# HSC Math Extension 2: Vectors Mastery

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# 1 Introduction

## 1.1 Project Overview

This booklet compiles high quality vector problems curated specifically for the HSC Mathematics Extension 2 syllabus. The collection covers all essential topics in 3D vectors including geometric proofs, line equations, dot and cross products, spheres, planes, and distance calculations. Each problem is selected to challenge students while building toward exam readiness.

## 1.2 Target Audience

These problems are designed for Extension 2 students who want to master vectors through systematic practice. Part 1 provides detailed solutions showing complete algebraic steps, geometric intuition, and key takeaways. Part 2 offers hints and concise solutions to encourage independent problem-solving.

## 1.3 How to Use This Booklet

- Read the vectors primer below to review fundamental concepts and notation.
- Attempt problems in Part 1 independently; study the detailed solutions to understand model approaches.
- Use the upside-down hints in Part 2 only after making a genuine attempt.
- Focus on understanding geometric meaning alongside algebraic manipulation.
- Revisit challenging problems after a few days to reinforce techniques.

## 1.4 Vectors Primer

### 1.4.1 3D Vectors & Coordinates

Extension 2 introduces the $z$-axis, moving from $(x, y)$ plane to $(x, y, z)$ space.

- **Component Form:** $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or column vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- **Magnitude:** $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$
- **Distance Formula:** Between $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$: $|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

### 1.4.2 The Dot Product (Scalar Product)

The dot product is the central algebraic tool for vectors in Extension 2.

- **Definition:** $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ and algebraically: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
- **Perpendicular Vectors:** $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$
- **Angle Between Vectors:** $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
- **Scalar Projection:** $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$
- **Vector Projection:** $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\mathbf{b}$

### 1.4.3 Vector Equation of a Line

Extension 2 uses vector form rather than $y = mx + b$.

- **Parametric Form:** $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where $\mathbf{a}$ is a point on the line, $\mathbf{b}$ is the direction vector, and $\lambda$ is a parameter
- **Parallel Lines:** Direction vectors are scalar multiples
- **Skew Lines:** In 3D, lines can be non-parallel and non-intersecting

### 1.4.4 Spheres & Planes

- **Sphere:** Vector form $|\mathbf{r} - \mathbf{c}| = R$ or Cartesian $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$
- **Plane (Point-Normal):** $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ where $\mathbf{n}$ is normal to the plane
- **Plane (Cartesian):** $ax + by + cz = d$ where $(a, b, c)$ is the normal vector

### 1.4.5 Geometric Proofs

Use pure vector logic (without coordinates) to prove theorems about parallelograms, triangles, centroids, medians, and 3D shapes like tetrahedrons.

### 1.4.6 The Cross Product (Optional Advanced Technique)

While not explicitly in the NESA syllabus, the cross product is a powerful tool for certain problems.

**Definition:** For $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

**Properties:**

- $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ (area of parallelogram)
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (anticommutative)

**Worked Example: Distance from Point to Line**

*Problem:* Find the distance from point $P(1,2,0)$ to the line $\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$.

*Solution using Cross Product:* Let $A(1,0,1)$ be a point on the line and $\mathbf{d} = \begin{pmatrix}0\\1\\1\end{pmatrix}$ be the direction vector.

The vector from $A$ to $P$ is $\vec{AP} = \begin{pmatrix}0\\2\\-1\end{pmatrix}$.

Using the cross product:

$$\vec{AP} \times \mathbf{d} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 2 & -1\\ 0 & 1 & 1\end{vmatrix} = 3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \begin{pmatrix}3\\0\\0\end{pmatrix}$$

Distance: $d = \frac{|\vec{AP}\times\mathbf{d}|}{|\mathbf{d}|} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$ units.

*Comparison with Dot Product Method:* The dot product method uses projection. The component of $\vec{AP}$ perpendicular to the line is found by:

$$\vec{AP}_\perp = \vec{AP} - \text{proj}_{\mathbf{d}}\vec{AP} = \vec{AP} - \frac{\vec{AP}\cdot\mathbf{d}}{\mathbf{d}\cdot\mathbf{d}}\mathbf{d}$$

Then $d = |\vec{AP}_\perp|$. Both methods yield the same result, but cross product is often more direct for distance calculations.

# 2 Part 1: Problems and Solutions (Detailed)

Part 1 contains 15 carefully selected problems covering all 10 topic areas. Each problem includes complete solutions with all algebraic steps, geometric diagrams where helpful, and key takeaways. Solutions use compact notation to fit on one A4 page while maintaining clarity.

## 2.1 Part 1 Basic Problems (Easy)

**Problem 2.1: Vector Projection Formula**

The vector $\mathbf{a}$ is $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and the vector $\mathbf{b}$ is $\begin{pmatrix}2\\0\\-4\end{pmatrix}$.

(i) Find $\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$.

(ii) Show that $\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$ is perpendicular to $\mathbf{b}$.

**Solution 2.1**

Given: $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

**(i)** Calculate the dot products:

$$\mathbf{a} \cdot \mathbf{b} = (1)(2) + (2)(0) + (3)(-4) = 2 - 12 = -10$$
$$\mathbf{b} \cdot \mathbf{b} = 4 + 0 + 16 = 20$$

Thus:

$$\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \frac{-10}{20}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

**(ii)** Let $\mathbf{v} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$. Then:

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

Check perpendicularity: $\mathbf{v} \cdot \mathbf{b} = (2)(2) + (2)(0) + (1)(-4) = 4 - 4 = 0 \quad \therefore$ perpendicular.

**Takeaways 2.1**

The expression $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$ is the **vector projection** of $\mathbf{a}$ onto $\mathbf{b}$. The remainder $\mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}$ is the component perpendicular to $\mathbf{b}$. This orthogonal decomposition is fundamental: any vector can be split into parallel and perpendicular components relative to another vector. Always verify perpendicularity by checking that the dot product equals zero.

**Problem 2.2: Shortest Distance from Point to Line**

$\mathbf{r}_1$ and $\mathbf{r}_2$ are two lines with vector equations:

$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$

(i) Show that these two lines intersect.

(ii) Find the angle between the lines.

(iii) Find the shortest distance from point $P(1, 2, 0)$ to line $\mathbf{r}_1$.

**Solution 2.2**

**(i)** Equate components: $1 = 2 + \mu$ gives $\mu = -1$. Then $\lambda = 3\mu = -3$. Check $z$: $1 + \lambda = 1 - 3 = -2$ and $2\mu = -2$. Consistent. Lines intersect at $(1, -3, -2)$.

**(ii)** Direction vectors: $\mathbf{d}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = 0 + 3 + 2 = 5$$
$$|\mathbf{d}_1| = \sqrt{2}, \quad |\mathbf{d}_2| = \sqrt{1 + 9 + 4} = \sqrt{14}$$
$$\cos\theta = \tfrac{5}{\sqrt{2}\sqrt{14}} = \tfrac{5}{2\sqrt{7}} \implies \theta \approx 19.1^\circ$$

**(iii)** Let $A(1, 0, 1)$ be on line $\mathbf{r}_1$, direction $\mathbf{d} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\vec{AP} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

Cross product: $\vec{AP} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}(2 + 1) = 3\mathbf{i}$

$$D = \tfrac{|\vec{AP} \times \mathbf{d}|}{|\mathbf{d}|} = \tfrac{3}{\sqrt{2}} = \tfrac{3\sqrt{2}}{2} \text{ units}$$

**Takeaways 2.2**

To find line intersection in 3D: equate components and solve the system (3 equations, 2 unknowns). Consistent solution means intersection; inconsistent means skew lines. The **cross product method** for point-to-line distance is efficient: $d = \frac{|\vec{AP} \times \mathbf{d}|}{|\mathbf{d}|}$ where $A$ is any point on the line. This uses the geometric interpretation that $|\vec{AP} \times \mathbf{d}|$ equals the area of a parallelogram with base $|\mathbf{d}|$ and height $d$.

**Problem 2.3: Parallelogram Area via Cross Product**

The adjacent sides of a parallelogram are represented by vectors $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Show that the area of the parallelogram is $6\sqrt{10}$ square units.

**Solution 2.3**

Area of parallelogram $= |\mathbf{a} \times \mathbf{b}|$. Calculate the cross product:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} \\
&= \mathbf{i}(3 \cdot 3 - (-1)(-1)) - \mathbf{j}(4 \cdot 3 - (-1)(2)) + \mathbf{k}(4(-1) - 3 \cdot 2) \\
&= \mathbf{i}(9 - 1) - \mathbf{j}(12 + 2) + \mathbf{k}(-4 - 6) \\
&= 8\mathbf{i} - 14\mathbf{j} - 10\mathbf{k}
\end{aligned}$$

Magnitude:

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{64 + 196 + 100} = \sqrt{360} = \sqrt{36 \cdot 10} = 6\sqrt{10}$$

**Takeaways 2.3**

The **cross product** $\mathbf{a} \times \mathbf{b}$ produces a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, with magnitude equal to the area of the parallelogram they span. Key properties: (1) anti-commutative: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$; (2) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$. For area calculations, only the magnitude matters. Remember the determinant pattern for $3 \times 3$ cross products.

**Problem 2.4: Cosine Difference Formula Proof**

Let $\mathbf{a}$ and $\mathbf{b}$ be 2-dimensional unit vectors, inclined to the $x$-axis at angles $\alpha$ and $\beta$ respectively. You may assume $\mathbf{a} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$ and $\mathbf{b} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$. Prove that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

**Solution 2.4**

**Method 1: Geometric dot product**

The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \alpha - \beta$. Since both are unit vectors:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = (1)(1)\cos(\alpha - \beta) = \cos(\alpha - \beta)$$

**Method 2: Algebraic dot product**

$$\mathbf{a} \cdot \mathbf{b} = (\cos\alpha)(\cos\beta) + (\sin\alpha)(\sin\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Equating the two expressions: $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad \square$

**Takeaways 2.4**

This elegant proof demonstrates the power of vectors in deriving trigonometric identities. The key insight: the dot product has both a **geometric definition** ($|\mathbf{a}||\mathbf{b}|\cos\theta$) and an **algebraic definition** (sum of component products). Equating these yields the cosine difference formula. Similar approaches can derive $\cos(\alpha+\beta)$, $\sin(\alpha\pm\beta)$, and other compound angle formulas. This vector method is often cleaner than traditional geometric proofs.

**Problem 2.5: Perpendicular Vectors Condition**

Consider two vectors $\mathbf{u} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = p\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. For what values of $p$ are $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ perpendicular?

**Solution 2.5**

First compute $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$:

$$\mathbf{u} - \mathbf{v} = (-2-p)\mathbf{i} + (-1-1)\mathbf{j} + (3-2)\mathbf{k} = (-2-p)\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{u} + \mathbf{v} = (-2+p)\mathbf{i} + (-1+1)\mathbf{j} + (3+2)\mathbf{k} = (-2+p)\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}$$

For perpendicularity, their dot product must equal zero:

$$\begin{aligned}
(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= 0 \\
(-2-p)(-2+p) + (-2)(0) + (1)(5) &= 0 \\
4 - p^2 + 5 &= 0 \\
9 - p^2 &= 0 \\
p^2 &= 9 \\
p &= \pm 3
\end{aligned}$$

**Takeaways 2.5**

Perpendicularity problems always reduce to setting the dot product equal to zero. This problem illustrates the algebraic identity: $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2$ (vector form of difference of squares). The general principle: two vectors are perpendicular if and only if the sum of products of corresponding components equals zero. When the problem involves a parameter, solve the resulting equation carefully—don't forget both positive and negative solutions for squared terms.

## 2.2 Part 1 Medium Problems

**Problem 2.6: Line Tangent to Sphere**

A sphere has centre at $(3, -3, 4)$ and radius 6 units.

A line has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$.

(i) Write down the vector equation of the sphere.

(ii) Determine whether the line is a tangent to the sphere, clearly justifying your conclusion.

**Solution 2.6**

**(i)** Let centre $\mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}$ and radius $R = 6$. Vector equation:

$$\left| \mathbf{r} - \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} \right| = 6$$

**(ii)** Substitute line into sphere. Cartesian form: $(x-3)^2 + (y+3)^2 + (z-4)^2 = 36$. Parametric line: $x = 1 + 2\lambda$, $y = 5 - \lambda$, $z = 4 - \lambda$.

$$\begin{aligned} (1 + 2\lambda - 3)^2 + (5 - \lambda + 3)^2 + (4 - \lambda - 4)^2 &= 36 \\ (2\lambda - 2)^2 + (8 - \lambda)^2 + (-\lambda)^2 &= 36 \\ 4\lambda^2 - 8\lambda + 4 + 64 - 16\lambda + \lambda^2 + \lambda^2 &= 36 \\ 6\lambda^2 - 24\lambda + 68 &= 36 \\ 6\lambda^2 - 24\lambda + 32 &= 0 \\ 3\lambda^2 - 12\lambda + 16 &= 0 \end{aligned}$$

Discriminant: $\Delta = (-12)^2 - 4(3)(16) = 144 - 192 = -48 < 0$
No real solutions $\implies$ **line does NOT intersect sphere** (not a tangent).

**Takeaways 2.6**

For line-sphere intersection: substitute parametric line equations into sphere equation to get a quadratic in $\lambda$. The discriminant determines the nature: $\Delta > 0$ (2 intersections, secant), $\Delta = 0$ (1 intersection, tangent), $\Delta < 0$ (no intersection). A tangent touches the sphere at exactly one point, requiring distance from centre to line equals radius. Always check discriminant rather than just counting solutions.

## Problem 2.7: Perpendicular Lines and Plane Equation

Consider lines $L_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(i) Show that $L_1$ and $L_2$ intersect and are perpendicular, stating the point of intersection.

(ii) Deduce that the plane containing both lines has equation $y + z = 1$.

(iii) Find the perpendicular distance from the origin to this plane.

## Solution 2.7

**(i)** Directions: $\mathbf{d}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Perpendicular: $\mathbf{d}_1 \cdot \mathbf{d}_2 = 2 - 1 - 1 = 0$ ✓

Intersection: Equate $(3 + 2\lambda, 2 - \lambda, -1 + \lambda) = (-1 + \mu, 1 + \mu, -\mu)$

$$\begin{aligned} x &: 3 + 2\lambda = -1 + \mu \implies \mu - 2\lambda = 4 \quad (1) \\ y &: 2 - \lambda = 1 + \mu \implies \mu + \lambda = 1 \quad (2) \\ z &: -1 + \lambda = -\mu \implies \mu + \lambda = 1 \quad (3) \end{aligned}$$

(1) − (2): $-3\lambda = 3 \implies \lambda = -1$, so $\mu = 2$. Point: $(3 - 2, 2 + 1, -1 - 1) = (1, 3, -2)$ ✓

**(ii)** Plane through $(1, 3, -2)$ spanned by $\mathbf{d}_1, \mathbf{d}_2$. Normal vector:

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 0\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

Simplified: $\mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Plane: $0x + y + z = k$. Using $(1, 3, -2)$: $3 + (-2) = 1 \implies k = 1$.

Thus $y + z = 1$.

**(iii)** Distance from origin $(0, 0, 0)$ to plane $y + z - 1 = 0$:

$$D = \frac{|0+0-1|}{\sqrt{0^2+1^2+1^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \text{ units}$$

## Takeaways 2.7

When two lines intersect, they determine a unique plane. To find the plane's Cartesian equation: (1) verify intersection, (2) compute normal via cross product of direction vectors, (3) use point-normal form. The cross product automatically gives a perpendicular vector. For distance from point $(x_0, y_0, z_0)$ to plane $Ax + By + Cz + D = 0$: $d = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$. Memorize this formula.

## Problem 2.8: Three Conditions on Vectors

Which of the following is a true statement about the lines
$\ell_1 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ and $\ell_2 = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$?

A. $\ell_1$ and $\ell_2$ are the same line.

B. $\ell_1$ and $\ell_2$ are not parallel and they intersect.

C. $\ell_1$ and $\ell_2$ are parallel and they do not intersect.

D. $\ell_1$ and $\ell_2$ are not parallel and they do not intersect.

## Solution 2.8

Direction vectors: $\mathbf{d}_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$

**Step 1: Check parallelism**

$$\mathbf{d}_2 = -1 \cdot \mathbf{d}_1$$

Lines are **parallel** (eliminates B and D).
**Step 2: Check if coincident** Test if point $P_1 = (-1, 2, 5)$ from $\ell_1$ lies on $\ell_2$:

$$\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

Component equations:

$$\begin{aligned} x: -1 = 3 + \mu &\implies \mu = -4 \\ y: 2 = -10 - 3\mu &\implies \mu = -4 \\ z: 5 = 1 - \mu &\implies \mu = -4 \end{aligned}$$

Consistent value $\mu = -4$ for all components $\implies$ point lies on $\ell_2$.
**Answer: A** (same line).

## Takeaways 2.8

For two lines to be identical, two conditions must hold: (1) direction vectors are scalar multiples (parallel), and (2) they share at least one common point. If parallel but don't share a point, they're distinct parallel lines. In 3D, non-parallel lines can be skew (no intersection). The systematic approach: first check parallelism via direction vectors, then test a point from one line on the other. This problem reinforces the distinction between "parallel" and "coincident."

## Problem 2.9: Projectile Motion Vector Proof

A particle is projected from the origin with initial velocity $u$ m/s at angle $\theta$ to the horizontal. The acceleration vector is $\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$.

(i) Show that the position vector is $\mathbf{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$.

(ii) Show that the Cartesian equation of the path is $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$.

(iii) Given $u^2 > gR$, prove there are two distinct values of $\theta$ for which the particle lands at $x = R$.

## Solution 2.9

**(i)** Initial velocity: $\mathbf{v}_0 = u\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. Integrate acceleration:

$$\mathbf{v}(t) = \mathbf{v}_0 + \int \mathbf{a}\,dt = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$$

Integrate velocity (starting from origin):

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$

**(ii)** From $x = ut\cos\theta$, we get $t = \frac{x}{u\cos\theta}$. Substitute into $y$:

$$\begin{aligned} y &= u\sin\theta \cdot \tfrac{x}{u\cos\theta} - \tfrac{1}{2}g\left(\tfrac{x}{u\cos\theta}\right)^2 \\ &= x\tan\theta - \tfrac{gx^2}{2u^2\cos^2\theta} \end{aligned}$$

**(iii)** At landing, $y = 0$ and $x = R$:

$$0 = R\tan\theta - \frac{gR^2}{2u^2\cos^2\theta}$$

Multiply by $\cos^2\theta$: $0 = R\sin\theta\cos\theta - \frac{gR^2}{2u^2}$
Using $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$\sin(2\theta) = \frac{gR}{u^2}$$

Since $u^2 > gR$, we have $\frac{gR}{u^2} < 1$. Thus $2\theta$ has two solutions in $[0, 180°]$: one acute, one obtuse. This gives two distinct values of $\theta$ in $[0, 90°]$.

## Takeaways 2.9

Projectile motion problems integrate naturally with vectors: acceleration $\rightarrow$ velocity $\rightarrow$ position. The key insight for part (iii): for a given range $R$, there are two launch angles (complementary angles to 45°) that achieve the same horizontal distance, provided the initial speed is sufficient. The condition $u^2 > gR$ ensures the target is within reach. The identity $\sin(2\theta) = 2\sin\theta\cos\theta$ is crucial for converting between different forms.

## Problem 2.10: Distance from Point to Line and Sphere

Consider point $B$ with position vector $\mathbf{b}$ and line $\ell : \mathbf{a} + \lambda\mathbf{d}$, where $|\mathbf{d}| = 1$ and $\lambda$ is a parameter. Let $f(\lambda)$ be the distance between a point on $\ell$ and point $B$.

(i) Find $\lambda_0$, the value of $\lambda$ that minimises $f$, in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{d}$.

(ii) Let $P$ be the point with position vector $\mathbf{a} + \lambda_0\mathbf{d}$. Show that $PB$ is perpendicular to the direction of $\ell$.

(iii) Hence find the shortest distance between line $\ell$ and the sphere of radius 1 centred at origin $O$, in terms of $\mathbf{d}$ and $\mathbf{a}$.

## Solution 2.10

**(i)** Minimize $S(\lambda) = f(\lambda)^2 = |(\mathbf{a} + \lambda\mathbf{d}) - \mathbf{b}|^2$:

$$\begin{aligned} S(\lambda) &= |\mathbf{a} - \mathbf{b}|^2 + 2\lambda(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} + \lambda^2|\mathbf{d}|^2 \\ \tfrac{dS}{d\lambda} &= 2(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} + 2\lambda = 0 \\ \lambda_0 &= -(\mathbf{a} - \mathbf{b}) \cdot \mathbf{d} = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{d} \end{aligned}$$

**(ii)** $\vec{PB} = \mathbf{b} - (\mathbf{a} + \lambda_0\mathbf{d}) = (\mathbf{b} - \mathbf{a}) - \lambda_0\mathbf{d}$ Check: $\vec{PB} \cdot \mathbf{d} = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{d} - \lambda_0(\mathbf{d} \cdot \mathbf{d}) = \lambda_0 - \lambda_0(1) = 0$ ✓
**(iii)** For origin (set $\mathbf{b} = \mathbf{0}$): $\lambda_{min} = -\mathbf{a} \cdot \mathbf{d}$. Point on line closest to $O$: $\mathbf{p} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{d})\mathbf{d}$

$$|\mathbf{p}|^2 = |\mathbf{a}|^2 - 2(\mathbf{a} \cdot \mathbf{d})^2 + (\mathbf{a} \cdot \mathbf{d})^2 = |\mathbf{a}|^2 - (\mathbf{a} \cdot \mathbf{d})^2$$

Shortest distance to sphere: $\sqrt{|\mathbf{a}|^2 - (\mathbf{a} \cdot \mathbf{d})^2} - 1$ or equivalently $|\mathbf{a} \times \mathbf{d}| - 1$.

## Takeaways 2.10

Point-to-line distance problems use calculus (minimize squared distance) or projection (subtract parallel component). The perpendicularity in part (ii) confirms the minimum—the shortest path is always perpendicular to the line. For sphere-line distance, first find line-to-centre distance, then subtract radius. The identity $|\mathbf{a} - (\mathbf{a} \cdot \mathbf{d})\mathbf{d}|^2 = |\mathbf{a}|^2 - (\mathbf{a} \cdot \mathbf{d})^2$ (when $|\mathbf{d}| = 1$) is equivalent to $|\mathbf{a} \times \mathbf{d}|^2$ via Lagrange's identity.

## 2.3 Part 1 Advanced Problems (Hard)

**Problem 2.11: Position Vector Ratios and Parallelogram Division**

Point $C$ divides interval $AB$ so that $\frac{CB}{AC} = \frac{m}{n}$. Position vectors of $A$ and $B$ are $\mathbf{a}, \mathbf{b}$.

(i) Show that $\vec{AC} = \frac{n}{m+n}(\mathbf{b} - \mathbf{a})$.

(ii) Prove that $\vec{OC} = \frac{m}{m+n}\mathbf{a} + \frac{n}{m+n}\mathbf{b}$.

Let $OPQR$ be a parallelogram with $\vec{OP} = \mathbf{p}$, $\vec{OR} = \mathbf{r}$. $S$ is the midpoint of $QR$, $T$ is the intersection of $PR$ and $OS$.

(iii) Show that $\vec{OT} = \frac{2}{3}\mathbf{r} + \frac{1}{3}\mathbf{p}$.

(iv) Prove that $T$ divides $PR$ in the ratio $2 : 1$.

**Solution 2.11**

**(i)** Given $\frac{CB}{AC} = \frac{m}{n}$, let $\vec{AC} = k(\mathbf{b} - \mathbf{a})$ for some $k$. Then $C = A + k(\mathbf{b} - \mathbf{a})$. Since $\vec{CB} = \mathbf{b} - \mathbf{c} = (1 - k)(\mathbf{b} - \mathbf{a})$ and $\vec{AC} = k(\mathbf{b} - \mathbf{a})$:

$$\frac{CB}{AC} = \frac{1-k}{k} = \frac{m}{n} \implies n(1-k) = mk \implies n = k(m+n) \implies k = \frac{n}{m+n}$$

Thus $\vec{AC} = \frac{n}{m+n}(\mathbf{b} - \mathbf{a})$.
**(ii)** $\vec{OC} = \vec{OA} + \vec{AC} = \mathbf{a} + \frac{n}{m+n}(\mathbf{b} - \mathbf{a}) = \mathbf{a}\left(1 - \frac{n}{m+n}\right) + \frac{n}{m+n}\mathbf{b} = \frac{m}{m+n}\mathbf{a} + \frac{n}{m+n}\mathbf{b}$
**(iii)** In parallelogram $OPQR$: $\vec{OQ} = \mathbf{p} + \mathbf{r}$. $S$ is midpoint of $QR$:

$$\vec{OS} = \frac{1}{2}(\vec{OQ} + \vec{OR}) = \frac{1}{2}(\mathbf{p} + \mathbf{r} + \mathbf{r}) = \frac{1}{2}\mathbf{p} + \mathbf{r}$$

$T$ lies on $OS$: $\vec{OT} = s(\frac{1}{2}\mathbf{p} + \mathbf{r})$ for some $s$. $T$ also lies on $PR$. Since $\vec{OP} = \mathbf{p}$ and $\vec{OR} = \mathbf{r}$:

$$\vec{OT} = (1-t)\mathbf{p} + t\mathbf{r}$$

for some $t \in [0, 1]$. Equating: $s \cdot \frac{1}{2}\mathbf{p} + s\mathbf{r} = (1-t)\mathbf{p} + t\mathbf{r}$
Since $\mathbf{p}, \mathbf{r}$ are independent: $\frac{s}{2} = 1 - t$ and $s = t$. From second: $s = t$. Substitute into first: $\frac{s}{2} = 1 - s \implies \frac{3s}{2} = 1 \implies s = \frac{2}{3}$. Thus: $\vec{OT} = \frac{2}{3}(\frac{1}{2}\mathbf{p} + \mathbf{r}) = \frac{1}{3}\mathbf{p} + \frac{2}{3}\mathbf{r}$ (or equivalently $\frac{2}{3}\mathbf{r} + \frac{1}{3}\mathbf{p}$).
**(iv)** From part (iii), $t = \frac{2}{3}$ means $T$ divides $PR$ such that $\vec{PT} = \frac{2}{3}\vec{PR}$, so $PT : TR = 2 : 1$.

**Takeaways 2.11**

The **section formula** $\vec{OC} = \frac{m\mathbf{a}+n\mathbf{b}}{m+n}$ (when $C$ divides $AB$ in ratio $n : m$ from $A$) is fundamental for position vectors. For geometric proofs: (1) express target point in terms of base vectors using two different paths, (2) equate and solve using linear independence. Parts (iii)-(iv) show how medians and diagonals in parallelograms create consistent ratios—here $T$ is the centroid-like point dividing both segments in ratio $2 : 1$.

## Problem 2.12: Triangle Inequality and Cauchy-Schwarz on Sphere

(i) Point $P(x, y, z)$ lies on the unit sphere centred at origin $O$. Using the triangle inequality, show that $|x| + |y| + |z| \geq 1$.

(ii) Given vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, show that $|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$.

(iii) As in part (i), point $P(x, y, z)$ lies on the unit sphere. Using part (ii), show that $|x| + |y| + |z| \leq \sqrt{3}$.

## Solution 2.12

**(i)** Position vector: $\vec{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with $|\vec{OP}| = 1$.
Triangle inequality: $|\mathbf{u} + \mathbf{v} + \mathbf{w}| \leq |\mathbf{u}| + |\mathbf{v}| + |\mathbf{w}|$

$$1 = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| \leq |x\mathbf{i}| + |y\mathbf{j}| + |z\mathbf{k}| = |x| + |y| + |z|$$

**(ii)** Cauchy-Schwarz from dot product: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ Taking absolute value: $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta| \leq |\mathbf{a}||\mathbf{b}|$ (since $|\cos\theta| \leq 1$). In components: $|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$

**(iii)** Let $\mathbf{a} = \begin{pmatrix} |x| \\ |y| \\ |z| \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Apply Cauchy-Schwarz:

$$|x| + |y| + |z| = |x|(1) + |y|(1) + |z|(1) \leq \sqrt{x^2 + y^2 + z^2}\sqrt{1^2 + 1^2 + 1^2}$$

Since $P$ is on unit sphere: $x^2 + y^2 + z^2 = 1$. Thus:

$$|x| + |y| + |z| \leq \sqrt{1} \cdot \sqrt{3} = \sqrt{3}$$

## Takeaways 2.12

This problem demonstrates two fundamental inequalities. The **triangle inequality** $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ gives lower bounds. The **Cauchy-Schwarz inequality** $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ gives upper bounds. Both arise from the dot product definition. Part (iii) uses a clever choice of vectors to convert a sum constraint into a dot product. Combined: $1 \leq |x| + |y| + |z| \leq \sqrt{3}$ for points on the unit sphere—tight bounds achieved at corners of inscribed cube.

## Problem 2.13: Tetrahedron Bimedians Equality

On triangular pyramid $ABCD$, $L, M, N, P, Q, R$ are midpoints of edges $AB, AC, AD, CD, BD, BC$ respectively. Let $\mathbf{b} = \vec{AB}$, $\mathbf{c} = \vec{AC}$, $\mathbf{d} = \vec{AD}$.

(i) Show that $\vec{LP} = \frac{1}{2}(-\mathbf{b} + \mathbf{c} + \mathbf{d})$.

(ii) Given that $\vec{MQ} = \frac{1}{2}(\mathbf{b} - \mathbf{c} + \mathbf{d})$ and $\vec{NR} = \frac{1}{2}(\mathbf{b} + \mathbf{c} - \mathbf{d})$, prove that $|AB|^2 + |AC|^2 + |AD|^2 + |BC|^2 + |BD|^2 + |CD|^2 = 4(|LP|^2 + |MQ|^2 + |NR|^2)$.

## Solution 2.13

**(i)** $L$ is midpoint of $AB$: $\vec{AL} = \frac{1}{2}\mathbf{b}$. $P$ is midpoint of $CD$: $\vec{AP} = \frac{1}{2}(\vec{AC} + \vec{AD}) = \frac{1}{2}(\mathbf{c} + \mathbf{d})$.

$$\vec{LP} = \vec{AP} - \vec{AL} = \frac{1}{2}(\mathbf{c} + \mathbf{d}) - \frac{1}{2}\mathbf{b} = \frac{1}{2}(-\mathbf{b} + \mathbf{c} + \mathbf{d})$$

**(ii) LHS:** Sum of squared edge lengths.

$$\begin{aligned}
\text{LHS} &= |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 + |\mathbf{c} - \mathbf{b}|^2 + |\mathbf{d} - \mathbf{b}|^2 + |\mathbf{d} - \mathbf{c}|^2 \\
&= b^2 + c^2 + d^2 + (c^2 + b^2 - 2\mathbf{b} \cdot \mathbf{c}) + (d^2 + b^2 - 2\mathbf{b} \cdot \mathbf{d}) \\
&\quad + (d^2 + c^2 - 2\mathbf{c} \cdot \mathbf{d}) \\
&= 3(b^2 + c^2 + d^2) - 2(\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d} + \mathbf{c} \cdot \mathbf{d})
\end{aligned}$$

**RHS:** Calculate $4|LP|^2$:

$$\begin{aligned}
|LP|^2 &= \frac{1}{4}(-\mathbf{b} + \mathbf{c} + \mathbf{d}) \cdot (-\mathbf{b} + \mathbf{c} + \mathbf{d}) \\
&= \frac{1}{4}(b^2 + c^2 + d^2 - 2\mathbf{b} \cdot \mathbf{c} - 2\mathbf{b} \cdot \mathbf{d} + 2\mathbf{c} \cdot \mathbf{d}) \\
4|LP|^2 &= b^2 + c^2 + d^2 - 2\mathbf{b} \cdot \mathbf{c} - 2\mathbf{b} \cdot \mathbf{d} + 2\mathbf{c} \cdot \mathbf{d}
\end{aligned}$$

Similarly:

$$\begin{aligned}
4|MQ|^2 &= b^2 + c^2 + d^2 - 2\mathbf{b} \cdot \mathbf{c} + 2\mathbf{b} \cdot \mathbf{d} - 2\mathbf{c} \cdot \mathbf{d} \\
4|NR|^2 &= b^2 + c^2 + d^2 + 2\mathbf{b} \cdot \mathbf{c} - 2\mathbf{b} \cdot \mathbf{d} - 2\mathbf{c} \cdot \mathbf{d}
\end{aligned}$$

Sum:

$$\begin{aligned}
\text{RHS} &= 4(|LP|^2 + |MQ|^2 + |NR|^2) \\
&= 3(b^2 + c^2 + d^2) + (-2 - 2 + 2)\mathbf{b} \cdot \mathbf{c} + (-2 + 2 - 2)\mathbf{b} \cdot \mathbf{d} \\
&\quad + (2 - 2 - 2)\mathbf{c} \cdot \mathbf{d} \\
&= 3(b^2 + c^2 + d^2) - 2(\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d} + \mathbf{c} \cdot \mathbf{d}) = \text{LHS} \quad \square
\end{aligned}$$

## Takeaways 2.13

**Bimedians** (segments joining midpoints of opposite edges) in a tetrahedron have remarkable properties. This identity relates all six edge lengths to the three bimedian lengths—a 3D analogue of the parallelogram law. The proof strategy: express everything in terms of base vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}$, expand dot products carefully, and verify algebraic cancellation. Note the symmetry in coefficients $(+2, -2, -2)$ cycling through the three cross terms. Such identities are useful in crystallography and structural analysis.

**Problem 2.14: Point-to-Line Distance and Sphere Intersection**

Consider the line $\ell$ with equation $\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$.

(i) Find the perpendicular distance from point $P(1,2,0)$ to the line $\ell$.

(ii) Find the shortest distance from the origin to the line $\ell$.

(iii) Determine the points where line $\ell$ intersects the sphere $x^2+y^2+z^2=4$.

**Solution 2.14**

**(i)** Let $A(1,0,1)$ be on $\ell$, direction $\mathbf{d} = \begin{pmatrix}0\\1\\1\end{pmatrix}$.

$$\vec{AP} = \begin{pmatrix}0\\2\\-1\end{pmatrix}$$

Cross product:

$$\vec{AP}\times\mathbf{d} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 2 & -1\\ 0 & 1 & 1\end{vmatrix} = \mathbf{i}(2+1) = 3\mathbf{i}$$

Distance: $d = \frac{|\vec{AP}\times\mathbf{d}|}{|\mathbf{d}|} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$ units.

**(ii)** Use projection method. Let $\vec{OA} = \begin{pmatrix}1\\0\\1\end{pmatrix}$. Projection of $\vec{OA}$ onto $\mathbf{d}$: $\text{proj}_{\mathbf{d}}\vec{OA} = \frac{\vec{OA}\cdot\mathbf{d}}{|\mathbf{d}|^2}\mathbf{d} = \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}$

Perpendicular component: $\vec{OA}_{\perp} = \vec{OA} - \text{proj}_{\mathbf{d}}\vec{OA} = \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$

Distance: $|\vec{OA}_{\perp}| = \sqrt{1+\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$ units.

**(iii)** Substitute line into sphere: $x=1$, $y=\lambda$, $z=1+\lambda$.

$$1+\lambda^2+(1+\lambda)^2 = 4$$

$$1+\lambda^2+1+2\lambda+\lambda^2 = 4$$

$$2\lambda^2+2\lambda-2=0 \implies \lambda^2+\lambda-1=0$$

$$\lambda = \frac{-1\pm\sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2}$$

Points: For $\lambda_1 = \frac{-1+\sqrt{5}}{2}$: $(1, \frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$
For $\lambda_2 = \frac{-1-\sqrt{5}}{2}$: $(1, \frac{-1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$

**Takeaways 2.14**

Three key distance techniques demonstrated: (1) **cross product** for point-to-line (geometric), (2) **projection** for point-to-line (algebraic), (3) **substitution** for line-sphere intersection. Both methods in (i)-(ii) should give same result (verify as practice). For intersections, substitute parametric equations into the surface equation to get a quadratic—the number of real solutions indicates the geometric relationship. The golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ appearing in $\lambda$ is a nice coincidence!

# 3 Part 2: Problems and Solutions (Concise + Hints)

Part 2 contains 45 additional problems distributed across three difficulty levels. Each problem includes an upside-down hint and a concise solution sketch to encourage independent thinking.

## 3.1 Part 2 Basic Problems

**Problem 3.1: Vector to Cartesian Line Equation**

What is the Cartesian equation of the line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \end{pmatrix}$?

**Hint:** Eliminate the parameter $\lambda$ by expressing it from one equation and substituting into the other.

**Solution 3.1: Sketch**

Write parametric equations: $x = 1 + 2\lambda$, $y = 3 + 4\lambda$. From the first equation, $\lambda = \frac{x-1}{2}$. Substitute into second: $y = 3 + 4 \cdot \frac{x-1}{2} = 3 + 2(x-1) = 2x + 1$. Therefore $y = 2x + 1$ or $2x - y + 1 = 0$.

**Problem 3.2: Sketch 3D Helix**

Sketch the curve described by $\mathbf{r}(t) = -5\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + t\,\mathbf{k}$ for $0 \leq t \leq 4\pi$.

**Hint:** The $x$ and $y$ components trace a circle while $z$ increases linearly with $t$.

**Solution 3.2: Sketch**

In the $xy$-plane, $x^2 + y^2 = 25\cos^2 t + 25\sin^2 t = 25$, which is a circle of radius 5. As $t$ increases from 0 to $4\pi$, the point moves around the circle twice while rising from $z = 0$ to $z = 4\pi$. This creates a helix spiraling upward around the $z$-axis.

## Problem 3.3: Angle Between 3D Vectors

Find the angle between vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ to the nearest degree.

**Hint:** Use $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

## Solution 3.3: Sketch

$\mathbf{a}\cdot\mathbf{b} = (2)(-3) + (0)(1) + (4)(2) = -6 + 8 = 2$. $|\mathbf{a}| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$, $|\mathbf{b}| = \sqrt{9+1+4} = \sqrt{14}$. Therefore $\cos\theta = \frac{2}{2\sqrt{5}\sqrt{14}} = \frac{1}{\sqrt{70}}$, so $\theta = \cos^{-1}(1/\sqrt{70}) \approx 83°$.

## Problem 3.4: Unit Vector in Direction

Given $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$, find a vector parallel to $\mathbf{u}$ with length 5 units.

**Hint:** Find the unit vector in the direction of $\mathbf{u}$, then scale by 5.

## Solution 3.4: Sketch

$|\mathbf{u}| = \sqrt{4+4+25} = \sqrt{33}$. Unit vector: $\hat{\mathbf{u}} = \frac{1}{\sqrt{33}}(2\mathbf{i} - 2\mathbf{j} - 5\mathbf{k})$. Required vector: $5\hat{\mathbf{u}} = \frac{5}{\sqrt{33}}(2\mathbf{i} - 2\mathbf{j} - 5\mathbf{k})$.

## Problem 3.5: Distance to Plane and Axis

Find the distance from point $P(1,4,3)$ to: (a) the $xz$-plane, (b) the $y$-axis.

**Hint:** (a) Distance to $xz$-plane is the $y$-coordinate. (b) Distance to $y$-axis uses $x$ and $z$ coordinates.

## Solution 3.5: Sketch

(a) The $xz$-plane has equation $y = 0$, so distance is $|y| = |4| = 4$. (b) The closest point on the $y$-axis is $(0,4,0)$, so distance is $\sqrt{1^2+3^2} = \sqrt{10}$.

## Problem 3.6: Unit Vector Perpendicular to Two Vectors

Find a unit vector perpendicular to both $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{k}$.

**Hint:**Use the cross product $\mathbf{u} \times \mathbf{v}$, then normalize.

## Solution 3.6: Sketch

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{i}(1) - \mathbf{j}(1) + \mathbf{k}(-1) = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Magnitude: $|\mathbf{w}| = \sqrt{1+1+1} = \sqrt{3}$.
Unit vector: $\frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$.

## Problem 3.7: Point on Line

Find $a$ and $b$ such that $(a, 1, b)$ lies on the line through $(0, 2, 1)$ and $(2, 7, 4)$.

**Hint:**Find parametric equations using direction vector $(2, 5, 3)$, then solve for the parameter when $y = 1$.

## Solution 3.7: Sketch

Direction vector: $(2, 5, 3)$. Parametric equations: $x = 2t$, $y = 2 + 5t$, $z = 1 + 3t$. When $y = 1$: $2 + 5t = 1 \Rightarrow t = -1/5$. Therefore $a = 2(-1/5) = -2/5$ and $b = 1 + 3(-1/5) = 2/5$.

## Problem 3.8: Equal Magnitude Perpendicular Vectors

Prove that if $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors of equal magnitude, then $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$.

**Hint:**Show that $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = 0$.

## Solution 3.8: Sketch

Expand the dot product: $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 - |\mathbf{v}|^2$. Since $|\mathbf{u}| = |\mathbf{v}|$, this equals 0, proving perpendicularity.

**Problem 3.9: Direction Cosines**

Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ for a position vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ making angles $\alpha, \beta, \gamma$ with the axes.

**Hint:** Use dot products with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to find the direction cosines.

**Solution 3.9: Sketch**

$\cos\alpha = \frac{a}{|\mathbf{r}|}$, $\cos\beta = \frac{b}{|\mathbf{r}|}$, $\cos\gamma = \frac{c}{|\mathbf{r}|}$. Therefore $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2+b^2+c^2}{|\mathbf{r}|^2} = \frac{|\mathbf{r}|^2}{|\mathbf{r}|^2} = 1$.

**Problem 3.10: Line Intersection by Components**

Find the intersection point of lines $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$.

**Hint:** Equate components and solve the system for $\lambda_1$ and $\lambda_2$.

**Solution 3.10: Sketch**

Equating: $3 + \lambda_1 = 3 - 2\lambda_2$, $-1 + 2\lambda_1 = -6 + \lambda_2$, $7 + \lambda_1 = 2 + 3\lambda_2$. From first equation: $\lambda_1 = -2\lambda_2$. Substitute into second: $-1 - 4\lambda_2 = -6 + \lambda_2 \Rightarrow \lambda_2 = 1$, so $\lambda_1 = -2$. Check third equation: $7 - 2 = 5 = 2 + 3$ ✓. Intersection point: $(1, -5, 5)$.

**Problem 3.11: Multiple Choice: Cartesian Equation**

What is the Cartesian equation of $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \end{pmatrix}$? Options: A. $2y + x = 7$, B. $y - 2x = -5$, C. $y + 2x = 5$, D. $2y - x = -1$.

**Hint:** Eliminate $\lambda$ from $x = 1 - 2\lambda$ and $y = 3 + 4\lambda$.

**Solution 3.11: Sketch**

From $x = 1 - 2\lambda$, get $\lambda = \frac{1-x}{2}$. Substitute: $y = 3 + 4 \cdot \frac{1-x}{2} = 3 + 2(1-x) = 5 - 2x$, giving $y + 2x = 5$. Answer: C.

## Problem 3.12: Closest Point on Line to Origin

Find the point on line $\mathbf{r} = \begin{pmatrix}1\\4\\6\end{pmatrix} + \lambda \begin{pmatrix}2\\1\\1\end{pmatrix}$ closest to the origin.

**Hint:** The closest point occurs when $\mathbf{r}$ is perpendicular to the direction vector.

## Solution 3.12: Sketch

Let $\mathbf{r} = \begin{pmatrix}1+2\lambda\\4+\lambda\\6+\lambda\end{pmatrix}$. For closest point: $\mathbf{r}\cdot \begin{pmatrix}2\\1\\1\end{pmatrix} = 0$. This gives $2(1+2\lambda)+(4+\lambda)+(6+\lambda) = 0 \Rightarrow 6\lambda + 12 = 0 \Rightarrow \lambda = -2$. Point: $(-3, 2, 4)$.

## Problem 3.13: Unit Vector Perpendicular to Two

Find a unit vector perpendicular to $\begin{bmatrix}3\\-2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\4\\-1\end{bmatrix}$.

**Hint:** Compute the cross product and normalize the result.

## Solution 3.13: Sketch

Cross product: $\mathbf{a} \times \mathbf{b} = \begin{bmatrix}(-2)(-1)-(1)(4)\\(1)(1)-(3)(-1)\\(3)(4)-(-2)(1)\end{bmatrix} = \begin{bmatrix}-2\\4\\14\end{bmatrix}$. Magnitude: $\sqrt{4+16+196} = 6\sqrt{6}$. Unit vector: $\frac{1}{6\sqrt{6}}\begin{bmatrix}-2\\4\\14\end{bmatrix} = \frac{1}{3\sqrt{6}}\begin{bmatrix}-1\\2\\7\end{bmatrix}$.

## Problem 3.14: Perpendicular Dot Product Proof

Prove that for non-zero vectors $\mathbf{a}, \mathbf{b}$, the equation $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2$ holds only if $\mathbf{a} \perp \mathbf{b}$.

**Hint:** Expand the left side and compare with the right side.

**Solution 3.14: Sketch**

Expanding: $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2+2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2$. Comparing with RHS: $|\mathbf{a}|^2+2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2 = |\mathbf{a}|^2+|\mathbf{b}|^2 \Rightarrow 2\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow \mathbf{a}\cdot\mathbf{b} = 0$, which means $\mathbf{a} \perp \mathbf{b}$.

## 3.2 Part 2 Medium Problems

**Problem 3.15: Line Intersection in 3D**

Find the intersection point of lines $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$.

**Hint:**Set up three equations by equating components, then solve for parameters.

**Solution 3.15: Sketch**

Equating components: $3+\lambda = 1-2\mu$, $-1+2\lambda = -6+\mu$, $7+\lambda = 2+3\mu$. From equations 1 and 3: $\lambda = -2\mu$ and $\lambda = -5+3\mu$. Solving: $\mu = 1$, $\lambda = -2$. Verify in equation 2. Intersection point: $(1, -5, 5)$.

**Problem 3.16: Perpendicular Intersecting Lines**

Two lines intersect and are perpendicular.

Find values of $p$ and $q$ for the line $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ q \\ -1 \end{pmatrix}$.

**Hint:**Use perpendicularity condition (dot product $= 0$) and intersection condition (solve system).

**Solution 3.16: Sketch**

Direction vectors must be perpendicular: $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$. Additionally, equate position vectors at intersection and solve the resulting system for $p$ and $q$. The specific values depend on the second line equation (not fully shown in sample).

**Problem 3.17: Tetrahedron Collinearity**

Given tetrahedron with vertices and specific vector relationships, prove collinearity of certain points using linear combinations of non-parallel vectors.

**Hint:** Express vectors in terms of base vectors and use uniqueness of coefficients for non-parallel vectors.

## Solution 3.17: Sketch

Show that if $\lambda\mathbf{a}+\mu\mathbf{b} = \vec{0}$ for non-parallel $\mathbf{a}, \mathbf{b}$, then $\lambda = \mu = 0$. Use this to prove position relationships, showing that $\overrightarrow{BL} = \frac{4}{7}\overrightarrow{BC}$ by equating coefficients.

## Problem 3.18: Parallelogram Proof

Show that quadrilateral $CDFE$ is a parallelogram given that $ABCD$ and $ABEF$ are parallelograms.

**Hint:** Use the fact that opposite sides of parallelograms are equal vectors.

## Solution 3.18: Sketch

From parallelogram $ABCD$: $\overrightarrow{AB} = \overrightarrow{DC}$. From parallelogram $ABEF$: $\overrightarrow{AB} = \overrightarrow{FE}$. Therefore $\overrightarrow{DC} = \overrightarrow{FE}$, which means $\overrightarrow{CD} = \overrightarrow{EF}$, proving $CDFE$ is a parallelogram.

## Problem 3.19: Force Vector Analysis

Force $\mathbf{F}_1$ has magnitude 12 N in direction $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Show $\mathbf{F}_1 = 8\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}$, find resultant with $\mathbf{F}_2 = -6\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$, and compute work done.

**Hint:** Find unit vector in given direction, multiply by magnitude, then add forces vectorially.

## Solution 3.19: Sketch

Direction vector has magnitude $\sqrt{4+4+1} = 3$. Unit vector: $\frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. Force: $12 \cdot \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 8\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}$. Resultant: $\mathbf{F}_3 = 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$. Work: $\mathbf{F}_3 \cdot \mathbf{d} = 2 + 4 + 16 = 22$.

## Problem 3.20: Double Angle with Vectors

For vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$ with acute angle $\theta$, find $\sin 2\theta$.

**Hint:** Find $\cos\theta$ from dot product, then $\sin\theta$ from Pythagorean identity, and use $\sin 2\theta = 2\sin\theta\cos\theta$.

### Solution 3.20: Sketch

$|\mathbf{a}| = 3$, $|\mathbf{b}| = 6$, $\mathbf{a}\cdot\mathbf{b} = 2$. Thus $\cos\theta = \frac{2}{18} = \frac{1}{9}$. Then $\sin\theta = \sqrt{1-\frac{1}{81}} = \frac{4\sqrt{5}}{9}$. Therefore $\sin 2\theta = 2\cdot\frac{4\sqrt{5}}{9}\cdot\frac{1}{9} = \frac{8\sqrt{5}}{81}$.

### Problem 3.21: Vector Projection

Find the integer value of $m$ such that the projection of $\mathbf{a} = 2\mathbf{i}-3\mathbf{j}+\mathbf{k}$ onto $\mathbf{b} = \mathbf{i}+m\mathbf{j}-\mathbf{k}$ equals $-\frac{11}{18}\mathbf{b}$.

**Hint:** Use projection formula: $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$.

### Solution 3.21: Sketch

$\mathbf{a}\cdot\mathbf{b} = 2-3m-1 = 1-3m$, $|\mathbf{b}|^2 = 1+m^2+1 = m^2+2$. Set $\frac{1-3m}{m^2+2} = -\frac{11}{18}$. Cross-multiply: $18(1-3m) = -11(m^2+2)$, giving $11m^2-54m+40 = 0$. Factor: $(11m-10)(m-4) = 0$. Since $m$ is an integer, $m = 4$.

### Problem 3.22: Perpendicular Vectors Condition

For $\mathbf{u} = -2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{v} = p\mathbf{i}+\mathbf{j}+2\mathbf{k}$, find values of $p$ such that $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are perpendicular.

**Hint:** Use $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$.

### Solution 3.22: Sketch

$|\mathbf{u}|^2 = 4+1+9 = 14$. $|\mathbf{v}|^2 = p^2+1+4 = p^2+5$. Setting equal: $14 = p^2+5 \Rightarrow p^2 = 9 \Rightarrow p = \pm 3$.

## Problem 3.23: Parallel and Perpendicular Lines

For lines with direction vectors $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} a-2 \\ -7 \\ 7 \end{pmatrix}$, find $a$ if they are: (a) parallel, (b) perpendicular.

**Hint:** Parallel: direction vectors are scalar multiples. Perpendicular: dot product equals zero.

## Solution 3.23: Sketch

(a) For parallel: $\frac{3}{a-2} = \frac{-3}{-7} = \frac{3}{7}$. Solving: $3 \cdot 7 = 3(a-2) \Rightarrow a = 9$. (b) For perpendicular: $3(a-2) + (-3)(-7) + 3(7) = 0 \Rightarrow 3a + 36 = 0 \Rightarrow a = -12$.

## Problem 3.24: Angle BCD Using Dot Product

Given position vectors $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{d} = a\mathbf{i} - 2\mathbf{j}$, and angle $BCD = \frac{\pi}{3}$, find $a$.

**Hint:** Find vectors $\overrightarrow{CB}$ and $\overrightarrow{CD}$, use dot product formula with $\cos(\pi/3) = 1/2$.

## Solution 3.24: Sketch

$\overrightarrow{CB} = -\mathbf{i} + \mathbf{k}$, $\overrightarrow{CD} = (a-2)\mathbf{i} - \mathbf{j} - \mathbf{k}$. Dot product: $-(a-2) - 1 = 1 - a$. Magnitudes: $|\overrightarrow{CB}| = \sqrt{2}$, $|\overrightarrow{CD}| = \sqrt{(a-2)^2 + 2}$. Equation: $1 - a = \frac{\sqrt{2}\sqrt{(a-2)^2+2}}{2}$. Squaring and solving: $a^2 = 4$, but checking sign constraint gives $a = -2$.

## Problem 3.25: Direction Cosines Sum

Prove that for a position vector making angles with coordinate axes, the sum of squares of direction cosines equals 1.

**Hint:** Express direction cosines as ratios of components to magnitude.

## Solution 3.25: Sketch

$\cos\alpha = \frac{a}{|\mathbf{r}|}$, $\cos\beta = \frac{b}{|\mathbf{r}|}$, $\cos\gamma = \frac{c}{|\mathbf{r}|}$. Sum: $\frac{a^2+b^2+c^2}{|\mathbf{r}|^2} = \frac{|\mathbf{r}|^2}{|\mathbf{r}|^2} = 1$.

## Problem 3.26: Section Formula Proof

Prove that if $C$ divides $AB$ in ratio $m : n$, then $\overrightarrow{OC} = \frac{m\mathbf{a}+n\mathbf{b}}{m+n}$ where $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$.

**Hint:** Use $\overrightarrow{AC} = \frac{n}{m+n}\overrightarrow{AB}$ and vector addition.

## Solution 3.26: Sketch

Since $\frac{CB}{AC} = \frac{m}{n}$, we have $\overrightarrow{AC} = \frac{n}{m+n}(\mathbf{b}-\mathbf{a})$. Then $\overrightarrow{OC} = \mathbf{a}+\overrightarrow{AC} = \mathbf{a}+\frac{n}{m+n}(\mathbf{b}-\mathbf{a}) = \frac{m\mathbf{a}+n\mathbf{b}}{m+n}$.

## Problem 3.27: Skew or Intersecting Lines

Determine whether lines $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -11 \\ 3 \end{pmatrix}$ are skew or intersecting.

**Hint:** Check if direction vectors are parallel. If not, solve system to see if it's consistent.

## Solution 3.27: Sketch

Direction vectors not parallel (check ratios). Solve system: from equations 1 and 3, find $\lambda = 6$, $\mu = 3$. Check equation 2: $4(6) + 11(3) = 57 \neq 9$. System inconsistent, so lines are skew.

## Problem 3.28: Line Through Points, Intersection Check

Line passes through $A(1, 3, -2)$ and $B(2, -1, 5)$. Does $C(3, 4, 9)$ lie on the line? Does it intersect another given line?

**Hint:** Find direction vector, write parametric equations, check if $C$ satisfies them.

## Solution 3.28: Sketch

Direction: $\overrightarrow{AB} = \mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$. Line: $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$. For $C$: $x$-component gives $\lambda = 2$, but $y$-component gives $-5 \neq 4$, so $C$ not on line. For intersection with second line, set up system and check consistency.

**Problem 3.29: Line-Sphere Intersection Angle**

Find intersection points of line and sphere with center $O$ and radius $\sqrt{10}$, then find angle $\angle AOB$ between the two intersection points.

**Hint:** Substitute parametric equations into sphere equation, solve quadratic for parameter values.

**Solution 3.29: Sketch**

Substitute line equations into $x^2 + y^2 + z^2 = 10$. Get quadratic in $\lambda$, solve to find $\lambda = 1$ and $\lambda = -2$. Points: $A(1, 3, 0)$ and $B(1, 0, 3)$. Find $\cos\theta = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|} = \frac{1}{10}$, so $\theta \approx 84°$.

**Problem 3.30: Linear Combination of Vectors**

Express $\mathbf{u} = 5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$ as $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are given non-parallel vectors.

**Hint:** Set up system of three equations by equating components, solve for $\lambda, \mu, \nu$.

**Solution 3.30: Sketch**

From component equations: $2\lambda + \mu - \nu = 5$, $3\lambda - \mu + 2\nu = 5$, $\lambda + 2\mu - \nu = 5$. Solve systematically: subtract equations to eliminate variables. Find $\lambda = \mu = \frac{15}{8}$, $\nu = \frac{5}{8}$.

**Problem 3.31: Parallelogram Fourth Vertex**

Three vertices of parallelogram are $O(0, 0, 0)$, $A(2, 2, 1)$, $B(1, 2, 2)$. Find possible positions of fourth vertex.

**Hint:** Consider three cases: which point is opposite to which, giving three parallelograms.

**Solution 3.31: Sketch**

Case 1: $C$ opposite to $O$: $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = (3, 4, 3)$. Case 2: $C$ opposite to $B$: $\overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OB} = (1, 0, -1)$. Case 3: $C$ opposite to $A$: $\overrightarrow{OC} = \overrightarrow{OB} - \overrightarrow{OA} = (-1, 0, 1)$.

## 3.3 Part 2 Advanced Problems

**Problem 3.32: Triangle Inequality and Cauchy-Schwarz**

For point $P(x, y, z)$ on unit sphere centered at origin, prove: (i) $|x| + |y| + |z| \geq 1$, (ii) Cauchy-Schwarz inequality, (iii) $|x| + |y| + |z| \leq \sqrt{3}$.

**Hint:**Use triangle inequality for part (i) and Cauchy-Schwarz with vector $(1, 1, 1)$ for part (iii).

**Solution 3.32: Sketch**

(i) By triangle inequality: $1 = |\mathbf{r}| \leq |x| + |y| + |z|$. (ii) From dot product: $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. (iii) Apply Cauchy-Schwarz with $\mathbf{a} = (|x|, |y|, |z|)$, $\mathbf{b} = (1, 1, 1)$: $|x| + |y| + |z| \leq \sqrt{x^2 + y^2 + z^2}\sqrt{3} = \sqrt{3}$.

**Problem 3.33: Tetrahedron Vector Collinearity**

In tetrahedron with vertices and given vector relations, prove points are collinear using unique representation with non-parallel vectors.

**Hint:**Show that if $\lambda\mathbf{u} + \mu\mathbf{v} = \vec{0}$ for non-parallel vectors, then both coefficients must be zero.

**Solution 3.33: Sketch**

Prove uniqueness: if $\lambda\mathbf{u} + \mu\mathbf{v} = \vec{0}$ and vectors non-parallel, then $\lambda = \mu = 0$. Use this to show $\overrightarrow{BL} = \frac{4}{7}\overrightarrow{BC}$ by expressing $\overrightarrow{OR}$ in two ways via different paths and equating coefficients of non-parallel vectors $\overrightarrow{SB}$ and $\overrightarrow{SC}$.

**Problem 3.34: Bimedians of Tetrahedron**

Show that opposite edges of tetrahedron satisfy certain equality relationship involving midpoint connections.

**Hint:**Express midpoint vectors in terms of vertex position vectors.

## Solution 3.34: Sketch

Let vertices be $A, B, C, D$ with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Midpoints of edges have position vectors like $\frac{\mathbf{a}+\mathbf{b}}{2}$. Show that vectors connecting midpoints of opposite edges have equal magnitudes by expressing them in terms of vertex vectors and using algebraic manipulation.

## Problem 3.35: Triangle Intersection Ratios

In triangle with specific point constructions, prove that intersection point divides segments in particular ratios.

**Hint:** Express position vectors through multiple paths and use uniqueness of representation.

## Solution 3.35: Sketch

Use vector methods to express $\overrightarrow{OR}$ via different routes. Equate coefficients of non-parallel base vectors to find parameters. Show systematic approach leads to specific ratio like $\overrightarrow{BL} = \frac{4}{7}\overrightarrow{BC}$. Verify point $P$ lies on line $AL$ by showing $\overrightarrow{AP} = -14\overrightarrow{AL}$.

## Problem 3.36: Circle Intersection of Sets

Two sets defined by perpendicularity condition and equidistance condition. Show intersection is a circle and find its radius.

**Hint:** $S_1$ is sphere with diameter $AB$; $S_2$ is perpendicular bisecting plane of segment $AM$.

## Solution 3.36: Sketch

$S_1$: sphere centered at midpoint $M$ with radius $\frac{|\overrightarrow{AB}|}{2}$. $S_2$: plane perpendicular to $AM$ at its midpoint. Distance from sphere center to plane: $d = \frac{|\overrightarrow{AB}|}{4}$. Circle radius: $r = \sqrt{R^2 - d^2} = \sqrt{\frac{|\overrightarrow{AB}|^2}{4} - \frac{|\overrightarrow{AB}|^2}{16}} = \frac{\sqrt{3}|\overrightarrow{AB}|}{4}$. Answer: D.

## Problem 3.37: Complex Numbers and Centroid

Using vectors and complex numbers on unit circle, prove that centroid of three points is never a cube root of their product.

**Hint:** Centroid has modulus less than 1, while cube roots of product have modulus exactly 1.

**Solution 3.37: Sketch**

Points $x, w, z$ on unit circle: $|x| = |w| = |z| = 1$. Centroid: $G = \frac{x+w+z}{3}$. Any cube root $K$ of $xwz$ satisfies $|K|^3 = |xwz| = 1$, so $|K| = 1$. By triangle inequality (strict since points distinct): $|x + w + z| < 3$, thus $|G| < 1$. Therefore $G \neq K$.

**Problem 3.38: Parallelogram Intersection Ratio**

In parallelogram $OPQR$ with diagonals intersecting at $T$, prove $T$ divides diagonal $PR$ in ratio $2 : 1$.

**Hint:** Express $\overrightarrow{OT}$ via midpoint $S$ of $QR$ using collinearity, then via diagonal $PR$.

**Solution 3.38: Sketch**

Find $\overrightarrow{OS} = \mathbf{r} + \frac{1}{2}\mathbf{p}$. Since $T$ on $OS$: $\overrightarrow{OT} = \lambda(\mathbf{r} + \frac{\mathbf{p}}{2})$. Since $T$ on $PR$: coefficients sum to 1. Solve: $\lambda(\frac{1}{2}) + \lambda = 1 \Rightarrow \lambda = \frac{2}{3}$. Thus $\overrightarrow{OT} = \frac{2}{3}\mathbf{r} + \frac{1}{3}\mathbf{p}$, giving ratio $PT : TR = 2 : 1$.

**Problem 3.39: Pyramid Centroid**

For pyramid with square base $ABCD$ and apex $S$, show sum of vectors from center of base to all vertices is zero, then find centroid $G$ of pyramid.

**Hint:** Diagonals of square bisect at $H$. Use symmetry to show $\overrightarrow{HA} + \overrightarrow{HC} = \vec{0}$ and $\overrightarrow{HB} + \overrightarrow{HD} = \vec{0}$.

**Solution 3.39: Sketch**

(i) $H$ is midpoint of diagonals: $\overrightarrow{HA} = -\overrightarrow{HC}$ and $\overrightarrow{HB} = -\overrightarrow{HD}$, so sum is zero. (ii) From $\sum \overrightarrow{GA_i} = \vec{0}$, get $4\overrightarrow{GH} + \overrightarrow{GS} = \vec{0}$. Therefore $\overrightarrow{HG} = \frac{1}{5}\overrightarrow{HS}$, giving $\lambda = \frac{1}{5}$.

## Problem 3.40: Point-to-Line Distance via Quadratic

Show that $|\overrightarrow{AB}|^2 = 6p^2 - 24p + 125$ where $B$ is on line through origin. Find shortest distance from $A$ to line.

**Hint:**Minimize the quadratic by completing the square or using calculus.

## Solution 3.40: Sketch

Express $\overrightarrow{AB} = \begin{pmatrix} p-8 \\ p+6 \\ 2p-5 \end{pmatrix}$. Square and sum: $|\overrightarrow{AB}|^2 = (p-8)^2 + (p+6)^2 + (2p-5)^2 = 6p^2 - 24p + 125$. Complete square: $6(p-2)^2 + 101$. Minimum at $p = 2$ gives $|\overrightarrow{AB}|^2_{\text{min}} = 101$, so distance $= \sqrt{101}$.

## Problem 3.41: Triangle Intersection with Complex Centroid

Points on triangle with various constructions. Show centroid lies on specific line and prove relationship.

**Hint:**Express vectors through different routes and use coefficient uniqueness.

## Solution 3.41: Sketch

Given $\overrightarrow{SK} = \frac{1}{4}\overrightarrow{SB} + \frac{1}{3}\overrightarrow{SC}$ and $L$ on $BC$. Show $\overrightarrow{OR} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ by expressing via paths through $P$ and $Q$. Find $k = \frac{1}{6}$. Centroid $G$ position: $\overrightarrow{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ lies on line $MC$ where $M$ is midpoint of $AB$.

## Problem 3.42: Line-Sphere Intersection Points

Find intersection points of line $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and sphere $(x-1)^2 + (y-3)^2 + (z+4)^2 = 81$.

**Hint:**Substitute parametric equations into sphere equation and solve resulting quadratic.

## Solution 3.42: Sketch

Parametric: $x = 1 + t$, $y = 3 + 2t$, $z = -4 + 2t$. Substitute: $(t)^2 + (2t)^2 + (2t)^2 = 81 \Rightarrow 9t^2 = 81 \Rightarrow t = \pm 3$. Points: $(4, 9, 2)$ when $t = 3$ and $(-2, -3, -10)$ when $t = -3$.

## Problem 3.43: Line Tangent to Sphere

Show line intersects sphere at exactly one point, proving tangency. Find all possible tangent points for lines parallel to given line.

**Hint:** Tangent line produces repeated root (discriminant = 0) in quadratic equation.

## Solution 3.43: Sketch

Substitute line equations into sphere equation. Get quadratic: $17\lambda_1^2 - 136\lambda_1 + 272 = 0 \Rightarrow (\lambda_1 - 4)^2 = 0$. Repeated root means tangency. Tangent point: $(3, -6, 5)$. For parallel tangent lines: radius vector perpendicular to direction vector forms a circle where plane $x + y + 2z - 1 = 0$ intersects sphere.

## Problem 3.44: Regular Octagon Vector Sum

In regular octagon with side length 4, find magnitude of sum of vectors from midpoint of one side to all vertices.

**Hint:** Use symmetry: sum of vectors from center to vertices is zero. Express via center.

## Solution 3.44: Sketch

$\sum \overrightarrow{AP_i} = 8\overrightarrow{AO}$ since $\sum \overrightarrow{OP_i} = \vec{0}$ by symmetry. Need apothem $|AO|$: in right triangle with half-side $= 2$ and half-central-angle $= 22.5°$, get $|AO| = 2\cot(22.5°) = 2(\sqrt{2} + 1)$. Therefore $|\sum \overrightarrow{AP_i}| = 8 \cdot 2(\sqrt{2} + 1) = 16(\sqrt{2} + 1)$.

## Problem 3.45: Two Lines Tangent to Sphere

Show two lines intersect and that intersection point is where first line is tangent to given sphere.

**Hint:** Find intersection by solving system. For tangency, show quadratic has discriminant zero.

## Solution 3.45: Sketch

Solve system to find intersection: $\lambda_1 = 4$, $\lambda_2 = -2$ gives point $(3, -6, 5)$. Substitute first line into sphere equation: get $17\lambda_1^2 - 136\lambda_1 + 272 = 0$, which factors as $(\lambda_1 - 4)^2 = 0$. Discriminant zero confirms tangency at $(3, -6, 5)$.

**Problem 3.46: Constant Expression with Position Vectors**

Given $\overrightarrow{KB} = 2\overrightarrow{AK}$, prove that $2|\overrightarrow{MA}|^2 + |\overrightarrow{MB}|^2 - 3|\overrightarrow{MK}|^2$ is constant for any point $M$.

**Hint:** Express vectors in terms of $\overrightarrow{MK}$ and use given relationship between $A$, $K$, $B$.

**Solution 3.46: Sketch**

Write $\overrightarrow{MA} = \overrightarrow{MK} + \overrightarrow{KA}$ and $\overrightarrow{MB} = \overrightarrow{MK} + \overrightarrow{KB}$. Expand magnitudes: $E = 2|\overrightarrow{MK} + \overrightarrow{KA}|^2 + |\overrightarrow{MK} + \overrightarrow{KB}|^2 - 3|\overrightarrow{MK}|^2$. Coefficient of $|\overrightarrow{MK}|^2$ is zero. Using $\overrightarrow{KB} = -2\overrightarrow{KA}$, the dot product term vanishes: $2\overrightarrow{KA} + \overrightarrow{KB} = \vec{0}$. Result: $E = 2|\overrightarrow{KA}|^2 + |\overrightarrow{KB}|^2$ (constant).

**Problem 3.47: Collinear Points Ratio Determination**

In triangle $ABC$ with $C$ on $OB$ at ratio $3 : 1$, $P$ on $AC$, and $Q$ on $AB$ such that $O, P, Q$ collinear. Find ratio $AQ : QB$.

**Hint:** Express $\overrightarrow{OQ}$ via collinearity with $\overrightarrow{OP}$ and via position on line $AB$. Equate coefficients.

**Solution 3.47: Sketch**

$\overrightarrow{OC} = \frac{3}{4}\mathbf{b}$. Assuming $P$ midpoint of $AC$: $\overrightarrow{OP} = \frac{1}{2}\mathbf{a} + \frac{3}{8}\mathbf{b}$. Collinearity: $\overrightarrow{OQ} = \lambda(\frac{1}{2}\mathbf{a} + \frac{3}{8}\mathbf{b})$. On $AB$: coefficients sum to 1, so $\frac{\lambda}{2} + \frac{3\lambda}{8} = 1 \Rightarrow \lambda = \frac{8}{7}$. Thus $\overrightarrow{OQ} = \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{b}$, giving $AQ : QB = 3 : 4$.

# 4 Conclusion

Vectors are a powerful tool in HSC Mathematics Extension 2, unifying geometry, algebra, and analytic methods. Mastery comes from understanding both the algebraic manipulation and the geometric intuition behind each technique. Use these 60 problems to develop fluency across all vector topics, from basic computations to sophisticated proofs.

Best of luck with your studies and the HSC examination!


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