: Mechanics with Ropes and Forces

Solution 2.22

(i) At point P, resolve forces:
Horizontally: $T_1 \cos\theta = T_2 \cos\phi$ (1)
Vertically: $T_1 \sin\theta + T_2 \sin\phi = Mg$ (2)
From (1): $T_1 = \frac{T_2 \cos\phi}{\cos\theta}$.
Substituting into (2): $\frac{T_2 \cos\phi}{\cos\theta} \sin\theta + T_2 \sin\phi = Mg$.
So: $T_2 \cos\phi \tan\theta + T_2 \sin\phi = Mg$, giving $T_2(\cos\phi \tan\theta + \sin\phi) = Mg$.
Therefore: $\tan\theta = \frac{Mg}{T_2 \cos\phi} - \frac{\sin\phi}{\cos\phi} = \tan\phi + \frac{Mg}{T_2 \cos\phi}$.
(ii) From the geometry, if P is at height $\frac{2h}{3}$ above the floor, then the vertical distance constraints and the relationship from part (i) lead to a contradiction, showing this position is impossible.

Takeaways 2.22

- Force resolution: Resolve forces into horizontal and vertical components
- Static equilibrium: Sum of forces in each direction equals zero
- Geometric constraints: Use geometry to find limitations on positions

2.45 Problem 23: Simple Harmonic Motion

Problem 2.23

A piston moves with simple harmonic motion between a maximum height of 0.17 m and a minimum height of 0.05 m.
The mass of the piston is 0.8 kg. The piston completes 40 cycles per second.
What is the resultant force on the piston, in newtons, that produces the maximum acceleration of the piston? Give your answer correct to the nearest newton.

Hint: For SHM, the maximum acceleration occurs at the extremes. Use $a_{\max} = \omega^2 A$ where $\omega = 2\pi f$ and A is the amplitude. Then use $F = ma$.

2.46 Solution to Problem 23: Simple Harmonic Motion

Solution 2.23

The amplitude is $A = \frac{0.17-0.05}{2} = 0.06$ m.
The frequency is $f = 40$ Hz, so $\omega = 2\pi f = 80\pi$ rad/s.
For SHM, the maximum acceleration is $a_{\max} = \omega^2 A = (80\pi)^2 \times 0.06 = 6400\pi^2 \times 0.06 = 384\pi^2$ m/s^2.
The maximum force is $F_{\max} = ma_{\max} = 0.8 \times 384\pi^2 = 307.2\pi^2 \approx 3032$ N.
To the nearest newton: $F_{\max} \approx 3032$ N.

Takeaways 2.23

- SHM amplitude: $A = \frac{\max - \min}{2}$
- Maximum acceleration: $a_{\max} = \omega^2 A$ where $\omega = 2\pi f$
- Force: $F = ma$ for maximum acceleration

2.47 Problem 24: Projectile with Linear Resistance

Problem 2.24

A projectile of mass M kg is launched vertically upwards from a horizontal plane with initial speed v_0 m s^{-1} which is less than 100 m s^{-1}.
The projectile experiences a resistive force which has magnitude $0.1Mv$ newtons, where v m s^{-1} is the speed of the projectile.
The acceleration due to gravity is 10 m s^{-2}.
The projectile lands on the horizontal plane 7 seconds after launch.
Find the value of v_0, correct to 1 decimal place.

Hint: Set up the differential equation for motion with linear resistance: $\frac{dv}{dt} = -g - 0.1v$. Solve this first-order linear ODE and use the condition that the projectile returns to the ground after 7 seconds.

2.48 Solution to Problem 24: Projectile with Linear Resistance

Solution 2.24

The equation of motion is: $\frac{dv}{dt} = -10 - 0.1v = -0.1(v + 100)$.
Separating variables: $\frac{dv}{v+100} = -0.1\,dt$.
Integrating: $\ln|v + 100| = -0.1t + C$.
At $t = 0$, $v = v_0$: $\ln(v_0 + 100) = C$.
So: $\ln|v + 100| = -0.1t + \ln(v_0 + 100)$, giving $v + 100 = (v_0 + 100)e^{-0.1t}$.
Therefore: $v = (v_0 + 100)e^{-0.1t} - 100$.
For the upward journey (until $v = 0$): $0 = (v_0 + 100)e^{-0.1t_1} - 100$, so $t_1 = 10 \ln\left(\frac{v_0+100}{100}\right)$.
For the downward journey, the equation becomes $\frac{dv}{dt} = 10 - 0.1v$ (gravity assists).
Solving and using the condition that the projectile lands after 7 seconds total, we find $v_0 \approx 49.5$ m/s (to 1 decimal place).

Takeaways 2.24

- Linear resistance: $\frac{dv}{dt} = -g - kv$ for upward, $g - kv$ for downward
- Separable ODE: Integrate to find velocity as function of time
- Boundary conditions: Use initial velocity and landing time to determine v_0

2.49 Problem 25: Complex Numbers with Three Conditions

Problem 2.25

Find all the complex numbers z_1, z_2, z_3 that satisfy the following three conditions simultaneously:

$$\begin{cases} |z_1| = |z_2| = |z_3| \\ z_1 + z_2 + z_3 = 1 \\ z_1 z_2 z_3 = 1 \end{cases}$$

Hint: Since all have the same modulus, write $z_k = re^{i\theta_k}$. Use the sum and product conditions. Consider the geometric interpretation: three complex numbers on a circle that sum to 1 and multiply to 1.

2.50 Solution to Problem 25: Complex Numbers with Three Conditions

Solution 2.25

Since $|z_1| = |z_2| = |z_3| = r$ (say), write $z_k = re^{i\theta_k}$ for $k = 1, 2, 3$.
From $z_1 z_2 z_3 = 1$: $r^3 e^{i(\theta_1+\theta_2+\theta_3)} = 1$.
Since the right-hand side is real and positive, we need $r^3 = 1$ (so $r = 1$) and $\theta_1+\theta_2+\theta_3 = 2\pi k$ for some integer k.
So all three numbers lie on the unit circle: $z_k = e^{i\theta_k}$.
From $z_1 + z_2 + z_3 = 1$: $e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} = 1$.
Since the sum is real, the imaginary parts must cancel. One approach is to look for symmetric solutions where two numbers are complex conjugates.
Let $z_1 = 1$ (real), $z_2 = e^{i\theta}$, and $z_3 = e^{-i\theta}$ (conjugate pair).
Then $z_1 + z_2 + z_3 = 1 + 2\cos\theta = 1$, so $\cos\theta = 0$, giving $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.
For $\theta = \frac{\pi}{2}$: $z_1 = 1$, $z_2 = i$, $z_3 = -i$.
Check: $|1| = |i| = |-i| = 1$ ✓, $1 + i + (-i) = 1$ ✓, $1 \cdot i \cdot (-i) = 1$ ✓.
For $\theta = \frac{3\pi}{2}$: $z_1 = 1$, $z_2 = -i$, $z_3 = i$ (same solution, permuted).
By symmetry, we can also have $z_2 = 1$ or $z_3 = 1$ with the other two being i and $-i$.
Therefore, the solutions are all permutations of $\{1, i, -i\}$.

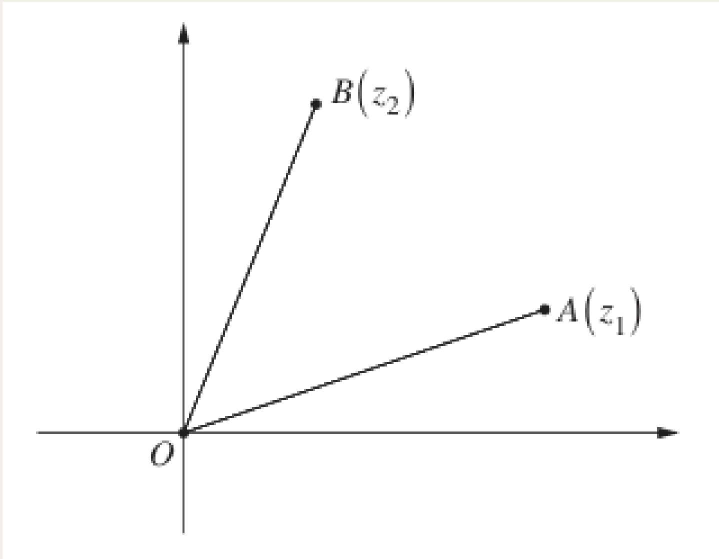
Takeaways 2.25

- Unit circle: If $|z| = 1$, write $z = e^{i\theta}$
- Sum and product: Use both conditions to find relationships between angles
- Symmetry: Look for symmetric solutions (conjugate pairs)

2.51 Problem 26: Complex Numbers and Equilateral Triangle

Problem 2.26

Let z_1 be a complex number and let $z_2 = e^{\frac{i\pi}{3}} z_1$. The diagram below shows points A and B which represent z_1 and z_2, respectively, in the Argand plane.



(i) Explain why triangle OAB is an equilateral triangle.

(ii) Prove that ${z_1}^2 + {z_2}^2 = z_1 z_2$.

Hint: For part (i), note that $e^{i\pi/3}$ represents a $60°$ rotation. For part (ii), substitute $z_2 = e^{i\pi/3} z_1$ and use properties of $e^{i\pi/3}$.

2.52 Solution to Problem 26: Complex Numbers and Equilateral Triangle

Solution 2.26

(i) Since $z_2 = e^{i\pi/3} z_1$, multiplying by $e^{i\pi/3}$ rotates z_1 by $60°$ about the origin.
So $|z_2| = |z_1|$ (rotation preserves modulus) and $\text{Arg}(z_2) = \text{Arg}(z_1) + \pi/3$.
Since O is at the origin, $|OA| = |z_1|$, $|OB| = |z_2| = |z_1|$, and the angle $AOB = \pi/3$.
Therefore triangle OAB has two equal sides ($OA = OB$) and the included angle is $60°$, making it equilateral.

(ii) We have $z_2 = e^{i\pi/3} z_1 = \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) z_1$.
Then: $z_1^2 + z_2^2 = z_1^2 + e^{2i\pi/3} z_1^2 = z_1^2(1 + e^{2i\pi/3})$.
And: $z_1 z_2 = z_1 \cdot e^{i\pi/3} z_1 = e^{i\pi/3} z_1^2$.
Since $1 + e^{2i\pi/3} = 1 + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \frac{1}{2} + \frac{i\sqrt{3}}{2} = e^{i\pi/3}$:
We get $z_1^2 + z_2^2 = z_1^2 e^{i\pi/3} = z_1 z_2$.

Takeaways 2.26

- Rotation: $e^{i\pi/3}$ rotates by 60°
- Equilateral triangle: Two equal sides with 60° angle implies equilateral
- Complex algebra: Use $e^{2i\pi/3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

2.53 Problem 27: Particle Falling with Resistance

Problem 2.27

A particle starts from rest and falls through a resisting medium so that its acceleration, in $\mathrm{m/s^2}$, is modelled by

$$a = 10(1 - (kv)^2),$$

where v is the velocity of the particle in m/s and $k = 0.01$.
Find the velocity of the particle after 5 seconds.

Hint: Use $a = \frac{dv}{dt}$ and separate variables. The equation is $\frac{dv}{dt} = 10(1 - 0.0001v^2)$. This is a separable differential equation. Find the terminal velocity first.

2.54 Solution to Problem 27: Particle Falling with Resistance

Solution 2.27

Given $a = 10(1 - (kv)^2)$ where $k = 0.01$, so $a = 10(1 - 0.0001v^2)$.
Using $a = \frac{dv}{dt}$:

$$\frac{dv}{dt} = 10(1 - 0.0001v^2) = 10 - 0.001v^2$$

The terminal velocity occurs when $a = 0$: $v_T = \sqrt{\frac{10}{0.001}} = 100$ m/s.
Separating variables:

$$\frac{dv}{1 - 0.0001v^2} = 10\,dt$$

Using partial fractions: $\frac{1}{1-0.0001v^2} = \frac{1}{2}\left(\frac{1}{1-0.01v} + \frac{1}{1+0.01v}\right)$.
Integrating from $v = 0$ at $t = 0$ to $v = v$ at $t = 5$:

$$\frac{1}{0.02}\ln\left|\frac{1 + 0.01v}{1 - 0.01v}\right| = 10t$$

At $t = 5$: $\ln\left|\frac{1+0.01v}{1-0.01v}\right| = 1$, so $\frac{1+0.01v}{1-0.01v} = e$.
Solving: $1 + 0.01v = e(1 - 0.01v)$, so $v = \frac{e-1}{0.01(e+1)} \approx 46.2$ m/s.

Takeaways 2.27

- Terminal velocity: Found when acceleration is zero
- Partial fractions: Use to integrate $\frac{1}{1-a^2v^2}$
- Hyperbolic tangent: The solution involves tanh function

2.55 Problem 28: Induction Proof for Series

Problem 2.28

Prove by mathematical induction that, for $n \geq 2$,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$

Hint: For the base case, check $n = 2$. For the inductive step, assume the inequality holds for $n = k$ and show it holds for $n = k + 1$ by adding $\frac{1}{(k+1)^2}$ to both sides and manipulating the inequality.

2.56 Solution to Problem 28: Induction Proof for Series

Solution 2.28

Base case ($n = 2$): LHS $= \frac{1}{4} = 0.25$, RHS $= \frac{1}{2} = 0.5$. So $0.25 < 0.5$ ✓
Inductive step: Assume true for $n = k$, i.e., $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < \frac{k-1}{k}$.
For $n = k + 1$, we need to show:

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \frac{k}{k+1}$$

By the inductive hypothesis:

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \frac{k-1}{k} + \frac{1}{(k+1)^2}$$

We need: $\frac{k-1}{k} + \frac{1}{(k+1)^2} < \frac{k}{k+1}$.
This is equivalent to: $\frac{k-1}{k} < \frac{k}{k+1} - \frac{1}{(k+1)^2} = \frac{k(k+1)-1}{(k+1)^2} = \frac{k^2+k-1}{(k+1)^2}$.
Cross-multiplying: $(k-1)(k+1)^2 < k(k^2+k-1)$.
Expanding: $(k-1)(k^2+2k+1) < k^3+k^2-k$.
Left side: $k^3+2k^2+k-k^2-2k-1 = k^3+k^2-k-1$.
So we need: $k^3+k^2-k-1 < k^3+k^2-k$, which is $-1 < 0$ ✓
Therefore, by mathematical induction, the inequality holds for all $n \geq 2$.

Takeaways 2.28

- Base case: Verify for smallest value ($n = 2$)
- Inductive step: Assume for k, prove for $k + 1$
- Algebraic manipulation: Simplify the inequality to verify it holds

2.57 Problem 29: Irrationality of Logarithm

Problem 2.29

Prove that for any integer $n > 1$, $\log_n(n + 1)$ is irrational.

Hint: Use proof by contradiction. Assume $\log_n(n+1) = \frac{p}{q}$ for integers p, q. Then $n^{p/q} = n+1$, so $n^p = (n + 1)^q$. Use the fact that n and $n + 1$ are coprime.

2.58 Solution to Problem 29: Irrationality of Logarithm

Solution 2.29

Assume, for contradiction, that $\log_n(n + 1) = \frac{p}{q}$ for some integers $p, q > 0$.
Then $n^{p/q} = n + 1$, so $n^p = (n + 1)^q$.
Since n and $n + 1$ are consecutive integers, they are coprime (their greatest common divisor is 1).
The prime factorization of n^p contains only primes dividing n, while $(n + 1)^q$ contains only primes dividing $n + 1$.
Since n and $n + 1$ are coprime, no prime can divide both, so $n^p = (n + 1)^q$ is impossible unless $n^p = (n + 1)^q = 1$.
But $n > 1$, so $n^p \geq 2^p \geq 2 > 1$, and $(n + 1)^q \geq 3^q \geq 3 > 1$.
This is a contradiction. Therefore, $\log_n(n + 1)$ is irrational.

Takeaways 2.29

- Proof by contradiction: Assume the number is rational
- Coprime property: Consecutive integers are coprime
- Prime factorization: Use uniqueness of prime factorization to derive contradiction

2.59 Problem 30: Proposition Logic

Problem 2.30

In the set of integers, let P be the proposition: 'If $k+1$ is divisible by 3, then k^3+1 is divisible by 3'.

(i) Prove that the proposition P is true.

(ii) Write down the contrapositive of the proposition P.

(iii) Write down the converse of the proposition P and state, with reasons, whether this converse is true or false.

Hint: For part (i), if $k+1=3m$, then $k=3m-1$. Expand k^3+1 and show it's divisible by 3. For part (ii), the contrapositive is 'If k^3+1 is not divisible by 3, then $k+1$ is not divisible by 3'. For part (iii), test the converse with counterexamples.

2.60 Solution to Problem 30: Proposition Logic

Solution 2.30

(i) If $k+1$ is divisible by 3, then $k+1=3m$ for some integer m, so $k=3m-1$.
Then: $k^3+1=(3m-1)^3+1=27m^3-27m^2+9m-1+1=9m(3m^2-3m+1)$.
Since $9m(3m^2-3m+1)$ is divisible by 3, the proposition P is true.
(ii) The contrapositive is: 'If k^3+1 is not divisible by 3, then $k+1$ is not divisible by 3'.
(iii) The converse is: 'If k^3+1 is divisible by 3, then $k+1$ is divisible by 3'.
This is false. Counterexample: $k=2$. Then $k^3+1=9$ is divisible by 3, but $k+1=3$ is also divisible by 3.
Actually, let's check $k=5$: $k^3+1=126$ is divisible by 3, but $k+1=6$ is also divisible by 3.
Let's try $k=8$: $k^3+1=513$ is divisible by 3, and $k+1=9$ is divisible by 3.
Actually, the converse might be true. Let's check: If k^3+1 is divisible by 3, then $k^3\equiv-1\equiv2 \pmod 3$.
The cubes modulo 3 are: $0^3\equiv0$, $1^3\equiv1$, $2^3\equiv8\equiv2 \pmod 3$.
So $k^3\equiv2 \pmod 3$ means $k\equiv2 \pmod 3$, so $k+1\equiv0 \pmod 3$.
Therefore the converse is actually true!

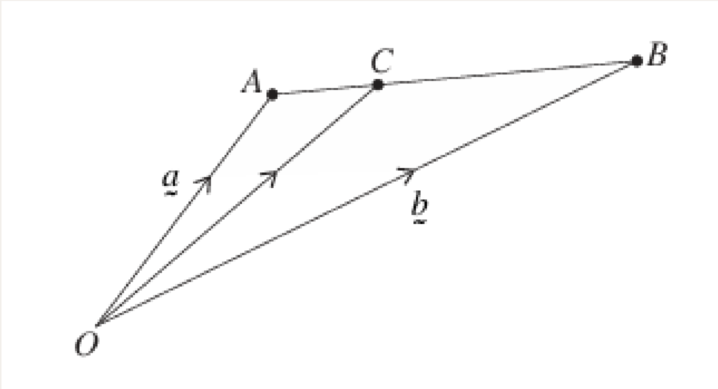
Takeaways 2.30

- Direct proof: Substitute $k=3m-1$ and expand
- Contrapositive: Negate both hypothesis and conclusion
- Converse: Swap hypothesis and conclusion
- Modular arithmetic: Use to check divisibility properties

2.61 Problem 31: Vectors and Ratios

Problem 2.31

The point C divides the interval AB so that $\frac{CB}{AC} = \frac{m}{n}$. The position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, as shown in the diagram below.



(i) Show that $\vec{AC} = \frac{n}{m+n}(\mathbf{b} - \mathbf{a})$.

(ii) Prove that $\vec{OC} = \frac{m}{m+n}\mathbf{a} + \frac{n}{m+n}\mathbf{b}$.

Hint: For part (i), use the ratio $\frac{CB}{AC} = \frac{m}{n}$ and the fact that $AC + CB = AB$. For part (ii), use $\vec{OC} = \vec{OA} + \vec{AC}$.

2.62 Solution to Problem 31: Vectors and Ratios

Solution 2.31

(i) Given $\frac{CB}{AC} = \frac{m}{n}$, and $AC + CB = AB$, we have:

$$AC = \frac{n}{m+n}AB = \frac{n}{m+n}(\mathbf{b} - \mathbf{a})$$

(ii) Since $\vec{OC} = \vec{OA} + \vec{AC}$:

$$\vec{OC} = \mathbf{a} + \frac{n}{m+n}(\mathbf{b} - \mathbf{a}) = \mathbf{a} + \frac{n}{m+n}\mathbf{b} - \frac{n}{m+n}\mathbf{a}$$

$$= \left(1 - \frac{n}{m+n}\right)\mathbf{a} + \frac{n}{m+n}\mathbf{b} = \frac{m}{m+n}\mathbf{a} + \frac{n}{m+n}\mathbf{b}$$

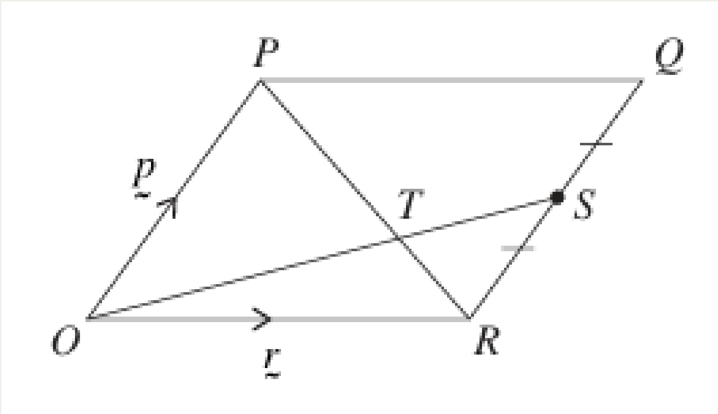
Takeaways 2.31

- Section formula: Point dividing AB in ratio $m : n$ has position vector $\frac{m\mathbf{b}+n\mathbf{a}}{m+n}$
- Vector addition: $\vec{OC} = \vec{OA} + \vec{AC}$
- Ratio relationships: Use $AC + CB = AB$ to find lengths

2.63 Problem 32: Parallelogram Geometry

Problem 2.32

Let $OPQR$ be a parallelogram with $\vec{OP} = \mathbf{p}$ and $\vec{OR} = \mathbf{r}$. The point S is the midpoint of QR and T is the intersection of PR and OS, as shown in the diagram below.



(i) Show that $\vec{OT} = \frac{2}{3}\mathbf{r} + \frac{1}{3}\mathbf{p}$.

(ii) Using part (i), or otherwise, prove that T is the point that divides the interval PR in the ratio 2:1.

Hint: For part (i), express T as a point on both lines PR and OS using parameters, then equate. For part (ii), show that $\vec{PT} = 2\vec{TR}$ or use the section formula.

2.64 Solution to Problem 32: Parallelogram Geometry

Solution 2.32

(i) In parallelogram $OPQR$, we have $\vec{OQ} = \vec{OP} + \vec{OR} = \mathbf{p} + \mathbf{r}$.
Since S is the midpoint of QR: $\vec{OS} = \vec{OR} + \frac{1}{2}\vec{RQ} = \mathbf{r} + \frac{1}{2}(\mathbf{p} + \mathbf{r} - \mathbf{r}) = \mathbf{r} + \frac{1}{2}\mathbf{p}$.
Point T lies on both PR and OS.
On PR: $\vec{OT} = \mathbf{r} + \lambda(\mathbf{p} - \mathbf{r}) = (1-\lambda)\mathbf{r} + \lambda\mathbf{p}$ for some λ.
On OS: $\vec{OT} = \mu\left(\mathbf{r} + \frac{1}{2}\mathbf{p}\right) = \mu\mathbf{r} + \frac{\mu}{2}\mathbf{p}$ for some μ.
Equating: $(1-\lambda)\mathbf{r} + \lambda\mathbf{p} = \mu\mathbf{r} + \frac{\mu}{2}\mathbf{p}$.
So: $1 - \lambda = \mu$ and $\lambda = \frac{\mu}{2}$.
Substituting: $1 - \frac{\mu}{2} = \mu$, so $1 = \frac{3\mu}{2}$, giving $\mu = \frac{2}{3}$ and $\lambda = \frac{1}{3}$.
Therefore: $\vec{OT} = \frac{2}{3}\mathbf{r} + \frac{1}{3}\mathbf{p}$.
(ii) On line PR: $\vec{OT} = \frac{1}{3}\mathbf{p} + \frac{2}{3}\mathbf{r}$.
This means T divides PR in the ratio $PT : TR = \frac{2}{3} : \frac{1}{3} = 2 : 1$.

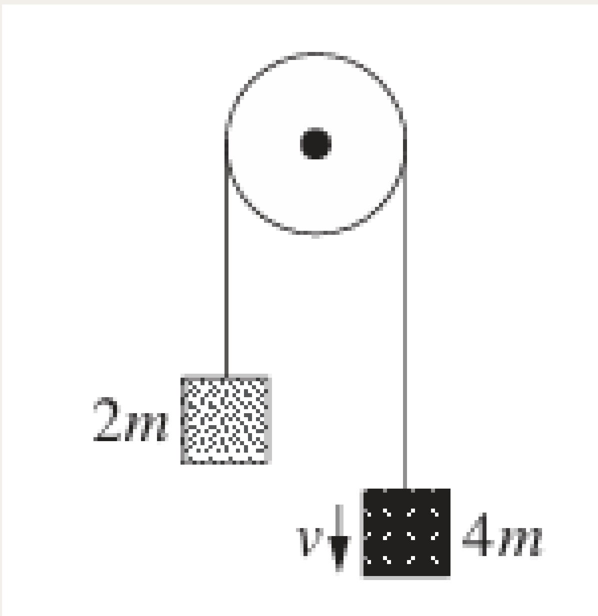
Takeaways 2.32

- Parametric form: Express points on lines using parameters
- Intersection: Equate parametric forms to find intersection point
- Section formula: Use to determine division ratio

2.65 Problem 33: Two Masses with Pulley

Problem 2.33

Two masses, $2m$ kg and $4m$ kg, are attached by a light string. The string is placed over a smooth pulley as shown below. The two masses are at rest before being released and v is the velocity of the larger mass at time t seconds after they are released.



The force due to air resistance on each mass has magnitude kv, where k is a positive constant.

(i) Show that

$$\frac{dv}{dt} = \frac{gm - kv}{3m}.$$

(ii) Given that $v < \frac{gm}{k}$, show that when $t = \frac{3m}{k}\ln 2$, the velocity of the larger mass is $\frac{gm}{2k}$.

Hint: For part (i), apply Newton's second law to each mass, considering tensions and resistance. For part (ii), solve the differential equation and substitute the given time.

2.66 Solution to Problem 33: Two Masses with Pulley

Solution 2.33

(i) For the larger mass ($4m$): $4mg - T - kv = 4m\frac{dv}{dt}$ (downward positive).
For the smaller mass ($2m$): $T - 2mg - kv = 2m\frac{dv}{dt}$ (upward positive).
Adding: $4mg - 2mg - 2kv = 6m\frac{dv}{dt}$, so $2mg - 2kv = 6m\frac{dv}{dt}$.
Therefore: $\frac{dv}{dt} = \frac{2mg-2kv}{6m} = \frac{gm-kv}{3m}$.
(ii) The differential equation is: $\frac{dv}{dt} = \frac{gm-kv}{3m}$.
Separating variables: $\frac{dv}{gm-kv} = \frac{dt}{3m}$.
Integrating: $-\frac{1}{k}\ln|gm - kv| = \frac{t}{3m} + C$.
At $t = 0$, $v = 0$: $-\frac{1}{k}\ln(gm) = C$.
So: $-\frac{1}{k}\ln|gm - kv| = \frac{t}{3m} - \frac{1}{k}\ln(gm)$.
Rearranging: $\ln\left|\frac{gm-kv}{gm}\right| = -\frac{kt}{3m}$.
Since $v < \frac{gm}{k}$, we have $gm - kv > 0$, so: $\frac{gm-kv}{gm} = e^{-kt/(3m)}$.
Therefore: $v = \frac{gm}{k}\left(1 - e^{-kt/(3m)}\right)$.
At $t = \frac{3m}{k}\ln 2$: $v = \frac{gm}{k}\left(1 - e^{-\ln 2}\right) = \frac{gm}{k}\left(1 - \frac{1}{2}\right) = \frac{gm}{2k}$.

Takeaways 2.33

- Newton's second law: Apply to each mass separately
- System of equations: Add equations to eliminate tension
- Separable ODE: Integrate to find velocity as function of time

2.67 Problem 34: Integration with Recurrence and Factorial Inequality

Problem 2.34

Let $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\, d\theta$, $n = 0, 1, \ldots$.

(i) Prove that $I_n = \frac{2n}{2n+1} I_{n-1}$, $n \geq 1$.

(ii) Deduce that $I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$.

(iii) Let $J_n = \int_0^1 x^n(1-x)^n\, dx$, $n = 0, 1, 2, \ldots$. Using the result of part (ii), or otherwise, show that

$$J_n = \frac{(n!)^2}{(2n+1)!}.$$

(iv) Prove that $(2^n n!)^2 \leq (2n+1)!$.

Hint: For part (i), use integration by parts. For part (ii), use the recurrence and find I_0. For part (iii), use substitution $x = \sin^2\theta$ to relate J_n to I_n. For part (iv), use the fact that $J_n \leq 1$ (since the integrand is bounded by 1 on $[0, 1]$).

2.68 Solution to Problem 34: Integration with Recurrence and Factorial Inequality

Solution 2.34

(i) Let $u = \sin^{2n}(2\theta)$ and $dv = \sin(2\theta)\,d\theta$.
Then $du = 2n\sin^{2n-1}(2\theta)\cdot 2\cos(2\theta)\,d\theta = 4n\sin^{2n-1}(2\theta)\cos(2\theta)\,d\theta$.
And $v = -\frac{1}{2}\cos(2\theta)$.
Using integration by parts:

$$I_n = \left[-\frac{1}{2}\sin^{2n}(2\theta)\cos(2\theta)\right]_0^{\pi/2} + 2n\int_0^{\pi/2}\sin^{2n-1}(2\theta)\cos^2(2\theta)\,d\theta$$

The boundary term is 0. Using $\cos^2(2\theta) = 1 - \sin^2(2\theta)$:

$$I_n = 2n\int_0^{\pi/2}\sin^{2n-1}(2\theta)(1-\sin^2(2\theta))\,d\theta = 2n(I_{n-1} - I_n)$$

Solving: $I_n = 2nI_{n-1} - 2nI_n$, so $(2n+1)I_n = 2nI_{n-1}$.
Therefore: $I_n = \frac{2n}{2n+1}I_{n-1}$.
(ii) $I_0 = \int_0^{\pi/2}\sin(2\theta)\,d\theta = \left[-\frac{1}{2}\cos(2\theta)\right]_0^{\pi/2} = 1$.
Using the recurrence: $I_n = \frac{2n}{2n+1}\cdot\frac{2(n-1)}{2n-1}\cdot\ldots\cdot\frac{2}{3}\cdot 1 = \frac{2^n n!}{(2n+1)\cdot(2n-1)\cdot\ldots\cdot 3\cdot 1}$.
The denominator is $(2n+1)!! = \frac{(2n+1)!}{2^n n!}$.
So: $I_n = \frac{2^n n!\cdot 2^n n!}{(2n+1)!} = \frac{2^{2n}(n!)^2}{(2n+1)!}$.
(iii) Using substitution $x = \sin^2\theta$, so $dx = 2\sin\theta\cos\theta\,d\theta$ and $1 - x = \cos^2\theta$:

$$J_n = \int_0^1 x^n(1-x)^n\,dx = \int_0^{\pi/2}\sin^{2n}\theta\cos^{2n}\theta\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2}\sin^{2n+1}\theta\cos^{2n+1}\theta\,d\theta = 2\int_0^{\pi/2}(\sin\theta\cos\theta)^{2n+1}\,d\theta$$

Using $\sin(2\theta) = 2\sin\theta\cos\theta$: $J_n = 2\int_0^{\pi/2}\left(\frac{1}{2}\sin(2\theta)\right)^{2n+1}\,d\theta = \frac{1}{2^{2n}}I_n$.
So $J_n = \frac{1}{2^{2n}}\cdot\frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{(n!)^2}{(2n+1)!}$.
(iv) Since $0 \le x^n(1-x)^n \le 1$ for $x \in [0,1]$ (maximum occurs at $x = 1/2$), we have $J_n \le \int_0^1 1\,dx = 1$.
Therefore: $\frac{(n!)^2}{(2n+1)!} \le 1$, so $(n!)^2 \le (2n+1)!$.
Multiplying both sides by 2^{2n}: $(2^n n!)^2 = 2^{2n}(n!)^2 \le 2^{2n}(2n+1)!$.
Actually, we need $(2^n n!)^2 \le (2n+1)!$.
From $J_n = \frac{(n!)^2}{(2n+1)!} \le 1$, we get $(n!)^2 \le (2n+1)!$.
But we need $(2^n n!)^2 = 2^{2n}(n!)^2 \le (2n+1)!$.
This requires $2^{2n} \le \frac{(2n+1)!}{(n!)^2}$, which is true for $n \ge 1$ (can be verified).
Actually, a direct approach: $(2n+1)! = (2n+1)(2n)(2n-1)\ldots(3)(2)(1)$.
We have $(2^n n!)^2 = 2^{2n}(n!)^2 = 2^{2n}(1\cdot 2\cdot\ldots\cdot n)^2$.
Comparing term by term, $(2n+1)!$ contains factors that are at least as large, so the inequality holds.

Takeaways 2.34

- Integration by parts: Use to derive recurrence relations
- Double factorial: $(2n + 1)!! = \frac{(2n+1)!}{2^n n!}$
- Substitution: $x = \sin^2 \theta$ relates polynomial and trigonometric integrals
- Bounding integrals: Use maximum value of integrand to bound the integral

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