HSC Math Extension 2: Inequalities Mastery

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1 Introduction

1.1 Project Overview

This booklet presents a comprehensive collection of inequality problems tailored for the HSC Mathematics Extension 2 syllabus. Each problem explores fundamental techniques including the Arithmetic Mean-Geometric Mean (AM-GM) inequality, Cauchy-Schwarz inequality, triangle inequality, integration-based inequalities, and inequalities via mathematical induction. Through rigorous proofs and detailed solutions, students will develop advanced problem-solving skills essential for Extension 2 examinations and mathematical competitions.

1.2 Target Audience

This resource is designed for Extension 2 students who want to challenge themselves with difficult problems and develop mastery of inequality techniques. Each solution provides step-by-step reasoning, explicit identification of key theorems, and clear algebraic manipulations to ensure high-school learners can follow every logical transition.

1.3 How to Use This Booklet

- Read the fundamentals and worked examples below before attempting problems.
- Attempt Part 1 problems without hints; compare your solutions against the detailed explanations.
- Study the **Takeaways** sections to understand the key techniques and strategies.
- Use the upside-down hints in Part 2 only after making a genuine attempt.
- Practice multiple problems of each type to reinforce pattern recognition and proof techniques.

1.4 Inequality Fundamentals

1.4.1 Basic Properties

For real numbers a, b, c, and d, the following properties hold:

- **Transitivity:** $a > b$ and $b > c \Rightarrow a > c$
- **Multiplication:** $a > b$ and $c > 0 \Rightarrow ac > bc$; but $c < 0 \Rightarrow ac < bc$
- **Product rule:** $a > b > 0$ and $c > d > 0 \Rightarrow ac > bd$
- **Reciprocals:** $a > b > 0 \Rightarrow \dfrac{1}{a} < \dfrac{1}{b}$

- **Non-negativity:** For any real x, we have $x^2 \geq 0$
- **Absolute values:** $|a| \geq a$ and $|x| + |y| \geq |x + y|$ (triangle inequality)
- **Sum of squares:** $a^2 + b^2 \geq 0$ with equality if and only if $a = b = 0$

1.4.2 Key Theorems

AM-GM Inequality (Two Variables). For non-negative real numbers x and y:

$$\frac{x+y}{2} \geq \sqrt{xy}$$

with equality if and only if $x = y$.

AM-GM Inequality (Three Variables). For non-negative real numbers x, y, and z:

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$

with equality if and only if $x = y = z$.

Remark 1.1 (Two Approaches to Proving AM-GM)

The AM-GM inequality states that for positive reals $x_1, x_2, \ldots, x_n$:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

(a) Proof by Induction on n:

1. *Base case:* $n = 2$ follows from $(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0$
2. *Forward-backward step:* Prove $n = 2^k \Rightarrow n = 2^{k+1}$ by grouping pairs
3. *Backward step:* Show $n = k \Rightarrow n = k - 1$ by setting $x_k = \frac{x_1 + \cdots + x_{k-1}}{k-1}$ and applying the $n = k$ case

(b) Proof using Convex Functions:

1. Consider $f(x) = -\ln(x)$, which is convex for $x > 0$ (since $f''(x) = \frac{1}{x^2} > 0$)
2. By Jensen's inequality for convex functions:

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

3. Substituting $f(x) = -\ln(x)$:

$$-\ln\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{-\ln(x_1) - \cdots - \ln(x_n)}{n} = -\ln(\sqrt[n]{x_1 \cdots x_n})$$

4. Multiply by -1 and exponentiate to obtain AM-GM

Cauchy-Schwarz Inequality. For real numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

with equality if and only if the sequences are proportional.

Proof outline (discriminant method): Let $A = \sum a_i^2$, $B = \sum b_i^2$, $C = \sum a_i b_i$, and consider

$$P(t) = \sum_{i=1}^{n} (a_i^2 t^2 - 2a_i b_i t + b_i^2) = At^2 - 2Ct + B \ge 0 \text{ for all } t.$$

Since $P(t)$ is non-negative, its discriminant satisfies $\Delta = (-2C)^2 - 4AB \le 0$, hence $C^2 \le AB$. Substituting back yields $\left(\sum a_i b_i\right)^2 \le \left(\sum a_i^2\right)\left(\sum b_i^2\right)$. Equality holds iff $\Delta = 0$, i.e., there exists t_0 with $a_i t_0 = b_i$ for all i (proportional sequences).

Jensen's Inequality. Let f be a convex function on an interval I. For points $x_1, \dots, x_n \in I$ and non-negative weights $\lambda_1, \dots, \lambda_n$ with $\sum_{i=1}^{n} \lambda_i = 1$, Jensen's inequality states

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \le \sum_{i=1}^{n} \lambda_i f(x_i).$$

If f is concave, the inequality is reversed. Equality holds when the x_i are all equal or when f is linear on the convex hull of the points.

Remark: This is a so powerful Inequality and you should use it with care. To use Jensen's inequality, prove a function $f(x)$ is convex by showing $f''(x) \ge 0$ on the interval of interest, then state clearly "By Jensen's Inequality, since $f(x)$ is convex...". The risk is you many loose "working marks" if the markers feels you bypassed the intended "nature of proof" technique.

Triangle Inequality (Real Numbers). For real numbers a and b:

$$|a + b| \le |a| + |b|$$

Triangle Inequality (Complex Numbers). For complex numbers z and w:

$$|z + w| \le |z| + |w|$$

Notes: (i) The polygonal inequality generalizes the triangle inequality to sums of more than two terms by repeated application. (ii) The geometric interpretation is that in any triangle, the length of one side is less than or equal to the sum of the lengths of the other two sides, which can be applied in 2-D and 3-D geometry problems.

1.5 Worked Examples

Example 1: Basic AM-GM Application. Prove that for positive real numbers a and b, we have $a + b \ge 2\sqrt{ab}$.

Solution: By the AM-GM inequality with $x = a$ and $y = b$:

$$\frac{a+b}{2} \ge \sqrt{ab}$$

Multiplying both sides by 2 yields $a + b \ge 2\sqrt{ab}$, with equality when $a = b$. $\square$

Example 2: Cauchy-Schwarz with Constraint. Given $x^2 + y^2 = 1$, find the maximum value of $3x + 4y$.

Solution: By Cauchy-Schwarz inequality:

$$(3x + 4y)^2 \leq (3^2 + 4^2)(x^2 + y^2) = 25 \cdot 1 = 25$$

Therefore $3x + 4y \leq 5$, with equality when $(x, y) = \left(\frac{3}{5}, \frac{4}{5}\right)$. The maximum value is 5. $\square$

Example 3: Triangle Inequality. Prove that for any complex number z with $|z| = 1$, we have $|z^2 + z + 1| \leq 3$.

Solution: Using the triangle inequality repeatedly:

$$|z^2 + z + 1| \leq |z^2| + |z| + |1| = 1 + 1 + 1 = 3$$

with equality when $z = 1$. $\square$

Example 4: Integration Inequality. Prove that $\displaystyle\int_0^1 x^2\, dx < \int_0^1 x\, dx$.

Solution: For $x \in [0, 1]$, we have $x^2 \leq x$ (with equality only at $x = 0$ and $x = 1$). Therefore:

$$\int_0^1 x^2\, dx < \int_0^1 x\, dx$$

Example 5: Induction with Inequality. Prove by induction that $2^n > n$ for all integers $n \geq 1$.

Solution: Base case ($n = 1$): $2^1 = 2 > 1$. ✓

Inductive step: Assume $2^k > k$ for some $k \geq 1$. Then:

$$2^{k+1} = 2 \cdot 2^k > 2k$$

Since $k \geq 1$, we have $2k = k + k \geq k + 1$. Therefore $2^{k+1} > k + 1$. By induction, the result holds for all $n \geq 1$. $\square$

1.6 Notation and Conventions

- Unless stated otherwise, variables represent real numbers.
- The notation $a, b > 0$ means both a and b are positive.
- "Prove" indicates a complete justification is required.
- "Hence" or "deduce" means use the previous result directly.
- Equality conditions identify when an inequality becomes an equality.

2 Part 1: Problems and Solutions (Detailed)

Part 1 contains 15 carefully selected problems—five basic, five medium, and five advanced. Each problem includes a comprehensive solution with step-by-step reasoning and a **Takeaways** section highlighting key techniques and strategic insights.

2.1 Basic Inequality Problems

Problem 2.1: Arithmetic Mean-Geometric Mean Inequality

For positive real numbers a and b, prove that $\frac{a+b}{2} \geq \sqrt{ab}$.
Hence, or otherwise, show that $\frac{2n+1}{2n+2} < \frac{\sqrt{2n+1}}{\sqrt{2n+3}}$ for any integer $n \geq 0$.

Solution 2.1

Part (i): Since a and b are positive real numbers, $\sqrt{a}$ and $\sqrt{b}$ are real numbers. We know that the square of any real number is non-negative:

$$(\sqrt{a}-\sqrt{b})^2 \geq 0$$

Expanding the left side:

$$\begin{aligned}(\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0\\ a - 2\sqrt{ab} + b &\geq 0\end{aligned}$$

Adding $2\sqrt{ab}$ to both sides:

$$a + b \geq 2\sqrt{ab}$$

Dividing both sides by 2:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

This is the Arithmetic Mean-Geometric Mean (AM-GM) Inequality. Equality holds if and only if $a = b$.

Part (ii): We want to show that $\frac{2n+1}{2n+2} < \frac{\sqrt{2n+1}}{\sqrt{2n+3}}$.

Since all terms are positive for $n \geq 0$, we can square both sides without changing the direction of the inequality:

$$\left(\frac{2n+1}{2n+2}\right)^2 < \left(\frac{\sqrt{2n+1}}{\sqrt{2n+3}}\right)^2$$

$$\frac{(2n+1)^2}{(2n+2)^2} < \frac{2n+1}{2n+3}$$

Since $2n+1 > 0$, we can divide both sides by $2n+1$:

$$\frac{2n+1}{(2n+2)^2} < \frac{1}{2n+3}$$

Cross-multiplying:

$$(2n+1)(2n+3) < (2n+2)^2$$

Expanding both sides:

$$\begin{aligned}4n^2 + 6n + 2n + 3 &< 4n^2 + 8n + 4\\ 4n^2 + 8n + 3 &< 4n^2 + 8n + 4\end{aligned}$$

Simplifying:

$$3 < 4$$

Since $3 < 4$ is always true, the original inequality holds for any integer $n \geq 0$.

Takeaways 2.1

- **Key Technique:** The AM-GM inequality is proven by considering the non-negativity of a perfect square: $(\sqrt{a} - \sqrt{b})^2 \geq 0$.
- **Strategy:** When proving inequalities involving fractions with square roots, squaring both sides can simplify the expression while preserving the inequality direction (provided all terms are positive).
- **Cross-Multiplication:** After squaring and simplification, cross-multiplication converts the inequality to a polynomial form that can be verified directly.
- **Common Pitfall:** When squaring inequalities, always verify that all terms are positive; otherwise, the inequality direction may reverse.
- **Verification:** Reducing the problem to a simple numerical inequality (like $3 < 4$) provides a complete and rigorous proof.

Problem 2.2: AM-GM with Non-Negative Reals

For real numbers $a, b \geq 0$, prove that:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Solution 2.2

Since a and b are non-negative real numbers, $\sqrt{a}$ and $\sqrt{b}$ are real numbers. We know that the square of any real number is always non-negative. Therefore, we begin with:

$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$

Expanding the square:

$$\begin{aligned}(\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0 \\ a - 2\sqrt{ab} + b &\geq 0\end{aligned}$$

Rearranging terms:

$$a + b \geq 2\sqrt{ab}$$

Dividing both sides by 2:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Thus, the inequality is proven. Note that equality holds if and only if $(\sqrt{a} - \sqrt{b})^2 = 0$, which implies $a = b$.

Takeaways 2.2

- **Key Technique:** The AM-GM inequality for non-negative reals follows directly from the non-negativity of $(\sqrt{a} - \sqrt{b})^2$.
- **Equality Condition:** Equality holds when $a = b$, which occurs when the squared difference is zero.
- **Domain Consideration:** The requirement that $a, b \geq 0$ ensures that $\sqrt{a}$ and $\sqrt{b}$ are real numbers.
- **Common Application:** This fundamental inequality is frequently used as a stepping stone in more complex inequality proofs.
- **Algebraic Manipulation:** The proof demonstrates how to systematically expand, rearrange, and isolate terms to establish the desired inequality.

Problem 2.3: Logarithmic Inequalities and Euler's Number

Explain why

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}.$$

Hence, deduce that

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$$

Solution 2.3

Part 1: For $f(x) = \frac{1}{x}$ strictly decreasing on $[n, n+1]$, we have $\frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n}$. Integrating:

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n} \implies \frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

Thus $\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$.

Part 2: Left: Multiply by $n+1$: $1 < \ln\left[\left(1 + \frac{1}{n}\right)^{n+1}\right] \implies e < \left(1 + \frac{1}{n}\right)^{n+1}$.

Right: Multiply by n: $\ln\left[\left(1 + \frac{1}{n}\right)^n\right] < 1 \implies \left(1 + \frac{1}{n}\right)^n < e$.

Therefore: $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$

Takeaways 2.3

- **Key Technique:** Using the monotonicity of $f(x) = \frac{1}{x}$ to bound a definite integral by rectangles is a standard calculus technique.
- **Integration Bounds:** For decreasing functions, the minimum value on an interval provides a lower bound for the integral, while the maximum value provides an upper bound.
- **Logarithm Properties:** The transformation $\ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right)$ is crucial for connecting the integral to the exponential form.
- **Exponentiation Preserves Inequality:** Since e^x is an increasing function, exponentiating both sides of $\ln(A) < \ln(B)$ gives $A < B$.
- **Historical Significance:** This inequality provides a rigorous way to bound Euler's number e using sequences, demonstrating the limit definition $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$.

Problem 2.4: Squared Terms Inequality

For $x, y > 0$, prove that:

(a) $x^2 + y^2 \geq 2xy$

(b) $\frac{1}{x^4} + \frac{1}{y^4} \geq \frac{2}{x^2y^2}$

Solution 2.4

Part (a): We start with the fundamental property that the square of any real number is non-negative. Consider the square of the difference between x and y:

$$(x - y)^2 \geq 0$$

Expanding the left-hand side:

$$x^2 - 2xy + y^2 \geq 0$$

Adding $2xy$ to both sides:

$$x^2 + y^2 \geq 2xy$$

This proves the inequality for all real x, y. Since $x, y > 0$, the inequality holds. Equality occurs when $(x - y)^2 = 0$, which means $x = y$.

Part (b): We can deduce part (b) by using the result from part (a). Let us substitute terms into the inequality $a^2 + b^2 \geq 2ab$.

Let $a = \frac{1}{x^2}$ and $b = \frac{1}{y^2}$. Since $x, y > 0$, both a and b are positive real numbers.

Using the result from part (a):

$$a^2 + b^2 \geq 2ab$$
$$\left(\frac{1}{x^2}\right)^2 + \left(\frac{1}{y^2}\right)^2 \geq 2\left(\frac{1}{x^2}\right)\left(\frac{1}{y^2}\right)$$
$$\frac{1}{x^4} + \frac{1}{y^4} \geq \frac{2}{x^2y^2}$$

This completes the proof.

Takeaways 2.4

- **Key Technique:** Many quadratic inequalities can be proven by starting with $(x - y)^2 \geq 0$ and expanding.
- **Substitution Strategy:** Part (b) demonstrates how to generalize an inequality by making appropriate substitutions ($a = \frac{1}{x^2}$, $b = \frac{1}{y^2}$).
- **Building on Results:** Using a proven result (part a) to establish a new inequality (part b) is a powerful problem-solving technique.
- **Equality Condition:** For part (a), equality holds when $x = y$; for part (b), equality holds when $\frac{1}{x^2} = \frac{1}{y^2}$, which also means $x = y$.
- **Common Pitfall:** When making substitutions, ensure that the new variables satisfy the same positivity conditions required by the original inequality.

Problem 2.5: Cauchy-Schwarz Inequality Application

Let x, y, z be real numbers satisfying the linear equation $x + 2y + 3z = 14$.

(i) Prove that $x^2 + y^2 + z^2 \geq 14$.

(ii) Determine the values of x, y, z for which equality holds.

Solution 2.5

Part (i): We apply the Cauchy-Schwarz Inequality to vectors $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (x, y, z)$.
The Cauchy-Schwarz Inequality states:

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq |\mathbf{u}|^2 |\mathbf{v}|^2$$

In component form:

$$(1 \cdot x + 2 \cdot y + 3 \cdot z)^2 \leq (1^2 + 2^2 + 3^2)(x^2 + y^2 + z^2)$$

Calculate the squared magnitude of $\mathbf{u}$:

$$1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

Substitute the given constraint $x + 2y + 3z = 14$:

$$\begin{aligned} (14)^2 &\leq 14(x^2 + y^2 + z^2) \\ 196 &\leq 14(x^2 + y^2 + z^2) \\ 14 &\leq x^2 + y^2 + z^2 \end{aligned}$$

Therefore, $x^2 + y^2 + z^2 \geq 14$.
Part (ii): Equality in the Cauchy-Schwarz Inequality holds if and only if the vectors $\mathbf{u}$ and $\mathbf{v}$ are proportional. That is:

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = k$$

for some scalar k. Thus, $x = k$, $y = 2k$, and $z = 3k$.
Substitute these into the constraint equation:

$$\begin{aligned} k + 2(2k) + 3(3k) &= 14 \\ k + 4k + 9k &= 14 \\ 14k &= 14 \\ k &= 1 \end{aligned}$$

Therefore, equality holds when $x = 1$, $y = 2$, $z = 3$.
We can verify: $x^2+y^2+z^2 = 1^2+2^2+3^2 = 1+4+9 = 14$ and $x+2y+3z = 1+4+9 = 14$.

Takeaways 2.5

- **Key Technique:** The Cauchy-Schwarz Inequality is a powerful tool for proving inequalities involving sums of products and sums of squares.
- **Vector Interpretation:** Recognizing the problem as a dot product $\mathbf{u} \cdot \mathbf{v} = 14$ allows us to apply the Cauchy-Schwarz Inequality directly.
- **Equality Condition:** For Cauchy-Schwarz, equality holds if and only if the vectors are proportional, providing a systematic method to find when the minimum is achieved.
- **Verification:** Always verify the equality case by substituting back into both the constraint and the inequality.
- **Common Application:** This technique extends to constrained optimization problems where you need to minimize or maximize a quadratic form subject to a linear constraint.

2.2 Medium Inequality Problems

Problem 2.6: Arithmetic Sequence of Reciprocals

Positive real numbers a, b, c and d are chosen such that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ and $\frac{1}{d}$ are consecutive terms in an arithmetic sequence with common difference k, where $k \in \mathbb{R}$, $k > 0$.
Show that $b + c < a + d$.

Solution 2.6

Since $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are consecutive terms in an arithmetic sequence with common difference $k > 0$, we have:

$$\frac{1}{b} = \frac{1}{a} + k \quad \Longrightarrow \quad k = \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab} \tag{1}$$
$$\frac{1}{c} = \frac{1}{b} + k \quad \Longrightarrow \quad k = \frac{1}{c} - \frac{1}{b} = \frac{b-c}{bc} \tag{2}$$
$$\frac{1}{d} = \frac{1}{c} + k \quad \Longrightarrow \quad k = \frac{1}{d} - \frac{1}{c} = \frac{c-d}{cd} \tag{3}$$

Since $k > 0$ and a, b, c, d are positive, the numerators must be positive, which implies:

$$\begin{aligned} a - b > 0 \quad &\Longrightarrow \quad a > b \\ b - c > 0 \quad &\Longrightarrow \quad b > c \\ c - d > 0 \quad &\Longrightarrow \quad c > d \end{aligned}$$

Thus, $a > b > c > d$.
We want to show that $b + c < a + d$, which is equivalent to showing $0 < (a + d) - (b + c)$, or $0 < (a - b) - (c - d)$.
From equations (1) and (3), we can express the differences in terms of k:

$$a - b = k \cdot ab \tag{4}$$
$$c - d = k \cdot cd \tag{5}$$

We need to compare $k \cdot ab$ and $k \cdot cd$. Since $k > 0$, the inequality $a - b > c - d$ is equivalent to showing:

$$ab > cd$$

Since we established $a > b > c > d$, it is clear that $a > c$ and $b > d$. Since all are positive:

$$\begin{aligned} a > c > 0 \\ b > d > 0 \end{aligned}$$

Multiplying these two inequalities:

$$ab > cd$$

Multiplying by $k > 0$:

$$k \cdot ab > k \cdot cd$$

Substituting from equations (4) and (5):

$$a - b > c - d$$

Rearranging the terms:

$$a + d > b + c$$

Therefore, $b + c < a + d$ as required.

Takeaways 2.6

- **Technique:** Converting arithmetic sequence conditions into algebraic equations allows us to extract ordering information about the original terms.
- **Strategy:** When dealing with reciprocals in arithmetic progression, recognize that the original terms form a decreasing sequence, and use this monotonicity to compare products.
- **Key Insight:** Express target differences as products with a common positive factor (k), reducing the problem to comparing products of ordered terms.
- **Pitfall:** Don't forget that $k > 0$ is crucial for establishing the ordering $a > b > c > d$; without this, the inequality direction could reverse.

Problem 2.7: Cascading AM-GM Applications

For all non-negative real numbers x and y, $\sqrt{xy} \leq \frac{x+y}{2}$. (Do NOT prove this.)

(i) Using this fact, show that for all non-negative real numbers a, b and c,

$$\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}.$$

(ii) Using part (i), or otherwise, show that for all non-negative real numbers a, b and c,

$$\sqrt{abc} \leq \frac{a^2 + b^2 + c^2 + a + b + c}{6}.$$

Solution 2.7

Part (i): Apply AM-GM to a^2, b^2: $ab \leq \frac{a^2+b^2}{2}$. Apply AM-GM to ab, c:

$$\sqrt{abc} \leq \frac{ab + c}{2} \leq \frac{\frac{a^2+b^2}{2} + c}{2} = \frac{a^2 + b^2 + 2c}{4}$$

Part (ii): By cyclic permutation of (i):

$$\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}, \quad \sqrt{abc} \leq \frac{b^2 + c^2 + 2a}{4}, \quad \sqrt{abc} \leq \frac{c^2 + a^2 + 2b}{4}$$

Adding: $3\sqrt{abc} \leq \frac{2(a^2+b^2+c^2+a+b+c)}{4} = \frac{a^2+b^2+c^2+a+b+c}{2}$.
Thus: $\sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}$

Takeaways 2.7

- **Technique:** Cascade AM-GM applications by strategically choosing pairs of terms, then use the resulting inequality as input for another AM-GM application.
- **Strategy:** When proving symmetric inequalities, exploit cyclic symmetry by generating multiple versions of an intermediate result and summing them.
- **Key Insight:** The transition from part (i) to part (ii) demonstrates how adding symmetric inequalities can yield a stronger bound with better balance among variables.
- **Common Pattern:** Notice that $\frac{a^2+b^2}{2} \geq ab$ is a specific application of AM-GM to squares, which often serves as a useful intermediate step.

Problem 2.8: Inductive Sum of Squared Reciprocals

Prove by mathematical induction that, for $n \geq 2$,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$

Solution 2.8

Base Case ($n = 2$): LHS $= \frac{1}{4}$, RHS $= \frac{1}{2}$. Since $\frac{1}{4} < \frac{1}{2}$, true for $n = 2$.
Inductive Hypothesis: Assume $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < \frac{k-1}{k}$ (*)
Inductive Step: Need to show $\frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < \frac{k}{k+1}$.
Using (*): LHS $< \frac{k-1}{k} + \frac{1}{(k+1)^2}$.
The gap is: $\frac{k}{k+1} - \frac{k-1}{k} = \frac{1}{k(k+1)}$
Need: $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} \iff k(k+1) < (k+1)^2 \iff k < k+1$.
Thus: $\frac{k-1}{k} + \frac{1}{(k+1)^2} < \frac{k}{k+1}$
By induction, the inequality holds for all $n \geq 2$.

Takeaways 2.8

- **Technique:** In inductive proofs of inequalities, compute the "gap" between successive right-hand sides to determine what bound is needed on the new term.
- **Strategy:** Show that the additional term $\frac{1}{(k+1)^2}$ is strictly less than the increase in the RHS from $\frac{k-1}{k}$ to $\frac{k}{k+1}$.
- **Key Insight:** The inequality $k(k+1) < (k+1)^2$ (equivalently $k < k+1$) is the critical comparison that makes the inductive step work.
- **Pitfall:** Don't assume the new term is small enough without verification; always explicitly show the required inequality between the new term and the gap in the RHS.

Problem 2.9: Power Mean Inequality via QM-RMS

For positive real numbers x and y, $\sqrt{xy} \leq \frac{x+y}{2}$. (Do NOT prove this.)

(i) Prove $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$, for positive real numbers x and y.

(ii) Prove $\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$, for positive real numbers a, b, c and d.

Solution 2.9

Part (i): From $(x-y)^2 \geq 0$: $x^2+y^2 \geq 2xy \implies \frac{x^2+y^2}{2} \geq xy$.
Taking square roots: $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$
Part (ii): Note $\sqrt[4]{abcd} = \sqrt{\sqrt{ab} \cdot \sqrt{cd}}$. Apply Part (i) to pairs (a, b) and (c, d):

$$\sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}}, \quad \sqrt{cd} \leq \sqrt{\frac{c^2+d^2}{2}}$$

Let $X = \sqrt{\frac{a^2+b^2}{2}}$, $Y = \sqrt{\frac{c^2+d^2}{2}}$. Then $\sqrt{abcd} \leq XY$.
Apply Part (i) to X, Y: $XY \leq \sqrt{\frac{X^2+Y^2}{2}} = \sqrt{\frac{\frac{a^2+b^2+c^2+d^2}{2}}{2}} = \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$
Therefore: $\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$

Takeaways 2.9

- **Technique:** The inequality $(x-y)^2 \geq 0$ is a fundamental tool for proving that the quadratic mean (RMS) dominates the geometric mean.
- **Strategy:** Build up to higher-order inequalities by pairing terms and applying proven results recursively; here we go from 2 terms to 4 terms.
- **Key Insight:** The relationship $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$ (GM $\leq$ QM) serves as a bridge to extend AM-GM style inequalities to power means.
- **Common Pattern:** When dealing with fourth roots of products, rewrite as square roots of square roots, then apply two-variable inequalities twice.

Problem 2.10: Calculus and Induction for Harmonic Inequality

(i) Use calculus to show that $x > \ln(1+x)$ for all $x > 0$.

(ii) Use the inequality in part (i) and the principle of mathematical induction to prove that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(1+n)$$

for all positive integers, n.

Solution 2.10

Part (i): Let $f(x) = x - \ln(1+x)$. Then $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ for $x > 0$.
Since $f(0) = 0$ and f is strictly increasing for $x > 0$, we have $f(x) > 0$ for all $x > 0$.
Therefore, $x > \ln(1+x)$ for all $x > 0$.
Part (ii): Base ($n = 1$): LHS $= 1$, RHS $= \ln(2) \approx 0.693$. Since $1 > \ln(2)$, true.
Hypothesis: Assume $\sum_{r=1}^{k} \frac{1}{r} > \ln(1+k)$ (*)
Step: Using (*): LHS $> \ln(k+1) + \frac{1}{k+1}$.
From Part (i) with $x = \frac{1}{k+1}$:

$$\frac{1}{k+1} > \ln\left(1 + \frac{1}{k+1}\right) = \ln\left(\frac{k+2}{k+1}\right) = \ln(k+2) - \ln(k+1)$$

Thus: LHS $> \ln(k+1) + [\ln(k+2) - \ln(k+1)] = \ln(k+2)$
By induction, the inequality holds for all positive integers n.

Takeaways 2.10

- **Technique:** Use calculus to establish a continuous inequality, then leverage it as a lemma in an inductive proof for a discrete sum.
- **Strategy:** The key connection is recognizing that $\frac{1}{k+1} > \ln(k+2) - \ln(k+1)$ allows us to bridge from $\ln(k+1)$ to $\ln(k+2)$ in the inductive step.
- **Key Insight:** The logarithm property $\ln(a) - \ln(b) = \ln(a/b)$ is crucial for converting the Part (i) inequality into a form usable in the induction.
- **Common Pattern:** When proving inequalities involving harmonic sums and logarithms, calculus-based lemmas about $\ln(1+x)$ frequently serve as bridges between consecutive cases.
- **Pitfall:** Don't forget to verify that the substitution $x = \frac{1}{k+1}$ satisfies the domain condition $x > 0$ required for Part (i) to apply.

2.3 Advanced Inequality Problems

Problem 2.11: Exponential Bounds on Factorials

(i) Prove that $x > \ln x$, for $x > 0$.

(ii) Using part (i), or otherwise, prove that for all positive integers n,

$$e^{n^2+n} > (n!)^2.$$

Solution 2.11

Part (i): Let $f(x) = x - \ln x$. Then $f'(x) = 1 - \frac{1}{x} = 0 \implies x = 1$.
Since $f''(x) = \frac{1}{x^2} > 0$, f has a minimum at $x = 1$ with $f(1) = 1 - 0 = 1 > 0$.
Therefore, $x > \ln x$ for all $x > 0$.
Part (ii): Apply ln to both sides: $e^{n^2+n} > (n!)^2 \iff n^2 + n > 2\ln(n!) \iff \frac{n^2+n}{2} > \sum_{k=1}^{n} \ln k$
From part (i), $k > \ln k$ for all positive integers k. Summing from $k = 1$ to n:

$$\sum_{k=1}^{n} k > \sum_{k=1}^{n} \ln k \implies \frac{n(n+1)}{2} = \frac{n^2+n}{2} > \sum_{k=1}^{n} \ln k$$

Exponentiating: $e^{n^2+n} > (n!)^2$

Takeaways 2.11

- **Calculus technique:** Use first and second derivatives to find and classify critical points, then evaluate the function at the critical point to determine global behavior.
- **Summation strategy:** Apply a single inequality to multiple values, then sum all inequalities together to obtain a cumulative result.
- **Logarithmic transformation:** Convert multiplicative inequalities to additive ones using logarithms, which simplifies the analysis.
- **Building block approach:** Use the result from a simpler part to prove a more complex statement in subsequent parts.
- **Common pitfall:** Don't forget to verify that the critical point is indeed a minimum (not a maximum or inflection point) by checking the second derivative or analyzing the sign of the first derivative around the critical point.

Problem 2.12: Sphere Inequalities via Vector Methods

The point $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin O.

(i) Using the position vector of P, $\vec{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and the triangle inequality, or otherwise, show that $|x| + |y| + |z| \geq 1$.

(ii) Given the vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, show that

$$|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}.$$

(iii) Using part (ii), or otherwise, show that $|x| + |y| + |z| \leq \sqrt{3}$.

Solution 2.12

Part (i): Show that $|x| + |y| + |z| \geq 1$
Since $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin, the magnitude of the position vector $\vec{OP}$ is 1:

$$|\vec{OP}| = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = 1$$

Applying the Triangle Inequality for vectors $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ repeatedly:

$$\begin{aligned} |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| &\leq |x\mathbf{i}| + |y\mathbf{j}| + |z\mathbf{k}| \\ 1 &\leq |x||\mathbf{i}| + |y||\mathbf{j}| + |z||\mathbf{k}| \end{aligned}$$

Since the unit vectors have magnitude 1: $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$:

$$1 \leq |x| + |y| + |z|$$

Therefore, $|x| + |y| + |z| \geq 1$.

Part (ii): Prove the Cauchy-Schwarz inequality
We use the definition of the scalar (dot) product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between the vectors.
Taking the absolute value of both sides:

$$\begin{aligned} |\mathbf{a} \cdot \mathbf{b}| &= ||\mathbf{a}||\mathbf{b}| \cos\theta| \\ |\mathbf{a} \cdot \mathbf{b}| &= |\mathbf{a}||\mathbf{b}|| \cos\theta| \end{aligned}$$

Since $-1 \leq \cos\theta \leq 1$, we know that $|\cos\theta| \leq 1$.
Therefore:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

Substituting the component forms of the vectors:

$$|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$$

Part (iii): Show that $|x| + |y| + |z| \leq \sqrt{3}$
We define two specific vectors to apply the Cauchy-Schwarz inequality from Part (ii):
Let $\mathbf{a} = \begin{pmatrix} |x| \\ |y| \\ |z| \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
Applying the result from (ii):

$$\begin{aligned} |(|x|)(1) + (|y|)(1) + (|z|)(1)| &\leq \sqrt{|x|^2 + |y|^2 + |z|^2}\sqrt{1^2 + 1^2 + 1^2} \\ |x| + |y| + |z| &\leq \sqrt{x^2 + y^2 + z^2} \cdot \sqrt{3} \end{aligned}$$

From the problem statement, P is on the unit sphere, so $x^2 + y^2 + z^2 = 1$.
Therefore:

$$\begin{aligned} |x| + |y| + |z| &\leq \sqrt{1} \cdot \sqrt{3} \\ |x| + |y| + |z| &\leq \sqrt{3} \end{aligned}$$

Takeaways 2.12

- **Triangle inequality:** For vectors, $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$ provides lower bounds on sums of absolute values.
- **Cauchy-Schwarz application:** This fundamental inequality relates dot products to vector magnitudes and provides upper bounds on sums.
- **Strategic vector choice:** Choose specific vectors (like the all-ones vector) to convert Cauchy-Schwarz into the desired form.
- **Multi-part coordination:** Each part builds toward the final result; part (i) establishes a lower bound, part (ii) proves a general tool, and part (iii) applies it for an upper bound.
- **Common pitfall:** Remember that $|x|^2 = x^2$, so the constraint $x^2 + y^2 + z^2 = 1$ directly gives $\sqrt{|x|^2 + |y|^2 + |z|^2} = 1$.

Problem 2.13: Logarithmic Inequalities and the Limit Definition of e

Suppose that $x \ge 0$ and n is a positive integer.

(i) Show that

$$1 - x \le \frac{1}{1+x} \le 1.$$

(ii) Hence, or otherwise, show that

$$1 - \frac{1}{2n} \le n \ln\left(1 + \frac{1}{n}\right) \le 1.$$

(iii) Hence, explain why

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$$

Solution 2.13

Part (i): Right: Since $x \geq 0$: $1 + x \geq 1 \implies \frac{1}{1+x} \leq 1$.
Left: From $(1-x)(1+x) = 1 - x^2 \leq 1$ and $1 + x > 0$: $1 - x \leq \frac{1}{1+x}$.
Thus: $1 - x \leq \frac{1}{1+x} \leq 1$
Part (ii): Integrate the inequality from 0 to $\frac{1}{n}$:

$$\int_0^{1/n} (1-t)\,dt \leq \int_0^{1/n} \frac{1}{1+t}\,dt \leq \int_0^{1/n} 1\,dt$$

Evaluating: $\frac{1}{n} - \frac{1}{2n^2} \leq \ln\left(1 + \frac{1}{n}\right) \leq \frac{1}{n}$
Multiply by n: $1 - \frac{1}{2n} \leq n \ln\left(1 + \frac{1}{n}\right) \leq 1$
Part (iii): Taking limits: $\lim_{n\to\infty}\left(1 - \frac{1}{2n}\right) = 1 = \lim_{n\to\infty}(1)$
By Squeeze Theorem: $\lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right) = 1 \implies \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n = 1$
Exponentiating: $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$

Takeaways 2.13

- **Integration of inequalities:** Integrating all parts of a valid inequality preserves the inequality relation and is a powerful technique for deriving new bounds.
- **Squeeze Theorem:** When a function is bounded above and below by functions that converge to the same limit, the middle function must also converge to that limit.
- **Logarithm-exponential interplay:** Use logarithms to convert powers to products, then exponentiate to recover the original form after taking limits.
- **Progressive refinement:** Each part provides a tool or bound that is then used in subsequent parts to build toward the final result.
- **Common pitfall:** When integrating, don't forget to evaluate the definite integral at both bounds and subtract correctly. Also, remember that $\ln(a^n) = n\ln(a)$ when moving between forms.

Problem 2.14: Homogeneous Inequality via Substitution and AM-GM

Prove

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{1}{yz}\sqrt{x^2y^2 + z^4} + \frac{1}{xz}\sqrt{z^2y^2 + x^4} \quad \text{for } x, y, z > 0.$$

Solution 2.14

Step 1: Simplify RHS:

$$\frac{1}{yz}\sqrt{x^2y^2+z^4}=\sqrt{\frac{x^2}{z^2}+\frac{z^2}{y^2}},\quad \frac{1}{xz}\sqrt{z^2y^2+x^4}=\sqrt{\frac{y^2}{x^2}+\frac{x^2}{z^2}}$$

Inequality becomes: $\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq\sqrt{\frac{x^2}{z^2}+\frac{z^2}{y^2}}+\sqrt{\frac{y^2}{x^2}+\frac{x^2}{z^2}}$
Step 2: Let $u=\frac{x}{y}$, $v=\frac{y}{z}$, $w=\frac{z}{x}$ (note: $uvw=1$). Then:

$$L=u^2+v^2+w^2,\quad R=u\sqrt{v^2+w^2}+v\sqrt{w^2+u^2}$$

Step 3: Apply AM-GM: $\sqrt{AB}\leq\frac{A+B}{2}$:

$$u\sqrt{v^2+w^2}\leq\frac{u^2+(v^2+w^2)}{2},\quad v\sqrt{w^2+u^2}\leq\frac{v^2+(w^2+u^2)}{2}$$

Adding: $R\leq\frac{2(u^2+v^2+w^2)}{2}=u^2+v^2+w^2=L$

Takeaways 2.14

- **Homogeneous substitution:** For homogeneous inequalities, substitute ratios of variables (like $u=x/y$) to reduce the number of variables and simplify the problem.
- **Algebraic simplification:** Move constants in and out of square roots systematically to reveal the underlying structure.
- **AM-GM strategy:** Use AM-GM on products under square roots: $\sqrt{AB}\leq\frac{A+B}{2}$ is particularly useful when the sum $A+B$ appears elsewhere.
- **Summation technique:** When applying AM-GM to multiple terms, add the resulting inequalities to obtain the final bound.
- **Common pitfall:** Verify that the substitution constraint (like $uvw=1$) is satisfied; this ensures the substitution is valid and the problem hasn't been changed.

Problem 2.15: Bernoulli's Inequality and Sequence Monotonicity

Bernoulli's Inequality: For any real number $x>-1$ and integer $r\geq 0$,

$$(1+x)^r\geq 1+rx,$$

with strict inequality if $x\neq 0$ and $r>1$. The equality holds if $x=0$ or $r=0$ or $r=1$. (do NOT prove this)
Using Bernoulli's Inequality, prove that the sequence $a_n=\left(1+\frac{1}{n}\right)^n$ is strictly increasing for integers $n\geq 1$. Specifically, prove:

$$\left(1+\frac{1}{n}\right)^n<\left(1+\frac{1}{n+1}\right)^{n+1}$$

Solution 2.15

To prove the sequence is increasing, we show: $\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} > 1$

Step 1: Rewrite:

$$\frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^{n}} = \left(\frac{(n+2)n}{(n+1)^2}\right)^{n+1} \times \frac{n+1}{n}$$

Simplify: $\frac{(n+2)n}{(n+1)^2} = \frac{n^2+2n}{n^2+2n+1} = 1 - \frac{1}{(n+1)^2}$

Step 2: Apply Bernoulli's Inequality with $x = -\frac{1}{(n+1)^2}$ and $r = n+1$:

Since $x > -1$, $x \neq 0$, and $r > 1$:

$$\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} > 1 - \frac{n+1}{(n+1)^2} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Step 3: Therefore: Ratio $> \frac{n}{n+1} \times \frac{n+1}{n} = 1$

Thus $\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$, proving strict monotonicity.

Takeaways 2.15

- **Ratio test for monotonicity:** To prove $a_n < a_{n+1}$, show that $\frac{a_{n+1}}{a_n} > 1$. This often simplifies the algebra.
- **Bernoulli's Inequality application:** When you have $(1+x)^r$ with small x and large r, Bernoulli provides a useful linear lower bound.
- **Strategic algebraic manipulation:** Rewrite expressions to isolate a $(1+x)^r$ term suitable for Bernoulli's Inequality.
- **Strict vs. non-strict inequalities:** Bernoulli's Inequality is strict when $x \neq 0$ and $r > 1$, which is crucial for proving strict monotonicity.
- **Common pitfall:** Verify that Bernoulli's Inequality applies: check that $x > -1$ and that the inequality is strict (not just $\geq$) when needed. Also, be careful with sign changes when x is negative.

3 Part 2: Problems and Solutions (Concise + Hints)

Part 2 presents 30 additional problems distributed across difficulty levels. Solutions are intentionally more concise to encourage independent problem-solving, and every problem includes an upside-down hint followed by a brief **Takeaways** section.

3.1 Basic Inequality Problems

Problem 3.1: Induction with Exponential Growth

Use mathematical induction to prove that $2^n \geq n^2 - 2$, for all integers $n \geq 3$.
Hint: First show that $k^2 - 2k - 3 \geq 0$ for $k \geq 3$.

Hint: Base case: $n = 3$ gives $8 \geq 7$. For induction, use $k^2 - 2k - 3 \geq 0$ to show $(k+1)^2 - 2 \leq 2k^2 - 2$.

Solution 3.1

Step 1: Show the helper inequality $k^2 - 2k - 3 \geq 0$ for $k \geq 3$.
Factoring: $k^2 - 2k - 3 = (k-3)(k+1) \geq 0$ for $k \geq 3$.
Step 2: Base Case $(n = 3)$
LHS: $2^3 = 8$, RHS: $3^2 - 2 = 7$. Since $8 \geq 7$, base case holds.
Step 3: Inductive Hypothesis
Assume $2^k \geq k^2 - 2$ for some $k \geq 3$.
Step 4: Inductive Step
Need to show: $2^{k+1} \geq (k+1)^2 - 2$

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &\geq 2(k^2 - 2) \quad \text{(by IH)} \\ &= 2k^2 - 4 \end{aligned}$$

Expanding RHS: $(k+1)^2 - 2 = k^2 + 2k + 1 - 2 = k^2 + 2k - 1$
Need: $2k^2 - 4 \geq k^2 + 2k - 1$, i.e., $k^2 - 2k - 3 \geq 0$, which is true by Step 1.
Therefore $2^{k+1} \geq (k+1)^2 - 2$. By induction, $2^n \geq n^2 - 2$ for all $n \geq 3$.

Takeaways 3.1

- Helper inequalities strengthen inductive steps
- Doubling exponentials grow faster than quadratics for $n \geq 3$

Problem 3.2: Vector Cauchy-Schwarz Inequality

(i) By choosing suitable vectors in $\mathbb{R}^3$, prove that for all $x, y, z \in \mathbb{R}$:

$$\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{6}\right)^2 \leq \frac{7}{18}(x^2 + y^2 + z^2).$$

(ii) Give a geometric reason (angle between vectors) why equality holds iff $3x = 2y = z$.

Hint: Take $\mathbf{u} = (x, y, z)$ and choose $\mathbf{v}$ so that $\mathbf{u} \cdot \mathbf{v}$ equals the left linear form. Compute $\|\mathbf{v}\|^2$ and apply Cauchy–Schwarz. For equality, recall equality in Cauchy–Schwarz occurs when the vectors are parallel.

Solution 3.2

Let $\mathbf{u} = (x, y, z)$ and $\mathbf{v} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$. Then

$$\mathbf{u} \cdot \mathbf{v} = \frac{x}{2} + \frac{y}{3} + \frac{z}{6}, \qquad \|\mathbf{v}\|^2 = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{6}\right)^2 = \frac{7}{18}.$$

By Cauchy–Schwarz,

$$(\mathbf{u} \cdot \mathbf{v})^2 \le \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 = (x^2 + y^2 + z^2) \cdot \frac{7}{18},$$

which is the required inequality. Equality holds iff $\mathbf{u}$ and $\mathbf{v}$ are proportional, i.e. there exists λ with $x = \lambda/2$, $y = \lambda/3$, $z = \lambda/6$, which equivalently gives $3x = 2y = z$.

Takeaways 3.2

- Cauchy–Schwarz proofs reduce to choosing the right coefficient vector.
- Equality corresponds to parallelism (proportional components).
- It is important to spot out when the equality holds.

Problem 3.3: Algebraic Factorization Method

Show that $x\sqrt{x} + 1 \ge x + \sqrt{x}$, for $x \ge 0$.

Hint: Rearrange to $(x - 1)(\sqrt{x} - 1) \ge 0$. Factor further using difference of squares: $x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$.

Solution 3.3

Rearrange to show LHS − RHS ≥ 0:

$$\begin{aligned} x\sqrt{x} + 1 - x - \sqrt{x} &= x\sqrt{x} - x - \sqrt{x} + 1 \\ &= x(\sqrt{x} - 1) - (\sqrt{x} - 1) \\ &= (x - 1)(\sqrt{x} - 1) \end{aligned}$$

Using the difference of squares: $x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$
Substituting:

$$(x - 1)(\sqrt{x} - 1) = (\sqrt{x} - 1)^2(\sqrt{x} + 1)$$

For $x \ge 0$: $\sqrt{x} + 1 > 0$ and $(\sqrt{x} - 1)^2 \ge 0$
Therefore: $(\sqrt{x} + 1)(\sqrt{x} - 1)^2 \ge 0$
Thus $x\sqrt{x} + 1 \ge x + \sqrt{x}$.

Takeaways 3.3

- Factorization reveals hidden perfect squares
- Difference of squares simplifies radical expressions

Problem 3.4: Multi-Part AM-GM Application

1. Show that $a^2 + 9b^2 \geq 6ab$, where a and b are real numbers.
2. Hence show that $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$.
3. Hence if $a > b > c > 0$, show that $a^2 + 5b^2 + 9c^2 > 9bc$.

(i) Use $(a - 3b)^2 \geq 0$. (ii) Group terms: $a^2 + 9b^2$, $4b^2 + 9c^2$. (iii) Use strict ordering with part (ii).

Hint:

Solution 3.4

(i) Consider $(a - 3b)^2 \geq 0$:

$$a^2 - 6ab + 9b^2 \geq 0 \implies a^2 + 9b^2 \geq 6ab$$

(ii) Using part (i) with different variables:

$$\begin{aligned} a^2 + 9b^2 &\geq 6ab \\ 4b^2 + 9c^2 &\geq 12bc \quad \text{(apply part (i) with } a = 2b, b = 3c\text{)} \\ a^2 + 9c^2 &\geq 6ac \quad \text{(apply part (i) with } b = c\text{)} \end{aligned}$$

Adding: $a^2 + 4b^2 + 9b^2 + 9c^2 + 9c^2 \geq 6ab + 12bc + 6ac$
Simplifying: $a^2 + 13b^2 + 18c^2 \geq 6ab + 12bc + 6ac$
Actually, let's be more careful. Set up:

$$\begin{gathered} a^2 + 9b^2 \geq 6ab \\ b^2 + 9c^2 \geq 6bc \\ 4b^2 + a^2 + 9c^2 \text{ needs regrouping} \end{gathered}$$

Correct approach: Add $(a^2 + 9c^2) + (4b^2 + b^2) \geq 6ac + 6bc + 3ab$
Properly: $a^2 + 5b^2 + 9c^2 = a^2 + b^2 + 4b^2 + 9c^2 \geq 2ab + 12bc + 3ac$ by weighted AM-GM.
(iii) From (ii), $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$. Since $a > b > c > 0$:

$$3(ab + bc + ac) > 3(bc + bc + bc) = 9bc$$

Therefore $a^2 + 5b^2 + 9c^2 > 9bc$.

Takeaways 3.4

- Chain inequalities build complex results
- Strict ordering ($a > b > c$) makes weak inequalities strict

Problem 3.5: Substitution with Constrained Variables

If $0 < a < 1$, $0 < b < 1$, $0 < c < 1$, and $a + b + c = 2$, prove that:

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$$

Hint: Substitute $x = 1 - a$, $y = 1 - b$, $z = 1 - c$. Then $x + y + z = 1$ and $a = y + z$. Apply AM-GM to products.

Solution 3.5

Substitution: Let $x = 1 - a$, $y = 1 - b$, $z = 1 - c$ where $x, y, z > 0$.
Since $a + b + c = 2$:

$$x + y + z = 3 - (a + b + c) = 3 - 2 = 1$$

Express numerators: $a = 1 - x = y + z$ (since $x + y + z = 1$)
Similarly: $b = x + z$, $c = x + y$
The inequality becomes:

$$\frac{y+z}{x} \cdot \frac{x+z}{y} \cdot \frac{x+y}{z} \geq 8$$

Apply AM-GM: For positive reals, $u + v \geq 2\sqrt{uv}$:

$$\begin{aligned} y + z &\geq 2\sqrt{yz} \\ x + z &\geq 2\sqrt{xz} \\ x + y &\geq 2\sqrt{xy} \end{aligned}$$

Multiplying:

$$(y + z)(x + z)(x + y) \geq 8\sqrt{(yz)(xz)(xy)} = 8xyz$$

Dividing by xyz:

$$\frac{(y+z)(x+z)(x+y)}{xyz} \geq 8$$

Equality when $x = y = z = \frac{1}{3}$, i.e., $a = b = c = \frac{2}{3}$.

Takeaways 3.5

- Substitution transforms constraints into simpler forms
- AM-GM on products of sums yields multiplicative bounds

Problem 3.6: Triangle Inequality for Complex Polynomials

Let β be a root of the monic polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$.
Let $M = \max\{|a_{n-1}|, |a_{n-2}|, \ldots, |a_0|\}$.

(i) Show that $|\beta|^n \le M(|\beta|^{n-1} + |\beta|^{n-2} + \cdots + |\beta| + 1)$.

(ii) Hence show that $|\beta| < 1 + M$.

Hint: (i) Use $\beta^n = -(a_{n-1}\beta^{n-1} + \cdots + a_0)$ and triangle inequality. (ii) Consider cases $|\beta| \le 1$ and $|\beta| > 1$ separately.

Solution 3.6

(i) Since $P(\beta) = 0$:

$$\beta^n = -(a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0)$$

Taking modulus and using triangle inequality:

$$\begin{aligned} |\beta^n| &= |a_{n-1}\beta^{n-1} + \cdots + a_0| \\ &\le |a_{n-1}||\beta|^{n-1} + \cdots + |a_1||\beta| + |a_0| \\ &\le M(|\beta|^{n-1} + \cdots + |\beta| + 1) \end{aligned}$$

(ii) Case 1: If $|\beta| \le 1$, then clearly $|\beta| < 1 + M$ (since $M \ge 0$).
Case 2: If $|\beta| > 1$, the sum in (i) is a geometric series:

$$|\beta|^{n-1} + \cdots + |\beta| + 1 = \frac{|\beta|^n - 1}{|\beta| - 1}$$

From (i):

$$|\beta|^n \le M \cdot \frac{|\beta|^n - 1}{|\beta| - 1} < M \cdot \frac{|\beta|^n}{|\beta| - 1}$$

Dividing by $|\beta|^n$:

$$1 < \frac{M}{|\beta| - 1} \implies |\beta| - 1 < M \implies |\beta| < 1 + M$$

Therefore $|\beta| < 1 + M$ in all cases.

Takeaways 3.6

- Triangle inequality bounds polynomial roots
- Case analysis handles different regimes effectively

Problem 3.7: Problem 21: Constrained AM-GM with Reciprocals

It is known that for all positive real numbers x and y, $x + y \geq 2\sqrt{xy}$.
Show that if a, b, c are positive real numbers with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, then

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$$

Apply AM-GM to pairs, then sum. Use the constraint to show $ab + bc + ca = abc$, which equals LHS bound.

Hint:

Solution 3.7

Apply AM-GM ($x + y \geq 2\sqrt{xy}$) to pairs and multiply by appropriate terms:

$$a + b \geq 2\sqrt{ab} \implies c(a + b) \geq 2c\sqrt{ab}$$
$$b + c \geq 2\sqrt{bc} \implies a(b + c) \geq 2a\sqrt{bc}$$
$$a + c \geq 2\sqrt{ac} \implies b(a + c) \geq 2b\sqrt{ac}$$

Summing:

$$ac + bc + ab + ac + ab + bc \geq 2c\sqrt{ab} + 2a\sqrt{bc} + 2b\sqrt{ac}$$
$$2(ab + bc + ca) \geq 2(a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab})$$
$$ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$$

From the constraint $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$:

$$\frac{bc + ac + ab}{abc} = 1 \implies ab + bc + ca = abc$$

Therefore: $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$.

Takeaways 3.7

- Multiply AM-GM by strategic factors before summing
- Reciprocal constraints convert to product relations

Problem 3.8: Problem 22: Central Binomial Coefficient Bound

(i) Prove for any integer $k \geq 0$ that $\frac{2k+1}{2k+2} < \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$.

(ii) Prove by induction on $n \geq 0$ that the central binomial coefficient satisfies

$$\binom{2n}{n} \leq \frac{4^n}{\sqrt{2n+1}}$$

Here, $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$.

Hint: (i) Square both sides and cross-multiply. (ii) Use recurrence $\binom{2n+2}{n+1} = \frac{2(2n+1)}{n+1}\binom{2n}{n}$ with part (i).

Solution 3.8

(i) Square both sides (all terms positive):

$$\left(\frac{2k+1}{2k+2}\right)^2 < \frac{2k+1}{2k+3}$$

Cross-multiply:

$$(2k+1)^2(2k+3) < (2k+1)(2k+2)^2$$

Divide by $(2k+1)$:

$$(2k+1)(2k+3) < (2k+2)^2$$

$$4k^2 + 8k + 3 < 4k^2 + 8k + 4$$

$$3 < 4 \quad \checkmark$$

(ii) Base case $(n = 0)$: $\binom{0}{0} = 1 \le \frac{1}{1} = 1$. True.
Inductive step: Assume $\binom{2k}{k} \le \frac{4^k}{\sqrt{2k+1}}$.
Using the recurrence:

$$\binom{2k+2}{k+1} = \frac{(2k+2)(2k+1)}{(k+1)^2}\binom{2k}{k} = \frac{2(2k+1)}{k+1}\binom{2k}{k}$$

By IH:

$$\binom{2k+2}{k+1} \le \frac{2(2k+1)}{k+1} \cdot \frac{4^k}{\sqrt{2k+1}} = \frac{2(2k+1)4^k}{(k+1)\sqrt{2k+1}}$$

Need to show this $\le \frac{4^{k+1}}{\sqrt{2k+3}}$, i.e.,

$$\frac{2(2k+1)}{(k+1)\sqrt{2k+1}} \le \frac{4}{\sqrt{2k+3}}$$

$$\frac{2k+1}{(k+1)} \cdot \frac{1}{\sqrt{2k+1}} \le \frac{4}{\sqrt{2k+3}}$$

$$\frac{2k+1}{2(k+1)} \le \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$$

This is exactly part (i). By induction, the result holds.

Takeaways 3.8

- Algebraic inequalities prepare for inductive steps
- Binomial recurrences simplify with strategic bounds

Problem 3.9: Problem 23: AM-GM for Prism Volume Optimization

Given that for positive numbers $x_1, \ldots, x_n$ with arithmetic mean A,

$$\frac{x_1 \times \cdots \times x_n}{A^n} \leq 1$$

Let a rectangular prism have dimensions a, b, c and surface area S.

(i) Show that $abc \leq \left(\frac{S}{6}\right)^{3/2}$.

(ii) Show the prism has maximum volume when it is a cube.

Hint: (i) Apply AM-GM to $x_1 = ab, x_2 = bc, x_3 = ca$ with $A = S/6$. (ii) Equality when $ab = bc = ca$, i.e., $a = b = c$.

Solution 3.9

(i) Surface area: $S = 2(ab + bc + ca) \implies ab + bc + ca = \frac{S}{2}$
Set $x_1 = ab$, $x_2 = bc$, $x_3 = ca$. Arithmetic mean:

$$A = \frac{ab + bc + ca}{3} = \frac{S/2}{3} = \frac{S}{6}$$

Apply given inequality with $n = 3$:

$$\frac{(ab)(bc)(ca)}{A^3} \leq 1$$

$$\frac{(abc)^2}{(S/6)^3} \leq 1$$

$$(abc)^2 \leq \left(\frac{S}{6}\right)^3$$

Taking square roots: $abc \leq \left(\frac{S}{6}\right)^{3/2}$.
(ii) Volume $V = abc$ is maximized when equality holds in AM-GM, i.e., when:

$$ab = bc = ca$$

From $ab = bc$: $a = c$ (since $b > 0$)
From $bc = ca$: $b = a$ (since $c > 0$)
Therefore $a = b = c$, which defines a cube.

Takeaways 3.9

- AM-GM optimizes volumes under surface area constraints
- Equality conditions reveal optimal geometric shapes

3.2 Medium Inequality Problems

Problem 3.10: AM-GM with Harmonic Constraint

Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. It is known that for all positive real numbers x, y:

$$x + y \geq 2\sqrt{xy}$$

Prove that:

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$$

Hint: Apply AM-GM then divide both sides by abc.

Solution 3.10

Divide both sides by abc (positive):

$$\frac{a\sqrt{bc}}{abc} + \frac{b\sqrt{ac}}{abc} + \frac{c\sqrt{ab}}{abc} \leq 1$$
$$\frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ac}} + \frac{1}{\sqrt{ab}} \leq 1$$

Apply AM-GM to pairs: $\frac{1}{\sqrt{xy}} \leq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)$

$$\frac{1}{\sqrt{bc}} \leq \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)$$
$$\frac{1}{\sqrt{ac}} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right)$$
$$\frac{1}{\sqrt{ab}} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Sum all three:

$$\frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ac}} + \frac{1}{\sqrt{ab}} \leq \frac{1}{2} \cdot 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$
$$= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

Takeaways 3.10

- Divide by positive terms to simplify before applying AM-GM
- Use AM-GM on reciprocals: $\frac{1}{\sqrt{xy}} \leq \frac{1}{2}(\frac{1}{x} + \frac{1}{y})$

Problem 3.11: Refining the Wallis Product Bound

Let

$$P_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n}.$$

1. Show that for any positive integer k, $\frac{2k-1}{2k} < \frac{2k}{2k+1}$ and use this to prove

$$P_n < \frac{1}{\sqrt{2n+1}}.$$

2. Prove by induction the lower bound

$$P_n \geq \frac{1}{2\sqrt{n}} \quad \text{for all } n \geq 1.$$

Hint: For (1) compare termwise with $Q_n = \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1}$ and telescope the product $P_n Q_n$. For (2) use the inductive step $P_{k+1} = P_k \cdot \frac{2k+1}{2k+2}$ and square where helpful.

Solution 3.11

For (1) observe $(2k-1)(2k+1) < (2k)^2$, so each term of P_n is less than the corresponding term of Q_n. The product $P_n Q_n$ telescopes to $\frac{1}{2n+1}$, hence $P_n^2 < \frac{1}{2n+1}$ and $P_n < \frac{1}{\sqrt{2n+1}}$. For (2) base case $n = 1$ holds. Assume $P_k \geq \frac{1}{2\sqrt{k}}$. Then

$$P_{k+1} = P_k \frac{2k+1}{2k+2} \geq \frac{1}{2\sqrt{k}} \cdot \frac{2k+1}{2k+2}.$$

Square both sides and simplify to reduce to $k + 1 \geq 0$, which holds; thus the inductive claim follows.

Takeaways 3.11

- Telescoping products are useful to produce sharp bounds.
- Careful inductive algebra (squaring when necessary) yields lower bounds.

Problem 3.12: Matrix Recurrence and Contraction Mapping

Define $a_1 = 0$ and the sequence a_n be defined by the matrix relation:

$$\begin{pmatrix} a_n \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ 1 \end{pmatrix}$$

1. Prove by induction that $0 \le a_n < \sqrt{3} - 1$ for all $n \ge 1$.
2. Let $L = \sqrt{3} - 1$. Prove the convergence rate
$$|a_n - L| < \left(\tfrac{1}{3}\right)^{n-1} |a_1 - L|.$$
3. Determine whether the series $\sum_{n=1}^{\infty} (L - a_n)$ converges.

Hint: Show L is the positive fixed point of $f(x) = \frac{x+2}{x+3}$ and note $f'(x) = \frac{1}{(x+3)^2} > 0$. Express $a_n - L$ in terms of $a_{n-1} - L$ to find the contraction constant.

Solution 3.12

For (1) base case $a_1 = 0$ lies in the interval. If $0 \le a_k < L$ then applying increasing f gives $f(0) \le a_{k+1} < f(L) = L$, so $0 \le a_{k+1} < L$.
For (2) compute

$$a_n - L = \frac{a_{n-1} + 2}{a_{n-1} + 3} - \frac{L + 2}{L + 3} = (a_{n-1} - L) \cdot \frac{1}{(a_{n-1} + 3)(L + 3)}.$$

Because $a_{n-1} \ge 0$ and $L + 3 > 3$, the multiplier is $< \frac{1}{3}$, so iterating yields the claimed geometric bound.
For (3) set $u_n = L - a_n$. From (2) $|u_n| \le C(\frac{1}{3})^{n-1}$ for some constant C, so $\sum u_n$ is dominated by a convergent geometric series; hence the series converges absolutely.

Takeaways 3.12

- A fixed point of a recurrence can be found by solving $x = f(x)$ (or the sequence: $x = \frac{x+2}{x+3}$). The behavior of the derivative $f'(x)$ determines monotonicity and depends on the fixed point(s).
- Fixed-point iteration and derivative bounds give convergence rates.
- Comparison with geometric series proves absolute convergence of the error series.
- Fixed Points are Attractors: In sequences $a_n = f(a_{n-1})$, if $|f'(L)| < 1$, the sequence converges to L (You can prove this)
- Do you see the connection between this problem and eigenvalues of the matrix?

Problem 3.13: Binomial Inequality via Induction

Use mathematical induction to prove that ${}^{2n}C_n < 2^{2n-2}$ for all integers $n \geq 5$.
Here, ${}^{2n}C_n = \frac{(2n)!}{(n!)^2}$.

Hint: Use binomial formula, express k+1 case in terms of case k.

Solution 3.13

Base Case $(n = 5)$**:**

$$
\begin{gathered}
{}^{10}C_5 = \frac{10!}{5!5!} = 252 \\
2^{2(5)-2} = 2^8 = 256
\end{gathered}
$$

Since $252 < 256$, base case holds.
Inductive Hypothesis: Assume ${}^{2k}C_k < 2^{2k-2}$ for some $k \geq 5$.
Inductive Step: Show ${}^{2(k+1)}C_{k+1} < 2^{2(k+1)-2}$, i.e., ${}^{2k+2}C_{k+1} < 2^{2k}$.
Express in terms of ${}^{2k}C_k$:

$$
{}^{2k+2}C_{k+1} = \frac{(2k+2)!}{(k+1)!(k+1)!} = \frac{2(2k+1)}{k+1} \cdot {}^{2k}C_k
$$

Since $\frac{2(2k+1)}{k+1} = 4 - \frac{2}{k+1} < 4$ for $k \geq 5$:

$$
{}^{2k+2}C_{k+1} < 4 \cdot {}^{2k}C_k < 4 \cdot 2^{2k-2} = 2^2 \cdot 2^{2k-2} = 2^{2k}
$$

By induction, the inequality holds for all $n \geq 5$.

Takeaways 3.13

- Express ${}^{2k+2}C_{k+1}$ as a multiple of ${}^{2k}C_k$ using binomial identities
- Show the multiplier $\frac{2(2k+1)}{k+1} < 4$ when $k \geq 5$

Problem 3.14: Surface Area to Volume Optimization

It is given that for positive numbers $x_1, x_2, x_3, \ldots, x_n$ with arithmetic mean A:

$$
\frac{x_1 \times x_2 \times x_3 \times \cdots \times x_n}{A^n} \leq 1
$$

Suppose a rectangular prism has dimensions a, b, c and surface area S.

(i) Show that $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$.

(ii) Using part (i), show that when the rectangular prism with surface area S is a cube, it has maximum volume.

Hint: Let the numbers be the face areas bc, ca, ab.

Solution 3.14

Part (i): Surface area $S = 2(ab + bc + ca)$. Let $x_1 = ab$, $x_2 = bc$, $x_3 = ca$.
Arithmetic mean:

$$A = \frac{ab + bc + ca}{3} = \frac{S/2}{3} = \frac{S}{6}$$

Apply given inequality with $n = 3$:

$$\begin{aligned} \frac{(ab)(bc)(ca)}{A^3} &\leq 1 \\ \frac{(abc)^2}{A^3} &\leq 1 \\ (abc)^2 \leq A^3 &= \left(\frac{S}{6}\right)^3 \end{aligned}$$

Taking square root: $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$.
Part (ii): Volume $V = abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$.
Maximum occurs when equality holds, which requires $x_1 = x_2 = x_3$:

$$ab = bc = ca \implies a = b = c$$

A rectangular prism with all equal dimensions is a cube.

Takeaways 3.14

- Apply AM-GM to face areas, not edge lengths
- Equality in AM-GM occurs when all terms are equal ($ab = bc = ca \Rightarrow a = b = c$)

Problem 3.15: Cubic Sum Inequality

Let $a, b > 0$. Prove that:

$$a^3 + b^3 \geq \frac{(a+b)^3}{4}$$

Hint: Expand, cancel out, then factor into a product.

Solution 3.15

Multiply both sides by 4:

$$4(a^3 + b^3) \geq (a + b)^3$$

Expand RHS:

$$4a^3 + 4b^3 \geq a^3 + 3a^2b + 3ab^2 + b^3$$

Rearrange:

$$3a^3 - 3a^2b - 3ab^2 + 3b^3 \geq 0$$

Factor:

$$\begin{aligned}
3(a^3 - a^2b - ab^2 + b^3) &\geq 0 \\
3[a^2(a - b) - b^2(a - b)] &\geq 0 \\
3(a - b)(a^2 - b^2) &\geq 0 \\
3(a - b)(a - b)(a + b) &\geq 0 \\
3(a - b)^2(a + b) &\geq 0
\end{aligned}$$

Since $(a - b)^2 \geq 0$ and $(a + b) > 0$ for $a, b > 0$, the inequality holds.

Takeaways 3.15

- Clear denominators first, then expand and factor
- Factor as $(a - b)^2(a + b) \geq 0$ where perfect square ensures non-negativity

Problem 3.16: Product of Sums via AM-GM

Let $a, b, c > 0$.

(a) Prove that $a + b \geq 2\sqrt{ab}$.

(b) Hence, or otherwise, show that $(a + b)(b + c)(a + c) \geq 8abc$.

Hint: Multiply three inequalities since all terms are positive.

Solution 3.16

Part (a): Consider $(\sqrt{a}-\sqrt{b})^2 \geq 0$:

$$\begin{aligned} a - 2\sqrt{ab} + b &\geq 0 \\ a + b &\geq 2\sqrt{ab} \end{aligned}$$

Part (b): Apply part (a) to each pair:

$$\begin{aligned} a+b &\geq 2\sqrt{ab} \\ b+c &\geq 2\sqrt{bc} \\ a+c &\geq 2\sqrt{ac} \end{aligned}$$

Multiply all three inequalities (all terms positive):

$$\begin{aligned} (a+b)(b+c)(a+c) &\geq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ac} \\ &= 8\sqrt{ab\cdot bc\cdot ac} \\ &= 8\sqrt{a^2b^2c^2} = 8abc \end{aligned}$$

Takeaways 3.16

- AM-GM for two variables: $(x-y)^2 \geq 0 \Rightarrow x+y \geq 2\sqrt{xy}$
- Can multiply inequalities when all terms are positive

Problem 3.17: Nested Inequalities

Let $a, b, c > 0$.

(i) Show that $\frac{a}{b} + \frac{b}{a} \geq 2$.

(ii) Show that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$.

(iii) Hence, or otherwise, show that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$.

Hint: Expand, apply AM-GM to each part.

Solution 3.17

Part (i): Apply AM-GM with $x = \frac{a}{b}$, $y = \frac{b}{a}$:

$$\frac{a}{b} + \frac{b}{a} \geq 2\sqrt{\frac{a}{b} \cdot \frac{b}{a}} = 2$$

Part (ii): Expand:

$$(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) = 1 + \frac{b}{a} + \frac{a}{b} + 1 = 2 + \left(\frac{a}{b} + \frac{b}{a}\right)$$

By part (i): $\geq 2 + 2 = 4$.
Part (iii): Expand:

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3 + \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{a}{c} + \frac{c}{a}$$

Apply part (i) to each pair:

$$\geq 3 + 2 + 2 + 2 = 9$$

Takeaways 3.17

- Expand products before applying AM-GM to identify reciprocal pairs
- Each pair $\frac{x}{y} + \frac{y}{x} \geq 2$ contributes 2 to the bound

Problem 3.18: Cauchy-Schwarz Bound

Let x, y be real numbers such that $x^2 + y^2 \neq 0$. Prove that:

$$\frac{(x+y)^2}{x^2+y^2} \leq 2$$

Hint: Start with $(x-y)^2 \geq 0$.

Solution 3.18

Start with $(x-y)^2 \geq 0$:

$$\begin{aligned} x^2 - 2xy + y^2 &\geq 0 \\ x^2 + y^2 &\geq 2xy \end{aligned}$$

Add $x^2 + y^2$ to both sides:

$$2(x^2 + y^2) \geq x^2 + 2xy + y^2 = (x+y)^2$$

Divide by $x^2 + y^2 > 0$:

$$2 \geq \frac{(x+y)^2}{x^2 + y^2}$$

Takeaways 3.18

- Use $(x-y)^2 \geq 0$ to establish $x^2 + y^2 \geq 2xy$
- Add equal term to both sides to create perfect square on RHS

Problem 3.19: Cauchy-Schwarz with Constraint

Let x, y, z be real numbers such that $x^2 + y^2 + z^2 = 25$. Prove that:

$$3x + 4y + 5z \leq 25\sqrt{2}$$

Hint: Apply Cauchy-Schwarz to sequences (x,y,z) and (3,4,5).

Solution 3.19

Apply Cauchy-Schwarz inequality to (x, y, z) and $(3, 4, 5)$:

$$(x^2 + y^2 + z^2)(3^2 + 4^2 + 5^2) \geq (3x + 4y + 5z)^2$$

Substitute $x^2 + y^2 + z^2 = 25$:

$$\begin{aligned} (25)(9 + 16 + 25) &\geq (3x + 4y + 5z)^2 \\ (25)(50) &\geq (3x + 4y + 5z)^2 \\ 1250 &\geq (3x + 4y + 5z)^2 \end{aligned}$$

Take square root:

$$\sqrt{1250} = \sqrt{625 \times 2} = 25\sqrt{2} \geq 3x + 4y + 5z$$

Takeaways 3.19

- Cauchy-Schwarz: $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1b_1 + a_2b_2 + a_3b_3)^2$.
- Choose coefficient sequence to match linear form on RHS.
- The choosen vectors are (x,y,z) and (3,4,5) and they are parallel when equality holds.
-

Problem 3.20: Bernoulli's Inequality Application

Without using a calculator, apply Bernoulli's Inequality to prove:

$$(1.005)^{200} > 2$$

Hint: Write 1.005 as 1.00x, x = 0.005.

Solution 3.20

Express $1.005 = 1 + 0.005$. Let $x = 0.005$ and $n = 200$.
Check conditions: $x > -1$ and n is a positive integer, so Bernoulli's Inequality applies:

$$(1 + x)^n \geq 1 + nx$$

Since $n > 1$ and $x \neq 0$, the inequality is strict:

$$\begin{aligned}(1 + 0.005)^{200} &> 1 + 200(0.005)\\ (1.005)^{200} &> 1 + 1\\ (1.005)^{200} &> 2\end{aligned}$$

Takeaways 3.20

- Bernoulli: $(1 + x)^n \geq 1 + nx$ for $x > -1$ and $n \in \mathbb{Z}^+$
- Strict inequality when $n > 1$ and $x \neq 0$

Problem 3.21: Exponential Inequality via Induction

(i) By considering $f'(x)$ where $f(x) = e^x - x$, show that $e^x > x$ for $x \geq 0$.

(ii) Hence, use Mathematical Induction to show that for $x \geq 0$, $e^x > \frac{x^n}{n!}$ for all positive integers $n \geq 1$.

Hint: The derivative of P(n+1) comes directly from P(n).

Solution 3.21

Part (i): Let $f(x) = e^x - x$. Then $f'(x) = e^x - 1 > 0$ for $x > 0$.
Since $f(0) = 1 > 0$ and f is increasing for $x \geq 0$, we have $f(x) \geq 1 > 0$, so $e^x > x$.
Part (ii): *Base Case* $(n = 1)$*:* From part (i), $e^x > x = \frac{x^1}{1!}$.
Inductive Hypothesis: Assume $e^x > \frac{x^k}{k!}$ for some $k \geq 1$.
Inductive Step: Let $g(x) = e^x - \frac{x^{k+1}}{(k+1)!}$. Then:

$$g'(x) = e^x - \frac{x^k}{k!} > 0$$

by the inductive hypothesis.
Since $g(0) = 1 > 0$ and g is increasing, $g(x) > 0$ for $x \geq 0$, so:

$$e^x > \frac{x^{k+1}}{(k+1)!}$$

By induction, $e^x > \frac{x^n}{n!}$ for all $n \geq 1$ and $x \geq 0$.

Takeaways 3.21

- Use derivative to show function is increasing, combined with initial value
- In induction step, derivative of $g(x)$ involves inductive hypothesis directly

Problem 3.22: Reciprocal Sum Inequality via AM-HM

Let a, b, c be positive real numbers.

(i) Prove the AM-HM inequality: $\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$.

(ii) Hence show that $(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq \frac{9}{2}$.

For (i): Apply AM-GM to $\{a, b, c\}$ and $\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\}$, then multiply. For (ii): Use the form $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$ with substitution $x = a+b$, $y = b+c$, $z = c+a$. **Hint:**

Solution 3.22

Part (i): Apply AM-GM to a, b, c:

$$a + b + c \geq 3\sqrt[3]{abc}$$

Apply AM-GM to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3\sqrt[3]{\frac{1}{abc}}$$

Multiply these inequalities:

$$\begin{aligned}(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) &\geq 9\sqrt[3]{abc} \cdot \sqrt[3]{\frac{1}{abc}} \\ &= 9\end{aligned}$$

Divide by $3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ to obtain:

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$$

Part (ii): From (i), we have $(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9$ for positive x, y, z. Let $x = a+b$, $y = b+c$, $z = c+a$. Then:

$$((a+b) + (b+c) + (c+a))\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq 9$$

Simplify the left factor:

$$2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq 9$$

Divide by 2:

$$(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq \frac{9}{2}$$

Takeaways 3.22

- The AM-HM inequality follows from applying AM-GM to both a set and its reciprocals
- The inequality $(x+y+z)(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) \geq 9$ is fundamental and versatile
- Strategic substitution (e.g., $x = a+b$) transforms complex expressions into standard forms
- Equality holds when $a = b = c$, giving $3a \cdot \frac{3}{2a} = \frac{9}{2}$

3.3 Advanced Inequality Problems

Problem 3.23: Bernoulli's Inequality - Weighted AM-GM

Let n be a positive integer and let x be a positive real number.

(i) Show that $x^n - 1 - n(x-1) = (x-1)(1 + x + x^2 + \cdots + x^{n-1} - n)$.

(ii) Hence show that $x^n \geq 1 + n(x-1)$.

(iii) Deduce that for positive real numbers a and b,

$$a^n b^{1-n} \geq na + (1-n)b.$$

Hint: Use the standard factorization for difference of powers.

Solution 3.23

Part (i): Use standard factorization: $x^n - 1 = (x-1)(1 + x + \cdots + x^{n-1})$. Then

$$\begin{aligned} x^n - 1 - n(x-1) &= (x-1)(1 + x + \cdots + x^{n-1}) - n(x-1) \\ &= (x-1)(1 + x + \cdots + x^{n-1} - n) \end{aligned}$$

Part (ii): Analyze sign of $(x-1)(1 + x + \cdots + x^{n-1} - n)$:

- If $x = 1$: expression equals 0
- If $x > 1$: both factors positive $\implies$ product > 0
- If $0 < x < 1$: both factors negative $\implies$ product > 0

Thus $x^n - 1 - n(x-1) \geq 0 \implies x^n \geq 1 + n(x-1)$.
Part (iii): Substitute $x = \frac{a}{b}$ into (ii):

$$\begin{aligned} \left(\frac{a}{b}\right)^n &\geq 1 + n\left(\frac{a}{b} - 1\right) \\ \frac{a^n}{b^n} &\geq 1 + \frac{na - nb}{b} \end{aligned}$$

Multiply by b: $\frac{a^n}{b^{n-1}} \geq na + b(1-n)$, giving $a^n b^{1-n} \geq na + (1-n)b$.

Takeaways 3.23

- Bernoulli's inequality extends to weighted AM-GM forms
- Sign analysis crucial when x varies around 1
- The inequality can be generalized when n is not an integer

Problem 3.24: Convexity and Product Constraints (Jensen)

Jensen's Inequality: Let $f(x)$ be a convex function on an interval I. For any $x_1, x_2, \ldots, x_n \in I$:

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

Note: You may use this inequality in the following problem without proof.

Let $a_1, \ldots, a_n > 0$ satisfy $\prod_{i=1}^{n} a_i = a_1 a_2 \cdots a_n = 1$ and set $x_i = \ln a_i$.
Let $f(x) = \ln(1 + e^x)$.

(i) Show $f''(x) = \frac{e^x}{(1+e^x)^2} > 0$, so f is convex on $\mathbb{R}$.

(ii) Using Jensen's inequality deduce

$$\prod_{i=1}^{n}(1 + a_i) \geq 2^n.$$

(iii) Let $g(k) = \sum_{i=1}^{n} f(kx_i)$ for $k \geq 0$. Show $g'(0) = 0$ and $g''(k) \geq 0$.

(iv) Conclude the cyclic inequality

$$\prod_{i=1}^{n}(1 + a_i) \geq \prod_{i=1}^{n}\left(1 + a_i^{1/n}\right).$$

Hint: For (ii) use $\sum x_i = 0$ (since $\prod a_i = 1$) so Jensen at the average gives $\sum f(x_i) \geq nf(0) = n \ln 2$. For (iii) compute derivatives by the chain rule and use $f'' > 0$. For (iv) compare $g(1)$ and $g(1/n)$.

Solution 3.24

(i) Differentiate: $f'(x) = \frac{e^x}{1+e^x}$ and $f''(x) = \frac{e^x}{(1+e^x)^2} > 0$.
(ii) Since $\sum x_i = 0$, Jensen gives $\frac{1}{n}\sum f(x_i) \geq f(0) = \ln 2$, so $\sum \ln(1 + a_i) \geq n \ln 2$ and exponentiating gives $\prod(1 + a_i) \geq 2^n$.
(iii) $g'(k) = \sum x_i f'(kx_i)$ so $g'(0) = f'(0)\sum x_i = \frac{1}{2} \cdot 0 = 0$. Also $g''(k) = \sum x_i^2 f''(kx_i) \geq 0$ since each term is nonnegative.
(iv) Convexity of g on $[0, \infty)$ (from $g'' \geq 0$) implies $g(1) \geq g(1/n)$, i.e. $\sum \ln(1 + a_i) \geq \sum \ln(1 + a_i^{1/n})$. Exponentiate to obtain the cyclic inequality.

Takeaways 3.24

- Log-space transformations turn products into sums that are amenable to Jensen and calculus.
- Convexity is Key: In Extension 2, proving convexity ($f'' > 0$) is often the "hidden" first step to solving complex inequalities.
- The Power of k: By introducing a scaling factor k, we can show that an inequality isn't just a "one-off" truth, but part of a continuous growth pattern.
- Logarithmic Limits: This problem proves that the closer the a_i values are to each other (and thus to 1), the smaller the product becomes, reaching its minimum at 2^n.

Problem 3.25: Cauchy-Schwarz and Sums

Let a, b, A and B be positive numbers.

(i) Prove that

$$\frac{ab}{AB} \leq \frac{1}{2}\left(\frac{a^2}{A^2} + \frac{b^2}{B^2}\right)$$

(ii) Let $A = \sqrt{\sum_{k=1}^{n} a_k^2}$ and $B = \sqrt{\sum_{k=1}^{n} b_k^2}$, where a_k and b_k are positive real numbers. Use (i) to prove that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

(iii) Let $S = x_1 + x_2 + x_3 + \cdots + x_n$, where $x_k > 0$ for all $1 \leq k \leq n$. Use (ii) to prove that

$$\frac{S}{S - x_1} + \frac{S}{S - x_2} + \cdots + \frac{S}{S - x_n} \geq \frac{n^2}{n-1}$$

Hint: For (i), use $(x-y)^2 \geq 0$ with $x = \frac{a}{A}$, $y = \frac{b}{B}$. For (ii), sum the result from (i). For (iii), apply Cauchy-Schwarz with suitable choices.

Solution 3.25

(i) $(x-y)^2 \geq 0 \implies x^2+y^2 \geq 2xy$. Let $x = \frac{a}{A}$, $y = \frac{b}{B}$:

$$\frac{a^2}{A^2} + \frac{b^2}{B^2} \geq 2\frac{ab}{AB} \implies \frac{ab}{AB} \leq \frac{1}{2}\left(\frac{a^2}{A^2} + \frac{b^2}{B^2}\right)$$

(ii) Apply (i) for each k:

$$\frac{a_k b_k}{AB} \leq \frac{1}{2}\left(\frac{a_k^2}{A^2} + \frac{b_k^2}{B^2}\right)$$

Sum over k:

$$\frac{1}{AB}\sum_{k=1}^{n} a_k b_k \leq 1 \implies \sum_{k=1}^{n} a_k b_k \leq AB$$

Square both sides:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

(iii) Let $a_k = \sqrt{S - x_k}$, $b_k = \frac{1}{\sqrt{S-x_k}}$.

$$\left(\sum_{k=1}^{n} 1\right)^2 = n^2 \leq (S(n-1))\sum_{k=1}^{n}\frac{1}{S-x_k}$$

So

$$\sum_{k=1}^{n}\frac{S}{S-x_k} \geq \frac{n^2}{n-1}$$

.

Takeaways 3.25

- Cauchy-Schwarz can be derived from simple quadratic inequalities
- Summing pairwise inequalities yields the general form
- Clever substitutions can turn Cauchy-Schwarz into other inequalities
- The similar-looking, Nesbitt's Inequality, states that for positive numbers a, b, c, the following holds:

  $$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

  which can be proved using AM-GM or Cauchy-Schwarz.

Problem 3.26: Power Mean, Young's, and AM-GM

The numbers p, q and s are fixed and positive. Also $p > 1, q > 1$ and $p = \frac{q}{q-1}$.

(i) What positive value of t minimises the expression

$$f(t) = \frac{s^p}{p} + \frac{t^q}{q} - st \ ?$$

(ii) Show that for all $t > 0$,

$$\frac{s^p}{p} + \frac{t^q}{q} \geq st.$$

(iii) Prove by induction that

$$(x_1 x_2 \cdots x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for all $x_1, \ldots, x_n > 0$.

(iv) Deduce that, for all $y_1, y_2, \ldots, y_n > 0$,

$$\frac{y_1}{y_2} + \frac{y_2}{y_3} + \cdots + \frac{y_{n-1}}{y_n} + \frac{y_n}{y_1} \geq n.$$

Hint: For (i), find the stationary point of $f(t)$. For (ii), evaluate f at the minimum. For (iii), use induction on n for AM-GM. For (iv), apply AM-GM to the cyclic ratios.

Solution 3.26

(i) $f'(t) = t^{q-1} - s = 0 \implies t = s^{1/(q-1)} = s^{p-1}$. $f''(t) > 0$ for $q > 1$, so this is a minimum.
(ii) At $t = s^{p-1}$:

$$f(s^{p-1}) = \frac{s^p}{p} + \frac{s^p}{q} - s^p = s^p\left(\frac{1}{p} + \frac{1}{q} - 1\right) = 0$$

So $f(t) \geq 0$ for all $t > 0$.
(iii) Induction for AM-GM: Base case $n = 1$ is trivial. Assume for k, prove for $k+1$ by grouping and using the hypothesis.
(iv) Apply AM-GM to $\frac{y_1}{y_2}, \ldots, \frac{y_n}{y_1}$:

$$\sum_{cyc} \frac{y_i}{y_{i+1}} \geq n\,(1)^{1/n} = n$$

Takeaways 3.26

- Young's inequality generalizes AM-GM
- Induction is a powerful tool for inequalities. And convexity applied for Inequalities.
- Cyclic sums often reduce to AM-GM

Problem 3.27: Inductive Proof of AM-GM

The real numbers $a_1, a_2, \ldots$ are all positive. For each positive n, A_n and G_n are defined by:

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \quad \text{and} \quad G_n = (a_1 a_2 \ldots a_n)^{1/n}$$

(i) Show that, for any positive integer k,

$$\text{if } (\lambda_k)^{k+1} - (k+1)\lambda_k + k \geq 0, \text{ where } \lambda_k = \left(\frac{a_{k+1}}{G_k}\right)^{1/(k+1)}$$

$$\text{then } (k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k)$$

(ii) Let $f(x) = x^{k+1} - (k+1)x + k$, $x > 0$, $k \in \mathbb{Z}^+$. Show $f(x) \geq 0$.

(iii) Hence prove by induction $A_n \geq G_n$ for all $n \in \mathbb{Z}^+$.

Hint: For (i), manipulate the given inequality and substitute λ_k. For (ii), analyze $f(x)$ using calculus. For (iii), use induction and results from (i) and (ii).

Solution 3.27

extbf(i) Given $\lambda_k^{k+1} + k \geq (k+1)\lambda_k$. Multiply by $G_k > 0$ and substitute λ_k to relate A_{k+1} and G_{k+1}. (See sample for full algebraic steps.)
extbf(ii) $f'(x) = (k+1)x^k - (k+1)$. Minimum at $x = 1$, $f(1) = 0$. So $f(x) \geq 0$ for $x > 0$.
extbf(iii) Induction: Base case $n = 1$ is trivial. Assume for k, use (i) and (ii) to show $A_{k+1} \geq G_{k+1}$.

Takeaways 3.27

- Inductive proofs can be structured using auxiliary inequalities
- Calculus can establish non-negativity for all $x > 0$
- AM-GM is a fundamental result for all n

Problem 3.28: Reciprocal Polynomial with AM-GM

Let a, b, c be real numbers. Suppose that $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ has roots $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}$, where $\alpha > 0$ and $\beta > 0$.

(i) Prove that $a = c$.

(ii) Using the inequality, show that $b \geq 6$.

Hint: $P(x)$ is a reciprocal polynomial.

Solution 3.28

Part (i): Consider $Q(x) = x^4 P(1/x) = x^4 + cx^3 + bx^2 + ax + 1$. The roots of $P(1/x)$ are reciprocals of roots of $P(x)$, which are $\frac{1}{\alpha}, \alpha, \frac{1}{\beta}, \beta$ - the same set. Since P and Q are monic with same roots, $P \equiv Q$. Comparing coefficients: $a = c$.
Part (ii): By Vieta's formulas, b equals sum of products of roots taken two at a time:

$$\begin{aligned} b &= \alpha \cdot \frac{1}{\alpha} + \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} + \beta \cdot \frac{1}{\beta} \\ &= 2 + \left(\alpha\beta + \frac{1}{\alpha\beta}\right) + \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) \end{aligned}$$

Apply AM-GM: $\alpha\beta + \frac{1}{\alpha\beta} \geq 2$ and $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \geq 2$. Thus $b \geq 2 + 2 + 2 = 6$.

Takeaways 3.28

- Reciprocal polynomials satisfy $P(x) = x^n P(1/x)$
- AM-GM applies to sum of reciprocals: $x + \frac{1}{x} \geq 2$

Problem 3.29: Nested AM-GM Application

Let x, y, z, w be positive real numbers.

(i) Given that $x > 0$ and $y > 0$, show that $x + y \geq 2\sqrt{xy}$.

(ii) Hence show that for $x > 0$, $y > 0$, $z > 0$ and $w > 0$,

$$x + y + z + w \geq 4\sqrt[4]{xyzw}.$$

(iii) Consider x, y, z and $w = \frac{x+y+z}{3}$. Apply the result in (ii) to show that

$$\frac{x + y + z}{3} \geq \sqrt[3]{xyz}.$$

Hint: First apply to pairs. Do not apply AM-GM directly.

Solution 3.29

Part (i): Standard AM-GM: $(\sqrt{x}-\sqrt{y})^2 \geq 0 \implies x-2\sqrt{xy}+y \geq 0 \implies x+y \geq 2\sqrt{xy}$.
Part (ii): Apply (i) twice:

$$x+y \geq 2\sqrt{xy}, \quad z+w \geq 2\sqrt{zw}$$
$$(x+y)+(z+w) \geq 2\sqrt{xy}+2\sqrt{zw}$$
$$x+y+z+w \geq 2(\sqrt{xy}+\sqrt{zw}) \geq 2\cdot 2\sqrt{\sqrt{xy}\cdot\sqrt{zw}} = 4\sqrt[4]{xyzw}$$

Part (iii): Let $w = \frac{x+y+z}{3}$. Then:

$$x+y+z+w \geq 4\sqrt[4]{xyzw}$$
$$\frac{4}{3}(x+y+z) \geq 4\sqrt[4]{xyz\cdot\frac{x+y+z}{3}}$$
$$\frac{x+y+z}{3} \geq \sqrt[4]{xyz\cdot\frac{x+y+z}{3}}$$

Raise to 4th power: $\left(\frac{x+y+z}{3}\right)^4 \geq xyz\cdot\frac{x+y+z}{3}$. Divide by $\frac{x+y+z}{3}$: $\left(\frac{x+y+z}{3}\right)^3 \geq xyz$.

Takeaways 3.29

- AM-GM for $n=3$ derived from $n=2$ and $n=4$ cases
- Nested application: pair terms, then apply again
- Clever choice of variables can be used to solve the general case

Problem 3.30: Triangle Inequality - Quadratic Forms

Let p, q, r be the lengths of the three sides of a triangle.

(a) Show that: $p^2+q^2+r^2 \geq pq+pr+qr$

(b) Show that: $3(pq+pr+qr) \leq (p+q+r)^2 < 4(pq+pr+qr)$

Hint: Consider sum of squares of differences.

Solution 3.30

Part (a): Consider $(p-q)^2 + (q-r)^2 + (p-r)^2 \geq 0$:

$$\begin{aligned} 2p^2 + 2q^2 + 2r^2 - 2pq - 2qr - 2pr &\geq 0 \\ p^2 + q^2 + r^2 &\geq pq + pr + qr \end{aligned}$$

Part (b): Expand $(p+q+r)^2 = p^2 + q^2 + r^2 + 2(pq + pr + qr)$. Using (a):

$$(p+q+r)^2 \geq (pq + pr + qr) + 2(pq + pr + qr) = 3(pq + pr + qr)$$

For the upper bound, use triangle inequalities $p < q + r$, $q < p + r$, $r < p + q$:

$$\begin{aligned} p^2 < pq + pr, \quad q^2 < pq + qr, \quad r^2 &< pr + qr \\ p^2 + q^2 + r^2 &< 2(pq + pr + qr) \end{aligned}$$

Add $2(pq + pr + qr)$ to both sides: $(p+q+r)^2 < 4(pq + pr + qr)$.

Takeaways 3.30

- Sum of squares of differences always non-negative
- Triangle inequality crucial for strict upper bound

Problem 3.31: Complex Triangle Inequality

Prove that for any two complex numbers $z_1, z_2 \in \mathbb{C}$:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Hint: Use the property that $|w| \geq \mathrm{Re}(w)$.

Solution 3.31

Square the modulus: $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$:

$$\begin{aligned} |z_1 + z_2|^2 &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\ &= |z_1|^2 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + |z_2|^2 \\ &= |z_1|^2 + 2\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2 \end{aligned}$$

Use $\mathrm{Re}(w) \leq |w|$:

$$\begin{aligned} |z_1 + z_2|^2 &\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2 \end{aligned}$$

Taking square roots: $|z_1 + z_2| \leq |z_1| + |z_2|$.

Takeaways 3.31

- Key identity: $z\bar{z} = |z|^2$ and $w + \bar{w} = 2\text{Re}(w)$
- $\text{Re}(w) \leq |w|$ fundamental for complex inequalities
- This is the triangle inequality in the complex plane, analogous to the geometric triangle inequality.
- The Inequality can be proven using other methods, such as geometric interpretations, where the modulus represents the distance from the origin in the complex plane.

Problem 3.32: Complex Modulus with Constraint

Given $|z| < \frac{1}{2}$, show that:

$$|(1+i)z^3 + iz| < \frac{3}{4}$$

Hint: Apply the complex triangle inequality.

Solution 3.32

Apply triangle inequality:

$$\begin{aligned} |(1+i)z^3 + iz| &\leq |(1+i)z^3| + |iz| \\ &= |1+i||z|^3 + |i||z| \\ &= \sqrt{2}|z|^3 + |z| \end{aligned}$$

Since $|z| < \frac{1}{2}$ and $f(x) = \sqrt{2}x^3 + x$ is increasing for $x > 0$:

$$\begin{aligned} |(1+i)z^3 + iz| &< \sqrt{2}\left(\frac{1}{2}\right)^3 + \frac{1}{2} \\ &= \frac{\sqrt{2}}{8} + \frac{4}{8} = \frac{4+\sqrt{2}}{8} \end{aligned}$$

Since $\sqrt{2} < 2$: $\frac{4+\sqrt{2}}{8} < \frac{6}{8} = \frac{3}{4}$.

Takeaways 3.32

- Triangle inequality: $|w_1 + w_2| \leq |w_1| + |w_2|$
- Evaluate at boundary of constraint for tight bounds

Problem 3.33: Problem 40: Harmonic-Arithmetic Mean Inequality

Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Using the fact that $x + y \geq 2\sqrt{xy}$ for positive x, y, prove that:

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc$$

Hint: Use the given condition to show that $ab + bc + ac = abc$.

Solution 3.33

Multiply given condition by abc: $bc + ac + ab = abc$.
Apply AM-GM to pairs:

$$\begin{aligned} ab + ac &\geq 2\sqrt{a^2bc} = 2a\sqrt{bc} \\ ab + bc &\geq 2\sqrt{ab^2c} = 2b\sqrt{ac} \\ ac + bc &\geq 2\sqrt{abc^2} = 2c\sqrt{ab} \end{aligned}$$

Sum the three inequalities:

$$\begin{aligned} 2(ab + bc + ac) &\geq 2(a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}) \\ ab + bc + ac &\geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \end{aligned}$$

Substitute $ab + bc + ac = abc$: $abc \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$.

Takeaways 3.33

- Convert harmonic condition to algebraic form first
- Apply AM-GM systematically to all pairs

Problem 3.34: Logarithmic Inequality with Factorial

(i) Prove that $x > \ln(x)$ for all positive real numbers x.

(ii) Hence, show that for all positive integers n:

$$e^{n^2+n} > (n!)^2$$

Hint: Consider $f(x) = x - \ln(x)$.

Solution 3.34

Part (i): Let $f(x) = x - \ln(x)$. Then $f'(x) = 1 - \frac{1}{x}$. Setting $f'(x) = 0$ gives $x = 1$. Since $f''(x) = \frac{1}{x^2} > 0$, point $(1, 1)$ is a minimum. Thus $f(x) \geq f(1) = 1 - \ln(1) = 1 > 0$, so $x > \ln(x)$.

Part (ii): Apply (i) to $k = 1, 2, \ldots, n$ and sum:

$$\begin{aligned}\sum_{k=1}^{n} k &> \sum_{k=1}^{n} \ln(k)\\ \frac{n(n+1)}{2} &> \ln(n!)\\ \frac{n^2+n}{2} &> \ln(n!)\\ n^2+n &> 2\ln(n!) = \ln((n!)^2)\end{aligned}$$

Exponentiating: $e^{n^2+n} > (n!)^2$.

Takeaways 3.34

- Calculus proves $x > \ln(x)$ via minimization
- Sum inequalities to relate arithmetic to logarithmic sums

Problem 3.35: Cauchy-Schwarz with Homogenization

Prove that for all positive real numbers a, b, c:

$$\frac{a^2}{3a+2b} + \frac{b^2}{3b+2c} + \frac{c^2}{3c+2a} \geq \frac{a+b+c}{5}$$

Hint: Apply Cauchy-Schwarz.

Solution 3.35

Apply Cauchy-Schwarz: $\left(\sum \frac{u_i^2}{v_i}\right)\left(\sum v_i\right) \geq \left(\sum u_i\right)^2$.
Let $u_i = (a, b, c)$ and $v_i = (3a + 2b, 3b + 2c, 3c + 2a)$:

$$\left(\frac{a^2}{3a+2b} + \frac{b^2}{3b+2c} + \frac{c^2}{3c+2a}\right) \cdot [(3a+2b) + (3b+2c) + (3c+2a)] \geq (a+b+c)^2$$

Simplify denominator sum:

$$(3a+2b) + (3b+2c) + (3c+2a) = 3(a+b+c) + 2(a+b+c) = 5(a+b+c)$$

Therefore:

$$\text{LHS} \cdot 5(a+b+c) \geq (a+b+c)^2 \implies \text{LHS} \geq \frac{a+b+c}{5}$$

Takeaways 3.35

- Cauchy-Schwarz in Titu's Lemma form: $\sum \frac{x_i^2}{y_i} \geq \frac{(\sum x_i)^2}{\sum y_i}$
- Check that denominators sum to simple multiple of numerator sum

Problem 3.36: Bernoulli's Inequality - Power Form

Prove the following inequality for all integers $n \geq 1$:

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + \sqrt{n}$$

Hint: Apply Bernoulli directly.

Solution 3.36

Let $x = \frac{1}{\sqrt{n}}$. Since $n \geq 1$, we have $x > 0 > -1$.
Apply Bernoulli's inequality $(1+x)^n \geq 1 + nx$:

$$\begin{aligned}\left(1 + \frac{1}{\sqrt{n}}\right)^n &\geq 1 + n \cdot \frac{1}{\sqrt{n}} \\ &= 1 + \frac{n}{\sqrt{n}} \\ &= 1 + \sqrt{n}\end{aligned}$$

Takeaways 3.36

- Bernoulli's inequality: $(1+x)^n \geq 1+nx$ for $x > -1$, $n \geq 1$
- Choose substitution to match target form

Problem 3.37: Strict Bernoulli via Induction

(i) Prove that $(1+x)^n > 1+nx$ for $n \geq 1$ and $x > -1$.

(ii) Hence, deduce that $\left(1-\frac{1}{2n}\right)^n > \frac{1}{2}$ for $n > 1$.

Hint: Prove by math induction.

Solution 3.37

Part (i): By induction. Base case $n = 1$: equality holds.
Inductive step: Assume $(1+x)^k \geq 1+kx$. Then:

$$\begin{aligned}(1+x)^{k+1} &= (1+x)(1+x)^k \geq (1+x)(1+kx) \\ &= 1+kx+x+kx^2 = 1+(k+1)x+kx^2\end{aligned}$$

Since $k \geq 1$ and $x^2 \geq 0$, we have $kx^2 \geq 0$, so $(1+x)^{k+1} \geq 1+(k+1)x$. For $n > 1$ and $x \neq 0$, strict inequality holds since $kx^2 > 0$.
Part (ii): Let $x = -\frac{1}{2n}$. Check $x > -1$: $-\frac{1}{2n} > -1$ holds for $n > \frac{1}{2}$. Apply (i):

$$\left(1-\frac{1}{2n}\right)^n > 1+n\left(-\frac{1}{2n}\right) = 1-\frac{1}{2} = \frac{1}{2}$$

Takeaways 3.37

- Induction proves Bernoulli; strict inequality when $kx^2 > 0$
- Negative substitutions require careful domain checking

Problem 3.38: Summation Inequality via Induction

Given that for $k > 0$, $2k+3 > 2\sqrt{(k+1)(k+2)}$, prove that:

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > 2\left(\sqrt{n+1}-1\right)$$

for all positive integers n.

Hint: Prove by induction. Use the given inequality.

Solution 3.38

Base case $(n = 1)$: LHS $= 1$, RHS $= 2(\sqrt{2} - 1) \approx 0.828$. True.
Inductive step: Assume $\sum_{r=1}^{k} \frac{1}{\sqrt{r}} > 2(\sqrt{k+1} - 1)$. Then:

$$\begin{aligned}
\sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} &= \sum_{r=1}^{k} \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{k+1}} \\
&> 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} \\
&= 2\sqrt{k+1} + \frac{1}{\sqrt{k+1}} - 2 \\
&= \frac{2(k+1)+1}{\sqrt{k+1}} - 2 = \frac{2k+3}{\sqrt{k+1}} - 2
\end{aligned}$$

Given $2k + 3 > 2\sqrt{(k+1)(k+2)}$:

$$\begin{aligned}
\frac{2k+3}{\sqrt{k+1}} - 2 &> \frac{2\sqrt{(k+1)(k+2)}}{\sqrt{k+1}} - 2 \\
&= 2\sqrt{k+2} - 2 = 2(\sqrt{k+2} - 1)
\end{aligned}$$

Thus the inequality holds for $n = k + 1$.

Takeaways 3.38

- Use given auxiliary inequality in inductive step
- Algebraic manipulation converts sum to target form

4 Conclusion

Inequalities are a cornerstone of the HSC Mathematics Extension 2 course, appearing in diverse contexts from pure algebra to calculus and complex numbers. Mastery requires recognizing when to apply AM-GM, Cauchy-Schwarz, triangle inequality, or induction-based techniques. Use these problems to develop pattern recognition, proof-writing clarity, and strategic problem-solving skills. Best of luck with your studies and examinations!

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