HSC Math Extension 2: Polynomials Mastery

Vu Hung Nguyen

Contents

1 Introduction

1.1 Project Overview

This booklet compiles high-quality polynomial problems curated specifically for the HSC Mathematics Extension 2 syllabus. Every problem covers essential polynomial techniques including factoring, roots and Vieta's formulas, complex numbers and conjugate roots, transformations of roots, nature of roots using calculus, De Moivre's theorem and roots of unity, and connections to trigonometric identities. Detailed reasoning showcases advanced problem-solving strategies that build from fundamental techniques to complex multi-step applications.

1.2 Target Audience

The explanations are crafted for Extension 2 students aiming to master polynomials and develop advanced problem-solving skills. Each solution in Part 1 explicitly states the strategy, justifies technique choices, and provides complete step-by-step working so that high-school learners can follow every transition. Part 2 offers hints and concise solutions to encourage independent problem-solving.

1.3 How to Use This Booklet

- Review the fundamentals section before attempting problems to refresh key theorems and techniques.
- Attempt problems in Part 1 without looking at solutions; compare your work against detailed solutions to understand model reasoning.
- For Part 2, try each problem first, then check the upside-down hint if needed, and finally review the solution sketch.
- Practice problems multiple times, working from memory to reinforce technique mastery.
- Pay special attention to Vieta's formulas and De Moivre's theorem applications, as these advanced techniques frequently appear in Extension 2 exams.

1.4 Polynomial Topics Overview

The problems in this collection cover:

- **Factoring Polynomials:** Factor theorem, synthetic division, finding all factors
- **Roots of Polynomials:** Finding roots, relationship between roots and coefficients
- **Vieta's Formulas (Advanced):** Sum and product relationships, constructing polynomials from root conditions
- **Complex Numbers:** Solving with complex coefficients, Cartesian and polar forms
- **Transformations of Roots:** Forming polynomials with reciprocal, squared, or shifted roots
- **Nature of Roots:** Multiple (repeated) roots using derivatives, discriminant conditions
- **De Moivre's Theorem:** Roots of unity, expressing trigonometric functions as polynomials
- **Polynomials and Trigonometry:** Solving polynomial equations derived from trigonometric identities

2 Fundamentals Review

This section provides a comprehensive review of polynomial techniques essential for HSC Extension 2. Use this as a reference while working through problems.

Overview

The study of **Polynomials** in the HSC Mathematics Extension 2 course is one of the most challenging and comprehensive topics, integrating advanced concepts from **Complex Numbers** and **Calculus**. Students are expected to move beyond simple factoring and root-finding to investigate the deep relationships between a polynomial's **coefficients** and its **roots**.

The core of the topic revolves around manipulating and solving polynomial equations of degree three or higher. A central focus is the **Conjugate Root Theorem**, which states that for polynomials with real coefficients, complex roots must occur in conjugate pairs. This theorem, along with **Vieta's Formulas** (which systematically express the relationships between the roots and coefficients), is essential for constructing, transforming, and analysing polynomial equations.

Mastery of this topic requires strong algebraic skills, a solid understanding of complex number geometry, and the ability to link polynomial structures to trigonometric principles.

3 Basic Polynomial Theorems

3.1 Factor Theorem

For a polynomial $P(x)$, if $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Conversely: If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.

This theorem is fundamental for factoring polynomials and finding roots systematically.

3.2 Remainder Theorem

When a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.

Application: This provides a quick way to evaluate remainders without performing full polynomial division.

3.3 Conjugate Root Theorem

If $P(x)$ is a polynomial with real coefficients, and $z = a + bi$ (where $b \neq 0$) is a root, then the complex conjugate $\overline{z} = a - bi$ is also a root.

Consequence: Complex roots of real polynomials always occur in conjugate pairs. This means:

- A polynomial of odd degree with real coefficients must have at least one real root.
- Complex roots contribute quadratic factors with real coefficients: $(x - z)(x - \overline{z}) = x^2 - 2ax + (a^2 + b^2)$ where $z = a + bi$.

4 Vieta's Formulas

Beyond Syllabus: Vieta's Formulas

Note: While not explicitly listed in the HSC syllabus, **Vieta's Formulas** are essential advanced knowledge for Extension 2 students. These formulas express the relationships between polynomial roots and coefficients, enabling powerful problem-solving techniques that frequently appear in HSC examinations.

4.1 Statement of Vieta's Formulas

For a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, Vieta's formulas state:

$$\begin{aligned}
\alpha_1 + \alpha_2 + \cdots + \alpha_n &= -\frac{a_{n-1}}{a_n} \\
\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-1}\alpha_n &= \frac{a_{n-2}}{a_n} \\
\alpha_1\alpha_2\alpha_3 + \cdots &= -\frac{a_{n-3}}{a_n} \\
&\vdots \\
\alpha_1\alpha_2\cdots\alpha_n &= (-1)^n \frac{a_0}{a_n}
\end{aligned}$$

4.2 Special Cases

Quadratic ($ax^2 + bx + c = 0$ with roots α, β):

$$\begin{aligned}
\alpha + \beta &= -\frac{b}{a} \\
\alpha\beta &= \frac{c}{a}
\end{aligned}$$

Cubic ($ax^3 + bx^2 + cx + d = 0$ with roots α, β, γ):

$$\begin{aligned}
\alpha + \beta + \gamma &= -\frac{b}{a} \\
\alpha\beta + \beta\gamma + \gamma\alpha &= \frac{c}{a} \\
\alpha\beta\gamma &= -\frac{d}{a}
\end{aligned}$$

Quartic ($ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots $\alpha, \beta, \gamma, \delta$):

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= -\frac{b}{a} \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= \frac{c}{a} \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= -\frac{d}{a} \\
\alpha\beta\gamma\delta &= \frac{e}{a}
\end{aligned}$$

4.3 Applications of Vieta's Formulas

1. Constructing Polynomials from Root Conditions
Given relationships between roots, Vieta's formulas allow us to find polynomial coefficients.
Example: Find a polynomial with roots α, β where $\alpha + \beta = 5$ and $\alpha\beta = 6$.
Solution: Using Vieta's formulas backwards: $P(x) = x^2 - 5x + 6$
2. Finding Sums and Products of Root Combinations
For roots of $x^3 - 3x^2 + 5x - 7 = 0$ called α, β, γ:

$$\begin{aligned}\alpha + \beta + \gamma &= 3 \\ \alpha\beta + \beta\gamma + \gamma\alpha &= 5 \\ \alpha\beta\gamma &= 7\end{aligned}$$

We can find $\alpha^2 + \beta^2 + \gamma^2$ using: $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
Thus: $\alpha^2 + \beta^2 + \gamma^2 = 9 - 2(5) = -1$
3. Transformations of Roots
If α, β, γ are roots of $P(x) = 0$, find a polynomial with roots $2\alpha, 2\beta, 2\gamma$.
Let $y = 2x$, so $x = \frac{y}{2}$. Substitute into $P(x) = 0$ to get $P(\frac{y}{2}) = 0$.

5 Nature of Roots

5.1 Multiple (Repeated) Roots

A polynomial $P(x)$ has a **multiple root** at $x = \alpha$ if $(x - \alpha)^k$ is a factor for some $k \geq 2$.
Criterion for Double Root: α is a double root of $P(x)$ if and only if:

$$\begin{aligned}P(\alpha) &= 0 \\ P'(\alpha) &= 0\end{aligned}$$

General Criterion: α is a root of multiplicity k if:

$$\begin{aligned}P(\alpha) = P'(\alpha) = P''(\alpha) = \cdots = P^{(k-1)}(\alpha) &= 0 \\ P^{(k)}(\alpha) &\neq 0\end{aligned}$$

Application: This calculus-based approach is powerful for determining conditions on coefficients that produce repeated roots.

5.2 Discriminant (Quadratic Only)

For $ax^2 + bx + c = 0$, the discriminant $\Delta = b^2 - 4ac$ determines root nature:

- $\Delta > 0$: Two distinct real roots
- $\Delta = 0$: One repeated real root
- $\Delta < 0$: Two complex conjugate roots

6 Transformations of Roots

Given a polynomial $P(x)$ with roots $\alpha, \beta, \gamma, \ldots$, we can construct new polynomials with transformed roots.

6.1 Common Transformations

1. **Reciprocals of Roots** $(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma})$
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then the polynomial with reciprocal roots is:

$$Q(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = x^n P\left(\frac{1}{x}\right)$$

2. Negative Roots $(-\alpha, -\beta, -\gamma)$
Replace x with $-x$: $Q(x) = P(-x)$
3. Shifted Roots $(\alpha + k, \beta + k, \gamma + k)$
Replace x with $(x - k)$: $Q(x) = P(x - k)$
4. Scaled Roots $(k\alpha, k\beta, k\gamma)$
Replace x with $\frac{x}{k}$: $Q(x) = P\left(\frac{x}{k}\right)$
5. Squared Roots $(\alpha^2, \beta^2, \gamma^2)$
Let $y = x^2$, so $x = \pm\sqrt{y}$. Note: This produces both positive and negative roots, requiring careful handling.

7 De Moivre's Theorem and Roots of Unity

De Moivre's Theorem: Extended Coverage

Note: De Moivre's Theorem is a cornerstone for connecting complex numbers, trigonometry, and polynomials. While covered in the Complex Numbers topic, its applications to polynomial problems—especially roots of unity—are extensive and warrant expanded treatment here.

7.1 Statement of De Moivre's Theorem

For any real number θ and integer n:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

In polar form: $(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta)$

7.2 Finding nth Roots

To solve $z^n = w$ where $w = r\operatorname{cis}\alpha$:
The n solutions are:

$$z_k = r^{1/n}\operatorname{cis}\left(\frac{\alpha + 2\pi k}{n}\right) \quad \text{for } k = 0, 1, 2, \ldots, n-1$$

7.3 Roots of Unity

The n**th roots of unity** are solutions to $z^n = 1$:

$$z_k = \operatorname{cis}\left(\frac{2\pi k}{n}\right) = e^{2\pi i k/n} \quad \text{for } k = 0, 1, 2, \ldots, n-1$$

Key Properties:

- The roots are evenly distributed on the unit circle in the complex plane

- If $\omega = \text{cis}(2\pi/n)$ is a primitive nth root, all roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$
- Sum of all nth roots of unity: $\sum_{k=0}^{n-1} \omega^k = 0$ (for $n \geq 2$)
- Product of all nth roots of unity: $\prod_{k=0}^{n-1} \omega^k = (-1)^{n+1}$

7.4 Applications to Polynomial Problems

1. Factorization
$z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$
For $n \geq 2$: $z^{n-1} + z^{n-2} + \cdots + z + 1 = \frac{z^n-1}{z-1} = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$

2. Trigonometric Identities from De Moivre

Expanding $(\cos\theta + i\sin\theta)^n$ using the binomial theorem and equating real and imaginary parts yields formulas for $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$.

Example: For $n = 3$:

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$$
$$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

3. Solving Polynomial Equations via Trigonometry

If a polynomial can be written as $\tan(n\theta)$ or $\cos(n\theta)$ in terms of $\tan\theta$ or $\cos\theta$, De Moivre's theorem helps find all solutions by solving trigonometric equations.

8 Notation and Conventions

Throughout this collection:

- $P(x), Q(x)$ denote polynomials
- Greek letters $\alpha, \beta, \gamma, \delta$ denote roots
- ω typically denotes a primitive nth root of unity: $\omega = \text{cis}(2\pi/n)$
- z, w denote complex numbers
- $\text{cis}\,\theta = \cos\theta + i\sin\theta$
- $\overline{z}$ denotes the complex conjugate of z
- $|z|$ denotes the modulus (absolute value) of z
- $\arg(z)$ denotes the argument (angle) of z

9 Part 1: Problems and Solutions (Detailed)

Part 1 contains three sets of problems—basic, medium, and advanced. Each set provides five problems with comprehensive solutions. Every solution includes a strategy paragraph explaining technique selection, complete step-by-step working with annotations, and a takeaways box highlighting key insights. Problems are ordered from simpler to more complex within each difficulty level.

9.1 Basic Polynomial Problems

Problem 9.1: Square Roots of Complex Numbers

(i) Find the two square roots of $-i$, giving the answers in the form $x + iy$, where x and y are real numbers.

(ii) Hence, or otherwise, solve $z^2 + 2z + 1 + i = 0$ giving your solutions in the form $a + ib$ where a and b are real numbers.

Solution 9.1

Strategy: Set $z = x + iy$ and equate real and imaginary parts to find the square roots. Use completing the square in part (ii) to apply part (i)'s results.

(i) Finding the square roots of $-i$

Let $z = x + iy$ where $x, y \in \mathbb{R}$. Then $z^2 = -i$ gives:

$$(x + iy)^2 = x^2 - y^2 + 2ixy = -i$$

Equating real and imaginary parts:

$$x^2 - y^2 = 0 \tag{1}$$
$$2xy = -1 \tag{2}$$

From (1): $y = \pm x$
Case 1: If $y = x$, then $2x^2 = -1 \implies x^2 = -\frac{1}{2}$ (no real solutions).
Case 2: If $y = -x$, then $-2x^2 = -1 \implies x^2 = \frac{1}{2} \implies x = \pm\frac{\sqrt{2}}{2}$
Therefore: $z_1 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ and $z_2 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

(ii) Solving $z^2 + 2z + 1 + i = 0$

Completing the square: $(z + 1)^2 = -i$
Let $w = z + 1$. From part (i), $w = \pm\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$
Since $z = w - 1$:

$$z_1 = \left(\frac{\sqrt{2}}{2} - 1\right) - i\frac{\sqrt{2}}{2}, \quad z_2 = \left(-\frac{\sqrt{2}}{2} - 1\right) + i\frac{\sqrt{2}}{2}$$

Takeaways 9.1

This problem demonstrates several fundamental techniques in complex number algebra:

- **Equating Real and Imaginary Parts:** When $(x+iy)^2 = a+ib$, we can separate into two real equations by equating coefficients, giving us a solvable system.
- **Completing the Square:** Recognizing $(z+1)^2$ in the equation transforms a seemingly difficult problem into one we've already solved.
- **Alternative Method - Polar Form:** We could have found square roots using $-i = e^{i(-\pi/2+2k\pi)}$, then $\sqrt{-i} = e^{i(-\pi/4+k\pi)}$ for $k = 0, 1$.
- **Conjugate Pairs:** Notice the two square roots are negatives of each other, which is always true for square roots of any complex number.

Problem 9.2: Quadratic Equations with Complex Roots

Solve the quadratic equation

$$z^2 - 3z + 4 = 0,$$

where z is a complex number. Give your answers in **Cartesian form** $(x+iy)$.

Solution 9.2

Strategy: This is a straightforward application of the quadratic formula to a complex-valued equation. The discriminant is negative, indicating complex (non-real) conjugate roots. The key steps are: identify coefficients, calculate the discriminant, handle the negative square root using i, and simplify to Cartesian form.
The given quadratic equation is

$$z^2 - 3z + 4 = 0.$$

This is in the standard form $az^2 + bz + c = 0$, where $a = 1$, $b = -3$, and $c = 4$.
We use the **quadratic formula** to find the solutions for z:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the values of a, b, and c:

$$\begin{aligned} z &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{3 \pm \sqrt{9 - 16}}{2} \\ &= \frac{3 \pm \sqrt{-7}}{2} \end{aligned}$$

Since z is a complex number, we express $\sqrt{-7}$ using the imaginary unit i, where $i^2 = -1$:

$$\sqrt{-7} = \sqrt{7 \times (-1)} = \sqrt{7}\sqrt{-1} = i\sqrt{7}$$

Substituting this back into the expression for z:

$$z = \frac{3 \pm i\sqrt{7}}{2}$$

We can write the two solutions in the required Cartesian form, $x + iy$:

$$z_1 = \frac{3}{2} + i\frac{\sqrt{7}}{2}$$

$$z_2 = \frac{3}{2} - i\frac{\sqrt{7}}{2}$$

Final Answer: The two solutions are $z = \frac{3}{2} + i\frac{\sqrt{7}}{2}$ and $z = \frac{3}{2} - i\frac{\sqrt{7}}{2}$.

Takeaways 9.2

This problem reinforces essential concepts in quadratic equations with complex numbers:

- **Discriminant Analysis:** When $\Delta = b^2 - 4ac < 0$, the equation has two complex conjugate roots. Here, $\Delta = -7$.
- **Complex Conjugate Pairs:** For quadratic equations with real coefficients, complex roots always come in conjugate pairs: $a + bi$ and $a - bi$.
- **Imaginary Unit:** $\sqrt{-n} = i\sqrt{n}$ for positive real n is a fundamental identity for working with complex numbers.
- **Geometric Interpretation:** These solutions are symmetric about the real axis in the complex plane, both at distance $\sqrt{(\frac{3}{2})^2 + (\frac{\sqrt{7}}{2})^2} = \sqrt{\frac{9+7}{4}} = 2$ from the origin.

Problem 9.3: Polynomial with Given Factor

Given that $(z + 2 - i)$ is a factor of $P(z) = z^4 + 4z^3 + 3z^2 - 8z - 10$, factorise $P(z)$ over the set of complex numbers.

Solution 9.3

Strategy: Use the Conjugate Root Theorem to find the conjugate factor, multiply to get a real quadratic, divide into $P(z)$, then factorize the quotient.
Since $P(z)$ has real coefficients and $(z + 2 - i)$ is a factor, the conjugate root theorem implies $(z + 2 + i)$ is also a factor.
Step 1: Find the quadratic from conjugate factors
The product gives a quadratic with real coefficients:

$$Q(z) = (z + 2 - i)(z + 2 + i) = ((z + 2) - i)((z + 2) + i) = (z + 2)^2 + 1 = z^2 + 4z + 5$$

Step 2: Perform polynomial division
Write $P(z) = (z^2 + 4z + 5)(az^2 + bz + c)$. Comparing coefficients:

- Leading term: $a = 1$
- Constant term: $5c = -10 \implies c = -2$
- Coefficient of z^3: $b + 4 = 4 \implies b = 0$

Thus $R(z) = z^2 - 2 = (z - \sqrt{2})(z + \sqrt{2})$.
Final Answer: $P(z) = (z + 2 - i)(z + 2 + i)(z - \sqrt{2})(z + \sqrt{2})$

Takeaways 9.3

This problem illustrates several key polynomial factorization techniques:

- **Conjugate Root Theorem:** For polynomials with real coefficients, complex roots always occur in conjugate pairs. If α is a root, so is $\bar{\alpha}$.
- **Product of Conjugate Factors:** $(z-(a+bi))(z-(a-bi)) = (z-a)^2+b^2$ always gives a quadratic with real coefficients.
- **Polynomial Division Strategy:** Compare leading and constant coefficients first for quick results, then work through middle terms.
- **Complete Factorization:** Over $\mathbb{C}$, every polynomial factors completely into linear factors. Over $\mathbb{R}$, we can have irreducible quadratics.
- **Verification:** We can verify by expanding: $(z^2+4z+5)(z^2-2)$ should give the original polynomial.

Problem 9.4: Finding Polynomial Coefficients from Roots

A cubic polynomial has the form

$$p(z) = z^3 + bz^2 + cz + d, \quad z \in \mathbb{C}, \quad \text{where } b, c, d \in \mathbb{R}.$$

Given that a solution of $p(z) = 0$ is $z_1 = 3-2i$ and that $p(-2) = 0$, find the values of b, c and d.

Solution 9.4

Strategy: Use the Conjugate Root Theorem to identify all three roots, then apply Vieta's formulas to relate roots to coefficients directly.
Since the coefficients are real, if $z_1 = 3-2i$ is a root, then $z_2 = 3+2i$ (conjugate) is also a root. Given $p(-2) = 0$, the three roots are: $z_1 = 3-2i$, $z_2 = 3+2i$, $z_3 = -2$.
Applying Vieta's formulas:
Finding b: $-b = z_1 + z_2 + z_3 = (3-2i) + (3+2i) + (-2) = 4 \implies b = -4$
Finding c: $c = z_1z_2 + z_1z_3 + z_2z_3$
Note $z_1z_2 = (3-2i)(3+2i) = 9+4 = 13$, so:

$$c = 13 + (3-2i)(-2) + (3+2i)(-2) = 13 - 12 = 1$$

Finding d: $-d = z_1z_2z_3 = 13 \cdot (-2) = -26 \implies d = 26$
Final Answer: $b = -4$, $c = 1$, $d = 26$. The polynomial is $p(z) = z^3 - 4z^2 + z + 26$.

Takeaways 9.4

This problem showcases efficient polynomial reconstruction techniques:

- **Vieta's Formulas:** These provide direct relationships between roots and coefficients, eliminating the need for expansion or division.
- **Product of Conjugates:** $(a+bi)(a-bi) = a^2+b^2$ is a key simplification. Here, $(3-2i)(3+2i) = 9+4 = 13$.
- **Imaginary Parts Cancel:** When adding conjugate pairs, imaginary parts always cancel: $(3-2i)+(3+2i) = 6$.
- **Efficient Calculation:** Notice how $z_1z_2 + z_1z_3 + z_2z_3 = 13 + (z_1+z_2)(z_3) = 13 + 6(-2) = 1$.
- **Verification Method:** We can verify by substituting back: $p(3-2i)$ should equal zero.

Problem 9.5: Polynomial with Real Parameter

Given that w is a root of the cubic equation $z^3 + iz^2 + ikz + 2i = 0$, where k is real, and $(1-i)w$ is real, find the possible value of k.

Solution 9.5

Strategy: Use the constraint that $(1-i)w$ is real to find the form of w, substitute into the cubic equation, then separate real and imaginary parts to solve for k.

Step 1: Determine the form of w

Let $w = x + iy$ where $x, y \in \mathbb{R}$. Since $(1-i)w = (1-i)(x+iy) = x + y + i(y-x)$ is real:

$$y - x = 0 \implies y = x \implies w = x(1+i), \quad x \neq 0$$

(Note: $x \neq 0$ since $w = 0$ gives $2i = 0$, a contradiction)

Step 2: Calculate powers and substitute

For $w = x(1+i)$:

$$w^2 = x^2(1+i)^2 = x^2(2i) = 2ix^2$$

$$w^3 = w \cdot w^2 = x(1+i) \cdot 2ix^2 = 2ix^3(1+i) = -2x^3 + 2ix^3$$

Substituting into $z^3 + iz^2 + ikz + 2i = 0$:

$$(-2x^3 + 2ix^3) + i(2ix^2) + ikx(1+i) + 2i = 0$$

$$(-2x^3 - 2x^2 - kx) + i(2x^3 + kx + 2) = 0$$

Step 3: Solve the system

Equating real and imaginary parts to zero:

$$-2x^3 - 2x^2 - kx = 0 \tag{3}$$

$$2x^3 + kx + 2 = 0 \tag{4}$$

From (3), divide by $-x$: $k = -2x^2 - 2x$

Substitute into (4): $2x^3 + (-2x^2 - 2x)x + 2 = 0 \implies -2x^2 + 2 = 0 \implies x = \pm 1$

Step 4: Find values of k

For $x = 1$: $k = -2(1)^2 - 2(1) = -4$ (root: $w = 1 + i$)

For $x = -1$: $k = -2(-1)^2 - 2(-1) = 0$ (root: $w = -1 - i$)

Final Answer: $k = -4$ or $k = 0$

Takeaways 9.5

This problem combines constraint analysis with polynomial root theory:

- **Complex Constraint Analysis:** The condition "$(1-i)w$ is real" translates to requiring the imaginary part to vanish, giving us $y = x$.
- **Parametric Form:** Expressing $w = x(1+i)$ reduces the problem from two unknowns (x, y) to one unknown (x).
- **Simultaneous Equations:** Separating complex equations into real and imaginary parts always yields a system of real equations.
- **Strategic Elimination:** Dividing the first equation by x (since $x \neq 0$) allows us to express k in terms of x, which we then substitute into the second equation.
- **Multiple Solutions:** The problem allows two values of k because different values of x satisfy the constraints.

9.2 Medium Polynomial Problems

Problem 9.6: Roots of Unity and Sum Relations

Let w be a complex number such that $1 + w + w^2 + \cdots + w^6 = 0$.

(i) Show that w is a 7th root of unity.

The complex number $\alpha = w + w^2 + w^4$ is a root of the equation $x^2 + bx + c = 0$, where b and c are real and α is not real.

(ii) Find the other root of $x^2 + bx + c = 0$ in terms of positive powers of w.

(iii) Find the numerical value of c.

Solution 9.6

Strategy: Use geometric series to show $w^7 = 1$, conjugate root theorem for the second root, and Vieta's formula with the sum relation for c.

(i) **Show that w is a 7th root of unity.**

The sum $S = 1 + w + w^2 + \cdots + w^6 = 0$ is a geometric series. For $w \neq 1$:

$$0 = \frac{w^7 - 1}{w - 1} \implies w^7 = 1$$

(Note: $w = 1$ gives $S = 7 \neq 0$, so $w \neq 1$)

(ii) **Find the other root of $x^2 + bx + c = 0$.**

Since b, c are real and α is not real, the other root is $\beta = \bar{\alpha}$.

For $\alpha = w + w^2 + w^4$ and using $\bar{w}^k = w^{-k}$ with $w^7 = 1$:

$$\beta = \bar{w} + \overline{w^2} + \overline{w^4} = w^{-1} + w^{-2} + w^{-4} = w^6 + w^5 + w^3$$

(iii) **Find the numerical value of c.**

By Vieta's formula: $c = \alpha\beta = (w + w^2 + w^4)(w^3 + w^5 + w^6)$

Expanding:

$$c = (w^4 + w^6 + w^7) + (w^5 + w^7 + w^8) + (w^7 + w^9 + w^{10})$$

Using $w^7 = 1$, $w^8 = w$, $w^9 = w^2$, $w^{10} = w^3$:

$$c = (w^4 + w^6 + 1) + (w^5 + 1 + w) + (1 + w^2 + w^3) = 3 + (w + w^2 + w^3 + w^4 + w^5 + w^6)$$

From part (i): $1 + w + w^2 + \cdots + w^6 = 0 \implies w + w^2 + \cdots + w^6 = -1$

Therefore: $c = 3 + (-1) = 2$

Final Answer: (i) $w^7 = 1$; (ii) $w^3 + w^5 + w^6$; (iii) $c = 2$

Takeaways 9.6

This problem demonstrates deep connections between roots of unity and polynomial theory:

- **Geometric Series Formula:** For $r \neq 1$, $\sum_{k=0}^{n-1} r^k = \frac{r^n-1}{r-1}$ is essential for proving root of unity properties.
- **Conjugate Properties for Unit Circle:** When $|w| = 1$, we have $\bar{w} = w^{-1}$, which is crucial for converting negative to positive exponents.
- **Cyclic Property:** $w^7 = 1$ means all exponents can be reduced modulo 7, simplifying calculations.
- **Sum of Roots of Unity:** The identity $1 + w + w^2 + \cdots + w^{n-1} = 0$ for primitive n-th roots of unity is fundamental.
- **Vieta's Formula Application:** Product of roots equals c in $x^2 + bx + c = 0$, providing a direct path to the answer.

Problem 9.7: Cube Roots and Trigonometric Products

The number $w = e^{\frac{2\pi i}{3}}$ is a complex cube root of unity. The number γ is a cube root of w.

(i) Show that $\gamma + \bar{\gamma}$ is a real root of $z^3 - 3z + 1 = 0$.

(ii) By using part (i) to find the exact value of $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}$, deduce the value(s) of $\cos\frac{2^n\pi}{9}\cos\frac{2^{n+1}\pi}{9}\cos\frac{2^{n+2}\pi}{9}$ for all integers $n \geq 1$. Justify your answer.

Solution 9.7

Strategy: Use binomial expansion of $(\gamma + \bar{\gamma})^3$ with $\gamma^3 = w$ to verify the polynomial. Apply Vieta's formulas for the product, and show the product is constant via modular periodicity.

Part (i): Show that $\gamma + \bar{\gamma}$ is a real root of $z^3 - 3z + 1 = 0$

Given $\gamma^3 = w = e^{\frac{2\pi i}{3}}$, the three cube roots are $\gamma = e^{\frac{2\pi i}{9}} e^{\frac{2k\pi i}{3}}$ for $k = 0, 1, 2$, giving $\gamma_0 = e^{\frac{2\pi i}{9}}$, $\gamma_1 = e^{\frac{8\pi i}{9}}$, $\gamma_2 = e^{-\frac{4\pi i}{9}}$.

Let $z = \gamma + \bar{\gamma} = 2\cos\theta$ (real by conjugate property). Expanding:

$$z^3 - 3z = (\gamma + \bar{\gamma})^3 - 3(\gamma + \bar{\gamma}) = \gamma^3 + \bar{\gamma}^3 + 3\gamma\bar{\gamma}(\gamma + \bar{\gamma}) - 3(\gamma + \bar{\gamma})$$

Since $|\gamma| = 1$ (as $|\gamma|^3 = |w| = 1$):

$$z^3 - 3z = \gamma^3 + \bar{\gamma}^3 = e^{\frac{2\pi i}{3}} + e^{-\frac{2\pi i}{3}} = 2\cos\frac{2\pi}{3} = -1$$

Therefore $z^3 - 3z + 1 = 0$, proving $\gamma + \bar{\gamma}$ is a real root.

Part (ii): Find $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}$ and deduce the general value

The three roots of $z^3 - 3z + 1 = 0$ are:

$$z_0 = 2\cos\frac{2\pi}{9}, \quad z_1 = 2\cos\frac{8\pi}{9}, \quad z_2 = 2\cos\frac{4\pi}{9}$$

By Vieta's formula, the product of roots is $-c = -1$:

$$8\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -1 \implies \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -\frac{1}{8}$$

Deduction: For $P_n = \cos\frac{2^n\pi}{9}\cos\frac{2^{n+1}\pi}{9}\cos\frac{2^{n+2}\pi}{9}$, note that $2^n \bmod 18$ has period 6 (since $2^7 \equiv 2 \pmod{18}$). Using the identity $\cos\theta\cos(2\theta)\cos(4\theta) = \frac{\sin(8\theta)}{8\sin\theta}$ and analyzing each case modulo the period shows that the product remains constant at $-\frac{1}{8}$ for all $n \geq 1$.

Final Answer: (i) Shown; (ii) $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -\frac{1}{8}$ for all $n \geq 1$

Takeaways 9.7

This elegant problem reveals deep connections in complex analysis and trigonometry:

- **Roots of Roots of Unity:** The n-th roots of a primitive m-th root of unity are primitive (nm)-th roots of unity.
- **Conjugate Sum Formula:** For $z = e^{i\theta}$, $z + \bar{z} = 2\cos\theta$ is a fundamental bridge between complex and trigonometric forms.
- **Vieta's Product Formula:** For $z^3 + az^2 + bz + c = 0$, the product of roots equals $-c/1$.
- **Modular Arithmetic in Trigonometry:** The periodicity of $2^n \bmod 18$ with period 6 explains why the product is constant.
- **Product-to-Sum Identity:** $\cos\theta\cos(2\theta)\cos(4\theta) = \frac{\sin(8\theta)}{8\sin\theta}$ is a powerful tool for evaluating such products.

Problem 9.8: Conjugate Root Theorem and Factorization

The complex number $2+i$ is a zero of the polynomial

$$P(z) = z^4 - 3z^3 + cz^2 + dz - 30$$

where c and d are real numbers.

(i) Explain why $2-i$ is also a zero of the polynomial $P(z)$.

(ii) Find the remaining zeros of the polynomial $P(z)$.

Solution 9.8

Strategy: Apply Conjugate Root Theorem for part (i), then construct the quadratic factor from the conjugate pair and use coefficient comparison to find remaining roots.
Part (i): By the **Conjugate Root Theorem**, if $P(z)$ has real coefficients and $2+i$ is a zero, then its conjugate $\overline{2+i} = 2-i$ must also be a zero. Since c and d are real, all coefficients of $P(z) = z^4 - 3z^3 + cz^2 + dz - 30$ are real.
Part (ii): Since $2+i$ and $2-i$ are zeros, the conjugate factor is:

$$(z-(2+i))(z-(2-i)) = ((z-2)-i)((z-2)+i) = (z-2)^2 + 1 = z^2 - 4z + 5$$

This is a factor of $P(z)$, so $P(z) = (z^2 - 4z + 5)(z^2 + Az + B)$ for some A, B.
Expanding: $(z^2-4z+5)(z^2+Az+B) = z^4+(A-4)z^3+(B-4A+5)z^2+(5A-4B)z+5B$
Comparing coefficients with $P(z)$:

- z^3: $A - 4 = -3 \implies A = 1$
- Constant: $5B = -30 \implies B = -6$

Therefore $Q(z) = z^2 + z - 6 = (z+3)(z-2)$, giving zeros $z = -3$ and $z = 2$.
Final Answer: (i) By Conjugate Root Theorem; (ii) -3 and 2

Takeaways 9.8

This problem reinforces polynomial factorization techniques with complex numbers:

- **Conjugate Root Theorem:** Essential for polynomials with real coefficients—complex roots always come in conjugate pairs.
- **Difference of Squares:** $(z-(a+bi))(z-(a-bi)) = ((z-a)-bi)((z-a)+bi) = (z-a)^2 + b^2$.
- **Strategic Coefficient Comparison:** Compare leading and constant terms first for immediate results, then middle terms.
- **Factoring Quadratics:** $z^2 + z - 6 = (z+3)(z-2)$ by inspection or quadratic formula.
- **Complete Factorization:** $P(z) = (z-2-i)(z-2+i)(z+3)(z-2) = (z^2 - 4z+5)(z+3)(z-2)$.

Problem 9.9: Verifying Complex Roots

Consider the polynomial:

$$P(z) = z^3 - z^2 - 7z + 15$$

1. Show that $z = 2 + i$ is a root of $P(z)$.
2. Find the other two roots of $P(z)$.
3. Hence express $P(z)$ as a product of factors with real coefficients.

Solution 9.9

Strategy: Direct substitution to verify root; apply Conjugate Root Theorem; use constant term comparison to find remaining root.
Part (a): Calculate $(2+i)^2 = 3+4i$ and $(2+i)^3 = (2+i)(3+4i) = 2+11i$. Then:

$$P(2+i) = (2+11i) - (3+4i) - 7(2+i) + 15 = 0$$

Thus $z = 2+i$ is a root.
Part (b): By Conjugate Root Theorem, $2-i$ is also a root. The conjugate factor is:

$$(z-(2+i))(z-(2-i)) = (z-2)^2 + 1 = z^2 - 4z + 5$$

Since $P(z)$ is cubic, $P(z) = (z^2-4z+5)(z-\alpha)$. Comparing constant terms:

$$5(-\alpha) = 15 \implies \alpha = -3$$

The other roots are $2-i$ and -3.
Part (c): $P(z) = (z^2-4z+5)(z+3)$
Final Answer: (a) Verified; (b) $2-i$ and -3; (c) $P(z) = (z^2-4z+5)(z+3)$

Takeaways 9.9

This problem demonstrates the process of working with complex polynomial roots:

- **Complex Arithmetic:** Careful calculation with $(a+bi)^2 = a^2 - b^2 + 2abi$ and $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$.
- **Verification by Substitution:** Always separate real and imaginary parts when substituting complex numbers.
- **Conjugate Factor Product:** $(z-(a+bi))(z-(a-bi)) = z^2 - 2az + (a^2+b^2)$ has only real coefficients.
- **Constant Term Method:** For $P(z) = Q(z)(z-\alpha)$, comparing constant terms gives constant of $Q \times (-\alpha) =$ constant of P.
- **Mixed Real and Complex Roots:** Cubic polynomials with real coefficients have either three real roots or one real and two complex conjugate roots.

Problem 9.10: Double Roots and Polynomial Structure

Consider the quintic polynomial:

$$P(x) = x^5 - 5x^4 + 12x^3 - 16x^2 + 12x - 4$$

1. Show that $x = 1 + i$ is a double root.
2. Hence, find the other 4 roots and write $P(x)$ as a product of real linear and quadratic factors.

Solution 9.10

Strategy: Use double root criterion ($P(\alpha) = P'(\alpha) = 0$) and Conjugate Root Theorem to establish $(x^2-2x+2)^2$ as a factor, then find remaining factor by coefficient comparison.
Part (a): A double root satisfies $P(\alpha) = 0$ and $P'(\alpha) = 0$. By Conjugate Root Theorem, if $1 + i$ is a root, then $1 - i$ is also a root, giving quadratic factor:

$$(x - (1 + i))(x - (1 - i)) = x^2 - 2x + 2$$

If $1+i$ is a double root (by conjugacy, $1-i$ is also double), then $(x^2-2x+2)^2$ divides $P(x)$. Since $P(x)$ is degree 5 and $(x^2 - 2x + 2)^2$ is degree 4, write $P(x) = (x^2 - 2x + 2)^2(ax + b)$ with $a = 1$ (monic).
Comparing constant terms: $4b = -4 \implies b = -1$, so $P(x) = (x^2 - 2x + 2)^2(x - 1)$.
Verify: Let $R(x) = x^2 - 2x + 2$. Then $P'(x) = 2R(x)R'(x)(x - 1) + R(x)^2$. At $x = 1 + i$, $R(1 + i) = 0$, so $P(1 + i) = 0$ and $P'(1 + i) = 0$
Check $P''(1 + i) \neq 0$: Since $R(1 + i) = 0$, $P''(1 + i) = 2[R'(1 + i)]^2(1 + i - 1) = 2(2i)^2 i = -8i \neq 0$
Part (b): From $P(x) = (x^2 - 2x + 2)^2(x - 1)$, the five roots are $1 + i$ (double), $1 - i$ (double), and 1 (simple).
Final Answer: (a) Verified via derivative test; (b) Roots: $1 + i$ (twice), $1 - i$ (twice), 1 (once); $P(x) = (x - 1)(x^2 - 2x + 2)^2$

Takeaways 9.10

This problem illustrates advanced polynomial root analysis:

- **Multiple Root Criterion:** α is a root of multiplicity k if $P(\alpha) = P'(\alpha) = \cdots = P^{(k-1)}(\alpha) = 0$ but $P^{(k)}(\alpha) \neq 0$.
- **Conjugate Multiplicity:** If $a+bi$ is a root of multiplicity k for a real polynomial, then $a - bi$ also has multiplicity k.
- **Factored Form Powers:** $(x^2 - 2x + 2)^2$ contributes four roots (two pairs of conjugates).
- **Coefficient Comparison:** Matching leading and constant terms quickly determines unknown factors.
- **Derivative Test:** Using $P'(\alpha) = 0$ confirms multiplicity without full factorization.

9.3 Advanced Polynomial Problems

Problem 9.11: Complex Solutions with Triangle Inequality

Consider the equation

$$z^n \cos(n\theta) + z^{n-1} \cos((n-1)\theta) + z^{n-2} \cos((n-2)\theta) + \cdots + z \cos(\theta) = 1$$

where $z \in \mathbb{C}$, $\theta \in \mathbb{R}$, and n is a positive integer.
Using a proof by contradiction and the triangle inequality, or otherwise, prove that all the solutions to the equation lie outside the circle $|z| = \frac{1}{2}$ on the complex plane.

Solution 9.11

Strategy: Use proof by contradiction: assume $|z_0| \leq \frac{1}{2}$ for a solution, apply triangle inequality to bound $|E|$, then show this contradicts $|E| = 1$.
Proof by Contradiction: Assume there exists a solution z_0 with $|z_0| \leq \frac{1}{2}$.
Let $E = \sum_{k=1}^{n} z^k \cos(k\theta)$. Since z_0 is a solution, $|E| = 1$.
By the triangle inequality:

$$|E| = \left| \sum_{k=1}^{n} z^k \cos(k\theta) \right| \leq \sum_{k=1}^{n} |z|^k |\cos(k\theta)| \leq \sum_{k=1}^{n} |z|^k$$

Using $|z_0| \leq \frac{1}{2}$ and $|\cos(k\theta)| \leq 1$:

$$|E| \leq \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}(1-(1/2)^n)}{1-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n < 1$$

This gives $|E| < 1$, contradicting $|E| = 1$.
Therefore, all solutions satisfy $|z| > \frac{1}{2}$.

Takeaways 9.11

This problem showcases sophisticated proof techniques in complex analysis:

- **Triangle Inequality:** $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$ is fundamental for bounding complex sums.
- **Proof by Contradiction:** Assume the negation, derive a logical impossibility, conclude the original statement is true.
- **Modulus Properties:** $|z^k| = |z|^k$ and $|z_1 z_2| = |z_1||z_2|$ simplify modulus calculations.
- **Bounded Trigonometric Functions:** $|\cos\theta| \leq 1$ for all real θ provides crucial bounds.
- **Geometric Series:** The formula $\sum_{k=1}^{n} r^k = r\frac{1-r^n}{1-r}$ is essential for summing powers.
- **Strict Inequality:** The key insight is that $\sum_{k=1}^{n}(1/2)^k < 1$ for all finite n, creating the necessary contradiction.

Problem 9.12: Equilateral Triangle in Complex Plane

Let w be the complex number $w = e^{\frac{2\pi i}{3}}$.

(i) Show that $1 + w + w^2 = 0$.

Three complex numbers a, b and c are represented in the complex plane by points A, B and C respectively.

(ii) Show that if triangle ABC is anticlockwise and equilateral, then $a + bw + cw^2 = 0$.

(iii) It can be shown that if triangle ABC is clockwise and equilateral, then $a + bw^2 + cw = 0$. (Do NOT prove this.)

Show that if ABC is an equilateral triangle, then

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

Solution 9.12

Strategy: Part (i) uses geometric series or factorization of $z^3 - 1$. Part (ii) exploits rotation property: $120°$ rotation in complex plane gives $a + bw + cw^2 = 0$. Part (iii) multiplies anticlockwise and clockwise conditions, then applies $w + w^2 = -1$.
Let $w = e^{\frac{2\pi i}{3}}$.

(i) Since $w^3 = e^{2\pi i} = 1$, we have $1 + w + w^2 = \frac{w^3-1}{w-1} = 0$. Alternatively, w satisfies $z^3 - 1 = (z-1)(z^2+z+1) = 0$, so $w^2 + w + 1 = 0$.

(ii) Rotating $\vec{BC}$ by $120°$ anticlockwise gives $\vec{CA}$: $a - c = (c - b)w$. Rearranging: $a + bw - c(1 + w) = 0$. Since $1 + w = -w^2$ from (i), we get $a + bw + cw^2 = 0$.

(iii) For anticlockwise: $a+bw+cw^2 = 0$. For clockwise: $a+bw^2+cw = 0$. Multiplying:

$$(a+bw+cw^2)(a+bw^2+cw) = a^2+b^2w^3+c^2w^3+ab(w+w^2)+ac(w+w^2)+bc(w+w^2)$$

Using $w^3 = 1$ and $w + w^2 = -1$:

$$a^2 + b^2 + c^2 - ab - ac - bc = 0$$

Therefore, $a^2 + b^2 + c^2 = ab + bc + ca$.

Final Answer: (i) $1 + w + w^2 = 0$ by geometric series or roots of unity; (ii) Rotation property gives $a + bw + cw^2 = 0$; (iii) $a^2 + b^2 + c^2 = ab + bc + ca$.

Takeaways 9.12

This beautiful problem connects complex numbers with geometry:

- **Roots of Unity Properties:** For $w = e^{2\pi i/3}$, we have $w^3 = 1$ and $1+w+w^2 = 0$.
- **Rotation in Complex Plane:** Multiplying by $e^{i\theta}$ rotates a complex number by angle θ counterclockwise.
- **Equilateral Triangle Condition:** The relation $a+bw+cw^2 = 0$ (or its variant) characterizes equilateral triangles.
- **Algebraic Identity:** $w+w^2 = -1$ is a key simplification that appears repeatedly.
- **Product of Conditions:** Multiplying the anticlockwise and clockwise conditions eliminates the orientation dependence.
- **Symmetric Functions:** The identity $a^2 + b^2 + c^2 = ab + bc + ca$ is a beautiful symmetric relation for equilateral triangles.

Problem 9.13: Fifth Roots of -1 and Trigonometric Values

Consider the equation $z^5 + 1 = 0$, where z is a complex number.

1. Solve the equation $z^5 + 1 = 0$ by finding the 5^{th} roots of -1.
2. Show that if z is a solution of $z^5 + 1 = 0$ and $z \neq -1$, then $u = z + \frac{1}{z}$ is a solution of $u^2 - u - 1 = 0$.
3. Hence find the exact value of $\cos \frac{3\pi}{5}$.

Solution 9.13

Strategy: Part (1) applies de Moivre's theorem for fifth roots. Part (2) divides by z^2 and substitutes $u = z + 1/z$ to derive a quadratic. Part (3) uses $u = 2\cos\theta$ and quadrant analysis to select the correct root.

1. Solving $z^5 = -1$ with $-1 = e^{i\pi}$, the fifth roots are $z_k = e^{i(2k+1)\pi/5}$ for $k = 0, 1, 2, 3, 4$, giving $\{e^{i\pi/5}, e^{i3\pi/5}, -1, e^{i7\pi/5}, e^{i9\pi/5}\}$.
2. For $z \neq -1$, factor $z^5+1 = (z+1)(z^4-z^3+z^2-z+1) = 0$, so $z^4-z^3+z^2-z+1 = 0$. Dividing by z^2:
$$z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0 \implies \left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 1 = 0$$
Let $u = z + \frac{1}{z}$. Then $u^2 = z^2 + 2 + \frac{1}{z^2}$, so $z^2 + \frac{1}{z^2} = u^2 - 2$. Substituting: $(u^2 - 2) - u + 1 = 0$, giving $u^2 - u - 1 = 0$.
3. For $z_1 = e^{i3\pi/5}$, we have $u = e^{i3\pi/5} + e^{-i3\pi/5} = 2\cos\frac{3\pi}{5}$. From part (ii), $u^2 - u - 1 = 0$ gives $u = \frac{1\pm\sqrt{5}}{2}$. Since $\frac{3\pi}{5}$ is in the second quadrant, $\cos\frac{3\pi}{5} < 0$. Thus $2\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{2}$, so $\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$.

Final Answer: (1) $z_k = e^{i(2k+1)\pi/5}$ for $k = 0, 1, 2, 3, 4$; (2) Shown by polynomial division and substitution; (3) $\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$.

Takeaways 9.13

This problem beautifully connects roots of unity with exact trigonometric values:

- **de Moivre's Theorem:** For finding n-th roots, use $z^n = re^{i\theta} \implies z = r^{1/n}e^{i(\theta+2k\pi)/n}$.
- **Polynomial Factorization:** $z^5+1 = (z+1)(z^4-z^3+z^2-z+1)$ separates the real root.
- **Clever Substitution:** Setting $u = z + 1/z$ reduces a quartic to a quadratic, a powerful technique.
- **Euler's Formula Bridge:** $z + \bar{z} = 2\cos\theta$ connects complex and trigonometric forms.
- **Quadrant Analysis:** Determining the sign of $\cos\theta$ from the quadrant is essential for selecting the correct root.
- **Golden Ratio Connection:** $(1+\sqrt{5})/2$ is the golden ratio ϕ, appearing naturally in pentagon geometry.

Problem 9.14: De Moivre's Theorem and Secant Value

1. Solve $z^5 + 1 = 0$ by de Moivre's theorem, leaving your solutions in modulus-argument form.
2. Prove that the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^5 + 1 = 0$.
3. Show that if $z^4 - z^3 + z^2 - z + 1 = 0$ where $z = \operatorname{cis}\,\theta$ then $4\cos^2\theta - 2\cos\theta - 1 = 0$.
4. Hence find the exact value of $\sec\frac{3\pi}{5}$.

Solution 9.14

Strategy: Part (i) applies de Moivre's theorem. Part (ii) uses factorization: $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$. Part (iii) divides by z^2 and substitutes $z + 1/z = 2\cos\theta$. Part (iv) solves the quadratic and rationalizes.

1. Solving $z^5 = -1$ with $-1 = \text{cis}(\pi)$ gives $z_k = \text{cis}((2k+1)\pi/5)$ for $k = 0, 1, 2, 3, 4$, yielding $\{\text{cis}(\pi/5), \text{cis}(3\pi/5), -1, \text{cis}(7\pi/5), \text{cis}(9\pi/5)\}$.

2. Factor $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$. The root $z = -1$ is real. The quartic factor gives the four non-real solutions z_0, z_1, z_3, z_4 with modulus 1 and non-zero arguments.

3. For $z^4 - z^3 + z^2 - z + 1 = 0$, divide by z^2: $z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0$. Group: $(z^2 + \frac{1}{z^2}) - (z + \frac{1}{z}) + 1 = 0$. For $z = \text{cis}\,\theta$, use $z + \frac{1}{z} = 2\cos\theta$ and $z^2 + \frac{1}{z^2} = (z + \frac{1}{z})^2 - 2 = 4\cos^2\theta - 2$. Substituting: $(4\cos^2\theta - 2) - 2\cos\theta + 1 = 0$, giving $4\cos^2\theta - 2\cos\theta - 1 = 0$.

4. The quadratic gives $\cos\theta = \frac{2\pm2\sqrt{5}}{8} = \frac{1\pm\sqrt{5}}{4}$. Since $\frac{3\pi}{5}$ is in the second quadrant, $\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$. Thus $\sec\frac{3\pi}{5} = \frac{4}{1-\sqrt{5}} = \frac{4(1+\sqrt{5})}{-4} = -(1+\sqrt{5})$.

Final Answer: (i) $z_k = \text{cis}((2k+1)\pi/5)$ for $k = 0, 1, 2, 3, 4$; (ii) Shown by factorization; (iii) $4\cos^2\theta - 2\cos\theta - 1 = 0$; (iv) $\sec\frac{3\pi}{5} = -(1+\sqrt{5})$.

Takeaways 9.14

This comprehensive problem ties together multiple advanced concepts:

- **Cis Notation:** $\text{cis}\,\theta = \cos\theta + i\sin\theta = e^{i\theta}$ is a compact notation for complex exponentials.
- **Polynomial Factorization:** Separating real from non-real roots via $(z+1)$ factor.
- **Trigonometric Substitution:** For $|z| = 1$, the substitution $z + 1/z = 2\cos\theta$ is fundamental.
- **Double Angle Formula:** $\cos(2\theta) = 2\cos^2\theta - 1$ can derive $z^2 + 1/z^2$.
- **Rationalizing Denominators:** Multiply by conjugate: $\frac{1}{1-\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} = \frac{1+\sqrt{5}}{-4}$.
- **Sign Determination:** Quadrant analysis is crucial for selecting correct values from quadratic formula.

Problem 9.15: Tangent Function and Product Identity

(i) Use De Moivre's theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$.
(ii) Hence show that $x^4 - 10x^2 + 5 = 0$ has roots $\pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$.
(iii) Deduce that $\tan\frac{\pi}{5} \cdot \tan\frac{2\pi}{5} \cdot \tan\frac{3\pi}{5} \cdot \tan\frac{4\pi}{5} = 5$.

Solution 9.15

Strategy: Part (i) expands $(\cos\theta + i\sin\theta)^5$ via binomial theorem and divides by $\cos^5\theta$. Part (ii) sets $\tan 5\theta = 0$ to find angles $\theta = k\pi/5$, yielding the quartic. Part (iii) applies Vieta's formula for product of roots.

(i) By de Moivre's theorem, $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$. Expanding and separating imaginary/real parts:

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

Dividing numerator and denominator by $\cos^5\theta$:

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}, \quad \text{where } t = \tan\theta$$

(ii) Setting $\tan 5\theta = 0$ gives $5\theta = k\pi$, so $\theta = \frac{k\pi}{5}$. For $0 < \theta < \pi$: $\theta = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$. The numerator vanishes: $t^5 - 10t^3 + 5t = 0$. Since $t \neq 0$, divide by t: $t^4 - 10t^2 + 5 = 0$. Using $\tan(\pi - A) = -\tan A$, the roots are $\pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$.

(iii) By Vieta's formula, for $x^4 - 10x^2 + 5 = 0$, the product of roots is $5/1 = 5$. Therefore:

$$\tan\frac{\pi}{5} \cdot \tan\frac{2\pi}{5} \cdot \tan\frac{3\pi}{5} \cdot \tan\frac{4\pi}{5} = 5$$

Final Answer: (i) $\tan 5\theta = \frac{5t-10t^3+t^5}{1-10t^2+5t^4}$ where $t = \tan\theta$; (ii) Roots are $\pm\tan\frac{\pi}{5}, \pm\tan\frac{2\pi}{5}$; (iii) Product equals 5.

Takeaways 9.15

This elegant problem demonstrates the power of combining complex analysis with algebra:

- **Binomial Expansion:** $(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta(i\sin\theta)^k$ generates multiple-angle formulas.
- **Separation Technique:** Dividing numerator and denominator by $\cos^n\theta$ converts to tangent form.
- **Zero Finding:** Setting $\tan n\theta = 0$ identifies specific angles whose tangents satisfy polynomial equations.
- **Supplementary Angle:** $\tan(\pi - \theta) = -\tan\theta$ explains why roots come in $\pm$ pairs.
- **Vieta's Product Formula:** For $a_n x^n + \cdots + a_0 = 0$, product of roots equals $(-1)^n a_0/a_n$.
- **Pentagon Connection:** These tangent values relate to regular pentagon geometry, where the number 5 appears naturally.

10 Part 2: Problems with Hints and Solutions (Concise)

Part 2 presents additional problems with upside-down hints. Try each problem first, then rotate the page to read the hint if needed. These 23 problems provide additional practice across all difficulty levels, ordered from simpler to more complex within each category.

10.1 Basic Polynomial Problems

Problem 10.1

State the Binomial Theorem for the expansion of $(x+a)^n$, where n is a positive integer. Define the binomial coefficient used in the expansion.

Hint:

The theorem expresses $(x+a)^n$ as a sum involving binomial coefficients $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Solution 10.1

The Binomial Theorem states:

$$(x+a)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} a^r$$

or equivalently:

$$(x+a)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} a + \binom{n}{2} x^{n-2} a^2 + \cdots + \binom{n}{n} a^n$$

The binomial coefficient is defined as:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n! = n \times (n-1) \times \cdots \times 2 \times 1$ and $0! = 1$.
Answer: Binomial Theorem with coefficients $\binom{n}{r}$.

Takeaways 10.1

Binomial coefficients count combinations; central to polynomial expansions.

Problem 10.2

Consider the integral:

$$I = \int \frac{1}{x^6 - x^4}\, dx$$

(i) Factorise the polynomial $P(x) = x^6 - x^4$ completely.

(ii) Decompose the integrand into partial fractions in simplest form.

$$\frac{1}{x^6 - x^4} = A\frac{1}{x^2} + B\frac{1}{x^4} + \frac{C}{x^2 - 1}$$

(iii) Hence, evaluate the integral I.

Hint:

Factor out the greatest common factor and apply difference of squares. For the decomposition, set $u = x^2$ and perform partial fractions in u. Use standard antiderivatives for power functions and the logarithmic form for $1/(x^2 - 1)$.

Solution 10.2

Factor: $x^6 - x^4 = x^4(x^2 - 1) = x^4(x-1)(x+1)$. Let $u = x^2$. Decompose:

$$\frac{1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1}$$

Solving gives $A = -1, B = -1, C = 1$, so returning to x:

$$\frac{1}{x^6 - x^4} = -\frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^2 - 1}$$

Integrating termwise:

$$I = \int \left(-x^{-2} - x^{-4} + \frac{1}{x^2 - 1}\right) dx = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

Takeaways 10.2

- We first factor the polynomial completely to identify the structure of the integrand.
- Partial-fraction substitution can simplify high-degree rational integrands; using $u = x^2$ is a useful trick when even powers appear.
- Recognise standard antiderivatives for quick evaluation.
- Try this technique on other rational functions with polynomial denominators.

$$J = \int \frac{1}{x^6 + x^4}\, dx$$

Problem 10.3

Solve for p, q, r over the complex numbers, given:

$$\begin{aligned} p+q+r &= 1 \\ pq+pr+qr &= 9 \\ pqr &= 9 \end{aligned}$$

Hint:

These are elementary symmetric polynomials. Construct cubic $P(x) = x^3 - x^2 + 9x - 9$ with roots p, q, r. Factor by grouping.

Solution 10.3

The cubic polynomial with roots p, q, r is:

$$P(x) = x^3 - (p+q+r)x^2 + (pq+pr+qr)x - pqr = x^3 - x^2 + 9x - 9$$

Factor by grouping:

$$x^3 - x^2 + 9x - 9 = x^2(x-1) + 9(x-1) = (x^2+9)(x-1) = 0$$

From $x - 1 = 0$: $x = 1$
From $x^2 + 9 = 0$: $x^2 = -9 \implies x = \pm 3i$
Answer: $\{p, q, r\} = \{1, 3i, -3i\}$ (in any order).

Takeaways 10.3

Vieta's formulas connect roots to coefficients; factorization reveals complex roots.

Problem 10.4

The complex roots of $iz^2 + \sqrt{3}z - 1 = 0$ are α and β.

1. Find α and β in Cartesian form.
2. Show that $\alpha^2\beta^2 + 1 = 0$.

Hint:

Use quadratic formula with $a = i$. Find $\sqrt{3-4i}$ by setting $\sqrt{3-4i} = x + iy$ and solving. Part (b): Use Vieta's formula $\alpha\beta = -1/i = i$.

Solution 10.4

(a) Using quadratic formula with $a = i, b = \sqrt{3}, c = -1$:

$$z = \frac{-\sqrt{3} \pm \sqrt{3-4i}}{2i}$$

To find $\sqrt{3-4i}$, let $x + iy = \sqrt{3-4i}$. Then $(x+iy)^2 = 3 - 4i$, giving:

$$x^2 - y^2 = 3, \quad 2xy = -4 \implies y = -2/x$$

Substituting: $x^2 - 4/x^2 = 3 \implies x^4 - 3x^2 - 4 = 0 \implies (x^2-4)(x^2+1) = 0$
Thus $x = 2, y = -1$, so $\sqrt{3-4i} = \pm(2-i)$.
Computing: $\alpha = -\frac{1}{2} + \frac{\sqrt{3}-2}{2}i$ and $\beta = \frac{1}{2} + \frac{\sqrt{3}+2}{2}i$
(b) By Vieta's formulas: $\alpha\beta = \frac{-1}{i} = i$
Therefore: $\alpha^2\beta^2 = (\alpha\beta)^2 = i^2 = -1$, so $\alpha^2\beta^2 + 1 = 0$.

Takeaways 10.4

Finding square roots of complex numbers requires solving simultaneous equations; Vieta's formulas simplify products.

Problem 10.5

Prove that the only integer solution to

$$(x-a)(x-b)(x-c)(x-d) - 4 = 0$$

is $x = \frac{a+b+c+d}{4}$, where a, b, c, d are unique integers.

Hint:

The product equals 4. For distinct integer factors, the only way is $\{1, 2, -1, -2\}$ with product 4. Their sum is 0.

Solution 10.5

Let $y_i = x - i$ for $i \in \{a, b, c, d\}$. Then $y_a y_b y_c y_d = 4$.
Since a, b, c, d are unique integers, the y_i are four distinct integers.
The only way to factor 4 into four distinct integers is $\{-2, -1, 1, 2\}$, since their product is $(-2)(-1)(1)(2) = 4$.
Therefore: $\{x-a, x-b, x-c, x-d\} = \{-2, -1, 1, 2\}$
Summing: $(x-a) + (x-b) + (x-c) + (x-d) = -2 - 1 + 1 + 2 = 0$
Thus: $4x - (a+b+c+d) = 0 \implies x = \frac{a+b+c+d}{4}$
Answer: $x = \frac{a+b+c+d}{4}$ is the unique integer solution.

Takeaways 10.5

Integer factorization constraints severely limit solutions; summing symmetric expressions reveals structure.

Problem 10.6

Without using the rational roots theorem, prove that there is no rational solution to the equation $x^3 + x + 1 = 0$. Hint: Assume there exists a rational root and consider whether the LHS is odd or even.

Hint:

If $x = p/q$ (coprime), then $p^3 + pq^2 + q^3 = 0$. Check all parity cases: (odd,odd), (odd,even), (even,odd). All lead to odd=even contradiction.

Solution 10.6

Assume $x = \frac{p}{q}$ where $\gcd(p, q) = 1$. Substituting into $x^3 + x + 1 = 0$ and multiplying by q^3:

$$p^3 + pq^2 + q^3 = 0$$

Case 1: p odd, q odd $\implies$ $p^3 + pq^2 + q^3 =$ odd + odd + odd = odd $\neq 0$ (even). Contradiction.
Case 2: p odd, q even $\implies$ $p^3 + pq^2 + q^3 =$ odd + even + even = odd $\neq 0$. Contradiction.
Case 3: p even, q odd $\implies$ $p^3 + pq^2 + q^3 =$ even + even + odd = odd $\neq 0$. Contradiction.
All cases lead to contradiction.
Answer: No rational solution exists.

Takeaways 10.6

Parity arguments provide elegant proofs without explicit factorization or rational root tests.

10.2 Medium Polynomial Problems

Problem 10.7

Suppose that $P(x) = x^3 - x^2 + mx + n$, where m and n are integers.

1. Show that $P(-i) = (1+n) + i(1-m)$.
2. When $P(x)$ is divided by x^2+1 the remainder is $6x-3$. Find the values of m and n.

Hint:

(a) Compute $(-i)^2 = -1, (-i)^3 = i$. (b) $P(-i)$ equals remainder evaluated at $-i$: $6(-i)-3 = -3-6i$. Equate real/imaginary parts.

Solution 10.7

(a) $P(-i) = (-i)^3 - (-i)^2 + m(-i) + n = i - (-1) - mi + n = (1+n) + i(1-m)$
(b) Since x^2+1 has root $-i$, by remainder theorem:

$$P(-i) = 6(-i) - 3 = -3 - 6i$$

Equating with part (a): $(1+n) + i(1-m) = -3 - 6i$
Real parts: $1 + n = -3 \implies n = -4$
Imaginary parts: $1 - m = -6 \implies m = 7$
Answer: $m = 7, n = -4$.

Takeaways 10.7

Remainder theorem applies to complex divisors; equate real and imaginary parts separately.

Problem 10.8

1. Show that $(1+i\tan\theta)^n + (1-i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ where $\cos\theta \neq 0$ and n is a positive integer.
2. Hence show that if z is purely imaginary, the roots of $(1+z)^4 + (1-z)^4 = 0$ are $z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$.

Hint:

(a) Rewrite $1 \pm i\tan\theta = \frac{\cos\theta \pm i\sin\theta}{\cos\theta}$, apply De Moivre. (b) Set $z = i\tan\theta$, equation becomes $\frac{2\cos 4\theta}{\cos^4\theta} = 0$, so $\cos 4\theta = 0$.

Solution 10.8

(a) $1 + i\tan\theta = \frac{\cos\theta + i\sin\theta}{\cos\theta} = \frac{e^{i\theta}}{\cos\theta}$
By De Moivre:

$$(1 + i\tan\theta)^n = \frac{e^{in\theta}}{\cos^n\theta}, \quad (1 - i\tan\theta)^n = \frac{e^{-in\theta}}{\cos^n\theta}$$

Sum: $\frac{e^{in\theta}+e^{-in\theta}}{\cos^n\theta} = \frac{2\cos n\theta}{\cos^n\theta}$
(b) Let $z = i\tan\theta$. Then $(1+z)^4 + (1-z)^4 = \frac{2\cos 4\theta}{\cos^4\theta} = 0$
Since $\cos\theta \neq 0$: $\cos 4\theta = 0 \implies 4\theta = \frac{\pi}{2} + k\pi \implies \theta = \frac{\pi}{8} + \frac{k\pi}{4}$
For $k = 0, 1, 2, 3$: $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$
Using $\tan(\pi - x) = -\tan x$: $z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$

Takeaways 10.8

De Moivre's theorem applies to expressions of form $\frac{e^{i\theta}}{\cos^n\theta}$; tangent values related by symmetry.

Problem 10.9

1. Show that $\dfrac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = i\cot\dfrac{\theta}{2}$.

2. Hence solve $\left(\dfrac{z-1}{z+1}\right)^8 = -1$.

Hint:

(a) Use half-angle formulas: $1+\cos\theta = 2\cos^2(\theta/2)$, $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$. *(b) Set* $w = (z-1)/(z+1)$, *find* $w^8 = -1$, *then* $z = \frac{1+w}{1-w}$.

Solution 10.9

(a) Using half-angle identities:

$$\frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = \frac{2\cos^2(\theta/2) + 2i\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2) - 2i\sin(\theta/2)\cos(\theta/2)}$$

$$= \frac{\cos(\theta/2)[\cos(\theta/2) + i\sin(\theta/2)]}{\sin(\theta/2)[\sin(\theta/2) - i\cos(\theta/2)]} = \cot(\theta/2)\cdot\frac{\cos(\theta/2) + i\sin(\theta/2)}{-i[\cos(\theta/2) + i\sin(\theta/2)]} = i\cot(\theta/2)$$

(b) Let $w = \frac{z-1}{z+1}$. Then $w^8 = -1 = e^{i\pi(2k+1)}$, so $w = e^{i\pi(2k+1)/8}$, $k = 0, \ldots, 7$.
From $w = \frac{z-1}{z+1}$: $z = \frac{1+w}{1-w} = \frac{1+\cos\alpha_k + i\sin\alpha_k}{1-\cos\alpha_k - i\sin\alpha_k} = i\cot(\alpha_k/2)$
where $\alpha_k = \frac{(2k+1)\pi}{8}$. Thus $z = i\cot(\frac{(2k+1)\pi}{16})$ for $k = 0, \ldots, 7$.
Answer: $z = \pm i\cot\frac{\pi}{16}, \pm i\cot\frac{3\pi}{16}, \pm i\cot\frac{5\pi}{16}, \pm i\cot\frac{7\pi}{16}$.

Takeaways 10.9

Half-angle formulas simplify complex fractions; eighth roots of -1 give eight solutions.

Problem 10.10

Prove that for $0 \le b < 1$:

$$\frac{1-b^{n+1}}{1-b} < n+1$$

where $n \in \mathbb{Z}^+$.

Hint:

Factor LHS as geometric series: $1 + b + b^2 + \cdots + b^n$. *Compare term-by-term with* $\underbrace{1+1+\cdots+1}_{n+1}$. *Each* $b^k < 1$ *for* $k \ge 1$.

Solution 10.10

LHS: $\frac{1-b^{n+1}}{1-b} = 1 + b + b^2 + \cdots + b^n$ (geometric series)
RHS: $n + 1 = \underbrace{1+1+\cdots+1}_{n+1 \text{ terms}}$
Term-by-term comparison: - First term: $b^0 = 1 = 1$ (equal) - Terms $k = 1, \ldots, n$: $b^k < 1$ since $0 \le b < 1$
Therefore: $1 + b + b^2 + \cdots + b^n < 1 + 1 + \cdots + 1 = n + 1$

Takeaways 10.10

Geometric series allow term-by-term comparison; strict inequality holds when $b < 1$.

Problem 10.11

Prove by induction:

$$x^n + x^{n-2} + x^{n-4} + \cdots + \frac{1}{x^{n-4}} + \frac{1}{x^{n-2}} + \frac{1}{x^n} \ge n+1$$

for $x > 0$ and $n \in \mathbb{Z}^+$. [Hint: separate base cases for n even or odd.]

Hint:

Base cases: $P(1)$*:* $x + 1/x \geq 2$ *(AM-GM);* $P(2)$*:* $x^2 + 1 + 1/x^2 \geq 3$*. Inductive step:* $P(k) \implies P(k+2)$*. Add* $(x^{k+2} + 1/x^{k+2}) \geq 2$*.*

Solution 10.11

Base cases: - $n = 1$: $x + \frac{1}{x} \geq 2$ by AM-GM: $\frac{x+1/x}{2} \geq \sqrt{x \cdot 1/x} = 1$. - $n = 2$: $x^2 + 1 + \frac{1}{x^2} \geq 3$. Since $x^2 + \frac{1}{x^2} \geq 2$ by AM-GM, LHS $\geq 2 + 1 = 3$.
Inductive step: Assume $P(k)$ holds. Prove $P(k+2)$:

$$\text{LHS}_{k+2} = (x^{k+2} + \frac{1}{x^{k+2}}) + \sum_{i=0}^{k} x^{k-2i}$$

By AM-GM: $x^{k+2} + \frac{1}{x^{k+2}} \geq 2$
By induction: $\sum_{i=0}^{k} x^{k-2i} \geq k+1$
Therefore: $\text{LHS}_{k+2} \geq 2 + (k+1) = k+3$, which equals RHS_{k+2}.

Takeaways 10.11

Two base cases handle parity; inductive step jumps by 2 to preserve parity structure.

Problem 10.12

Prove that $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ is an integer for all integers $n \geq 1$.

Hint:

Combine over common denominator 30: $\frac{6n^5+15n^4+10n^3-n}{30} = \frac{n(6n^4+15n^3+10n^2-1)}{30}$*. Show numerator divisible by 30 using Fermat's Little Theorem.*

Solution 10.12

Combine: $P(n) = \frac{6n^5+15n^4+10n^3-n}{30} = \frac{n(6n^4+15n^3+10n^2-1)}{30}$
Note: $P(n) = \sum_{k=1}^{n} k^4$ (sum of fourth powers formula).
Alternatively, show $n^5 - n$ divisible by 30: - Divisible by 2: $(n-1)n(n+1)$ contains even number - Divisible by 3: $(n-1)n(n+1)$ contains multiple of 3 - Divisible by 5: Fermat's Little Theorem gives $n^5 \equiv n \pmod 5$
Since $n^5 - n \equiv 0 \pmod{30}$ and remaining terms also yield multiples of 30, $P(n)$ is an integer.

Takeaways 10.12

Fermat's Little Theorem handles prime divisibility; consecutive integers ensure small prime factors.

Problem 10.13

Let α, β, γ be roots of $x^3 + Ax^2 + Bx + 8 = 0$ (where A, B are real numbers). Given $\alpha^2 + \beta^2 = 0$ and $\beta^2 + \gamma^2 = 0$:

1. Explain why β is real and α, γ are not real.
2. Show α, γ are purely imaginary.
3. Find A and B.

Hint:

(a) If β non-real, conjugate must be root, but constraints force contradictions. (b) From $\alpha^2 = -\beta^2 < 0$. (c) Let $\alpha = bi, \gamma = -bi, \beta = b$. Check product=$-8$ gives $b^2 = -8$ (contradiction: typo in problem?).

Solution 10.13

(a) From conditions: $\alpha^2 = -\beta^2$ and $\gamma^2 = -\beta^2$, so $\alpha^2 = \gamma^2$.
If β non-real, its conjugate $\bar{\beta}$ is also a root. But then α or γ must be real. If γ real, $\gamma^2 \geq 0$, but $\beta^2 = -\gamma^2 \leq 0$ implies β purely imaginary. Testing: if $\beta = bi$, then $\alpha^2 = -\beta^2 = -(-b^2) = b^2 > 0$, making α real. But three roots must include conjugate pair. Contradiction forces β real.
(b) Since β real and $\beta \neq 0$, $\alpha^2 = -\beta^2 < 0$, so $\alpha = \pm i|\beta|$ (purely imaginary). Similarly for γ.
(c) Let $\alpha = bi, \beta = b, \gamma = -bi$. By Vieta: $\alpha\beta\gamma = -8 \implies bi \cdot b \cdot (-bi) = b^2 = -8$.
This is impossible for real b. [Note: Problem likely has typo; constant should be -8 not $+8$.]
If equation is $x^3 + Ax^2 + Bx - 8 = 0$: $b^2 = 8 \implies b = \pm 2\sqrt{2}$, giving $A = \mp 2\sqrt{2}, B = 8$.
Answer: (Assuming corrected problem) $A = \pm 2\sqrt{2}, B = 8$.

Takeaways 10.13

Conjugate root theorem constrains complex roots; Vieta's formulas connect roots to coefficients.

Problem 10.14

1. Given z is a root of $az^3 + bz^2 + cz + d = 0$ (real a, b, c, d), prove $\bar{z}$ is also a root.
2. Find all roots of $z^3 - 6z^2 + 13z - 20 = 0$ given $1 + 2i$ is one root.

Hint:

(a) Take conjugate of $P(z) = 0$: $\overline{P(z)} = \bar{0} = 0$. Since coefficients real, $\overline{P(z)} = P(\bar{z})$. (b) If $z_1 = 1 + 2i$, then $z_2 = 1 - 2i$. Use sum of roots $= -(-6)/1 = 6$.

Solution 10.14

(a) If $P(z) = az^3 + bz^2 + cz + d = 0$, take conjugate:

$$\overline{P(z)} = \bar{a}\overline{z^3} + \bar{b}\overline{z^2} + \bar{c}\bar{z} + \bar{d} = 0$$

Since a, b, c, d real: $\bar{a} = a$, etc. Thus $P(\bar{z}) = 0$, so $\bar{z}$ is a root.
(b) Given $z_1 = 1 + 2i$, by part (a): $z_2 = 1 - 2i$.
Sum of roots: $z_1 + z_2 + z_3 = 6$

$$(1 + 2i) + (1 - 2i) + z_3 = 6 \implies 2 + z_3 = 6 \implies z_3 = 4$$

Answer: Roots are $1 + 2i, 1 - 2i, 4$.

Takeaways 10.14

Conjugate root theorem: complex roots of real polynomials come in conjugate pairs.

Problem 10.15

The roots of $z^n = 1$ are $z_k = e^{2\pi ik/n}$, $k = 1, \ldots, n$. If z_k^m generates all roots for $m = 1, \ldots, n$, then z_k is a primitive root.

1. Show z_1 is a primitive root of $z^n = 1$.
2. Show z_5 is a primitive root of $z^6 = 1$.
3. If $\gcd(n, k) = h$, show z_k primitive implies $h = 1$. Here gcd is the greatest common divisor.

Hint:

(a) $z_1^m = e^{2\pi im/n}$ gives all n roots. (b) For $z_5^6 = 1$: $5m \bmod 6$ gives $5, 4, 3, 2, 1, 0$. (c) $z_k^m = z_{km}^1$; generates all roots iff $\gcd(k, n) = 1$.

Solution 10.15

(a) $z_1^m = e^{2\pi i m/n}$ for $m = 1, \ldots, n$ generates all n-th roots.
(b) $z_5 = e^{10\pi i/6}$. Powers: $z_5^m = e^{10\pi i m/6}$. Values $5m \bmod 6$: $5, 10 \equiv 4, 15 \equiv 3, 20 \equiv 2, 25 \equiv 1, 30 \equiv 0$. These are $\{5, 4, 3, 2, 1, 0\}$—all residues, so all roots generated.
(c) z_k generates all roots iff powers $z_k^m = z_1^{km}$ hit all roots. This requires $km \bmod n$ to cover all residues $0, \ldots, n-1$, which happens iff $\gcd(k, n) = 1$. Thus $h = 1$.

Takeaways 10.15

Primitive roots generate all roots of unity; requires coprimality of index and order.

Problem 10.16

Given $z = \cos\alpha + i \sin\alpha$ with $\sin\alpha \neq 0$:

1. Prove that $\frac{1}{1 - z\cos\alpha} = 1 + i\cot\alpha$
2. Hence, by considering $\sum_{k=0}^{\infty} (z\cos\alpha)^k$, deduce:

$$\sum_{k=1}^{\infty} \sin k\alpha \cos^k \alpha = \cot\alpha$$

Hint:

(a) $1 - z\cos\alpha = \sin^2\alpha - i\sin\alpha\cos\alpha = \sin\alpha(\sin\alpha - i\cos\alpha)$. Rationalize. (b) Geometric series $= 1 + i\cot\alpha$; extract imaginary part.

Solution 10.16

(a) $1 - z\cos\alpha = 1 - \cos^2\alpha - i\sin\alpha\cos\alpha = \sin^2\alpha - i\sin\alpha\cos\alpha = \sin\alpha(\sin\alpha - i\cos\alpha)$
Rationalize:

$$\frac{1}{\sin\alpha(\sin\alpha - i\cos\alpha)} \cdot \frac{\sin\alpha + i\cos\alpha}{\sin\alpha + i\cos\alpha} = \frac{\sin\alpha + i\cos\alpha}{\sin\alpha(\sin^2\alpha + \cos^2\alpha)} = 1 + i\cot\alpha$$

(b) Geometric series: $\sum_{k=0}^{\infty} (z\cos\alpha)^k = \frac{1}{1 - z\cos\alpha} = 1 + i\cot\alpha$
Expand LHS using $z^k = \cos k\alpha + i\sin k\alpha$:

$$1 + \sum_{k=1}^{\infty} (\cos k\alpha + i\sin k\alpha)\cos^k\alpha = 1 + (\text{real part}) + i\sum_{k=1}^{\infty} \sin k\alpha \cos^k\alpha$$

Equating imaginary parts: $\sum_{k=1}^{\infty} \sin k\alpha \cos^k\alpha = \cot\alpha$

Takeaways 10.16

Geometric series with complex terms; separate real/imaginary parts to extract trigonometric identities.

Problem 10.17

Let β be a root of $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with $M = \max(|a_{n-1}|, \ldots, |a_0|)$.

1. Show that $|\beta|^n \leq M(|\beta|^{n-1} + \cdots + |\beta| + 1)$.
2. Hence show for any root β: $|\beta| < 1 + M$.

Hint:

(a) $P(\beta) = 0 \implies \beta^n = -(a_{n-1}\beta^{n-1} + \cdots + a_0)$. *Triangle inequality:* $|\beta|^n \leq |a_{n-1}||\beta|^{n-1} + \cdots \leq M(\cdots)$. *(b) Assume* $|\beta| \geq 1 + M$, *get contradiction.*

Solution 10.17

(a) From $P(\beta) = 0$: $\beta^n = -(a_{n-1}\beta^{n-1} + \cdots + a_0)$
Triangle inequality: $|\beta|^n \leq |a_{n-1}||\beta|^{n-1} + \cdots + |a_0| \leq M(|\beta|^{n-1} + \cdots + 1)$
(b) Assume $|\beta| \geq 1 + M$. Then $|\beta| - 1 \geq M$.
From (a) with geometric series (for $|\beta| > 1$):

$$|\beta|^n \leq M\frac{|\beta|^n - 1}{|\beta| - 1} \leq M\frac{|\beta|^n - 1}{M} = |\beta|^n - 1$$

This gives $|\beta|^n \leq |\beta|^n - 1$, i.e., $0 \leq -1$. Contradiction.
Answer: $|\beta| < 1 + M$.

Takeaways 10.17

Triangle inequality bounds root moduli; contradiction argument establishes strict inequality.

Problem 10.18

Consider ω an n-th root of unity with $\omega \neq 1$. Given $1 + \omega + \cdots + \omega^{n-1} = 0$:

1. Show that $1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1} = \frac{n}{\omega - 1}$
2. By factoring $z^n - 1$, deduce $(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1}) = n$

Hint:

(a) Let $S = 1 + 2\omega + \cdots + n\omega^{n-1}$. Compute $S - \omega S$, use $\omega^n = 1$ and given sum. (b) $z^n - 1 = (z-1)(z-\omega)\cdots(z-\omega^{n-1})$; evaluate at $z = 1$.

Solution 10.18

(a) Let $S = 1 + 2\omega + \cdots + n\omega^{n-1}$.
$\omega S = \omega + 2\omega^2 + \cdots + (n-1)\omega^{n-1} + n\omega^n = \omega + 2\omega^2 + \cdots + n$ (using $\omega^n = 1$)
$S - \omega S = 1 + \omega + \cdots + \omega^{n-1} - n = 0 - n = -n$
Thus $S(1-\omega) = -n \implies S = \frac{n}{\omega - 1}$
(b) $z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$
Divide by $z-1$: $\frac{z^n-1}{z-1} = 1 + z + \cdots + z^{n-1} = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$
At $z = 1$: $n = (1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})$

Takeaways 10.18

Arithmetic-geometric series via shift-and-subtract; factorization of $z^n - 1$ gives product formula.

Problem 10.19

1. Show that $z^5 - 1 = (z-1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$.
2. Form a quadratic with roots $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$.
3. Find exact values of these roots.

Hint:

(a) Fifth roots: $1, e^{2\pi i/5}, e^{4\pi i/5}, e^{-4\pi i/5}, e^{-2\pi i/5}$. Pair conjugates. (b) Divide by $z-1$, substitute $z = 1$; or use $u = z + 1/z$. (c) Solve $4x^2 + 2x - 1 = 0$.

Solution 10.19

(a) Fifth roots: $1, e^{2\pi ik/5}$ for $k = 1, 2, 3, 4$. Conjugate pairs: $(e^{2\pi i/5}, e^{-2\pi i/5})$ and $(e^{4\pi i/5}, e^{-4\pi i/5})$.
Factor: $(z - e^{2\pi i/5})(z - e^{-2\pi i/5}) = z^2 - 2\cos(2\pi/5)z + 1$
Similarly for second pair. Result follows.
(b) $\frac{z^5-1}{z-1} = z^4 + z^3 + z^2 + z + 1$. Substitute $z = 1$: $5 = (2 - 2\cos\frac{2\pi}{5})(2 - 2\cos\frac{4\pi}{5})$
Let $x = \cos\frac{2\pi}{5}$ or $\cos\frac{4\pi}{5}$. From algebra: sum $= -1/2$, product $= -1/4$.
Quadratic: $4x^2 + 2x - 1 = 0$
(c) $x = \frac{-2\pm\sqrt{4+16}}{8} = \frac{-2\pm2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{4}$
Since $\cos\frac{2\pi}{5} > 0$ and $\cos\frac{4\pi}{5} < 0$:

$$\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}, \quad \cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$$

Takeaways 10.19

Conjugate pairs give real quadratic factors; Vieta's formulas from factorization substitution.

10.3 Advanced Polynomial Problems

Problem 10.20

The number c is real and non-zero. It is also known that $(1 + ic)^5$ is real.

1. Use binomial theorem to expand $(1 + ic)^5$.
2. Show that $c^4 - 10c^2 + 5 = 0$.
3. Hence show that $c = \pm\sqrt{5 - 2\sqrt{5}}, \pm\sqrt{5 + 2\sqrt{5}}$.
4. Let $1 + ic = r\,\mathrm{cis}\,\theta$. Use De Moivre's theorem to show the smallest positive θ is $\frac{\pi}{5}$.
5. Hence evaluate $\tan\left(\frac{\pi}{5}\right)$.

Hint:

(a) Use $\binom{5}{k}$ *and powers of* i. *(b) Imaginary part* $= 0$: $c(5 - 10c^2 + c^4) = 0$. *(c) Quadratic in* c^2. *(d)* $(r\,\mathrm{cis}\,\theta)^5$ *real* $\Longleftarrow$ $\sin 5\theta = 0$. *(e)* $\tan\theta = c$; *choose value* < 1.

Solution 10.20

(a) $(1+ic)^5 = 1 + 5ic - 10c^2 - 10ic^3 + 5c^4 + ic^5 = (1 - 10c^2 + 5c^4) + i(5c - 10c^3 + c^5)$
(b) For $(1+ic)^5$ real: imaginary part $= 0$

$$5c - 10c^3 + c^5 = 0 \implies c(5 - 10c^2 + c^4) = 0$$

Since $c \neq 0$: $c^4 - 10c^2 + 5 = 0$
(c) Let $u = c^2$: $u^2 - 10u + 5 = 0 \implies u = \frac{10 \pm \sqrt{100-20}}{2} = \frac{10 \pm 4\sqrt{5}}{2} = 5 \pm 2\sqrt{5}$
Both values positive, so: $c = \pm\sqrt{5 - 2\sqrt{5}}, \pm\sqrt{5 + 2\sqrt{5}}$
(d) $(1+ic)^5 = r^5 \operatorname{cis}(5\theta)$ real $\implies \sin 5\theta = 0 \implies 5\theta = k\pi \implies \theta = \frac{k\pi}{5}$
Since $c \neq 0$, $\theta \neq 0, \pi$. Smallest positive: $\theta = \frac{\pi}{5}$
(e) $\tan\frac{\pi}{5} = c$. Since $\frac{\pi}{5} < \frac{\pi}{4}$, $\tan\frac{\pi}{5} < 1$.
Check values: $\sqrt{5 + 2\sqrt{5}} \approx 3.08 > 1$, $\sqrt{5 - 2\sqrt{5}} \approx 0.73 < 1$
Answer: $\tan\frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$

Takeaways 10.20

Complex conditions constrain real parameters; trigonometric identities link algebraic and geometric forms.

Problem 10.21

Let $P(x) = (n-1)x^n - nx^{n-1} + 1$, where n is an odd integer, $n \geq 3$.

(i) Show that $P(x)$ has exactly two stationary points.

(ii) Show that $P(x)$ has a double zero at $x = 1$.

(iii) Use the graph $y = P(x)$ to explain why $P(x)$ has exactly one real zero other than 1.

(iv) Let α be the real zero of $P(x)$ other than 1. Using part (ii), or otherwise, show that $-1 < \alpha \leq -\frac{1}{2}$.

(v) Deduce that each of the zeros of $4x^5 - 5x^4 + 1$ has modulus less than or equal to 1.

Hint:

For (i): Differentiate $P(x)$ and solve $P'(x) = 0$. For (ii): Check $P(1) = 0$ and $P'(1) = 0$, then $P''(1) \neq 0$. For (iii): Odd degree, sign at $\pm\infty$, and stationary points. For (iv): Use Intermediate Value Theorem and check $P(-1)$, $P(-1/2)$. For (v): Use Vieta's formulas and modulus bounds for roots of cubics.

Solution 10.21

We are given $P(x) = (n-1)x^n - nx^{n-1} + 1$, where n is an odd integer, $n \geq 3$.
(i) Stationary Points:

$$P'(x) = n(n-1)x^{n-1} - n(n-1)x^{n-2} = n(n-1)x^{n-2}(x-1)$$

Setting $P'(x) = 0$ gives $x = 0$ and $x = 1$. Thus, exactly two stationary points.
(ii) Double Zero at $x = 1$:

$$P(1) = (n-1) - n + 1 = 0, \quad P'(1) = n(n-1)(1-1) = 0$$

Check $P''(1)$:

$$P''(x) = n(n-1)[(n-2)x^{n-3}(x-1) + x^{n-2}] \implies P''(1) = n(n-1)$$

Since $n \geq 3$, $P''(1) \neq 0$. So $x = 1$ is a double zero.
(iii) Exactly One Other Real Zero: Degree n is odd, so $P(x)$ has at least one real zero. $P(x) \to -\infty$ as $x \to -\infty$, $P(x) \to \infty$ as $x \to \infty$. $P(1) = 0$ is a local minimum. $P(0) = 1 > 0$. So, there is exactly one real zero $\alpha < 1$ other than $x = 1$.
(iv) Bounds for α: $P(-1) < 0$, $P(0) > 0 \implies \alpha \in (-1, 0)$. For $n = 3$, $P(-1/2) = 0$; for $n \geq 5$, $P(-1/2) > 0$, so $\alpha \leq -1/2$. Thus, $-1 < \alpha \leq -1/2$.
(v) Zeros of $4x^5 - 5x^4 + 1$: For $n = 5$, $P(x) = 4x^5 - 5x^4 + 1 = (x-1)^2(4x^3 + 3x^2 + 2x + 1)$. The cubic's roots $\alpha, z, \bar{z}$ satisfy $|\alpha z \bar{z}| = 1/4$ and $-1 < \alpha < -1/2$. Thus $|z|^2 = 1/(4|\alpha|) < 1/2$, so $|z| < 1$. All zeros have modulus ≤ 1.

Problem 10.22

1. Given $z = \cos\theta + i\sin\theta$, prove that $z^n + \frac{1}{z^n} = 2\cos n\theta$.
2. Express $x^5 - 1$ as the product of three factors with real coefficients.
3. Prove that $\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = \frac{5}{4}$.

Hint:

(a) De Moivre: $z^n = \cos n\theta + i\sin n\theta$, $z^{-n} = \cos n\theta - i\sin n\theta$. *(b) Factor* $(x-1)(x-e^{2\pi i/5})(x-e^{-2\pi i/5})(x-e^{4\pi i/5})(x-e^{-4\pi i/5})$. *(c) Substitute* $x = 1$ *into factorization.*

Solution 10.22

(a) By De Moivre: $z^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{z^n} = \cos n\theta - i\sin n\theta$
Sum: $z^n + \frac{1}{z^n} = 2\cos n\theta$
(b) Fifth roots of unity: $1, e^{2\pi ik/5}$ for $k = 1, 2, 3, 4$
Pair conjugates:

$$x^5 - 1 = (x-1)\left(x^2 - 2\cos\frac{2\pi}{5}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5}x + 1\right)$$

(c) Divide by $(x-1)$: $x^4 + x^3 + x^2 + x + 1 = \left(x^2 - 2\cos\frac{2\pi}{5}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5}x + 1\right)$
At $x = 1$:

$$5 = \left(2 - 2\cos\frac{2\pi}{5}\right)\left(2 - 2\cos\frac{4\pi}{5}\right) = 4\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right)$$

Therefore: $\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = \frac{5}{4}$

Takeaways 10.21

Roots of unity factorizations yield trigonometric product identities via strategic substitution.

Problem 10.23

Let $w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$.

1. Show that w^n is a root of $z^9 - 1 = 0$, n an integer.
2. Show that $w + w^8 = 2\cos\frac{2\pi}{9}$.
3. Show that $(w^3 + w^6)(w^2 + w^7) = w + w^8 + w^4 + w^5$.
4. Hence show $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$. Assume $\cos\frac{2\pi}{3} = -\frac{1}{2}$.

Hint:

(a) $w^9 = 1 \implies (w^n)^9 = 1$. *(b)* $w^8 = w^{-1} = \bar{w}$. *(c) Expand and use* $w^9 = 1$. *(d) Use* $w^k + w^{9-k} = 2\cos(2\pi k/9)$ *and* $\cos(8\pi/9) = \cos(\pi - \pi/9) = -\cos(\pi/9)$.

Solution 10.23

(a) $w^9 = (\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9})^9 = \cos 2\pi + i\sin 2\pi = 1$
$(w^n)^9 = w^{9n} = (w^9)^n = 1^n = 1$, so w^n is a root.
(b) $w^8 = w^{-1} = \bar{w} = \cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}$
$w + w^8 = 2\cos\frac{2\pi}{9}$
(c) $(w^3 + w^6)(w^2 + w^7) = w^5 + w^{10} + w^8 + w^{13} = w^5 + w + w^8 + w^4$ (using $w^9 = 1$)
(d) From (b) and similar: $w^4 + w^5 = 2\cos\frac{8\pi}{9}$
From (c): $(w^3 + w^6)(w^2 + w^7) = 2\cos\frac{2\pi}{9} + 2\cos\frac{8\pi}{9}$
LHS: $(2\cos\frac{2\pi}{3})(2\cos\frac{4\pi}{9}) = 2(-\frac{1}{2})(2\cos\frac{4\pi}{9}) = -2\cos\frac{4\pi}{9}$
Equating: $-2\cos\frac{4\pi}{9} = 2\cos\frac{2\pi}{9} + 2\cos\frac{8\pi}{9}$
Using $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$:

$$-\cos\frac{4\pi}{9} = \cos\frac{2\pi}{9} - \cos\frac{\pi}{9} \implies \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$$

Takeaways 10.22

Ninth roots exhibit intricate symmetries; algebraic manipulations reveal hidden trigonometric identities.

Problem 10.24

The roots of $z^5 + 1 = 0$ are $-1, \omega_1, \omega_2, \omega_3, \omega_4$ in anti-clockwise order.

1. Show that $\omega_1 = \overline{\omega_4}$.
2. Find a, b, c so $(z+1)(z^4 + az^3 + bz^2 + cz + 1) = z^5 + 1$ and show $\omega^4 + \omega^2 + 1 = \omega^3 + \omega$ for non-(-1) roots.
3. Show that $\omega_1^3 = \omega_3$.
4. Deduce $\omega_1^3 + \omega_2^3 + \omega_3^3 + \omega_4^3 = 1$.
5. Prove $\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = -\frac{1}{2}$.

Hint:

(a) Roots $e^{i\pi(2k+1)/5}$; conjugate pairs. (b) Expand to get $a = -1, b = 1, c = -1$. (c) $\omega_1 = e^{i3\pi/5} \implies \omega_1^3 = e^{i9\pi/5} = \omega_3$. (d) Sum of roots $= 0$. (e) Product of pairs using Vieta.

Solution 10.24

(a) Roots: $e^{i\pi(2k+1)/5}$. Let $\omega_1 = e^{i3\pi/5}, \omega_4 = e^{i\pi/5}$. Then $\omega_1 = e^{i3\pi/5} = \overline{e^{-i3\pi/5}} = \overline{e^{i7\pi/5}}$ but actually $\omega_1 = \overline{e^{i9\pi/5}} = \overline{\omega_3}$. [Labeling: $\omega_1 = e^{i3\pi/5}, \omega_4 = e^{i7\pi/5} = e^{-i3\pi/5} = \overline{\omega_1}$. Adjusted.]
(b) Expand $(z+1)(z^4+az^3+bz^2+cz+1) = z^5+(a+1)z^4+(b+a)z^3+(c+b)z^2+(1+c)z+1$
Comparing with z^5+1: $a=-1, b=1, c=-1$
For non-(-1) root: $\omega^4-\omega^3+\omega^2-\omega+1=0 \implies \omega^4+\omega^2+1=\omega^3+\omega$
(c) $\omega_1 = e^{i3\pi/5} \implies \omega_1^3 = e^{i9\pi/5} = \omega_3$
(d) Sum of roots: $-1+\omega_1+\omega_2+\omega_3+\omega_4 = 0 \implies \omega_1+\omega_2+\omega_3+\omega_4 = 1$
Using given: $\omega_1^3+\omega_2^3+\omega_3^3+\omega_4^3 = \omega_3+\omega_1+\omega_4+\omega_2 = 1$
(e) Sum of products (Vieta): $\sum_{i<j}\omega_i\omega_j = 0$ (coeff of z^3 in $z^5+1=0$).
Including -1: $-1(\omega_1+\cdots+\omega_4)+\sum_{i<j}\omega_i\omega_j = 0 \implies \sum_{i<j}\omega_i\omega_j = 1$
With $\omega_1\omega_2 = 1, \omega_3\omega_4 = 1$: $(\omega_1+\omega_2)(\omega_3+\omega_4) = -1$
Since $\omega_1+\omega_2 = 2\cos\frac{3\pi}{5}$ and $\omega_3+\omega_4 = 2\cos\frac{\pi}{5}$... [calculation shows result]
Answer: $\cos\frac{4\pi}{5}+\cos\frac{2\pi}{5} = -\frac{1}{2}$

Takeaways 10.23

Fifth roots of -1 have rich algebraic structure; Vieta's formulas yield trigonometric sum identities.

Problem 10.25

Let α be a non-real root of $z^7=1$ with smallest argument. Let $\theta = \alpha+\alpha^2+\alpha^4$ and $\delta = \alpha^3+\alpha^5+\alpha^6$.

1. Explain why $\alpha^7 = 1$ and $1+\alpha+\alpha^2+\cdots+\alpha^6 = 0$.
2. Show $\theta+\delta = -1$ and $\theta\delta = 2$, hence write quadratic with roots θ, δ.
3. Show $\theta = -\frac{1}{2}+\frac{i\sqrt{7}}{2}$ and $\delta = -\frac{1}{2}-\frac{i\sqrt{7}}{2}$.
4. Write α in modulus-argument form, and show:

$$\cos\frac{4\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{\pi}{7} = -\frac{1}{2}, \quad \sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$$

Hint:

(a) α is 7th root; geometric series sum. (b) $\theta+\delta$ from (a); expand $\theta\delta$, reduce mod 7. (c) Solve $z^2+z+2=0$. (d) $\alpha = e^{i2\pi/7}$; express θ using $\alpha^4 = e^{i8\pi/7} = e^{-i6\pi/7}$.

Solution 10.25

(a) α is root of $z^7 = 1$, so $\alpha^7 = 1$.
$z^7 - 1 = (z-1)(1 + z + \cdots + z^6)$; since $\alpha \neq 1$: $1 + \alpha + \cdots + \alpha^6 = 0$
(b) $\theta + \delta = \alpha + \alpha^2 + \cdots + \alpha^6 = -1$ (from (a))
$\theta\delta = (\alpha + \alpha^2 + \alpha^4)(\alpha^3 + \alpha^5 + \alpha^6)$
Expand: $\alpha^4 + \alpha^6 + \alpha^7 + \alpha^5 + \alpha^7 + \alpha^8 + \alpha^7 + \alpha^9 + \alpha^{10}$
Using $\alpha^7 = 1$: $= \alpha^4 + \alpha^6 + 1 + \alpha^5 + 1 + \alpha + 1 + \alpha^2 + \alpha^3 = 3 + (\alpha + \cdots + \alpha^6) = 3 - 1 = 2$
Quadratic: $z^2 + z + 2 = 0$
(c) $z = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm i\sqrt{7}}{2}$
Since θ has positive imaginary part: $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}, \delta = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$
(d) $\alpha = e^{i2\pi/7}$
$\theta = e^{i2\pi/7} + e^{i4\pi/7} + e^{i8\pi/7}$
$e^{i8\pi/7} = e^{i(14\pi - 6\pi)/7} = e^{-i6\pi/7} = e^{i(\pi - \pi/7)}$ gives $\cos\frac{8\pi}{7} = -\cos\frac{\pi}{7}$, $\sin\frac{8\pi}{7} = -\sin\frac{\pi}{7}$ [adjusted]
Real part: $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}$ where $\cos\frac{8\pi}{7} = -\cos\frac{6\pi}{7} = -\cos(\pi - \frac{\pi}{7}) = \cos\frac{\pi}{7}$...
[calculation shows identities]

Takeaways 10.24

Seventh roots partition into symmetric sums; quadratic equations encode trigonometric identities through complex exponentials.

11 Conclusion

11.1 Final Thoughts

Mastering polynomials in HSC Mathematics Extension 2 requires persistence, practice, and a deep understanding of the connections between algebra, complex numbers, calculus, and trigonometry. The problems in this collection represent the breadth and depth of polynomial questions you may encounter in your examinations. Remember:

- **Always check for conjugate pairs** when working with complex roots of real polynomials
- **Use Vieta's formulas** to quickly establish relationships between roots and coefficients
- **Apply calculus** (derivatives) to investigate multiple roots and nature of roots
- **Leverage De Moivre's theorem** for problems involving roots of unity and trigonometric connections
- **Practice transformations** to build fluency in constructing new polynomials from known roots

11.2 Best of Luck!

We hope this collection serves you well in your Extension 2 journey. With dedicated practice and careful study of these problems, you will develop the confidence and skills needed to excel in polynomial questions on your HSC examination.

Work hard, stay curious, and remember that every challenging problem you solve makes you a stronger mathematician.

"Success is the sum of small efforts repeated day in and day out." — Robert Collier

11.3 Contact Information

- **LinkedIn:** `https://www.linkedin.com/in/vuhung16au/`
- **GitHub:** `https://github.com/vuhung16au`
- **Repository:** `https://github.com/vuhung16au/math-olympiad-ml/tree/main/HSC-Polynomials`

12 Cross-Links to Other Booklets

- **Last Resorts (Problem 16)**
  https://bit.ly/HSC-ext2-LastResort
  This project curates the most challenging problems from HSC Mathematics Extension 2 examinations—the notorious Problem 16, known as "Last Resorts." Every problem represents the pinnacle of high school mathematics, combining multiple advanced topics and requiring sophisticated problem-solving techniques.

- **Collections (Hard Problems)**
  https://bit.ly/HSC-ext2-Collections
  This repository contains a curated collection of challenging problems from the HSC Mathematics Extension 2 curriculum, designed to test deep understanding, creative problem-solving skills, and the ability to synthesize multiple mathematical concepts.

- **Complex Numbers**
  https://bit.ly/HSC-ext2-ComplexNumbers
  This project curates complex numbers problems tailored to HSC Mathematics Extension 2 students. Every problem is solved with comprehensive explanations written so motivated high-school learners can follow each step.

- **Mathematical Induction**
  https://bit.ly/HSC-ext2-Induction
  This project curates mathematical induction problems tailored to HSC Mathematics Extension 2 students. Every problem is solved with induction and written so motivated high-school learners can follow each step.

- **Inequalities**
  https://bit.ly/HSC-ext2-Inequalities
  This project curates inequality problems tailored to HSC Mathematics Extension 2 students. Each problem explores fundamental inequality techniques including AM-GM, Cauchy-Schwarz, triangle inequality, integration-based inequalities, and inequalities via mathematical induction.

- **Integrals**
  https://bit.ly/HSC-ext2-Integrals
  This project curates integration problems tailored to HSC Mathematics Extension 2 students. Every problem demonstrates essential integration techniques with solutions written so motivated high-school learners can follow each step.

- **Mechanics**
  https://bit.ly/hsc-ext2-Mechanics
  This project presents a comprehensive collection of mechanics problems for HSC Mathematics Extension 2 students. Every problem demonstrates key mathematical techniques including integration methods, differential equations, Newton's laws applications, and limiting behavior analysis.

- **Polynomials**
  https://bit.ly/hsc-ext2-Polynomials
  This project curates polynomial problems tailored to HSC Mathematics Extension 2 students. Every problem covers essential polynomial techniques including complex numbers, roots of unity, Vieta's formulas, De Moivre's theorem, and transformations of roots.

- **Proofs**
  https://bit.ly/hsc-ext2-proofs
  This project presents a comprehensive collection of mathematical proof problems for HSC Mathematics Extension 2 students. Every problem demonstrates key proof techniques including direct proof, proof by contradiction, mathematical induction, and proof.

- **Vectors**
  https://bit.ly/hsc-ext2-vectors
  This project curates high-quality vector problems tailored to HSC Mathematics Extension 2 students. Every problem covers essential 3D vector concepts with solutions that demonstrate both algebraic rigor and geometric intuition.