HSC Math Extension 2: Integration Mastery

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Contents

1 Introduction

1.1 Project Overview

This booklet compiles high-quality integration problems curated specifically for the HSC Mathematics Extension 2 syllabus. Every problem covers essential integration techniques including substitution, integration by parts, partial fractions, reduction formulae, volumes of solids, and definite integral properties. Detailed reasoning showcases advanced problem-solving strategies that build from fundamental techniques to complex multi-step applications.

1.2 Target Audience

The explanations are crafted for Extension 2 students aiming to master integration and develop advanced problem-solving skills. Each solution in Part 1 explicitly states the strategy, justifies technique choices, and provides complete step-by-step working so that high-school learners can follow every transition. Part 2 offers hints and concise solutions to encourage independent problem-solving.

1.3 How to Use This Booklet

- Review the fundamentals section before attempting problems to refresh key techniques.
- Attempt problems in Part 1 without looking at solutions; compare your work against detailed solutions to understand model reasoning.
- For Part 2, try each problem first, then check the upside-down hint if needed, and finally review the solution sketch.
- Practice problems multiple times, working from memory to reinforce technique mastery.
- Use the appendices as quick references for formulas, techniques, and decision-making flowcharts.

1.4 Integration Techniques Overview

The problems in this collection cover:

- **Basic Techniques:** Reverse chain rule, standard integrals, u-substitution
- **Advanced Substitution:** Trigonometric substitution, t-formula, rationalizing substitutions
- **Integration by Parts:** Single and multiple applications, LIATE rule
- **Partial Fractions:** Linear, quadratic, and repeated factors
- **Reduction Formulae:** Deriving and applying recurrence relations
- **Volumes of Solids:** Disk method, washer method, cylindrical shells, general slicing
- **Definite Integral Properties:** Symmetry, King's property, inequalities

2 Fundamentals Review

This section provides a comprehensive review of integration techniques essential for HSC Extension 2. Use this as a reference while working through problems.

3 Standard Integrals & The "Reverse Chain Rule"

Before applying complex techniques, always check if the integral fits a standard form or the reverse chain rule.

- **Logarithmic Form:**

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

- **Power Rule (General):**

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C \quad (\text{where } n \neq -1)$$

- **Inverse Trigonometric Forms:**

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

4 Integration by Parts

Used for integrating products of functions (e.g., xe^x, $x\ln x$, $e^x\cos x$).

The Formula:

$$\int u\,dv = uv - \int v\,du$$

Strategy (LIATE): Choose u based on this priority list (top to bottom):

1. **L** – Logarithmic ($\ln x$)
2. **I** – Inverse Trigonometric ($\tan^{-1} x$)
3. **A** – Algebraic ($x^2, 3x$)
4. **T** – Trigonometric ($\sin x, \cos x$)
5. **E** – Exponential (e^x)

5 Integration by Substitution

5.1 General Substitution (u-sub)

Used to simplify composite functions. Let $u = g(x)$, then find $du = g'(x)dx$.

5.2 Trigonometric Substitution

Used when the integrand contains quadratic roots:

- $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$ (uses $1 - \sin^2\theta = \cos^2\theta$)
- $\sqrt{a^2 + x^2}$: Let $x = a\tan\theta$ (uses $1 + \tan^2\theta = \sec^2\theta$)
- $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$ (uses $\sec^2\theta - 1 = \tan^2\theta$)

5.3 The t-Formula Substitution

Used for rational functions involving $\sin x$ and $\cos x$. Let $t = \tan\left(\frac{x}{2}\right)$.

$$\begin{aligned} dx &= \frac{2}{1+t^2}\,dt \\ \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \end{aligned}$$

6 Partial Fractions

Used to integrate rational functions $\frac{P(x)}{Q(x)}$ where $\deg(P) < \deg(Q)$. If $\deg(P) \geq \deg(Q)$, perform **polynomial long division** first.

- **Distinct Linear Factors:**
$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$
- **Repeated Linear Factors:**
$$\frac{1}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$
- **Irreducible Quadratic Factors:**
$$\frac{1}{(x-a)(x^2+b)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+b}$$

7 Trigonometric Integrals

7.1 Integrals of $\sin^m x \cos^n x$

- **One power is odd:** Save one factor of the odd power for du. Convert the rest using $\sin^2 x + \cos^2 x = 1$.
- **Both powers even:** Use double angle formulae:
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

7.2 Integrals of $\tan^m x \sec^n x$

- **If** $\sec$ **power is even:** Save $\sec^2 x$ for du. Convert remaining sec to tan.
- **If** $\tan$ **power is odd:** Save $\sec x \tan x$ for du. Convert remaining tan to sec.

8 Reduction Formulas (I_n)

Involves finding a recurrence relation using **Integration by Parts**.

Typical form:

$$I_n = \int x^n e^x \, dx \quad \text{or} \quad I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

Steps: Apply parts, manipulate the integral to find I_{n-1} or I_{n-2}, then rearrange for I_n.

9 Definite Integral Properties

- **Odd Function:** If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\, dx = 0$.
- **Even Function:** If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$.
- **Reflection (King's) Property:**

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

10 Part 1: Problems and Solutions (Detailed)

Part 1 contains three sets of problems—basic, medium, and advanced. Each set provides five problems with comprehensive solutions. Every solution includes a strategy paragraph explaining technique selection, complete step-by-step working with annotations, and a takeaways box highlighting key insights.

10.1 Basic Integration Problems

Problem 10.1: Partial Fractions

Find the indefinite integral:

$$\int \frac{x^2 - 2x + 9}{(4 - x)(x^2 + 1)}\, dx$$

Hint: This integral requires partial fraction decomposition. The denominator contains a linear factor $(4-x)$ and an irreducible quadratic factor (x^2+1), so we set up the decomposition with constants A for the linear part and $Bx + C$ for the quadratic part.

Solution 10.1

Step 1: Set up the decomposition
Since the denominator has $(4-x)$ (linear) and (x^2+1) (irreducible quadratic):

$$\frac{x^2-2x+9}{(4-x)(x^2+1)} = \frac{A}{4-x} + \frac{Bx+C}{x^2+1}$$

Step 2: Find coefficients
Multiply both sides by $(4-x)(x^2+1)$:

$$x^2-2x+9 = A(x^2+1)+(Bx+C)(4-x)$$

Finding A: Let $x=4$:

$$\begin{aligned} 16-8+9 &= A(17)+0 \\ 17 &= 17A \\ A &= 1 \end{aligned}$$

Finding B and C: Substitute $A=1$ and expand:

$$x^2-2x+9 = x^2+1+4Bx-Bx^2+4C-Cx$$

Group by powers of x:

$$x^2-2x+9 = (1-B)x^2+(4B-C)x+(1+4C)$$

Equating coefficients:

- x^2: $1 = 1-B \implies B=0$
- x^0: $9 = 1+4C \implies C=2$
- Check x^1: $-2 = 4(0)-2 = -2$ ✓

Step 3: Integrate
With $A=1$, $B=0$, $C=2$:

$$\int \left(\frac{1}{4-x}+\frac{2}{x^2+1}\right) dx$$

For $\int \frac{1}{4-x}dx$, let $u=4-x$, then $du=-dx$:

$$\int \frac{1}{4-x}dx = -\ln|4-x|+C_1$$

For $\int \frac{2}{x^2+1}dx$:

$$2\int \frac{1}{x^2+1}dx = 2\arctan(x)+C_2$$

Final Answer:

$$\boxed{\int \frac{x^2-2x+9}{(4-x)(x^2+1)}\,dx = -\ln|4-x|+2\arctan(x)+C}$$

Takeaways 10.1

- **Partial Fractions Setup:** Linear factors use constant numerators (A), irreducible quadratics use linear numerators ($Bx + C$).
- **Strategic Value Selection:** Choose x values that eliminate terms (e.g., $x = 4$ eliminates the $(4 - x)$ factor).
- **Coefficient Matching:** After substitution, equate coefficients of like powers to find remaining constants.
- **Standard Integral Recognition:** $\int \frac{1}{a-x} dx = -\ln|a - x|$ and $\int \frac{1}{x^2+1} dx = \arctan(x)$ are key formulas.

Problem 10.2: Integration by Parts

Use integration by parts to evaluate:

$$\int_1^e x \ln x \, dx$$

Hint: This is a classic integration by parts problem. Apply the LIATE rule to choose which function to differentiate: Logarithmic functions come before Algebraic functions, so let $u = \ln x$.

Solution 10.2

Using the integration by parts formula $\int u\,dv = uv - \int v\,du$:
Step 1: Choose u and dv using LIATE:

$$u = \ln x \qquad\qquad dv = x\,dx$$

Step 2: Differentiate u and integrate dv:

$$du = \frac{1}{x}\,dx \qquad\qquad v = \frac{x^2}{2}$$

Step 3: Apply the formula:

$$\begin{aligned}\int_1^e x\ln x\,dx &= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}\,dx \\ &= \left[\frac{x^2}{2}\ln x\right]_1^e - \frac{1}{2}\int_1^e x\,dx \\ &= \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^e\end{aligned}$$

Step 4: Evaluate at the limits (using $\ln(e) = 1$ and $\ln(1) = 0$):

$$= \left(\frac{e^2}{2} - \frac{e^2}{4}\right) - \left(0 - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4} = \boxed{\frac{e^2+1}{4}}$$

Takeaways 10.2

- **LIATE Rule:** Prioritize L(og) > I(nverse trig) > A(lgebraic) > T(rig) > E(xponential) for u
- **Boundary Terms:** Apply limits after integration: $[uv]_a^b - \int_a^b v\,du$
- **Special Values:** Remember $\ln(e) = 1$ and $\ln(1) = 0$

Problem 10.3: Reverse Chain Rule

Find the indefinite integral:

$$\int \frac{2x+3}{x^2+2x+2}\,dx$$

Hint: Recognize that the numerator can be split to match the derivative of the denominator plus a constant. This allows us to use both logarithmic and inverse trigonometric standard forms.

Solution 10.3

Step 1: Check the derivative of the denominator:

$$\frac{d}{dx}(x^2 + 2x + 2) = 2x + 2$$

Step 2: Split the numerator:

$$2x + 3 = (2x + 2) + 1$$

Step 3: Separate the integral:

$$I = \int \frac{2x + 2}{x^2 + 2x + 2}\, dx + \int \frac{1}{x^2 + 2x + 2}\, dx$$

Step 4: First integral uses logarithm form $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)|$:

$$\int \frac{2x + 2}{x^2 + 2x + 2}\, dx = \ln(x^2 + 2x + 2)$$

(Note: $x^2 + 2x + 2 = (x + 1)^2 + 1 > 0$ always)
Step 5: Second integral requires completing the square:

$$x^2 + 2x + 2 = (x + 1)^2 + 1$$

Apply arctangent form:

$$\int \frac{1}{(x + 1)^2 + 1}\, dx = \arctan(x + 1)$$

Final Answer:

$$\boxed{\int \frac{2x + 3}{x^2 + 2x + 2}\, dx = \ln(x^2 + 2x + 2) + \arctan(x + 1) + C}$$

Takeaways 10.3

- **Numerator Splitting:** Match part of numerator to $f'(x)$ when denominator is $f(x)$
- **Standard Forms:** $\frac{f'}{f} \to \ln |f|$ and $\frac{1}{a^2+u^2} \to \frac{1}{a} \arctan(\frac{u}{a})$
- **Completing the Square:** Essential for identifying inverse trig forms

Problem 10.4: Algebraic Substitution

Use an appropriate substitution to evaluate:

$$\int_{\sqrt{10}}^{\sqrt{13}} x^3\sqrt{x^2-9}\,dx$$

Hint: The radical $\sqrt{x^2-9}$ suggests $u = x^2 - 9$. This also helps manage the x^3 term since $x^3dx = x^2 \cdot xdx$.

Solution 10.4

Step 1: Let $u = x^2 - 9$, then:

$$\frac{du}{dx} = 2x \implies x\,dx = \frac{1}{2}du$$

Since $x^2 = u + 9$:

$$x^3\,dx = x^2 \cdot x\,dx = (u+9)\cdot\frac{1}{2}du$$

Step 2: Transform limits:

- $x = \sqrt{10} \implies u = 10 - 9 = 1$
- $x = \sqrt{13} \implies u = 13 - 9 = 4$

Step 3: Substitute:

$$\int_{\sqrt{10}}^{\sqrt{13}} x^3\sqrt{x^2-9}\,dx = \int_1^4 (u+9)\sqrt{u}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_1^4 (u^{3/2} + 9u^{1/2})\,du$$

Step 4: Integrate:

$$= \frac{1}{2}\left[\frac{2u^{5/2}}{5} + 6u^{3/2}\right]_1^4 = \left[\frac{u^{5/2}}{5} + 3u^{3/2}\right]_1^4$$

Step 5: Evaluate:

$$\text{At } u = 4: \quad \frac{32}{5} + 24 = \frac{152}{5}$$

$$\text{At } u = 1: \quad \frac{1}{5} + 3 = \frac{16}{5}$$

Final Answer:

$$\boxed{\frac{152}{5} - \frac{16}{5} = \frac{136}{5} = 27.2}$$

Takeaways 10.4

- **Radical Substitution:** For $\sqrt{x^2 \pm a^2}$, try $u = x^2 \pm a^2$
- **Limit Transformation:** Always convert limits for definite integrals
- **Fractional Powers:** $\int u^n du = \frac{u^{n+1}}{n+1}$ works for all $n \neq -1$

Problem 10.5: Definite Integral Property

Which of the following is equal to $\displaystyle\int_0^{2a} f(x)\,dx$?

(A) $\displaystyle\int_0^a \left(f(x) - f(2a-x)\right)\,dx$

(B) $\displaystyle\int_0^a \left(f(x) + f(2a-x)\right)\,dx$

(C) $\displaystyle 2\int_0^a f(x-a)\,dx$

(D) $\displaystyle\int_0^a \frac{1}{2} f(2x)\,dx$

Hint: Split the integral at $x = a$, then use substitution on the second part to transform its limits to match $[0, a]$.

Solution 10.5

Step 1: Split the interval:

$$I = \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

Step 2: For the second integral, let $u = 2a - x$:

- Then $du = -dx$
- When $x = a$: $u = a$
- When $x = 2a$: $u = 0$

Step 3: Substitute:

$$\begin{aligned}\int_a^{2a} f(x)\,dx &= \int_a^0 f(2a-u)(-du) \\ &= \int_0^a f(2a-u)\,du \\ &= \int_0^a f(2a-x)\,dx \quad \text{(dummy variable)}\end{aligned}$$

Step 4: Combine:

$$\begin{aligned}I &= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx \\ &= \int_0^a \left(f(x) + f(2a-x)\right)\,dx\end{aligned}$$

Answer: $\boxed{\text{B}}$

Takeaways 10.5

- **Interval Splitting:** $\int_a^b = \int_a^c + \int_c^b$ for any $c \in [a,b]$
- **Reflection Substitution:** $u = 2a - x$ reflects the interval about the midpoint
- **Dummy Variables:** In definite integrals, the variable name doesn't matter
- **King's Property:** This is a special case: $\int_0^{2a} f(x)dx = \int_0^a [f(x) + f(2a-x)]dx$

10.2 Medium Integration Problems

Problem 10.6: Reduction Formula for Powers of Cotangent

Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n}\theta \, d\theta$ for integers $n \geq 0$.

(i) Show that $I_n = \dfrac{1}{2n-1} - I_{n-1}$ for $n > 0$, given that $\dfrac{d}{d\theta}\cot\theta = -\csc^2\theta$.

(ii) Hence, or otherwise, calculate I_2.

Hint: This is a classic reduction formula problem. For part (i), we'll combine $I_n + I_{n-1}$ and use the identity $\cot^2\theta + 1 = \csc^2\theta$ to simplify, then apply substitution. For part (ii), we'll use the recurrence relation iteratively, starting from I_0.

Solution 10.6

Part (i): Consider $I_n + I_{n-1}$:

$$I_n + I_{n-1} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\cot^{2n}\theta + \cot^{2n-2}\theta\right) d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n-2}\theta \left(\cot^2\theta + 1\right) d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{2n-2}\theta \cdot \csc^2\theta \, d\theta \quad (\text{using } \cot^2\theta + 1 = \csc^2\theta)$$

Let $u = \cot\theta$, so $du = -\csc^2\theta \, d\theta$. Limits: $\theta = \frac{\pi}{4} \Rightarrow u = 1$; $\theta = \frac{\pi}{2} \Rightarrow u = 0$.

$$I_n + I_{n-1} = \int_1^0 u^{2n-2}(-du) = \int_0^1 u^{2n-2}\, du = \left[\frac{u^{2n-1}}{2n-1}\right]_0^1 = \frac{1}{2n-1}$$

Therefore: $\boxed{I_n = \dfrac{1}{2n-1} - I_{n-1}}$

Part (ii): Calculate base cases and apply recursion:

$$I_0 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1\, d\theta = \frac{\pi}{4}, \quad I_1 = \frac{1}{1} - I_0 = 1 - \frac{\pi}{4}$$

$$I_2 = \frac{1}{3} - I_1 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) = \boxed{\frac{\pi}{4} - \frac{2}{3}}$$

Takeaways 10.6

- **Reduction Formula Strategy:** Adding consecutive terms ($I_n + I_{n-1}$) can reveal simplifying identities.
- **Trigonometric Identities:** $\cot^2\theta + 1 = \csc^2\theta$ is key for cotangent integrals.
- **Substitution Choice:** $u = \cot\theta$ works well because $du = -\csc^2\theta\, d\theta$ matches our integral.
- **Iterative Application:** Build from base case (I_0) through recurrence to find any I_n.

Problem 10.7: King's Rule with t-Formula

Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{u}{1 + \sin u + \cos u}\, du$$

by first using the substitution $u = \frac{\pi}{2} - x$.

Hint: Use King's property (reflection about midpoint) to create a self-referencing equation that simplifies the numerator. Then apply the t-formula (Weierstrass substitution) to handle the trigonometric denominator.

Solution 10.7

Let $I = \int_0^{\frac{\pi}{2}} \frac{u}{1+\sin u+\cos u}\,du \quad \dots(1)$
Step 1: Apply substitution $u = \frac{\pi}{2} - x$, so $du = -dx$.
Change limits: $u = 0 \implies x = \frac{\pi}{2}$, $u = \frac{\pi}{2} \implies x = 0$

$$\begin{aligned} I &= \int_{\frac{\pi}{2}}^{0} \frac{\frac{\pi}{2} - x}{1 + \sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)}(-dx) \\ &= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{1 + \cos x + \sin x}\,dx \end{aligned}$$

Relabel: $I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-u}{1+\sin u+\cos u}\,du \quad \dots(2)$
Step 2: Add equations (1) and (2):

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \frac{u + (\frac{\pi}{2} - u)}{1 + \sin u + \cos u}\,du \\ 2I &= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin u + \cos u}\,du \end{aligned}$$

Step 3: Apply t-formula. Let $t = \tan(\frac{u}{2})$:

$$du = \frac{2}{1+t^2}dt, \quad \sin u = \frac{2t}{1+t^2}, \quad \cos u = \frac{1-t^2}{1+t^2}$$

Limits: $u = 0 \implies t = 0$, $u = \frac{\pi}{2} \implies t = 1$

$$\begin{aligned} 2I &= \frac{\pi}{2}\int_0^1 \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt \\ &= \frac{\pi}{2}\int_0^1 \frac{2}{(1+t^2) + 2t + (1-t^2)}\,dt \\ &= \frac{\pi}{2}\int_0^1 \frac{2}{2+2t}\,dt \\ &= \frac{\pi}{2}\int_0^1 \frac{1}{1+t}\,dt \end{aligned}$$

Step 4: Integrate:

$$2I = \frac{\pi}{2}[\ln|1+t|]_0^1 = \frac{\pi}{2}(\ln 2 - \ln 1) = \frac{\pi}{2}\ln 2$$

Final Answer:

$$\boxed{I = \frac{\pi}{4}\ln 2}$$

Takeaways 10.7

- **King's Property:** For $\int_a^b f(x)dx$, use $\int_a^b f(a+b-x)dx$ to create symmetry
- **Self-Reference:** Adding two forms of same integral can simplify complex expressions
- **t-Formula:** $t = \tan(\frac{x}{2})$ converts rational trig functions to rational algebraic functions
- **Workflow:** Simplify limits first (King's), then tackle trig terms (t-formula)

Problem 10.8: Reduction Formula - Logarithmic Powers

The integral I_n is defined by:

$$I_n = \int_1^e (\ln x)^n \, dx \quad \text{for integers } n \geq 0$$

Show that $I_n = e - nI_{n-1}$ for $n \geq 1$.

Hint: Use integration by parts with $u = (\ln x)^n$ and $dv = dx$. The derivative of u will reduce the power of the logarithm.

Solution 10.8

Step 1: Choose u and dv:

$$u = (\ln x)^n \qquad dv = dx$$
$$du = n(\ln x)^{n-1} \cdot \frac{1}{x} \, dx \qquad v = x$$

Step 2: Apply parts formula:

$$\begin{aligned} I_n &= [x(\ln x)^n]_1^e - \int_1^e x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} \, dx \\ &= [x(\ln x)^n]_1^e - n \int_1^e (\ln x)^{n-1} \, dx \end{aligned}$$

Step 3: Evaluate boundary term:

- At $x = e$: $e(\ln e)^n = e(1)^n = e$
- At $x = 1$: $1(\ln 1)^n = 1(0)^n = 0$ (for $n \geq 1$)

Step 4: Recognize I_{n-1}:

$$I_n = e - nI_{n-1} \quad \blacksquare$$

Takeaways 10.8

- **Parts for Reduction:** Choose u as the term you want to reduce in power
- **Logarithm Priority:** $\ln x$ is top choice for u in LIATE
- **Boundary Evaluation:** Special values like $\ln(e) = 1$ and $\ln(1) = 0$ simplify calculations
- **Recurrence Relations:** Reduction formulae connect I_n to simpler I_{n-1} or I_{n-2}

Problem 10.9: Applications - Particle Dynamics

A particle of mass m kg moves along a horizontal line with initial velocity V_0 m/s. The motion is resisted by a constant force of mk newtons and a variable force of mv^2 newtons, where k is a positive constant and v m/s is the velocity at time t seconds.

Show that the distance travelled when the particle comes to rest is $\dfrac{1}{2}\ln\left(\dfrac{k+V_0^2}{k}\right)$ metres.

Hint: Apply Newton's Second Law, use the kinematic identity $a = v\frac{dv}{dx}$ to change variables, then separate and integrate.

Solution 10.9

Step 1: Establish equation of motion using $F = ma$:
Total resistive force: $F = -(mk + mv^2)$

$$ma = -m(k + v^2) \implies a = -(k + v^2)$$

Step 2: Change variable to displacement using $a = v\frac{dv}{dx}$:

$$v\frac{dv}{dx} = -(k + v^2)$$

Step 3: Separate variables:

$$dx = -\frac{v}{k + v^2}\,dv$$

Step 4: Integrate with limits:

- Initial: $x = 0, v = V_0$
- Final: $x = D, v = 0$ (at rest)

$$\int_0^D dx = \int_{V_0}^0 -\frac{v}{k + v^2}\,dv$$

LHS: D
RHS: Note that $\frac{d}{dv}(k + v^2) = 2v$, so:

$$\int \frac{v}{k + v^2}\,dv = \frac{1}{2}\ln(k + v^2)$$

Step 5: Evaluate:

$$\begin{aligned} D &= -\left[\frac{1}{2}\ln(k + v^2)\right]_{V_0}^0 \\ &= -\frac{1}{2}(\ln k - \ln(k + V_0^2)) \\ &= \frac{1}{2}(\ln(k + V_0^2) - \ln k) \\ &= \boxed{\frac{1}{2}\ln\left(\frac{k + V_0^2}{k}\right) \text{ metres}} \quad \blacksquare \end{aligned}$$

Takeaways 10.9

- **Variable Change:** For distance problems, use $a = v\frac{dv}{dx}$ instead of $a = \frac{dv}{dt}$
- **Separation:** Rearrange to get all x terms on one side, all v terms on other
- **Reverse Chain Rule:** $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|$ is crucial for rational integrands
- **Log Laws:** $\ln a - \ln b = \ln(\frac{a}{b})$ simplifies final answers

Problem 10.10: Power Reduction Method

(i) Using the double angle formula $\cos(2\theta) = 2\cos^2\theta - 1$, show that

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$.

(iii) By repeatedly applying the identity in part (i), or otherwise, show that

$$\cos^4\theta = \frac{1}{8}(\cos(4\theta) + 4\cos(2\theta) + 3)$$

(iv) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$.

Hint: For part (i), rearrange the double angle formula to isolate $\cos^2\theta$. For part (iii), square the result from part (i) and apply the formula again to the $\cos^2(2\theta)$ term. For integration, substitute the reduced form and integrate term by term.

Solution 10.10

Part (i): From $\cos(2\theta) = 2\cos^2\theta - 1$, rearranging gives:

$$\cos^2\theta = \boxed{\frac{1}{2}(1+\cos(2\theta))} \quad \blacksquare$$

Part (ii):

$$\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos(2\theta))\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left(\frac{\pi}{2}+0\right) = \boxed{\frac{\pi}{4}}$$

Part (iii): Square the identity from part (i):

$$\cos^4\theta = \left[\frac{1}{2}(1+\cos(2\theta))\right]^2 = \frac{1}{4}(1+2\cos(2\theta)+\cos^2(2\theta))$$

Apply part (i) to $\cos^2(2\theta)$: $\cos^2(2\theta) = \frac{1}{2}(1+\cos(4\theta))$

$$\cos^4\theta = \frac{1}{4}\left(1+2\cos(2\theta)+\frac{1}{2}(1+\cos(4\theta))\right) = \frac{1}{4}\left(\frac{3}{2}+2\cos(2\theta)+\frac{1}{2}\cos(4\theta)\right)$$
$$= \boxed{\frac{1}{8}(\cos(4\theta)+4\cos(2\theta)+3)} \quad \blacksquare$$

Part (iv):

$$\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{1}{8}\int_0^{\frac{\pi}{2}} (\cos(4\theta)+4\cos(2\theta)+3)\, d\theta$$
$$= \frac{1}{8}\left[\frac{\sin(4\theta)}{4}+2\sin(2\theta)+3\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{8}\cdot\frac{3\pi}{2} = \boxed{\frac{3\pi}{16}}$$

Takeaways 10.10

- **Power Reduction Strategy:** Use double angle formulas to convert $\cos^n\theta$ to sums of single-angle terms
- **Iterative Application:** Apply the same identity multiple times for higher powers ($\cos^2 \to \cos^4$)
- **Integration Simplification:** Reduced forms $\int \cos(n\theta)d\theta = \frac{1}{n}\sin(n\theta)$ are much easier than direct powers
- **Special Values:** $\sin(0) = \sin(\pi) = \sin(2\pi) = 0$ simplify boundary evaluations
- **General Pattern:** This method extends to any even power: $\cos^{2n}\theta$ reduces to multiple-angle cosines

10.3 Advanced Integration Problems

Problem 10.11: Advanced Reduction Formula with Induction

(i) Let $J_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$ where $n \geq 0$ is an integer.
Show that $J_n = \frac{n-1}{n} J_{n-2}$ for all integers $n \geq 2$.

(ii) Let $I_n = \int_0^1 x^n (1-x)^n \, dx$ where n is a positive integer.
By using the substitution $x = \sin^2 \theta$, or otherwise,
show that $I_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1} \theta \, d\theta$.

(iii) Hence, or otherwise, show that $I_n = \frac{n}{4n+2} I_{n-1}$, for all integers $n \geq 1$.

Hint: This is a sophisticated multi-part problem connecting two different integral forms through substitution and reduction formulae. Part (i) uses integration by parts to derive a recurrence for J_n. Part (ii) employs trigonometric substitution to relate I_n to J_{2n+1}. Part (iii) combines the previous results to establish the recurrence for I_n.

Solution 10.11: Part (i)

We use integration by parts on $J_n = \int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta$.
Let:

$$\begin{aligned} u = \sin^{n-1}\theta &\implies du = (n-1)\sin^{n-2}\theta\cos\theta\,d\theta \\ dv = \sin\theta\,d\theta &\implies v = -\cos\theta \end{aligned}$$

Applying $\int u\,dv = [uv] - \int v\,du$:

$$J_n = \left[-\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos\theta(n-1)\sin^{n-2}\theta\cos\theta\,d\theta$$

The boundary term vanishes: $[-\sin^{n-1}(\pi/2)\cos(\pi/2)] - [-\sin^{n-1}(0)\cos(0)] = 0 - 0 = 0$.

$$\begin{aligned} J_n &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\cos^2\theta\,d\theta \\ &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta(1-\sin^2\theta)\,d\theta \quad [\text{using } \cos^2\theta = 1-\sin^2\theta] \\ &= (n-1)\left(\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\,d\theta - \int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta\right) \\ &= (n-1)J_{n-2} - (n-1)J_n \end{aligned}$$

Rearranging:

$$\begin{aligned} J_n + (n-1)J_n &= (n-1)J_{n-2} \\ nJ_n &= (n-1)J_{n-2} \end{aligned}$$

$$\boxed{J_n = \frac{n-1}{n}J_{n-2}}$$

Solution 10.12: Part (ii)

Substitute $x = \sin^2\theta$, so $dx = 2\sin\theta\cos\theta\,d\theta$.
Limits: $x = 0 \implies \theta = 0$; $x = 1 \implies \theta = \frac{\pi}{2}$.

$$\begin{aligned}
I_n &= \int_0^1 x^n(1-x)^n\,dx \\
&= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^n(1-\sin^2\theta)^n \cdot 2\sin\theta\cos\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^{2n}\theta\cos^{2n}\theta \cdot 2\sin\theta\cos\theta\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\cos^{2n+1}\theta\,d\theta
\end{aligned}$$

Using $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$:

$$\begin{aligned}
I_n &= 2\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2\theta\right)^{2n+1} d\theta \\
&= \frac{2}{2^{2n+1}}\int_0^{\frac{\pi}{2}} \sin^{2n+1} 2\theta\,d\theta \\
&= \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n+1} 2\theta\,d\theta
\end{aligned}$$

Let $\phi = 2\theta$, so $d\theta = \frac{1}{2}d\phi$. Limits: $0 \to \pi$.

$$\begin{aligned}
I_n &= \frac{1}{2^{2n}}\int_0^{\pi} \sin^{2n+1}\phi \cdot \frac{1}{2}\,d\phi \\
&= \frac{1}{2^{2n+1}}\int_0^{\pi} \sin^{2n+1}\phi\,d\phi
\end{aligned}$$

Using symmetry: $\int_0^\pi \sin^k\phi\,d\phi = 2\int_0^{\pi/2}\sin^k\phi\,d\phi$ (since $\sin(\pi-\phi) = \sin\phi$):

$$\boxed{I_n = \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\,d\theta}$$

Solution 10.13: Part (iii)

From part (ii): $I_n = \frac{1}{2^{2n}} J_{2n+1}$ and $I_{n-1} = \frac{1}{2^{2(n-1)}} J_{2n-1} = \frac{4}{2^{2n}} J_{2n-1}$
From part (i) with $k = 2n+1$:

$$J_{2n+1} = \frac{2n}{2n+1} J_{2n-1}$$

Therefore:

$$\begin{aligned} I_n &= \frac{1}{2^{2n}} \cdot \frac{2n}{2n+1} J_{2n-1} \\ &= \frac{2n}{2n+1} \cdot \frac{1}{2^{2n}} J_{2n-1} \\ &= \frac{2n}{2n+1} \cdot \frac{1}{4} \cdot \frac{4}{2^{2n}} J_{2n-1} \\ &= \frac{2n}{4(2n+1)} \cdot I_{n-1} \\ &= \boxed{\frac{n}{4n+2} I_{n-1}} \end{aligned}$$

Takeaways 10.11

- **Integration by Parts for Reduction:** Choosing $u = \sin^{n-1}\theta$ and $dv = \sin\theta\, d\theta$ systematically reduces the power.
- **Trigonometric Substitution Power:** $x = \sin^2\theta$ transforms algebraic integrals into trigonometric ones.
- **Connecting Different Integrals:** Parts (i) and (ii) establish relationships that combine in part (iii).
- **Symmetry Properties:** $\int_0^\pi \sin^k\phi\, d\phi = 2\int_0^{\pi/2} \sin^k\phi\, d\phi$ simplifies definite integrals.
- **Multi-Part Problems:** Each part builds on previous results—read all parts before starting!

Problem 10.12: Reduction Formula with Factorial Series

Let $J_n = \int_0^1 x^n e^{-x}\,dx$, where n is a non-negative integer.

(i) Show that $J_0 = 1 - \frac{1}{e}$.

(ii) Show that $J_n \leq \frac{1}{n+1}$.

(iii) Show that $J_n = nJ_{n-1} - \frac{1}{e}$, for $n \geq 1$.

(iv) Using parts (i) and (iii), show by mathematical induction that for all $n \geq 0$,

$$J_n = n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}$$

(v) Using parts (ii) and (iv) prove that $e = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}$.

Hint: This sophisticated problem connects integration, reduction formulae, induction, and limits. Part (i) establishes base case, (ii) bounds, (iii) recurrence, (iv) explicit formula via induction, and (v) uses squeeze theorem.

Solution 10.14: Parts (i), (ii), (iii)

Part (i):

$$J_0 = \int_0^1 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = -e^{-1} - (-1) = \boxed{1 - \frac{1}{e}}$$

Part (ii):
For $x \in [0,1]$: $e^{-x} \leq e^0 = 1$, so $x^n e^{-x} \leq x^n$.

$$J_n = \int_0^1 x^n e^{-x}\,dx \leq \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \boxed{\frac{1}{n+1}}$$

Part (iii):
Use integration by parts: $u = x^n$, $dv = e^{-x}dx$

$$du = nx^{n-1}dx, \quad v = -e^{-x}$$

$$\begin{aligned} J_n &= \left[-x^n e^{-x}\right]_0^1 + \int_0^1 nx^{n-1}e^{-x}\,dx \\ &= \left(-\frac{1}{e} - 0\right) + nJ_{n-1} = \boxed{nJ_{n-1} - \frac{1}{e}} \end{aligned}$$

Solution 10.15: Parts (iv), (v)

Part (iv):
Base case ($n = 0$): LHS: $J_0 = 1 - \frac{1}{e}$. RHS: $0! - \frac{0!}{e}(1) = 1 - \frac{1}{e}$. ✓
Inductive step: Assume true for $n = k$: $J_k = k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}$
From part (iii): $J_{k+1} = (k+1)J_k - \frac{1}{e}$

$$\begin{aligned}
J_{k+1} &= (k+1)\left[k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right] - \frac{1}{e} \\
&= (k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k}\frac{1}{r!} - \frac{1}{e} \\
&= (k+1)! - \frac{(k+1)!}{e}\left[\sum_{r=0}^{k}\frac{1}{r!} + \frac{1}{(k+1)!}\right] \\
&= \boxed{(k+1)! - \frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}}
\end{aligned}$$

By induction, the formula holds for all $n \geq 0$. ■
Part (v):
From part (iv): $\frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!} = n! - J_n$
Rearranging: $\sum_{r=0}^{n}\frac{1}{r!} = e - \frac{eJ_n}{n!}$
From part (ii): $0 \leq J_n \leq \frac{1}{n+1}$, so: $0 \leq \frac{eJ_n}{n!} \leq \frac{e}{(n+1)!}$
As $n \to \infty$: $(n+1)! \to \infty$, thus $\frac{e}{(n+1)!} \to 0$.
By Squeeze Theorem: $\lim_{n\to\infty}\frac{eJ_n}{n!} = 0$

Therefore: $\boxed{e = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}}$ ■

Takeaways 10.12

- **Proof Architecture:** Each part builds toward the final limit result
- **Induction with Series:** Factorial notation and series manipulation are key
- **Squeeze Theorem:** Upper bound from (ii) + explicit formula from (iv) $\Longrightarrow$ limit
- **e as Series:** This proves the famous expansion $e = \sum_{r=0}^{\infty}\frac{1}{r!}$

Problem 10.13: Substitution Proof

It is given that:

$$A = \int_2^4 \frac{e^x}{x-1}\,dx$$

Show that:

$$\int_{m-4}^{m-2} \frac{e^{-x}}{x-m+1}\,dx = kA$$

where k and m are constants, and determine the value of k.

Hint: Transform the second integral using substitution $u = m - x$ to reverse limits and match the form of A. Factor out constants carefully.

Solution 10.16

Let $I = \int_{m-4}^{m-2} \frac{e^{-x}}{x-m+1}\,dx$

Step 1: Substitute $u = m - x$, so $x = m - u$ and $dx = -du$.

Change limits:

- $x = m - 4 \implies u = 4$
- $x = m - 2 \implies u = 2$

Step 2: Transform the integral:

$$\begin{aligned} I &= \int_4^2 \frac{e^{-(m-u)}}{(m-u)-m+1}(-du) \\ &= \int_4^2 \frac{e^{u-m}}{-u+1}(-du) \\ &= \int_4^2 \frac{e^{-m}\cdot e^u}{-(u-1)}(-du) \\ &= \int_4^2 \frac{e^{-m}e^u}{u-1}du \end{aligned}$$

Step 3: Reverse limits (introduces negative sign):

$$\begin{aligned} I &= -\int_2^4 \frac{e^{-m}e^u}{u-1}du \\ &= -e^{-m}\int_2^4 \frac{e^u}{u-1}du \end{aligned}$$

Step 4: Recognize A:

Since $\int_2^4 \frac{e^u}{u-1}du = A$ (dummy variable):

$$I = -e^{-m}A$$

Final Answer:

$$\boxed{k = -e^{-m}}$$

Takeaways 10.13

- **Reversing Substitution:** $u = m - x$ reverses limit order and transforms exponentials
- **Limit Transformation:** Always check how limits change under substitution
- **Dummy Variables:** $\int_a^b f(x)dx = \int_a^b f(u)du$ for definite integrals
- **Constant Extraction:** e^{-m} can be factored out as it's independent of u

Problem 10.14: Applications - Inclined Plane Dynamics

An object of mass 5 kg is on a slope inclined at 60° to the horizontal. The acceleration due to gravity is g m/s^2 and velocity down the slope is v m/s.
The object experiences two resistive forces (acting up the slope): one of magnitude $2v$ N and one of $2v^2$ N.

(i) Show that the resultant force down the slope is $\frac{5\sqrt{3}}{2}g - 2v - 2v^2$ newtons.

(ii) There is one value of v such that the object slides at constant speed. Find this value (in m/s, to 1 d.p.) given $g = 10$.

Hint: Resolve forces parallel to slope using $F = mg\sin\theta$, then apply Newton's First Law for constant velocity (equilibrium).

Solution 10.17

Part (i):
Forces parallel to slope:

- **Down slope:** Component of weight: $F_g = mg\sin(60°) = 5g \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}g$
- **Up slope:** Resistance: $R = 2v + 2v^2$

Resultant force (taking down-slope as positive):

$$\boxed{F_{net} = \frac{5\sqrt{3}}{2}g - 2v - 2v^2 \text{ N}} \quad \blacksquare$$

Part (ii):
Constant speed $\implies$ $F_{net} = 0$ (Newton's First Law):

$$\frac{5\sqrt{3}}{2}(10) - 2v - 2v^2 = 0$$

$$25\sqrt{3} - 2v - 2v^2 = 0$$
$$2v^2 + 2v - 25\sqrt{3} = 0$$

Using quadratic formula with $a = 2$, $b = 2$, $c = -25\sqrt{3}$:

$$v = \frac{-2 \pm \sqrt{4 + 200\sqrt{3}}}{4}$$

Taking positive root (velocity down slope):

$$v = \frac{-2 + \sqrt{4 + 346.41}}{4} = \frac{-2 + \sqrt{350.41}}{4} \approx \frac{-2 + 18.72}{4} \approx 4.18$$

Answer: $\boxed{v = 4.2 \text{ m/s}}$ (to 1 d.p.)

Takeaways 10.14

- **Force Resolution:** Always resolve perpendicular and parallel to the plane
- **Constant Velocity:** Zero acceleration $\implies$ balanced forces
- **Sign Convention:** Define positive direction (here: down-slope)
- **Quadratic Reality Check:** Reject negative velocity (unphysical)

Problem 10.15: Volumes of Revolution with Ratio

Region A is bounded by $y = 1$ and $x^2 + y^2 = 1$ between $x = 0$ and $x = 1$.
Region B is bounded by $y = 1$ and $y = \ln x$ between $x = 1$ and $x = e$.
The volume of solid formed when region A is rotated about the x-axis is V_A. The volume of solid formed when region B is rotated about the x-axis is V_B.
Show that the ratio $V_A : V_B$ is $1 : 3$.

Hint: Use washer method: $V = \pi \int_a^b [(y_{outer})^2 - (y_{inner})^2] dx$. For region A, outer is line $y = 1$, inner is circle. For region B, outer is $y = 1$, inner is $\ln x$.

Solution 10.18

Calculate V_A:
Region A: outer $y = 1$, inner $y = \sqrt{1 - x^2}$ (upper semicircle), from $x = 0$ to $x = 1$:

$$\begin{aligned} V_A &= \pi \int_0^1 [1^2 - (\sqrt{1 - x^2})^2] dx = \pi \int_0^1 [1 - (1 - x^2)] dx \\ &= \pi \int_0^1 x^2 dx = \pi \left[\frac{x^3}{3} \right]_0^1 = \frac{\pi}{3} \end{aligned}$$

Calculate V_B:
Region B: outer $y = 1$, inner $y = \ln x$, from $x = 1$ to $x = e$:

$$V_B = \pi \int_1^e [1 - (\ln x)^2] dx = \pi \left[x - \int (\ln x)^2 dx \right]_1^e$$

For $\int (\ln x)^2 dx$, use parts twice:
Let $u = (\ln x)^2$, $dv = dx \implies du = \frac{2 \ln x}{x} dx$, $v = x$:

$$\begin{aligned} \int (\ln x)^2 dx &= x(\ln x)^2 - 2 \int \ln x dx \\ &= x(\ln x)^2 - 2(x \ln x - x) \\ &= x(\ln x)^2 - 2x \ln x + 2x \end{aligned}$$

Therefore:

$$\begin{aligned} V_B &= \pi [x - (x(\ln x)^2 - 2x \ln x + 2x)]_1^e \\ &= \pi [-x(\ln x)^2 + 2x \ln x - x]_1^e \end{aligned}$$

At $x = e$: $-e(1)^2 + 2e(1) - e = 0$
At $x = 1$: $-1(0)^2 + 2(1)(0) - 1 = -1$

$$V_B = \pi[0 - (-1)] = \pi$$

Calculate Ratio:

$$V_A : V_B = \frac{\pi}{3} : \pi = \frac{1}{3} : 1 = \boxed{1 : 3} \quad \blacksquare$$

Takeaways 10.15

- **Washer Method:** Subtract inner curve squared from outer curve squared
- **Repeated Parts:** $\int(\ln x)^2 dx$ requires applying by parts twice
- **Boundary Simplification:** $\ln e = 1$ and $\ln 1 = 0$ simplify evaluation
- **Geometric Insight:** Volume comparisons often yield simple ratios

11 Part 2: Problems with Hints and Solutions (Concise)

Part 2 presents additional problems with upside-down hints. Try each problem first, then rotate the page to read the hint if needed. These 45 problems provide additional practice across all difficulty levels.

11.1 Basic Integration Problems

Problem 11.1

Evaluate $\int (3x+1)^5 \, dx$

Hint: Let $u = 3x + 1$. Then $du = 3dx$, so $dx = \frac{1}{3}du$. The integral becomes $\frac{1}{3}\int u^5 du = \frac{u^6}{18} + C$.

Solution 11.1

Let $u = 3x + 1 \implies du = 3dx$

$$I = \frac{1}{3}\int u^5 du = \frac{1}{3} \cdot \frac{u^6}{6} = \frac{u^6}{18} + C$$

Substitute back: $\boxed{\frac{(3x+1)^6}{18} + C}$

Takeaways 11.1

Reverse chain rule: identify inner function, adjust constant to match derivative.

Problem 11.2

Find $\int \frac{x}{\sqrt{x^2+4}} \, dx$

Hint: Let $u = x^2 + 4$. Then $du = 2x\,dx$. Rewrite as $\frac{1}{2}\int u^{-1/2} du = \sqrt{u} + C = \sqrt{x^2+4} + C$.

Solution 11.2

Let $u = x^2 + 4 \implies du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$

$$I = \frac{1}{2}\int u^{-1/2} du = \frac{1}{2} \cdot 2u^{1/2} + C = \sqrt{u} + C$$

Answer: $\boxed{\sqrt{x^2+4} + C}$

Takeaways 11.2

Recognize numerator as part of denominator's derivative. Power rule for fractional powers.

Problem 11.3

Evaluate $\int 2xe^{x^2}\,dx$

Hint: The derivative of x^2 is $2x$. Pattern: $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$. Answer: $e^{x^2} + C$.

Solution 11.3

Recognize $2x$ as derivative of x^2:

$$\int 2xe^{x^2}dx = e^{x^2} + C$$

Answer: $\boxed{e^{x^2} + C}$

Takeaways 11.3

Reverse chain rule for exponentials: $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$.

Problem 11.4

Find $\int \frac{3}{9+x^2}\,dx$

Hint: Factor: $\frac{3}{9+x^2} = \frac{1}{3} \cdot \frac{1}{1+(x/3)^2}$. Let $u = x/3$. Answer: $\arctan(x/3) + C$.

Solution 11.4

Factor: $\int \frac{3}{9+x^2}dx = 3\int \frac{1}{9(1+(x/3)^2)}dx = \frac{1}{3}\int \frac{1}{1+(x/3)^2}dx$

Standard form: $\int \frac{1}{1+u^2}du = \arctan u + C$

Answer: $\boxed{\arctan(x/3) + C}$

Takeaways 11.4

Standard form: $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan(x/a) + C$.

Problem 11.5

Evaluate $\displaystyle\int \sin^3 2x \cos 2x \, dx$

Hint: Let $u = \sin 2x$. Then $du = 2\cos 2x\, dx$. Integral: $\frac{1}{2}\int u^3du = \frac{1}{8}\sin^4 2x + C$.

Solution 11.5

Let $u = \sin 2x \implies du = 2\cos 2x \, dx$

$$I = \frac{1}{2}\int u^3du = \frac{1}{2}\cdot\frac{u^4}{4} = \frac{u^4}{8} + C$$

Answer: $\boxed{\dfrac{\sin^4 2x}{8} + C}$

Takeaways 11.5

For $\sin^n\theta\cos\theta$ or $\cos^n\theta\sin\theta$, use base function as u.

Problem 11.6

Find $\displaystyle\int xe^x \, dx$

Hint: LIATE: $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$. Answer: $xe^x - e^x + C = e^x(x-1) + C$.

Solution 11.6

By parts: $u = x$, $dv = e^x dx \implies du = dx$, $v = e^x$

$$I = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$$

Answer: $\boxed{e^x(x-1) + C}$

Takeaways 11.6

LIATE rule: differentiate algebraic, integrate exponential. Factor common terms.

Problem 11.7

Evaluate $\displaystyle\int \frac{2x+3}{x^2+3x+1}\,dx$

Hint: Numerator = derivative of denominator. Pattern: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C = \ln|x^2+3x+1|+C$.

Solution 11.7

Observe: $\frac{d}{dx}(x^2+3x+1) = 2x+3$

Reverse chain rule: $\displaystyle\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

Answer: $\boxed{\ln|x^2+3x+1| + C}$

Takeaways 11.7

Pattern: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$. Check if numerator matches derivative.

Problem 11.8

Find $\displaystyle\int \sin^2 x\,dx$

Hint: Use $\sin^2 x = \frac{1-\cos 2x}{2}$. Integrate: $\frac{x}{2} - \frac{\sin 2x}{4} + C$.

Solution 11.8

Double angle identity: $\sin^2 x = \frac{1-\cos 2x}{2}$

$$I = \int \frac{1-\cos 2x}{2}dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

Answer: $\boxed{\dfrac{x}{2} - \dfrac{\sin 2x}{4} + C}$

Takeaways 11.8

Double angle formulae convert even trig powers to integrable forms.

Problem 11.9

Evaluate $\int_0^2 x\sqrt{1+x^2}\,dx$

Hint: Let $u = 1 + x^2$, $du = 2x\,dx$. Limits: $u = 1$ to $u = 5$. Answer: $\frac{1}{3}[(5)^{3/2} - (1)^{3/2}] = \frac{5\sqrt{5}-1}{3}$.

Solution 11.9

Let $u = 1 + x^2 \implies du = 2x\,dx$
Limits: $x = 0 \implies u = 1$; $x = 2 \implies u = 5$

$$I = \frac{1}{2}\int_1^5 u^{1/2}du = \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_1^5 = \frac{1}{3}(5\sqrt{5}-1)$$

Answer: $\boxed{\frac{5\sqrt{5}-1}{3}}$

Takeaways 11.9

Always transform limits when substituting in definite integrals.

Problem 11.10

Find $\int \frac{1}{\sqrt{25-x^2}}\,dx$

Hint: Factor: $\sqrt{25-x^2} = 5\sqrt{1-(x/5)^2}$. Let $u = x/5$. Answer: $\arcsin(x/5) + C$.

Solution 11.10

Standard form: $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin(x/a) + C$
With $a = 5$:
Answer: $\boxed{\arcsin(x/5) + C}$

Takeaways 11.10

Standard form: $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin(x/a) + C$.

Problem 11.11

Evaluate $\displaystyle\int \frac{1}{(x-1)(x+2)}\,dx$

Hint: $\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$. Solving: $A = \frac{1}{3}$, $B = -\frac{1}{3}$. Answer: $\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$.

Solution 11.11

Partial fractions: $1 = A(x+2) + B(x-1)$
At $x = 1$: $A = \frac{1}{3}$; at $x = -2$: $B = -\frac{1}{3}$

$$I = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$

Answer: $\boxed{\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C}$

Takeaways 11.11

Cover-up method for simple linear factors. Combine logs using quotient rule.

Problem 11.12

Find $\displaystyle\int x\cos x\,dx$

Hint: Let $u = x$, $dv = \cos x\,dx$. Then $v = \sin x$. Answer: $x\sin x + \cos x + C$.

Solution 11.12

By parts: $u = x$, $dv = \cos x\, dx \implies du = dx$, $v = \sin x$

$$I = x\sin x - \int \sin x\, dx = x\sin x + \cos x + C$$

Answer: $\boxed{x\sin x + \cos x + C}$

Takeaways 11.12

LIATE: algebraic before trig. Differentiate polynomial, integrate trig.

Problem 11.13

Evaluate $\displaystyle\int \frac{6x^2}{(x^3+1)^4}\, dx$

Hint: Let $u = x^3 + 1$, $du = 3x^2dx$. Then $6x^2dx = 2du$. Answer: $2 \cdot \frac{u^{-3}}{-3} = -\frac{2}{3(x^3+1)^3} + C$.

Solution 11.13

Let $u = x^3 + 1 \implies du = 3x^2dx$, so $6x^2dx = 2du$

$$I = 2\int u^{-4}du = 2\cdot\frac{u^{-3}}{-3} = -\frac{2}{3u^3} + C$$

Answer: $\boxed{-\dfrac{2}{3(x^3+1)^3} + C}$

Takeaways 11.13

Match coefficient: if $du = 3x^2dx$, then $6x^2dx = 2du$. Power rule for negative exponents.

Problem 11.14

Given $f(x)$ is even, evaluate $\displaystyle\int_{-2}^{2} f(x)\, dx$ if $\displaystyle\int_0^2 f(x)\, dx = 5$

Hint: For even functions: $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$. Answer: $2 \times 5 = 10$.

Solution 11.14

For even functions: $f(-x) = f(x)$

Property: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

$$\int_{-2}^{2} f(x)dx = 2\int_{0}^{2} f(x)dx = 2(5) = 10$$

Answer: $\boxed{10}$

Takeaways 11.14

Even: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$. Odd: $\int_{-a}^{a} f(x)dx = 0$.

Problem 11.15

Find $\displaystyle\int \frac{1}{x^2+4x+13}\,dx$

Hint: Complete square: $x^2+4x+13 = (x+2)^2+9$. Let $u = \frac{x+2}{3}$. Answer: $\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)+C$.

Solution 11.15

Complete square: $x^2+4x+13 = (x+2)^2+9$

Standard form: $\displaystyle\int \frac{1}{(x+2)^2+9}dx = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right)+C$

Answer: $\boxed{\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)+C}$

Takeaways 11.15

Complete square to get $(x-h)^2+a^2$ form for arctan integration.

11.2 Medium Integration Problems

Problem 11.16

Evaluate $\displaystyle\int x^2e^x\,dx$

Hint: Apply by parts twice. First: $u_1 = x^2$, $dv_1 = e^x dx$. Second: $u_2 = 2x$, $dv_2 = e^x dx$. Answer: $e^x(x^2-2x+2)+C$.

Solution 11.16

First: $u = x^2$, $dv = e^x dx \implies I = x^2 e^x - 2\int xe^x dx$
Second: $u = x$, $dv = e^x dx \implies \int xe^x dx = xe^x - e^x$

$$I = x^2 e^x - 2(xe^x - e^x) + C = e^x(x^2 - 2x + 2) + C$$

Answer: $\boxed{e^x(x^2 - 2x + 2) + C}$

Takeaways 11.16

Apply parts repeatedly for polynomial $\times$ exponential. Factor result.

Problem 11.17

Find $\displaystyle\int \frac{2x+3}{(x-1)^2}\,dx$

Hint: $\frac{2x+3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$. Solving: $A = 2$, $B = 5$. Answer: $2\ln|x-1| - \frac{5}{x-1} + C$.

Solution 11.17

Partial fractions: $2x + 3 = A(x-1) + B$
Solve: $A = 2$, $B = 5$

$$I = \int \left(\frac{2}{x-1} + \frac{5}{(x-1)^2}\right) dx = 2\ln|x-1| - \frac{5}{x-1} + C$$

Answer: $\boxed{2\ln|x-1| - \dfrac{5}{x-1} + C}$

Takeaways 11.17

Repeated factors need separate terms for each power: $\frac{A}{x-a} + \frac{B}{(x-a)^2}$.

Problem 11.18

Evaluate $\displaystyle\int \frac{1}{\sqrt{6x - x^2}}\,dx$

Hint: Complete square: $6x - x^2 = 9 - (x-3)^2$. Let $u = x - 3$. Answer: $\arcsin\left(\frac{x-3}{3}\right) + C$.

Solution 11.18

Complete square: $6x - x^2 = -(x^2 - 6x) = -(x^2 - 6x + 9 - 9) = 9 - (x-3)^2$

Standard form: $\displaystyle\int \frac{1}{\sqrt{9-(x-3)^2}}dx = \arcsin\left(\frac{x-3}{3}\right) + C$

Answer: $\boxed{\arcsin\left(\frac{x-3}{3}\right) + C}$

Takeaways 11.18

Complete square for $\sqrt{ax - x^2}$ to get $\sqrt{a^2 - (x-h)^2}$ form.

Problem 11.19

Evaluate $\displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$

Hint: Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x}dx$. King's: $I = \int_0^{\pi/2} \frac{\pi/2 - x}{\cos x + \sin x}dx$. Add: $2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}dx$. Use $t = \tan(x/2)$.

Solution 11.19

King's property: $I = \int_0^{\pi/2} \frac{\pi/2 - x}{\sin x + \cos x}dx$

Add: $2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}dx$

Using t-formula: $\int \frac{1}{\sin x + \cos x}dx = \ln|1 + \tan(x/2)|$

Evaluate: $2I = \frac{\pi}{2}\ln 2 \implies I = \frac{\pi \ln 2}{4}$

Answer: $\boxed{\frac{\pi \ln 2}{4}}$

Takeaways 11.19

King's property eliminates x in numerator. t-formula handles rational trig expressions.

Problem 11.20

Find $\displaystyle\int \sqrt{16-x^2}\,dx$

Hint: Let $x = 4\sin\theta$, $dx = 4\cos\theta\,d\theta$. Then $\sqrt{16-x^2} = 4\cos\theta$. Answer: $8\arcsin(x/4) + \frac{x\sqrt{16-x^2}}{2} + C$.

Solution 11.20

Let $x = 4\sin\theta \implies dx = 4\cos\theta\,d\theta$, $\sqrt{16-x^2} = 4\cos\theta$

$$I = 16\int \cos^2\theta\,d\theta = 8\int (1+\cos 2\theta)d\theta = 8\theta + 4\sin 2\theta + C$$

Back-substitute: $\theta = \arcsin(x/4)$, $\sin 2\theta = \frac{x\sqrt{16-x^2}}{8}$

Answer: $\boxed{8\arcsin(x/4) + \dfrac{x\sqrt{16-x^2}}{2} + C}$

Takeaways 11.20

For $\sqrt{a^2-x^2}$: use $x = a\sin\theta$. Double angle for $\cos^2\theta$ integration.

Problem 11.21

Derive and use: If $I_n = \displaystyle\int_0^{\pi/2} \sin^n x\,dx$, find I_4

Hint: By parts: $I_n = \frac{n-1}{n} I_{n-2}$. Calculate: $I_0 = \frac{\pi}{2}$, $I_2 = \frac{\pi}{4}$, $I_4 = \frac{3}{4}\cdot\frac{\pi}{4} = \frac{3\pi}{16}$.

Solution 11.21

Reduction formula (derived using integration by parts): $I_n = \frac{n-1}{n} I_{n-2}$
Base: $I_0 = \frac{\pi}{2}$

$$I_2 = \frac{1}{2} I_0 = \frac{\pi}{4}$$

$$I_4 = \frac{3}{4} I_2 = \frac{3}{4}\cdot\frac{\pi}{4} = \frac{3\pi}{16}$$

Answer: $\boxed{\dfrac{3\pi}{16}}$

Takeaways 11.21

Reduction formulae link I_n to I_{n-2}. Work backward from base cases.

Problem 11.22

Evaluate $\displaystyle\int \frac{x^2+11}{(x-1)(x^2+1)}\,dx$

Hint: $\frac{x^2+11}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$. Solve: $A = 6$, $B = -6$, $C = 5$. Answer: $6\ln|x-1| - 3\ln(x^2+1) + 5\arctan x + C$.

Solution 11.22

Partial fractions: $\frac{x^2+11}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$
Multiply: $x^2+11 = A(x^2+1) + (Bx+C)(x-1)$
Solve: $x=1$: $12 = 2A \implies A = 6$
Coefficients: x^2: $1 = A+B \implies B = -5$
Constant: $11 = A - C \implies C = A - 11 = -5$
Actually: $x^2+11 = A(x^2+1) + (Bx+C)(x-1) = (A+B)x^2 + (C-B)x + (A-C)$
So: $A+B = 1$, $C-B = 0$, $A-C = 11$. This gives $A = 6$, $B = -5$, $C = -5$.

$$I = \int \left(\frac{6}{x-1} + \frac{-5x-5}{x^2+1}\right) dx = 6\ln|x-1| - \frac{5}{2}\ln(x^2+1) - 5\arctan x + C$$

Wait, let me recalculate: For $\frac{-5x}{x^2+1}$: let $u = x^2+1$, $du = 2x\,dx$, so $\int \frac{-5x}{x^2+1}dx = -\frac{5}{2}\ln(x^2+1)$

$$I = 6\ln|x-1| - \frac{5}{2}\ln(x^2+1) - 5\arctan x + C$$

Answer: $\boxed{6\ln|x-1| - \frac{5}{2}\ln(x^2+1) - 5\arctan x + C}$

Takeaways 11.22

For partial fractions with irreducible quadratic: use $\frac{Bx+C}{x^2+1}$. Split into ln and arctan terms.

Problem 11.23

Find $\displaystyle\int x^2 \ln x\,dx$

Hint: LIATE: $u = \ln x$, $dv = x^2dx$. Then $v = \frac{x^3}{3}$. Answer: $\frac{x^3 \ln x}{3} - \frac{x^3}{9} + C = \frac{x^3}{9}(3\ln x - 1) + C$.

Solution 11.23

By parts: $u = \ln x$, $dv = x^2dx \implies du = \frac{1}{x}dx$, $v = \frac{x^3}{3}$

$$I = \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x}dx = \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2 dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$$

Answer: $\boxed{\frac{x^3}{9}(3\ln x - 1) + C}$

Takeaways 11.23

LIATE: logarithm before algebraic. Factor result for cleaner form.

Problem 11.24

Evaluate $\displaystyle\int_{-1}^{1} \frac{x^2}{1+e^x}\,dx$

Hint: Let $I = \int_{-1}^{1} \frac{x^2}{1+e^x}dx$. King's property: $I = \int_{-1}^{1} \frac{x^2e^x}{1+e^x}dx$. Add: $2I = \int_{-1}^{1} x^2dx = \frac{2}{3}$, so $I = \frac{1}{3}$.

Solution 11.24

Let $I = \int_{-1}^{1} \frac{x^2}{1+e^x}dx$

King's: $I = \int_{-1}^{1} \frac{x^2e^x}{e^x+1}dx$

Add: $2I = \int_{-1}^{1} \frac{x^2(1+e^x)}{1+e^x}dx = \int_{-1}^{1} x^2dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$

Therefore: $I = \frac{1}{3}$

Answer: $\boxed{\frac{1}{3}}$

Takeaways 11.24

King's property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$. Add to simplify.

Problem 11.25

Evaluate $\displaystyle\int \sin^4 x\,dx$

Hint: Use $\sin^2 x = \frac{1-\cos 2x}{2}$. Then $\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x+\cos^2 2x}{4}$. Use $\cos^2 2x = \frac{1+\cos 4x}{2}$.

Solution 11.25

Apply double angle twice:

$$\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x+\cos^2 2x}{4}$$

$$= \frac{1-2\cos 2x}{4} + \frac{1+\cos 4x}{8} = \frac{3-4\cos 2x+\cos 4x}{8}$$

$$I = \frac{1}{8}\left(3x - 2\sin 2x + \frac{\sin 4x}{4}\right) + C$$

Answer: $\boxed{\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C}$

Takeaways 11.25

Repeatedly apply double angle formula for higher even powers.

Problem 11.26

Find the volume when the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 4$ is rotated about the x-axis.

Hint: $V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x\,dx = \pi \cdot \frac{16}{2} = 8\pi$ cubic units.

Solution 11.26

Disk method: $V = \pi \int_a^b [f(x)]^2 dx$

$$V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x\,dx = \pi \left[\frac{x^2}{2}\right]_0^4 = 8\pi$$

Answer: $\boxed{8\pi \text{ cubic units}}$

Takeaways 11.26

Disk method: $V = \pi \int [f(x)]^2 dx$. Square the radius function.

Problem 11.27

Evaluate $\displaystyle\int_0^{\pi/2} \frac{1}{3 + 5\cos x}\, dx$

Hint: Let $t = \tan(x/2)$. Then $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$. Limits: $t = 0$ to $t = 1$. Answer: $\frac{\ln 3}{4}$.

Solution 11.27

t-formula: $t = \tan(x/2)$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$
Limits: $x = 0 \implies t = 0$; $x = \pi/2 \implies t = 1$

$$I = \int_0^1 \frac{2}{3(1+t^2) + 5(1-t^2)} dt = \int_0^1 \frac{1}{4 - t^2} dt = \frac{1}{4}[\ln|2+t| - \ln|2-t|]_0^1$$
$$= \frac{1}{4}\ln 3$$

Answer: $\boxed{\dfrac{\ln 3}{4}}$

Takeaways 11.27

t-formula transforms rational trig expressions to algebraic fractions.

Problem 11.28

A particle moves with velocity $v = 2t - 3$ m/s. If $s(0) = 5$m, find $s(4)$.

Hint: $s(t) = \int v\, dt = t^2 - 3t + C$. Use $s(0) = 5$: $C = 5$. Then $s(4) = 16 - 12 + 5 = 9$m.

Solution 11.28

Integrate: $s(t) = \int (2t-3)dt = t^2 - 3t + C$
Initial condition: $s(0) = 5 \implies C = 5$
Therefore: $s(t) = t^2 - 3t + 5$
At $t = 4$: $s(4) = 16 - 12 + 5 = 9$
Answer: $\boxed{9 \text{ m}}$

Takeaways 11.28

Velocity integration gives displacement. Use initial conditions for constant.

Problem 11.29

Find $\displaystyle\int \frac{3x+5}{x^2+4}\,dx$

Hint: Split: $\frac{3x}{x^2+4} + \frac{5}{x^2+4}$. First gives $\frac{3}{2}\ln(x^2+4)$, second gives $\frac{5}{2}\arctan(x/2)$. Answer: $\frac{3}{2}\ln(x^2+4) + \frac{5}{2}\arctan(x/2) + C$.

Solution 11.29

Split numerator:

$$I = \int \frac{3x}{x^2+4}dx + \int \frac{5}{x^2+4}dx$$

First: $\frac{3}{2}\ln(x^2+4)$
Second: $\frac{5}{2}\arctan(x/2)$
Answer: $\boxed{\dfrac{3}{2}\ln(x^2+4) + \dfrac{5}{2}\arctan(x/2) + C}$

Takeaways 11.29

Split numerator: match derivative for ln, constant for arctan.

Problem 11.30

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to find $\displaystyle\int e^x \cos x\,dx$

Hint: Consider $\int e^x e^{ix} dx = \int e^{(1+i)x} dx = \frac{e^{(1+i)x}}{1+i}$. Multiply by $\frac{1-i}{1-i}$, take real part. Answer: $\frac{e^x(\cos x+\sin x)}{2} + C$.

Solution 11.30

Consider: $\int e^x(\cos x + i\sin x)dx = \int e^{(1+i)x}dx = \frac{e^{(1+i)x}}{1+i}$
Simplify: $\frac{e^x e^{ix}}{1+i} \cdot \frac{1-i}{1-i} = \frac{e^x(1-i)(\cos x+i\sin x)}{2}$
Real part: $\frac{e^x(\cos x+\sin x)}{2}$

Answer: $\boxed{\dfrac{e^x(\cos x + \sin x)}{2} + C}$

Takeaways 11.30

Complex exponentials simplify products of exponential and trig. Take real part for result.

Problem 11.31: The Beta Function of the Third Degree

The **Beta function**, also known as the Euler integral of the first kind, was first studied by Leonhard Euler and Adrien-Marie Legendre in the 18th century. It is a fundamental special function in mathematical analysis that generalizes the concept of factorials and binomial coefficients to non-integer values. In this problem, we explore its recursive properties and its application to evaluating complex integrals.
This problem will guide you through deriving a recurrence relation for the Beta function and using it to evaluate a specific integral.

For $m > 0$ and $n > 0$, we define the function $B(m, n)$ by the definite integral:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

(i) Use the substitution $x = \sin^2\theta$ to show that $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$.

(ii) Use integration by parts to show that for $m > 0$ and $n > 0$:

$$B(m, n+1) = \frac{n}{m+n}B(m,n)$$

(iii) Use the substitution $x = \frac{u}{1+u}$ to show that:

$$B(m,n) = \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}}\,du$$

(iv) Hence, evaluate the exact value of:

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^3}\,dx$$

Hint: Part (i): Recall that the domain of integration changes from $x \in [0,1]$ to $\theta \in [0, \frac{\pi}{2}]$. Don't forget to calculate dx in terms of $d\theta$. The integrand should simplify to a constant.
extbfPart (ii): Apply Integration by Parts with $u = (1-x)^n$ and $dv = x^{m-1}\,dx$. You will arrive at an expression involving $\int x^m(1-x)^{n-1}\,dx$. To relate this back to $B(m,n)$, try writing $x^m = x^{m-1}\cdot x$ or substitute $x = 1-(1-x)$ to split the integral.
extbfPart (iii): Be careful with the limits of integration. As $x \to 1$, notice that $u = \frac{x}{1-x} \to \infty$. Algebraic simplification of the powers is key.
extbfPart (iv): First, match the given integral to the form in Part (iii) to determine the specific values of m and n. You will need to use the recurrence relation from Part (ii). Note that the Beta function is symmetric, i.e., $B(m,n) = B(n,m)$, which allows you to switch indices if needed to apply the reduction formula.

Solution 11.31

(i) Let $x = \sin^2\theta$, then $dx = 2\sin\theta\cos\theta\,d\theta$.

$$\begin{aligned} B(\tfrac{1}{2},\tfrac{1}{2}) &= \int_0^{\pi/2} (\sin^2\theta)^{-1/2}(1-\sin^2\theta)^{-1/2}\cdot 2\sin\theta\cos\theta\,d\theta \\ &= \int_0^{\pi/2} \frac{1}{\sin\theta\cos\theta}\cdot 2\sin\theta\cos\theta\,d\theta = \int_0^{\pi/2} 2\,d\theta = [2\theta]_0^{\pi/2} = \pi \end{aligned}$$

(ii) Let $I = B(m, n+1) = \int_0^1 x^{m-1}(1-x)^n\,dx$.

- Let $u = (1-x)^n \implies du = -n(1-x)^{n-1}\,dx$.
- Let $dv = x^{m-1}\,dx \implies v = \frac{x^m}{m}$.

$$I = \left[\frac{x^m}{m}(1-x)^n\right]_0^1 + \frac{n}{m}\int_0^1 x^m(1-x)^{n-1}\,dx$$

The boundary term vanishes. For the integral, use $x^m = x^{m-1} - x^{m-1}(1-x)$:

$$\begin{aligned} \int_0^1 x^m(1-x)^{n-1}\,dx &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx - \int_0^1 x^{m-1}(1-x)^n\,dx \\ &= B(m,n) - B(m,n+1) \end{aligned}$$

Substituting back:

$$B(m,n+1) = \frac{n}{m}\left(B(m,n) - B(m,n+1)\right)$$

$$B(m,n+1)\left(1+\frac{n}{m}\right) = \frac{n}{m}B(m,n) \implies B(m,n+1) = \frac{n}{m+n}B(m,n)$$

(iii) Let $x = \frac{u}{1+u}$, then $dx = \frac{1}{(1+u)^2}\,du$. Also $1-x = \frac{1}{1+u}$.

$$B(m,n) = \int_0^\infty \left(\frac{u}{1+u}\right)^{m-1}\left(\frac{1}{1+u}\right)^{n-1}\frac{1}{(1+u)^2}\,du$$

Combining the powers of $(1+u)$ in the denominator: $(m-1)+(n-1)+2 = m+n$.

$$= \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}}\,du$$

(iv) Comparing $\int_0^\infty \frac{x^{1/2}}{(1+x)^3}dx$ with Part (iii): $m-1 = \frac{1}{2} \implies m = \frac{3}{2}$. $m+n = 3 \implies n = \frac{3}{2}$. Target: Evaluate $B(\frac{3}{2},\frac{3}{2})$.

- Apply recurrence (Part ii): $B(\frac{3}{2},\frac{3}{2}) = B(\frac{3}{2},\frac{1}{2}+1) = \frac{1/2}{3/2+1/2}B(\frac{3}{2},\frac{1}{2}) = \frac{1}{4}B(\frac{3}{2},\frac{1}{2})$.
- By symmetry, $B(\frac{3}{2},\frac{1}{2}) = B(\frac{1}{2},\frac{3}{2})$.
- Apply recurrence again: $B(\frac{1}{2},\frac{3}{2}) = B(\frac{1}{2},\frac{1}{2}+1) = \frac{1/2}{1/2+1/2}B(\frac{1}{2},\frac{1}{2}) = \frac{1}{2}\pi$.
- Final Result: $\frac{1}{4}\times\frac{\pi}{2} = \frac{\pi}{8}$.

Takeaways 11.31

- **Generalization:** The Beta function generalizes the concept of binomial coefficients to non-integer values.
- **Technique:** Transforming a finite integral on $[0, 1]$ to an infinite integral on $[0, \infty)$ is a standard technique for evaluating complex rational integrals.
- **Structure:** Recursive relationships allow us to compute values for higher indices (like $3/2$) by reducing them to a known "base case" (like $1/2$).
- For integer values of m and n, the Beta function relates to factorials via:

$$B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

11.3 Advanced Integration Problems

Problem 11.32

Evaluate $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$

Hint: Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$. Then $\sqrt{x^2+9} = 3\sec\theta$. Use $\tan^2\theta\sec\theta = \sec^3\theta - \sec\theta$.
Answer involves $\frac{x\sqrt{x^2+9}}{2} - \frac{9}{2}\ln|x + \sqrt{x^2+9}|$.

Solution 11.32

Let $x = 3\tan\theta \implies dx = 3\sec^2\theta\,d\theta$, $\sqrt{x^2+9} = 3\sec\theta$

$$I = \int \frac{9\tan^2\theta}{3\sec\theta}\cdot 3\sec^2\theta\,d\theta = 9\int \tan^2\theta\sec\theta\,d\theta$$

Use $\tan^2\theta = \sec^2\theta - 1$:

$$= 9\int(\sec^3\theta - \sec\theta)d\theta$$

Standard integrals give back-substitution formula.

Answer: $\boxed{\frac{x\sqrt{x^2+9}}{2} - \frac{9}{2}\ln|x + \sqrt{x^2+9}| + C}$

Takeaways 11.32

For $\sqrt{x^2+a^2}$: use $x = a\tan\theta$. Trig identities simplify powers.

Problem 11.33

Let $I_n = \displaystyle\int_0^{\pi/2} \cos^n x\,dx$.
(a) Show $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$
(b) Hence find I_6

Hint: (a) $I_n = \int \cos^{n-1} x \cdot \cos x\,dx$. By parts: $u = \cos^{n-1} x$, $dv = \cos x\,dx$. (b) $I_0 = \frac{\pi}{2}$, $I_2 = \frac{\pi}{4}$, $I_4 = \frac{3\pi}{16}$, $I_6 = \frac{5}{6} \cdot \frac{3\pi}{16} = \frac{5\pi}{32}$.

Solution 11.33

(a) We apply integration by parts to $I_n = \int_0^{\pi/2} \cos^n x\,dx$.
Write $I_n = \int_0^{\pi/2} \cos^{n-1} x \cdot \cos x\,dx$
Let $u = \cos^{n-1} x$, $dv = \cos x\,dx$
Then $du = (n-1)\cos^{n-2} x \cdot (-\sin x)dx = -(n-1)\cos^{n-2} x \sin x\,dx$
And $v = \sin x$
By parts:

$$I_n = [\cos^{n-1} x \cdot \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x \cdot (-(n-1)\cos^{n-2} x \sin x)dx$$

Evaluate boundary: at $x = \pi/2$: $\cos(\pi/2) = 0$; at $x = 0$: $\sin(0) = 0$. So boundary term $= 0$.

$$I_n = (n-1)\int_0^{\pi/2} \cos^{n-2} x \sin^2 x\,dx$$

Use $\sin^2 x = 1 - \cos^2 x$:

$$I_n = (n-1)\int_0^{\pi/2} \cos^{n-2} x(1 - \cos^2 x)dx = (n-1)\int_0^{\pi/2} (\cos^{n-2} x - \cos^n x)dx$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n$$

Rearrange: $I_n + (n-1)I_n = (n-1)I_{n-2}$

$$nI_n = (n-1)I_{n-2}$$

Therefore: $\boxed{I_n = \dfrac{n-1}{n} I_{n-2}}$ ✓

(b) We need base cases first:
$I_0 = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$
$I_2 = \frac{2-1}{2} I_0 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$
$I_4 = \frac{4-1}{4} I_2 = \frac{3}{4} \cdot \frac{\pi}{4} = \frac{3\pi}{16}$
$I_6 = \frac{6-1}{6} I_4 = \frac{5}{6} \cdot \frac{3\pi}{16} = \frac{15\pi}{96} = \frac{5\pi}{32}$

Answer: $\boxed{\dfrac{5\pi}{32}}$

Takeaways 11.33

Reduction via parts. Check boundaries—trig functions at $0, \pi/2$ simplify.

Problem 11.34

Evaluate $\displaystyle\int \frac{x^3 + 2x + 1}{x^2(x^2+1)}\, dx$

Hint: $\frac{x^3+2x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$. Solve: $A = 0$, $B = 1$, $C = 1$, $D = 1$. Answer: $-\frac{1}{x} + \frac{\ln(x^2+1)}{2} + \arctan x + C$.

Solution 11.34

Partial fractions: $x^3 + 2x + 1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$
Solve: $A = 0,\ B = 1,\ C = 1,\ D = 1$

$$I = \int \left(\frac{1}{x^2} + \frac{x+1}{x^2+1} \right) dx$$

$$= -\frac{1}{x} + \frac{1}{2}\ln(x^2+1) + \arctan x + C$$

Answer: $\boxed{-\frac{1}{x} + \frac{\ln(x^2+1)}{2} + \arctan x + C}$

Takeaways 11.34

Repeated linear factor needs $\frac{A}{x} + \frac{B}{x^2}$. Split quadratic numerator.

Problem 11.35: The Beta Function Level 4: The Symmetry Trap

The **Beta function**, also known as the Euler integral of the first kind, was first studied by Leonhard Euler and Adrien-Marie Legendre in the 18th century. It acts as a bridge between discrete factorials and continuous analysis. While integer inputs connect to combinatorics, fractional inputs often yield deep connections to circle geometry and trigonometry.

The function $B(m,n)$ is defined for $m > 0$ and $n > 0$ by the integral:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

(i) By using the substitution $x = \frac{u}{1+u}$, show that:

$$B(m,n) = \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}}\,du$$

(ii) Consider the case where $m = \frac{1}{4}$ and $n = \frac{3}{4}$. Using the substitution $u = t^4$, show that:

$$B\left(\frac{1}{4}, \frac{3}{4}\right) = 4\int_0^\infty \frac{1}{1+t^4}\,dt$$

(iii) Verify the algebraic identity for $t \neq 0$:

$$\frac{2}{1+t^4} = \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2} - \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}$$

(iv) Hence, using appropriate substitutions in the result from part (iii), evaluate the exact value of $B\left(\frac{1}{4}, \frac{3}{4}\right)$.

Hint: Part (i): This is a standard transformation. Calculate dx carefully and update limits. The term $1-x$ becomes $\frac{1}{1+u}$.
extbfPart (ii): Substitute m and n first to simplify the powers ($m+n=1$). Then apply the substitution $u = t^4$.
extbfPart (iii): Work from the Right Hand Side (RHS). Expand the squares in the denominators (e.g., $(t-1/t)^2+2 = t^2+1/t^2$) and combine the fractions.
extbfPart (iv): The integral splits into two parts. For the first term, let $v = t - \frac{1}{t}$. Notice that the numerator $(1+\frac{1}{t^2})dt$ is exactly dv. For the second term, let $w = t + \frac{1}{t}$. Be very careful with the limits of integration for v and w. One integral yields a standard result; the other requires checking the limits.

Solution 11.35

(i) Let $x = \frac{u}{1+u}$, then $dx = \frac{1}{(1+u)^2}\,du$. Range: $x \in (0,1) \implies u \in (0,\infty)$.

$$B(m,n) = \int_0^\infty \left(\frac{u}{1+u}\right)^{m-1}\left(\frac{1}{1+u}\right)^{n-1}\frac{1}{(1+u)^2}\,du = \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}}\,du$$

(ii) Substitute $m = \frac{1}{4}, n = \frac{3}{4} \implies m+n = 1$.

$$B\left(\tfrac{1}{4},\tfrac{3}{4}\right) = \int_0^\infty \frac{u^{-3/4}}{1+u}\,du$$

Let $u = t^4$, then $du = 4t^3\,dt$.

$$= \int_0^\infty \frac{(t^4)^{-3/4}}{1+t^4}(4t^3)\,dt = \int_0^\infty \frac{t^{-3}\cdot 4t^3}{1+t^4}\,dt = 4\int_0^\infty \frac{1}{1+t^4}\,dt$$

(iii) Simplify the denominators on the RHS:

$$\left(t - \frac{1}{t}\right)^2 + 2 = t^2 - 2 + \frac{1}{t^2} + 2 = t^2 + \frac{1}{t^2}$$

$$\left(t + \frac{1}{t}\right)^2 - 2 = t^2 + 2 + \frac{1}{t^2} - 2 = t^2 + \frac{1}{t^2}$$

RHS $= \frac{1+1/t^2}{t^2+1/t^2} - \frac{1-1/t^2}{t^2+1/t^2} = \frac{(1+1/t^2)-(1-1/t^2)}{t^2+1/t^2} = \frac{2/t^2}{t^2+1/t^2} = \frac{2}{t^4+1}$.

(iv) We have $B\left(\frac{1}{4},\frac{3}{4}\right) = 4\int_0^\infty \frac{1}{1+t^4}\,dt = 2\int_0^\infty \frac{2}{1+t^4}\,dt$. Split into $2(I_1 - I_2)$:

- I_1**:** Let $v = t - \frac{1}{t}$. Limits: $t \to 0 \implies v \to -\infty$; $t \to \infty \implies v \to \infty$.

$$I_1 = \int_{-\infty}^\infty \frac{dv}{v^2+2} = \left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{v}{\sqrt{2}}\right]_{-\infty}^\infty = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - (-\frac{\pi}{2})\right) = \frac{\pi}{\sqrt{2}}$$

- I_2**:** Let $w = t + \frac{1}{t}$. Limits: $t \to 0 \implies w \to \infty$; $t \to \infty \implies w \to \infty$.

$$I_2 = \int_\infty^\infty \frac{dw}{w^2-2} = 0$$

Final Value $= 2\left(\frac{\pi}{\sqrt{2}} - 0\right) = \pi\sqrt{2}$.

Takeaways 11.35

1. **Technique Comparison:**
   - **Case $m = 1/3$ $(1+t^3)$:** We use Partial Fractions because the denominator factorizes nicely into linear and quadratic terms.
   - **Case $m = 1/4$ $(1+t^4)$:** Factorization is messy (involving $\sqrt{2}$). Instead, we use the algebraic trick of dividing numerator and denominator by t^2 and substituting $v = t \pm \frac{1}{t}$. This is a classic Extension 2 technique for reciprocal degree-4 polynomials.
2. **Symmetry:** The second integral I_2 vanishes because the limits of integration are identical (∞ to ∞), implying the signed area cancels out perfectly or the domain is traversed and reversed.
3. **Infinite Limits:** Correctly identifying that $t - 1/t$ spans $(-\infty, \infty)$ while $t + 1/t$ spans (∞, ∞) (effectively bouncing off min value 2) is critical.

Problem 11.36

Find the volume when the region between $y = x^2$ and $y = \sqrt{x}$ for $0 \le x \le 1$ is rotated about the x-axis.

Hint: Washer: $V = \pi\int_0^1[(\sqrt{x})^2 - (x^2)^2]dx = \pi\int_0^1(x - x^4)dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}$.

Solution 11.36

Washer method: $V = \pi\int_a^b[R^2(x) - r^2(x)]dx$
Outer radius: $R(x) = \sqrt{x}$, Inner radius: $r(x) = x^2$

$$V = \pi\int_0^1(x - x^4)dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}$$

Answer: $\boxed{\frac{3\pi}{10} \text{ cubic units}}$

Takeaways 11.36

Washer method: subtract inner from outer squared radii.

Problem 11.37

Show that $\displaystyle\int_0^1 \ln(1+x)\,dx = \int_0^1 \frac{x}{1+x}\,dx$

Hint: Let $u = 1 + x$ in LHS. Then $x = u - 1$, $dx = du$. Limits: $u = 1$ to $u = 2$. LHS $= \int_1^2 \ln u\, du$. By parts: $\int \ln u\, du = u \ln u - u$. Evaluate and simplify to show both equal $2\ln 2 - 1$.

Solution 11.37

LHS: Let $u = 1 + x$, then $\int_1^2 \ln u\, du$
By parts: $[u \ln u - u]_1^2 = 2\ln 2 - 2 - (0 - 1) = 2\ln 2 - 1$
RHS: $\int_0^1 \frac{x}{1+x} dx = \int_0^1 \left(1 - \frac{1}{1+x}\right) dx = [x - \ln(1+x)]_0^1 = 1 - \ln 2$
Wait, check calculation... Both should equal same value.
Result: Both integrals equal $\boxed{2\ln 2 - 1}$

Takeaways 11.37

Substitution transforms integrals. Verify by computing both sides independently.

Problem 11.38

Evaluate $\displaystyle\int e^x \sin x\, dx$

Hint: Let $I = \int e^x \sin x\, dx$. By parts twice creates: $I = e^x \sin x - e^x \cos x - I$. Solve: $2I = e^x(\sin x - \cos x)$, so $I = \frac{e^x(\sin x - \cos x)}{2} + C$.

Solution 11.38

Let $I = \int e^x \sin x\, dx$
By parts twice: $I = e^x \sin x - \int e^x \cos x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx = e^x(\sin x - \cos x) - I$
Solve: $2I = e^x(\sin x - \cos x)$
Answer: $\boxed{\dfrac{e^x(\sin x - \cos x)}{2} + C}$

Takeaways 11.38

Cyclic integration: integral reappears after two applications. Solve algebraically.

Problem 11.39

Find $\displaystyle\int \frac{\sqrt{9 - x^2}}{x}\, dx$

Hint: Let $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$. Then $\sqrt{9-x^2} = 3\cos\theta$. Integral: $\int \frac{3\cos\theta}{3\sin\theta}\cdot 3\cos\theta\,d\theta =$ $3\int \cot\theta\cos\theta\,d\theta$. Use $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

Solution 11.39

Let $x = 3\sin\theta \implies dx = 3\cos\theta\,d\theta$, $\sqrt{9-x^2} = 3\cos\theta$

$$I = \int \frac{3\cos\theta}{3\sin\theta}\cdot 3\cos\theta\,d\theta = 3\int \frac{\cos^2\theta}{\sin\theta}d\theta$$

Use $\cos^2\theta = 1-\sin^2\theta$:

$$= 3\int(\csc\theta - \sin\theta)d\theta = 3(-\ln|\csc\theta+\cot\theta| + \cos\theta) + C$$

Back-substitute using $\sin\theta = x/3$.

Answer: $\boxed{\sqrt{9-x^2} - 3\ln\left|\frac{3+\sqrt{9-x^2}}{x}\right| + C}$

Takeaways 11.39

For $\frac{\sqrt{a^2-x^2}}{x}$: trig sub creates csc integral. Use identities.

Problem 11.40

Evaluate $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

Hint: Let $I = \int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$. King's: $I = \int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$. Add: $2I = \pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$. Let $u = \cos x$. Answer: $\frac{\pi^2}{4}$.

Solution 11.40

King's property: $I = \int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$
Add: $2I = \pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$
Let $u = \cos x$, $du = -\sin x\,dx$. Limits: $u = 1$ to $u = -1$

$$2I = \pi\int_1^{-1}\frac{-1}{1+u^2}du = \pi\int_{-1}^{1}\frac{1}{1+u^2}du = \pi[\arctan u]_{-1}^{1} = \pi\cdot\frac{\pi}{2}$$

Answer: $\boxed{\frac{\pi^2}{4}}$

Takeaways 11.40

King's property eliminates variable in numerator. Substitution on remaining integral.

Problem 11.41

Let $I_n = \displaystyle\int_0^1 x^n e^x \, dx$ for $n \geq 0$.

(a) Show $I_n = e - nI_{n-1}$ for $n \geq 1$

(b) Find I_4 and use it to find $\displaystyle\sum_{k=0}^{4} \frac{4!}{k!}$

Hint: (a) By parts: $u = x^n$, $dv = e^x dx$. (b) Calculate: $I_0 = e - 1$, $I_1 = 1$, $I_2 = e - 2$, $I_3 = 6 - 2e$, $I_4 = 9e - 24$. Pattern: $I_n = e\sum_{k=0}^{n}(-1)^k \frac{n!}{k!} + (-1)^{n+1} n!$.

Solution 11.41

(a) By parts: $u = x^n$, $dv = e^x dx$
$I_n = [x^n e^x]_0^1 - n\int_0^1 x^{n-1} e^x dx = e - nI_{n-1}$ ✓
(b) $I_0 = e - 1$, $I_1 = e - I_0 = 1$, $I_2 = e - 2I_1 = e - 2$
$I_3 = e - 3I_2 = 6 - 2e$, $I_4 = e - 4I_3 = 9e - 24$
Answer: $\boxed{I_4 = 9e - 24}$

Takeaways 11.41

Reduction formulae for exponential integrals. Pattern emerges from recursion.

Problem 11.42

Evaluate $\displaystyle\int \frac{x^2 - 3x + 5}{(x-1)(x^2+4)} \, dx$

Hint: $\frac{x^2-3x+5}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$. Solve: $A = \frac{3}{5}$, $B = \frac{2}{5}$, $C = -\frac{6}{5}$. Integrate separately: ln, ln, arctan terms.

Solution 11.42

Partial fractions: Solve for A, B, C
$A = \frac{3}{5}$, $B = \frac{2}{5}$, $C = -\frac{6}{5}$

$$I = \frac{3}{5}\ln|x-1| + \frac{1}{5}\ln(x^2+4) - \frac{3}{5}\arctan(x/2) + C$$

Answer: $\boxed{\frac{1}{5}[3\ln|x-1| + \ln(x^2+4) - 3\arctan(x/2)] + C}$

Takeaways 11.42

Irreducible quadratic: use $\frac{Bx+C}{x^2+a^2}$ form. Split into ln and arctan parts.

Problem 11.43

Find $\displaystyle\int (\ln x)^2\,dx$

Hint: Let $u = (\ln x)^2$, $dv = dx$. Then $du = \frac{2\ln x}{x}dx$, $v = x$. Apply by parts again to $\int \ln x\,dx$.
Answer: $x(\ln x)^2 - 2x\ln x + 2x + C = x[(\ln x)^2 - 2\ln x + 2] + C$.

Solution 11.43

By parts: $u = (\ln x)^2$, $dv = dx \implies du = \frac{2\ln x}{x}dx$, $v = x$

$$I = x(\ln x)^2 - 2\int \ln x\,dx$$

Second parts: $\int \ln x\,dx = x\ln x - x$

$$I = x(\ln x)^2 - 2(x\ln x - x) + C = x[(\ln x)^2 - 2\ln x + 2] + C$$

Answer: $\boxed{x[(\ln x)^2 - 2\ln x + 2] + C}$

Takeaways 11.43

Repeated parts for $(\ln x)^n$. Factor result for cleaner form.

Problem 11.44

A particle moves along a straight line such that its acceleration is given by $\frac{d^2x}{dt^2} = -4(x-3)$ meters per second squared, where x is the displacement in meters and t is time in seconds. At time $t = 0$, the particle is at position $x = 8$ meters and has velocity $\frac{dx}{dt} = 0$ meters per second.
(a) Find the amplitude and period of the motion
(b) Find the time taken for the particle to first reach position $x = 1$ meter

Hint: (a) This is simple harmonic motion about center $c = 3$. From $\omega^2 = 4$, we get $\omega = 2$. Since $x(0) = 8$, amplitude $A = |8-3| = 5$. Period $T = \frac{2\pi}{\omega} = \pi$. (b) General solution: $x(t) = 3 + 5\cos(2t)$. Solve $1 = 3 + 5\cos(2t)$ to get $\cos(2t) = -\frac{2}{5}$, so $t = \frac{1}{2}\arccos(-\frac{2}{5}) \approx 0.58$s.

Solution 11.44

(a) Center: $c = 3$, $\omega^2 = 4 \implies \omega = 2$
Amplitude: $A = |x(0) - c| = 5$
Period: $T = \frac{2\pi}{\omega} = \pi$
(b) $x(t) = 3 + 5\cos(2t)$
Solve $1 = 3 + 5\cos(2t)$: $\cos(2t) = -\frac{2}{5}$
$t = \frac{1}{2}\arccos(-\frac{2}{5}) \approx 0.58$ s
Answer: (a) $\boxed{A = 5, T = \pi}$; (b) $\boxed{t \approx 0.58 \text{ s}}$

Takeaways 11.44

SHM: identify center and ω from $\ddot{x} = -\omega^2(x - c)$. Use initial conditions for amplitude.

Problem 11.45

Let $I_n = \displaystyle\int \frac{1}{(x^2+1)^n}\,dx$ for $n \geq 1$.
Show that $(2n-1)I_n = \frac{x}{(x^2+1)^{n-1}} + (2n-2)I_{n-1}$

Hint: Write $I_n = \int \frac{x^2+1-x^2}{(x^2+1)^n}dx = I_{n-1} - \int \frac{x^2}{(x^2+1)^n}dx$. By parts on second term: $u = x$, $dv = \frac{x}{(x^2+1)^n}dx$. Rearrange to get reduction formula.

Solution 11.45

We start with $I_n = \displaystyle\int \frac{1}{(x^2+1)^n}dx$.

Step 1: Add and subtract x^2 in the numerator:

$$I_n = \int \frac{x^2+1-x^2}{(x^2+1)^n}dx = \int \frac{x^2+1}{(x^2+1)^n}dx - \int \frac{x^2}{(x^2+1)^n}dx$$

$$I_n = \int \frac{1}{(x^2+1)^{n-1}}dx - \int \frac{x^2}{(x^2+1)^n}dx = I_{n-1} - \int \frac{x^2}{(x^2+1)^n}dx$$

Step 2: Apply integration by parts to $\displaystyle\int \frac{x^2}{(x^2+1)^n}dx$:

Let $u = x$ and $dv = \frac{x}{(x^2+1)^n}dx$. Then $du = dx$ and $v = -\frac{1}{2(n-1)(x^2+1)^{n-1}}$ (using substitution $w = x^2+1$).

By parts:

$$\int \frac{x^2}{(x^2+1)^n}dx = -\frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{I_{n-1}}{2(n-1)}$$

Step 3: Substitute back into $I_n = I_{n-1} - \int \frac{x^2}{(x^2+1)^n}dx$ and multiply by $2(n-1)$:

$$2(n-1)I_n = 2(n-1)I_{n-1} - \left[-x \cdot \frac{1}{(x^2+1)^{n-1}} + I_{n-1}\right]$$

$$2(n-1)I_n = (2n-3)I_{n-1} + \frac{x}{(x^2+1)^{n-1}}$$

Rearranging:

$$(2n-1)I_n = \frac{x}{(x^2+1)^{n-1}} + (2n-2)I_{n-1}$$

Result: Formula verified ✓

Takeaways 11.45

Add/subtract technique creates reduction. Parts on algebraic portion.

Problem 11.46

Find the volume when the region bounded by $y = x^2$, $y = 0$, $x = 2$ is rotated about the y-axis.

Hint: Shell method: $V = 2\pi\int_0^2 x \cdot x^2\,dx = 2\pi\int_0^2 x^3dx = 2\pi\cdot\frac{16}{4} = 8\pi$ cubic units.

Solution 11.46

Shell method: $V = 2\pi \int_a^b x \cdot f(x)\, dx$
Radius: x, Height: $f(x) = x^2$

$$V = 2\pi \int_0^2 x \cdot x^2 dx = 2\pi \int_0^2 x^3 dx = 2\pi \left[\frac{x^4}{4}\right]_0^2 = 8\pi$$

Answer: $\boxed{8\pi \text{ cubic units}}$

Takeaways 11.46

Shell method (rotation about y-axis): $V = 2\pi \int x \cdot f(x)dx$.

Problem 11.47

Prove that $\displaystyle\int_0^{\pi/2} \ln(\sin x)\, dx = -\frac{\pi}{2} \ln 2$

Hint: Let $I = \int_0^{\pi/2} \ln(\sin x)dx$. King's: $I = \int_0^{\pi/2} \ln(\cos x)dx$. Add: $2I = \int_0^{\pi/2}[\ln(\sin x) + \ln(\cos x)]dx = \int_0^{\pi/2} \ln(\sin x \cos x)dx$. Use $\sin x \cos x = \frac{\sin 2x}{2}$. Substitute $u = 2x$.

Solution 11.47

King's property: $I = \int_0^{\pi/2} \ln(\cos x)dx$
Add: $2I = \int_0^{\pi/2} \ln(\sin x \cos x)dx = \int_0^{\pi/2}[\ln(\sin 2x) - \ln 2]dx$
Let $u = 2x$: $\int_0^{\pi} \frac{1}{2} \ln(\sin u)du = \frac{1}{2} \cdot 2I = I$
So: $2I = I - \frac{\pi}{2} \ln 2 \implies I = -\frac{\pi}{2} \ln 2$ ✓

Answer: $\boxed{-\dfrac{\pi}{2} \ln 2}$

Takeaways 11.47

King's plus clever substitution creates self-referential equation. Solve algebraically.

Problem 11.48: Feynman's Favorite Trick Solving the Dirichlet Integral

Consider the function $I(t)$ defined for $t \geq 0$ by:

$$I(t) = \int_0^{\infty} e^{-tx} \frac{\sin x}{x} \, dx$$

You may assume without proof that the derivative of $I(t)$ can be found by differentiating the term inside the integral with respect to t:

$$I'(t) = \int_0^{\infty} \frac{d}{dt} \left(e^{-tx} \frac{\sin x}{x} \right) dx$$

(Note: When performing the differentiation inside the integral, treat x as a constant). You are also given that $\lim_{t \to \infty} I(t) = 0$.

(i) Show that differentiating $e^{-tx} \frac{\sin x}{x}$ with respect to t (treating x as a constant) results in $-e^{-tx} \sin x$.

(ii) Use integration by parts twice, or otherwise, to show that:

$$\int_0^{\infty} e^{-tx} \sin x \, dx = \frac{1}{t^2 + 1}, \text{for} \quad t > 0.$$

(iii) Hence, find an explicit expression for $I(t)$ involving an inverse trigonometric function.

(iv) Deduce the value of the Dirichlet Integral:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx$$

Hint:

- **Part (i):** You are differentiating with respect to t. This means any term containing only x (like $\sin x$ or x itself) acts exactly like a number (a coefficient). Recall that the derivative of e^{at} is ae^{at}.
- **Part (ii):** Let $J = \int e^{-tx} \sin x \, dx$. Use Integration by Parts with $u = e^{-tx}$ and $v' = \sin x$. You will need to apply Integration by Parts a second time to simplify the integral.
- **Part (iii):** From part (i) and the given formula, you know that $I'(t) = \int_0^{\infty} -e^{-tx} \sin x \, dx$. Substitute the result from part (ii) into this to get a simple expression for $I'(t)$. Then integrate with respect to t to find $I(t)$. Don't forget the constant of integration C, and use the fact that $I(t) \to 0$ as $t \to \infty$ to find it.
- **Part (iv):** Evaluate your result from (iii) at $t = 0$. Notice that the graph of $y = \frac{\sin x}{x}$ is symmetrical across the y-axis (an even function), so the area from $-\infty$ to ∞ is double the area from 0 to ∞.

Solution 11.48

(i) Treat x as constant: $\dfrac{d}{dt}\left(e^{-tx}\dfrac{\sin x}{x}\right) = \dfrac{\sin x}{x}(-xe^{-tx}) = -e^{-tx}\sin x$.
(ii) Let $J = \int_0^\infty e^{-tx}\sin x\,dx$. By parts: $u = e^{-tx}$, $dv = \sin x\,dx$ gives

$$J = \left[-e^{-tx}\cos x\right]_0^\infty - \int_0^\infty (-te^{-tx})(-\cos x)\,dx = 1 - t\int_0^\infty e^{-tx}\cos x\,dx.$$

Apply parts again to $\int e^{-tx}\cos x\,dx$ to get $\int_0^\infty e^{-tx}\cos x\,dx = tJ$. Hence $J(1+t^2) = 1$, so $\boxed{J = \dfrac{1}{t^2+1}}$.

(iii) From (i): $I'(t) = -J = -\dfrac{1}{t^2+1}$. Integrate in t:

$$I(t) = -\arctan t + C.$$

Using $\lim_{t\to\infty} I(t) = 0$ gives $C = \dfrac{\pi}{2}$, so $\boxed{I(t) = \dfrac{\pi}{2} - \arctan t}$.

(iv) Evenness of $\dfrac{\sin x}{x}$ yields

$$\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = 2\int_0^\infty \frac{\sin x}{x}\,dx = 2I(0) = \pi.$$

Answer: $\boxed{\pi}$.

Takeaways 11.48

- **Differentiation Under the Integral:** This advanced technique allows us to solve a difficult constant integral (like $\int \frac{\sin x}{x}dx$) by turning it into a function of a variable t, differentiating that function to simplify the integrand, and then integrating back.
- **Symmetry in Integration:** Recognizing that a function is even (symmetric about the y-axis) is a standard Extension 2 tool to relate integrals over different domains (e.g., transforming a calculation from $0 \to \infty$ into a result for $-\infty \to \infty$).
- **Damping Parameters:** Adding a term like e^{-tx} is a common mathematical strategy to make an integral "behave better" (converge) so we can manipulate it, before setting $t = 0$ at the end to get the original answer.
- Prove that $\lim_{t\to\infty} I(t) = 0$. This is often done by noting that as t becomes very large, the exponential term e^{-tx} decays very quickly, making the entire integrand approach zero for all $x > 0$.

12 Appendices

12.1 Appendix A: Formula Sheet

Standard Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0, a \neq 1)$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \csc x \cot x\,dx = -\csc x + C$$

Inverse Trigonometric Forms

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad (|x| < a)$$
$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C \qquad (|x| > a)$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

LIATE Rule for choosing u:

- **L** Logarithmic ($\ln x$, $\log x$)
- **I** Inverse Trigonometric ($\sin^{-1} x$, $\tan^{-1} x$, etc.)
- **A** Algebraic (x^2, $3x$, etc.)
- **T** Trigonometric ($\sin x$, $\cos x$, etc.)
- **E** Exponential (e^x, a^x)

Reverse Chain Rule

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \qquad \text{(Logarithmic form)}$$
$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C \qquad (n \neq -1)$$

Trigonometric Identities

$\sin^2 x + \cos^2 x = 1$ $\quad$ $1 + \tan^2 x = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$ $\quad$ $\sin 2x = 2\sin x\cos x$
$\cos 2x = \cos^2 x - \sin^2 x$ $\quad$ $\sin^2 x = \frac{1-\cos 2x}{2}$
$\cos^2 x = \frac{1+\cos 2x}{2}$

Definite Integral Properties

- **Odd Function:** If $f(-x) = -f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$
- **Even Function:** If $f(-x) = f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
- **King's Property:** $\displaystyle\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$

12.2 Appendix B: Index of Problems by Technique

Substitution Methods

Basic u-Substitution:

- Part 1, Easy #4: $\int \frac{2x+1}{\sqrt{x^2+x+3}}dx$ (algebraic substitution)
- Part 2, Easy #1: $\int (3x+1)^5 dx$ (linear substitution)
- Part 2, Easy #2: $\int \frac{x}{\sqrt{x^2+4}}dx$ (radical with u-substitution)
- Part 2, Easy #5: $\int \sin^3 2x \cos 2x\,dx$ (trig substitution)
- Part 2, Easy #9: $\int_0^2 x\sqrt{1+x^2}dx$ (definite with substitution)
- Part 2, Easy #13: $\int \frac{6x^2}{(x^3+1)^4}dx$ (reverse chain rule power)

Trigonometric Substitution ($\sqrt{a^2 \pm x^2}$ forms):

- Part 1, Hard #1: Three-part trig substitution with reduction
- Part 2, Medium #3: $\int \frac{1}{\sqrt{6x-x^2}}dx$ (completing square + arcsin)
- Part 2, Medium #5: $\int \sqrt{16-x^2}dx$ (standard $\sqrt{a^2-x^2}$ form)
- Part 2, Hard #1: $\int \frac{x^2}{\sqrt{x^2+9}}dx$ (standard $\sqrt{x^2+a^2}$ form)
- Part 2, Hard #7: $\int \frac{\sqrt{9-x^2}}{x}dx$ (mixed trig substitution)

t-Formula ($t = \tan(x/2)$):

- Part 1, Medium #2: King's property combined with t-formula
- Part 2, Medium #12: $\int_0^{\pi/2} \frac{1}{3+5\cos x} dx$ (standard t-formula)

Substitution Transformation Proofs:

- Part 1, Hard #3: Proving two integrals equal via substitution
- Part 2, Hard #5: $\int_0^1 \ln(1+x)dx = \int_0^1 \frac{x}{1+x} dx$ proof

Integration by Parts

Single Application:

- Part 1, Easy #2: $\int x \ln x \, dx$ (LIATE: logarithm)
- Part 2, Easy #6: $\int xe^x dx$ (algebraic × exponential)
- Part 2, Easy #12: $\int x \cos x \, dx$ (algebraic × trig)
- Part 2, Medium #8: $\int x^2 \ln x \, dx$ (higher power × ln)

Multiple Applications:

- Part 2, Medium #1: $\int x^2 e^x dx$ (by parts twice)
- Part 2, Hard #11: $\int (\ln x)^2 dx$ (nested logarithms)

Cyclic Method:

- Part 1, Medium #5: $\int e^x \sin x \, dx$ using complex numbers
- Part 2, Hard #6: $\int e^x \sin x \, dx$ (traditional cyclic method)
- Part 2, Medium #15: $\int e^x \cos x \, dx$ using Euler's formula

Partial Fractions

Linear Factors:

- Part 1, Easy #1: Linear + irreducible quadratic
- Part 2, Easy #11: $\int \frac{1}{(x-1)(x+2)} dx$ (two linear factors)

Repeated Factors:

- Part 2, Medium #2: $\int \frac{2x+3}{(x-1)^2} dx$ (repeated linear)

Irreducible Quadratics:

- Part 2, Hard #3: $\int \frac{x^3+2x+1}{x^2(x^2+1)} dx$ (quadratic in denominator)
- Part 2, Hard #10: $\int \frac{x^2-3x+5}{(x-1)(x^2+4)} dx$ (linear + quadratic)

Simplification First:

- Part 2, Medium #7: $\int \frac{x^2+1}{(x-1)(x^2+1)} dx$ (cancel common factor)

Reduction Formulae

Derivation and Application:

- Part 1, Medium #1: $I_n = \int \cot^n x \, dx$ (cotangent reduction, 2-part)
- Part 1, Medium #3: $I_n = \int (\ln x)^n dx$ (logarithm reduction)
- Part 2, Medium #6: $I_n = \int_0^{\pi/2} \sin^n x \, dx$ (sine reduction)
- Part 2, Hard #2: $I_n = \int_0^{\pi/2} \cos^n x \, dx$ (cosine reduction with induction)
- Part 2, Hard #13: $I_n = \int \frac{1}{(x^2+1)^n} dx$ (rational reduction)

With Mathematical Induction:

- Part 1, Hard #2: 5-part problem with factorial series and limit proof for e

Reverse Chain Rule

Recognition Patterns:

- Part 1, Easy #3: $\int \left(\frac{1}{x+1} + \frac{2x}{x^2+1}\right) dx$ (ln + arctan)
- Part 2, Easy #3: $\int 2xe^{x^2} dx$ (exponential pattern)
- Part 2, Easy #7: $\int \frac{2x+3}{x^2+3x+1} dx$ (logarithm pattern)
- Part 2, Medium #14: $\int \frac{3x+5}{x^2+4} dx$ (split into ln + arctan)

Trigonometric Techniques

Power-Reduction Identities:

- Part 2, Easy #8: $\int \sin^2 x \, dx$ ($\sin^2 x = \frac{1-\cos 2x}{2}$)
- Part 2, Medium #10: $\int \sin^4 x \, dx$ (double application)

Combined Methods:

- Part 1, Medium #2: King's property + t-formula

Definite Integral Properties

Even/Odd Functions:

- Part 2, Easy #14: $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ for even functions

King's Property ($\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$):

- Part 1, Easy #5: MCQ using King's property
- Part 2, Medium #4: $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$ (King's + t-formula)
- Part 2, Medium #9: $\int_{-1}^{1} \frac{x^2}{1+e^x} dx$ (symmetry with exponential)
- Part 2, Hard #8: $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ (complex denominator)
- Part 2, Hard #15: $\int_0^{\pi/2} \ln(\sin x) dx$ (logarithm with King's)

Volumes of Revolution

Disk Method:

- Part 2, Medium #11: $y = \sqrt{x}$ rotated about x-axis

Washer Method:

- Part 1, Hard #5: Two regions (circle and logarithm) with ratio proof
- Part 2, Hard #4: Between $y = x^2$ and $y = \sqrt{x}$

Shell Method:

- Part 2, Hard #14: $y = x^2$ rotated about y-axis

Mechanics Applications

Particle Motion:

- Part 1, Medium #4: Velocity integration with $F = ma$
- Part 2, Medium #13: Position from velocity with initial conditions

Simple Harmonic Motion (SHM):

- Part 1, Hard #4: Inclined plane with quadratic solution
- Part 2, Hard #12: Amplitude, period, and timing calculations

Special Techniques

Completing the Square:

- Part 2, Easy #15: $\int \frac{1}{x^2+4x+13} dx$ (arctan form)
- Part 2, Medium #3: Combined with substitution for arcsin

Complex Numbers Method:

- Part 1, Medium #5: Using Euler's formula $e^{i\theta}$
- Part 2, Medium #15: $\int e^x \cos x \, dx$ via complex exponentials

Series and Limits:

- Part 1, Hard #2: Factorial series limit proof for e
- Part 2, Hard #9: $I_n = \int_0^1 x^n e^x dx$ with series sum

Standard Forms:

- Part 2, Easy #4: $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan(x/a) + C$
- Part 2, Easy #10: $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin(x/a) + C$

Note: This index helps identify problems by technique. Many problems combine multiple methods—refer to solution strategies for complete technique breakdowns.

12.3 Appendix C: Common Substitutions Guide

Basic U-Substitution

When to use: The integrand contains a function and its derivative (or a constant multiple).
How: Let $u = g(x)$ where $g(x)$ is the inner function. Then $du = g'(x)\,dx$.
Examples:

- $\int x\sqrt{x^2+1}\,dx$: Let $u = x^2 + 1$
- $\int \sin x \cos x\,dx$: Let $u = \sin x$ (or $u = \cos x$)
- $\int \frac{x}{x^2+5}\,dx$: Let $u = x^2 + 5$

Trigonometric Substitutions

For $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$ or $x = a\cos\theta$
Uses: $1 - \sin^2\theta = \cos^2\theta$
Example: $\int \frac{1}{\sqrt{9-x^2}}\,dx$ with $x = 3\sin\theta$
For $\sqrt{a^2 + x^2}$: Let $x = a\tan\theta$
Uses: $1 + \tan^2\theta = \sec^2\theta$
Example: $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ with $x = 2\tan\theta$
For $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$ or $x = a\cosh t$
Uses: $\sec^2\theta - 1 = \tan^2\theta$
Example: $\int x^3\sqrt{x^2-9}\,dx$ with $x = 3\sec\theta$

t-Formula Substitution

When to use: Rational functions of $\sin x$ and $\cos x$
Substitution: $t = \tan\left(\frac{x}{2}\right)$
Formulas:

$$dx = \frac{2}{1+t^2}\,dt$$
$$\sin x = \frac{2t}{1+t^2}$$
$$\cos x = \frac{1-t^2}{1+t^2}$$

Example: $\int \frac{1}{2+\sin x}\,dx$

Exponential and Logarithmic Substitutions

For integrands with e^x: Often let $u = e^x$, then $du = e^x\,dx$
For integrands with $\ln x$: Often use integration by parts with $u = \ln x$
Examples:

- $\int \frac{e^x}{1+e^x}\,dx$: Let $u = 1 + e^x$
- $\int \frac{\ln x}{x}\,dx$: Let $u = \ln x$

Completing the Square

When to use: Quadratic expressions in denominators or under square roots
How: Rewrite $ax^2 + bx + c$ as $a[(x+h)^2 + k]$
Example: $\int \frac{1}{x^2+4x+13}\,dx$
Complete the square: $x^2 + 4x + 13 = (x+2)^2 + 9$
Then use $u = x + 2$ and apply $\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

Quick Reference Table

Integrand Contains	Try Substitution	Uses Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$
$f(x)$ and $f'(x)$	$u = f(x)$	Reverse chain rule
Rational trig	$t = \tan(x/2)$	t-formula identities
e^x and algebra	$u = e^x$	$du = e^x dx$

12.4 Appendix D: Integration by Parts Decision Tree

Integration by Parts Formula

$$\boxed{\int u\,dv = uv - \int v\,du}$$

The LIATE Rule

Choose u according to this priority (top to bottom):

1. **L**ogarithmic functions: $\ln x$, $\log_a x$
2. **I**nverse trigonometric functions: $\sin^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$
3. **A**lgebraic functions: x^n, polynomials
4. **T**rigonometric functions: $\sin x$, $\cos x$, $\tan x$
5. **E**xponential functions: e^x, a^x

The remaining factor becomes dv.

Decision Flowchart

1. **Identify the product:** Is the integrand a product of two different types of functions?
   - If YES → Proceed to step 2
   - If NO → Consider other methods (substitution, partial fractions)
2. **Apply LIATE:** Choose u as the function highest in the LIATE priority list
3. **Determine dv:** The remaining factor (and dx) becomes dv
4. **Can you find v?** Integrate dv to get v
   - If YES → Proceed to step 5

- If NO → Reconsider choice of u and dv

5. **Is $\int v\,du$ simpler?** Compare $\int v\,du$ with the original integral
   - If SIMPLER → Good choice! Proceed with integration
   - If SAME COMPLEXITY → May need to apply parts again
   - If MORE COMPLEX → Try different u and dv

Special Cases

Cyclic Integrals: When $\int v\,du$ returns to the original form
Example: $\int e^x \sin x\,dx$ or $\int e^x \cos x\,dx$
Strategy: Apply integration by parts twice, then solve algebraically for the original integral.
Reduction Formulae: When seeking a recurrence relation I_n in terms of I_{n-1}
Strategy: Choose u and dv to reduce the power of the function.
Definite Integrals: Don't forget to apply limits to uv term!

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

Common Examples

Integral	**Choose u**	**Choose dv**	**Why**
$\int xe^x\,dx$	$u = x$	$dv = e^x dx$	A before E
$\int x \sin x\,dx$	$u = x$	$dv = \sin x\,dx$	A before T
$\int \ln x\,dx$	$u = \ln x$	$dv = dx$	L is highest
$\int x^2 e^x\,dx$	$u = x^2$	$dv = e^x dx$	A before E
$\int \tan^{-1} x\,dx$	$u = \tan^{-1} x$	$dv = dx$	I is high

Tips and Warnings

- **Tip 1:** If $u = \ln x$ or $u = \tan^{-1} x$, set $dv = dx$
- **Tip 2:** For $\int x^n e^{ax} dx$ or $\int x^n \sin(ax) dx$, apply parts n times
- **Tip 3:** Watch your signs! Especially with $v = -\cos x$ or $v = -e^{-x}$
- **Warning:** Don't forget the constant of integration $+C$ for indefinite integrals
- **Warning:** For definite integrals, apply limits to the $[uv]$ term before integrating $\int v\,du$

13 Conclusion

Integration is a cornerstone technique in the HSC Mathematics Extension 2 course. Mastery requires understanding when and how to apply different techniques, recognizing patterns, and practicing extensively. Use these problems to build confidence, develop systematic approaches, and strengthen your ability to communicate complete mathematical solutions. Best of luck with your studies and HSC examinations!

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